MATH 5302 Elementary Analysis II - Homework 1

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Problem 1

Consider $f(x) = 2x + 1$ over the interval $[1, 3]$. Let P be the partition $\{1, 1.5, 2, 3\}$.

a)

Problem: Compute $L(f, P), U(f, P)$, and $U(f, P) - L(f, P)$

Definition 1. *Define bounded function $f : [a, b] \to \mathbb{R}$ and set $S \subseteq [a, b]$.*
Let $M(f, S) := \sup f(x) \ : \ x \in S$ and $m(f, S) = \inf f(x) \ : \ x \in S$
Define the partition P of $[a, b]$ as

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

The Upper Darboux Sum *$U(f, P)$ for f w.r.t. P is defined as*

$$U(f, P) = \sum_{i=1}^{n} M(f, [x_{i-1}, x_i]) \cdot (x_i - x_{i-1})$$

The Lower Darboux Sum *$L(f, P)$ for f w.r.t. P is defined as*

$$L(f, P) = \sum_{i=1}^{n} m(f, [x_{i-1}, x_i]) \cdot (x_i - x_{i-1})$$

Solution: Let $f : [1, 3] \to \mathbb{R}$ defined by

$$f(x) = 2x + 1$$

and partition P of $[1, 3]$ defined as

$$\{1, 1.5, 2, 3\}$$

$$\begin{aligned}
L(f, P) &= \sum_{i=1}^{n} m(f, [x_{i-1}, x_i]) \cdot (x_i - x_{i-1}) \\
&= \sum_{i=1}^{3} m(2x + 1, [x_{i-1}, x_i]) \cdot (x_i - x_{i-1}) \\
&= m(2x + 1, [1, 1.5]) * (1.5 - 1) + m(2x + 1, [1.5, 2]) * (2 - 1.5) + m(2x + 1, [2, 3]) * (3 - 2) \\
&= 3(0.5) + 4(0.5) + 5(1)
\end{aligned}$$

$$\boxed{L(f, P) = 8.5}$$

$$
\begin{aligned}
U(f,P) &= \sum_{i=1}^{n} M(f,[x_{i-1},x_i])\cdot(x_i-x_{i-1}) \\
&= \sum_{i=1}^{3} M(2x+1,[x_{i-1},x_i])\cdot(x_i-x_{i-1}) \\
&= M(2x+1,[1,1.5])*(1.5-1)+M(2x+1,[1.5,2])*(2-1.5)+M(2x+1,[2,3])*(3-2) \\
&= 4(0.5)+5(0.5)+7(1)
\end{aligned}
$$

$$\boxed{L(f,P)=11.5}$$

$$\boxed{U(f,p)-L(f,p)=11.5-8.5=3}$$

b)

Problem: What happens to the value of $U(f,P)-L(f,P)$?
Solution: it gets smaller
Proof:

$$
\begin{aligned}
L(f,P) &= \sum_{i=1}^{n} m(f,[x_{i-1},x_i])\cdot(x_i-x_{i-1}) \\
&= \sum_{i=1}^{4} m(2x+1,[x_{i-1},x_i])\cdot(x_i-x_{i-1}) \\
&= m(2x+1,[1,1.5])*(1.5-1)+m(2x+1,[1.5,2])*(2-1.5) \\
&\quad + m(2x+1,[2,2.5])*(2.5-2)+m(2x+1,[2.5,3])*(3-2.5) \\
&= 3(0.5)+4(0.5)+5(0.5)+6(0.5)
\end{aligned}
$$

$$\boxed{L(f,P)=9}$$

$$
\begin{aligned}
U(f,P) &= \sum_{i=1}^{n} M(f,[x_{i-1},x_i])\cdot(x_i-x_{i-1}) \\
&= \sum_{i=1}^{3} M(2x+1,[x_{i-1},x_i])\cdot(x_i-x_{i-1}) \\
&= M(2x+1,[1,1.5])*(1.5-1)+M(2x+1,[1.5,2])*(2-1.5) \\
&\quad + M(2x+1,[2,2.5])*(2.5-2)+M(2x+1,[2.5,3])*(3-2.5) \\
&= 4(0.5)+5(0.5)+6(0.5)+7(0.5)
\end{aligned}
$$

$$\boxed{L(f,P)=11}$$

$$\boxed{U(f,p)-L(f,p)=11-9=2}$$

c)

Problem: Find a partition P' of $[1,3]$ for which $U(L,P') - L(f,P') < 2$
Solution: Let

$$P' = \{1, 1.4, 1.8, 2.2, 2.6, 3\}$$

$$\begin{aligned}
L(f,P') &= \sum_{i=1}^{n} m(f,[x_{i-1},x_i]) \cdot (x_i - x_{i-1}) \\
&= \sum_{i=1}^{5} m(2x+1,[x_{i-1},x_i]) \cdot (x_i - x_{i-1}) \\
&= m(2x+1,[1,1.4]) * (1.4-1) \\
&\quad + m(2x+1,[1.4,1.8]) * (1.8-1.4) \\
&\quad + m(2x+1,[1.8,2.2]) * (2.2-1.8) \\
&\quad + m(2x+1,[2.2,2.6]) * (2.6-2.2) \\
&\quad + m(2x+1,[2.6,3]) * (3-2.6) \\
&= 3(0.4) + 3.8(0.4) + 4.6(0.4) + 5.4(0.4) + 6.2(0.4)
\end{aligned}$$

$$\boxed{L(f,P') = 9.2}$$

$$\begin{aligned}
U(f,P') &= \sum_{i=1}^{n} M(f,[x_{i-1},x_i]) \cdot (x_i - x_{i-1}) \\
&= \sum_{i=1}^{5} M(2x+1,[x_{i-1},x_i]) \cdot (x_i - x_{i-1}) \\
&= M(2x+1,[1,1.4]) * (1.4-1) \\
&\quad + M(2x+1,[1.4,1.8]) * (1.8-1.4) \\
&\quad + M(2x+1,[1.8,2.2]) * (2.2-1.8) \\
&\quad + M(2x+1,[2.2,2.6]) * (2.6-2.2) \\
&\quad + M(2x+1,[2.6,3]) * (3-2.6) \\
&= 3.8(0.4) + 4.6(0.4) + 5.4(0.4) + 6.2(0.4) + 7(0.4)
\end{aligned}$$

$$\boxed{U(f,P') = 10.8}$$

$$\boxed{U(f,p) - L(f,p) = 10.8 - 9.2 = 1.6}$$

Problem 2

Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational on } [0,1] \\ 0 & \text{if } x \text{ is irrational on } [0,1] \end{cases}$$

a)

Problem: Find the upper and lower Darboux integrals for f on the interval $[0, 1]$.

Definition 2. *Define bounded function $f : [a, b] \to \mathbb{R}$ and set $S \subseteq [a, b]$.*
Let $M(f, S) := \sup f(x) \ : \ x \in S$ and $m(f, S) = \inf f(x) \ : \ x \in S$
Let $U(f, P)$ and $L(f, P)$ for f w.r.t. P be defined by

$$U(f, P) = \sum_{i=1}^{n} M(f, [x_{i-1}, x_i]) \cdot (x_i - x_{i-1})$$

and

$$L(f, P) = \sum_{i=1}^{n} m(f, [x_{i-1}, x_i]) \cdot (x_i - x_{i-1})$$

The Upper Darboux Integral *$U(f)$ for f over $[a, b]$ is defined as*

$$U(f) = \inf \{U(f, P) \ : \ P = \{a = x_0 < x_1 < \cdots < x_n = b\}\}$$

The Lower Darboux Integral *$L(f)$ for f over $[a, b]$ is defined as*

$$L(f) = \sup \{L(f, P) \ : \ P = \{a = x_0 < x_1 < \cdots < x_n = b\}\}$$

Solution: Let $P_1 = 0, 1$,

$$\begin{aligned} U(f, P_1) &= \sum_{i=1}^{n} M(f, [x_{i-1}, x_i]) \cdot (x_i - x_{i-1}) \\ &= M(f, [0, 1]) * (1 - 0) \\ &= 1 * 1 \\ U(f, P_1) &= 1 \end{aligned}$$

$$\begin{aligned} L(f, P_1) &= \sum_{i=1}^{n} m(f, [x_{i-1}, x_i]) \cdot (x_i - x_{i-1}) \\ &= m(f, [0, 1]) * (1 - 0) \\ &= 0 * 1 \\ L(f, P_1) &= 0 \end{aligned}$$

$U(f)$ is bounded by $U(f, P_1)$,

$$U(f) = \inf_P \{U(f, P)\} \leq U(f, P_1) = 1$$

Similarly, $L(f)$ is bounded by $L(f, P_1)$,

$$L(f) = \sup_P \{L(f, P)\} \geq L(f, P_1) = 0$$

Let $P_n = \left\{\frac{i}{n}, \forall_{i=0,\ldots,n}\right\}$,

$$\begin{aligned}
U(f,P_n) &= \sum_{i=1}^{n} M(f,[x_{i-1},x_i]) \cdot (x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} M(f,[\frac{i-1}{n},\frac{i}{n}]) * (\frac{i}{n} - \frac{i-1}{n}) \\
&= \sum_{i=1}^{n} \frac{i}{n} * (\frac{1}{n}) \\
&= \frac{1}{n^2} \sum_{i=1}^{n} i \\
&= \frac{1}{n^2} \frac{n(n+1)}{2} \\
&= \frac{n+1}{2n} \\
U(f,P_n) &= \frac{1}{2} + \frac{1}{2n}
\end{aligned}$$

$$\begin{aligned}
L(f,P_n) &= \sum_{i=1}^{n} m(f,[x_{i-1},x_i]) \cdot (x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} m(f,[\frac{i-1}{n},\frac{i}{n}]) * (\frac{i}{n} - \frac{i-1}{n}) \\
&= \sum_{i=1}^{n} 0 * (\frac{1}{n}) \\
L(f,P_n) &= 0
\end{aligned}$$

$U(f)$ is bounded by $U(f,P_n)$,

$$U(f) = \inf_P \{U(f,P)\} \leq U(f,P_n) = \frac{1}{2} + \frac{1}{2n}$$

which when taking n to the limit results in

$$\boxed{U(f) \leq \lim_{n\to\infty} \frac{1}{2} + \frac{1}{2n} = \frac{1}{2}}$$

$L(f)$ is bounded by $L(f,P_n)$,

$$L(f) = \sup_P \{L(f,P)\} \geq L(f,P_n) = 0$$

For f, the definitions of $L(f,P)$ and $m(f,[a,b])$ actually demonstrate that $L(f,P) = 0 \forall_P$. The definition of $L(f)$ then implies

$$\boxed{L(f) = \sup_P (L(f,P) = 0) = 0}$$

b)

Problem: Is f integrable on $[0,1]$?

Definition 3. *f is* Darboux Integrable *on $[a,b]$ iff $L(f) = U(f)$. i.e.*

$$\int_a^b f = \int_a^b f(x)\mathrm{d}x = L(f) = U(f)$$

Answer: No
Proof:

$$L(f) = 0 \neq \frac{1}{2} = U(f)$$

Problem 3

Let

$$f(x) = \begin{cases} 1 & \text{if } \exists_{n\in\mathbb{N}} \,:\, x = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

Show that f is integrable on $[0,1]$ and compute $\int_0^1 f$.
Solution: Let $P_n = \left\{\frac{i}{n}, \forall_{i=0,\ldots,n}\right\}$,

$$\begin{aligned} U(f,P_n) &= \sum_{i=1}^{n} M(f,[x_{i-1},x_i]) \cdot (x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} M(f,[\frac{i-1}{n},\frac{i}{n}]) * (\frac{i}{n} - \frac{i-1}{n}) \\ &= \frac{1}{n}\sum_{i=1}^{n} M(f,[\frac{i-1}{n},\frac{i}{n}]) \end{aligned}$$

$M(f,[a,b])$ is 0 unless $\left\{\frac{1}{n} \,:\, \exists_{n\in\mathbb{N}}\right\} \cap [a,b] \neq \emptyset$

$$= \frac{1}{n}\sum_{i=1}^{n} \begin{cases} 1 & \exists_{k\in\mathbb{N}} \,:\, \frac{1}{k} \in [\frac{i-1}{n},\frac{i}{n}] \\ 0 & \text{otherwise} \end{cases}$$

As n increases, fewer portions of the partition have $M(f,[x_{i-1},x_i]) = 1$. Since as n increases, the size of each partition also gets smaller,

$$\lim_{n\to\infty} U(f,P_n) = 0$$

Therefore, by definition,

$$\boxed{U(f) = \inf_P \{U(f,P)\} = 0}$$

$$\begin{aligned} L(f,P_n) &= \sum_{i=1}^{n} m(f,[x_{i-1},x_i]) \cdot (x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} m(f,[\frac{i-1}{n},\frac{i}{n}]) * (\frac{i}{n} - \frac{i-1}{n}) \\ &= \sum_{i=1}^{n} 0 * (\frac{1}{n}) \\ L(f,P_n) &= 0 \end{aligned}$$

$L(f)$ is bounded by $L(f,P_n)$,

$$L(f) = \sup_P \{L(f,P)\} \geq L(f,P_n) = 0$$

The definition of $L(f)$ then implies

$$\boxed{L(f) = \sup_P (L(f,P) = 0) = 0}$$

Answer: f is integrable on $[0,1]$ since $L(f) = U(f)$.
The integral is given as:

$$\int_0^1 f = L(f) = U(f) = 0$$

Problem 4

Problem: Let f be a bounded function on $[a,b]$. Supposes there exist sequences (U_n) and (L_n) of upper and lower Darboux sums for f such that $\lim_{n\to\infty}(U_n - L_n) = 0$. Show that f is integrable and $\int_a^b f = \lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n$.

Theorem 1. *For bounded function $f : [a,b] \to \mathbb{R}$, if*

$$\exists_{(U_n),(L_n)} \ : \ \lim_{n\to\infty}(U_n - L_n) = 0$$

then f is integrable and

$$\int_a^b f = \lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n$$

Proof. By Definition 3, f is integrable iff $L(f) = U(f)$.
Fundamentally this is proven by the squeeze theorem. From Definition 2,

$$L(f, P_l) \leq L(f) = \int_a^b f = U(f) \leq U(f, P_u)$$

for all partitions of $[a,b]$, P_l and P_u.
The convergence of Darboux sum sequences U_n and L_n to the same point,

$$\lim_{n\to\infty}(U_n - L_n) = 0$$

"squeezes" the Darboux integrals to the same point.

$$L_n = L(f, P_n) \leq L(f) = \int_a^b f = U(f) \leq U(f, P_n) = U_n$$
$$\lim_{n\to\infty} \{L_n \leq L(f) = U(f) \leq U_n\}$$

Since $\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n$,

$$\boxed{\lim_{n\to\infty} L_n = L(f) = \int_a^b f = U(f) = \lim_{n\to\infty}}$$

□

Problem 5

Let f be integrable on $[a, b]$, and suppose g is a function on $[a, b]$ such that $g(x) = f(x)$ except for finitely many x in $[a, b]$. Show that g is integrable and $\int_a^b f = \int_a^b g$.

Theorem 2. *For integrable function $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$,*

$$f(x) = g(x) \forall_{x \in [a,b] \setminus S}$$

where S is finite, then

a. g is Darboux integrable

b. $\int_a^b f = \int_a^b g$

Proof. By Definition 3, f is integrable iff $L(f) = U(f)$. In addition, Definition 2 implies

$$L(f, P_l) \leq L(f) = \int_a^b f = U(f) \leq U(f, P_u)$$

for all partitions of $[a, b]$, P_l and P_u.
From Definition 1, for partition $P = \{a = x_0 < \cdots < x_n = b\}$,

$$L(f, P) = \sum_{i=1}^{n} m(f, [x_{i-1}, x_i]) \cdot (x_i - x_{i-1})$$

and

$$U(f, P) = \sum_{i=1}^{n} M(f, [x_{i-1}, x_i]) \cdot (x_i - x_{i-1})$$

When comparing the $L(f, P)$ to $L(g, p)$ or $U(f, P)$ to $U(g, P)$, we can decompose the partition segment with any differences between $m(\cdot)$ and $M(\cdot)$.

$$\begin{aligned} L(g, P_n) &= \sum_{i=1}^{n} m(g, [x_{i-1}, x_i]) \cdot (x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} (m(f, [x_{i-1}, x_i]) + \Delta_i) \cdot (x_i - x_{i-1}) \\ &= L(f, P) + \sum_{i=1}^{n} \Delta_i \cdot (x_i - x_{i-1}) \end{aligned}$$

Since S is a finite set of points in which $f(x) \neq g(x)$,

$$\lim_{n\to\infty} L(g, P_n) = \lim_{n\to\infty} L(f, P_n) + \lim_{n\to\infty} \sum_{i=1}^{n} \Delta_i \cdot (x_i - x_{i-1})$$

$$\boxed{L(g) = \lim_{n\to\infty} L(g, P_n) = \lim_{n\to\infty} L(f, P_n) + 0 = L(f)}$$

$$\begin{aligned} LU(g, P_n) &= \sum_{i=1}^{n} M(g, [x_{i-1}, x_i]) \cdot (x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} (M(f, [x_{i-1}, x_i]) + \Delta_i) \cdot (x_i - x_{i-1}) \\ &= U(f, P) + \sum_{i=1}^{n} \Delta_i \cdot (x_i - x_{i-1}) \end{aligned}$$

Since S is a finite set of points in which $f(x) \neq g(x)$,

$$\lim_{n\to\infty} U(g, P_n) = \lim_{n\to\infty} U(f, P_n) + \lim_{n\to\infty} \sum_{i=1}^{n} \Delta_i \cdot (x_i - x_{i-1})$$

$$\boxed{U(g) = \lim_{n\to\infty} U(g, P_n) = \lim_{n\to\infty} U(f, P_n) + 0 = U(f)}$$

Since $L(g) = L(f)$, $U(g) = U(f)$, and $L(f) = \int_a^b f = U(f)$,

$$L(g) = L(f) = \int_a^b f = U(f) = U(g)$$

These equivalences mean that g is Darboux integrable since

$$\boxed{L(g) = \int_a^b g = U(g)}$$

and that

$$\boxed{\int_a^b f = \int_a^b g}$$

$\square$