MATH 5302 Elementary Analysis II - Homework 7

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Preliminaries

Definition 1. <u>n-dimensional Euclidean norm-space:</u> *Let $\mathbb{R}^n$ be defined as*

$$\mathbb{R}^n := \{x = (x_1, x_2, \dots, x_n) \ : \ x_j \in \mathbb{R}\}$$

Let $A, B \subseteq \mathbb{R}^n$.

a. <u>Compliment of A</u>
$$A^c := \{x \in \mathbb{R}^n \ : \ x \neq A\}$$

b. <u>Union of A and B</u>
$$A \cup B := \{x \in \mathbb{R}^n \ : \ x \in A \vee x \in B\}$$

c. <u>Intersection of A and B</u>
$$A \cap B := \{x \in \mathbb{R}^n \ : \ x \in A \wedge x \in B\}$$

d. <u>Difference of A and B</u>
$$A \backslash B = A \cap B^c := \{x \in \mathbb{R}^n \ : \ x \in A \wedge x \neq B\}$$

e. <u>Closure of A</u>
$$\overline{A} := \{x \in \mathbb{R}^n \ : \ x \in A \vee x = \lim_{k \to \infty} x_k \ : \ [x_k] \in A\}$$

f. <u>Euclidean norm on $\mathbb{R}^n$</u>
$$\|x\| := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

with triangular inequality:
$$\|x + y\| \leq \|x\| + \|y\|$$

g. <u>Metric on $\mathbb{R}^n$</u>
$$d(x, y) = \|x - y\|$$

with properties

(a) $d(x, y) \geq 0$

(b) $d(x, y) = 0 \iff x = y$

(c) $d(x, y) = d(y, x)$

(d) $d(x, y) \leq d(x, z) + d(z, y)$

h. *A is considered* <u>bounded</u> *if every point in A is bounded:*

$$\exists_{M > 0} \ : \ \forall_{x \in A} \|x\| \leq M$$

Definition 2. Open and Closed Sets: *Let $A \subseteq \mathbb{R}^n$ and $x \in A$.*

a. *Denote the* open ball *centered at* x *of radius* r *as:*

$$B(x,r) := \{y \in \mathbb{R}^n \ : \ d(x,y) < r\}$$

b. x *is considered an* interior point of A *if :*

$$\exists_{r>0} \ : \ B(x,r) \subseteq A$$

c. A *is considered* open *if every point* $x \in A$ *is an interior point of* A*:*

$$\forall_{x \in A} \exists_{r>0} \ : \ B(x,r) \subseteq A$$

The following properties exist for open sets:

(a) $\emptyset$ *is open*
(b) $\mathbb{R}^n$ *is open*
(c) *Union of any collection of open sets is open*
(d) *Intersection of any finite collection of open sets is open*
(e) *Any open ball is an open set*

d. *The* Interior of A *is the set of all interior points of* A

$$A^\circ := \{x \ : \ x \textit{is an interior point of } A\}$$

Properties of A°

(a) A *open* $\iff A^\circ = A$
(b) A° *open*
(c) $(A^\circ)^\circ = A^\circ$
(d) $(A \cap B)^\circ = A^\circ \cap B^\circ$
(e) $A^\circ \cup B^\circ$ *not generally equal to* $(A \cup B)^\circ$
(f) A° *is the union of all open subsets of* A
(g) A° *is the largest open subset of* A

e. *The* Boundary of A *is defined as the closure minus the Interior of* A*:*

$$\partial A := \overline{A} \backslash A^\circ$$

f. A *is considered* closed *if* A^c *is open. Properties of closed sets*

(a) $\mathbb{R}^n$ *is closed*
(b) $\emptyset$ *is closed*
(c) *The intersection of any collection of closed sets is closed*
(d) *The union of any finite collection of closed sets is closed*

Definition 3. Compact set: *Let* $A \subseteq \mathbb{R}^n$.

a.

b. A *is called* compact *if every open cover of* A *has a finite subcover.*

c. *Properties of compact sets*

(a) $\emptyset$ *is compact*
(b) *Any finite set is compact*

(c) A and B compact $\implies A \cup B$ compact
(d) Any finite union of compact sets is compact
(e) $B(x, r)$ is not compact
(f) $\mathbb{R}^n$ is not compact
(g) If A is compact then A is closed and bounded.

Definition 4. Lebesgue Measure: *Let $A \subseteq \mathbb{R}^n$.* Note: *For various n, a Lebesgue Measure is essentially:*

a. *For $n = 1$, $\lambda(A)$ is a length*

b. *For $n = 2$, $\lambda(A)$ is an area*

c. *For $n = 3$, $\lambda(A)$ is a volume*

Each of the following stages define the Lebesgue measure in increasing complexity.

a. **Empty Set:**

$$\lambda(\emptyset) = 0$$

b. **Special Rectangles:**

Let

$$I = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

then

$$\lambda(I) = (b_1 - a_1) \cdots (b_n - a_n)$$

c. **Special Polygons:** *Special polygons are a finite union of special rectangles. They are closed and bounded subsets and therefore compact.*

Let P be a special polygon, decomposed into the following union of nonoverlaping special rectangles:

$$P = \bigcup_{k=1}^{N} I_k$$

The Lebesgue Measure for the special polygon is defined as the sum of the Lebesgue Measures of the special rectangles:

$$\lambda(P) = \sum_{k=1}^{N} \lambda(I_k)$$

The following are a few properties of the Lebesgue Measure for Special Polygons:

(a) $P_1 \subseteq P_2 \implies \lambda(P_1) \leq \lambda(P_2)$
(b) $P_1 \cap P_2 = \emptyset \implies \lambda(P_1 \cup P_2) = \lambda(P_1) + \lambda(P_2)$

d. **Open Sets:** *Let $G \subseteq \mathbb{R}^n$ be open and $G \neq \emptyset$.*

The Lebesgue Measure of an open set is defined as

$$\lambda(G) = \sup\{\lambda(P) \ : \ P \subset G, \ P \text{ is a special polygon}\}$$

The following are some properties of the Lebesgue Measure for open sets:

(a) $0 \leq \lambda(G) \leq \infty$
(b) $\lambda(G) = 0 \iff G = \emptyset$
(c) $\lambda(\mathbb{R}^n) = \infty$
(d) $G_1 \subseteq G_2 \implies \lambda(G_1) \leq \lambda(G_2)$
(e) $\lambda(\bigcup_{k=1}^{\infty} G_k) \leq \sum_{k=1}^{\infty} \lambda(G_k)$

(f) $\bigcap_{k=1}^{\infty} G_k = \emptyset \implies \lambda(\bigcup_{k=1}^{\infty} G_k) = \sum_{k=1}^{\infty} \lambda(G_k)$

(g) *If P is a special polygon, then* $\lambda(P) = \lambda(P^\circ)$

e. **Compact Sets:** *Let $K \subseteq \mathbb{R}^n$ be a compact set.*

The Lebesgue Measure of a compact set is defined as

$$\lambda(K) = \inf\{\lambda(G) \ : \ K \subset G, \ G \text{ is an open set}\}$$

The following are some properties of the Lebesgue Measure for compact sets:

(a) $0 \leq \lambda(K) \leq \infty$

(b) $K_1 \subseteq K_2 \implies \lambda(K_1) \leq \lambda(K_2)$

(c) $\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2)$

(d) $K_1 \cap K_2 = \emptyset \implies \lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$

Problem 1

Problem:

Let $A \subset \mathbb{R}^n$ be a compact set. Show that A is bounded.

Solution:

Theorem 1. *Let $A \subset \mathbb{R}^n$. If A is a compact set then A is also bounded.*

Proof. For $x \in A$, let
$$G_i = B(x, r_i) = \{y \in \mathbb{R}^n \ : \ d(x,y) < r_i\}, \forall_{i \in \mathbb{N}}$$
with $r_i \leq r_{i+1}$, meaning $G_i \subset G_{i+1}$ and also that
$$A \subseteq \bigcup_{i=1}^{\infty} G_i$$
A compact means that the finite subcover exists
$$\exists_{i_1, i_2, \ldots, i_N} \ : \ A \subseteq \bigcup_{i=1}^{N} G_{i_k}$$
and therefore
$$\exists_k \ : \ A \subseteq G_{i_k}$$
which means that
$$\exists_{r_k} \ : \ \forall_{y \in A} d(x,y) \leq r_k$$
which is the definition of a bounded set. $\square$

Problem 2

Problem:

Let G be a nonempty subset of $\mathbb{R}^n$. If G is open and P is a special polygon with $P \subset G$, prove there exists a special polygon P' such that $P \subset P' \subset G$ and $\lambda(P) < \lambda(P')$. (Hint: consider $G \backslash P$).

Solution:

Theorem 2. *Let $G \neq \emptyset \subset \mathbb{R}^n$ and $P \subset G$. If G open and P is a special polygon, then*

$$\exists_{P'} \ : \ P \subset P' \subset G \wedge \lambda(P) < \lambda(P')$$

Proof. By definition, P is composed of a finite collection of nonoverlaping special rectangles:

$$P = \bigcup_{k=1}^{N} I_k$$

Since G is open and $P \subset G$,

$$G \backslash P \neq \emptyset \implies \exists_{I_{N+1} \text{ special rectangle}} \subset G \backslash P$$

I_{N+1} is then another nonoverlaping special rectangle so that

$$P' = P \cup I_{N+1} = \bigcup_{k=1}^{N+1} I_k \subset G$$

Since $\lambda(P) = \sum_{k=1}^{N} \lambda(I_k)$,

$$\lambda(P') = \lambda(P) + \lambda(I_k) > P$$

therefore

$$P \subset P' \subset G \wedge \lambda(P) < \lambda(P')$$

□

Problem 3

Problem:

Use the definition of Lebesgue measure, $\lambda(G)$, of an open set $G \subset \mathbb{R}^n$ to prove the following statements:

a. If G is a bounded open set, then $\lambda(G) < \infty$.

b. Let
$$G = \{(x, y) \in \mathbb{R}^2 \ : \ 0 < x < 1 \wedge 0 < y < x^2\}$$
Then $\lambda(G) = \frac{1}{3}$.

(Hint: relate $\lambda(G)$ to the lower and upper Darboux sums of the function $f(x) = x^2$ on $[0, 1]$. However, you cannot use the methods of calculus to the extent that $\lambda(G) = \int_0^1 x^2 \, \mathrm{d}x = \frac{1}{3}$. You must use the actual definition of $\lambda(G)$).

a) If G is a bounded open set, then $\lambda(G) < \infty$.

Theorem 3. *Let $G \subset \mathbb{R}^n$. If G is a bounded open set then $\lambda(G) < \infty$.*

Proof. The definition of $\lambda(G)$ as an open set is

$$\lambda(G) = \sup\{\lambda(P) \ : \ P \subset G, \ P \text{ is a special polygon}\}$$

Let P_k be defined as a sequence with $P_k = \cup_{i=1}^k I_i \subset P_{k+1} \subset G \forall_{k \in \mathbb{N}}$. Since G is bounded, every $P_k \subset G$ would also be bounded (which is enough to conclude that $\lambda(P_k) \forall_k$ is bounded). We have $\lambda(P_k) \leq \lambda(P_{k+1})$, and more specifically,
$$\lambda(P_{k+1}) = \lambda(P_k) + \lambda(I_{k+1})$$
Since $I_{k+1} \cup G \backslash P_{k+1} = G \backslash P_k \subset G \backslash P_{k-1}$,

$$\lambda(G \backslash P_k) = \lambda(G \backslash P_{k+1}) + \lambda(I_{k+1})$$

and therefore $\lambda(I_{k+1}) < \lambda(I_k)$. This means that a finite upper bound exists on P_k as

$$\sup P_k = \lim_{k \to \infty} \sum_{i=1}^{k} I_i$$

and thus $\lambda(G) = \sup P_k < \infty$. $\square$

b) Let $G = \{(x, y) \in \mathbb{R}^2 \ : \ 0 < x < 1 \wedge 0 < y < x^2\}$ Then $\lambda(G) = \frac{1}{3}$.

Example 1. *Let*
$$G = \{(x, y) \in \mathbb{R}^2 \ : \ 0 < x < 1 \wedge 0 < y < x^2\}$$
$\lambda(G) = \frac{1}{3}$.

Proof. Consider $f : [0, 1] \to \mathbb{R}$ defined as
$$f(x) = x^2$$
Defining a partition of $[0, 1]$
$$\mathcal{P}_k = \{0 = x_0 < x_1 < \cdots < x_k = 1\}$$
Since f is increasing $\forall_{x \in [0,1]}$, special polygons $P_{l_k} \subset G \subset P_{u_k}$ can be defined as the union of special rectangles:

$$P_{l_k} = \bigcup_{i=1}^{k} I_{l_i}, \ I_i = [x_{i-1}, x_i] \times [0, f(x_{i-1})]$$

and

$$P_{u_k} = \bigcup_{i=1}^{k} I_{u_i},\ \ I_i = [x_{i-1}, x_i] \times [0, f(x_i)]$$

We then have the Lebesgue Measures for the upper and lower special polygons as

$$\lambda(P_{l_k}) = \sum_{i=1}^{k} \lambda(I_{l_k}) = \sum_{i=1}^{k} (f(x_{i-1}) - 0) \cdot (x_i - x_{i-1}) = L(f, P_k)$$

and

$$\lambda(P_{u_k}) = \sum_{i=1}^{k} \lambda(I_{u_k}) = \sum_{i=1}^{k} (f(x_i) - 0) \cdot (x_i - x_{i-1}) = U(f, P_k)$$

where each are equivalent to their respective Darboux sums. Therefore,

$$\lambda(P_{l_k}) = L(f, P_k) \leq \sup_{P_k} \lambda(P_k) = L(f) \leq U(f) = \inf_{P_k} \lambda(P_k) \leq U(f, P_k) = \lambda(P_{u_k})$$

Since we know that these Darboux sums result in equivalent Darboux integrals, $L(f) = U(f) = \mathcal{R}\int_0^1 f = \frac{1}{3}$, we can say that

$$\lambda(P_{l_\infty}) = \lambda(G) = \lambda(P_{u_\infty}) = \frac{1}{3}$$

$\square$

Problem 4

Problem:

Prove that every nonempty open subset of $\mathbb{R}$ can be expressed as a countable disjoint union of open intervals:

$$G = \bigcup_k (a_k, b_k)$$

where the range on k can be finite or infinite. Furthermore, show that this expression is unique except for the number of the component intervals. (Hint: for any $x \in G$, show that there exist a largest open interval A_x such that $x \in A_x$ and $A_x \subseteq G$. Also note that the set of rational numbers is countable and dense in $\mathbb{R}$.)

Solution:

Theorem 4. *Let $G \in \mathbb{R}$ be nonempty and open. G can be expressed as a countable disjoint union of open intervals:*

$$G = \bigcup_k (a_k, b_k)$$

Proof. Let $x \in G$. The largest open interval within G containing x is defined as

$$A_x := (a_x, b_x) \subseteq G \ : \ x \in (a_x, b_x) \wedge \left(\forall_{(a'_x, b'_x) \subseteq G} x \in (a'_x, b'_x) \implies (a'_x, b'_x) \subset (a_x, b_x)\right)$$

From this definition we have the following for $A_x = (a_x, b_x)$,

$$A_x = (a_x, b_x) \subseteq G \implies a_x \in \overline{G} \wedge b_x \in \overline{G}$$

Furthermore, the definition that A_x is the maximum possible subset implies that

$$a_x \in \partial G \wedge b_x \in \partial G$$

therefore

$$a_x < x < b_x \ : \ a_x, b_x \in \partial G \wedge$$

Since $G \subseteq \mathbb{R}$, these boundary points will be unique. i.e.

$$\forall_{x \in G} \exists_{\text{unique } a_x, b_x \in \partial G} \ : \ a_x < x < b_x$$

Consider $K = \{x_k \in \mathbb{Q}\}$. Since $\mathbb{Q}$ is complete in $\mathbb{R}$, and therefore complete in G, we have that

$$\exists_{K \subset \mathbb{Q}} \bigcup_{x_k \in K} A_{x_k} = G$$

This is because every open region $A_k = (a_k, b_k) \in G$ will contain a point $x_k \in K$ which will result in the union of all these countable open regions to be equivalent to G. $\square$

Problem 5

Problem:

In the notation of Problem 4, prove that $\lambda(G) = \sum_k (b_k - a_k)$.

Solution:

Theorem 5. *Let $G \subseteq \mathbb{R}$ be nonempty and open. The Lebesgue Measure of G is defined as*

$$\lambda(G) = \sum_k (b_k - a_k)$$

Proof. From Theorem 4 we have that

$$G = \bigcup_k (a_k, b_k)$$

Since $(a_k, b_k) \subset \mathbb{R}$, we know that

$$\lambda((a_k, b_k)) = b_k - a_k$$

Since G is composed of a collection of disjoint open sets, we have that

$$\lambda(\bigcup_k (a_k, b_k)) = \sum_k \lambda((a_k, b_k))$$

Therefore,

$$\lambda(G) = \sum_k (b_k - a_k)$$

$\square$

Problem 6

Problem:

Let C be the Cantor Set. Show that $\frac{1}{4} \in C$ and that $\frac{1}{4}$ is not an end point of any of the intervals in the G_k's.

Preliminaries:

Definition 5. *The* <u>Cantor set</u> C *is defined by*

$$C = [0,1] \setminus \bigcup_{k=1}^{\infty} G_k$$

where G_k *are defined as*

$$\bigcup_{i=0}^{3^{k-1}-1} \left(\frac{3i+1}{3^{i+1}}, \frac{3i+2}{3^{n+1}} \right)$$

Solution:

Example 2. *Let C be the Cantor Set. Show that $\frac{1}{4} \in C$ and $\frac{1}{4} \notin \partial G_k \forall_{G_k}$.*

Proof. For $n = 1$, we have $C_n = C_1 = [0,1] \setminus (\frac{1}{3}, \frac{2}{3})$ $\frac{1}{4}$ is clearly in C_1 since $\frac{1}{4} \in [0,1] \wedge \frac{1}{4} \notin (\frac{1}{3}, \frac{2}{3})$. Generally for n, $\frac{1}{4} \notin G_n$ since $\forall_{i \in \mathbb{N}} \frac{1}{4} \notin \left(\frac{3i+1}{3^{i+1}}, \frac{3i+2}{3^{i+1}} \right)$. Therefore, if $\frac{1}{4} \in C_n$ then $\frac{1}{4} \in C_{n+1}$ since

$$\frac{1}{4} \in C_n \implies \left(\frac{1}{4} \in [0,1] \setminus \cup_{k=1}^{n} G_k \right) \wedge \frac{1}{4} \notin G_{n+1}$$

Therefore, by induction, $\frac{1}{4} \in C$.

Additionally, since $\forall_{i \in \mathbb{N}} \neg 3^i \vdots 4$, $\frac{1}{4} \neq \frac{3i+1}{3^{i+1}} \neq \frac{3i+2}{3^{i+1}}$. Therefore, $\frac{1}{4}$ is not an endpoint of an of the intervals of G_k. $\square$