MATH 5302 Elementary Analysis II - Homework 5

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Preliminaries

Definition 1. Darboux-Stieltjes Integral *Let $f : [a, b] \to \mathbb{R}$ and $\alpha : [a, b] \to \mathbb{R}$, with f bounded and α increasing on $[a, b]$. Let partition P be defined as*

$$P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

Let

$$M(f, [x_1, x_2]) = \sup_{x \in [x_1, x_2]} f(x)$$

and

$$m(f, [x_1, x_2]) = \inf_{x \in [x_1, x_2]} f(x)$$

a. *The upper and lower* Darboux-Stieltjes Sums *are defined*

$$U(f, \alpha, P) = \sum_{i=1}^{n} M(f, [x_{i-1}, x_i]) \cdot \Delta_i \alpha$$

and

$$L(f, \alpha, P) = \sum_{i=1}^{n} m(f, [x_{i-1}, x_i]) \cdot \Delta_i \alpha$$

respectively with

$$\Delta_i \alpha = \alpha(x_i) - \alpha(x_{i-1})$$

A more general sum $S(f, \alpha, P)$ is when $f(x_i^*)$ for $x_i^* \in [x_{i-1}, x_i]$ is used instead.

Note:

$$m(f, [a, b]) \cdot (\alpha(b) - \alpha(a)) \leq L(f, \alpha, P) \leq U(f, \alpha, P) \leq M(f, [a, b]) \cdot (\alpha(b) - \alpha(a))$$

b. *The upper and lower* Darboux-Stieltjes Integrals *are defined*

$$U(f, \alpha) = \inf_{P \text{ partition of } [a,b]} U(f, \alpha, P)$$

and

$$L(f, \alpha) = \sup_{P \text{ partition of } [a,b]} U(f, \alpha, P)$$

respectively.

Note:

$$L(f, \alpha) \leq L(f, \alpha, P) \leq U(f, \alpha, P) \leq U(f, \alpha)$$

for any P partition of $[a, b]$.

c. *f is called* <u>Darboux-Stieltjes Integrable</u> *with respect to α if and only if*

$$\forall_{\epsilon>0}\exists_{P=\{a=x_0<x_1<\cdots<x_{n-1}<x_n=b\}} : U(f,\alpha,P)-L(f,\alpha,P)<\epsilon$$

in which case the <u>Darboux-Stieltjes Integral</u> *with respect to α is defined as*

$$\mathcal{DS}\int_a^b f\,\mathrm{d}\alpha = U(f,\alpha) = L(f,\alpha)$$

<u>**Note:**</u> *If f is also continuous on $[a,b]$ then f is Riemann-Stieljes integrable which implies f is Darboux-Stieljes integrable.*

<u>**Properties:**</u> *When f is Darboux-Stieltjes integrable on $[a,b]$ and α is increasing on $[a,b]$ then*

a. *$|f|$ is Darboux-Stieltjes integrable on $[a,b]$ and*

$$\mathcal{DS}\int_a^b f\,\mathrm{d}\alpha \le \mathcal{DS}\int_a^b |f|\,\mathrm{d}\alpha$$

b. *f^2 is Darboux-Stieltjes integrable on $[a,b]$.*

c. *If g is also Darboux-Stieltjes integrable on $[a,b]$, then fg is Darboux-Stieltjes integrable on $[a,b]$.*

d. *For α_1 and α_2 also increasing on $[a,b]$ and f is Darboux-Stieltjes integrable with respect to α_1 and α_2, then f is Darboux-Stieltjes integrable with respect to α_1 and α_2. Additionally,*

$$\begin{aligned}&\mathcal{DS}\int_a^b f(x)\,\mathrm{d}\alpha_1(x) + \mathcal{DS}\int_a^b f(x)\,\mathrm{d}\alpha_2(x)\\ =&\mathcal{DS}\int_a^b f(x)\,\mathrm{d}\alpha(x)+\alpha_2(x)\end{aligned}$$

e. *For $a<c<b$, f is Darboux-Stieltjes integrable with respect to α on $[a,b]$ if and only if f is Darboux-Stieltjes integrable with respect to α on $[a,c]$ and $[c,b]$. Furthermore,*

$$\mathcal{DS}\int_a^b f(x)\,\mathrm{d}\alpha(x) = \mathcal{DS}\int_a^c f(x)\,\mathrm{d}\alpha(x) + \mathcal{DS}\int_c^b f(x)\,\mathrm{d}\alpha(x)$$

Definition 2. ***Continuity:*** *Let $f:[a,b]\to\mathbb{R}$.*

a. *f is* <u>Lipschitz Continuous</u> *on $[a,b]$ if*

$$\exists_C : \forall_{x,y\in[a,b]}|f(x)-f(y)| \le |x-y|$$

b. *f is* <u>Absolutely Continuous</u> *on $[a,b]$ if*

$$\forall_{\epsilon>0}\exists_{\delta>0}\forall_{finite\ collection\ \{(x,x')\}\ of\ nonoverlaping\ intervals: \sum_{i=1}^n |x_i'-x_i|<\delta} \sum_{i=1}^n |f(x_i')-f(x_i)|<\epsilon$$

c. *f is* <u>uniformly continuous</u> *on $[a,b]$ if*

$$\forall_{\epsilon>0}\exists_{\delta>0} : (x,y\in[a,b]) \wedge \{|x-y|<\delta\} \implies |f(x)-f(y)|<\epsilon$$

d. *f is* <u>continuous</u> *on $[a,b]$ if f is continuous at all $x_0\in[a,b]$. i.e.*

$$\forall_{\epsilon>0}\exists_{\delta>0} : \forall_{x\in[a,b]} \wedge |x-x_0|<\delta \implies |f(x)-f(x_0)|<\epsilon$$

Properties:

a. *f continuous on closed $[a, b]$, then f is uniformly continuous on $[a, b]$.*

b. *f differentiable at $x \in [a, b]$ implies Locally Lipschitz continuous at x.*

c. *$C^1[a, b]$ is the set of differentiable functions with continuous derivatives on $[a, b]$.*

d. *$C^1[a, b] \subset$ differentiable functions with bounded derivatives*

e. *Differentiable with bounded derivatives $\implies$ Lipschitz continuous $\implies$ Absolutely continuous $\implies$ uniformly continuous $\implies$ continuous*

Problem 1

Problem:

Assume f is a real-valued function defined on $[a, b]$ and f is Lipschitz continuous on $[a, b]$. Show that f is absolutely continuous on $[a, b]$.

Solution:

Proposition 1. *Let $f : [a, b] \to \mathbb{R}$. If f is Lipschitz continuous on $[a, b]$ then f is also absolutely continuous on $[a, b]$.*

Proof. Consider the finite collection of nonoverlaping intervals,

$$\mathcal{I} = \{(x_1, x_1'), (x_2, x_2'), \ldots, (x_n, x_n')\}$$

where $x_1 = a$, $x_N' = b$, $x_i < x_i' \forall_{i=1,\ldots,n}$, and $x_i' \leq x_{i+1} \forall_{j=1,\ldots,n-1}$. By definition, f being Lipschitz continous on $[a, b]$ means:

$$\exists_M : \forall_{x,y \in [a,b]} |f(x) - f(y)| \leq M|x - y|$$

Therefore,

$$|f(x_i') - f(x_i)| < M|x_i' - x_i|, \ \forall_{(x_i, x_i') \in \mathcal{I}}$$

Equivalently,

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < M \sum_{i=1}^{n} |x_i' - x_i|$$

Taking $\delta = M\epsilon$, we have

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta = M\epsilon \implies \sum_{i=1}^{n} |f(x_i') - f(x_i)| < \epsilon$$

Therefore,

$$\forall_{\epsilon > 0} \exists_{\delta = M\epsilon} \forall_{\mathcal{I} \text{ nonoverlaping } : \sum_{i=1}^{n} |x_i' - x_i| < \delta} \sum_{i=1}^{n} |f(x_i') - f(x_i)| < \epsilon$$

which is the definition of absolutely continuous. $\square$

Problem 2

If f is continuous and α is of bounded variation on $[a,b]$, then f is Riemann-Stieltjes integrable with respect to α on $[a,b]$. Let $\beta(x) = V_a^x(\alpha)$ and $\gamma(x) = \beta(x) - \alpha(x)$, $x \in [a,b]$. Show that

a)

Problem:

$$\left| \int_a^b f(x) \, \mathrm{d}\alpha(x) \right| \le \int_a^b |f(x) \, \mathrm{d}\beta(x)| \le \max_{x \in [a,b]} |f| V_a^b(\alpha).$$

Solution:

By definition,

$$\beta(x) = V_a^x(\alpha) = |\alpha(x) - \alpha(a)| = \alpha(x) + \gamma(x)$$

and

$$\beta(x_k) - \beta(x_{k-1}) = |\alpha(x_k) - \alpha(a)| - |\alpha(x_{k-1}) - \alpha(a)| = \alpha(x_k) - \alpha(x_{k-1})$$

Note that $V_a^{x_k} \ge V_a^{x_{k-1}} \ge 0$, so

$$\beta(x_k) - \beta(x_{k-1}) > 0 \implies \beta(x_k) - \beta(x_{k-1}) = |\beta(x_k) - \beta(x_{k-1})|$$

Additionally, since $\alpha(x)$ is of bounded variation,

$$\Delta_k(\alpha) \le V_a^{x_k}(\alpha) \le V_a^b(\alpha)$$

We have that

$$\int_a^b f(x) \, \mathrm{d}\alpha(x) = \lim_{\mathrm{mesh}(P) \to 0} \sum_{i=1}^n f(x_i^*)(\alpha(x_i) - \alpha(x_{i-1}))$$

therefore,

$$\begin{aligned}
\left| \int_a^b f(x) \, \mathrm{d}\alpha(x) \right| &= \lim_{\mathrm{mesh}(P) \to 0} \left| \sum_{i=1}^n f(x_i^*)(\alpha(x_i) - \alpha(x_{i-1})) \right| \\
&= \lim_{\mathrm{mesh}(P) \to 0} \sum_{i=1}^n |f(x_i^*)(\alpha(x_i) - \alpha(x_{i-1}))| \\
&\le \lim_{\mathrm{mesh}(P) \to 0} \sum_{i=1}^n |f(x_i^*)||(\alpha(x_i) - \alpha(x_{i-1}))| \\
&= \lim_{\mathrm{mesh}(P) \to 0} \sum_{i=1}^n |f(x_i^*)||\beta(x_k) - \beta(x_{k-1})| \\
&= \lim_{\mathrm{mesh}(P) \to 0} \sum_{i=1}^n |f(x_i^*)|(\beta(x_k) - \beta(x_{k-1})) \\
&= \int_a^b |f(x)| \, \mathrm{d}\beta(x) \\
&= \lim_{\mathrm{mesh}(P) \to 0} \sum_{i=1}^n |f(x_i^*)| V_{x_k}^{x_{k-1}}(\alpha) \\
&\le \lim_{\mathrm{mesh}(P) \to 0} \sum_{i=1}^n |f(x_i^*)| V_a^b(\alpha) \\
&= \max_{x \in [a,b]} |f| V_a^b(\alpha)
\end{aligned}$$

Thus,

$$\left| \int_a^b f(x) \, \mathrm{d}\alpha(x) \right| \leq \int_a^b |f(x) \, \mathrm{d}\beta(x)| \leq \max_{x \in [a,b]} |f| V_a^b(\alpha).$$

b)

Problem:

The function α is Riemann-Stieltjes integrable with respect to f on $[a, b]$.

Solution:

Since α of bounded variation and f is continuous on $[a, b]$,

$$\int_a^b \alpha(x) \, \mathrm{d}f(x) = \lim_{\mathrm{mesh}(P) \to 0} \sum_{i=1}^n \alpha(x_i^*)(f(x_i) - f(x_{i-1}))$$

which clearly converges since $\alpha(x_{i-1}) \leq \alpha(x_i^*) \leq \alpha(x_i)$ and $f(x_i) - f(x_{i-1}) \to 0$.

Problem 3

Problem:

Given a positive integer n and numbers $c_0, c_1, c_2, \dots, c_n$, let α be the step function defined on $[0,1]$ by

$$\begin{aligned}
&\alpha(0) = 0, \\
&\alpha(x) = c_0 \text{ for } 0 < x < \frac{1}{n}, \\
&\alpha(x) = \sum_{i=0}^{k-1} c_i \text{ for } \frac{k-1}{n} < x < \frac{k}{n}, k = 2, 3, \dots, n, \\
&\alpha(1) = \sum_{i=0}^{n} c_i
\end{aligned}$$

Show that $V_0^1(\alpha) \leq \sum_{i=0}^{n} |c_i|$. (Hint: Use Riemann-Stieltjes Integral to estimate the variation.)

Solution:

Let partition P be defined as

$$P = \left\{ 0 = x_0 < x_1 < \cdots < x_{k_1} = \frac{k_1}{n} < \cdots < x_N = 1 \right\}$$

where $x_k = \frac{k_1}{n}$. Then we have

$$V_a^b(f) = \sup_P V_a^b(f, P)$$

and

$$V_a^b(f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

which is very similar to the definition of a RS sum with $f(x) = 1$:

$$\sum_{i=1}^{n} f(x_i^*) \cdot \Delta_i \alpha$$

$$\begin{aligned}
V_0^1(\alpha) &= \sup_P V_0^1(\alpha, P) \\
&\approx \mathcal{RS} \int_0^1 1 \, \mathrm{d}\alpha(x) \\
&= \lim_{N \to \infty} \sum_{i=1}^{N} (1)(\alpha(x_i) - \alpha(x_{i-1})
\end{aligned}$$

since $\alpha(x_i) - \alpha(x_{i-1})$ is only nonzero whenever $i = k_j \forall_{j=1,\dots,n}$

$$\begin{aligned}
&= \sum_{i=1}^{n} (\alpha(x_{k_i}) - \alpha(x_{k_j - 1}) \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{i} c_j - \sum_{j=1}^{i-1} c_j \right) \\
&= \sum_{i=1}^{n} c_i \\
&\leq \sum_{i=1}^{n} |c_i|
\end{aligned}$$

Therefore,

$$V_0^1(\alpha) \leq \sum_{i=1}^{n} |c_i|$$

Problem 4

Problem:

Let

$$f(x) = \begin{cases} x^2 & \text{if} -1 \le x \le 0; \\ x^3 & \text{if} 0 < x \le 1; \end{cases} \text{ and } \alpha(x) = \begin{cases} 1 & \text{if} x = -1; \\ 2x^2 & \text{if} -1 < x < 1; \\ -1 & \text{if} x = 1. \end{cases}$$

Evaluate the Darboux-Stieltjes integral $\int_{-1}^{1} f(x)\,\mathrm{d}\alpha(x)$.

Solution:

Example 1. *Let*

$$f(x) = \begin{cases} x^2 & -1 \le x \le 0 \\ x^3 & 0 < x \le 1 \end{cases}$$

and

$$\alpha(x) = \begin{cases} 1 & x = -1 \\ 2x^2 & -1 < x < 1 \\ -1 & x = 1 \end{cases}$$

Evaluate the Darboux-Stieltjes integral $\int_{-1}^{1} f(x)\,\mathrm{d}\alpha(x)$.

The definition of a Darboux-Stieltjes integral is

$$\mathcal{DS}\int_a^b f\,\mathrm{d}\alpha = U(f,\alpha) = \inf_P U(f,\alpha,P) = \sup_P L(f,\alpha,P) = L(f,\alpha)$$

where the upper Darboux-Stieltjest sum is

$$U(f,\alpha,P) = \sum_{i=1}^{n} M(f,[x_{i-1},x_i]) \cdot \Delta_i\alpha = \sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} f(x) \cdot (\alpha(x_{i-1}) - \alpha(x_i)$$

and the lower Darboux-Stieltjest sum is

$$U(f,\alpha,P) = \sum_{i=1}^{n} m(f,[x_{i-1},x_i]) \cdot \Delta_i\alpha = \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} f(x) \cdot (\alpha(x_{i-1}) - \alpha(x_i)$$

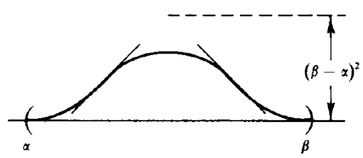
These definitions could be used to both directly compute the sums and that the integral exists; however, we can take the conclusion that $\int_{-1}^{1} f(x)\,\mathrm{d}\alpha(x)$ exists since $f(x)$ is continuous and $\alpha(x)$ is differentiable apart from two finite points (at a and b).

Taking a few jumps we have that

$$\begin{aligned}
\int_{-1}^{1} f(x)\,\mathrm{d}\alpha(x) &= f(-1)(\alpha(-1^+) - \alpha(-1^-)) + \int_{-1}^{0} f(x)\frac{\mathrm{d}\alpha}{\mathrm{d}x} + \int_{0}^{1} f(x)\frac{\mathrm{d}\alpha}{\mathrm{d}x} + f(1)(\alpha(1^+) - \alpha(1^-)) \\
&= (1)(2-1) + \int_{-1}^{0} (x^2)(4x)\,\mathrm{d}x + \int_{0}^{1} (x^3)(4x)\,\mathrm{d}x + (1)(-1-2) \\
&= (1)(1) + \int_{-1}^{0} 4x^3\,\mathrm{d}x + \int_{0}^{1} 4x^4\,\mathrm{d}x + (1)(-3) \\
&= 1 + x^4\Big|_{-1}^{0} + \frac{4}{5}x^5\Big|_{0}^{1} - 3 \\
&= 1 - 3 + (0^4 - (-1)^4) + \frac{4}{5}(1^5 - (0)^5) \\
&= -2 - 1 + \frac{4}{5} \\
&= \frac{-11}{5} = -2.2
\end{aligned}$$

Problem 5

Let C be the Cantor set in $[0,1]$. The Cantor set C is created by iteratively deleting the open middle third from a set of non-overlapping closed intervals. One starts by deleting the open middle third $(\frac{1}{3}, \frac{2}{3})$ from the interval $[0,1]$, leaving two closed intervals: $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Next, the open middle third of each of these remaining intervals is deleted, leaving four closed intervals: $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$, and $[\frac{8}{9}, 1]$. Continue this process forever. The Cantor set contains all points in the interval $[0,1]$ that are not deleted at any step in this infinite process. Let D be the open set deleted. Then $C = [0,1] \sim D$.
A continuous function f is defined to be zero on C and on each component interval (α, β) of D to have its graph as shown in the figure. The exact equation is not important, but on (α, β), f' is continuous, $f'(\alpha^+) = f'(\beta^-) = 0$, $\max_{x \in (\alpha,\beta)} |f'(x)| = 1$, and $\max_{x \int(\alpha,\beta)} f(x) \leq (\beta - \alpha)^2$. Show that the Riemann integral $\int_{0,1} f'(x)\,\mathrm{d}x$ doesn't exist even though $f'(x)$ exists and are bounded on $[0,1]$.



Example 2. *Let $C \in [0,1]$ be the Cantor set and $D = C'$. Let $f : [0,1] \to \mathbb{R}$ be a continuous function defined as 0 $\forall_{x \in C}$ and on each interval of D, (α, β), we have*

$$f'(\alpha^+) = f'(\beta^-) = 0 \wedge \max_{x \in (\alpha,\beta)} |f'(x)| = 1 \wedge \max_{x \in (\alpha,\beta)} f(x) \leq (\beta - \alpha)^2$$

However, the Riemann integral $\mathcal{R}\int_0^1 f'(x)\,\mathrm{d}x$ does not exist.

Consider the partition P defined by

$$P = \{0 = x_0 < x_1 < \cdots < x_N = 1\}$$

with mesh size $\operatorname{mesh}(P) < \delta$. Let $(\alpha_0, \beta_0) \in [0,1]$ describe an arbitrary interval of D.
The lower Riemann (Darboux) sum is defined as

$$L(f,P) = \sum_{k=1}^{N} m(f, [x_{k-1}, x_k])$$

The upper Riemann (Darboux) sum is defined as

$$U(f,P) = \sum_{k=1}^{N} M(f, [x_{k-1}, x_k])$$

When $\operatorname{mesh}(P) > (\alpha_0 - \beta_0)$, $\forall_{k=1,\dots,N}$, $m(f, [x_{k-1}, x_k]) = 0$ since $\exists_{x \in [x_{k-1}, x_k] : x \in C}$ and $0 < M(f, [x_{k-1}, x_k]) \leq (\beta_0 - \alpha_0)^2$ since $\exists_{x \in [x_{k-1}, x_k] : x \in D}$.
This means that $\forall_P$,

$$L(f,P) = \sum_{k=1}^{N} (m(f, [x_{k-1}, x_k] = 0) = 0$$

and

$$U(f,P) = \sum_{k=1}^{N} (M(f, [x_{k-1}, x_k]) > 0) > 0$$

Thus,

$$L(f) = 0 < U(f)$$

and therefore

$$L(f) \neq U(f)$$

which means that f is not Riemann Integrable.