MATH 5302 Elementary Analysis II - Homework 2

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Problem 1

Complete the proof of Theorem 2.3 in the lecture notes by showing that a decreasing function on $[a, b]$ is integrable.

Definition 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function.*

a. f is <u>strictly increasing</u> *over interval I if*

$$\forall_{x_1,x_2 \in I} x_1 < x_2 \implies f(x_1) < f(x_2)$$

b. f is <u>strictly decreasing</u> *over interval I if*

$$\forall_{x_1,x_2 \in I} x_1 < x_2 \implies f(x_1) > f(x_2)$$

c. f is <u>increasing</u> *over interval I if*

$$\forall_{x_1,x_2 \in I} x_1 < x_2 \implies f(x_1) \leq f(x_2)$$

d. f is <u>decreasing</u> *over interval I if*

$$\forall_{x_1,x_2 \in I} x_1 < x_2 \implies f(x_1) \geq f(x_2)$$

Definition 2. *$f : \mathbb{R} \to \mathbb{R}$ is* <u>monotone</u> *over interval I if f is either increasing or decreasing over interval I.*

Theorem 1. *Theorem 1.4 states:*
A bounded function $f : [a, b] \to \mathbb{R}$ is <u>integrable</u> *iff*

$$\forall_{\epsilon>0} \exists_{P=\{a=x_0<x_1<\cdots<x_n=b\}} \ : \ U(f, P) - L(f, P) < \epsilon$$

Theorem 2. *Every monotone function f on $[a, b]$ is integrable.*

Proof.

Lemma 1. *Every increasing function f on $[a, b]$ is integrable.*

Proof. Proved in class □

Lemma 2. *Every decreasing function f on $[a, b]$ is integrable.*

Proof. Let f be decreasing on $[a, b]$. Let $\epsilon > 0$. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition on $[a, b]$ with mesh less than $\dfrac{\epsilon}{f(a) - f(b)}$.

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{i=1}^{n} M(f, [x_{x-1}, x_i])(x_i - x_{i-1}) - \sum_{i=1}^{n} m(f, [x_{x-1}, x_i])(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})](x_i - x_{i-1}) \\
&< \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]\left(\frac{\epsilon}{f(a) - f(b)}\right) \\
&= \sum_{i=1}^{n} -[f(x_{i-1}) - f(x_i)]\left(\frac{\epsilon}{f(a) - f(b)}\right) \\
&= (f(a) - f(b))\left(\frac{\epsilon}{f(a) - f(b)}\right) = \epsilon
\end{aligned}$$

□

□

Problem 2

Let f be a bounded function on $[a,b]$, so that there exists $B>0$ such that $|f(x)| \leq B$ for all $x \in [a,b]$.

a)

Show

$$U(f^2,P) - L(f^2,P) \leq 2B[U(f,P) - L(f,P)]$$

for all partitions P of $[a,b]$. Hint: $f^2(x) - f^2(y) = (f(x)+f(y))(f(x)-f(y))$

Theorem 3.

$$U(f^2,P) - L(f^2,P) \leq 2B[U(f,P) - L(f,P)]$$

for all partitions $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ *of* $[a,b]$.

Proof.

$$\begin{aligned}
U(f^2,P) - L(f^2,P) &= \sum_{i=1}^{n} M(f^2,[x_{x-1},x_i])(x_i - x_{i-1}) - \sum_{i=1}^{n} m(f^2,[x_{x-1},x_i])(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} f^2(x_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} f^2(x_{i-1})(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} [f^2(x_i) - f^2(x_{i-1})](x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} [(f(x_i) + f(x_{i-1}))(f(x_i) - f(x_{i-1}))](x_i - x_{i-1})
\end{aligned}$$

Since $|f(x)| \leq B \forall_{x \in [a,b]}$,

$$\begin{aligned}
&\leq \sum_{i=1}^{n} [(B+B))(f(x_i) - f(x_{i-1}))](x_i - x_{i-1}) \\
&= 2B \sum_{i=1}^{n} [(f(x_i) - f(x_{i-1}))](x_i - x_{i-1}) \\
&= 2B \left[\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) \right] \\
&= 2B \left[\sum_{i=1}^{n} M(f,[x_{x-1},x_i])(x_i - x_{i-1}) - \sum_{i=1}^{n} m(f,[x_{x-1},x_i])(x_i - x_{i-1}) \right] \\
&= 2B[U(f,P) - L(f,P)]
\end{aligned}$$

□

b)

Show that if f is integrable on $[a, b]$, then f^2 is also integrable on $[a, b]$.

Theorem 4. *f integrable on $[a, b]$ $\implies$ f^2 integrable on $[a, b]$.*

Proof. From Theorem 3, we have that

$$U(f^2, P) - L(f^2, P) \leq 2B[U(f, P) - L(f, P)]$$

for all partitions $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$.
f integrable on $[a, b]$ implies

$$\exists_{\{P_k\}} \; : \; \overset{k\to\infty}{\lim}[U(f, P) - L(f, P)] = 0$$

Therefore,

$$\exists_{\{P_k\}} \; : \; \overset{k\to\infty}{\lim}[U(f^2, P) - L(f^2, P)] \leq 2B[U(f, P) - L(f, P)] = 0$$

Which means that the lower and upper Darboux integrals are equal, $U(f^2) = L(f^2)$ and by definition this means f^2 is Darboux integrable. $\square$

Problem 3

Let f be a bounded function on $[a, b]$. Suppose f^2 is integrable on $[a, b]$. Must f also be integrable on $[a, b]$?
Answer: No.
A modification of the rational number indicator function can be shown as a counter example:
Let

$$f(x) = \begin{cases} 1 & \text{if} x \in \mathbb{Q} \\ -1 & \text{if} x \notin \mathbb{Q} \end{cases}$$

which is clearly not integrable due to the infinite number of discontinuities.
However, f^2 would be defined by

$$f^2(x) = 1$$

which is clearly integrable.

Problem 4

Suppose that f and g are integrable on $[a, b]$. Show that $\max(f, g)$ is also integrable on $[a, b]$. Hint: Derive and apply the formula:

$$\max(f, g) = \frac{1}{2}(f + g + |f - g|)$$

Theorem 5. *For all functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ that are integrable on $[a, b]$, $\max(f, g)$ is also integrable on $[a, b]$.*

Proof. Let function h be defined as

$$h(x) := \max(f(x), g(x))$$

which is the same as $\max(f, g)$.

$$\begin{aligned}
\max(f, g)(x) = h(x) &= \max(f(x), g(x)) \\
&= \begin{cases} f(x) & f(x) \geq g(x) \\ g(x) & g(x) < f(x) \end{cases} \\
&= \begin{cases} g(x) + [f(x) - g(x)] & f(x) \geq g(x) \\ f(x) + [g(x) - f(x)] & g(x) < f(x) \end{cases} \\
&= \begin{cases} g(x) + |f(x) - g(x)| & f(x) \geq g(x) \\ f(x) + |g(x) - f(x)| & g(x) < f(x) \end{cases} \\
&= \frac{1}{2} \begin{cases} f(x) + g(x) + |f(x) - g(x)| & f(x) \geq g(x) \\ f(x) + g(x) + |g(x) - f(x)| & g(x) < f(x) \end{cases} \\
&= \frac{f(x) + g(x) + |f(x) - g(x)|}{2}
\end{aligned}$$

Since f and g are integrable on $[a, b]$, the following is true for some $\epsilon > 0$ and partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$.

a. $U(f) = L(f)$

b. $U(g) = L(g)$

c. $U(f, P) - L(f, P) < \epsilon_f$

d. $U(g, P) - L(g, P) < \epsilon_g$

e. $|f|$ is integrable.

f. $|g|$ is integrable.

g. $U(|f|, P) - L(|f|, P) < \epsilon_{abs(f)} < \frac{\epsilon}{2}$

h. $U(|g|, P) - L(|g|, P) < \epsilon_{abs(g)} < \frac{\epsilon}{2}$

Also, by the triangular inequality,

$$|f(x) - g(x)| \leq |f(x)| + |g(x)|$$

and therefore

$$U(|f(x) - g(x)|) - L(|f(x) - g(x)|) < \epsilon_{abs(f-g)} < \epsilon$$

$$
\begin{aligned}
U(h,P)-L(h,P) &= \sum_{i=1}^{n} M(h,[x_{x-1},x_i])(x_i-x_{i-1}) - \sum_{i=1}^{n} m(h,[x_{x-1},x_i])(x_i-x_{i-1}) \\
&= \sum_{i=1}^{n} h(x_i)(x_i-x_{i-1}) - \sum_{i=1}^{n} h(x_{i-1})(x_i-x_{i-1}) \\
&= \sum_{i=1}^{n} [h(x_i)-h(x_{i-1})](x_i-x_{i-1}) \\
&= \sum_{i=1}^{n} \left[\frac{(f(x_i)+g(x_i)+|f(x_i)-g(x_i)|)-(f(x_{i-1})+g(x_{i-1})+|f(x_{i-1})-g(x_{i-1})|)}{2}\right](x_i-x_{i-1}) \\
&= \frac{1}{2}\sum_{i=1}^{n} \left[(f(x_i)-f(x_{i-1}))\right](x_i-x_{i-1}) \\
&\quad + \left[g(x_i)-g(x_{i-1})\right](x_i-x_{i-1}) \\
&\quad + \left[|f(x_i)-g(x_i)|-|f(x_{i-1})-g(x_{i-1})|\right](x_i-x_{i-1}) \\
&= \frac{1}{2}\sum_{i=1}^{n} \left[f(x_i)-f(x_{i-1})\right](x_i-x_{i-1}) \\
&\quad + \frac{1}{2}\sum_{i=1}^{n} \left[g(x_i)-g(x_{i-1})\right](x_i-x_{i-1}) \\
&\quad + \frac{1}{2}\sum_{i=1}^{n} \left[|f(x_i)-g(x_i)|-|f(x_{i-1})-g(x_{i-1})|\right](x_i-x_{i-1}) \\
&= \frac{1}{2}(U(f,P)-L(f,P)) + \frac{1}{2}(U(g,P)-L(g,P)) + \frac{1}{2}(U(|f-g|,P)-L(|f-g|,P)) \\
U(h,P)-L(h,P) &< \frac{1}{2}\left(\frac{\epsilon}{2}\right) + \frac{1}{2}\left(\frac{\epsilon}{2}\right) + \frac{1}{2}(\epsilon) = \epsilon
\end{aligned}
$$

Since ϵ bounds $U(h,P)-L(h,P)$,

$$U(h) = L(h)$$

and therefore h is integrable. This implies that the maximum of two integrable functions, $\max f, g$, will always be integrable. □

Problem 5

Suppose f and g are continuous functions on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Prove there exists x in (a, b) such that $f(x) = g(x)$.

Theorem 6. *If functions* $f : [a, b] \to \mathbb{R}$ *and* $g : [a, b] \to \mathbb{R}$ *are continuous on* $[a, b]$*, and* $\int_a^b f = \int_a^b g$*, then* $\exists_{x \in (a,b)} \ : \ f(x) = g(x)$.

Proof. Let $h : [a, b] \to \mathbb{R}$ be defined by

$$h(x) = g(x) - f(x)$$

which is known to be continuous and bounded since both f and g are.
The equivalency of the integrals becomes

$$\int_a^b f = \int_a^b g \implies \int_a^b g - f = \int_a^b h = 0$$

Proof by contradiction: First, assume $g(x) > f(x) \forall_{x \in [a,b]}$. This implies $h(x) > 0 \forall_{x \in [a,b]}$ and therefore $\int_a^b h > 0$ which is a contradiction.
Similarly, assume $g(x) < f(x) \forall_{x \in [a,b]}$. This implies $h(x) < 0 \forall_{x \in [a,b]}$ and therefore $\int_a^b h < 0$ which is a contradiction.
Therefore, in order for $\int_a^b h = 0$,

$$\exists_{x_1, x_2 \in (a,b)} \ : \ h(x_1) \leq 0 \wedge h(x_2) \geq 0$$

Since h is continuous, this means that

$$\exists_{x_0 \in (a,b)} \ : \ h(x_0) = 0$$

which implies

$$\exists_{x_0 \in (a,b)} \ : \ g(x_0) = f(x_0)$$

□