Assignment 5 of Math 5302

Due Date: March 25, 2022 at 11:59pm

1. Let f be a real-valued bounded function on $[-1, 1]$. Let

$$\alpha(x) = \begin{cases} 0 & \text{if } -1 \le x < 0; \\ 2 & \text{if } 0 \le x \le 1. \end{cases}$$

Assume f is Riemann-Stieltjes integrable with respect to α on $[-1, 1]$. Show that
(a) f is continuous at 0 from the left.
(b) $\int_{-1}^{1} f(x) d\alpha(x) = 2f(0)$.

2. Let f and α be a real-valued bounded functions on $[a, b]$ and α is increasing. Let $L(f, \alpha)$ and $U(f, \alpha)$ represent the lower and upper Darboux-Stieltjes integral of f with respect to α on $[a, b]$, respectively.
(a) Show that $U(f, \alpha) \le U(|f|, \alpha)$.
(b) Is it true that $L(f, \alpha) \le L(|f|, \alpha)$?

3. Let α be a bounded real-valued increasing function on $[a, b]$. Assume $a < c < b$ and α is continuous at c. Let

$$f(x) = \begin{cases} 1 & \text{if } x = c; \\ 0 & \text{if } x \ne c. \end{cases}$$

Show directly that f is Darboux-Stieltjes integrable on $[a, b]$ and $\int_a^b f(x) d\alpha(x) = 0$. (Do not use Theorem 8.16.)

4. Let f and α be real-valued bounded functions on $[a, b]$ and α is increasing on $[a, b]$. Assume f is Darboux-Stieltjes integrable with respect to α on $[a, b]$. Let $[c, d] \subset [a, b]$. Show that f is Darboux-Stieltjes integrable with respect to α on $[c, d]$.

5. Let α be a real-valued bounded function on $[a, b]$ and α is increasing with $\alpha(a) < \alpha(b)$. Let

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that if α is continuous on $[a, b]$, then f is not Darboux-Stieltjes integrable with respect to α on $[a, b]$.