Assignment 10 of Math 5302

Due Date: May 4, 2022 at 10am

1. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^n$ such that $f_k \geq f_{k+1}$ for all $k \in \mathbb{N}$, and $f = \lim_{k\to\infty} f_k$. Assume $\int f_1 \, d\lambda < \infty$. Prove that $\int f d\lambda = \lim\limits_{k\to\infty} \int f_k d\lambda$.

2. Assume $f \in L^1(\mathbb{R}^n)$. Define

$$f_k(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq k \text{ and } |x| \leq k; \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $\lim\limits_{k\to\infty} \int f_k d\lambda = \int f d\lambda$.

3. (A generalized Dominated Convergence Theorem)
Let $\{f_k\}$ be a sequence of measurable functions on in $\mathbb{R}^n$ such that
a. $\lim\limits_{k\to\infty} f_k(x) = f(x)$ a.e.
b. there exists a sequence of nonnegative functions $\{g_k\}$ such that $g_k \in L^1(\mathbb{R}^n)$ and $|f_k| \leq g_k$ a.e. for all $k \in \mathbb{N}$.
c. $\lim\limits_{k\to\infty} g_k(x) = g(x)$ a.e. and $g \in L^1(\mathbb{R}^n)$.
d. $\int g \, d\lambda = \lim\limits_{k\to\infty} \int g_k \, d\lambda$.
Prove that $f \in L^1(\mathbb{R}^n)$ and $\int f \, d\lambda = \lim_{k\to\infty} \int f_k \, d\lambda$.
(Hint: Imitate the proof of Dominated Convergence Theorem.)

4. Assume $\lambda(E) = 0$. Prove that every function defined on E is measurable and that $\int_E f \, d\lambda = 0$ for all f.