Assignment 2 of Math 5302

Due Date: Feb. 9, 2022 at 11:59pm

1. Complete the proof of Theorem 2.3 in the lecture notes by showing that a decreasing function on $[a, b]$ is integrable.

2. Let f be a bounded function on $[a, b]$, so that there exists $B > 0$ such that $|f(x)| \leq B$ for all $x \in [a, b]$.

(a) Show

$$U(f^2, P) - L(f^2, P) \leq 2B[U(f, P) - L(f, P)]$$

for all partitions P of $[a, b]$. Hint: $f^2(x) - f^2(y) = (f(x) + f(y))(f(x) - f(y))$.

(b) Show that if f is integrable on $[a, b]$, then f^2 is also integrable on $[a, b]$.

3. Let f be a bounded function on $[a, b]$. Suppose that f^2 is integrable on $[a, b]$. Must f also be integrable on $[a, b]$?

4. Suppose that f and g are integrable on $[a, b]$. Show that $\max(f, g)$ is also integrable on $[a, b]$.

Hint: Derive and apply the formula

$$\max(f, g) = \frac{1}{2}(f + g + |f - g|).$$

5. Suppose f and g are continuous functions on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Prove there exists x in (a, b) such that $f(x) = g(x)$.