Assignment 9 of Math 5302

Due Date: April 27, 2022 at 11:59pm

1. a) Write down all possible σ-algebras on $X = \{1, 2, 3, 4\}$ which contains the element $\{1\}$.
   b) Let $\mathcal{A} = \{\emptyset, \{1\}, \{2, 3\}, X\}$ and $\mathcal{B} = \{\emptyset, \{2\}, \{1, 3\}, X\}$.
   Find $\mathcal{A} \cup \mathcal{B}$. Is $\mathcal{A} \cup \mathcal{B}$ a σ-algebra?
   c) Find $\mathcal{A} \cap \mathcal{B}$. Is $\mathcal{A} \cup \mathcal{B}$ a σ-algebra?

2. Prove that the intersection of any family of σ-algebras on X is a σ-algebra.

3. Prove that if N is a null set in $\mathbb{R}^n$, then there exists a Borel null set N' such that $N' \subseteq N$. Prove that N' may be chosen to be a "G_δ" set, a countable intersection of open sets.

4. Prove Property MF2: If $f : X \to \mathbb{R}$ is $\mathcal{M}$-measurable, and $f \neq 0$, then $\frac{1}{f}$ is $\mathcal{M}$-measurable.

5. Let $E \subseteq \mathbb{R}$ be a set which is not Lebesgue measurable. Let

$$f(x) = \begin{cases} e^x & \text{if } x \in E; \\ -e^x & \text{if } x \in E^c. \end{cases}$$

   (a) Prove that f is not Lebesgue measurable.
   (b) Prove that for all t, $f^{-1}(\{t\})$ is Lebesgue measurable.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove that the derivative f' is Borel measurable. (Be careful that f' may not be continuous.)