Assignment 1 of Math 5302

Due Date: Feb. 2, 2022 at 11:59pm

1. Consider $f(x) = 2x + 1$ over the interval $[1, 3]$. Let P be the partition consisting of the points $\{1, 1.5, 2, 3\}$.

(a) Compute $L(f, P)$, $U(f, P)$ and $U(f, P) - L(f, P)$.

(b) What happens to the value of $U(f, P) - L(f, P)$ when we add the point 2.5 to the partition?

(c) Find a partition P' of $[1, 3]$ for which $U(L, P') - L(f, P') < 2$.

2. Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational on } [0,1]; \\ 0 & \text{if } x \text{ is irrational on } [0,1]. \end{cases}$$

(a) Find the upper and lower Darboux integrals for f on the interval $[0, 1]$.

(b) Is f integrable on $[0, 1]$?

3. Let

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}; \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is integrable on $[0, 1]$ and compute $\int_0^1 f$.

4. Let f be a bounded function on $[a, b]$. Suppose there exist sequences (U_n) and (L_n) of upper and lower Darboux sums for f such that $\lim\limits_{n\to\infty} (U_n - L_n) = 0$. Show f is integrable and $\int_a^b f = \lim\limits_{n\to\infty} U_n = \lim\limits_{n\to\infty} L_n$.

5. Let f be integrable on $[a, b]$, and suppose g is a function on $[a, b]$ such that $g(x) = f(x)$ except for finitely many x in $[a, b]$. Show g is integrable and $\int_a^b f = \int_a^b g$.