MATH 5302 Elementary Analysis II - Homework 5

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Preliminaries

Definition 1. Darboux-Stieltjes Integral *Let* $f : [a, b] \to \mathbb{R}$ *and* $\alpha : [a, b] \to \mathbb{R}$*, with* f *bounded and* α *increasing on* $[a, b]$*. Let partition* P *be defined as*

$$P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

Let

$$M(f, [x_1, x_2]) = \sup_{x \in [x_1, x_2]} f(x)$$

and

$$m(f, [x_1, x_2]) = \inf_{x \in [x_1, x_2]} f(x)$$

a. The upper and lower Darboux-Stieltjes Sums *are defined*

$$U(f, \alpha, P) = \sum_{i=1}^{n} M(f, [x_{i-1}, x_i]) \cdot \Delta_i \alpha$$

and

$$L(f, \alpha, P) = \sum_{i=1}^{n} m(f, [x_{i-1}, x_i]) \cdot \Delta_i \alpha$$

respectively with

$$\Delta_i \alpha = \alpha(x_i) - \alpha(x_{i-1})$$

A more general sum $S(f, \alpha, P)$ *is when* $f(x_i^*)$ *for* $x_i^* \in [x_{i-1}, x_i]$ *is used instead.*

Note:

$$m(f, [a, b]) \cdot (\alpha(b) - \alpha(a)) \leq L(f, \alpha, P) \leq U(f, \alpha, P) \leq M(f, [a, b]) \cdot (\alpha(b) - \alpha(a))$$

b. The upper and lower Darboux-Stieltjes Integrals *are defined*

$$U(f, \alpha) = \inf_{P \text{ partition of } [a,b]} U(f, \alpha, P)$$

and

$$L(f, \alpha) = \sup_{P \text{ partition of } [a,b]} U(f, \alpha, P)$$

respectively.

Note:

$$L(f, \alpha) \leq L(f, \alpha, P) \leq U(f, \alpha, P) \leq U(f, \alpha)$$

for any P *partition of* $[a, b]$*.*

c. *f is called* Darboux-Stieltjes Integrable *with respect to* α *if and only if*

$$\forall_{\epsilon>0}\exists_{P=\{a=x_0<x_1<\cdots<x_{n-1}<x_n=b\}} : U(f,\alpha,P) - L(f,\alpha,P) < \epsilon$$

in which case the Darboux-Stieltjes Integral *with respect to* α *is defined as*

$$\mathcal{DS}\int_a^b f\,\mathrm{d}\alpha = U(f,\alpha) = L(f,\alpha)$$

Note: *If f is also continuous on $[a,b]$ then f is Riemann-Stieljes integrable which implies f is Darboux-Stieljes integrable.*

Properties: *When f is Darboux-Stieltjes integrable on $[a,b]$ and α is increasing on $[a,b]$ then*

a. *$|f|$ is Darboux-Stieltjes integrable on $[a,b]$ and*

$$\mathcal{DS}\int_a^b f\,\mathrm{d}\alpha \leq \mathcal{DS}\int_a^b |f|\,\mathrm{d}\alpha$$

b. *f^2 is Darboux-Stieltjes integrable on $[a,b]$.*

c. *If g is also Darboux-Stieltjes integrable on $[a,b]$, then fg is Darboux-Stieltjes integrable on $[a,b]$.*

d. *For α_1 and α_2 also increasing on $[a,b]$ and f is Darboux-Stieltjes integrable with respect to α_1 and α_2, then f is Darboux-Stieltjes integrable with respect to α_1 and α_2. Additionally,*

$$\begin{aligned}&\mathcal{DS}\int_a^b f(x)\,\mathrm{d}\alpha_1(x) + \mathcal{DS}\int_a^b f(x)\,\mathrm{d}\alpha_2(x)\\ =&\mathcal{DS}\int_a^b f(x)\,\mathrm{d}\alpha(x) + \alpha_2(x)\end{aligned}$$

e. *For $a<c<b$, f is Darboux-Stieltjes integrable with respect to α on $[a,b]$ if and only if f is Darboux-Stieltjes integrable with respect to α on $[a,c]$ and $[c,b]$. Furthermore,*

$$\mathcal{DS}\int_a^b f(x)\,\mathrm{d}\alpha(x) = \mathcal{DS}\int_a^c f(x)\,\mathrm{d}\alpha(x) + \mathcal{DS}\int_c^b f(x)\,\mathrm{d}\alpha(x)$$

Definition 2. ***Continuity:*** *Let $f:[a,b]\to\mathbb{R}$.*

a. *f is* Lipschitz Continuous *on $[a,b]$ if*

$$\exists_C : \forall_{x,y\in[a,b]}|f(x)-f(y)| \leq |x-y|$$

b. *f is* Absolutely Continuous *on $[a,b]$ if*

$$\forall_{\epsilon>0}\exists_{\delta>0}\forall_{\textit{finite collection } \{(x,x')\} \textit{ of nonoverlaping intervals}:\sum_{i=1}^n |x_i'-x_i|<\delta}\sum_{i=1}^n |f(x_i')-f(x_i)| < \epsilon$$

c. *f is* uniformly continuous *on $[a,b]$ if*

$$\forall_{\epsilon>0}\exists_{\delta>0} : (x,y\in[a,b]) \wedge \{|x-y|<\delta\} \implies |f(x)-f(y)| < \epsilon$$

d. *f is* continuous *on $[a,b]$ if f is continuous at all $x_0\in[a,b]$. i.e.*

$$\forall_{\epsilon>0}\exists_{\delta>0} : \forall_{x\in[a,b]} \wedge |x-x_0| < \delta \implies |f(x)-f(x_0)| < \epsilon$$

Properties:

a. *f continuous on closed $[a, b]$, then f is uniformly continuous on $[a, b]$.*

b. *f differentiable at $x \in [a, b]$ implies Locally Lipschitz continuous at x.*

c. *$C^1[a, b]$ is the set of differentiable functions with continuous derivatives on $[a, b]$.*

d. *$C^1[a, b] \subset$ differentiable functions with bounded derivatives*

e. *Differentiable with bounded derivatives $\implies$ Lipschitz continuous $\implies$ Absolutely continuous $\implies$ uniformly continuous $\implies$ continuous*

Problem 1

Let f be a real-valued bounded function on $[-1,1]$. Let

$$\alpha(x) = \begin{cases} 0 & \text{if } -1 \leq x < 0; \\ 2 & \text{if } 0 \leq x \leq 1. \end{cases}$$

Assume f is Riemann-Stieljes integrable with respect to α on $[-1,1]$. Show that

a. f is continuous at 0 from the left.

b. $\int_{-1}^{1} f(x)\, \mathrm{d}\alpha(x) = 2f(0)$.

a) f is continuous at 0 from the left

Example 1. *Let $f : [-1,1] \to \mathbb{R}$ bounded. Let*

$$\alpha(x) = \begin{cases} 0 & -1 \leq x < 0 \\ 2 & 0 \leq x < 1 \end{cases}$$

If f is Riemann-Stieljes integrable w.r.t. α on $[-1,1]$, then f is continuous at 0 from the left.

Proof. f is Riemann-Stieljes integrable w.r.t. α on $[-1,1]$ means that

$$\forall_{\epsilon>0} \exists_{\gamma} \exists_{\delta>0} : \forall_{P:\mathrm{mesh}(P)<\delta} \implies |S(f,\alpha,P) - \epsilon|$$

For

$$P = \{-1 = x_0 < x_1 < \cdots < x_{k-1} = -\gamma < x_k = 0 < x_{k-1} < \cdots x_n = 1\}$$

with $\mathrm{mesh}(P) < \delta$. For x s.t. $-\gamma < x < 0$ Select $x_i^* \in [x_i, x_{i+1}]$ and $x_{k-1}^* = x$.

$$\begin{aligned} \sum_{i=1}^{n} f(x_i^*)\Delta_i \alpha &= f(x)(\alpha(x_k) - \alpha(x_{k-1})) \\ &= f(x)(2-0) = 2f(x) \\ \sum_{i=1}^{n} f(x_i^*)\Delta_i \alpha &= 2f(x) \end{aligned}$$

Therefore,

$$|f(x) - \gamma| < \epsilon$$

and

$$\lim_{x \to 0^-} f(x) = \gamma$$

Then by taking $x_{k-1}^* = 0$, the Riemann-Stieltjes sum $S = f(0)$, therefore

$$|f(x=0) - \gamma| < \epsilon,\ \forall_{\epsilon>0}$$

which is the definition of continuity meaning that f is continuous at 0 from the left. □

b) $\int_{-1}^{1} f(x)\, \mathrm{d}\alpha(x) = 2f(0)$

Example 2. *Let $f : [-1,1] \to \mathbb{R}$ bounded. Let*

$$\alpha(x) = \begin{cases} 0 & -1 \leq x < 0 \\ 2 & 0 \leq x < 1 \end{cases}$$

If f is Riemann-Stieljes integrable w.r.t. α on $[-1,1]$, then $\int_{-1}^{1} f(x)\, \mathrm{d}\alpha(x) = 2f(0)$

Proof. Let

$$P = \{-1 = x_0 < x_1 < \cdots < x_{k-1} = -\gamma < x_k = 0 < x_{k-1} < \cdots x_n = 1\}$$

with $\text{mesh}(P) < \delta$. By definition,

$$\mathcal{DS} \int_a^b f(x) \, \mathrm{d}\alpha(x) = \lim_{\delta \to 0} S(f, \alpha, P)$$

and

$$\begin{aligned} S(f, \alpha, P) &= \sum_{i=1}^{n} f(x_i^*) \Delta_i \alpha \\ &= \sum_{i=1}^{n} f(x_i^*)(\alpha(x_i) - \alpha(x_{i-1})) \end{aligned}$$

since $\alpha(x_i) - \alpha(x_{i-1}) = 0 \forall_{i \neq k}$

$$\begin{aligned} &= f(x_k)(\alpha(x_k) - \alpha(x_{k-1})) \\ &= f(x_k)(2 - 0) = 2f(x_k) = 2f(0) \end{aligned}$$

$$\boxed{\mathcal{DS} \int_a^b f(x) \, \mathrm{d}\alpha(x) = 2f(0)}$$

$\square$

Problem 2

Let f and α be real-valued bounded functions on $[a, b]$ and α is increasing. Let $L(f, \alpha)$ and $U(f, \alpha)$ represents the lower and upper Darboux-Stieltjes integral of f with respect to α on $[a, b]$, respectively,

a. Show that $U(f, \alpha) \leq U(|f|, \alpha)$.

b. Is it true that $L(f, \alpha) \leq L(|f|, \alpha)$?

Example 3. *Let $f : [a, b] \to \mathbb{R}$ and $\alpha : [a, b] \to \mathbb{R}$, with f and α both bounded with α also increasing on $[a, b]$. Let partition P be defined as*

$$P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

Let

$$M(f, [x_1, x_2]) = \sup_{x \in [x_1, x_2]} f(x)$$

and

$$m(f, [x_1, x_2]) = \inf_{x \in [x_1, x_2]} f(x)$$

Then by definition,

$$U(f, \alpha, P) = \sum_{i=1}^{n} M(f, [x_{i-1}, x_i]) \cdot \Delta_i \alpha$$

and

$$L(f, \alpha, P) = \sum_{i=1}^{n} m(f, [x_{i-1}, x_i]) \cdot \Delta_i \alpha$$

respectively with

$$\Delta_i \alpha = \alpha(x_i) - \alpha(x_{i-1})$$

Additionally,

$$U(f, \alpha) = \inf_{P \text{ partition of } [a,b]} U(f, \alpha, P)$$

and

$$L(f, \alpha) = \sup_{P \text{ partition of } [a,b]} U(f, \alpha, P)$$

a. **Show that $U(f, \alpha) \leq U(|f|, \alpha)$.**

Proof. Since f is a real-valued function, $\forall_{x \in [a,b]} f(x) \leq |f(x)|$. For every $[x_1, x_2] \in [a, b]$,

$$M(f, [x_1, x_2]) = \sup_{x \in [x_1, x_2]} f(x) \leq \sup_{x \in [x_1, x_2]} |f(x)| = M(|f|, [x_1, x_2])$$

Then for every P,

$$U(f, \alpha, P) = \sum_{i=1}^{n} M(f, [x_{i-1}, x_i]) \cdot \Delta_i \alpha \leq \sum_{i=1}^{n} M(|f|, [x_{i-1}, x_i]) \cdot \Delta_i \alpha = U(|f|, \alpha, P)$$

Therefore,

$$U(f, \alpha) = \inf_{P \text{ partition of } [a,b]} U(f, \alpha, P) \leq \inf_{P \text{ partition of } [a,b]} U(|f|, \alpha, P) = U(|f|, \alpha)$$

□

b. **Is it true that $L(f, \alpha) \leq L(|f|, \alpha)$?** *Yes, by the same logic as the previous statement. Essentially since every value of $f \leq |f|$, the same progression is true.*

Proof. Since f is a real-valued function, $\forall_{x\in[a,b]} f(x) \leq |f(x)|$. For every $[x_1, x_2] \in [a, b]$,

$$m(f, [x_1, x_2]) = \inf_{x\in[x_1,x_2]} f(x) \leq \inf_{x\in[x_1,x_2]} |f(x)| = m(|f|, [x_1, x_2])$$

Then for every P,

$$L(f, \alpha, P) = \sum_{i=1}^{n} m(f, [x_{i-1}, x_i]) \cdot \Delta_i\alpha \leq \sum_{i=1}^{n} m(|f|, [x_{i-1}, x_i]) \cdot \Delta_i\alpha = L(|f|, \alpha, P)$$

Therefore,

$$L(f, \alpha) = \sup_{P \text{ partition of } [a,b]} L(f, \alpha, P) \leq \sup_{P \text{ partition of } [a,b]} L(|f|, \alpha, P) = L(|f|, \alpha)$$

$\square$

Problem 3

Let α be a bounded real-valued increasing function on $[a, b]$. Assume $a < c < b$ and α is continuous at c. Let

$$f(x) = \begin{cases} 1 & \text{if } x = c; \\ 0 & \text{if } x \neq c. \end{cases}$$

Show directly that f is Darboux-Stieltjes integrable on $[a, b]$ and $\int_a^b f(x)\,\mathrm{d}\alpha(x) = 0$. (Do not use Theorem 8.16.)

Example 4. *Let $\alpha : [a, b] \to \mathbb{R}$ bounded and increasing. Let $c \in (a, b)$ such that α is continuous at c. Let*

$$f(x) = \begin{cases} 1 & \textit{if } x = c; \\ 0 & \textit{if } x \neq c. \end{cases}$$

a. **f is Darboux-Stieltjes integrable on $[a, b]$.**

Proof. By definition, f is Darboux-Stieltjes integrable with respect to α on $[a, b]$ if and only if

$$\forall_{\epsilon>0}\exists_{P=\{a=x_0<x_1<\cdots<x_{n-1}<x_n=b\}} : U(f, \alpha, P) - L(f, \alpha, P) < \epsilon$$

Let $\epsilon > 0$. Let partition P be defined as

$$P = \{a = x_0 < x_1 < \cdots < x_{k-1} < x_k = c < x_{k-1} < \cdots < x_{n-1} < x_n = b\}$$

such that $\mathrm{mesh}(P) < \delta$ for some $\delta > 0$. The lower Darboux-Stieltjes sum is given by

$$\begin{aligned} L(f, \alpha, P) &= \sum_{i=1}^{n} m(f, [x_{i-1}, x_i])\Delta_i\alpha \\ &= \sum_{i=1}^{n} (\inf_{x\in[x_{i-1},x_i]} f(x))\Delta_i\alpha \end{aligned}$$

since $f(x) = 0\forall_i \neq k$,

$$= \sum_{i=1}^{k-1}(0)\Delta_i\alpha + \sum_{i=k+2}^{n}(0)\Delta_i\alpha + \inf_{x\in[x_{k-1},x_k]} f(x)\Delta_k\alpha + \inf_{x\in[x_k,x_{k-1}]} f(x)\Delta_{k+1}\alpha$$

which still results in 0 terms

$$\begin{aligned} &= 0 + 0 + 0(\Delta_k\alpha) + 0(\Delta_{k+1}\alpha) \\ L(f, \alpha, P) &= 0 \end{aligned}$$

The upper Darboux-Stieltjes sum is given by

$$\begin{aligned} U(f, \alpha, P) &= \sum_{i=1}^{n} M(f, [x_{i-1}, x_i])\Delta_i\alpha \\ &= \sum_{i=1}^{n} (\sup_{x\in[x_{i-1},x_i]} f(x))\Delta_i\alpha \end{aligned}$$

since $f(x) = 0 \forall_i \neq k$,

$$\begin{aligned}
&= \sum_{i=1}^{k-1} (0) \Delta_i \alpha \\
&\quad + \sum_{i=k+2}^{n} (0) \Delta_i \alpha \\
&\quad + \sup_{x \in [x_{k-1}, x_k]} f(x) \Delta_k \alpha \\
&\quad + \sup_{x \in [x_k, x_{k-1}]} f(x) \Delta_{k+1} \alpha \\
&= 0 + 0 + 1(\Delta_k \alpha) + 1(\Delta_{k+1} \alpha) \\
&= (\alpha(x_k) - \alpha(x_{k-1})) + (\alpha(x_{k+1}) - \alpha(x_k)) \\
&= \alpha(x_{k+1}) - \alpha(x_{k-1}
\end{aligned}$$

since $\text{mesh}(P) < \delta$,

$$\begin{aligned}
&\leq 2\delta \\
U(f, \alpha, P) &\leq 2\delta
\end{aligned}$$

Thus, setting $\delta < \frac{\epsilon}{2}$ results in

$$U(f, \alpha, P) - L(f, \alpha, P) \leq 2\delta < \epsilon$$

$\square$

b. $\underline{\int_a^b f(x) \, \mathrm{d}\alpha(x) = 0}$

Proof. From above we have $L(f, \alpha, P) = 0$ and $U(f, \alpha, P) \leq 2\delta$ for $\text{mesh}(P) < \delta$. So we have $L(f, \alpha) = \sup_P L(f, \alpha, P) = 0$. Taking the limit as $\delta \to 0$, we have

$$U(f, \alpha) = \lim_{\delta \to 0} U(f, \alpha, P) \leq \lim_{\delta \to 0} 2\delta = 0$$

so

$$\mathcal{DS} \int_a^b f \, \mathrm{d}\alpha = U(f, \alpha) = L(f, \alpha) = 0$$

$\square$

Problem 4

Let f and α be real-valued bounded functions on $[a, b]$ and α is increasing on $[a, b]$. Assume f is Darboux-Stieltjes integrable with respect to α on $[a, b]$. Let $[c, d] \subset [a, b]$. Show that f is Darboux-Stieltjes integrable with respect to α on $[c, d]$.

Example 5. *Let $f : [a, b] \to \mathbb{R}$ and $\alpha : [a, b] \to \mathbb{R}$, with f bounded on $[a, b]$ and α bounded and increasing on $[a, b]$. Let partition P be defined as*

$$P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

Assume f is Darboux-Stieltjes integrable with respect to α on $[a, b]$.
f is is Darboux-Stieltjes integrable with respect to α on all $[c, d] \subset [a, b]$

Proof. f is Darboux-Stieltjes integrable with respect to α on $[a, b]$ means that

$$\forall_{\epsilon>0} \exists_{P=\{a=x_0<x_1<\cdots<x_{n-1}<x_n=b\}} : U(f, \alpha, P) - L(f, \alpha, P) < \epsilon$$

when taking the limit of epsilon to 0. For this P, find cl, cu, dl, and du so that

$$P = \{a = x_0 < x_1 < \cdots < x_{cl} < c < x_{cu} < \cdots < x_{dl} < d < x_{du} < \cdots < x_n = b\}$$

Take

$$P_{cd} = \{x_{cl} < c < x_c < \cdots < x_{dl} = d < x_{du}\}$$

$$\begin{aligned}
U(f, \alpha, P) - L(f, \alpha, P) &= \sum_{i=1}^{n} M(f, [x_{i-1}, x_i])\Delta_i\alpha \\
&\quad - \sum_{i=1}^{n} m(f, [x_{i-1}, x_i])\Delta_i\alpha && < \epsilon \\
&= \sum_{i=1}^{cl-1} M(f, [x_{i-1}, x_i])\Delta_i\alpha \\
&\quad + \sum_{i=cl}^{du} M(f, [x_{i-1}, x_i])\Delta_i\alpha \\
&\quad + \sum_{i=du+1}^{n} M(f, [x_{i-1}, x_i])\Delta_i\alpha \\
&\quad - \sum_{i=1}^{cl-1} m(f, [x_{i-1}, x_i])\Delta_i\alpha \\
&\quad - \sum_{i=cl}^{du} m(f, [x_{i-1}, x_i])\Delta_i\alpha \\
&\quad - \sum_{i=du+1}^{n} m(f, [x_{i-1}, x_i])\Delta_i\alpha && < \epsilon
\end{aligned}$$

Observing the bounded sum

$$\sum_{i=cl}^{du} M(f, [x_{i-1}, x_i])\Delta_i\alpha - \sum_{i=cl}^{du} m(f, [x_{i-1}, x_i])\Delta_i\alpha < \epsilon - (***)$$

with $***$ being all the other bounded terms transferred to the RHS. This is a sum that bounds the $U(f,\alpha,P_{cd}) - L(f,\alpha,P_{cd})$ from above by potentially including the regions $[x_{cl}, c)$ and $(d, x_{du}]$ which would have an additional bounded term. Taking $\text{mesh}(P) \to 0$ we cause $x_{cl} \to c$ and $x_{du} \to d$ so that

$$U(f,\alpha,P_{cd}) - L(f,\alpha,P_{cd}) = \sum_{i=cl}^{du} M(f,[x_{i-1},x_i]) - m(f,[x_{i-1},x_i])\Delta_i\alpha < \epsilon_{cd} = \epsilon - (***)$$

and therefore since $\epsilon - (***)$ is bounded for all δ,

$$\forall_{\epsilon>0}\exists_\delta : \text{mesh}(P) < \delta \implies U(f,\alpha,P_{cd}) - L(f,\alpha,P_{cd}) < \epsilon_{cd}$$

$\square$

Problem 5

Let α be a real-valued bounded function on $[a, b]$ and α is increasing with $\alpha(a) < \alpha(b)$. Let

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that if α is continuous on $[a, b]$, then f is not Darboux-Stieltjes integrable with respect to α on $[a, b]$.

Example 6. *Let $\alpha : [a, b] \to \mathbb{R}$ bounded and increasing with $\alpha(a) < \alpha(b)$. Let*

$$f(x) = \begin{cases} 1 & \textit{if } x \textit{ is rational;} \\ 0 & \textit{if } x \textit{ is irrational.} \end{cases}$$

If α is continuous on $[a, b]$ then f is not Darboux-Stieltjes integrable with respect to α on $[a, b]$.

Proof. By definition, α continuous on the closed interval $[a, b]$ implies α is uniformly continuous on $[a, b]$. This means that

$$\forall_{\epsilon_1} \exists_{\delta > 0} : a \leq x < y \leq b,\ y - x < \delta \implies \alpha(y) - \alpha(x) < \epsilon_1$$

In order for f to be Darboux-Stieltjes integrable with respect to α on $[a, b]$, then for every ϵ_2 there must exists a partition P of $[a, b]$,

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

so that

$$U(f, \alpha, P) - L(f, \alpha, P) < \epsilon_2$$

For this example, we have

$$U(f, \alpha, P) - L(f, \alpha, P) = \sum_{i=1}^{n} M(f, [x_{i-1}, x_i])\Delta_i\alpha - \sum_{i=1}^{n} m(f, [x_{i-1}, x_i])\Delta_i\alpha$$

Since α is continuous and increasing, $\Delta_i\alpha = \alpha(x_{i-1}) - \alpha(x_i) > 0$

$$= \sum_{i=1}^{n} M(f, [x_{i-1}, x_i])(\alpha(x_{i-1}) - \alpha(x_i)) - \sum_{i=1}^{n} m(f, [x_{i-1}, x_i])(\alpha(x_{i-1}) - \alpha(x_i))$$

For f we have that $M(f, [x_1, x_2]) = 1$ and $m(f, [x_1, x_2]) = 0$ $\forall_{x_1 \neq x_2 \in [a,b]}$

$$\begin{aligned} &= \sum_{i=1}^{n} (1)(\alpha(x_{i-1}) - \alpha(x_i)) - \sum_{i=1}^{n} (0)(\alpha(x_{i-1}) - \alpha(x_i)) \\ &= \sum_{i=1}^{n} \alpha(x_{i-1}) - \alpha(x_i) \\ &= \alpha(x_n) - \alpha(x_0) = \alpha(b) - \alpha(a) > 0 \end{aligned}$$

Which contradicts the conditions needed for f to be Darboux-Stieltjes integrable with respect to α on $[a, b]$. □