MATH 5302 Elementary Analysis II - Homework 2

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Problem 1

Show that

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$$

is well-defined for $\alpha > 0$ and $\beta > 0$.

Definition 1. *The improper integral*

$$\int_0^a f(x)\,\mathrm{d}x$$

is <u>well-defined</u> *iff*

$$\lim_{\epsilon \to 0} \int_0^a f(x)\,\mathrm{d}x$$

exists.

Definition 2. *The* <u>gamma function</u> $\Gamma(\alpha)$ *is defined as*

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\,\mathrm{d}x$$

for $0 < \alpha < \infty$.

Definition 3. *The* <u>beta function</u> $B(\alpha, \beta)$ *is defined as*

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

for $\alpha > 0$ *and* $\beta > 0$.

Theorem 1. ***Limit Comparison Test:*** *Let* $f, g : [a, b) \to \mathbb{R}$ *be two functions such that (i)* $f(x)$ *and* $g(x)$ *are integrable on* $[a, A] \subset [a, b)$*, for* $a < A < b$*; (ii) There exists* $a \leq K \leq b$ *such that* $\lim_{x \to b^-} \frac{f(x)}{g(x)} = K$*. Then,*

a. If $0 < K < \infty$*, then* $\int_a^b g(x)\,\mathrm{d}x$ *converges iff* $\int_a^b f(x)$ *converges.*

b. If $K = 0$*, then* $\int_a^b g(x)$ *converges implies* $\int_a^b f(x)\,\mathrm{d}x$ *converges.*

c. If $K \infty 0$*, then* $\int_a^b g(x)$ *divergent implies* $\int_a^b f(x)\,\mathrm{d}x$ *divergent.*

Theorem 2. *The improper integral that defines the beta function*

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$$

is well-defined for $\alpha > 0$ *and* $\beta > 0$.

Proof. The integrand of $B(\alpha, \beta)$,

$$b(\alpha, \beta) = x^{\alpha-1}(1-x)^{\beta-1}$$

is not strictly bounded $\forall_{\alpha,\beta>0}$, but this is not necessary for convergence. $\forall \alpha, \beta \in [0, \infty)$ the $b(\alpha, \beta)$ is bounded. This makes $B(\alpha, \beta)$ a proper integral which is therefore convergent.
In the other case the integrand is not bounded, but the improper integral still converges. $\forall_{\alpha \in (0,1)}$ then $b(\alpha, \beta)$ is unbounded at $x = 0$. Similarly, $\forall_{\beta \in (0,1)}$ then $b(\alpha, \beta)$ is unbounded at $x = 1$.
The beta function can instead be split up into two parts:

$$B(\alpha, \beta) = \int_0^c x^{\alpha-1}(1-x)^{\beta-1} \, \mathrm{d}x + \int_c^1 x^{\alpha-1}(1-x)^{\beta-1} \, \mathrm{d}x$$

where $c \in (0, 1)$.
For the first improper integral, $\int_0^c x^{\alpha-1}(1-x)^{\beta-1} \, \mathrm{d}x$ a discontinuity exists at $x = 0$ for $\alpha \in (0, 1)$. Using the Limit Comparison Test from Theorem 1 with $g(x) = x^{\alpha-1}$,

$$\begin{aligned} \lim_{x \to 0^+} \frac{f(x)}{g(x)} &= \lim_{x \to 0^+} \frac{x^{\alpha-1}(1-x)^{\beta-1}}{x^{\alpha-1}} \\ &= \lim_{x \to 0^+} (1-x)^{\beta-1} \\ &= 1 \neq 0 \end{aligned}$$

Which then implies that $\int_0^c x^{\alpha-1}(1-x)^{\beta-1} \, \mathrm{d}x$ converges $\forall_{\alpha,\beta>0}$.
For the second improper integral, $\int_c^1 x^{\alpha-1}(1-x)^{\beta-1} \, \mathrm{d}x$ a discontinuity exists at $x = 1$ for $\beta \in (0, 1)$. Using the Limit Comparison Test from Theorem 1 with $g(x) = (1-x)^{\beta-1}$,

$$\begin{aligned} \lim_{x \to 1^-} \frac{f(x)}{g(x)} &= \lim_{x \to 1^-} \frac{x^{\alpha-1}(1-x)^{\beta-1}}{(1-x)^{\beta-1}} \\ &= \lim_{x \to 1^-} x^{\alpha-1} \\ &= 1 \neq 0 \end{aligned}$$

Which then implies that $\int_c^1 x^{\alpha-1}(1-x)^{\beta-1} \, \mathrm{d}x$ converges $\forall_{\alpha,\beta>0}$.
Together, the convergence of $\int_0^c x^{\alpha-1}(1-x)^{\beta-1} \, \mathrm{d}x$ and $\int_c^1 x^{\alpha-1}(1-x)^{\beta-1} \, \mathrm{d}x$ implies that $B(\alpha, \beta)$ converges $\forall_{\alpha,\beta>0}$ and therefore $B(\alpha, \beta)$ is well defined. $\square$

Problem 2

Show that f if Riemann integrable on $[a, b]$, then

$$\lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

Definition 4. *Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$.*

a. *A <u>Partition</u> of $[a, b]$ is any ordered $P \subset [a, b]$ given as*

$$P = \{a = x_0 < x_1 < \cdots < x_n < b\}$$

b. *A <u>Mesh</u> of partition P, mesh(P), is the maximum length of the subintervals in P. (i.e) For $P = \{a = x_0 < x_1 < \cdots < x_n < b\}$,*

$$mesh(P) = \max[\{x_i - x_{i-1} \ : \ i = 1, 2, \ldots, n\}]$$

c. *A <u>Riemann Sum</u> of f associated with partition P, $S(f, P)$, is the sum defined as*

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

where the specific $x_i^* \in [x_{i-1}, x_i]$ is arbitrary.

d. *f is considered <u>Riemann Integrable</u> on $[a, b]$ if*

$$\exists_r \forall_{\epsilon>0} \exists_{\delta>0} \ : \ \forall_{S(f,P) \ : \ mesh(P)<\delta} \implies |S(f, P) - r| < \epsilon$$

where the number r is considered the <u>Riemann Integral</u> of f on $[a, b]$, $\mathcal{R}\int_a^b f$.

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$. f being Riemann integrable on $[a, b]$ implies that*

$$\lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

Proof. By the definition of a function being Riemann Integrable, Definition 4, it is known that f must be bounded. From this fact, the limit described will always exists and an asymptote at the boundary would not be a concern. Using the construction of Riemann sum itself,

$$\begin{aligned}\lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,\mathrm{d}x &= \lim_{\epsilon \to 0^+} r \ : \ \forall_{P=\{a=x_0<x_1<\cdots<x_n<b-\epsilon\}} \forall_{\epsilon_0>0} \exists_{\delta>0} \ : \ \forall_{S(f,P) \ : \ \mathrm{mesh}(P)<\delta} \implies |S(f, P) - r| < \epsilon_0 \\ &= r \ : \ \forall_{P=\{a=x_0<x_1<\cdots<x_n<b\}} \forall_{\epsilon_0>0} \exists_{\delta>0} \ : \ \forall_{S(f,P) \ : \ \mathrm{mesh}(P)<\delta} \implies |S(f, P) - r| < \epsilon_0\end{aligned}$$

$$\boxed{\lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,\mathrm{d}x = \mathcal{R}\int_a^b f = \int_a^b f(x)\,\mathrm{d}x}$$

$\square$

Problem 3

Evaluate $\int_0^1 (1-x^{\frac{2}{3}})^{\frac{3}{2}}\,\mathrm{d}x$. Hint: Express the integral in terms of the gamma function first.

Example 1. *Let*

$$F(x) = \int_0^1 (1-x^{\frac{2}{3}})^{\frac{3}{2}}\,\mathrm{d}x$$

This can be simplified using u-substitution. Let

$$u = x^{\frac{2}{3}}$$

then

$$\mathrm{d}u = \frac{2}{3}x^{-\frac{1}{3}}\,\mathrm{d}x$$

and

$$\mathrm{d}x = \frac{3}{2}x^{\frac{1}{3}}\,\mathrm{d}u$$

The bounds are found as

$$0 = u(a) = a^{\frac{2}{3}} \implies a = 0^{\frac{3}{2}} = 0$$

and

$$1 = u(b) = b^{\frac{2}{3}} \implies b = 1^{\frac{3}{2}} = 1$$

$$\begin{aligned} F(x) &= \int_0^1 \left(1-x^{\frac{2}{3}}\right)^{\frac{3}{2}} \left(\frac{3}{2}x^{\frac{1}{3}}\,\mathrm{d}u\right) \\ &= \frac{3}{2}\int_0^1 (1-u)^{\frac{3}{2}} u^{\frac{1}{2}}\,\mathrm{d}u \\ &= \frac{3}{2}\int_0^1 u^{\frac{3}{2}-1}(1-u)^{\frac{5}{2}-1}\,\mathrm{d}u \end{aligned}$$

which is of the form of the beta function as defined in Definition 3

$$\begin{aligned} &= \frac{3}{2}B(\frac{3}{2},\frac{5}{2}) \\ &= \frac{3}{2}\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(\frac{3}{2}+\frac{5}{2})} \\ &= \frac{3\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{2\Gamma(4)} \\ &= \frac{3\left(\frac{\sqrt{\pi}}{2}\right)\left(\frac{3\sqrt{\pi}}{4}\right)}{2(3!)} \\ &= \frac{\frac{9\pi}{8}}{(2)(3)(2)(1)} \\ &= \frac{9\pi}{96} \\ &\boxed{F(x) = \frac{3\pi}{32} \approx 0.29452} \end{aligned}$$

Problem 4

Let

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } 0 < x \leq 1; \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f is bounded and continuous on $[0, 1]$, but not of bounded variation on $[0, 1]$.

Definition 5. *$f : (a, b) \to \mathbb{R}$ is a* <u>bounded function</u> *iff*

$$\exists_{N\in\mathbb{R}} \;:\; \forall_{x\in(a,b)}|f(x)| < N$$

Definition 6. *$f : (S_1, d_1) \to (S_2, d_2)$ is a* <u>continuous function</u> *iff*

$$\forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x, y) < \delta \implies d_2(f(x), f(y)) < \epsilon$$

Definition 7. *For function $f : [a, b] \to \mathbb{R}$ and partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$:*

a. the <u>variaton of f over P</u> *is defined as*

$$V_a^b(f, P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

b. the <u>variaton of f from a to b</u> *is defined as*

$$V_a^b(f) = \sup_P V_a^b(f, P)$$

c. f is considered of <u>bounded variation</u> *on $[a, b]$ if $V_a^b(f)$ is finite.*

d. the <u>family of functions of bounded variaton on $[a, b]$</u> *is denoted as BV_a^b.*

Example 2. *Let*

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

a) $f(x)$ is bounded

Proof. For $x \in \{0\}$, $f(x) = 0$. For $x \in (0, 1]$,

$$f(x) = x\sin\left(\frac{1}{x}\right) \leq x(1) \leq 1$$

which is bounded. Therefore, $f(x)$ is bounded $\forall_{x\in[0,1]}$. □

b) $f(x)$ is continuous

Proof. For $x \in \{0\}$, $f(x) = 0$. This means that

$$\lim_{x\to 0^-} f(x) = 0$$

For $x \in (0, 1]$,

$$f(x) = x\sin\left(\frac{1}{x}\right)$$

which is continuous on $(0, 1]$. Additionally, this results in

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right) = 0$$

Therefore, $f(x)$ is continuous $\forall_{x\in[0,1]}$. □

c) $f(x)$ is not of bounded variation on $[0, 1]$.

Proof. In order for f to be of bounded variation on $[a, b]$, the variation

$$V_a^b(f) = \sup_P V_a^b(f, P)$$

must be finite. The existence of this bound can be demonstrated with the following counter-example: Let

$$\{a\}_k := \left\{ a_k = (2k+1)\frac{\pi}{2} \forall_{k=0,\dots,N-1} \right\}$$

For size N, define partition of $[0, 1]$

$$P_N = \left\{ 0 = x_0 = 0 < x_1 = \frac{1}{a_{N-1}} < x_2 = \frac{1}{a_{N-2}} < \dots < x_{n-1} = \frac{1}{a_0} < x_n = 1 \right\}$$

which can be used to construct a sequence, but that's not the point.
The variation $V_a^b(f, P_N)$ is finite only for bounded N. i.e. $\exists_{0 < M_N < \infty}$ that bounds the variation $V_a^b(f, P_N)$ for a given N:

$$\begin{aligned}
V_a^b(f, P_N) &= \sum_{i=1}^{N} |f(x_i) - f(x_i - 1)| \\
&= \left| \frac{1}{a_{N-1}} \sin(a_{N-1}) \right| + \sum_{i=2}^{N} \left| \frac{1}{a_{N-i}} \sin(a_{N-i}) - \frac{1}{a_{N-i-1}} \sin(a_{N-i-1}) \right| \\
&= \left| \frac{1}{a_{N-1}} \right| + \sum_{i=1}^{N} \left| \frac{1}{a_{N-i}} - \frac{1}{a_{N-i-1}} \right| |(1) - (-1)|^1 \\
&= \left| \frac{1}{a_{N-1}} \right| + 2 \sum_{i=1}^{N} \left| \frac{a_{N-i} - a_{N-i-1}}{a_{N-i} a_{N-i-1}} \right|
\end{aligned}$$

However, this variation over $[a, b]$ is since the sum does not converge as $N \to \infty$. □

Problem 5

Assume f is differentiable on $[a, b]$ with $|f'(x)| \leq M < \infty$ for $a \leq x \leq b$. Show that f is of bounded variation and $V_a^b(f) \leq M(b-a)$. (Hint: Use Mean Value Theorem)

Theorem 4. ***Mean Value Theorem:*** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). There exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 5. *Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. If the derivative is bounded $\exists_{M>0} : \forall_{x\in[a,b]} |f'(x)| \leq M < \infty$, then f will have a bounded variation with $V_a^b(f) \leq M(b-a)$.*

Proof. From Definition 7, we have the following: The variation of f associated with P is

$$V_a^b(f, P) = \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})|$$

In order for f to be of bounded variation on $[a, b]$, the variation

$$V_a^b(f) = \sup_P V_a^b(f, P)$$

must be finite.
We also refer to the principles underlying Theorem 4, which ultimately states that

$$\exists_{c\in(a,b)} \ : \ f'(c) = \frac{f(b) - f(a)}{b - a}$$

which represents the mean of the the derivative overall.
Since $f'(x)$ has a bound, $|f'(x)| \leq M$, $V_a^b(f, P)$ for a given P will also be bounded.

$$\begin{aligned}
V_a^b(f, P) &= \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| \\
&\leq \sum_{i=1}^{N} |M(x_i - xi - 1)| \\
&= \sum_{i=1}^{N} |M||x_i - xi - 1| \\
&= M \sum_{i=1}^{N} x_i - x_{i-1} \\
&= M(x_1 - x_0 + x_2 - x_1 + \cdots + x_{N-1} - x_{N-2} + x_N - x_{N-1}) \\
&= M(x_1 - x_1 + x_2 - x_2 + \cdots + x_{N-1} - x_{N-1} + x_N - x_0) \\
&= M(x_N - x_0) \\
&= M(b - a)
\end{aligned}$$

This means that

$$V_a^b(f, P) \leq M(b-a)$$

forall partitions of $[a, b]$. Therefore, by definition, the variation of f on $[a, b]$ is

$$\boxed{V_a^b(f) \leq M(b-a)}$$

□