MATH 5302 Elementary Analysis II - Homework 5

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Preliminaries

Definition 1. Darboux-Stieltjes Integral *Let $f : [a,b] \to \mathbb{R}$ and $\alpha : [a,b] \to \mathbb{R}$, with f bounded and α increasing on $[a,b]$. Let partition P be defined as*

$$P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

Let

$$M(f, [x_1, x_2]) = \sup_{x \in [x_1, x_2]} f(x)$$

and

$$m(f, [x_1, x_2]) = \inf_{x \in [x_1, x_2]} f(x)$$

a. The upper and lower Darboux-Stieltjes Sums *are defined*

$$U(f, \alpha, P) = \sum_{i=1}^{n} M(f, [x_{i-1}, x_i]) \cdot \Delta_i \alpha$$

and

$$L(f, \alpha, P) = \sum_{i=1}^{n} m(f, [x_{i-1}, x_i]) \cdot \Delta_i \alpha$$

respectively with

$$\Delta_i \alpha = \alpha(x_i) - \alpha(x_{i-1})$$

Note:

$$m(f, [a,b]) \cdot (\alpha(b) - \alpha(a)) \leq L(f, \alpha, P) \leq U(f, \alpha, P) \leq M(f, [a,b]) \cdot (\alpha(b) - \alpha(a))$$

b. The upper and lower Darboux-Stieltjes Integrals *are defined*

$$U(f, \alpha) = \inf_{P \text{ partition of } [a,b]} U(f, \alpha, P)$$

and

$$L(f, \alpha) = \sup_{P \text{ partition of } [a,b]} U(f, \alpha, P)$$

respectively.

Note:

$$L(f, \alpha) \leq L(f, \alpha, P) \leq U(f, \alpha, P) \leq U(f, \alpha)$$

for any P partition of $[a,b]$.

c. f is called <u>Darboux-Stieltjes Integrable</u> *with respect to α if and only if*

$$\forall_{\epsilon>0}\exists_{P=\{a=x_0<x_1<\cdots<x_{n-1}<x_n=b\}} : U(f,\alpha,P) - L(f,\alpha,P) < \epsilon$$

in which case the <u>Darboux-Stieltjes Integral</u> *with respect to α is defined as*

$$\mathcal{DS}\int_a^b f\,\mathrm{d}\alpha = U(f,\alpha) = L(f,\alpha)$$

<u>Note:</u> *If f is also continuous on $[a,b]$ then f is Riemann-Stieljes integrable which implies f is Darboux-Stieljes integrable.*

<u>Properties:</u> *When f is Darboux-Stieltjes integrable on $[a,b]$ and α is increasing on $[a,b]$ then*

a. $|f|$ is Darboux-Stieltjes integrable on $[a,b]$ and

$$\mathcal{DS}\int_a^b f\,\mathrm{d}\alpha \leq \mathcal{DS}\int_a^b |f|\,\mathrm{d}\alpha$$

b. f^2 is Darboux-Stieltjes integrable on $[a,b]$.

c. If g is also Darboux-Stieltjes integrable on $[a,b]$, then fg is Darboux-Stieltjes integrable on $[a,b]$.

d. For α_1 and α_2 also increasing on $[a,b]$ and f is Darboux-Stieltjes integrable with respect to α_1 and α_2, then f is Darboux-Stieltjes integrable with respect to α_1 and α_2. Additionally,

$$\begin{aligned}&\mathcal{DS}\int_a^b f(x)\,\mathrm{d}\alpha_1(x) + \mathcal{DS}\int_a^b f(x)\,\mathrm{d}\alpha_2(x)\\ =&\mathcal{DS}\int_a^b f(x)\,\mathrm{d}\alpha(x) + \alpha_2(x)\end{aligned}$$

e. For $a<c<b$, f is Darboux-Stieltjes integrable with respect to α on $[a,b]$ if and only if f is Darboux-Stieltjes integrable with respect to α on $[a,c]$ and $[c,b]$. Furthermore,

$$\mathcal{DS}\int_a^b f(x)\,\mathrm{d}\alpha(x) = \mathcal{DS}\int_a^c f(x)\,\mathrm{d}\alpha(x) + \mathcal{DS}\int_c^b f(x)\,\mathrm{d}\alpha(x)$$

Definition 2. ***Continuity:*** *Let $f:[a,b]\to\mathbb{R}$.*

a. f is <u>Lipschitz Continuous</u> *on $[a,b]$ if*

$$\exists_C : \forall_{x,y\in[a,b]}|f(x)-f(y)| \leq |x-y|$$

b. f is <u>Absolutely Continuous</u> *on $[a,b]$ if*

$$\forall_{\epsilon>0}\exists_{\delta>0}\forall_{\textit{finite collection } \{(x,x')\} \textit{ of nonoverlaping intervals}:\sum_{i=1}^n |x_i'-x_i|<\delta} \sum_{i=1}^n |f(x_i')-f(x_i)| < \epsilon$$

c. f is <u>uniformly continuous</u> *on $[a,b]$ if*

$$\forall_{\epsilon>0}\exists_{\delta>0} : (x,y\in[a,b]) \wedge \{|x-y|<\delta\} \implies |f(x)-f(y)| < \epsilon$$

d. f is <u>continuous</u> *on $[a,b]$ if f is continuous at all $x_0\in[a,b]$. i.e.*

$$\forall_{\epsilon>0}\exists_{\delta>0} : \forall_{x\in[a,b]} \wedge |x-x_0| < \delta \implies |f(x)-f(x_0)| < \epsilon$$

Problem 1

Let f be a real-valued bounded function on $[-1, 1]$. Let

$$\alpha(x) = \begin{cases} 0 & \text{if } -1 \leq x < 0; \\ 2 & \text{if } 0 \leq x \leq 1. \end{cases}$$

Assume f is Riemann-Stieljes integrable with respect to α on $[-1, 1]$. Show that

a. f is continuous at 0 from the left.

b. $\int_{-1}^{1} f(x)\, \mathrm{d}\alpha(x) = 2f(0)$.

a) f is continuous at 0 from the left

Example 1. *Let $f : [-1, 1] \to \mathbb{R}$ bounded. Let*

$$\alpha(x) = \begin{cases} 0 & -1 \leq x < 0 \\ 2 & 0 \leq x < 1 \end{cases}$$

If f is Riemann-Stieljes integrable w.r.t. α on $[-1, 1]$, then f is continuous at 0 from the left.

Proof. $\square$

Problem 2

Let f and α be real-valued bounded functions on $[a, b]$ and α is increasing. Let $L(f, \alpha)$ and $U(f, \alpha)$ represents the lower and upper Darboux-Stieltjes integral of f with respect to α on $[a, b]$, respectively,

a. Show that $U(f, \alpha) \leq U(|f|, \alpha)$.

b. Is it true that $L(f, \alpha) \leq L(|f|, \alpha)$?

Problem 3

Let α be a bounded real-0valued increasing function on $[a, b]$. Assume $a < c < b$ and α is continuous at c. Let

$$f(x) = \begin{cases} 1 & \text{if } x = c; \\ 0 & \text{if } x \neq c. \end{cases}$$

Show directly that f is Darboux-Stieltjes integrable on $[a, b]$ and $\int_a^b f(x)\,\mathrm{d}\alpha(x) = 0$. (Do not use Theorem 8.16.)

Problem 4

Let f and α be real-valued bounded functions on $[a, b]$ and α is increasing on $[a, b]$. Assume f is Darboux-Stieltjes integrable with respect to α on $[a, b]$. Let $[c, d] \subset [a, b]$. Show that f is Darboux-Stieltjes integrable with respect to α on $[c, d]$.

Problem 5

Let α be a real-valued bounded function on $[a, b]$ and α is increasing with $\alpha(a) < \alpha(b)$. Let

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that if α is continuous on $[a, b]$, then f is not Darboux-Stieltjes integrable with respect to α on $[a, b]$.