Assignment 3 of Math 5302

Due Date: Feb. 16, 2022 at 11:59pm

1. Let

$$f(t) = \begin{cases} 0 & \text{if } t < 0; \\ t & \text{if } 0 \le t \le 1; \\ 4 & \text{if } t > 1. \end{cases}$$

Let $F(x) = \int_0^x f(t)dt$.
(a) Find $F(x)$.
(b) Where is F continuous?
(c) Where is F differentiable? Calculate F' at the points of differentiability.

2. Let f be a continuous function on $\mathbb{R}$ and define

$$F(x) = \int_0^{\sin x} f(t)dt \text{ for } x \in \mathbb{R}.$$

Show that F is differentiable on $\mathbb{R}$ and compute F'.

3. Let

$$F(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}) & \text{if } 0 < x \le 1; \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Show that F has a derivative at every $x \in [0, 1]$.
(b) Show that F' is not Riemann integrable on $[0, 1]$. (So F is not the integral of its derivative.)

4. Show that for each $p > 0$, $\int_1^\infty \frac{\sin x}{x^p} dx$ converges. Hint: For $0 < p < 1$, you may find it helpful to use integration by parts.

5. Consider $\int_1^\infty \frac{x^p}{1+x^q} dx$.
(a) For what values of p and q are the integral convergent?
(b) For what values of p and q are the integral absolutely convergent?