

MATH 5302 Elementary Analysis II - Homework 9

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2022, April 27th

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Preliminaries

Definition 1. n -dimensional Euclidean norm-space: Let \mathbb{R}^n be defined as

$$\mathbb{R}^n := \{x = (x_1, x_2, \dots, x_n) : x_j \in \mathbb{R}\}$$

Let $A, B \subseteq \mathbb{R}^n$.

1. Compliment of A

$$A^c := \{x \in \mathbb{R}^n : x \notin A\}$$

2. Union of A and B

$$A \cup B := \{x \in \mathbb{R}^n : x \in A \vee x \in B\}$$

3. Intersection of A and B

$$A \cap B := \{x \in \mathbb{R}^n : x \in A \wedge x \in B\}$$

4. Difference of A and B

$$A \setminus B = A \cap B^c := \{x \in \mathbb{R}^n : x \in A \wedge x \notin B\}$$

5. Closure of A

$$\overline{A} := \{x \in \mathbb{R}^n : x \in A \vee x = \lim_{k \rightarrow \infty} x_k : [x_k] \in A\}$$

6. Euclidean norm on \mathbb{R}^n

$$\|x\| := \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

with triangular inequality:

$$\|x + y\| \leq \|x\| + \|y\|$$

7. Metric on \mathbb{R}^n

$$d(x, y) = \|x - y\|$$

with properties

$$(a) \ d(x, y) \geq 0$$

$$(b) \ d(x, y) = 0 \iff x = y$$

$$(c) \ d(x, y) = d(y, x)$$

$$(d) \ d(x, y) \leq d(x, z) + d(z, y)$$

8. A is considered bounded if every point in A is bounded:

$$\exists_{M>0} : \forall_{x \in A} \|x\| \leq M$$

Definition 2. Open and Closed Sets: Let $A \subseteq \mathbb{R}^n$ and $x \in A$.

1. Denote the open ball centered at x of radius r as:

$$B(x, r) := \{y \in \mathbb{R}^n : d(x, y) < r\}$$

2. x is considered an interior point of A if:

$$\exists_{r>0} : B(x, r) \subseteq A$$

3. A is considered open if every point $x \in A$ is an interior point of A :

$$\forall_{x \in A} \exists_{r>0} : B(x, r) \subseteq A$$

The following properties exist for open sets:

- (a) \emptyset is open
 - (b) \mathbb{R}^n is open
 - (c) Union of any collection of open sets is open
 - (d) Intersection of any finite collection of open sets is open
 - (e) Any open ball is an open set
4. The Interior of A is the set of all interior points of A

$$A^\circ := \{x : x \text{ is an interior point of } A\}$$

Properties of A°

- (a) $A \text{ open} \iff A^\circ = A$
 - (b) A° open
 - (c) $(A^\circ)^\circ = A^\circ$
 - (d) $(A \cap B)^\circ = A^\circ \cap B^\circ$
 - (e) $A^\circ \cup B^\circ$ not generally equal to $(A \cup B)^\circ$
 - (f) A° is the union of all open subsets of A
 - (g) A° is the largest open subset of A
5. The Boundary of A is defined as the closure minus the Interior of A :

$$\partial A := \overline{A} \setminus A^\circ$$

6. A is considered closed if A^c is open. Properties of closed sets
- (a) \mathbb{R}^n is closed
 - (b) \emptyset is closed
 - (c) The intersection of any collection of closed sets is closed
 - (d) The union of any finite collection of closed sets is closed

Definition 3. Compact set: Let $A \subseteq \mathbb{R}^n$.

1. A is called compact if every open cover of A has a finite subcover.
2. Properties of compact sets
 - (a) \emptyset is compact
 - (b) Any finite set is compact
 - (c) A and B compact $\implies A \cup B$ compact
 - (d) Any finite union of compact sets is compact
 - (e) $B(x, r)$ is not compact
 - (f) \mathbb{R}^n is not compact
 - (g) If A is compact then A is closed and bounded.

Definition 4. Lebesgue Measure: Let $A \subseteq \mathbb{R}^n$. Note: For various n , a Lebesgue Measure is essentially:

1. For $n = 1$, $\lambda(A)$ is a length
2. For $n = 2$, $\lambda(A)$ is an area
3. For $n = 3$, $\lambda(A)$ is a volume

Each of the following stages define the Lebesgue measure in increasing complexity.

0. Empty Set:

$$\lambda(\emptyset) = 0$$

1. Special Rectangles:

Let

$$I = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

then

$$\lambda(I) = (b_1 - a_1) \cdots (b_n - a_n)$$

2. Special Polygons: *Special polygons are a finite union of special rectangles. They are closed and bounded subsets and therefore compact.*

Let P be a special polygon, decomposed into the following union of nonoverlapping special rectangles:

$$P = \bigcup_{k=1}^N I_k$$

The Lebesgue Measure for the special polygon is defined as the sum of the Lebesgue Measures of the special rectangles:

$$\lambda(P) = \sum_{k=1}^N \lambda(I_k)$$

The following are a few properties of the Lebesgue Measure for Special Polygons:

- (a) $P_1 \subseteq P_2 \implies \lambda(P_1) \leq \lambda(P_2)$
- (b) $P_1 \cap P_2 = \emptyset \implies \lambda(P_1 \cup P_2) = \lambda(P_1) + \lambda(P_2)$

3. Open Sets: *Let $G \subseteq \mathbb{R}^n$ be open and $G \neq \emptyset$.*

The Lebesgue Measure of an open set is defined as

$$\lambda(G) = \sup\{\lambda(P) : P \subset G, P \text{ is a special polygon}\}$$

The following are some properties of the Lebesgue Measure for open sets:

- (a) $0 \leq \lambda(G) \leq \infty$
- (b) $\lambda(G) = 0 \iff G = \emptyset$
- (c) $\lambda(\mathbb{R}^n) = \infty$
- (d) $G_1 \subseteq G_2 \implies \lambda(G_1) \leq \lambda(G_2)$
- (e) $\lambda(\bigcup_{k=1}^{\infty} G_k) \leq \sum_{k=1}^{\infty} \lambda(G_k)$
- (f) $\bigcap_{k=1}^{\infty} G_k = \emptyset \implies \lambda(\bigcup_{k=1}^{\infty} G_k) = \sum_{k=1}^{\infty} \lambda(G_k)$
- (g) If P is a special polygon, then $\lambda(P) = \lambda(P^\circ)$

4. Compact Sets:

Let $K \subseteq \mathbb{R}^n$ be a compact set.

The Lebesgue Measure of a compact set is defined as

$$\lambda(K) = \inf\{\lambda(G) : K \subset G, G \text{ is an open set}\}$$

The following are some properties of the Lebesgue Measure for compact sets:

- (a) $0 \leq \lambda(K) \leq \infty$
- (b) $K_1 \subseteq K_2 \implies \lambda(K_1) \leq \lambda(K_2)$

- (c) $\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2)$
- (d) $K_1 \cap K_2 = \emptyset \implies \lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$

5. **Finite Outer Measure:**

Definition 5. Let $A \subseteq \mathbb{R}^n$. The outer measure of A is defined as

$$\lambda^*(A) = \inf \{ \lambda(G) : A \subseteq G \wedge G \text{ open} \}$$

The inner measure of A is defined as

$$\lambda_*(A) = \sup \{ \lambda(K) : A \supseteq K \wedge K \text{ compact} \}$$

The following are properties of the outer and inner measures:

- (a) $\lambda_*(A) \leq \lambda^*(A)$
- (b) $A \subseteq B \implies \lambda^*(A) \leq \lambda^*(B) \wedge \lambda_*(A) \leq \lambda_*(B)$
- (c) $\lambda^*(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \lambda^*(A_k)$
- (d) If all A_k 's are disjoint, then

$$\lambda_*\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \sum_{k=1}^{\infty} \lambda^*(A_k)$$

- (e) If A is open or compact, then $\lambda^*(A) = \lambda_*(A) = \lambda(A)$

Definition 6. Let $A \subseteq \mathbb{R}^n$ be a set with finite outer measure: $\lambda^*(A) < \infty$. We say that $A \in \mathcal{L}_0$ if $\lambda^*(A) = \lambda_*(A)$. We define \mathcal{L}_0 as

$$\mathcal{L}_0 = \{ A \subseteq \mathbb{R}^n : \lambda^*(A) = \lambda_*(A) < \infty \}$$

If $A \in \mathcal{L}_0$ then the measure of A is

$$\lambda(A) = \lambda^*(A) = \lambda_*(A)$$

Problem 1

a)

Problem:

b)

Problem:

Problem 2

a)

Problem:

b)

Problem:

Problem 3

Problem:

Problem 4

a)

Problem:

Problem 5

Problem: