MATH 5302 Elementary Analysis II - Homework 2

Jonas Wagner

2022, Febuary 9^{th}

Problem 1

Let

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \leq t \leq 1 \\ 4 & \text{if } t > 1 \end{cases}$$

Let $F(x) = \int_0^x f(t)\mathrm{d}t$.

a. Find $F(x)$

b. Where is F continuous?

c. Where is F differentiable? Calculate F' at the points of differentiability.

Definition 1. *Let $f : [a, b) \to \mathbb{R}$ be integrable $\forall_[a, A] \subset [a, b)$. If*

$$\lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,\mathrm{d}x$$

exists, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,\mathrm{d}x$$

is an improper integral *from a to b.*

a. If $\int_a^b f(x)\,\mathrm{d}x$ is finite, then the improper integral converges.

b. Otherwise $\int_a^b f(x)\,\mathrm{d}x$ diverges, and thus the improper integral diverges.

Definition 2. *A function $f : (a, b) \to \mathbb{R}$ is continuous on $[a, b]$ if*

$$\forall_{x \in (a,b)} \forall_{\epsilon > 0} \exists_{\delta(x,\epsilon) > 0} \forall_{y \in \mathbb{R}} |y - x| < \epsilon \implies |f(y) - f(x)| < \delta$$

Definition 3. *Let $f : (a, b) \to \mathbb{R}$ be a function.*

a. The derivative of the function at point x_0 *is defined as*

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

b. If the derivative is defined at x_0, then it is differentiable at x_0.

c. If the derivative is defined for all $x_0 \in (a, b)$, then the function f is said to be differentialble.

d. When f is differentiable, the <u>derivative</u> *of $f(x)$ is defined as:*

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 1. *Let $f : \mathbb{R} \to \mathbb{R}$, be defined as*

$$f(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 4 & t > 1 \end{cases}$$

a) Find $F(x)$

Define the integral $F : \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$\boxed{F(t) = \begin{cases} \frac{1}{2}t^2 & 0 \leq t \leq 1 \\ \frac{1}{2} + 4(t-1) & t > 1 \end{cases}}$$

Proof. For $0 < t < 1$,

$$\begin{aligned} F(t) &= \int_0^t f(x)\,\mathrm{d}x \\ &= \int_0^t x\,\mathrm{d}x \end{aligned}$$

which is monotopically increasing, therefore:

$$\begin{aligned} &= \frac{1}{2}x^2 \bigg|_0^t \\ &= \frac{1}{2}t^2 - 0 \\ &= \frac{t^2}{2} \end{aligned}$$

For $t > 1$,

$$\begin{aligned} F(t) &= \int_0^t f(x)\,\mathrm{d}x \\ &= \int_0^1 x\,\mathrm{d}x + \int_1^t 4\,\mathrm{d}x \\ &= \frac{1}{2}x^2 \bigg|_0^1 + 4x \bigg|_1^t \\ &= \frac{1^2}{2} - 0 + 4(t) - 4(1) \\ &= \frac{1}{2} + 4(t-1) \end{aligned}$$

For $t = 1$ and $t \geq 1$, $1 \in [0, t)$ is a discontinuity within f; however, F remains continuous but not differentiable at the discontinuity point.

$$F(t \to 1^-) = lim_{t \to 1^1} \frac{t^2}{2} = \frac{1}{2} = \lim_{t \to 1^+} \frac{1}{2} + 4(t-1) = F(t \to 1^+)$$

$\square$

b) Where is F continuous?

$F(t)$ is continuous for the entire domain, $[0, \infty)$. i.e.

$$\forall_{x \in [0,\infty)} \forall_{\epsilon > 0} \exists_{\delta(x,\epsilon) > 0} \; : \; \forall_{y \in \mathbb{R}} |y - x| < \epsilon \implies |f(y) - f(x)| < \delta$$

Proof. (not required by problem...) Essentially, this is proven by demonstrating that

$$\lim_{t \to 1^-} F(t) = \lim_{t \to 1^+} F(t)$$

□

c) Where is F differentiable? Calculate F' at the points of differentiability.

$F(t)$ is differentiable in $(a, 1) \cup (1, \infty)$ which excludes 2 points from the domain: 0 and 1.

$$F' = \begin{cases} t & 0 < t < 1 \\ 4 & t > 1 \\ \textbf{\textit{Undefined}} & t \in \{0, 1\} \end{cases}$$

Proof. (not required by problem...) Essentially, this is proven by demonstrating that

$$\forall_{x \in (a,1) \cup (1,\infty)} \lim_{t \to x^-} F'(t) \neq \lim_{t \to x^+} F'(t)$$

This is also true since on regions $(a, 1)$ and $(1, \infty)$, $F(t)$ is smooth continuous which by definition implies differentiability. However, this is not the case for the boundary, $x = 1$:

$$\lim_{t \to 1^-} F'(t) \neq \lim_{t \to 1^+} F'(t)$$

which by definition means that $F(x)$ not differentiable at $x = 1$. □

Problem 2

Let f be a continuous function on $\mathbb{R}$ and define

$$F(x) = \int_0^{\sin x} f(t)\,\mathrm{d}t$$

Show that F is differentiable on $\mathbb{R}$ and compute F'.

Theorem 1. *If g is a continuous function on $[a, b]$ that is differentiable on (a, b), and if g' is integrable on $[a, b]$, then*

$$\int_a^b g' = g(b) - g(a)$$

Theorem 2. *If u and v are continuous function on $[a, b]$ that are differentiable on (a, b), and if u' and v' are integrable on $[a, b]$, then*

$$\int_a^b u(x)v'(x)\,\mathrm{d}x + \int_a^b u'(x)v(x)\,\mathrm{d}x = \int_a^b u(x)v(x)\,\mathrm{d}x = u(b)v(b) - u(a)v(a)$$

Theorem 3. *Let $u : J \to l$ be differentiable with u' continuous. If f continuous on l, then $f \odot u$ is continuous on J and*

$$\int_a^b f \odot u(x)u'(x)\,\mathrm{d}x = \int_{u(a)}^{ub} f(u)\,\mathrm{d}u$$

for $a, b \in J$.

Example 2. *Let*

$$F(x) = \int_0^{\sin x} f(t)\,\mathrm{d}t$$

where f is some continuous function on $\mathbb{R}$.

a) Show that F is differentiable on $\mathbb{R}$

Let $u(x) = \sin x$. This definition results in $u'(x) = \cos x$. Applying Theorem 3, we have

$$u(a) = \sin a = 0 \implies a = 0$$

and

$$u(b) = \sin b = \sin x \implies b = x$$

which defines the necessary conditions for differentiability according to the theorem.

b) Compute F'.

Following,

$$\begin{aligned} F(x) &= \int_0^{\sin x} f(t)\,\mathrm{d}t \\ &= \int_{u(a)}^{u(b)} f(t)\,\mathrm{d}t \\ &= \int_a^b f(u(x))u'(x)\,\mathrm{d}x \\ &= \int_0^x f(\sin(x))\cos(x)\,\mathrm{d}x \end{aligned}$$

Therefore, by the Fundamental Theorem of Calculus (Theorem 1),

$$\boxed{F'(x) = f(\sin(x))\cos(x)}$$

Problem 3

Let

$$F(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & \text{if } 0 < x \leq 1; \\ 0 & \text{if } x = 0. \end{cases}$$

a. Show that F has a derivative at every $x \in [0, 1]$.

b. Show that F' is not Riemann Integrable on $[0, 1]$. (So F is not the integral of its derivative.)

Definition 4. *Define bounded function $f : [a, b] \to \mathbb{R}$ and set $S \subseteq [a, b]$. Let $M(f, S) := \sup f(x) \ : \ x \in S$ and $m(f, S) = \inf f(x) \ : \ x \in S$ Let $U(f, P)$ and $L(f, P)$ for f w.r.t. P be defined by*

$$U(f, P) = \sum_{i=1}^{n} M(f, [x_{i-1}, x_i]) \cdot (x_i - x_{i-1})$$

and

$$L(f, P) = \sum_{i=1}^{n} m(f, [x_{i-1}, x_i]) \cdot (x_i - x_{i-1})$$

The Upper Darboux Integral *$U(f)$ for f over $[a, b]$ is defined as*

$$U(f) = \inf \{U(f, P) \ : \ P = \{a = x_0 < x_1 < \cdots < x_n = b\}\}$$

The Lower Darboux Integral *$L(f)$ for f over $[a, b]$ is defined as*

$$L(f) = \sup \{L(f, P) \ : \ P = \{a = x_0 < x_1 < \cdots < x_n = b\}\}$$

Definition 5. *f is* Darboux Integrable *on $[a, b]$ iff $L(f) = U(f)$. i.e.*

$$\int_a^b f = \int_a^b f(x)\mathrm{d}x = L(f) = U(f)$$

Example 3. *Let $F : [0, 1] \to \mathbb{R}$ be defined as*

$$F(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & x \in (0, 1] \\ 0 & x = 0; \end{cases}$$

a) Show that F has a derivative at every $x \in [0, 1]$.

From Definition 3, the derivative of f at point x_0, $f'(x_0)$ is defined as:

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

The derivative $F'(x_0)$ is defined $\forall_{x_0 \in [0,1]}$.

Proof. For $x_0 \in (0,1)$,

$$
\begin{aligned}
F'(x_0) &= \lim_{x\to x_0} \frac{x^2 \sin\left(\frac{1}{x^2}\right) - (x_0)^2 \sin\left(\frac{1}{x_0^2}\right)}{x - x_0} \\
F'(x) &= \lim_{h\to 0} \frac{(x+h)^2 \sin\left(\frac{1}{(x+h)^2}\right) - (x)^2 \sin\left(\frac{1}{x^2}\right)}{h} \\
&= \lim_{h\to 0} \frac{\left(x^2 + 2xh + h^2\right) \sin\left(\frac{1}{(x+h)^2}\right) - (x)^2 \sin\left(\frac{1}{x^2}\right)}{h} \\
&= \lim_{h\to 0} \frac{(x)^2 \left(\sin\left(\frac{1}{(x+h)^2}\right) - \sin\left(\frac{1}{x^2}\right)\right) + (2xh + h^2) \sin\left(\frac{1}{(x+h)^2}\right)}{h} \\
&= \lim_{h\to 0} \frac{x^2 \left(\sin\left(\frac{1}{(x+h)^2}\right) - \sin\left(\frac{1}{x^2}\right)\right)}{h} \\
&\quad + \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{(x+h)^2}\right)}{h} \\
&\quad + \lim_{h\to 0} \frac{2xh \sin\left(\frac{1}{(x+h)^2}\right)}{h} \\
&= x^2 \lim_{h\to 0} \frac{\sin\left(\frac{1}{x+h}^2\right) - \sin\left(\sin 1x^2\right)}{h} + 0 + \lim_{h\to 0} 2x \sin\left(\frac{1}{(x+h)^2}\right) \\
&= 2x \sin\left(\frac{1}{x^2}\right) + x^2 \frac{\mathrm{d}}{\mathrm{d} \sin\left(\frac{1}{x^2}\right)} \\
&= 2x \sin\left(\frac{1}{x^2}\right) + x^2 \left(-2\frac{1}{x^3} \cos\left(\frac{1}{x^2}\right)\right) \\
F'(x) &= 2x \sin\left(\frac{1}{x^2}\right) - 2\frac{1}{x} \cos\left(\frac{1}{x^2}\right)
\end{aligned}
$$

Which means that $F(x)$ is differentiable $\forall_{x_0 \in (0,1)}$.
This result can be expanded to the closed interval by taking the limit of $F(x)$ to the boundaries which also exist; therefore, $F(x)$ is differentiable $\forall_{x_0 \in [0,1]}$. □

b) Show that F' is not Riemann Integrable on $[0,1]$. (So F is not the integral of its derivative.)

Note: *Stating F' is not Riemann Integrable is equivalent to saying F' is not Darboux Integrable.*

Problem 4

Show that for each $p > 0$, $\int_1^\infty \frac{\sin(x)}{x^p}\,\mathrm{d}x$ converges. Hint: For 0¡p¡1, you may find it helpful to use integration by parts.

Theorem 4. *For all $p > 0$, the integral*

$$\int_1^\infty \frac{\sin(x)}{x^p}$$

converges.

Proof. First, letting $p = 1$, $\square$

Problem 5

Consider $\int_1^\infty \frac{x^p}{1+x^q}$.

a. For what values of p and q are the integral convergent?

b. For what values of p and q are the integral absolutely convergent?

Example 4. *Define the integral*

$$\int_1^\infty \frac{x^p}{1+x^q}$$