Introduction to

Mathematical Analysis

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with collaboration of

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Series of Lecture Notes on Nonlinear Analysis

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Part I

PRELIMINARIES AND METRIC SPACES

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Basic Set Theory, Logic and Introduction to Proofs

1.1 Sets

We will use the word "set" as an everyday word, whose meaning everybody knows. Sometimes, we will use other synonyms of "set" such as "class" or "collection" (in particular when we will be talking about a set composed of other sets).

The important question is: *What are members (or elements) of a set?* For a given set A, we will use the notation "$x \in A$" to say that "x is an element (or member) of A." The symbol "$\in$" stands for "in," i.e. "$x \in A$" should be read "x in A." If x is not an element of A, then we will write $x \notin A$, and read it "x not in A."

A set can be described by enumerating its elements within a pair of braces. For example,

$$A := \{2, 3, 5, 7\}, \quad B := \left\{\frac{3}{4}, \frac{5}{8}\right\},$$
$$C := \{1, 2, 3, \dots, 1000\},$$
$$D := \{n : n \text{ is an integer and } 1 \leq n \leq 1000\}.$$

In particular, we will be using the following standard notation:

(a) $\mathbb{N}$ — the set of all natural numbers $\{1, 2, 3, 4, \dots\}$;
(b) $\mathbb{Z}$ — the set of all integer numbers $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$;
(c) $\mathbb{Q}$ — the set of all rational numbers $\left\{\frac{m}{n} : m,\, n \in \mathbb{Z},\ n \neq 0\right\}$;
(d) $\mathbb{R}$ — the set of all real numbers,which will be explained later;
(e) $\emptyset$ — the empty set, i.e. the set which has no elements at all. It is important to point out that there is exactly one empty set $\emptyset$.

A set can also be a collection of other sets. For example, the set $\{\emptyset\}$ is not empty, it contains one element, which is the empty set; the set $\{\emptyset, \{\emptyset\}\}$ contains two elements, one of them is the empty set, another the set $\{\emptyset\}$. Therefore, one should not confuse the set with its elements; for example, the symbols $\{x\}$ and x are not the same; here x is an element of the set $\{x\}$, i.e. $x \in \{x\}$.

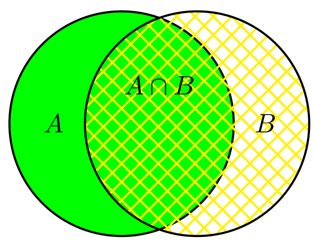
1.1.1 Subsets

We say that A is *included* in B, and write $A \subset B$, if every element of A is an element of B. In such a case, we will also say that A is *contained* in B or B *contains* A. We will use the symbol $A \subset B$ (or $A \subseteq B$) to write that A is included in B. If A and B are the sets satisfying $A \subset B$ and $B \subset A$, we will simply write $A = B$, i.e. A and B stand for the same set. If $A \subset B$, but $A \neq B$, then we will write $A \subsetneq B$. If $A \subset B$, then we will say that A is a *subset* of B. Notice that $\emptyset$ is a subset of every set, i.e. $\emptyset \subset A$ for every set A, and A is a subset of itself, i.e. $A \subset A$.

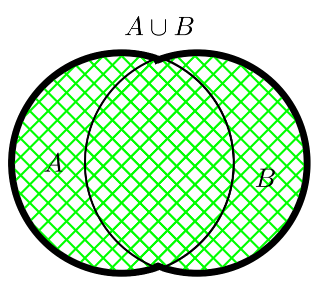
1.1.2 Operations on Sets

The fundamental operations on sets are *intersection* and *union*.

The *intersection* of two sets A and B is the set of all elements belonging to both A and B. The intersection of A and B is denoted by $A \cap B$. Notice that $x \in A \cap B$ if and only if $x \in A$ and $x \in B$. The following Venn diagram illustrates the intersection of two sets A and B.



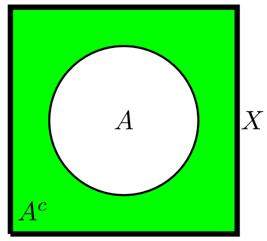
The *union* of two sets A and B is the set of all elements belonging to A or B. The union of A and B is denoted by $A \cup B$. The following Venn diagram illustrates the union $A \cup B$ of A and B.



1.1.3 Complementation of Sets

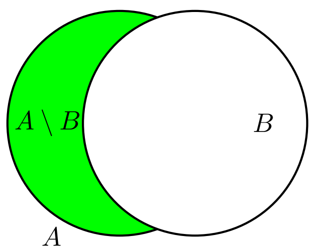
We introduce an additional operation: complementation. In order to do so, we have to make an agreement that everything is taking place inside some big fixed set X, i.e. we are interested only in elements and subsets of the set X. In such a case, the set X will be called a *space*. Given a set A with

$A \subset X$, we define A^c, the *complement* of A, to be the set of all elements in X but not in A (see the Venn diagram below).



1.1.4 Subtraction of Sets

Given two sets A and B we define $A \setminus B$ to be the set composed of all elements x such that $x \in A$ and $x \notin B$. In other words, the set $A \setminus B$ is the set of all elements of A which do not belong to B (see the diagram below).



1.2 Elements of Logic

1.2.1 Formal Statements

We have so far introduced fundamental notions of the set theory and, where necessary, described the basic operations on sets. Now, we are ready to turn to the derivation of elementary results describing the properties of these operations. Before we commence, we must agree about "the rules of our game," i.e. we have to describe what is suggested to be a rightful way of obtaining results. Results are usually formulated as *statements* —declarative sentences that are either true or false (the latter two notions are indefinable). Not all declarative sentences are statements. For example, sentences such as "I am tired" or "Calgary is far away" could be true or false depending upon the context or circumstances. Thus, they are not statements as we have formally defined this term, even though they are statements in the everyday meaning of that word. We shall denote statements by small Latin letters p, q or r.

1.2.2 Connectives: Elementary Notions of Logic

Statements may be used to form new statements by means of operations that are called *sentential connectives*. For the sake of brevity, each connective is denoted by a certain symbol. These notations

Name	**Symbolic Representation**	**Meaning**	**Other Common Notation**
Negation	$\sim p$	"NOT p "	$\neg p$
Conjunction	$p \wedge q$	"p AND q"	$p \cdot q$
Disjunction	$p \vee q$	"p OR q"	$p + q$
Implication	$p \Rightarrow q$	"p IMPLIES q"	$p \to q$, $p \equiv q$
Biconditional	$p \Leftrightarrow q$	"p IF AND ONLY IF q"	$p \leftrightarrow q$ "p IFF q"

Table 1.1. Sentential Connectives

are not universal, i.e. they may differ from text to text. The sentential connectives are listed in Table 1.1.

More precisely:

- For a given statement p, its *negation* $\sim p$ is a statement meaning (consisting in) the falsity of p. For example, for the statement p : "it is raining" we have the negation $\sim p$:"it is not raining," or if q : "m is even" then we have $\sim q$: "m is not even";
- *Disjunction* is the statement formed from two given statements by the connective "or" (which we denote by $\vee$), thereby asserting the truth of at least one of the given statements.
- *Conjunction* is the proposition formed from two given statements by the connective "and" (symbol $\wedge$); it is true if and only if both given statements are true.
- For two given statements p and q, the statement "if p then q" (symbol $p \Rightarrow q$) is called *implication* or *conditional* statement. Although it might be argued that other interpretations make equally good sense, mathematically $p \Rightarrow q$ is false only when p is true and q is false. The following expressions mean exactly the same as $p \Rightarrow q$:

  p implies q
  p only if q
  q if p
  q provided that p
  p is sufficient condition for q
  q is necessary condition for q

- The statement "p if and only if q" is the conjunction of two implications $p \Rightarrow q$ and $q \Rightarrow p$. It is called an *equivalence* and is denoted by $p \Leftrightarrow q$.

There two logical values of a sentence: Truth=1 or False=0. We have the following Truth Table for connectives:

TRUE TABLE FOR CONNECTIVES

p	q	$\sim p$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
0	0	1	0	0	1	1
0	1	1	0	1	1	0
1	0	0	0	1	0	0
1	1	0	1	1	1	1

1.2.3 Tautologies

It is not difficult to find out that some compound statements are true in all cases. For instance $p \vee \sim p$ or $(p \Rightarrow q) \Leftrightarrow (\sim p \vee q)$.

Example 1.1. Let us verify that the statement

$$p \;\Rightarrow\; q \quad\Longleftrightarrow\quad \sim p \;\vee\; q \tag{1.1}$$

is always true. Indeed, we have the following true-false table for this statement:

p	q	$p \Rightarrow q$	$\sim p \vee q$	$p \Rightarrow q \iff \sim p \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	1
1	1	1	1	1

Such an *always true* statement is called a *tautology*. Tautologies are useful in changing a statement from one form into an equivalent statement in a different form. In Table 1.2 we present several important tautologies, which are sometimes called *laws of logic*.

1.	Double Negation:	$\sim\sim p \Leftrightarrow p$	11.	Indempotent:	$p \vee p \Leftrightarrow p$
2.	De Morgan's:	$\sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q$	12.		$p \wedge p \Leftrightarrow p$
3.		$\sim (p \wedge q) \Leftrightarrow \sim p \vee \sim q$	13.	Identity:	$p \vee 0 \Leftrightarrow p$
4.	Commutative:	$p \vee q \Leftrightarrow q \vee p$	14.		$p \wedge 1 \Leftrightarrow p$
5.		$p \wedge q \Leftrightarrow q \wedge p$	15.	Inverse:	$p \vee \sim p \Leftrightarrow 1$
6.	Associative:	$(p \vee q) \vee s \Leftrightarrow p \vee (q \vee s)$	16.		$p \wedge \sim p \Leftrightarrow 0$
7.		$(p \wedge q) \wedge s \Leftrightarrow p \wedge (q \wedge s)$	17.	Domination:	$p \vee 1 \Leftrightarrow 1$
8.	Distributive:	$p \vee (q \wedge s) \Leftrightarrow (p \vee q) \wedge (p \vee s)$	18.		$p \wedge 0 \Leftrightarrow 0$
9.		$p \wedge (q \vee s) \Leftrightarrow (p \wedge q) \vee (p \wedge s)$	19.	Absorption:	$p \vee (p \wedge q) \Leftrightarrow p$
10.	Contrapositive:	$p \Rightarrow q \Leftrightarrow \sim q \Rightarrow \sim p$	20.		$p \wedge (p \vee q) \Leftrightarrow p$

Table 1.2. Law of Logic

Exercise 1.2. Use true tables to prove that all the compound statements listed in Table 1.2 are tautologies.

1.2.4 Quantifiers

A sentence of the type

$$p(x): \quad x^2 + 2x - 10 > 0, \tag{1.2}$$

involves a variable x which should be considered in an appropriate context in order to become a statement. For the above sentence, the variable x is a real number. However, depending on the value of x the statement $p(x)$ can be either true or false. For example $p(1)$ is false and $p(3)$ is true. Within this context the ambiguity of $p(x)$ can be removed by using a *quantifier*. The sentence

$$\text{for every } \ x, \quad x^2 + 2x - 10 > 0,$$

is a false statement, while the sentence

$$\text{there exists an } \ x \ \text{ such that } \ x^2 + 2x - 10 > 0$$

is true. The *universal quantifier* $\forall$ is read "for every," "for all," for each," or any similar equivalent phrase. The *existential quantifier* $\exists$ is read "there exists," "there is at least one," or something equivalent. Using the quantifiers, the above two sentences can be rewritten as

$$\forall_x \ x^2 + 2x - 10 > 0, \quad \text{and} \quad \exists_x \ x^2 + 2x - 10 > 0.$$

Example 1.3. A sequence $\{a_n\}$ of real numbers is said to be convergent to a limit value a (i.e. $a = \lim_{n\to\infty} a_n$) if the following condition is satisfied:
For every positive number ε there exists a positive integer N such that for all $n \geq N$ we have $|a_n - a| < \varepsilon$.
This sentence can be rewritten using the quantifiers:

$$\forall_{\varepsilon>0} \ \exists_{N\in\mathbb{N}} \ \forall_{n\geq N} \ \ |a_n - a| < \varepsilon.$$

The laws of logic for the quantified statements are listed in Table 1.3.

Notice that in the Table 1.3, for the properties 5 and 9, there is only an implication $\Leftarrow$ instead of the equivalence $\Leftrightarrow$. It should be pointed out that in general, we have

$$\exists_x \ (p(x) \wedge q(x)) \quad \not\Leftarrow \quad \exists_x \ p(x) \wedge \exists_x \ q(x), \tag{1.3}$$

$$\forall_x \ (p(x) \vee q(x)) \quad \not\Rightarrow \quad \forall_x \ p(x) \vee \forall_x \ q(x), \tag{1.4}$$

$$\exists_y \forall_x \ p(x, y) \quad \not\Leftarrow \quad \forall_x \exists_y \ p(x, y). \tag{1.5}$$

Indeed, notice that the statement $\exists_x \ x > 0 \wedge x < 0$ is false, while the statement $\exists_x \, x > 0 \ \wedge \ \exists_x \, x < 0$ is true. Therefore, in general, **we do not have the implication**

$$\exists_x \ p(x) \wedge \exists_x \ q(x), \ \Rightarrow \ \exists_x \ (p(x) \wedge q(x))$$

Exercise 1.4. Find similar examples for (1.4) and (1.5).

1.	$\sim \exists_x\, p(x)$	$\Leftrightarrow$	$\forall_x \sim p(x)$
	$\sim \forall_x\, p(x)$	$\Leftrightarrow$	$\exists_x \sim p(x)$
2.	$\exists_x \exists_y\, p(x,y)$	$\Leftrightarrow$	$\exists_y \exists_x\, p(x,y)$
	$\forall_x \forall_y\, p(x,y)$	$\Leftrightarrow$	$\forall_y \forall_x\, p(x,y)$
3.	$\exists_x\, p(x)$	$\Leftrightarrow$	$\exists_y\, p(y)$
	$\forall_x\, p(x)$	$\Leftrightarrow$	$\forall_y\, p(y)$
4.	$\exists_x\, (p(x) \vee q(x))$	$\Leftrightarrow$	$\exists_x\, p(x) \vee \exists_x\, q(x)$
	$\forall_x\, (p(x) \wedge q(x))$	$\Leftrightarrow$	$\forall_x\, p(x) \wedge \forall_x\, q(x)$
5.	$\exists_x\, (p(x) \wedge q(x))$	$\Rightarrow$	$\exists_x\, p(x) \wedge \exists_x\, q(x)$
	$\forall_x\, p(x) \vee \forall_x\, q(x)$	$\Rightarrow$	$\forall_x\, (p(x) \vee q(x))$
6.	$\exists_x\, (p \vee q(x))$	$\Leftrightarrow$	$p \vee \exists_x\, q(x)$
	$\exists_x\, (p \wedge q(x))$	$\Leftrightarrow$	$p \wedge \exists_x\, q(x)$
7.	$\forall_x\, (p \vee q(x))$	$\Leftrightarrow$	$p \vee \forall_x\, q(x)$
	$\forall_x\, (p \wedge q(x))$	$\Leftrightarrow$	$p \wedge \forall_x\, q(x)$
8.	$\exists_x\, (p \Rightarrow q(x))$	$\Leftrightarrow$	$p \Rightarrow \exists_x\, q(x)$
	$\forall_x\, (p \Rightarrow q(x))$	$\Leftrightarrow$	$p \Rightarrow \forall_x\, q(x)$
9.	$\exists_y \forall_x\, p(x,y)$	$\Rightarrow$	$\forall_x \exists_y\, p(x,y)$
10.	$\exists_x\, (p(x) \Rightarrow q)$	$\Leftrightarrow$	$\Big(\exists_x\, p(x)\Big) \Rightarrow q$
	$\forall_x\, (p(x) \Rightarrow q)$	$\Leftrightarrow$	$\Big(\forall_x\, p(x)\Big) \Rightarrow q$

Table 1.3. Laws of Logic for Quantified Statements

Definition 1.5. In order to simplify logical notation, the following convention is often applied:

$$\forall_{p(x)}\ q(x) \overset{\text{def}}{\Longleftrightarrow} \forall_x\ \Big(p(x)\ \Rightarrow\ q(x)\Big),$$
$$\exists_{p(x)}\ q(x) \overset{\text{def}}{\Longleftrightarrow} \exists_x\ \Big(p(x)\ \wedge\ q(x)\Big).$$

Example 1.6. Let us illustrate how to apply the laws listed in Tables 1.2 and 1.3 to derive new logical rules. Since (by (1.1)), $p(x)\ \Rightarrow\ q(x)$ is equivalent to $\sim p(x)\ \vee\ q(x)$ we have the following tautologies

$$\forall_x\ \Big(p(x)\ \Rightarrow\ q(x)\Big)\ \Longleftrightarrow\ \forall_x\ \Big(\sim p(x)\ \vee\ q(x)\Big), \tag{1.6}$$
$$\exists_x\ \Big(p(x)\ \Rightarrow\ q(x)\Big)\ \Longleftrightarrow\ \exists_x\ \Big(\sim p(x)\ \vee\ q(x)\Big). \tag{1.7}$$

On the other hand, if we denote by P the statement $\forall_x\ p(x)$ and by Q the statement $\forall_x\ q(x)$, then by (1.1) we get

$$\Big(\forall_x\ p(x)\ \Rightarrow\ \forall_x\ q(x)\Big)\ \Longleftrightarrow\ \Big(\sim \forall_x\ p(x)\ \vee\ \forall_x\ q(x)\Big). \tag{1.8}$$

Similarly, we also have the following tautologies

$$\Big(\forall_x\ p(x)\ \Rightarrow\ \exists_x\ q(x)\Big)\ \Longleftrightarrow\ \Big(\sim \forall_x\ p(x)\ \vee\ \exists_x\ q(x)\Big) \tag{1.9}$$
$$\Big(\exists_x\ p(x)\ \Rightarrow\ \forall_x\ q(x)\Big)\ \Longleftrightarrow\ \Big(\sim \exists_x\ p(x)\ \vee\ \forall_x\ q(x)\Big) \tag{1.10}$$
$$\Big(\exists_x\ p(x)\ \Rightarrow\ \exists_x\ q(x)\Big)\ \Longleftrightarrow\ \Big(\sim \exists_x\ p(x)\ \vee\ \exists_x\ q(x)\Big). \tag{1.11}$$

Let us show how to negate one of those statements:

$$\begin{aligned}
\sim \forall_x \Big(p(x) \;\Rightarrow\; q(x)\Big) &\iff \sim \forall_x \Big(\sim p(x) \;\vee\; q(x)\Big) \quad \text{by (1.6)}\\
&\iff \exists_x \sim \Big(\sim p(x) \;\vee\; q(x)\Big) \quad \text{by Table 1.3, (1)}\\
&\iff \exists_x \Big(\sim\sim p(x) \wedge \sim q(x)\Big) \quad \text{by Table 1.2, (2)}\\
&\iff \exists_x \Big(p(x) \wedge \sim q(x)\Big) \quad \text{by Table 1.2, (1)}
\end{aligned}$$

That means we have the following tautology

$$\sim \forall_x \Big(p(x) \;\Rightarrow\; q(x)\Big) \iff \exists_x \Big(p(x) \wedge \sim q(x)\Big).$$

If we use the notation introduced in Definition 1.5, the last equivalence can be written as

$$\sim \forall_{p(x)}\, q(x) \iff \exists_{p(x)} \sim q(x). \tag{1.12}$$

On the other hand, since

$$\exists_x \Big(p(x) \wedge \sim q(x)\Big) \implies \exists_x\, p(x) \;\wedge\; \exists \sim q(x) \quad \text{by Table 1.3, (4)}$$

thus we also have

$$\sim \forall_x \Big(p(x) \;\Rightarrow\; q(x)\Big) \iff \exists_x\, p(x) \;\wedge\; \exists \sim q(x).$$

Notice that in a similar way we can also prove that

$$\sim \exists_{p(x)}\, q(x) \iff \forall_{p(x)} \sim q(x). \tag{1.13}$$

Theorem 1.7. *The laws of logic for the generalized quantified statements are listed in Table 1.3 can be generalized to the following statements*

1.	$\sim \exists_{r(x)}\, p(x)$	$\Leftrightarrow$	$\forall_{r(x)} \sim p(x)$
	$\sim \forall_{r(x)}\, p(x)$	$\Leftrightarrow$	$\exists_{r(x)} \sim p(x)$
2.	$\exists_{r(x)}\exists_{s(y)}\, p(x,y)$	$\Leftrightarrow$	$\exists_{s(y)}\exists_{r(x)}\, p(x,y)$
	$\forall_{r(x)}\forall_{s(y)}\, p(x,y)$	$\Leftrightarrow$	$\forall_{s(y)}\forall_{r(x)}\, p(x,y)$
3.	$\exists_{r(x)}\, p(x)$	$\Leftrightarrow$	$\exists_{r(y)}\, p(y)$
	$\forall_{r(x)}\, p(x)$	$\Leftrightarrow$	$\forall_{r(y)}\, p(y)$
4.	$\exists_{r(x)}\, (p(x) \vee q(x))$	$\Leftrightarrow$	$\exists_{r(x)}\, p(x) \vee \exists_{r(x)}\, q(x)$
	$\forall_{r(x)}\, (p(x) \wedge q(x))$	$\Leftrightarrow$	$\forall_{r(x)}\, p(x) \wedge \forall_{r(x)}\, q(x)$
5.	$\exists_{r(x)}\, (p(x) \wedge q(x))$	$\Rightarrow$	$\exists_{r(x)}\, p(x) \wedge \exists_{r(x)}\, q(x)$
	$\forall_{r(x)}\, p(x) \vee \forall_{r(x)}\, q(x)$	$\Rightarrow$	$\forall_{r(x)}\, (p(x) \vee q(x))$
6.	$\exists_{r(x)}\, (p \vee q(x))$	$\Leftrightarrow$	$p \vee \exists_{r(x)}\, q(x)$
	$\exists_{r(x)}\, (p \wedge q(x))$	$\Leftrightarrow$	$p \wedge \exists_{r(x)}\, q(x)$
7.	$\forall_{r(x)}\, (p \vee q(x))$	$\Leftrightarrow$	$p \vee \forall_{r(x)}\, q(x)$
	$\forall_{r(x)}\, (p \wedge q(x))$	$\Leftrightarrow$	$p \wedge \forall_{r(x)}\, q(x)$
8.	$\exists_{r(x)}\, (p \Rightarrow q(x))$	$\Leftrightarrow$	$p \Rightarrow \exists_{r(x)}\, q(x)$
	$\forall_{r(x)}\, (p \Rightarrow q(x))$	$\Leftrightarrow$	$p \Rightarrow \forall_{r(x)}\, q(x)$
9.	$\exists_{s(y)}\forall_{r(x)}\, p(x,y)$	$\Rightarrow$	$\forall_{r(x)}\exists_{s(y)}\, p(x,y)$
10.	$\exists_{r(x)}\, (p(x) \Rightarrow q)$	$\Leftrightarrow$	$\Big(\exists_{r(x)}\, p(x)\Big) \Rightarrow q$
	$\forall_{r(x)}\, (p(x) \Rightarrow q)$	$\Leftrightarrow$	$\Big(\forall_{r(x)}\, p(x)\Big) \Rightarrow q$

Example 1.8. Let us use the generalized quantifier laws to prove the following equivalence

$$\exists_{r(x)}\Big(p(x) \Rightarrow q(x)\Big) \Leftrightarrow \Big(\forall_{r(x)}\ p(x) \Rightarrow \exists_{r(x)}\ q(x)\Big)$$

Notice that we have

$$\begin{aligned}
\exists_{r(x)}\Big(p(x) \Rightarrow q(x)\Big) &\iff \exists_{r(x)}\Big(\sim p(x)\ \vee\ q(x)\Big) \\
&\iff \exists_{r(x)}\ \sim p(x)\ \vee\ \exists_{r(x)}\ q(x) \\
&\iff \sim \big(\forall_{r(x)}\ p(x)\big)\ \vee\ \exists_{r(x)}\ q(x) \\
&\iff \forall_{r(x)}\ p(x)\ \Rightarrow\ \exists_{r(x)}\ q(x).
\end{aligned}$$

1.3 Laws of Logic and Operations on Sets

We will illustrate now how the laws of logic allow us to make conclusions about the other properties of sets. Let us begin with definitions of the operations on sets:

$$\begin{gathered}
x \in A \cap B \ \Leftrightarrow\ x \in A \wedge x \in B \\
x \in A \cup B \ \Leftrightarrow\ x \in A \vee x \in B \\
x \in A \setminus B \ \Leftrightarrow\ x \in A \wedge x \notin B \\
A \subset B \ \Leftrightarrow\ \forall_x\ (x \in A \Rightarrow x \in B) \\
x \in A^c \ \Leftrightarrow\ x \notin A \\
A = \emptyset\ \Leftrightarrow \sim \exists_x\, x \in A\ \Leftrightarrow\ \forall_x\, x \notin A \\
A = X\ \Leftrightarrow\ \forall_x\, x \in A,
\end{gathered}$$

where we assume that all the elements x are the members of the space X.

By applying the laws of logic, one can easily prove the following properties of sets:

$$A \cap B = B \cap A \tag{1.14}$$
$$(A \cap B) \cap C = A \cap (B \cap C) \tag{1.15}$$
$$A \cap B = A\ \Leftrightarrow\ A \subset B \tag{1.16}$$
$$A \cap \emptyset = \emptyset \tag{1.17}$$
$$A \cup B = B \cup A \tag{1.18}$$
$$(A \cup B) \cup C = A \cup (B \cup C) \tag{1.19}$$
$$A \cup B = A\ \Leftrightarrow\ B \subset A \tag{1.20}$$
$$A \cup \emptyset = A \tag{1.21}$$

Example 1.9. As an example we will prove the following identity:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \tag{1.22}$$

Notice that (1.22) can be written as

$$x \in A \cap (B \cup C) \;\Leftrightarrow\; x \in A \cap B \,\vee\, x \in A \cap C. \tag{1.23}$$

By definition of the intersection, we have

$$\begin{gathered} x \in A \cap B \,\vee\, x \in A \cap C \;\Leftrightarrow \\ (x \in A \,\wedge x \in B) \,\vee\, (x \in A \,\wedge\, x \in C). \end{gathered}$$

Next, by the distributive law (see Table 1.2, law 9)

$$\begin{gathered} (x \in A \,\wedge x \in B) \,\vee\, (x \in A \,\wedge\, x \in C) \;\Leftrightarrow \\ x \in A \,\wedge\, (x \in B \,\vee\, x \in C), \end{gathered}$$

so (by the definition of the union)

$$x \in A \,\wedge\, (x \in B \,\vee\, x \in C) \;\Leftrightarrow\; x \in A \,\wedge\, (x \in C \cup B)$$

and, consequently (by the definition of the intersection)

$$x \in A \,\wedge\, (x \in C \cup B) \;\Leftrightarrow\; x \in A \cap (B \cup C),$$

which implies (1.23). In a similar way, one can show that

$$x \in A \cup (B \cap C) \;\Leftrightarrow\; x \in A \cup B \,\wedge\, x \in A \cup C, \tag{1.24}$$

which means

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \tag{1.25}$$

By applying the laws of logic, it is easy to show that the operation of complement satisfies the following properties

$$(A^c)^c = A, \tag{1.26}$$
$$A \cap A^c = \emptyset, \tag{1.27}$$
$$A \cup A^c = X, \tag{1.28}$$
$$(A \cap B)^c = A^c \cup B^c, \tag{1.29}$$
$$(A \cup B)^c = A^c \cap B^c. \tag{1.30}$$

Example 1.10. We will use the laws of logic (without referring to the identity (1.25)) to show that

$$[A \cup (B \cap C)]^c = [(A \cup B) \cap (A \cup C)]^c. \tag{1.31}$$

Notice that (1.31) can be written as

$$x \in [A \cup (B \cap C)]^c \;\Leftrightarrow\; x \in [(A \cup B) \cap (A \cup C)]^c. \tag{1.32}$$

We have

$$x \in [A \cup (B \cap C)]^c \;\Leftrightarrow \sim \{x \in (A \cup (B \cap C)\},$$

and, by the definition of the union, we have

$$\sim \{x \in (A \cup (B \cap C)\} \;\Leftrightarrow \sim \{x \in A \vee x \in B \cap C\}.$$

By de Morgan's law (see Table 1.2, law 2) we have

$$\sim \{x \in A \vee x \in B \cap C\} \;\Leftrightarrow$$
$$\sim \{x \in A\} \wedge \sim \{x \in B \cap C\},$$

and, by the definition of the intersection, we obtain

$$\sim \{x \in A\} \wedge \sim \{x \in B \cap C\} \;\Leftrightarrow$$
$$\sim \{x \in A\} \wedge \sim \{x \in B \wedge x \in C\}.$$

Again, by de Morgan's law (see Table 1.2, law 3)

$$\sim \{x \in A\} \wedge \sim \{x \in B \wedge x \in C\} \;\Leftrightarrow$$
$$\sim \{x \in A\} \wedge (\sim \{x \in B\} \vee \sim \{x \in C\}).$$

By distributive law (see Table 1.2, law 9)

$$\sim \{x \in A\} \wedge \Big(\sim \{x \in B\} \vee \sim \{x \in C\} \Big) \;\Leftrightarrow$$
$$\Big(\sim \{x \in A\} \wedge \sim \{x \in B\} \Big) \vee \Big(\sim \{x \in A\} \wedge \sim \{x \in C\} \Big).$$

By de Morgan's law (see Table 1.2, law 2), we obtain

$$\Big(\sim \{x \in A\} \wedge \sim \{x \in B\} \Big)$$
$$\vee \Big(\sim \{x \in A\} \wedge \sim \{x \in C\} \Big)$$
$$\Leftrightarrow \sim \Big\{ x \in A \vee x \in B \Big\} \vee \sim \Big\{ x \in A \vee x \in C \Big\}$$

and again by de Morgan's law (see Table 1.2, law 3),

$$\sim \Big\{ x \in A \vee x \in B \Big\} \vee \sim \Big\{ x \in A \vee x \in C \Big\} \;\Leftrightarrow$$
$$\sim \Big\{ (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \Big\}.$$

By applying the definitions of the union and intersection, we obtain

$$\sim \Big\{ (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \Big\}$$
$$\Leftrightarrow \sim \Big\{ x \in A \cup B \wedge x \in A \cup C \Big\}$$
$$\Leftrightarrow \sim \Big\{ x \in (A \cup B) \cap (A \cup C) \Big\},$$

and consequently

$$\sim \Big\{ x \in (A \cup B) \cap (A \cup C) \Big\} \;\Leftrightarrow\; x \in [(A \cup B) \cap (A \cup C)]^c.$$

1.3.1 Product of Sets, Relations and Notion of a Function

Let X and Y be two sets and assume that $x \in X$ and $y \in Y$. The *ordered pair* (x, y) is defined as the set $\{x, \{x, y\}\}$. The product $X \times Y$ of sets X and Y is defined as the set of all ordered pairs (x, y), whose first element belongs to X and the second one to Y, i.e.

$$X \times Y := \{(x, y) : x \in X \quad \wedge \quad y \in Y\}.$$

The product $X \times Y$ is also called *Cartesian product* of X and Y.

Definition 1.11. Let X and Y be two sets. By a *relation* between elements of X and Y we understand a subset $\mathcal{R} \subset X \times Y$ and we say that $x \in X$ and $y \in Y$ is in relation $\mathcal{R}$ (notation $x\mathcal{R}y$) if $(x, y) \in \mathcal{R}$.

One of the most important kind of relations is the so-called *equivalence relation*.

Definition 1.12. Let X be a set and $\mathcal{R} \subset X \times X$ a relation. The relation $\mathcal{R}$ is called an *equivalence relation* on X if the following conditions are satisfied:

(e1)$\mathcal{R}$ is a *reflexive relation*:

$$\forall_{x \in X} \quad x\mathcal{R}x;$$

(e2)$\mathcal{R}$ is a *symmetric relation*:

$$\forall_{x,y \in X} \;\; x\mathcal{R}y \implies y\mathcal{R}x.$$

(e2)$\mathcal{R}$ is *transitive relation*:

$$\forall_{x,y \in X} \quad x\mathcal{R}y \wedge y\mathcal{R}z \implies x\mathcal{R}z$$

For a given equivalence relation $\mathcal{R}$ on X and $x \in X$ we denote by

$$[x]_{\mathcal{R}} := \{y \in X : x\mathcal{R}y\}$$

the *equivalence class* of x. One can easily verify that the equivalence classes satisfy the following properties

(a) $\forall_x \in X$ we have by reflexivity of $\mathcal{R}$ that $x \in [x]_{\mathcal{R}}$, i.e. in particular $[x]_{\mathcal{R}} \neq \emptyset$ and

$$X = \bigcup_{x \in X} [x]_{\mathcal{R}}.$$

(b) $\forall_{x,y \in X}$ we have by transitivity and symmetry of $\mathcal{R}$ that if $[x]_{\mathcal{R}} \cap [y]_{\mathcal{R}} \neq \emptyset$ then $[x]_{\mathcal{R}} = [y]_{\mathcal{R}}$. That means that two equivalence classes $[x]_{\mathcal{R}}$ and $[y]_{\mathcal{R}}$ either are disjoint or coincide.

We denote by $X/\mathcal{R}$ the set of all equivalence classes in X, i.e.

$$X/\mathcal{R} := \{[x]_{\mathcal{R}} : x \in X\}.$$

Example 1.13. Let X be a set and $A \subset X$ a subset. We introduce the following relation $\mathcal{R} \subset X \times X$ on X

$$\forall_{x,y\in X} \quad x\mathcal{R}y \quad \overset{\text{def}}{\iff} \quad x = y \;\text{ or }\; \{x, y\} \subset A$$

Clearly, the above relation is reflexive and symmetric. Since $\{x, y\} \subset A$ and $\{y, z\} \subset A$ implies that $\{x, z\} \subset A$, it is also transitive. Therefore, $\mathcal{R}$ is an equivalence relation on X and the set $X/\mathcal{R}$ is denoted by X/A and is called a *quotient set*. To be more precise, consider a concrete example of a set

$$X = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, r, s, t, u, v, w, x, y, z\}$$

and assume that $A = \{a, b, c, d, e, f, g, h, i, j, k\}$. Then, the quotient set X/A can be identified with the set

$$X/A = \{A, l, m, n, o, p, r, s, t, u, v, w, x, y, z\}.$$

Definition 1.14. If a relation $\mathcal{R}$ on a set X is *reflexive*, *transitive* and also satisfies the condition

(e4)$\mathcal{R}$ is *anti-symmetric*:

$$\forall_{x,y\in X} \quad x\mathcal{R}y \;\wedge\; y\mathcal{R}x \implies x = y,$$

then $\mathcal{R}$ is usually denoted by '$\leq$' (or '$\geq$') and is called a *partial order* relation on X. If the partial order relation '$\leq$' also satisfies the condition that

$$\forall_{x,y\in X} \quad x \leq y \;\vee\; y \leq x,$$

then the order relation '$\leq$' is called *total order*. Moreover, given the order relation '$\leq$', one can also define the *strict order* relation '$<$' by

$$\forall_{x,y\in X} \quad x < y \iff x \leq y \;\wedge\; x \neq y.$$

Example 1.15. Assume that $X := \mathbb{Z}$ and define a relation $\mathfrak{R}$ on $\mathbb{Z}$ as follows: two integers n, $m \in \mathbb{Z}$ are in relation $\mathfrak{R}$ if and only if n divides m, i.e. there is $k \in \mathbb{Z}$ such that $m = m \cdot k$ (we will use the notation $n|m$ to indicate that n divides m). In a symbolic way the relation $\mathfrak{R}$ can be defined by

$$\forall_{n,m\in\mathbb{Z}} \; n\mathfrak{R}m \iff n|m$$

Notice that this relation is reflexive: $\forall_{n\in\mathbb{Z}} \; n = n \cdot 1$, i.e. n always divides n. Notice that according to our definition $0|0$ (because $0 = 0 \cdot k$ for every $k \in \mathbb{Z}$), but it doesn't mean that $\frac{0}{0}$ is defined. Notice that the relation $\mathfrak{R}$ is transitive. Indeed, assume that $n|m$ and $m|k$, Then we have

$$\begin{cases} n|m \;\Leftrightarrow\; \exists_{s\in\mathbb{Z}} \; m = n \cdot s \\ m|k \;\Leftrightarrow\; \exists_{t\in\mathbb{Z}} \; k = m \cdot t \end{cases} \Rightarrow \; \exists_{s\cdot t\in\mathbb{Z}} \; k = n \cdot (s \cdot t) \;\Leftrightarrow\; n|k \;\Leftrightarrow\; n\mathfrak{R}m.$$

Notice that the relation $\mathfrak{R}$ is not anti-symmetric. Indeed, notice that $(-1)|1$ and $1|(-1)$ but $1 \neq -1$. Therefore, the relation $\mathfrak{R}$ is *not* an order relation.

However, if we put $X = \mathbb{N}$ and use the same definition

$$\forall_{n,m\in\mathbb{N}} \; n\mathfrak{R}m \iff n|m$$

gives us an order relation on $\mathbb{N}$. Clearly, this order on $\mathbb{N}$ is *not total*.

Suppose that X is a set. We will denote by $\mathcal{P}(X)$ the set, which is called the *power set*, of all subsets of X (sometimes the set of all subsets of X is also denoted by 2^X). It is easy to verify that if X has exactly n elements (in our example X has 4 elements), then the set $\mathcal{P}(X)$ has 2^n elements (in our example $\mathcal{P}(X)$ has $2^4 = 16$ elements), which can be used as a kind of explanation for this choice of the notation 2^X.

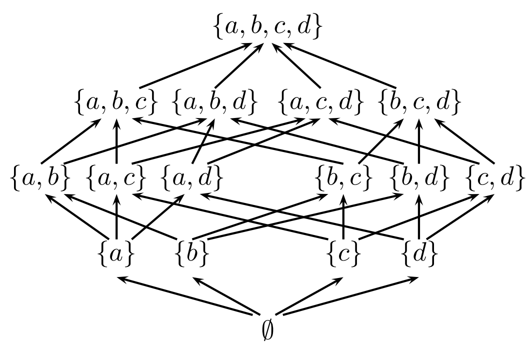
Example 1.16. Let X be a set and consider the set $\mathcal{P}(X)$ of all subsets of X. We define the following relation '$\le$' defined by

$$\forall_{A,B\in\mathcal{B}(X)} \quad A \le B \quad \Longleftrightarrow \quad A \subset B.$$

One can easily verify that this relation is a partial order relation. For example if $X = \{a, b, c, d\}$ then we have

$$\begin{aligned}\mathcal{P}(X) =& \Big\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\},\\ &\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\Big\}\end{aligned}$$

and we can represent this order relation by the following diagram, where the arrow from a subset A to the subset B means that $A < B$.



Lattice representing the partial order in the set $\mathcal{P}(X)$

A function from X to Y is commonly understood as a some kind of a 'rule' f assigning to each element $x \in X$ exactly one element $y \in Y$. This definition is not precise and very ambiguous. In order to give a precise meaning to the notion of a function, it is convenient to define a function f as a relation by identifying it with its graph (i.e. the set $\{(x, y) \in X \times Y : y = f(x)\}$).

Definition 1.17. Let X and Y be two sets. We say that the subset $f \subset X \times Y$ is a *function* from X to Y if

(i) $\forall_{x\in X} \;\; \exists_{y\in Y} \;\; (x, y) \in f$;
(ii) $\forall_{x\in X} \;\; (x, y) \in f \;\wedge\; (x, y') \in f \;\;\Rightarrow\;\; y = y'$.

In addition, we define $y = f(x) \;\; \overset{\text{def}}{\Longleftrightarrow} \;\; (x, y) \in f$. We will also write $f : X \to Y$ to indicate that f is function from X to Y.

Definition 1.18. Let $f : X \to Y$ be a function. Then f is said to be

(i) *one-to-one* or *injective* if

$$\forall_{x_1,x_2\in X}\ f(x_1)=f(x_2)\ \Rightarrow\ x_1=x_2;$$

(ii) *onto* or *surjective* if

$$\forall_{y\in Y}\ \exists_{x\in X}\ f(x)=y;$$

(iii)*bijective* if it is injective and surjective.

Definition 1.19. Given a function $f:X\to Y$ and $A\subset X$. The set $f(A):=\{y\in Y:\exists_{x\in A}\ \ f(x)=y\}$ is called *image* of A under f. For $B\subset Y$ the set $f^{-1}(B):=\{x\in X:\exists_{y\in B}\ \ f(x)=y\}$ is called *pre-image* (or *inverse image*) of B under f.

Proposition 1.20. *Let $f:X\to Y$ be a function, $B\subset Y$ and $A\subset X$. Then we have*

(a) $x\in f^{-1}(B)\ \ \Leftrightarrow\ \ f(x)\in B$;
(b) $f(f^{-1}(B))\subset B$ *and* $f^{-1}(f(A))\supset A$.

Proof: Under the above assumptions, we have
(a): for $x\in X$

$$x\in f^{-1}(B)\ \ \Leftrightarrow\ \ \exists_{y\in B} y=f(x)\ \ \Leftrightarrow\ \ \exists_y\ \ y\in B\ \wedge\ y=f(x)\ \ \Leftrightarrow\ \ f(x)\in B.$$

(b): We have for $y\in Y$

$$\begin{aligned} y\in f(f^{-1}(B))\ &\Leftrightarrow\ \exists_{x\in X}\ x\in f^{-1}(B)\ \wedge\ f(x)=y\\ &\Leftrightarrow\ \exists_{x\in X}\exists_{y'\in Y}\ y'\in B\ \wedge\ f(x)=y'\wedge\ f(x)=y. \end{aligned}$$

Since f is a function thus $f(x)=y'$ and $f(x)=y$ implies $y=y'$, thus

$$\begin{aligned} \exists_{x\in X}\exists_{y'\in Y}\ y'\in B\ \wedge\ f(x)=y'\wedge\ f(x)=y\ &\Rightarrow\ \exists_{x\in X}\exists_{y'\in Y}\ y'\in B\ \wedge\ y=y'\\ &\Rightarrow\ \exists_{x\in X}\exists_{y'\in Y}\ y\in B\ \wedge\ y=y'\\ &\Rightarrow\ \ y\in B. \end{aligned}$$

This implies that $f(f^{-1}(B)\subset B$.

In order to prove the second inclusion, we notice since f is a function $\forall_{x\in X}\exists_{y\in Y}\ \ y=f(x)$, thus we have for $x\in X$

$$\begin{aligned} x\in A\ &\Rightarrow\ \exists_{y\in Y}\ y=f(x)\ \wedge\ x\in A\\ &\Rightarrow\ \exists_{y\in Y}\ \ y\in f(A)\ \wedge\ y=f(x)\\ &\Rightarrow\ x\in f^{-1}(f(A)). \end{aligned}$$

Therefore, $A\subset f^{-1}(f(A))$. □

Proposition 1.21. *Let $f:X\to Y$ be a function and $A,\,B\subset Y$. Then we have*

(a) $f^{-1}(A\cup B)=f^{-1}(A)\cup f^{-1}(B)$,
(b) $f^{-1}(A\cap B)=f^{-1}(A)\cap f^{-1}(B)$,

(c) $f^{-1}(B^c) = (f^{-1}(B))^c$.

Proof: We have for $x \in X$
(a): by Proposition 1.20(a)

$$\begin{aligned} x \in f^{-1}(A \cup B) &\iff f(x) \in A \cup B \iff f(x) \in A \;\vee\; f(x) \in B \\ &\iff x \in f^{-1}(A) \;\vee\; x \in f^{-1}(B) \iff x \in f^{-1}(A) \cup f^{-1}(B). \end{aligned}$$

The proof of (b) is similar to (a) (see Problem 7(c)).
(c):

$$\begin{aligned} x \in f^{-1}(B^c) &\iff f(x) \in B^c \iff f(x) \notin B \\ &\iff \sim f(x) \in B \Leftrightarrow \sim x \in f^{-1}(B) \iff x \in (f^{-1}(B))^c. \end{aligned}$$

□

1.3.2 Generalized Unions and Intersections

Let Λ be a nonempty set and suppose that for every $\lambda \in \Lambda$ there is associated a set $A_\lambda \subset X$. In other words, we simply are dealing here with a function $\alpha : \Lambda \to \mathcal{P}(X)$, where $A_\lambda = \alpha(\lambda)$ for all $\lambda \in \Lambda$. We will use the notation $\{A_\lambda\}_{\lambda \in \Lambda}$ to denote this function and we will call it a *family of sets in X indexed by Λ*.

Definition 1.22. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of sets in X indexed by Λ. The *generalized union* $\bigcup_{\lambda\in\Lambda} A_\lambda$ of the sets A_λ is defined by the following statement:

$$x \in \bigcup_{\lambda\in\Lambda} A_\lambda \overset{\text{def}}{\Longrightarrow} \exists_{\lambda\in\Lambda}\; x \in A_\lambda.$$

The *generalized intersection* $\bigcap_{\lambda\in\Lambda} A_\lambda$ of the sets A_λ is defined by the following statement:

$$x \in \bigcap_{\lambda\in\Lambda} A_\lambda \overset{\text{def}}{\Longrightarrow} \forall_{\lambda\in\Lambda}\; x \in A_\lambda.$$

One can use logical laws for quantified statements to prove several simple properties of generalized unions and intersections:

Proposition 1.23. *Let $\{A_\lambda\}_{\lambda\in\Lambda}$ and $\{B_\lambda\}_{\lambda\in\Lambda}$ be two families of sets in X indexed by Λ, A, $B \subset X$. Then we have*

(a) $\forall_{\lambda_o\in\Lambda}\; A_{\lambda_o} \subset \bigcup_{\lambda\in\Lambda} A_\lambda$ *and* $\forall_{\lambda_o\in\Lambda}\; \bigcap_{\lambda\in\Lambda} A_\lambda \subset A_{\lambda_o}$;
(b) $\forall_{\lambda\in\Lambda}\; A_\lambda \subset A \implies \bigcup_{\lambda\in\Lambda} A_\lambda \subset A$;
(c) $\exists_{\lambda\in\Lambda}\; A_\lambda \subset A \implies \bigcap_{\lambda\in\Lambda} A_\lambda \subset A$;

(d) $\forall_{\lambda\in\Lambda}\ A_\lambda \subset B_\lambda \implies \bigcup_{\lambda\in\Lambda} A_\lambda \subset \bigcup_{\lambda\in\Lambda} B_\lambda$;

(e) $\forall_{\lambda\in\Lambda}\ A_\lambda \subset B_\lambda \implies \bigcap_{\lambda\in\Lambda} A_\lambda \subset \bigcap_{\lambda\in\Lambda} B_\lambda$;

(f) $A \cup \bigcup_{\lambda\in\Lambda} A_\lambda = \bigcup_{\lambda\in\Lambda} (A \cup A_\lambda)$;

(g) $A \cap \bigcup_{\lambda\in\Lambda} A_\lambda = \bigcup_{\lambda\in\Lambda} A \cap A_\lambda)$;

(h) $A \cup \bigcap_{\lambda\in\Lambda} A_\lambda = \bigcap_{\lambda\in\Lambda} (A \cup A_\lambda)$;

(i) $A \cap \bigcap_{\lambda\in\Lambda} A_\lambda = \bigcap_{\lambda\in\Lambda} (A \cap A_\lambda)$;

(j) $\bigcup_{\lambda\in\Lambda} (A_\lambda \cap B_\lambda) \subset \bigcup_{\lambda\in\Lambda} A_\lambda \cap \bigcup_{\lambda\in\Lambda} B_\lambda$;

(k) $\bigcap_{\lambda\in\Lambda} A_\lambda \cup \bigcap_{\lambda\in\Lambda} B_\lambda \subset \bigcap_{\lambda\in\Lambda} (A_\lambda \cup B_\lambda)$;

(l) $A \setminus \bigcup_{\lambda\in\Lambda} A_\lambda = \bigcap_{\lambda\in\Lambda} (A \setminus A_\lambda)$;

(m) $A \setminus \bigcup_{\lambda\in\Lambda} A_\lambda = \bigcap_{\lambda\in\Lambda} (A \setminus A_\lambda)$;

1.4 Cardinality and Countable Sets

The concept of *cardinality* is a 'measure' of elements in a set. More precisely, two sets X and Y are said to have the same *cardinality* if there exists a bijective function $f : X \to Y$. In such a case we write that $|X| = |Y|$. For the purpose of avoiding possible contradictions or well-known paradoxes of the set theory, we will assume that we only consider the sets that are subsets of a certain 'large' (unspecified) set. Clearly, the cardinality relation is an equivalence relation on the collection of all sets and its equivalence classes are called *cardinal numbers*. By obvious reasons, for a finite set X containing n elements, its cardinal number $|X|$ is exactly the number n, i.e. $|X| = n$. The cardinality of the set $\mathbb{N}$ is denoted by $\aleph_0$ (*Aleph zero* cardinality) and the cardinality of the real numbers $\mathbb{R}$ is denoted by $\mathfrak{c}$ (*continuum* cardinality).

It is possible to compare the cardinalities of the sets X and Y. More precisely, we have that $|X| \le |Y|$ id there exists an injective function $f : X \to Y$. Clearly, this condition implies that $|X| = |f(X)|$, and since $f(X) \subset Y$, in such a case it is obvious to expect that Y not less elements than $f(X)$. It is interesting that a proper subset of Y may have the same cardinality as Y. Indeed, consider $Y := \mathbb{N}$ and $X = \{2n : n \in \mathbb{N}\}$. Then, $X \subsetneq Y$ and $|X| = |Y|$, because $f : Y \to X$ defined by $f(n) = 2n$ is a bijection.

The following result, known as *Cantor-Bernstein Theorem*, shows that the relation '$\le$' on the cardinal numbers is in fact an order relation:

Theorem 1.24. (Cantor-Bernstein Theorem) *Let X and Y be two sets such that $|X| \le |Y|$ and $|Y| \le |X|$. Then $|X| = |Y|$.*

Proof: Since $|X| \le |Y|$, there exists an injective function $f : X \to Y$, and since $|Y| \le |X|$, there exists also an injective function $g : Y \to X$. Since $f : X \to f(X)$ is bijective, the inverse function $f^{-1} : f(X) \emptyset X$ is well-defined. Similarly, we have $g^{-1} : g(Y) \to Y$. For an element $x \in X$, an element $y_1 \in Y$ such that $g(y_1) = x$, i.e. $y_1 = g^{-1}(x)$, is called an *ancestor* of x of degree one. Similarly, for an element $y \in Y$, $x_1 \in X$, such that $x_1 = f^{-1}(y)$ is called an ancestor of y of degree one. If $y_1 = g^{-1}(x) \in f(X)$ and $x_2 = f^{-1}(y_1) = f^{-1}(g^{-1}(x))$, then x_2 will be called an ancestor of x of degree two. By following this idea we can say that y_3 is an ancestor of x of degree three, if $x_2 \in g(Y)$ and $y_3 := g^{-1}(x_2) = g^{-1}(f^{-1}(g^{-1}(x)))$. More precisely, $x_{2n} \in X$ is an ancestor of x of degree $2n$ if

$$(g \circ f) \underset{n\text{-times}}{\circ \cdots \circ} (g \circ f)(x_{2n}) = x,$$

and $y_{2n+1} \in Y$ is an ancestor of x of degree $2n+1$ if

$$(g \circ f) \underset{n\text{-times}}{\circ \cdots \circ} (g \circ f) \circ g(y_{2n+1}) = x.$$

In the same way, $y_{2n} \in Y$ is an ancestor of y of degree $2n$ if

$$(f \circ g) \underset{n\text{-times}}{\circ \cdots \circ} (f \circ g)(y_{2n}) = y,$$

and $x_{2n+1} \in X$ is an ancestor of y of degree $2n+1$ if

$$(f \circ g) \underset{n\text{-times}}{\circ \cdots \circ} (f \circ g) \circ f(x_{2n+1}) = y.$$

We will also say that $x \in X$ (or $y \in Y$) is an ancestor of itself of degree zero. Then, for $x \in X$ we put $\deg(x) = \infty$ if the set $\{n \in \mathbb{N} \cup \{0\} : x \text{ has an ancestor of degree } n\}$ is unbounded, and

$$\deg(x) := \sup\{n \in \mathbb{N} \cup \{0\} : x \ \text{ an ancestor of degree } \ n\},$$

otherwise. Similarly, for $y \in Y$ we put $\deg(y) = \infty$ if the set $\{n \in \mathbb{N} \cup \{0\} : y \text{ has an ancestor of degree } n\}$ is unbounded, and

$$\deg(y) := \sup\{n \in \mathbb{N} \cup \{0\} : x \ \text{ an ancestor of degree } \ n\},$$

otherwise. Then we define

$$\begin{aligned} X_\infty &:= \{x \in X : \deg(x) = \infty\}, & Y_\infty &:= \{y \in Y : \deg(y) = \infty\}, \\ X_{\text{even}} &:= \{x \in X : \deg(x) = \text{even}\}, & Y_{\text{even}} &:= \{y \in Y : \deg(y) = \text{even}\}, \\ X_{\text{odd}} &:= \{x \in X : \deg(x) = \text{odd}\}, & Y_{\text{odd}} &:= \{y \in Y : \deg(y) = \text{odd}\}. \end{aligned}$$

One can easily verify that $f(X_\infty) = Y_\infty$, $g(Y_\infty) = X_\infty$, $f(X_{\text{even}}) = Y_{\text{odd}}$, $g(Y_{\text{even}}) = X_{\text{odd}}$, $f(X_{\text{odd}}) = Y_{\text{even}}$, and $g(Y_{\text{odd}}) = X_{\text{even}}$. Therefore, we can define $h : X \to Y$ by putting

$$h(x) := \begin{cases} f(x), & \text{if } \ x \in X_\infty \cup X_{\text{even}}, \\ g^{-1}(x), & \text{if } \ x \in X_{\text{odd}} = g(Y_{\text{even}}). \end{cases}$$

We claim that h is bijective. First we show that h is surjective. Let $y \in Y$. We have three possible cases: (a): $y \in Y_\infty$, then there exists $x \in X_\infty$ such that $f(x) = y$, which implies that $h(x) = y$; (b)

$y \in Y_{\text{odd}} = f(X_{\text{even}})$, thus there exists $x \in X_\infty$ such that $f(x) = y$, which implies that $h(x) = y$; (c) $y \in Y_{\text{odd}} = g^{-1}(X_{\text{even}})$, thus there is $x \in X_{\text{even}}$ such that $h(x) = g^{-1}(x) = y$.

To prove injectivity of h, it suffice to notice that since $f|_{X_\infty \cup X_{\text{even}}}$ and $g^{-1}|_{X_{\text{odd}}}$ are injective, h is injective. □

The following property of sets, known as the *Axiom of Choice* plays important role when dealing with infinite sets.

Axiom of Choice:

Let $\mathcal{A}$ be a collection of nonempty sets. Then, there exists a set B such that it contains one element from each of the sets $A \in \mathcal{A}$, i.e.

$$\forall_{A \in \mathcal{A}} \;\; \exists_{x_A \in A} \quad B = \{x_A : A \in \mathcal{A}\}.$$

The Axiom of Choice simply claims that for any collection of nonempty sets $\mathcal{A}$ one can always choose one element x_A from each of those sets A and create a new set B. To be more precise, assume that the collection $\mathcal{A}$ is given as an indexed family of sets, say $\mathcal{A} := \{A_\lambda\}_{\lambda \in \Lambda}$, then the *Axiom of Choice* claims that there exists a function

$$x : \Lambda \to \bigcup_{\lambda \in \Lambda} A_\lambda, \qquad \text{such that} \qquad \forall_{\lambda \in \Lambda} \;\; x(\lambda) \in A_\lambda.$$

Then, the set B with the required property is simply given by

$$B = \{x(\lambda) : \lambda \in \Lambda\}.$$

Axiom of Choice allows us to prove the following statement

Proposition 1.25. *Let X and Y be two nonempty sets. Then $|X| \leq |Y|$ if and only if there exists a surjective function $g : Y \to X$.*

Proof: Assume that $|X| \leq |Y|$. Then by the definition, there exists an injective function $f : X \to Y$. Chose any element $x_o \in X$ and define $g : Y \to X$ by

$$g(y) := \begin{cases} f^{-1}(y) & \text{if } \; y \in f(X), \\ x_o & \text{if } \; y \in Y \setminus f(X). \end{cases}$$

Clearly, g is surjective.

Assume now that there is a surjective map $g : Y \to X$, i.e.

$$\forall_{x \in X} \;\; A_x := g^{-1}(x) \neq \emptyset.$$

In other words,

$$Y = g^{-1}(X) = \bigcup_{x \in X} g^{-1}(x) = \bigcup_{x \in X} A_x.$$

By applying the Axiom of Choice to the collection of sets $\mathcal{A} := \{A_x\}_{x \in X}$, we obtain that for every $x \in X$ we can chose a unique $y_x \in A_x$, and consequently we define

$$f(x) = y_x \in A_x, \quad x \in X.$$

Notice that for $x \neq x'$, $g^{-1}(x) \cap g^{-1}(x') = \emptyset$, thus $y_x \neq y_{x'}$, therefore f is injective.

□

Definition 1.26. A set X said to be *countable* if it is finite or $|X| = \aleph_0$. In other words, a set X is countable if there exists a surjection $g : \mathbb{N} \to X$.

It follows from the definition, that for an infinite countable set X one can arrange all elements of the set X into a sequence. More precisely, if $f : \mathbb{N} \to X$ is a bijection, one can write $x_n := f(n)$, $n \in \mathbb{N}$, and we have

$$X := \{x_n : n \in \mathbb{N}\}.$$

Example 1.27. The set of all rational numbers $\mathbb{Q}$ is countable.

Proof: Step 1: $|\mathbb{N}| = |\mathbb{Z}|$. We define the function $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = (-1)^n \left\lfloor \frac{n}{2} \right\rfloor, \quad n \in \mathbb{N},$$

where $\lfloor x \rfloor$ denotes the largest integer m such that $m \leq x$. Clearly, f is bijective thus $|\mathbb{N}| = |\mathbb{Z}|$.
Step 2: If $|X| = |Y| = \aleph_0$ then $|X \times Y| = \aleph_0$. We assume that

$$X = \{x_n : n \in \mathbb{N}\}, \quad \text{and} \quad Y = \{y_n : n \in \mathbb{N}\}.$$

$$\begin{array}{ccccccc}
(x_1, y_1) & \rightarrow (x_1, y_2) & (x_1, y_3) & \rightarrow (x_1, y_4) & (x_1, y_5) & \rightarrow (x_1, y_7) & \cdots \\
(x_2, y_1) & (x_2, y_2) & (x_2, y_3) & (x_2, y_4) & (x_2, y_5) & (x_2, y_7) & \cdots \\
(x_3, y_1) & (x_3, y_2) & (x_3, y_3) & (x_3, y_4) & (x_3, y_5) & (x_3, y_7) & \cdots \\
(x_4, y_1) & (x_4, y_2) & (x_4, y_3) & (x_4, y_4) & (x_4, y_5) & (x_4, y_7) & \cdots \\
(x_5, y_1) & (x_5, y_2) & (x_5, y_3) & (x_5, y_4) & (x_5, y_5) & (x_5, y_7) & \cdots \\
(x_6, y_1) & (x_6, y_2) & (x_6, y_3) & (x_6, y_4) & (x_6, y_5) & (x_6, y_7) & \cdots \\
(x_7, y_1) & (x_7, y_2) & (x_7, y_3) & (x_7, y_4) & (x_7, y_5) & (x_1 7 y_7) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

We can arrange the elements $(x_n, y_k) \in X \times Y$ into an infinite array (see the picture above) and by moving along the diagonals of this array it is possible to arrange all the elements in one sequence. Then we have

$$\begin{aligned} X \times Y = \{ & \underbrace{(x_1, y_1)}_{1^{st}\text{ diag}}, \underbrace{(x_1, y_2), (x_2, y_1)}_{2^{nd}\text{ diag}}, \underbrace{(x_3, y_1), (x_2, y_2), (x_1, y_3)}_{3^{rd}\text{ diag}}, \\ & \underbrace{(x_1, y_4), (x_2, y_3), (x_3, y_2), (x_4, y_1)}_{4^{th}\text{ diag}}, \underbrace{(x_5, y_1), (x_4, y_2), (x_3, y_3), (x_2, y_4), (x_1, y_5)}_{5^{th}\text{ diag}}, \dots \}. \end{aligned}$$

Step 3: $|\mathbb{Q}| = |\mathbb{N}|$. Clearly (by Step 2) the set $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is countable, i.e. there exists a bijection $f : \mathbb{N} \to \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. On the other hand, there is a surjection $g : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{Q}$ defined by

$$g(m, n) = \frac{m}{n}, \quad (m, n) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}),$$

and consequently the function $g \circ f : \mathbb{N} \to \mathbb{Q}$ is a surjection. Therefore $|\mathbb{Q}| \leq |\mathbb{N}|$, and since $\mathbb{N} \subset \mathbb{Q}$, we get $|\mathbb{Q}| = |\mathbb{N}| = \aleph_0$. $\square$

Example 1.28. We have that $\mathfrak{c} = |\mathbb{R}| > |\mathbb{N}| = \aleph_0$.

Proof: In order to prove this statement we will present a proof by contradiction (see the next section for more explanations). For this purpose we assume that the above statement is not true and will show that it leads to a contradiction. Suppose then that $|\mathbb{R}| = \aleph_0$. Since the interval $[0, 1]$ is a subset of $\mathbb{R}$, thus $|[0, 1]| = |\mathbb{R}|$, which implies that $|[a, b]| = \aleph_0$, which means that one can arrange all elements of the interval $[0, 1]$ in a sequence

$$[0, 1] = \{x_n : n \in \mathbb{N}\}.$$

In order to show that this is impossible, we will use the fact that every real number x from the interval $[0, 1]$ can be represented in a decimal form

$$x = 0.a_1a_2a_3\ldots a_n\ldots, \quad a_1, a_2, a_3, \ldots, a_n, \cdots \in \{0,1,2,3,4,5,6,7,8,9\}. \tag{1.33}$$

In order to assure that the representation (1.33) is unique, we consider only infinite decimal representations, i.e. instead of a decimal representation $x = 0.3427$, which is finite, we consider $x = 0.34269999\ldots$. Since the numbers from $[0,1]$ are assumed to be countable, we can arrange them into a sequence $\{x_n\ n \in \mathbb{N}\}$, for which

$$\begin{aligned}
x_1 &= 0.a_1^1a_2^1a_3^1a_4^1a_5^1a_6^1a_7^1a_8^1\ldots\\
x_2 &= 0.a_1^2a_2^2a_3^2a_4^2a_5^2a_6^2a_7^2a_8^2\ldots\\
x_3 &= 0.a_1^3a_2^3a_3^3a_4^3a_5^3a_6^3a_7^3a_8^3\ldots\\
x_4 &= 0.a_1^4a_2^4a_3^4a_4^4a_5^4a_6^4a_7^4a_8^4\ldots\\
x_5 &= 0.a_1^5a_2^5a_3^5a_4^5a_5^5a_6^5a_7^5a_8^5\ldots\\
\vdots \quad & \quad \vdots \qquad \vdots
\end{aligned}$$

We define the following real number

$$x = 0.a_1a_2a_3a_4\ldots a_n\ldots$$

by the formula

$$a_n := \begin{cases} 1 & \text{if } a_n^n = 2\\ 2 & \text{if } a_n^n \neq 2. \end{cases}$$

Clearly, $x \in [0,1]$, and therefore, there should exist $k \in \mathbb{N}$ such that $x_k = x$. However, this is impossible, because

$$x_k = 0.a_1^ka_2^ka_3^k\ldots a_k^k\ldots$$

and $a_k^k \neq a_k$. That means, the number x doesn't belong to the sequence $\{x_n : n \in \mathbb{N}\}$. Consequently, we got a contradiction with the assumption that all real numbers from $[0,1]$ can be arranged in a sequence. □

1.5 Techniques of Proof

1.5.1 Inductive and Deductive Reasoning; Axioms and Propositions

The next question is: how can we establish whether a statement is true or false? First of all we can use induction — a method of reasoning, which obtains general laws from the observation of particular facts or examples. This is how we have established our *axioms*, statements that are believed to be true. It would be appropriate to remind here that axioms are introduced to the theory to describe properties of our undefined notions, basic notions of the given field of knowledge.

There is a set of requirements imposed on systems of axioms in fundamental sciences. Of course, these requirements are axioms themselves; they may be called logical axioms. Most commonly these requirements are *consistency, independence, completeness.*

A set of axioms is *consistent* if the axioms do not contradict each other and it is impossible to derive from these axioms results that will contradict each other.

An axiom of a set is said to be *independent* if it cannot be concluded from some other axioms of the set. This property ensures that a theory is built on a minimal number of axioms. It is important for the clearness of the theory and for the real understanding of what the theory is built on.

The *completeness* guarantees that a set of axioms is sufficient to obtain all possible results concerning a given field of knowledge. The latter means the aggregate of all statements concerning the original set of undefined terms. In other words, a set of axioms is complete if it is impossible to add an additional consistent and independent axiom without introducing additional undefined terms.

Once a set of axioms is established, all statements of the theory can be deduced from it. For this purpose they use deduction — reasoning from general laws to particular cases. Deduced, or proved statements are usually called *propositions*. This type of proofs was already illustrated in Examples 1.9 and 1.10. The most important propositions are called *theorems*. They form a "skeleton" of a theory.

If a theorem is auxiliary in its nature, i.e. it is mostly used to simplify the proof of some basic theorem, it is called a *lemma*. A theorem that follows immediately (i.e. its proof is very short and straightforward) from some preceding axiom or proposition is usually called a *corollary*.

The division of theorems into theorems themselves, lemmas and corollaries is, of course, conventional.

For instance, such fundamental results as *Schur's Lemma* in Algebra or *Lebesgue Lemma* in Analysis are more important than many theorems following from them. They are called lemmas rather by historical reasons.

1.5.2 Theorems: the Structure and Proofs

Conditional Form; Direct Proofs

Each theorem consists of two parts: the *hypothesis* (or *condition*) and the *conclusion*. The hypothesis (condition) determines what is suggested as given. The conclusion is something that will be shown to follow from the hypothesis, or to be an implication of the condition. In other words, a theorem asserts that, given the hypothesis, the conclusion will follow. Thus, the structure of a theorem may be represented by the diagram:

$$\begin{gathered}\textit{If }[\text{hypothesis}]\textit{, then }[\text{conclusion}]\textit{, or}\\ [\text{hypothesis}] \implies [\text{conclusion}].\end{gathered}$$

In some cases a theorem may state only the conclusion; the axioms of the system are then implicit (assumed) as the hypothesis. If a theorem is not written in the aforesaid conditional form, it can nevertheless be translated into that form.

A typical direct proof consists of a chain of statements inferred one from another, starting with the hypothesis and ending with the conclusion of the theorem (see Examples 1.9 and 1.10). Each step in the proof (each "link of the chain") is justified by means of laws of logic and axioms or definitions. In other cases (e.g., more advanced theorems) justifications may also rely on previously proved propositions, theorems, lemmas or preceding steps in the proof.

Inductive Proof

In the case we are dealing with a statement $p(n)$ referring to a natural number n (i.e. $n \in \mathbb{N}$), one can apply the *Principle of Mathematical Induction*:

If the following two statements are true

- $p(1)$,
- $p(n) \Rightarrow p(n+1)$,

then the statement $p(n)$ is true for every natural number n.

Example 1.29. We will prove, by applying the Principle of Mathematical Induction, that the following statement

$$p(n) : (n+3)! > 2^{n+3}, \tag{1.34}$$

is true for every natural number n.

First, notice that $p(1)$ is true. Indeed, we have

$$p(1) : 4! = 24 > 16 = 2^4.$$

On the other hand, in order to show that the implication $p(n) \Rightarrow p(n+1)$ is true, we assume that $p(n)$ is true (notice that in the case $p(n)$ is false, the implication $p(n) \Rightarrow p(n+1)$ is always true), i.e.

$$p(n) : (n+3)! > 2^{n+3}, \quad \text{for some } n \in \mathbb{N}$$

is true. Then we need to show that

$$p(n+1) : (n+4)! > 2^{n+4}$$

is also true. Indeed, we have

$$\begin{aligned} (n+4)! &= (n+3)!(n+4) \\ &> 2^{n+3}(n+4) \qquad \text{since } p(n) \text{ is true} \\ &> 2^{n+3} \cdot 2 = 2^{n+4} \qquad \text{since } n+4 > 2 \end{aligned}$$

and therefore $p(n+1)$ is true, and it follows that the implication $p(n) \Rightarrow p(n+1)$ is also true. Consequently, by the Principle of Mathematical Induction, the inequality (1.34) is true for all natural numbers n.

Proof by Contradiction

Among indirect proofs the most frequently used one is the proof by contradiction. It is also called Reductio Ad Absurdum (abbreviated RAA).

To describe this method we need to use the notion of negation of a statement. For a given statement p its negation is a statement meaning (consisting in) the falsity of p and it is denoted $\sim p$. For example, for the sentence p : "This animal is a dog" we have the negation $\sim p$: "This animal is not a dog," or for the sentence q : "x is a negative number", its negation $\sim q$ is "x is positive or zero."

One of the laws of logic, called the *inverse law* (see Table 1.2, law 15), asserts that for any statement p (consider, e.g., the above examples), either the statement p itself or its negation $\sim p$ is true. Therefore, if $\sim p$ is not true, then p is true. To prove the conclusion of a theorem using RAA, one assumes the negation (denial) of the conclusion to be true and deduces an absurd statement using the hypothesis of the theorem in the deduction. Thus, the proof shows that the hypothesis and the negation of the conclusion are contradictory. Then, the negation of the conclusion is false, i.e. (according to the inverse law) the conclusion is true.

As we have already mentioned, each theorem can be formulated in the "*If ..., then ...*" form that is also called an implication or a conditional statement. Denoting the hypothesis (condition) of a theorem by h and the conclusion by c, we can symbolically represent the "*If h, then c*" theorem as "$h \Rightarrow c$". It is read as c follows from h or h implies c. Mathematicians also say: *"h is sufficient for c"* or *"c is necessary upon h."*

The "*if*" clause h is called the *antecedent* and the "*then*" clause c, the *consequent*.

The theorem formed by interchanging the hypothesis (antecedent) and the conclusion (consequent) of a given direct theorem, is called the converse of the given theorem. For example, "*if* $a > 0$ *or* $b > 0$ *then* $a + b > 0$" is the converse of "*if* $a + b > 0$ *then* $a > 0$ *or* $b > 0$." As one can easily notice, it is not always the case that the converse of a true statement (theorem) is true. For example, the statement "*I live in Dallas* $\Rightarrow$ *I live in Texas*" is true, but the converse statement '*I live in Texas* $\Rightarrow$ *I live in Dallas*" is evidently wrong.

If the converse of a true theorem is true, as for instance, "$a > b \;\Rightarrow\; a + c > b + c$," in this case one says that the hypothesis and the conclusion of the theorem are *logically equivalent* (the latter is a short notation for: "*h is true if and only if c is true*," or symbolically "$h \Leftrightarrow c$"). The *inverse* of a given theorem is formed by negating both the hypothesis (antecedent) and the conclusion (consequent): that means: *if the hypothesis is false, then the conclusion is false.*

For the conditional statement: "*I live in Dallas* $\Rightarrow$ *I live in Texas*," the inverse would be: "*I do not live in Dallas* $\Rightarrow$ *I do not live in Texas*," that is, of course, false. This example shows that in general the inverse of a true statement is not true.

It should be noticed here that when mathematicians say "is not true," they mean "is not true in general" or "is not necessarily true." However, "is not true" in mathematical sense does not exclude the option "sometimes is true." Thus, some inverses of valid statements are true. For instance, the inverse of "$a > b \Rightarrow a + c > b + c$, is "$a + c \leq b + c \Rightarrow a \leq b$," that is, of course, true. Another example: a mathematician would say that *an integer is not even* (in general), however one would agree that *some integers are even.*

One can also observe another important feature of deductive (logical) systems: one can disprove a statement by one *counterexample* (that is how we have shown that the inverse, also the converse, of a valid statement is not valid). Still one cannot prove a statement by adducing examples in favor of this statement. E.g., one can show many people who don't live in Dallas and don't live in Texas, and yet that does not mean that I do not live in Dallas implies I do not live in Texas.

It is easy to see that for a given conditional statement (no matter whether it is true or false) its converse and inverse are logically equivalent, i.e. if the converse is true, then the inverse is true and vice versa. Really, let us suppose (using our standard notation) that for some proposition "$h \Rightarrow c$" (that may be true or false), its converse is true: "$c \Rightarrow h$." Then, if h is not true, c cannot be true, because c necessarily implies h (h is necessary if c is true). Translating the latter into formal notation, we obtain: "*h is not true* $\Rightarrow$ *c is not true*," or "$\sim h \Rightarrow \sim c$" the inverse statement. Similarly, one can prove that the inverse implies the converse, and thus, the inverse and the converse are equivalent.

Finally, the *contrapositive* of a given conditional statement $h \Rightarrow c$ is formed by interchanging the hypothesis (antecedent) and the conclusion (consequent) and negating both of them, i.e. "$\sim c \Rightarrow \sim h$." For our example "*I live in Dallas* $\Rightarrow$ *I live in Texas*," the contrapositive will be: "*I do not live in Texas* $\Rightarrow$ *I do not live in Dallas.* "

One can easily show (for example by using the truth table) that a *direct statement* is logically equivalent to its *contrapositive*, i.e. we have

$$(h \Rightarrow c) \;\Leftrightarrow\; (\sim c \Rightarrow \sim h).$$

This property of conditional statements is called *contrapositive reasoning.* This is the kind of reasoning we use when proving conditional statements by contradiction (RAA method). Let us see how it works. Suppose, we have to show that *if h is true, then c is true*: $h \Rightarrow c$. We suggest that $\sim c$ is true, which is equivalent to c is false and arrive to the conclusion that h cannot be true, i.e. $\sim h$ is true; thus h implies c. Since the contrapositive of $h \Rightarrow c$ is true, the direct statement is also true.

Contrapositive reasoning is often unconsciously used in our everyday life. For example, when looking on a shelf for a certain book with a green cover, one would skip (deny) all books of other colors (non-green).

Let us denote the statements: h : "*this is the required book*;" c : "*this book has a green cover.*" The respective negations are: $\sim h$: "*this is not the required book;*" $\sim c$: "*the cover of this book is not green.*" It is known that "$h \Rightarrow c$" ("*if it is the needed book, then it is green*"). Then we start searching. The cover of the first book is blue (not green), hence it is not what we need: "$\sim c \Rightarrow \sim h$" — the second book is red (not green), so it is not the one we need: "$\sim c \Rightarrow \sim h$.' The twenty seventh book is green — let us look at the title, it may be the one we are looking for. Let us notice that the converse of "$h \Rightarrow c$" is not necessarily true, therefore the green book is not necessarily the one we need.

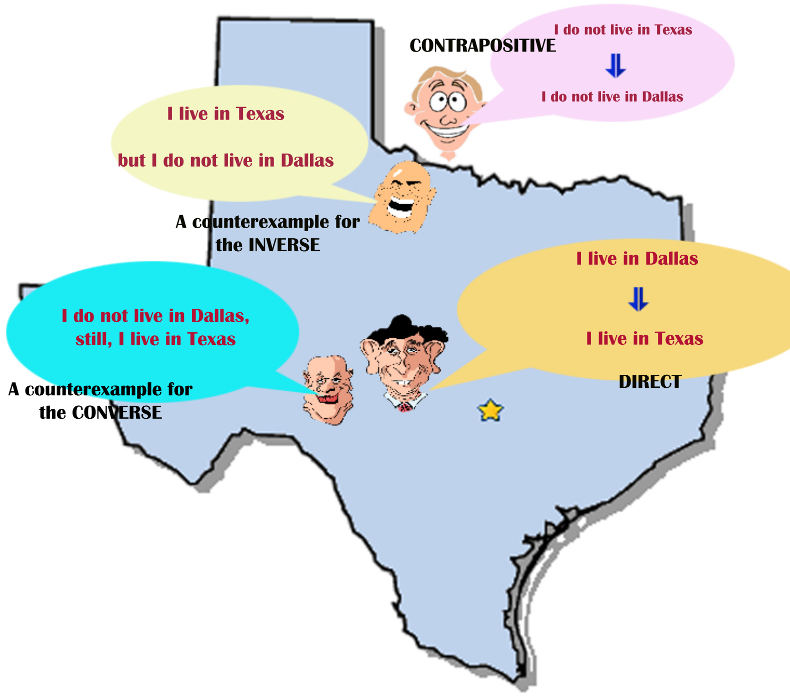


Fig. 1.1.

The properties of conditional statements are conveniently visualized by means of Venn diagrams, where statements are represented by points in the corresponding plane regions. For our "territorial" example: "*If I live in Dallas, then I live in Texas*," the diagram emerges naturally (see Figure 1.1). A plane bounded domain represents Texas, and everyone within this domain lives in Texas. Dallas is shown as a rectangle located approximately in the middle of the province. A person inside this rectangle illustrates the direct statement, the inhabitants of the Northern and Southern Texas disprove, respectively, the converse and inverse, and a resident of Oklahoma supports the contrapositive statement.

1.6 Problems

1. Use the Truth table to show that the following statements are tautologies:

(a) $[(p \Rightarrow q) \Rightarrow p] \Rightarrow p$ (Pierce's Law);
(b) $(\sim p \Rightarrow p) \Rightarrow p$ (Clavius Law);
(c) $[p \Rightarrow (q \Rightarrow r)] \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$ (Law of Conditional Syllogism);
(d) $(p \wedge q \Rightarrow r) \Leftrightarrow [p \Rightarrow (q \Rightarrow r)]$.

2. Check if the following statements are tautologies:

(a) $(p \Rightarrow q) \Rightarrow [p \Rightarrow (q \vee r)]$;
(b) $p \vee [(\sim p \wedge q) \vee (\sim p \wedge \sim q)]$;
(c) $[(p \Rightarrow q) \wedge (q \Rightarrow p)] \Rightarrow (p \vee q)$.

3. Check whether the following statements are true or false:

(a) *"If a is a multiple of 2 and is also a multiple of 7, then if a is not a multiple of 7 implies that a is a multiple of 3;"*
(b) *"If it is not true that the line l is parallel to the line m or the line p is not parallel to the line m, then the line l is not parallel to the line m or the line p is parallel to the line m;"*
(c) *"If James doesn't know analysis, then if James knows analysis implies that James was born in the 2nd century B.C.."*

4. Check if the following quantified statements are true or false:

(a) $\exists_x \Big(p(x) \Rightarrow q(x)\Big) \Rightarrow \Big[\exists_x\, p(x) \Rightarrow \exists_x\, q(x)\Big]$;
(b) $\exists_x\, p(x) \wedge \exists_x\, q(x) \;\Rightarrow\; \exists_x \Big(p(x) \wedge q(x)\Big)$.

5. Prove the following identities for the sets:

(a) $\displaystyle\bigcup_{t\in T} \big(A_t \cup B_t\big) = \bigcup_{t\in T} A_t \cup \bigcup_{t\in T} B_t$;
(b) $\displaystyle\bigcap_{t\in T} \big(A_t \cap B_t\big) = \bigcap_{t\in T} A_t \cap \bigcap_{t\in T} B_t$;
(c) $\displaystyle\bigcup_{t\in T} \big(A_t \cap B_t\big) \subset \bigcup_{t\in T} A_t \cap \bigcup_{t\in T} B_t$;

6. Let A, B, C be sets. Check if the following equalities are true:

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
(b) $(A \setminus B) \cup C = [(A \cup C) \setminus B] \cup (B \cap C)$;
(c) $A \setminus [B \setminus C] = (A \setminus B) \cup (A \cap C)$

7. Let $f : X \to Y$ be a function from X into Y. Show that if A and B are subsets of X, then

(a) $f(A \cap B) \subset f(A) \cap f(B)$:
(b) $(A' \subset B') \;\Rightarrow\; (f^{-1}(A') \subset f^{-1}(B'))$.
(c) $f^{-1}(A' \cap B') = f^{-1}(A') \cap f^{-1}(B')$.

8. Show that the function $f : X \to Y$ is

(a) injective if and only if

$$\forall_{A \subset X} \;\; f^{-1}(f(A)) \subset A.$$

(b) surjective if and only if

$$\forall_{B' \subset Y} \;\; f(f^{-1}(B')) = B'.$$

(c) (b) Show that f is bijective if and only if

$$(f^{-1}(f(A)) = A) \;\wedge\; (f(f^{-1}(B')) = B')$$

for all $A \subset X$ and $B' \subset Y$.

9. Let $f : X \to Y$ be a function. Write the logic negation to each of the following statements:

(a) f is surjective;
(b) f is injective;
(c) f is bijective.

2

Real Numbers

2.1 Fields

2.1.1 Axioms of Field

field, axioms Let $\mathbb{F}$ be a set equipped with one or more algebraic operations, denoted usually as

$$\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}, \quad + : \mathbb{F} \times \mathbb{F} \to \mathbb{F},$$

and suppose that a and $b \in \mathbb{F}$, $a \neq b$, are two elements, which are associated with special properties of the operations $\cdot$ and $+$. Then the set $\mathbb{F}$ is called an *algebraic structure* based on the set $\mathbb{F}$ and these two operations $\cdot$ and $+$ (and the elements a and b), and it is denoted by $\langle \mathbb{F}, \cdot, a, +, b \rangle$. The operations $\cdot$ is called commonly called a *multiplication* and $+$ an *addition* in the set $\mathbb{F}$. For two elements x, $y \in \mathbb{F}$ the symbol $x \cdot y$ is used to denote the *product* of x and y (the result of multiplication) and $x + y$ denotes the *sum* (the result of addition). The elements a and b are usually the *neutral* elements for the multiplication and addition, i.e. $a \cdot x = x$, and $b + x = x$ for $x \in \mathbb{F}$. Very often the neutral element for multiplication is denoted simply by 1 and for the addition 0, however, one should not forget that these are just the symbols, which may have absolutely nothing in common with the numbers 1 and 0.

One of the simplest algebraic structure is a *group*.

Definition 2.1. Let G be a set, $1 \in G$, and $\cdot : G \times G \to G$ an algebraic operation on G satisfying the properties

(g1)(ASSOCIATIVITY) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all x, y, $z \in G$;
(g2)(NEUTRAL ELEMENT) $e \cdot x = x \cdot e = x$ for all $x \in G$;
(g3)(INVERSE ELEMENT) For every $x \in G$ there is an element $-x \in G$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$.

Then the algebraic structure $\langle G, \cdot, e \rangle$ is called a *group*. If in addition, the operation $\cdot$ satisfies the property

(g4)(COMMUTATIVITY) $x \cdot y = y \cdot x$ for all x, $y \in G$,

then the group $\langle G, \cdot, e \rangle$ is called *commutative* or *abelian*.

Of course the simplest example of a group is the so-called *trivial group*, which is a set composed of one element $\{e\}$ where the operation of multiplication is defined by $e \cdot e = e$.

Remark 2.2. Let us point out that in the definition of a group we used the operation of multiplication. However, in the case of an abelian group, it is a common practice to use the addition instead of multiplication.

Let us show some simple properties, which follow directly from the definition of a group.

Proposition 2.3. *Let $\langle G, \cdot, e\rangle$ be a group. then we have:*

(a) (Uniqueness of Neutral Element) *If there is an element $e' \in G$ such that $e' \cdot x = x \cdot e' = x$ for all $x \in G$, then $e = e'$. In other words in a group there is always exactly one neutral element.*
(b) (Uniqueness of Inverse) *If $y \in G$ is such that $y \cdot x = x \cdot y = e$ for some element $x \in G$, then $y = x^{-1}$.*

Proof: (a): Assume that e, $e' \in G$ are two element such that

$$\forall_{x \in G} \quad e \cdot x = x \cdot e = x \quad \text{and} \quad e' \cdot x = x \cdot e' = x.$$

Then in particular, by taking $x = e'$ for the first equality and $x = e$ for the second one, we obtain

$$e \cdot e' = e' \cdot e = e' \quad \text{and} \quad e' \cdot e = e \cdot e' = e,$$

which implies that $e = e'$.

(b): Since $y \cdot x = x \cdot y = e$, then by the property (g1) and the property (g2) we have

$$x^{-1} = x^{-1} \cdot e = x^{-1} \cdot (x \cdot y) = (x^{-1} \cdot x) \cdot y = e \cdot y = y.$$

□

An algebraic structure of special interest for us is the so-called *field*.

Definition 2.4. Let $\mathbb{F}$ be a set and 0, $1 \in \mathbb{F}$ two distinct (i.e. $0 \neq 1$) elements of $\mathbb{F}$. Assume that $\mathbb{F}$ is equipped with two algebraic operations $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ satisfying the properties

(f1)$\forall_{x,y,z \in \mathbb{F}} \quad (x + y) + z = x + (y + z)$ (Associativity of Addition);
(f2)$\forall_{x \in \mathbb{F}} \quad x + 0 = 0 + x = x$ (Neutral Element of Addition);
(f3)$\forall_{x \in \mathbb{F}} \ \exists_{-x \in \mathbb{F}} \quad x + (-x) = (-x) + x = 0$ (Additive Inverse);
(f4)$\forall_{x,y \in \mathbb{F}} \quad x + y = y + x$ (Commutativity of Addition);
(f5)$\forall_{x,y,z \in \mathbb{F}} \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)$ (Associativity of Multiplication);
(f6)$\forall_{x \in \mathbb{F}} \quad x \cdot 1 = 1 \cdot x = x$ (Neutral Element of Multiplication);
(f7)$\forall_{x \in \mathbb{F}} \ x \neq 0 \ \Rightarrow \ \exists_{x^{-1} \in \mathbb{F}} \quad x \cdot x^{-1} = x^{-1} \cdot x = 1$ (Multiplicative Inverse);
(f8)$\forall_{x,y \in \mathbb{F}} \quad x \cdot y = y \cdot x$ (Commutativity of Multiplication);
(f9)$\forall_{x,y,z \in \mathbb{F}} \quad x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ (Distributivity of Addition).

Then the algebraic structure $\langle \mathbb{F}, +, 0, \cdot, 1\rangle$ is called *field*.

Remark 2.5. If $\langle \mathbb{F}, +, 0, \cdot, 1\rangle$ then it is clear that $\langle \mathbb{F}, +, 0\rangle$ is an abelian (additive) group (it satisfies the properties (f1)–(f4)) and $\langle \mathbb{F} \setminus \{0\}, \cdot, 1\rangle$ is also an abelian (multiplicative) group (it satisfies the properties (f5)–(f8)).

Proposition 2.6. *Let $\langle \mathbb{F}, +, 0, \cdot, 1\rangle$ be a field and $a, b \in \mathbb{F}$. Then we have*

(f10) $-(-a) = a$;
(f11) $-(a+b) = (-a) + (-b)$;
(f12) *if* $a \neq 0$ *then* $(a^{-1})^{-1} = a$;
(f13) *if* $a \neq 0 \neq b$ *then* $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$;
(f14) $0 \cdot a = 0$;
(f15) $a \cdot (-b) = -(a \cdot b) = (-a) \cdot b$;
(f16) $(-a) \cdot (-b) = a \cdot b$;
(f17) *if* $a \cdot b = 0$ *then* $a = 0$ *or* $b = 0$;

Proof: (f10): Since $-(-a) + (-a) = 0$ and $a + (-a) = 0$, by the uniqueness of the additive inverse (cf. Proposition 2.3) the element $-(-a)$ is an inverse of $-a$, thus $a = -(-a)$.

(f11): Notice that, by (f2), (f3) and (f1) we have $0 = 0 + 0 = (a + (-a)) + (b + (-b)) = (a + b) + ((-a) + (-b))$ thus $(-a) + (-b)$ is the inverse element of $a + b$, which, again by the uniquness of the additive inverse, $-(a + b) = (-a) + (-b)$.

(f12): The proof is similar to the proof of (f10);

(f13): The proof is similar to the proof of (f11);

(f14): Since (by (f2) and (f9)) we have $0 \cdot a = (0 + 0) \cdot a = (0 \cdot a) + (0 \cdot a)$. On the other hand, since $(0 \cdot a) + (-0 \cdot a) = 0$, we get from the last equality (by (f1), (f3) and (f2))

$$\begin{aligned} 0 &= (0 \cdot a) + (-0 \cdot a) = \big((0 \cdot a) + (0 \cdot a)\big) + (-0 \cdot a) \\ &= (0 \cdot a) + \big((0 \cdot a) + (-0 \cdot a)\big) = (0 \cdot a) + 0 \\ &= 0 \cdot a. \end{aligned}$$

(f15): By (f9) and (f14) we have

$$0 = a \cdot 0 = a \cdot (b + (-b)) = (a \cdot b) + (a \cdot (-b)),$$

which implies (by the uniqness of additive inverse) that $a \cdot (-b)$ is the inverse of $a \cdot b$, i.e. $-(a \cdot b) = a \cdot (-b)$;

(f16): By (f15) and (f10) we have

$$(-a) \cdot (-b) = -(a \cdot (-b)) = -(-(a \cdot b)) = a \cdot b.$$

(f17):By (f14) $0 \cdot 0 = 0$, assume, therefore, that $b \neq 0$ but $a \cdot b = 0$. Then (by (f6)) there is b^{-1} satisfying $b \cdot b^{-1} = 1$, and we have (by (f14), (f1), (f5), (f7) and (f6))

$$0 = 0 \cdot b^{-1} = (a \cdot b) \cdot b^{1} = a \cdot (b \cdot b^{-1}) = a \cdot 1 = a,$$

which proves that $a = 0$.

□

In what follows we will write (for simplicity) ab instead of $a \cdot b$ and we introduce the following two operations on $\mathbb{F}$, namely, the *subtraction* and *division*, which are defined by

$$a - b := a + (-b), \quad \text{and} \quad \frac{a}{b} := ab^{-1} \ \ (b \neq 0),$$

Then we have

Proposition 2.7. *Let $\langle \mathbb{F}, +, 0, \cdot, 1\rangle$ be a field and a, b, c, $d \in \mathbb{F}$, $b \neq 0 \neq d$. Then we have*

(f18) $\frac{a}{1} = a$;
(f19) $\frac{b}{b} = 1$;
(f20) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$;
(f21) $\frac{a}{b} = \frac{ad}{bd}$;
(f22) $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$;
(f23) If in addition $c \neq 0$, *then* $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$;
(f24) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$;

Proof: All these properties can be easily proved by applying the definitions of the division and the previously proved properties. □

Let us present some of typical examples of fields:

Example 2.8. (a) Let p be a prime number. We consider the set $\mathbb{Z}_p := \{0, 1, 2, \dots, p-1\}$ equipped with the multiplication $\odot$ and addition $\oplus$ modulo 2. More precisely, for two numbers m, $n \in \mathbb{Z}_p$ the element $m \odot n$ is the *remainder* of mn divided by p, and $m \oplus n$ is the *remainder* of $m+n$ divided by p. One can verify that these two algebraic operations satisfy the properties (f1)–(f9), and consequently, $\langle \mathbb{Z}_p, \odot, 1, \oplus, 0\rangle$ is a field. As an example, let us show that for every non-zero element $m \in \mathbb{Z}_p$, there exists an element $n \in \mathbb{Z}_p$ such that $m \odot n = 1$, or in other words, $mn = kp + 1$ for some $k \in \mathbb{N}$. Suppose that k, $l \in \{1, 2, \dots, p-\}$ be such that $k \neq l$, then the remainders r_k and r_l of km and lm, divided by p, are also not equal. Indeed, if $km = ap + r_k$, $lm = bp + rl$, and $r_k = r_l$, then $m(k-l) = p(a-b)$, which means that $m(k-l)$ is divisible by p, but this is impossible ($m < p$ and $|k-l| < p$). Thus, the remainders of m, $2m$, …, $(p-1)m$, divided by p, are all different elements from the set $\{1, 2, \dots, p-1\}$, and in particular, one of them must be 1.

Let us illustrate the addition and the multiplication tables for the field $\langle \mathbb{Z}_5, \oplus, 0, \odot, 1\rangle$:

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\odot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

(b) Let us consider the set of all rational numbers (fractions), which we denote by $\mathbb{Q}$, i.e.

$$\mathbb{Q} := \left\{ \frac{m}{n} : m,\, n \in \mathbb{Z},\ n \neq 0 \right\},$$

with the usual addition + and multiplication · operations. Of course, these two operations satisfy the properties (f1)–(f9), so $\langle \mathbb{Q}, +, 0, \cdot, 1\rangle$ is a field.

(c) Assume that p is a prime number and consider the set

$$\mathbb{Q}(\sqrt{p}) := \{r + s\sqrt{p} : r,\, p \in \mathbb{Q}\},$$

equipped with the usual operations of addition $+$ and multiplication $\cdot$. Then, it is easy to notice that for two element $a,\, b \in \mathbb{Q}(\sqrt{p})$ we have $a + b,\ a \cdot b \in Q(\sqrt{p})$. Indeed, if $a = r + s\sqrt{p}$ and $b = r' + s'\sqrt{p}$ then $a+b = (r+r') + (s+s')\sqrt{p}$ and $a \cdot b = (rr' + pss') + (rs' + r's)\sqrt{p}$. It is obvious that addition and multiplication satisfy the properties (f1)–(f9). Therefore, $\langle \mathbb{Q}(\sqrt{p}), +, 0, \cdot, 1\rangle$ is a field.

(d) We denote by $\mathbb{R}$ the set commonly known as the set of all real numbers equipped with standard addition and multiplication operations. Then $\langle \mathbb{R}, +, 0, \cdot, 1\rangle$ is a field. Unfortunately, the real numbers can not be easily described. Thus, in order to be able to use them correctly, it is important to know the additional properties (axioms) of the real numbers providing a complete characterization of $\mathbb{R}$.

(e) The set $\mathbb{C} := \mathbb{R}^2$ equipped with the operation of addition being the vector addition and the multiplication $\cdot$ defined by

$$(x, y) \cdot (x', y') = (xx' - yy', xy' + x'y), \quad (x, y),\, (x', y') \in \mathbb{R}^2,$$

also satisfies the properties (f1)–(f9) (see the Appendix 1). Therefore $\langle \mathbb{C}, +, 0, \cdot, 1\rangle$ is a field.

In Example 2.8 all the listed fields satisfy the inclusions

$$\mathbb{Q} \subset \mathbb{Q}(\sqrt{p}) \subset \mathbb{R} \subset \mathbb{C},$$

while the field $\mathbb{Z}_p$ is not related to these fields. Our objective is to describe the field $\mathbb{R}$ of real numbers by identifying all the properties, which can be used as the axioms of the field of real numbers.

2.1.2 Ordered Field

There is a definite difference between the fields $\mathbb{Q}$, $\mathbb{Q}(\sqrt{p})$ and $\mathbb{R}$ and the fields $\mathbb{Z}_p$ and $\mathbb{C}$. These fields are ordered by the *order relation* $>$.

Definition 2.9. Let $\langle \mathbb{F}, +, 0, \cdot, 1\rangle$ be a field and let $>$ be the relation defined on the elements of $\mathbb{F}$ satisfying the properties:

(o1)(Trichotomy) For all elements $a,\, b \in \mathbb{F}$ only one relation $a > b$ or $b > a$ or $a = b$ is satisfied;
(o2)(Transitivity) For all elements $a,\, b,\, c \in \mathbb{F}$ if $a > b$ and $b > c$ then $a > c$;
(o3)(Compatibility with Addition) For all elements $a,\, b,\, c \in \mathbb{F}$ if $a > b$ then $a + c > b + c$;
(o4)(Compatibility with Multiplication) For all elements $a,\, b,\, c \in \mathbb{F}$ if $a > b$ and $c > 0$ then $ac > bc$;

Then the field $\mathbb{F}$ is called an *ordered field*.

In what follows we will use the following notation:

$$a < b \text{ if } b > a \quad \text{and} \quad a \geq b \text{ if } a > b \text{ or } a = b \quad a \leq b \text{ if } b \geq a.$$

Proposition 2.10. *Let $\mathbb{F}$ be an ordered field and $a, b, c \in \mathbb{F}$. Then*

(o5) If $a < b$ then $-a > -b$;
(o6) $a > 0$ then $-a < 0$;
(o7) $a < b$ and $c < 0$ then $ac > bc$;
(o8) $1 > 0$.

Proof: (o5): If $a < b$ then by (o3)

$$a + (-b) < b + (-b) = 0 \;\Rightarrow\; a + (-b) < 0 \;\Rightarrow\; (-a) + a + (-b) < 0 + (-a)$$

which implies $-b < -a$.

(o6): Since $-0 = 0$, thus if $a > 0$ then $-a < 0$.

(o7): If $c < 0$ then $-c > 0$, so by (o4), if $a < b$ then $-(ac) = a(-c) < b(-c) = -(bc)$, and by (o5) $ac > bc$.

(o8): Assume for contradiction that $1 < 0$. Then by (o7) $1 = 1 \cdot 1 > 0 \cdot 1 = 0$, which is contradiction with (o1). □

Definition 2.11. Let $\mathbb{F}$ be an ordered field. A set $A \subset \mathbb{F}$ is said to be *bounded from above* if

$$\exists_{\alpha \in \mathbb{F}} \; \forall_{a \in A} \quad a \leq \alpha,$$

and A is said to be *bounded from below* if

$$\exists_{\beta \in \mathbb{F}} \; \forall_{a \in A} \quad a \geq \beta.$$

In addition, A is called *bounded* if it is bounded from below and above.

For an ordered field $\mathbb{F}$ and $a \in \mathbb{F}$ we define the so-called *absolute value* $|a|$ of a by the formula

$$|a| = \begin{cases} -a & \text{if } \; a < 0 \\ a & \text{if } \; a \geq 0. \end{cases}$$

The most important properties of absolute value are listed in the following

Proposition 2.12. *Let $\mathbb{F}$ be an ordered field, $a, b \in \mathbb{F}$. Then, we have*

(a) $a \leq |a|$;
(b) $|a + b| \leq |a| + |b|$ and $|a - b| \leq |a| + |b|$;
(c) $|a + b| \geq \big||a| - |b|\big|$ and $|a - b| \geq \big||a| - |b|\big|$;
(d) $|ab| = |a|\,|b|$;
(e) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, where $b \neq 0$;
(f) $|a| \leq b \iff -b \leq a \leq b$.

Proof: In order to show (i), notice that the inequality $|x| \leq a$ means that $x \leq a$ and $-x \leq a$, thus $-a \leq x \leq a$.

We will only show, as an example, how to prove (ii) and (iii) and the rest of the proofs are left to the reader. We have that $a \leq |a|$ and $b \leq |b|$, thus $a + b \leq |a| + |b|$. On the other hand $-|a| \leq a$ and $-|b| \leq b$ thus $-(|a| + |b|) \leq a + b$ and so we get $|a + b| \leq |a| + |b|$. Since the previous inequality is valid for all real numbers a and b, we get $|b| = |a+(-a+b)| \leq |a|+|a-b|$ and $|a| = |b+(a-b)| \leq |b|+|a-b|$, thus $-|a - b| \leq |a| - |b| \leq |a - b|$, and we obtain $||a| - |b|| \leq |a - b|$. □

Definition 2.13. Let $\mathbb{F}$ be an ordered field and $P \subset \mathbb{F}$. The set P is called *inductive* if

(i) $1 \in P$;
(ii) if $a \in P$ then $a + 1 \in P$.

Proposition 2.14. *Let $\{P_i\}_{i \in I}$ be a family of inductive sets in the ordered field $\mathbb{F}$. Then the intersection $P := \bigcap_{i \in I} P_i$ is also an inductive set.*

Proof: Since $1 \in P_i$ for every $i \in I$, $1 \in \bigcap_{i \in I} P_i$. On the other hand, if $a \in P$, then $a \in P_i$ for every $i \in I$, and since P_i is inductive $a + 1 \in P_i$ for every $i \in I$. Thus $a + 1 \in \bigcap_{i \in I} P_i$. Consequently, the set P satisfies the conditions (i) and (ii) of Definition 2.13, so it is inductive. □

In an ordered field $\mathbb{F}$ the set of natural numbers $\mathbb{N}$ is defined as the intersection of all inductive sets $P \subset \mathbb{F}$, or in other words, $\mathbb{N}$ is the *smallest inductive* subset of $\mathbb{F}$. The elements of $\mathbb{N}$ will be denoted by the symbols m, n, k, or l. Since $\mathbb{F}$ is a field, we can define the fractions $\frac{m}{n}$, m, $n \in \mathbb{N}$, $n \neq 0$. One can easily verify that the set of all fractions, which for obvious reasons we will denote by $\mathbb{Q}$, is itself an order field. In this way we obtain that $\mathbb{Q} \subset \mathbb{F}$ for any ordered field $\mathbb{F}$.

It is therefore clear that additional requirement are needed in order to distinguish the field of real numbers from other ordered field (for example, such as $\mathbb{Q}$). For this purpose, we introduce the notions of *supremum* and *infimum*.

Definition 2.15. Let $\mathbb{F}$ be an ordered field and A a bounded from above set. We say that $c \in \mathbb{F}$ is a *supremum* or *least upper bound* of A (we will use the notation $c = \sup A$) if the following condition is satisfied:

$$c = \sup A \quad \Longleftrightarrow \quad \begin{cases} \text{(i)} & \forall_{a \in A} \;\; a \leq c \\ \text{(ii)} & \forall_{\varepsilon > 0} \, \exists_{a \in A} \;\; c - \varepsilon < a. \end{cases}$$

If A is bounded from below, then an element $d \in \mathbb{F}$ is called *infimum* or *greatest lower bound* (we will use notation $d- = \inf A$) if and only if

$$d = \inf A \quad \Longleftrightarrow \quad \begin{cases} \text{(i)} & \forall_{a \in A} \;\; a \geq d \\ \text{(ii)} & \forall_{\varepsilon > 0} \, \exists_{a \in A} \;\; d + \varepsilon > a. \end{cases}$$

The following axiom completes the axiomatic properties of the field of real numbers:

Definition 2.16. Let $\langle \mathbb{F}, +, 0, \cdot, 1 \rangle$ be an ordered field. We say that $\mathbb{F}$ satisfies the *Completeness Axiom* or the *Least Upper Bound Axiom* if the following condition is satisfied:

(C)For every bounded from above subset A of $\mathbb{F}$ there exists $c \in \mathbb{F}$ such that $c = \sup A$.

Definition 2.17. The set of *real numbers* $\mathbb{R}$ is an ordered field satisfying the (C) axiom.

Remark 2.18. One can show that any ordered field $\mathbb{F}$ satisfying the Completeness Axiom (C) can be identified with the field of real numbers, i.e. it is *isomorphic* to $\mathbb{R}$, which was properly introduced only at the end of the nineteenth century by Richard Dedekind using the idea of the so-called *Dedekind cuts*. Although this method is presently considered as a standard way of formally introducing the real numbers, it is rarely included in the university curricula for math majors. Instead, it is a common practice to assume that the real numbers $\mathbb{R}$ indeed exist, and use the axioms of a field (f1)–(f9), axioms of order (o1)–(o4) and the Completeness axiom (C) to derive all needed properties of the real numbers.

As an example we will derive from the axioms of $\mathbb{R}$ the so-called *Archimedes' Axiom*:

Proposition 2.19. (ARCHIMEDES' AXIOM) *For every real number $c > 0$ there exists a natural number n such that $n > c$.*

Proof: First notice that if $a > 0$ then by (o3) $2a = a + a > a$ and by (o4), $a > \frac{a}{2}$. We will show that it is impossible that there exists a real number c which is greater than or equal to any natural number n, i.e $c \geq n$ for all n in $\mathbb{N}$. Indeed, if such a c exists, then it is an upper bound for the set $\mathbb{N}$, thus $\mathbb{N}$ is bounded above and the Least Upper Bound Axiom (Completeness Axiom) implies that $\mathbb{N}$ has a least upper bound $a = \sup \mathbb{N}$. Since $a \geq n$ for all $n \in \mathbb{N}$, it also satisfies $a \geq 2n$ for all $n \in \mathbb{N}$. Thus by applying (o4) we obtain $\frac{a}{2} \geq n$ for all $n \in \mathbb{N}$. In particular $a > \frac{a}{2} \geq n$ for all $n \in \mathbb{N}$, so a can not be the least upper bound. $\square$

Sometimes it is convienient to write $a < b$ instead $b > a$ and $a \leq b$ instead $b \geq a$.

Example 2.20. Notice that the above axioms imply that if $a < b$ then $-a > -b$. Indeed, by (o3) we can obtain from $a < b$ that $a - b = a + (-b) < b + (-b) = 0$, then $-b = a - b + (-a) < 0 + (-a) = -a$. Consequently, it follows that if $a < b$ and $c < 0$ then $ac > bc$. Indeed, by (O4) $-ac = a(-c) < b(-c) = -bc$, thus by the previous result $ac > bc$.

2.2 Principle of Mathematical Induction

It was Augustus de Morgan (1806-1871), who in 1838 defined and introduced the term 'mathematical induction' putting a process that had been used without clarity on a rigorous level. De Morgan was an unusually gifted English mathematician with wide range of interest and skills in popularizing the science. He was also a brilliant but eccentric teacher. The principle of Mathematical Induction follows immediately from the property that $\mathbb{N}$ is the smallest inductive set in $\mathbb{R}$.

Theorem 2.21. *(Principle of Mathematical Induction) Let P be a subset of natural numbers $\mathbb{N}$ such that following two properties are satisfied:*

(i) $1 \in P$;

(ii) if $n \in P$ then $n+1 \in P$.

Then the set P contains all the natural numbers, i.e. $P = \mathbb{N}$.

Proof: Since P is an inductive set, clearly $\mathbb{N} \subset P$. $\square$

The symbol $\sum$ is used to simplify notation relating to sums of a large number of reals. We write $\sum\limits_{k=1}^{n} a_k$ to denote the sum $a_1 + a_2 + \cdots + a_{n-1} + a_n$ of n real numbers $a_1, a_2, \ldots, a_n$.

As an example of an application of Mathematical Induction we will prove Newton's *Binomial Formula*:

Proposition 2.22. (Binomial Formula) *Let n be a natural number. Then for any two real numbers a and b we have the following formula*

$$\begin{aligned}(a+b)^n &= a^n + \binom{n}{1} a^{n-1}b + \cdots + \binom{n}{k} a^{n-k}b^k + \cdots + \binom{n}{n-1} ab^{n-1} + b^n \\ &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k \end{aligned} \tag{2.1}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, $k! = 1 \cdot 2 \cdot \cdots \cdot k$ *for* $k \geq 1$ *and* $0! = 1$.

Proof: We denote by P the set of all natural numbers n such that the binomial formula (2.1) is true. For $n = 1$ the formula take the form

$$(a+b)^1 = a^1 + b^1,$$

and it is clearly true, thus $1 \in P$. Assume that the formula is true for a fixed natural number $n \geq 1$, i.e. we have the equality

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k$$

Then,

$$\begin{aligned}(a+b)^{n+1} &= (a+b)^n(a+b) = a(a+b)^n + b(a+b)^n \quad \text{by the assumption} \\ &= a\sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k + b\sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k \\ &= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n-k+1}b^k + \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k}b^{k+1} + b^{n+1} \\ &= a^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1}\right] a^{n-(k-1)}b^k + b^{n+1} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k}b^k \end{aligned}$$

and consequently the number $n+1$ also belongs to the set P. This shows that both the properties (i) and (ii) of the Principle of Mathematical Induction (Theorem 2.21) are satisfied, so $P = \mathbb{N}$ and the binomial formula is true for all natural numbers. $\square$

Example 2.23. (a) We will apply the mathematical induction to show that for every natural number n and the real numbers $x_1, x_2, \dots, x_n$, we have the inequality

$$|x_1 + \cdots + x_n| \le |x_1| + \cdots + |x_n| \tag{2.2}$$

Indeed, let $\mathcal{P} := \{n \in \mathbb{N} : \forall_{x_1,\dots,x_n} \;\; |x_1+\cdots+x_n| \le |x_1|+\cdots+|x_n|\}$. We will show that (i) $1 \in \mathcal{P}$, and that (ii) if $n \in \mathcal{P}$ then $n+1 \in \mathcal{P}$. Indeed, for (i) we notice that the inequality (2.2) is trivially satisfied for $n = 1$. Indeed, we have $|x_1| \le |x_1|$. Thus $1 \in \mathcal{P}$. For (ii), assume that for some n the inequality (2.2) is satisfied (i.e. $n \in \mathcal{P}$). We want to show that (2.2) also holds for $n+1$, i.e we want to show the following

$$|x_1 + \cdots + x_n + x_{n+1}| \le |x_1| + \cdots + |x_n| + |x_{n+1}|. \tag{2.3}$$

In order to show (2.3), we notice that by Proposition 2.12, we have $|x+y| \le |x| + |y|$, for all x, $y \in \mathbb{R}$. Consequently we have

$$\begin{aligned} |\underbrace{x_1 + \cdots + x_n}_{x} + \underbrace{x_{n+1}}_{y}| &\le |\underbrace{x_1 + \cdots + x_n}_{x}| + |\underbrace{x_{n+1}}_{y}| && \text{by Prop. 2.12(ii)} \\ &\le |x_1| + \cdots + |x_n| + |x_{n+1}| && \text{by induction, i.e. (2.2)} \end{aligned}$$

Consequently, the inequality (2.2) holds by the principle of mathematical induction.

(b) We will show that for every natural number n and the real numbers $x_1, x_2, \dots, x_n$, we have the inequality

$$|x + x_1 + \cdots + x_n| \ge |x| - (|x_1| + \cdots + |x_n|). \tag{2.4}$$

Again, by Proposition 2.12(iii) and by (a), we have for every $n \in \mathbb{N}$

$$\begin{aligned} |x + \underbrace{x_1 + \cdots + x_n}_{y}| &\le |x| - |\underbrace{x_1 + \cdots + x_n}_{y}| && \text{by Prop. 2.12, (iii)} \\ &\le |x| - (|x_1| + \cdots + |x_n|) && \text{by inequality (2.2) in (a)} \end{aligned}$$

and the statement follows.

(c) We can use mathematical induction to show that for a natural number n

(i) $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$;

(ii) $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$;

(iii) $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$.

(i): We prove the inequality

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \tag{2.5}$$

by induction. First we show that (2.5) is true for $n = 1$. Indeed, we have that $1 = \frac{1\cdot 2}{2} = 1$. Assume that the equality (2.5) is true for n. We will show that it is also true for $n+1$, i.e. we need to show that the following equality

$$1+2+\dots+n+(n+1)=\frac{(n+1)(n+2)}{2} \tag{2.6}$$

is true. Indeed, we have

$$\begin{aligned}1+2+\dots+n+(n+1)&=\frac{n(n+1)}{2}+(n+1) \qquad \text{induction assumption}\\ &=\frac{n(n+1)}{2}+\frac{2(n+1)}{2}=\frac{(n+1)(n+2)}{2}\end{aligned}$$

and (2.6) follows.

(ii): Again, we use the mathematical induction to show that

$$1^2+2^2+\dots+n^2=\frac{n(n+1)(2n+1)}{6}. \tag{2.7}$$

First, we notice that (2.7) is true for $n=1$. Indeed, we have $1=1^2=\frac{1\cdot 2\cdot 3}{2}$. Assume that the equality (2.7) is true for n. We need to show that (2.7) is also true for $n+1$. We have

$$\begin{aligned}1^2+2^2+\dots+n^2+(n+1)^2&=\frac{n(n+1)(2n+1)}{6}+(n+1)^2 \qquad \text{by induction assumption}\\ &=\frac{n(n+1)(2n+1)}{6}+\frac{6(n+1)^2}{6}\\ &=\frac{(n+1)(2n^2+n+6n+6)}{6}=\frac{(n+1)(2n^2+7n+6)}{6}\\ &=\frac{(n+1)(n+2)(2n+3)}{6}=\frac{(n+1)\Big((n+1)+1\Big)\Big(2(n+1)+1\Big)}{6}\end{aligned}$$

and the proof is completed.

(iii): Since by (i) $1+2+\dots+n=\frac{n(n+1)}{2}$ it is sufficient to show that

$$1^3+2^3+\dots+n^3=\frac{n^2(n+1)^2}{4}. \tag{2.8}$$

Again, the equality (2.8) is trivially satisfied for $n=1$. Indeed, we have $1=1^3=\frac{1^2\cdot 2^2}{4}=1$. Assume that (2.8) is true for n. We will show that it is also true for $n+1$. We have

$$\begin{aligned}1^3+2^3+\dots n^3+(n+1)^3&=\frac{n^2(n+1)^2}{4}+(n+1)^3 \qquad \text{by induction assumption}\\ &=\frac{n^2(n+1)^2}{4}+\frac{4(n+1)^3}{4}=\frac{(n+1)^2(n^2+4n+4)}{4}\\ &=\frac{(n+1)^2(n+2)^2}{4}=\frac{(n+1)^2\Big((n+1)+1\Big)^2}{4}\end{aligned}$$

and therefore (2.8) is true for all natural numbers n.

(d) Let a be a real number such that $a>-1$ and let n be a natural number. We will prove the following *Bernoulli's inequality*:

$$(1+a)^n\ge 1+na. \tag{2.9}$$

It is clear that for $n=1$ the inequality (2.9) is satisfied. Indeed, $1+a=(1+a)^1\ge 1+1\cdot a$. Assume that for certain $n\ge 1$ the inequality (2.9) is satisfied. We need to show that (2.9) is also true for $n+1$, i.e. the following inequality

$$(1+a)^{n+1} \geq 1+(n+1)a, \tag{2.10}$$

holds. In order to prove (2.10) we notice that

$$\begin{aligned}(1+a)^{n+1} &= (1+a)^n(1+a) \geq (1+na)(1+a) \qquad \text{by induction assumption}\\ &= 1+na+a+na^2 = 1+(n+1)a+na^2 \geq 1+(n+1)a,\end{aligned}$$

and the inequality (2.10) follows. Thus (2.9) is always true for any natural number n.

(e) We will show that for every natural number n

$$2^3+4^3+\cdots+(2n)^3 = 2(2+4+\cdots+2n)^2 = 2n^2(n+1)^2. \tag{2.11}$$

Indeed, we notice that

$$\begin{aligned}2^3+4^3+\cdots+(2n)^3 &= 2^3\cdot 1^3+2^3\cdot 2^3+\cdots+2^3\cdot n^3\\ &= 2^3(1^3+2^3+\cdots+n^3)\\ &= 2^3(1+2+\cdots+n)^2 \qquad \text{by Example 2.23, (iii)}\\ &= 2\Big(2(1+2+\cdots+n)\Big)^2 = 2(2+4+\cdots+2n)^2,\end{aligned}$$

and the first equality follows. In order to show the second equality we notice that

$$\begin{aligned}2^3+4^3+\cdots+(2n)^3 &= 2^3(1+2+\cdots+n)^2 \qquad \text{as it was shown above}\\ &= 2^3\left[\frac{n(n+1)}{2}\right]^2 = \frac{2^3n^2(n+1)^2}{2^2} = 2n^2(n+1)^2.\end{aligned}$$

(f) We will use the mathematical induction to show that for $0 \leq k \leq n$ the number $\binom{n}{k}$ is an integer. Indeed, for $n=1$ we have that both numbers $\binom{1}{1}=1$ and $\binom{1}{0}=1$ are integers. Assume that for a certain $n \geq 1$ the numbers $\binom{n}{k}$ are integers for all values of k such that $0 \leq k \leq n$. Consider the number $\binom{n+1}{k}$, where $0 \leq k \leq n+1$. If $k=0$ then $\binom{n+1}{0}=1$ and if $k=n+1$ we have that $\binom{n+1}{n+1}=1$ so the statement is true for these two values of k. Therefore, we can assume that $1 \leq k \leq n$. We have that

$$\binom{n+1}{k} = \binom{n}{k-1}+\binom{n}{k}.$$

Indeed,

$$\begin{aligned}\binom{n}{k-1}+\binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!}+\frac{n!}{k!(n-k)!}\\ &= \frac{n!k}{k!(n-k+1)!}+\frac{n!(n-k+1)}{k!(n-k+1)!}\\ &= \frac{n!k+n!(n-k+1)}{k!(n-k+1)!} = \frac{n!(n+1)}{k!(n-k+1)!}\\ &= \frac{(n+1)!}{k!\big((n+1)-k\big)!} = \binom{n+1}{k}.\end{aligned}$$

Therefore, we can apply the induction assumption to both numbers $\binom{n}{k-1}$ and $\binom{n}{k}$, i.e. they are integers by the induction assumption, and since a sum of two integers is an integer, the number $\binom{n+1}{k}$ is an integer. Consequently, the statement follows from the principle of mathematical induction.

2.3 Powers and Logarithms

Let α be a real number and n a natural number. Recall that a real number a is called an n-th root of α if $a^n = \alpha$. We will show that for a positive real number α there exists a unique positive n-th root of α. Indeed, for uniqueness we notice that if a_1 and a_2 are *two* positive n-th roots of α then either $a_1 < a_2$, $a_2 < a_1$ or $a_1 = a_2$. Since the first two inequalities imply $a_1^n \neq a_2^n$, one of the numbers a_1 or a_2 can not be the n-th root of α. Consequently $a_1 = a_2$ and the uniqueness follows.

In order to show the existence of a positive real number a such that $a^n = \alpha$, we consider the set $\mathcal{X} = \{x \in \mathbb{R} : x^n \geq \alpha, \quad x > 0\}$. The set $\mathcal{X}$ is non-empty. Indeed, by Archimedes axiom there exists a natural number m such that $m > \alpha$ and since $m^n > m$ we have $m^n > \alpha$ so $m \in \mathcal{X}$. It is also clear that $\mathcal{X}$ is bounded below (since all the elements of $\mathcal{X}$ are positive). Therefore, it follows from the Completeness Axiom (C) that there exists $a = \inf \mathcal{X}$, i.e. (i) $\forall_{x\in\mathcal{X}} \;\; a \leq x$, (ii) $\forall_{\varepsilon>0}\exists_{y\in\mathcal{X}} \;\; a+\varepsilon > y$. We claim that $a^n = \alpha$. $(a+\varepsilon)^n > y^n \geq \alpha$ and since $\varepsilon > 0$ can be an arbitrarily small number, it follows that $a^n \geq \alpha$. We will show that $a^n = \alpha$. Assume that $a^n > \alpha$. By the binomial formula (Proposition 2.22) we have that for $1 > \delta > 0$

$$\begin{aligned}(a-\delta)^n &= \sum_{k=0}^{n} \binom{n}{k} a^k(-\delta)^{n-k} \geq a^n - \delta \sum_{k=0}^{n-1} \binom{n}{k} a^k \delta^{n-k-1} \\ &\geq a^n - \delta \sum_{k=0}^{n-1} \binom{n}{k} a^k\end{aligned}$$

Consequently, if we assume $\delta < \frac{a^n-\alpha}{\sum_{k=0}^{n-1}\binom{n}{k}a^k}$ we obtain

$$(a-\delta)^n \geq a^n - (a^n - \alpha) = \alpha. \tag{2.12}$$

Since $a - \delta < a$, the inequality (2.12) contradicts (i). Therefore, $a^n > \alpha$ is impossible, so $a^n = \alpha$.

Definition 2.24. Let $\alpha > 0$ be a real number and n a natural number. The *power* $\alpha^{\frac{1}{n}}$ is defined to be the unique positive root x of the equation $x^n = \alpha$. For a positive rational number $r = \frac{m}{n}$ we define the r-th power of α by $\alpha^r = (\alpha^{\frac{1}{n}})^m$ and for a negative rational number r' we put $\alpha^{r'} = \frac{1}{a^{-r'}}$. We also put $\alpha^0 = 1$.

Notice that for m, $n \in \mathbb{N}$

$$\left(\alpha^{\frac{1}{n}} \cdot \alpha^{\frac{1}{m}}\right)^{m\cdot n} = \alpha^m \cdot \alpha^n = \alpha^{m+n}$$

thus

$$\alpha^{\frac{1}{n}} \cdot \alpha^{\frac{1}{m}} = \alpha^{\frac{1}{n}+\frac{1}{m}},$$

and consequently, for all rational numbers r, $r' \in \mathbb{Q}$ we have

$$\alpha^r \cdot \alpha^{r'} = \alpha^{r+r'}.$$

Similarly, we have

$$\alpha \cdot \beta = \left(\alpha^{\frac{1}{n}}\right)^n \cdot \left(\beta^{\frac{1}{n}}\right)^n = \left(\alpha^{\frac{1}{n}} \cdot \beta^{\frac{1}{n}}\right)^n,$$

thus

$$(\alpha \cdot \beta)^{\frac{1}{n}} = \alpha^{\frac{1}{n}} \cdot \beta^{\frac{1}{n}}.$$

Consequently, we have that for every rational number $r \in \mathbb{Q}$

$$(\alpha \cdot \beta)^r = \alpha^r \cdot \beta^r.$$

We have the following

Proposition 2.25. *Let α and β be two positive real numbers and r and r' two rational numbers. Then we have*

(i) $\alpha^r \cdot \alpha^{r'} = \alpha^{r+r'}$;
(ii)$\alpha^r / \alpha^{r'} = \alpha^{r-r'}$;
(iii)$(\alpha^r)^{r'} = \alpha^{rr'}$;
(iv)$(\alpha\beta)^r = \alpha^r \cdot \beta^r$;
(v) $\left(\frac{\alpha}{\beta}\right)^r = \frac{\alpha^r}{\beta^r}$;
(vi)For $\alpha > 1$ and $r < r'$ $\alpha^r < \alpha^{r'}$.

Let b be a real number and α a positive real number. We define the power α^b as the infimum of the set $\mathcal{X} := \{\alpha^r : b < r, r \in \mathbb{Q}\}$.

Proposition 2.26. *Let $\alpha > 1$ and b be two real numbers. Then $\gamma = \alpha^b$ is the unique number satisfying the property*
(*) If r and r' are two rational numbers such that $r < b < r'$ then $\alpha^r < \gamma < \alpha^{r'}$.

Lemma 2.27. *Let $r < r'$ be two real numbers such that $r' - r < \frac{1}{n}$ for some natural number n. Then for every $\alpha > 1$*

$$0 < \alpha^{r'} - \alpha^r < \alpha^r \frac{\alpha - 1}{n}. \tag{2.13}$$

Proof: It follows from the properties of the powers (Proposition 2.25) that

$$0 < \alpha^{r'} - \alpha^r = \alpha^r(\alpha^{r'-r} - 1) < \alpha^r(\alpha^{\frac{1}{n}} - 1).$$

Put $a = \alpha^{\frac{1}{n}}$. By Bernoulli's inequality (2.9) (see Example 2.23), we have that

$$\alpha = (\alpha^{\frac{1}{n}})^n = (1 + a)^n \geq 1 + na = 1 + n(\alpha^{\frac{1}{n}} - 1),$$

and the inequality (2.13) follows. □

Now, by using the definition of the power with real exponent we will introduce the notion of the *logarithm* $\log_\alpha \gamma$ (more precisely *the logarithm of* γ *to the base* α) for every positive number γ and $\alpha > 1$ (in fact it is only needed to assume that $\alpha > 0$ and $\alpha \neq 1$).

If there exists a rational number r such that $\alpha^r = \gamma$ then the exponent r is exactly $\log_\alpha \gamma$. Suppose now that there is no such rational number r. We define the set $A = \{a : \alpha^a > \gamma \;\; a \in \mathbb{Q}\}$ (where $\mathbb{Q}$ denotes the set of rational numbers) and we claim that the set A is non-empty (and evidently bounded below). Indeed, by the Bernoulli's inequality (2.9), for a natural number $m > \frac{\gamma}{\alpha-1}$, we have $\alpha^m > \gamma$ and thus the number m belongs to A. By Proposition 2.25, there exists a real number $\beta = \inf A$. We claim that $\alpha^\beta = \gamma$. Indeed, since β is the infimum of A, by Proposition 2.25, (vi), $\alpha^\beta \geq \gamma$. In order to show that $\alpha^\beta = \gamma$ assume (in order to show that it is impossible) that $\alpha^\beta > \gamma$. Let n be a large natural number such that $\gamma\frac{\alpha-1}{n} < \frac{\gamma-\alpha^\beta}{2}$. We chose two rational numbers r, r' such that $r < \beta < r'$ and $r' - r < \frac{1}{n}$. Then we have by Lemma 2.27

$$\begin{aligned} \alpha^{r'} - \alpha^r < \alpha^r \frac{\alpha - 1}{n} &\leq \gamma \frac{\alpha - 1}{n} \\ &\leq \frac{\alpha^{r'} - \alpha^r}{2}, \end{aligned}$$

which is a contradiction. Therefore α^β must be equal to γ.

We summarize the properties of logarithms in the following

Theorem 2.28. *Let $\alpha > 1$, γ, $\gamma' > 0$ and r be real numbers. Then we have:*

(i) $\log_\alpha(\alpha^r) = r$ *and* $\alpha^{\log_\alpha \gamma} = \gamma$;
(ii) $\log_\alpha(\gamma\gamma') = \log_\alpha \gamma + \log_\alpha \gamma'$;
(iii) $\log_\alpha(\gamma/\gamma') = \log_\alpha \gamma - \log_\alpha \gamma'$;
(iv) $\log_\alpha(\gamma^r) = r \log_\alpha \gamma$;
(v) $\log_\alpha 1 = 0$.

Proof: Property (i) is evident from the definition of the logarithm. Properties (ii), (iii) and (iv) follow from Proposition 2.25, (i), (ii) and (iii). Property (v) is evident. □

2.4 Problems

1. Use the axioms of an ordered field $\mathbb{F}$ to show that the following properties are always satisfied in $\mathbb{F}$:

(a) $\forall_{a,b,c,d\in\mathbb{F}}\ (a < b \wedge c < d \implies a + c < b + d)$;
(b) $\forall_{a,b,c,d\in\mathbb{F}}\ (a > b > 0 \wedge c > d > 0 \implies ac > bd)$;
(c) $\forall_{a,b\in\mathbb{F}}\ \left(b > a > 0 \implies \frac{1}{a} > \frac{1}{b}\right)$;
(d) $\forall_{x,a\in\mathbb{F}}\ (|x| \leq a \iff -a \leq x \leq a)$;
(e) $\forall_{x,y\in\mathbb{F}}\ (|x + y| \leq |x| + |y|)$;

(f) $\forall_{a,b\in\mathbb{F}} \ \Big(|a-b| \geq \Big||a|-|b|\Big|\Big)$;

2. Show that, just like the set $\mathbb{Q}$ of rational numbers, the set $\mathbb{Q}(\sqrt{n})$ of numbers of the form $a+b\sqrt{n}$, where a, $b \in \mathbb{Q}$ and n is a fixed natural number that is not the square of any integer, is an ordered set (field) satisfying the principle of Archimedes but not the axiom of completeness.

3. Determine which axioms for the real numbers do not hold for $\mathbb{Q}(\sqrt{n})$ if the standard arithmetic operations are retained in $\mathbb{Q}(\sqrt{n})$ but order is defined by the rule $(a+b\sqrt{n}) < (a'+b'\sqrt{n})$ iff $b < b')$ or $(b = b') \ \vee \ (b = b' \ \wedge \ a < a')$. Will $\mathbb{Q}(\sqrt{n})$ now satisfy the principle of Archimedes?

4. Let us denote by $\mathbb{Q}^c$ the complement of the set of rational numbers $\mathbb{Q}$ in $\mathbb{R}$, i.e. $\mathbb{Q}^c$ stands for the set of *irrational numbers*. Show the following properties

(a) $\forall_{a,b\in\mathbb{R}} \ (a \in \mathbb{Q} \wedge b \in \mathbb{Q}^c \implies a+b \in \mathbb{Q}^c)$;
(b) $\forall_{a,b\in\mathbb{R}} \ (0 \neq a \in \mathbb{Q} \wedge b \in \mathbb{Q}^c \implies ab \in \mathbb{Q}^c)$.

5. Use the axioms and the already proved properties of real numbers $\mathbb{R}$ to show that the following statements are true:

(a) $\forall_{a\in\mathbb{R}} \ \big[a > 0 \implies \exists_{n\in\mathbb{N}} \ a > \frac{1}{n} > 0\big]$;
(b) $\forall_{a,b\in\mathbb{R}} \ [a < b \implies \exists_{q\in\mathbb{Q}} \ a < q < b]$.

6. Show that the following numbers are irrational:

(a) $[\sqrt{2}-1]^n$, where $n \in \mathbb{N}$;
(b) $\sqrt{x}$, where $x > 0$ is an irrational number;
(c) $\sqrt{p}$, where $p \in \mathbb{N}$ is a prime number.

7. Check if the following sets are bounded from above and/or below. For each bounded from above (resp, below) set, find its supremum (resp. infimum):

$$\begin{aligned} D &= \Big\{x \in \mathbb{R} : \Big||x-1|-1\Big| < 1\Big\}, \\ E &= \big\{k + \tfrac{1}{n} : k \in \{0,1,2\}, \ \ n \in \mathbb{N}\big\}, \\ G &= \big\{1 + \tfrac{1}{2^n} : n \in \mathbb{N}\big\}, \\ I &= \Big\{\tfrac{n^2+2n-3}{n+1} : n \in \mathbb{N}\Big\} \end{aligned}$$

8. Let $A+B$ be the set of numbers of form $a+b$, $A \cdot B$ the set of numbers of form $a \cdot b$, and $-A$ the set of numbers $-a$, where $a \in A \subset \mathbb{R}$ and $b \in B \subset \mathbb{R}$.
Determine whether it is always true that

(a) $\sup(A+B) = \sup A + \sup B$
(b) $\sup(A \cdot B) = \sup A \cdot \sup B$

(c) $\sup(-A) = -\inf A$.

9. Verify that $\mathbb{Z}$ and $\mathbb{Q}$ are inductive sets.

10. Give examples of inductive sets different from $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$.

11. Show that an inductive set is not bounded above.

12. Show that for every natural number n the number $5^n + 2 \cdot 3^{n-1} + 1$ is a multiple of 8.

13. Show that for every $n \in \mathbb{N}$ we have

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

14. Show that for every natural number $n \geq 2$ the number $2^{2^n} - 6$ is a multiple of 10.

15. Find the number of diagonals in a convex polygon with n sides.

16. Use the fact that if a product $m \cdot n$ of natural numbers is divisible by a prime number p, then either m or n is divisible by p, to show that if k^m (k, $m \in \mathbb{N}$) is divisible by a prime number p, then k is divisible by p.

17. Show that for every natural number $n \geq 2$ we have the inequality

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

18. Show that for every natural number $n \geq 2$ we have the inequality

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1).$$

19. Show that for every natural number n we have the inequality

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1.$$

20. Show that for $n \in \mathbb{N}$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

21. Let $n \geq 2$ be a natural number. Show that

$$n! < \left(\frac{n+1}{2}\right)^n.$$

22. Show that for $n \in \mathbb{N}$ we have

$$\sqrt[n]{n} \leq 1 + \sqrt{\frac{2}{n}}.$$

23. Show that

(a) every infinite set contains a countable subset;

(b) the set of even integers has the same cardinality as the set of all natural numbers.

(c) the union of an infinite set and an at most countable set has the same cardinality as the original set.

3

Elementary Theory of Metric Spaces

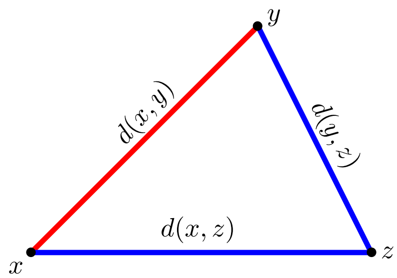
3.1 Metric Spaces and Basic Topological Concepts

The idea of a metric space come from the (un)real world. In many practical problems we are dealing with situations, where it is necessary to determine the distance between certain real (or imaginary) objects. The most common example of such a situation is the problem of finding the distance between two locations v and w in the space. Of course, one can always use the rectangular coordinate system in order to specify the coordinates $v = (x_1, y_2, z_1)$ and $w = (x_2, y_2, z_2)$ of these two places, and then apply the formula for the distance in the three-dimensional space $\mathbb{R}^3$:

$$d(v,w) := \sqrt{|x_1 - x_2|^2 + |y_1 + y_2|^2 + |z_1 - z_2|^2}.$$

However, there may sometimes exist certain conditions, which could create additional complications. For example, in an urban area divided into rectangular blocks, it is impossible to move in diagonal directions. A person, who needs to reach a certain destination, has to move following the streets and avenues. Also, topographical diversity of a terrain may also impose other restrictions.

The idea of a *metric* is expressing the most basic, but still *universal properties* of a function, which is used to measure *distance* between points, independently of the particularities of the considered space. One can ask a question: *what are the most obvious (from the common sense point of view) properties of a distance function (i.e. the function measuring distance between two points)?* Clearly, distance is never negative and if it is zero, than we do not need to move at all to reach the other point, so we are dealing here with two locations, which are identical (this property is called *Positive Definiteness of Metric*). On the other hand, the distance from one location to another is the same, doesn't matter from which of these two locations it is measured. For example the distance from Huston to Dallas is the same as from Dallas to Huston (this property is called *Symmetry of Metric*). Finally, it seems quite obvious that by going directly from one location to another (by following the shortest path) one covers smaller (or equal) distance than by passing through a certain specified third location (e.g. using direct flights always takes less time than using connecting flights). This property, which has a very clear geometrical meaning, is called *Triangle Inequality* (see the picture below).



The formal definition of a metric space is introduced below:

Definition 3.1. Let S be a set and $d : S \times S \to \mathbb{R}$ a function satisfying the conditions

(D1) (Positive Definiteness) $d(x,y) > 0$ if $x \neq y$ and $d(x,x) = 0$;
(D2) (Symmetry) $d(x,y) = d(y,x)$;
(D3) (Triangle Inequality) $d(x,y) \leq d(x,z) + d(z,y)$.

for all x, y, $z \in S$. Then the function d is called a *metric* (or *distance function*) on S and the pair (S,d) is called a *metric space*.

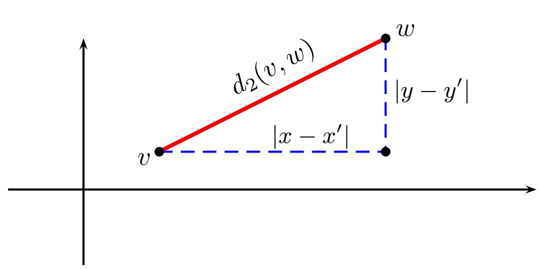
3.1.1 Examples of Metric Spaces

To give some flavor to the theory of metric spaces, we will discuss several interesting examples of metrics on the plane $\mathbb{R}^2$, which can also illustrate how this idea can be connected to the (un)real world.

Example 3.2. (a) **By-Air-Metric** or **Euclidean Metric**: We consider the space $S_a := \mathbb{R}^2$. One can imagine that when traveling by air from a point $v = (x,y)$ to a point $w = (x',y')$, one can follow the shortest path between these two locations, i.e. the straight line joining v and w. That means, for such a traveller, the distance between (x,y) and (x',y') is $\sqrt{(x-x')^2 + (y-y')^2}$, which leads to the following definition of *by-air-metric*:

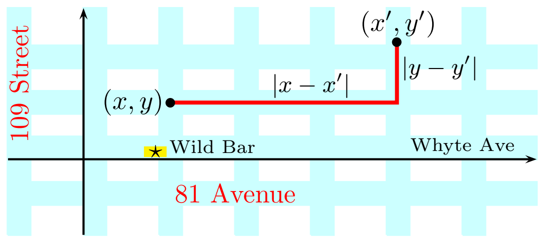
$$d_2(v,w) = \sqrt{(x-x')^2 + (y-y')^2}. \tag{3.1}$$

The space $S_a = \mathbb{R}^2$ equipped with the metric d_2 (which is called the *Euclidean metric*) becomes a metric space. The distance $d_2(v,w) = \|v-w\|_2$ between two points $v = (x,y)$, $w = (x',y') \in \mathbb{R}^2$, given by (3.1) represents the length of the shortest path joining v with w (see the picture below).



From the point of view of the real world, we have to consider that a distance between two locations should be measured in practical terms: the *distance travelled* from one place to another. The distance described by the metric d_2 refers to the *air travel*, when it is possible to avoid all the physical obstacles on the way. However, this is not always the case.

(b) **Urban Metric**: Assume that $S_u := \mathbb{R}^2$. Imagine an urban area, represented bt S_u, divided into blocks by horizontal and vertical streets (see the picture below).

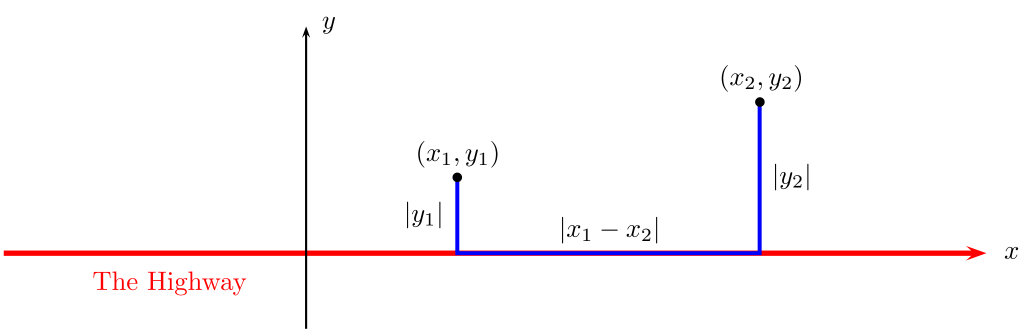


From this point of view, the metric function $d_1 : \mathbb{R}^2 \to \mathbb{R}$, which is defined by

$$d_1(v, w) = d_1((x, y), (x', y')) = |x - x'| + |y - y'| \tag{3.2}$$

represents the distance travelled in an urban area, following vertical streets and horizontal avenues, from one location with coordinates (x, y) to another location with coordinates (x', y'). We can refer to this metric as the *urban metric*. The space S_u equipped with the metric d_1 is a metric space.

(c) **Highway Metric:** Let $S_h := \mathbb{R}^2$. We can imagine that the x-axis is a local highway to which are connected many vertical rural roads. In order to get from one place to another (not on the same rural road) it is necessary to go to the local highway first, then travel on the highway till you reach another vertical rural road leading to the destination point. The actual distance traveled is considered to be the distance between two points with respect to this metric. See the picture below:



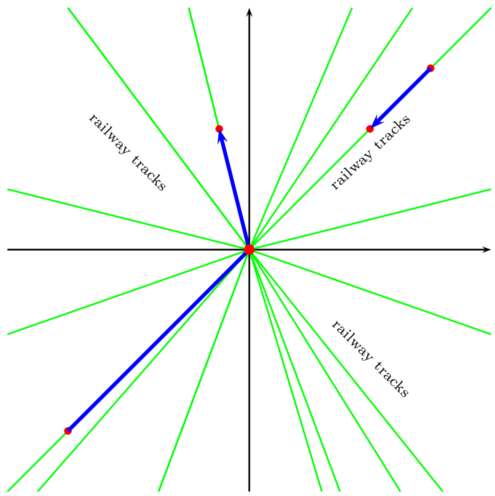
It is easy to verify this metric, which we denote by d_h and call it the *highway metric*, is defined by the formula:

$$d_h((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2, \\ |y_1| + |x_1 - x_2| + |y_2| & \text{if } x_1 \neq x_2. \end{cases} \tag{3.3}$$

Then (S, d_h) is a metric space, called the *highway metric space*.

(d) **Railway Metric**: We consider the space $S_r := \mathbb{R}^2$, this time equipped with the so-called *railway metric* $d_r : S_r \times S_r \to \mathbb{R}$, defined by

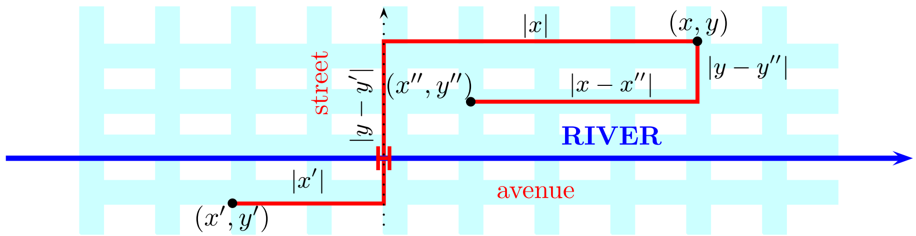
$$d_r((x_1,y_1),(x_2,y_2)) = \begin{cases} d_2((x_1,y_1),(x_2,y_2) & \text{if } \exists_{t\in\mathbb{R}}\ (x_1,y_1) = t(x_2,y_2), \\ & \quad\text{or } t(x_1,y_1) = (x_2,y_2) \\ \sqrt{|x_1|^2+|y_1|^2} + \sqrt{|x_2|^2+|y_2|^2} & \text{otherwise.} \end{cases} \tag{3.4}$$



One can imagine that the origin $(0,0)$ represents here a central railway station, to which converge the railway tracks from all directions (see picture above). If the two points are located on the same track, then it is sufficient to take one train and travel from one place to another one, following the shortest path between these two points (i.e. the Euclidean distance). However, if two points do not belong to the same railway track, one has to travel first to the central station, then change the train, and go to the another point.

(d) **Urban Bridge Metric:** Imagine that a city, partitioned into blocks by horizontal and vertical streets, is divided into two parts by a horizontal river passing through its center. These two parts are connected by a single bridge, which was build exactly in the center of the city (i.e. we will assume that this is the origin of the coordinate system). Of course, if two points belong to the same side of the river, the distance between them is the urban distance, which was explained in Example (b). However, if they belong to different sides of the river, then in order to reach another point, the traveller has to pass through the bridge (see the picture below). This metric, which we denote by d_b is defined on the set $S_b := \{(x,y) \in \mathbb{R}^2 : y \neq 0\} \cup \{(0,0)\}$. We will call it the *Bridge Metric*. One can easily check that for $v = (x,y)$ and $w = (x',y')$ in S_b we have

$$d_b(v,w) = d_b((x,y),(x',y')) = \begin{cases} |x-x'| + |y-y'|, & \text{if } \ yy' \geq 0, \\ |x| + |y-y'| + |x'|, & \text{if } \ yy' < 0. \end{cases} \tag{3.5}$$



(e) **Ghost Stories** and **Discrete Metric Space**: Imagine that you believe in ghosts that can instantaneously move from one place to another, just by using one thought "I want to be there." If the set S represents all the possible locations such a ghost is able to visit, then the metric $d_d : S \times S \to \mathbb{R}$ describing the distance in S is defined by

$$d_d(x, y) = \begin{cases} 1, & \text{if} \quad x \neq y, \\ 0, & \text{if} \quad x = y. \end{cases} \tag{3.6}$$

The metric space (S, d_d) is called a *discrete metric space*. Notice that, this example shows that you can define such a discrete metric d_d on any set S.

Remark 3.3. One can easily imagine other interesting examples of metric spaces. For instance, it is well-known that it is possible to identify any person by his/her fingerprints. In this case, we could think about the set S composed of all individuals who can be identified by their fingerprints (i.e. with hands). The problem of measuring differences between two sets of fingerprints is rather complicated, however such methods exists. Any kind of of such a measure is a metric function. Other patterns, like DNA sequences, or facial characteristics, provide other practical examples of metric spaces, which are used for example in criminology. An interesting example of a metric, is the cost of a plane ticket. It turns out that the fare for a trip is not proportional to the distance travelled, i.e. the airfare for a destination that is twice far away than another one is not twice higher! Having a formula translating the distance into the actual cost of air travel is also a metric function.

Definition 3.4. Let (S, d) be a metric space and $A \subset S$ a subset. Define $d_A : A \times A \to \mathbb{R}$ as a restriction of the metric d to the set A, i.e.

$$d_A(x, y) := d(x, y), \quad (x, y) \in A \times A.$$

Then the metric space (A, d_A) is called a *subspace* of (S, d).

Remark 3.5. Since the metric function reflects the topographical reality of the space in terms of distance between points in this space, the area of mathematics, which deals with such objects, is called *topology*. The idea of a metric space allows us to forget the specific type of objects we are dealing with, and based just on the properties of the metric and other definitions we can make statements and conclusions, which can help us to analyze the situation described by this space.

3.1.2 Normed Spaces

A large class of examples of metric spaces come from linear algebra. They are the *normed vector spaces*. More precisely, we have the following definition:

Definition 3.6. Let V be a vector space and let $\|\cdot\| : V \to \mathbb{R}$ be a function satisfying the following properties:

(N1)(Positive Definiteness) $\|\vec{v}\| > 0$ if $\vec{v} \neq \vec{0}$ and $\|\vec{0}\| = 0$;
(N2)(Homogeneity) $\|\alpha\,\vec{v}\| = |\alpha|\,\|\vec{v}\|$;
(N3)(Triangle Inequality) $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$,

for all $\vec{v}$, $\vec{w} \in V$ and $\alpha \in \mathbb{R}$. Then the function $\|\cdot\|$ is called a *norm* (or *length function*) on V, and the pair $(V, \|\cdot\|)$ is called a *normed space*.

Proposition 3.7. *Let (V, d) be a normed space and $d : V \times V \to \mathbb{R}$ be defined by $d(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\|$, $\vec{v}$, $\vec{w} \in V$. Then d is a metric on V and (V, d) is a metric space.*

Proof: Notice that (by (N1)) $d(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\| = 0$ if and only if $\vec{v} - \vec{w} = \vec{0}$, i.e. $\vec{v} = \vec{w}$, so d satisfies the property (D1). The property (D2) is also satisfied, since by (N2),

$$d(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\| = \|(-1)(\vec{w} - \vec{v})\| = |-1|\,\|\vec{w} - \vec{v})\| = \|\vec{w} - \vec{v})\| = d(\vec{w}, \vec{v}).$$

Finally, we have (by (N3))

$$d(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\| = \|(\vec{v} - \vec{z}) + (\vec{z} - \vec{w})\| \leq \|\vec{v} - \vec{z}\| + \|\vec{z} - \vec{w}\| = d(\vec{v}, \vec{z}) + d(\vec{z}, \vec{w}).$$

□

Example 3.8. Let us present several examples of metric spaces obtained from normed metric spaces.

(a) The simplest possible normed space is the space $V = \mathbb{R}$ equipped with the norm being the absolute value $|x|$, $x \in \mathbb{R}$. It is clear that the function $|\cdot|$ is a norm on $\mathbb{R}$, thus we obtain the metric space $S = \mathbb{R}$ equipped with the metric function $d(x, y) = |x - y|$, x, $y \in \mathbb{R}$;
(b) The space $V = \mathbb{R}^n$ can be equipped with the standard *Euclidean* norm $\|\cdot\|_2$, defined for $\vec{v} = (x_1, x_2, \dots, x_n)$ by
$$\|\vec{v}\|_2 = \left[\sum_{k=1}^{n} x_k^2\right]^{\frac{1}{2}}, \quad \vec{v} \in \mathbb{R}^n,$$
is a normed space.
(c) It is possible to defined other norms on the space $V = \mathbb{R}^n$. Let us consider a number $p \geq 1$ and define the so-called *p-norm* $\|\cdot\|_p : V \to \mathbb{R}$ by
$$\|\vec{v}\|_p = \left[\sum_{k=1}^{n} |x_k|^p\right]^{\frac{1}{p}}, \quad \vec{v} \in \mathbb{R}^n,$$
and in addition we define the norm $\|\cdot\|_\infty : V \to \mathbb{R}$ by
$$\|\vec{v}\|_\infty = \max\{|x_k| : k = 1, 2, \dots, n\}, \quad \vec{v} \in \mathbb{R}^n.$$

(d) As our first example of an infinite-dimensional normed vector space, we consider $p \geq 1$ and define $V := l^p$ to be the set of all infinite sequences $\vec{v} = (x_1, x_2, \dots, x_k, \dots)$ such that $\sum_{k-=1}^{\infty} |x_k|^p < \infty$. We can introduce, by a similar formula as in (c), the *p-norm* on l^p, by

$$\|\vec{v}\|_p = \left[\sum_{k=1}^{\infty} |x_k|^p\right]^{\frac{1}{p}}, \quad \vec{v} = (x_1, x_2, \dots, x_k, \dots) \in l^p,$$

which also satisfies the properties (N1)–(N3). Moreover, one can also introduce the space l^∞ consisting of all bounded sequences $\vec{v} = (x_1, x_2, \dots, x_k, \dots)$, i.e. $\vec{v} \in l^\infty$ if there is a constant $M > 0$ such that $|x_k| < M$ for all $k = 1, 2, 3, \dots$. Then, the norm $\|\cdot\|_\infty : l^2 \to \mathbb{R}$ is defined by

$$\|\vec{v}\|_\infty := \sup\{|x_k| : k = 1, 2, 3, \dots\}, \quad \vec{v} = (x_1, x_2, \dots, x_k, \dots) \in l^\infty.$$

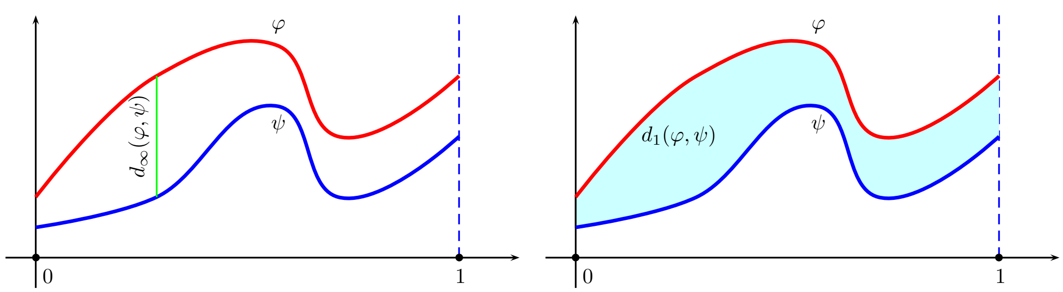
(e) Although we did not yet properly introduce the concept of continuity, we should mention the vector space of all continuous functions $\varphi : [0, 1] \to \mathbb{R}$, which is denoted by $C[0, 1]$. It is possible to define a norm $\|\cdot\|_\infty : C[0, 1] \to \mathbb{R}$, which is called the *sup-norm* on $C[0, 1]$, by the following formula

$$\|\varphi\|_\infty := \sup\{|\varphi(t)\| : t \in [0, 1]\}, \quad \varphi \in C[0, 1].$$

Notice that $\|\varphi\|_\infty$ is well-defined because φ is bounded on $[0, 1]$ due to its continuity. One can easily verify, using the properties of sup that $\|\cdot\|_\infty$ is a norm on $C[0, 1]$. Then we can also define the distance between two function φ, $\psi : [0, 1] \to \mathbb{R}$ as

$$d_\infty(\varphi, \psi) = \|\varphi - \psi\|_\infty$$

(see the picture below).



(f) There are also other norms which could be defined on the space $V = C[0, 1]$, for example, $\|\cdot\|_p : V \to \mathbb{R}$, $p \geq 1$, with

$$\|\varphi\|_p := \left[\int_0^1 |\varphi(t)|^p\right]^{\frac{1}{p}}, \quad \varphi \in C[0, 1].$$

We leave the verification that $\|\cdot\|_p$ is indeed a norm (i.e. satisfies the conditions (N1)–(N3) for later). Let us give a geometric interpretation of the distance $d_1(\varphi, \psi) := \|\varphi - \psi\|_1$ between two functions $\varphi, \psi : [0, 1] \to \mathbb{R}$, $\varphi, \psi \in V$. One can easily verify, that the number $d_1(\varphi, \psi)$ represents the area of the region contained in between the graphs of the functions φ and ψ (see picture above).

3.2 Elements of Point-Set Topology

In this section we will introduce the fundamental topological notions of *open* and *closed* sets in a metric space (S, d) and illustrate them on several examples.

Suppose that (S, d) is a metric space. In what follows, for the sake of simplicity, we will simply say that S is a metric space (instead (S, d)).

Definition 3.9. Let S be a metric space, $x_o \in S$ and $r > 0$. An *open ball* centered at x_o with radius r is the set

$$B_r(x_o) := \{x \in S : d(x_o, x) < r\},$$

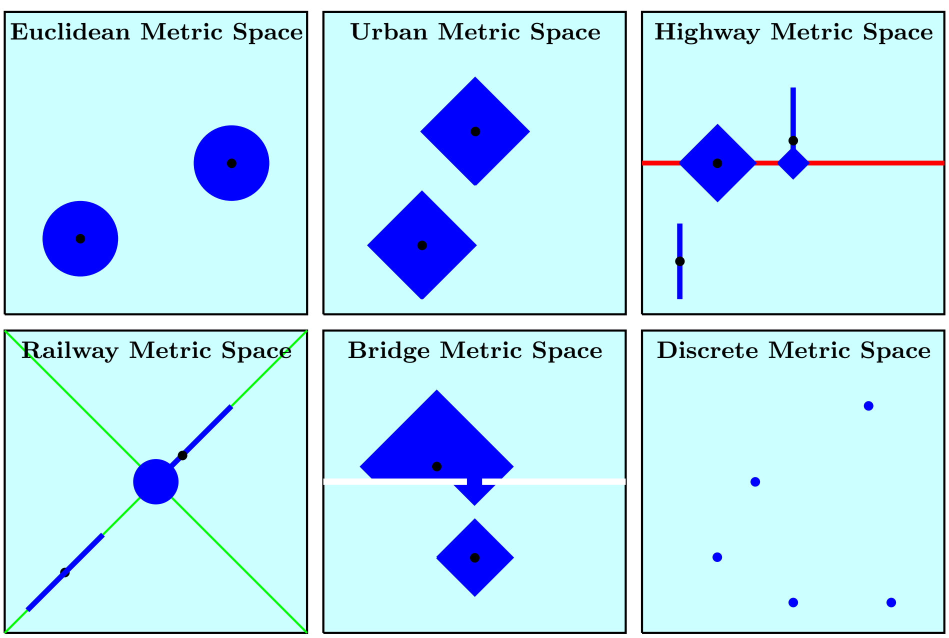
and a *closed ball* centered at x_o with radius r is the set

$$\overline{B}_r(x_o) := \{x \in S : d(x_o, x) \leq r\}.$$

From the intuitive point of view, the concept of an open ball $B_\varepsilon(x_o)$ is very simple: it is a set surrounding the point x_o, which depending on the value $\varepsilon > 0$, can be made arbitrarily close to x_o. We will sometimes call $B_\varepsilon(x_o)$ a ε-neighborhood of x_o.

The following examples show how open balls may look in different metric spaces.

Example 3.10. Let us look at different metric spaces, which were discussed in Example 3.2: Euclidean, urban, highway, railway, bridge, and discrete metric spaces. In the picture below, we show the shape of open balls in these metric spaces.



Remark 3.11. Let (S, d) be a metric space and (A, d_A) a subspace of (S, d) (see Definition 3.4). Then for every $x \in A$. We can consider the open ball $B_r^A(x)$ in A of radius $r > 0$ centered at x, i.e.

$$B_r^A(x) := \{y \in A : d_A(x, y) < r\}.$$

Since $A \subset X$, we also have the open ball in S of radius r that is centered at centered at x

$$B_r(x) := \{y \in S : d(x, y) < r\}.$$

Since for x, $y \in A$ we have $d_A(x, y) = d(x, y)$, it is easy to notice that

$$B_r^A(x) = B_r(x) \cap A.$$

3.2.1 Open and Closed Sets

Definition 3.12. Let (S, d) be a metric space and $U \subset S$. The set U is called *open* if the following condition is satisfied

$$\forall_{x\in U}\ \exists_{\varepsilon>0}\ B_\varepsilon(x) \subset U \qquad \Longleftrightarrow \qquad \forall_{x\in U}\ \exists_{\varepsilon>0}\ \forall_{y\in S}\ \ d(x, y) < \varepsilon \ \Rightarrow\ y \in U. \tag{3.7}$$

Definition 3.13. The family of all open sets U in the metric space S is called the *topology* of S and it is denoted by $\mathcal{T} := \mathcal{T}(S)$.

Proposition 3.14. *Let S be a metric space and $\mathcal{T}$ be the topology of S. Then we have*

(a) $\emptyset$, $S \in \mathcal{T}$, i.e. the empty set $\emptyset$ and the whole space S are open sets;
(b) $\{U_i\}_{i\in I} \subset \mathcal{T} \quad \Rightarrow \quad \bigcup_{i\in I} U_i \in \mathcal{T}$, i.e. any union of open sets is an open set;
(c) $U_1, U_2, \dots, U_k \in \mathcal{T} \quad \Rightarrow \quad U_1 \cap U_2 \cap \dots \cap U_k \in \mathcal{T}$, i.e. any intersection of finitely many open sets is an open set.

Proof: (a): is obvious.
(b): Let $U := \bigcup_{i\in I} U_i$. We will show that U satisfies condition (3.7). We need to show that $\forall_{x\in U}$ (which means we can choose $x \in U$ to be an arbitrary point) $\exists_{\varepsilon>0}$ (and we need to specify what is the value of $\varepsilon > 0$ for the chosen $x \in U$) such that $B_\varepsilon(x) \subset U$. Suppose thus that $x \in U$ is an arbitrary point. Since

$$x \in U = \bigcup_{i\in I} U_i \iff \exists_{i\in I}\ x \in U_i. \tag{3.8}$$

and since (by assumption) all the sets U_i are open, i.e.

$$x \in U_i \implies \exists_{\varepsilon>0}\ \ B_\varepsilon(x) \subset U_i. \tag{3.9}$$

By combining (3.8) with (3.9) we obtain

$$\forall_{x\in U}\ \exists_{i\in I}\ x \in U_i\ \exists_{\varepsilon>0}\ B_\varepsilon(x) \subset U_i. \tag{3.10}$$

On the other hand, by (3.10) and

$$\forall_{i\in I}\, U_i \subset U = \bigcup_{i\in I} U_i \tag{3.11}$$

we obtain (by combining (3.10) and (3.11))

$$\forall_{x\in U}\ \exists_{i\in I}\ x \in U_i\ \exists_{\varepsilon>0}\ B_\varepsilon(x) \subset U_i \subset U \implies \forall_{x\in U}\ \exists_{\varepsilon>0}\ B_\varepsilon(x) \subset U. \tag{3.12}$$

(c): Let us put $U := U_1 \cap U_2 \cap \dots \cap U_k$. We need to show that U is an open set. Assume that $x \in U$ is an arbitrary point. Then

$$x \in U \iff \forall_{j=1,2,\dots,k}\ x \in U_j, \tag{3.13}$$

and since U_j, $j = 1, 2, \dots, k$ are open

$$\forall_{j=1,2,\dots,k}\ \exists_{\varepsilon_j>0}\ \ B_{\varepsilon_j}(x) \subset U_j. \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\forall_{x\in U} \exists_{\varepsilon=\min\{\varepsilon_1,\dots,\varepsilon_k\}>0}\ B_\varepsilon(x) \subset U := U_1 \cap U_2 \cap \dots \cap U_k.$$

□

Proposition 3.15. *An open ball $B_r(x_o)$ in a metric space S is an open set.*

Proof: Let $x \in B_r(x_o)$ be an arbitrary point, i.e. $d(x_o, x) < r$. Put $\varepsilon := r - d(x_o, x)$. Then for every $x' \in B_\varepsilon(x)$ (i.e. $d(x, x') < \varepsilon$) we have

$$d(x_o, x') \le d(x_o, x) + d(x, x') < d(x_o, x) + \varepsilon = d(x_o, x) + r - d(x_o, x) = r,$$

i.e. $B_\varepsilon(x) \subset B_r(x_o)$, which proves that $B_r(x_o)$ is an open set. □

Definition 3.16. A set A in a metric space S is *closed* if its complement $A^c := S \setminus A$ is open.

The following properties of closed set follow immediately from Proposition 3.14:

Proposition 3.17. *Let S be a metric space. Then we have*

(a) $\emptyset$ and the whole space S are closed;
(b) $\{A_i\}_{i\in I}$ is a family of closed sets in S, then $\bigcap_{i\in I} A_i$ is closed i.e. any intersection of closed sets is a closed set;
(c) If A_1, A_2, ..., A_k are closed sets, then $A_1 \cup A_2 \cap \dots \cup A_k$ is a closed set, i.e. any union of finitely many closed sets is a closed set.

Proposition 3.18. *Let (S, d) be a metric space and (A, d_A) a subspace of (S, d). Then we have*

(a) A set $V \subset A$ is open in (A, d_A) if and only if there exists an open set U in (S, d) such that $V = U \cap A$;
(b) A set $P \subset A$ is closed in (A, d_A) if and only if there exists a closed set Q in (S, d) such that $P = P \cap Q$.

Proof: (a): Assume that $V \subset A$ is open in the subspace (A, d_A) (see Definition 3.4). Then, by the definition

$$\forall_{x\in V}\ \exists_{r_x>0}\ \ B^A_{r_x}(x) \subset V.$$

Then clearly,

$$V = \bigcup_{x\in V}\{x\} \subset \bigcup_{x\in V} B^A_{r_x}(x) \subset \bigcup_{x\in V} V = V,$$

which implies that

$$V = \bigcup_{x\in V} B^A_{r_x}(x).$$

On the other hand, since the balls $B_{r_x}(x)$ are open in (S, d), then by property (c) in Proposition 3.14 the set

$$U := \bigcup_{x\in V} B_{r_x}(x)$$

is open in (S, d) and by Remark 3.11

$$A \cap U = A \cap \bigcup_{x\in V} B_{r_x}(x) = \bigcup_{x\in V} \Big(A \cap B_{r_x}(x)\Big) = \bigcup_{x\in V} B^A_{r_x}(x) = V.$$

Conversely, if U is open in (S, d), then for every $x \in A\cap U$, there exists $r_x > 0$ such that $B_{r_x}(x) \subset U$. Put $V := A \cap U$. Then

$$\forall_{x\in V}\ \exists_{r_x>0}\ \ B^A_{r_x}(x) = A \cap B_{r_x}(x) \cap A \subset U \cap A = V,$$

which implies that V is open in (A, d_A).

(b): A set P is closed in (A, d_A) if and only if $V := A \setminus P$ is open in (A, d_A), which is equivalent (by (a)) to the condition that there exists open set U in (S, d) such that $V = U \cap A$, and that means that $Q := S \setminus U$ is closed (S, d). However,

$$A \cap Q = A \cap (S \setminus U) = A \cap U^c = A \setminus U = A \setminus V = P.$$

□

Definition 3.19. Let (S, d) be a metric space. A set A in S is called *bounded* if there exist $x_o \in S$ and $R > 0$ such that $A \subset B_R(x_o)$.

For a set A in a metric space (S, d) we define the *diameter* $\operatorname{diam}(A)$ of A by

$$\operatorname{diam}(A) := \sup\{d(x, y) : x,\ y \in A\}.$$

It is easy to observe that a set A is bounded if and only if $\operatorname{diam}(A) < \infty$.

3.2.2 Interior, Closure and Boundary of a Set

Definition 3.20. Let A be a set in a metric space S. We define the following sets:

(a) the *interior* $\operatorname{int}(A)$ of A by

$$x \in \operatorname{int}(A) \iff \exists_{\varepsilon>0}\ B_\varepsilon(x) \subset A \iff \exists_{\varepsilon>0}\ \forall_{x'\in S}\ d(x, x') < \varepsilon \ \Rightarrow\ x' \in A. \tag{3.15}$$

(b) the *closure* $\overline{A}$ of A by

$$x \in \overline{A} \iff \forall_{\varepsilon>0}\ B_\varepsilon(x) \cap A \neq \emptyset \iff \forall_{\varepsilon>0}\ \exists_{x'\in A}\ d(x,x') < \varepsilon. \tag{3.16}$$

(c) the *boundary* ∂A of A by

$$\begin{aligned} x \in \partial A &\iff \forall_{\varepsilon>0}\ B_\varepsilon(x) \cap A \neq \emptyset \ \wedge\ B_\varepsilon(x) \cap (S \setminus A) \neq \emptyset \\ &\iff \forall_{\varepsilon>0}\ \exists_{x'\in A}\ \exists_{x''\notin A}\ d(x,x') < \varepsilon \ \wedge\ d(x,x'') < \varepsilon. \end{aligned} \tag{3.17}$$

The most important properties of operations $\operatorname{int}(A)$, $\overline{A}$ and ∂A are summarized in the following

Theorem 3.21. *Let A be a set in a metric space S. Then*

(a) The set $\operatorname{int}(A)$ is an open set such that

$$\operatorname{int}(A) := \bigcup\{U : U \subset A, \quad \text{and } U \text{ open}\}, \tag{3.18}$$

i.e. $\operatorname{int}(A)$ is the largest open set contained in A;

(b) The set $\overline{A}$ is a closed set such that $\overline{A} = S \setminus \operatorname{int}(A^c) = (\operatorname{int}(A^c))^c$, where $A^c := S \setminus A$ denotes the complement of A, and

$$\overline{A} := \bigcap\{C : A \subset C, \quad \text{and } C \text{ closed}\},$$

i.e. $\overline{A}$ is the smallest closed set containing A;

(c) ∂A is a closed set such that

$$\partial A = \overline{A} \cap \overline{A^c},$$

i.e. the boundary of the set A coincides with the boundary of the complement A^c.

Proof: (a): In order to show that $\operatorname{int}(A)$ is open, suppose that $x \in \operatorname{int}(A)$, then $\exists_{\varepsilon>0}\ B_\varepsilon(x) \subset A$. Since $B_\varepsilon(x)$ is an open set,

$$\forall_{x'\in B_\varepsilon(x)}\ \exists_{\delta>0}\ B_\delta(x') \subset B_\varepsilon(x) \subset A,$$

which implies that

$$\forall_{x'\in B_\varepsilon(x)}\ B_\delta(x') \subset A \implies \forall_{x'\in B_\varepsilon(x)}\ x' \in \operatorname{int}(A),$$

i.e. $B_\varepsilon(x) \subset \operatorname{int}(A)$, and consequently, $\operatorname{int}(A)$ is open. We will show equality (3.18). Put $V := \bigcup\{U : U \subset A, \ \text{and } U \text{ open}\}$, Since for every open set $U \subset A$, we have

$$\forall_{x\in U}\ \exists_{\varepsilon>0}\ B_\varepsilon(x) \subset U \subset A \implies U \subset \operatorname{int}(A),$$

thus $V \subset \operatorname{int}(A)$. Since we already know that $\operatorname{int}(A)$ is an open set, thus $\operatorname{int}(A) \subset V$, i.e. $\operatorname{int}(A) = V$.
(b): It follows from (3.16) that

$$\begin{aligned} x \in (\overline{A})^c &\iff \sim \big(x \in \overline{A}\big) \\ &\iff \sim \big(\forall_{\varepsilon>)}\ B_\varepsilon(x) \cap A \neq \emptyset\big) \\ &\iff \exists_{\varepsilon>0}\ B_\varepsilon(x) \cap A = \emptyset \\ &\iff \exists_{\varepsilon>0}\ B_\varepsilon(x) \subset A^c \\ &\iff x \in \operatorname{int}(A^c) \end{aligned}$$

and consequently

$$(\overline{A})^c = \operatorname{int}(A^c) \iff \overline{A} = (\operatorname{int}(A^c))^c.$$

Since $\operatorname{int}(A^c)$ is open, it follows that $\overline{A} = S \setminus \operatorname{int}(A^c)$ is closed. On the other hand, by (a) and the above equality, we have

$$\begin{aligned} x \in \overline{A} &\iff \sim (x \in \operatorname{int}(A^c)) \\ &\iff \sim \Big(x \in \bigcup\{U : U \subset A^c \text{ and } U \text{ open}\}\Big) \\ &\iff x \notin \bigcup\{U : U \subset A^c \text{ and } U \text{ open}\} \\ &\iff x \in \bigcap\{U^c : U \subset A^c \text{ and } U \text{ open}\} \\ &\iff x \in \bigcap\{C : C \supset A \text{ and } C \text{ closed}\} \end{aligned}$$

(c): Follows immediately from (a) and (b). □

Corollary 3.22. *Let A be set in a metric space S. Then*

$$\overline{A} = \operatorname{int}(A) \cup \partial A, \quad \operatorname{int}(A) \cap \partial A = \emptyset.$$

The proofs of the following two propositions are straight forward. We leave them as exercises.

Proposition 3.23. *Let A and B be two sets in a metric space S. Then we have*

(i) A is open if and only if $\operatorname{int}(A) = A$*;*
(ii)$\operatorname{int}\operatorname{int}(A) = \operatorname{int}(A)$*;*
(iii)$\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$*;*
(iv)If $A \subset B$ then $\operatorname{int}(A) \subset \operatorname{int}(B)$.

Proposition 3.24. *Let A and B be two sets in a metric space S. Then we have*

(i) A is closed if and only if $\overline{A} = A$;
(ii)$\overline{\overline{A}} = \overline{A}$;
(iii)$\overline{A \cup B} = \overline{A} \cup \overline{B}$;
(iv)If $A \subset B$ then $\overline{A} \subset \overline{B}$.

Definition 3.25. Let A be a set in a metric space (S, d) and $x_o \in S$. We say that x_o is a *limit point* of the set A if

$$\forall_{\varepsilon>0}\ \exists_{x\in A}\ \ 0 < d(x, x_o) < \varepsilon. \tag{3.19}$$

We will denote the set of all limit points of A by A'.

Proposition 3.26. *Let A be a set in a metric space S. Then we have*

$$\overline{A} = A \cup A'.$$

Consequently, a set A is closed if and only if $A' \subset A$.

Definition 3.27. Let A be a set in a metric space and $x_o \in A$. We say that x_o is an *isolated point* in A if

$$\exists_{\delta>0}\ \ B_\delta(x_o) \cap A = \{x_o\} \quad \iff \quad \exists_{\delta>0}\forall_{x\in A}\ \ x \neq x_o \ \Rightarrow\ \ d(x, x_o) \geq \delta. \tag{3.20}$$

3.2.3 Sequences in a Metric Space

Definition 3.28. Let (S,d) be a metric space. A function $a:\mathbb{N}\to S$ is called a *sequence* in S, and it is denoted by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$, where $a_n := a(n)$. If A is a set in S and $a:\mathbb{N}\to A$, then we will write that $\{a_n\}\subset A$.

Definition 3.29. Let S be a metric space, $\{x_n\}$ a sequence in S and $a\in S$. We say that the sequence $\{x_n\}$ converges to a (we will use the notation $\lim\limits_{n\to\infty} x_n = a$) if the following condition is satisfied

$$\lim_{n\to\infty} x_n = a \quad \overset{\text{def}}{\iff} \quad \forall_{\varepsilon>0}\ \exists_{N>0}\ \forall_{n>N}\ \ d(x_n,a)<\varepsilon. \tag{3.21}$$

Definition 3.30. Let $k:\mathbb{N}\to\mathbb{N}$ be a increasing function, i.e. if $n<m$ then $k(n)<k(m)$, and $\{x_n\}$ be a sequence in a metric space S (i.e. $x:\mathbb{N}\to S$). Then the sequence $\{x_{k(n)}\}$, which is the composition of the function k and x, i.e.

$$\mathbb{N}\xrightarrow{k}\mathbb{N}\xrightarrow{x} S$$

is called a *subsequence* of $\{x_n\}$.

Proposition 3.31. *Let A be a set in a metric space S and $x_o\in S$. Then x_o is a limit point of A if and only if there exists a sequence $\{x_n\}\subset A\setminus\{x_o\}$ such that $\lim\limits_{n\to\infty} x_n = x_o$.*

Proof: Since x_o is a limit point of A, thus

$$\forall_{\varepsilon>0}\ \exists_{x\in A}\ \ 0<d(x,x_o)<\varepsilon.$$

In particular, take $\varepsilon=\frac{1}{n}$. then there exists $x_n\in A$ such that $0<d(x_n,x_o)<\frac{1}{n}$. The obtained in this way sequence $\{x_n\}\subset A\setminus\{x_o\}$ converges to x_o, i.e. $\lim_{n\to\infty} x_n = x_o$. The converse statement is trivial. □

Corollary 3.32. *Consider a set A in a metric space S. Then the following conditions are equivalent:*

(a) A is closed
(b) The set A contains all its limit points

Corollary 3.33. *Let A be a set in a metric space (S,d). Then we have*

$$x_o\in\overline{A} \quad\iff\quad \exists_{\{x_n\}\subset A}\ \ \lim_{n\to\infty} x_n = x_o. \tag{3.22}$$

Definition 3.34. Let S be a metric space. A sequence $\{x_n\}$ in S is called a *Cauchy sequence* if

$$\forall_{\varepsilon>0}\ \ \exists_{N>0}\ \ \forall_{n,m\geq N}\quad d(x_n,x_m)<\varepsilon. \tag{3.23}$$

Remark 3.35. It is easy to notice that every convergent sequence is a Cauchy sequence. Indeed, assume that $\{a_n\}$ converges to a certain limit x_o in S, then

$$\forall_{\varepsilon>0}\ \exists_{N>0}\ \forall_{n>N}\ \ d(a_n, x_o) < \frac{\varepsilon}{2}. \tag{3.24}$$

In particular, if n, $m \geq N$ then it follows from (3.24) that

$$d(a_n, a_m) \leq d(a_n, x_o) + d(a_m, x_o) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

However the converse statement is not true. Consider for example the space $S = (0,1)$ equipped with the usual metric $d(x,y) = |x-y|$, x, $y \in (0,1)$. Then the sequence $x_n = \frac{1}{n}$ is a Cauchy sequence (indeed, it is a convergent to zero sequence in $\mathbb{R}$), but since $0 \notin S$, thus it does not converge in S.

Lemma 3.36. *Let S be metric space and $\{x_n\}$ a Cauchy sequence in S. If $\{x_n\}$ contains a convergent subsequence then $\{x_n\}$ converges.*

Proof: Suppose that $\{x_{k(n)}\}$ is a subsequence of $\{x_n\}$ such that $\lim\limits_{n\to\infty} x_{k(n)} = x_o$, i.e.

$$\forall_{\varepsilon>0}\ \exists_{M_1>0}\ \forall_{n\geq M_1}\ \ d(x_{k(n)}, x_o) < \frac{\varepsilon}{2}. \tag{3.25}$$

On the other hand, since $\{x_n\}$ is Cauchy, by (3.23)

$$\forall_{\varepsilon>0}\ \exists_{N_1>0}\ \forall_{k,l\geq N_1}\ \ d(x_k, x_l) < \frac{\varepsilon}{2}. \tag{3.26}$$

We put $N := \max\{N_1, k(M_1)\}$ and let $n_o \in \mathbb{N}$ be such that $k(n_o) > N$, then by (3.25) and (3.26) we have

$$\forall_{\varepsilon>0}\ \ \exists_{N>0}\ \ \exists_{n_o\in\mathbb{N}}\ \ \forall_{n\geq N}\ \ d(x_n, x_o) \leq d(x_n, x_{k(n_o)}) + d(x_{k(n_o)}, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \tag{3.27}$$

which implies that

$$\lim_{n\to\infty} x_n = x_o.$$

$\square$

3.2.4 Complete Metric Space

Definition 3.37. We say that a metric space (S,d) is a *complete metric space* if every Cauchy sequence in S converges. Similarly, we will say that a set A in S is *complete* if every Cauchy sequence in A converges to a limit in A.

Remark 3.38. The idea of a Cauchy sequence is related to the complications involved in checking out that a given sequence is or not convergent. In order to know that a sequence is convergent, we need to find first a candidate for the limit, and only then we are able to verify the condition for convergence. However, the main problem with the limits is to know in advance that it exists, so a various techniques can be applied in order to identify it. The Cauchy sequence provides a condition which is easy to verify and if the space is complete, it is equivalent to the convergence of a sequence.

Theorem 3.39. *The set of real numbers* $\mathbb{R}$ *equipped with the usual metric* $d(x, y) = |x - y|$, *is a complete metric space.*

Proof: We will prove that if a sequence of real numbers $\{a_n\}$ satisfies the Cauchy Condition (3.23), then $\{a_n\}$ converges. Our first goal is to find a candidate for the limit of $\{a_n\}$. Let $\mathcal{X}$ be the set of all $y \in \mathbb{R}$ such that $a_n < y$ for only finitely many integers n. If $y \in \mathcal{X}$ and $z < y$, then evidently $z \in \mathcal{X}$. Consequently, if $y \in \mathcal{X}$ then $\mathcal{X}$ includes the entire interval $(-\infty, y]$. We will show that $\mathcal{X}$ is non-empty. Since $\{a_n\}$ satisfies (3.23), for an $\varepsilon > 0$ we can find N such that $|a_n - a_m| < \varepsilon$ whenever $m, n \geq N$, so that all but finitely many terms of the sequence $\{a_n\}$ lie in the interval $(a_N - \varepsilon, a_N + \varepsilon)$. In particular, $a_N - \varepsilon \in \mathcal{X}$, so that $\mathcal{X}$ is not empty. Also, no $t \geq a_N + \varepsilon$ belongs to $\mathcal{X}$, so that $a_N + \varepsilon$ is an upper bound of $\mathcal{X}$. By the least Upper Bound Axiom, there exists $x_o = \sup \mathcal{X}$. We will show that x_o is the limit of $\{a_n\}$. Since $a_N + \varepsilon$ is an upper bound, $x_o \leq a_N + \varepsilon$. Since $a_N - \varepsilon \in \mathcal{X}$, $a_N - \varepsilon \leq x_o$. Hence $|a_N - x_o| \leq \varepsilon$. If $n \geq N$, then

$$|a_n - x_o| \leq |a_n - a_N| + |a_N - x_o| \leq \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ is an arbitrary number and $|a_n - x_o| < 2\varepsilon$ for all $n \geq N$, and it follows that the sequence $\{a_n\}$ converges to x_o. $\square$

Corollary 3.40. *The Euclidean space* $(\mathbb{R}^n, d_2)$ *is a complete metric space.*

Lemma 3.41. *Let* (S, d) *be a metric space and* $\{x_n\}$ *be a Cauchy sequence in* S. *If* $\{x_n\}$ *contains a convergent subsequence, then* $\{x_n\}$ *converges.*

Proof: Suppose that $\{x_n\}$ is a Cauchy sequence, i.e.

$$\forall_{\varepsilon>0}\ \exists_{N>0}\ \forall_{n,m\geq N}\ \ d(x_n, x_m) < \frac{\varepsilon}{2},$$

and assume that $\{x_{n_k}\}$ is a convergent subsequence of $\{x_n\}$, i.e. there is $a \in S$ such that

$$\forall_{\varepsilon>0}\ \exists_{M>0}\ \forall_{k\geq M}\ \ d(a, x_{n_k}) < \frac{\varepsilon}{2}.$$

Let $N_1 = \max\{N, n_M\}$. Then

$$\begin{aligned}\forall_{\varepsilon>0}\ \exists_{N_1>0}\ \forall_{n>N_1}\ \exists_{k\geq M}\ \ n_k \geq N_1\ \wedge\ \ d(a, x_n) &\leq d(a, x_{n_k}) + d(x_{n_k}, x_n)\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

$\square$

Example 3.42. Consider the space $S_h = \mathbb{R}^2$ to be equipped with the highway metric

$$d_h((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2| & \text{if } \ x_1 = x_2, \\ |y_1| + |x_1 - x_2| + |y_2| & \text{if } \ x_1 \neq x_2. \end{cases}$$

Notice that (S_h, d_h) is a complete metric space. Indeed, suppose that $\{(x^n, y^n)\}$ is a Cauchy sequence in (S_h, d_h), i.e.

$$\forall_{\varepsilon>0}\ \ \exists_N\ \ \forall_{n,m\geq N}\ \ d_h((x^n, y^n), (x^m, y^m)) < \varepsilon.$$

Suppose that there exists a subsequence $(x^{n_k}, y^{n_k}$ such that $x^o := x^{n_k} = x^{n_m}$ for all $n, m \in \mathbb{N}$. In such a case, we have that $|y^{n_k} - y^{n_m}| = d_h((x^{n_k}, y^{n_k}), (x^{n_m}, y^{n_m}))$, thus $\{y^{n_k}\}$ is Cauchy in $\mathbb{R}$, and consequently it converges to y^o. It is easy to verify that the subsequence $\{(x^{n_k}, y^{n_k})\}$ converges to (x^o, y^o) in (S_h, d_h). Therefore, by Lemma 3.36, the sequence $\{(x^n, y^n)\}$ converges to (x^o, y^o) in (S_h, d_h). On the other hand, if such a sequence doesn't exists, then there is a subsequence $\{(x^{n_k}, y^{n_k})\}$ such that $x^{n_k} \neq x^{n_m}$ for all $k \neq m$. Consequently,

$$|y^{n_k}| + |x^{n_k} - x^{n_m}| + |y^{n_m}| = d_h((x^{n_k}, y^{n_k}), (x^{n_m}, y^{n_m})),$$

which implies that $\{(x^{n_k}, y^{n_k})\}$ converges to $(0, 0)$ in (S_h, d_h). Therefore, again by Lemma 3.36, the sequence $\{(x^n, y^n)\}$ converges to $(0, 0)$ in (S_h, d_h). This implies that (S_h, d) is complete.

3.2.5 Banach Contraction Principle

The Banach contraction principle is a simple but also fundamental result in fixed point theory. It is based on an iteration process which allows to construct an approximation sequence converging to a fixed point of a given contractive map.

Definition 3.43. Let (S_1, d_1) and (S_2, d_2) be two metric spaces. Assume that $F : S_1 \to S_2$ is a map satisfying the condition

$$\exists_{M>0}\ \forall_{x,y\in S_1}\quad d_2(F(x), F(y)) \leq M d_1(x, y). \tag{3.28}$$

Then the map F is called *Lipschitzian*. We also put

$$L(F) = \inf\{M > 0 : \forall_{x,y\in S_1}\ \ d_2(F(x), F(y)) \leq M d_1(x, y)\}.$$

The constant $L(F)$ (which is the smallest M satisfying (3.28)) is called the *Lipschitz constant* of F. The map F is called *contractive* if it is Lipschitz and $L(F) < 1$.

Let us notice that any Lipschitzian map $F : S_1 \to S_2$ is continuous. Indeed, assume that F satisfies the condition (3.28). Then we have for every $x \in S_1$

$$\forall_{\varepsilon>0}\ \exists_{\delta=\frac{\varepsilon}{M}>0}\ \forall_{y\in S_2}\ \ d_1(x, y) < \delta \ \ \Rightarrow\ \ d_2(F(x), F(y)) \leq M d_1(x, y) < M\delta = M\frac{\varepsilon}{M} = \varepsilon.$$

Theorem 3.44. (Banach Contraction Principle) *Let (S, d) be a complete metric space and $F : S \to S$ a contractive map. Then F has a unique fixed point, i.e. there exists a unique point $x_o \in S$ such that $F(x_o) = x_o$. Moreover, for every $y \in S$ we have that the sequence of iteration $F^n(y)$ converges to x_o, i.e.*

$$\lim_{n\to\infty} F^n(y) = x_o.$$

Proof: Assume that $F : S \to S$ is contractive and put $\alpha := L(F)$, i.e. $\alpha < 1$. We will show first that there at most one fixed point of F. Indeed, suppose that $F(x_o) = x_o$ and $F(y_o) = y_o$ with $y_o \neq x_o$, then the inequality

$$0 < d(x_o, y_o) = d(F(x_o), F(y_o)) \leq \alpha d(x_o, y_o) < d(x_o, y_o)$$

gives the contradiction.

To prove the existence, we will show that for any given $y \in S$ the sequence $\{F^n(y)\}$ of iterates (where $F^1(y) = F(y)$ and $F^{n+1}(y) := F(F^n(y))$ for $n > 1$) converges to a fixed point. Observe that

$$d(F(y), F^2(y)) \leq \alpha\, d(y, F(y)),$$

thus by induction

$$d(F^n(y), F^{n+1}(y)) \leq \alpha\, d(F^{n-1}(y), F^n(y))| \leq \cdots \leq \alpha^n\, d(y, F(y)).$$

Therefore, for any n, $k \in \mathbb{N}$ we have

$$\begin{aligned} d(F^n(y), F^{n+k}(y) &\leq \sum_{l=n}^{n+k-1} d(F^l(y), F^{l+1}(y)) \\ &\leq (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{n+k-1}) d(y, F(y)) \\ &\leq \frac{\alpha^n}{1-\alpha} d(y, F(y)). \end{aligned}$$

Since $\alpha < 1$, so $\alpha^n \to 0$ ad $n \to \infty$, it follows that the sequence $\{F^n(y)\}$ is Cauchy. Because (S, d) is complete, there exists $x_o \in S$ such that $\lim\limits_{n\to\infty} F^n(y) = x_o$. On the other hand F is continuous (see the next section; because it is Lipschitzian) therefore we have

$$F(x_o) = \lim_{n\to\infty} F(F^n(y)) = \lim_{n\to\infty} F^{n+1}(y) = \lim_{n\to\infty} F^n(y) = x_o,$$

which implies that x_o is a fixed point of F. Without using explicitly the continuity of F, one can also show that $F(x_o) = x_o$. Indeed, notice that we have

$$\lim_{n\to\infty} F^n(y) = x_o = \lim_{n\to\infty} F^{n+1}(y),$$

Thus

$$\forall_{\varepsilon>0}\ \exists_N\ \forall_{n\geq N} \quad d(x_o, F^n(y)) < \varepsilon,$$

and

$$\forall_{\varepsilon>0}\ \exists_N\ \forall_{n\geq N}\ \ d(F(x_o), F^{n+1}(y)) \leq \alpha d(x_o, F^n(y)) \leq \alpha\varepsilon < \varepsilon,$$

which implies that $\lim\limits_{n\to\infty} F^{n+1}(y) = F(x_o)$, therefore $F(x_o) = x_o$ by uniqueness of the limit.

Let us establish an error estimates for this approximation sequence $\{F^n(y)\}$ of the fixed point x_o. Observe that

$$d(F^n(y), F^{n+k}(y)) \leq \frac{\alpha^n}{1-\alpha}\, d(y, F(y)), \quad \forall_{k\in\mathbb{N}}$$

thus, by passing to the limit for $k \to \infty$, we obtain

$$d(F^n(y), x_o) = \lim_{k\to\infty} d(F^n(y), F^{n+k}(y)) \leq \frac{\alpha^n}{1-\alpha}\, d(y, F(y)).$$

$\square$

Corollary 3.45. *Let (S,d) be a complete metric space and $B = B(y_o, r) = \{x : d(x, y_o) < r\}$ the open ball centered at y_o of radius $r > 0$. Suppose that $F : B \to S$ is a contractive map with constant $\alpha < 1$. If*

$$d(F(y_o), y_o) < (1-\alpha)r,$$

then F has a fixed point.

Proof: Choose $0 < \varepsilon < r$ so that $d(F(y_o), y_o) \le (1-\alpha)\varepsilon < (1-\alpha)r$. Let $D := \{x : d(x, y_o) \le \varepsilon\}$ denote the closed ball centered at y_o of radius ε. We will show that F maps D into itself. Indeed, for every $x \in D$ we have

$$d(F(x), y_o) \le d(F(x), F(y_o)) + d(F(y_o), y_o) \le \alpha\, d(x, y_o) + (1-\alpha)\varepsilon \le \varepsilon.$$

Since D is complete (a closed subset of a complete metric space is complete), the conclusion follows from the Banach Contraction Principle (cf. Theorem 3.44). □

3.3 Cantor's Intersection Theorem and Baire's Lemma

Definition 3.46. Let (X,d) be a metric space and $A \neq \emptyset$ a set in X. The *diameter* of the set A is defined as

$$\operatorname{diam}(A) = \sup\{d(x,y) : x, y \in A\}. \tag{3.29}$$

Theorem 3.47. (Cantor's Intersection Theorem) *Let (X,d) be a complete metric space and $\{F_n\}_{n=1}^\infty$ be a sequence of sets in X such that*

(a) F_n is closed and $F_n \neq \emptyset$ for all $n = 1, 2, 3, \dots$;
(b) $F_1 \supset F_2 \supset \cdots \supset F_n \supset F_{n+1} \supset \dots$;
(c) $\lim\limits_{n\to\infty} \operatorname{diam}(F_n) = 0$.

Then there exists a (unique) point $x_o \in X$ such that

$$\bigcap_{n=1}^{\infty} F_n = \{x_o\}.$$

Proof: Since by (a) each of the sets F_n is nonempty, thus we can choose $x_n \in F_n$. We claim that the constructed in this way sequence $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. Indeed, let $\varepsilon > 0$ be an arbitrary number. By (c),

$$\exists_N\ \forall_{n\ge N}\ \ \operatorname{diam}(F_n) < \varepsilon.$$

Thus, by (3.29) and the definition of supremum,

$$\exists_N \forall_{n\ge N}\, \forall_{x,y\in F_n}\ \ d(x,y) \le \operatorname{diam}(F_n) < \varepsilon.$$

On the other hand, by (b), for all $n, m \ge N$, $x_n \in F_n \subset F_N$ and $x_m \in F_m \subset F_N$, thus

$$d(x_n, x_m) \le \operatorname{diam}(F_N) < \varepsilon,$$

therefore, we get

$$\forall_{\varepsilon>0}\ \exists_N\ \forall_{n,m\geq N}\ \ d(x_n,x_m)<\varepsilon,$$

which implies that $\{x_n\}$ is a Cauchy sequence. Since (X,d) is a complete metric space, there exists $x_o\in X$ such that $\lim_{n\to\infty} x_n = x_o$. Notice that for every $n=1,2,3,\dots$, if $m\geq n$ then $x_m\in F_m\subset F_n$, i.e. $\{x_m\}_{m=n}^{\infty}\subset F_n$, and therefore $x_o=\lim_{m\to\infty} x_m$ is the limit point of F_n. However, by (a), F_n is closed for all $n\in\mathbb{N}$, thus

$$\forall_{n\in\mathbb{N}}\ x_o\in F_n \quad\Leftrightarrow\quad x_o\in\bigcap_{n=1}^{\infty}F_n,$$

which proves that the intersection $\bigcap_{n=1}^{\infty}F_n$ is nonempty. Suppose now that $x_o'\in X$ such that

$$x_o'\in\bigcap_{n=1}^{\infty}F_n \quad\Leftrightarrow\quad \forall_{n\in\mathbb{N}}\ x_o'\in F_n,$$

thus, by (c), for every $\varepsilon>0$ there exists N such that $\operatorname{diam}(F_N)<\varepsilon$, which implies

$$\forall_{\varepsilon>0}\ \exists_N\ \ d(x_o,x_o')\leq \operatorname{diam}(F_N)<\varepsilon,$$

and consequently $x_o=x_o'$. Therefore,

$$\bigcap_{n=1}^{\infty}F_n=\{x_o\}.$$

□

Definition 3.48. Let A be a set in a metric space (X,d). The set A is called *dense* in X if and only if $\overline{A}=X$.

Proposition 3.49. *Let A be a set in (X,d). Then we have*

$$A \text{ is dense in } X \quad\Leftrightarrow\quad \forall_{\emptyset\neq U\subset X}\ \ U \text{ is open } \ \Rightarrow U\cap A\neq\emptyset.$$

Proof: $\Rightarrow$: Assume that A is a dense set, i.e.

$$\forall_{x_o\in X}\ \ x_o\in\overline{A} \quad\Leftrightarrow\quad \forall_{x_o\in X}\ \forall_{\varepsilon>0}\ \ B_\varepsilon(x_o)\cap A\neq\emptyset. \tag{3.30}$$

Let $U\subset X$ be a nonempty open set. Then, there exist $x_o\in U$ and $\varepsilon>0$ such that $B_\varepsilon(x_o)\subset U$. Therefore, by (3.30), $B_\varepsilon(x_o)\cap A\neq\emptyset$, and therefore

$$\emptyset\neq B_\varepsilon(x_o)\cap A\subset U\cap A.$$

$\Leftarrow$: Assume now that for every nonempty open set U we have $U\cap A\neq\emptyset$. Then, since for every $x_o\in X$ and $\varepsilon>0$ the ball $B_\varepsilon(x_o)$ is a nonempty open set, we have $B_\varepsilon(x_o)\cap A\neq\emptyset$, which implies $x_o\in\overline{A}$. Consequently, A is dense. □

Theorem 3.50. (Baire's Lemma) *Let (X,d) be a complete metric space and $\{U_n\}_{n=1}^{\infty}$ a sequence of open dense sets in X. Then the set $\bigcap_{n=1}^{\infty}U_n$ is also dense.*

Proof: Put $A := \bigcap_{n=1}^{\infty} U_n$. By Proposition 3.49, we need to show that for every nonempty open set U we have $U \cap A \neq \emptyset$. Assume therefore that U is an arbitrary open set such that $x_o \in U$. Since U is open, there exists $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(x_o) \subset U$. Since, by assumption, the set U_1 is dense, there exists $x_1 \in B_{\varepsilon_0}(x_o) \cap U_1$. Since $B_{\varepsilon_0}(x_o) \cap U_1$ is open, there exists $\tilde{\varepsilon}_1 > 0$ such that $\tilde{\varepsilon} \leq \varepsilon_0$ and $B_{\tilde{\varepsilon}_1}(x_1) \subset B_{\varepsilon_0}(x_o) \cap U_1$. Put $\varepsilon_1 := \frac{\tilde{\varepsilon}_1}{2}$. Then the closed ball

$$F_1 := \{x \in X : d(x_1, x) \leq \varepsilon_1\}$$

is a nonempty ($x_1 \in F_1$) closed set such that

$$B_{\varepsilon_1}(x_1) \subset F_1 \subset B_{\tilde{\varepsilon}_1}(x_1) \subset B_{\varepsilon_0}(x_o) \cap U_1,$$

and $\operatorname{diam}(F_1) \leq 2 \cdot \varepsilon_1 \leq \varepsilon_o$. Since U_2 is also open and dense, therefore there exists $x_2 \in B_{\varepsilon_1}(x_1) \cap U_2$. Again, since $B_{\varepsilon_1}(x_1) \cap U_2$ is open, there exists $\tilde{\varepsilon}_2 > 0$ such that $\tilde{\varepsilon}_2 \leq \frac{\varepsilon_1}{2}$ and $B_{\tilde{\varepsilon}_2}(x_2) \subset B_{\varepsilon_1}(x_1) \cap U_2$. Put $\varepsilon_2 := \frac{\tilde{\varepsilon}_2}{2}$. Then the closed ball

$$F_2 := \{x \in X : d(x_2, x) \leq \varepsilon_2\}$$

is a nonempty ($x_2 \in F_1$) closed set such that

$$B_{\varepsilon_2}(x_1) \subset F_2 \subset B_{\tilde{\varepsilon}_2}(x_2) \subset F_1,$$

and $\operatorname{diam}(F_2) \leq 2 \cdot \varepsilon_2 \leq \frac{\varepsilon_o}{2}$. Next, we proceed by induction. Assume that we have constructed for $m \leq n$ the sets $F_m = \{x : d(x_m, x) \leq \varepsilon_m\}$ satisfying the properties:

$$B_{\varepsilon_m}(x_m) \subset F_m \subset B_{\tilde{\varepsilon}_m}(x_m) \subset F_{m-1},$$

where $\tilde{\varepsilon}_m \leq \frac{\varepsilon_{m-1}}{2}$, $\varepsilon_m = \frac{\tilde{\varepsilon}_m}{2}$. Clearly, we have

$$F_1 \supset F_2 \supset \cdots \supset F_{n-1} \supset F_n$$

and diam $(F_m) \leq \frac{\varepsilon_o}{2^{m-1}}$, $m = 1, 2, \ldots, n$. Now, since the set U_{n+1} is also dense, there exists x_{n+1} such that

$$x_{n+1} \in B_{\varepsilon_n}(x_n) \cap U_{n+1}.$$

Again, since U_{n+1} is open, there exists $\tilde{\varepsilon}_{n+1} > 0$ such that $\tilde{\varepsilon}_{n+1} \leq \frac{\varepsilon_n}{2}$ and

$$B_{\tilde{\varepsilon}_{n+1}}(x_{n+1}) \subset B_{\varepsilon_n}(x_n) \cap U_{n+1}.$$

We put, $\varepsilon_{n+1} := \frac{\tilde{\varepsilon}_{n+1}}{2}$ and $F_{n+1} : \{x : d(x_{n+1}, x) \leq \varepsilon_{n+1}\}$. Then F_{n+1} is nonempty, closed, $\operatorname{diam}(F_{n+1}) \leq 2\varepsilon_{n+1} \leq \tilde{\varepsilon}_{n+1} \leq \frac{\varepsilon_o}{2^n}$ and

$$B_{\varepsilon_{n+1}}(x_{n+1}) \subset F_{n+1} \subset B_{\tilde{\varepsilon}_{n+1}}(x_{n+1}) \subset F_n.$$

Therefore, the sequence $\{F_n\}$ satisfies the assumptions of the Cantor's Intersection Theorem (cf. Theorem 3.47), which implies that for every $m = 1, 2, 3, 4, \ldots$

$$\emptyset \neq \bigcap_{n=1}^{\infty} F_n \subset F_{m+1} \subset B_{\varepsilon_m}(x_m) \cap U_m \subset U_m,$$

therefore

$$\emptyset \neq \bigcap_{n=1}^{\infty} F_n \subset \bigcap_{n=1}^{\infty} U_n,$$

and the conclusion follows from Proposition 3.49. □

Corollary 3.51. *The set $\mathbb{R}$ of real numbers is not countable.*

Proof: Assume for contradiction that $\mathbb{R}$ is countable, i.e $\mathbb{R} = \{x_1, x_2, x_3, \dots\}$. Define $U_n := \mathbb{R} \setminus \{x_n\}$. Clearly, U_n is open and $\overline{U_n} = \mathbb{R}$, i.e. U_n is dense in $\mathbb{R}$. Since, by Theorem 3.39, $\mathbb{R}$ is complete, it follows from Theorem 3.50 that $\bigcap_{n=1}^{\infty} U_n$ has to be dense in $\mathbb{R}$. But, by the definition,

$$\bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} (\mathbb{R} \setminus \{x_n\}) = \mathbb{R} \setminus \bigcup_{n=1}^{\infty} \{x_n\} = \mathbb{R} \setminus \mathbb{R} = \emptyset,$$

which implies a contradiction. Therefore, $\mathbb{R}$ cannot be countable. □

3.4 Limits and Continuity

Definition 3.52. Let (X, d_X) and (Y, d_Y) be two metric spaces and let $f : X \setminus \{a\} \to Y$ be a function. Assume that a is a limit point of X. Than we say that f *has a limit* $b \in Y$ at the point a if

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x\in X}\ \ 0 < d_X(a, x) < \delta \ \ \Rightarrow \ \ d_Y(f(x), b) < \varepsilon. \tag{3.31}$$

In such a case, we will use the notation

$$\lim_{x\to a} f(x) = b.$$

Proposition 3.53. *Let a be a limit point of X and $f : X \setminus \{a\} \to Y$ a function. Then*

$$\lim_{x\to a} f(x) = b \ \ \Leftrightarrow \ \ \forall_{\{x_n\}\subset X\setminus\{a\}}\ \lim_{n\to\infty} x_n = a \ \ \Rightarrow \ \ \lim_{n\to\infty} f(x_n) = b. \tag{3.32}$$

Proof: $\Longrightarrow$: Assume that $\lim_{x\to a} f(x) = b$, and assume that $\lim_{x\to a} x_n = a$, i.e.

$$\forall_{\delta>0}\ \exists_{N_x>0}\ \forall_{n\geq N_x}\ \ d_X(x_n, a) < \delta. \tag{3.33}$$

We need to prove that

$$\lim_{n\to\infty} f(x_n) = b \ \ \Leftrightarrow \ \ \forall_{\varepsilon>0}\ \exists_{N_Y>0}\ \forall_{n\geq N_Y}\ \ d_Y(f(x_n), b) < \varepsilon. \tag{3.34}$$

For this purpose, assume that $\varepsilon > 0$ is an arbitrary number (i.e. $\forall_{\varepsilon>0}$). Then, by (3.32)

$$\begin{aligned} &\exists_{\delta>0}\ \forall_{x\in X}\ \ 0 < d_X(a, x) < \delta \ \ \Rightarrow \ \ d_Y(f(x), b) < \varepsilon \quad \text{and by (3.33)} \\ &\exists_{N_x>0}\ \forall_{n\geq N_x}\ \ d_X(x_n, a) < \delta, \end{aligned}$$

therefore, we obtain

$$\forall_{\varepsilon>0}\ \exists_{N_y=N_x>0}\ \forall_{n\geq N_y}\ \ d_Y(f(x_n),b)<\varepsilon,$$

which concludes the proof of (3.34).

$\Longleftarrow$: Assume for contradiction that $\lim\limits_{x\to a} f(x)\neq b$, which means (by applying the negation to (3.31))

$$\exists_{\varepsilon>0}\ \forall_{\delta>0}\ \exists_{x\in X}\ \ 0<d_X(x,a)<\delta \ \ \wedge\ \ d_Y(f(x),b)\geq\varepsilon. \tag{3.35}$$

Consequently, by taking $\delta=\frac{1}{n}$ in (3.35), we obtain

$$\begin{aligned}&\exists_{\varepsilon>0}\ \forall_{n\in\mathbb{N}}\ \exists_{x_n\in X}\ \ 0<d_X(x_n,a)<\frac{1}{n}\ \ \wedge\ \ d_Y(f(x_n),b)\geq\varepsilon\\ &\exists_{\{x_n\}\subset X\setminus\{a\}}\ \ \lim_{n\to\infty}x_n=a\ \ \wedge\ \ \lim_{n\to\infty}f(x_n)\neq b.\end{aligned}$$

This completes the proof. □

In what follows we will assume that (X,d_X) and (Y,d_Y) are two metric spaces.

Definition 3.54. Let $f:X\to Y$ be a function and $a\in X$. We say that f is continuous at $a\in X$ if and only if

(a) if a is an isolated point in X, or,
(b) $\lim\limits_{x\to a} f(x)=f(a)$.

Theorem 3.55. *Let $f:X\to Y$ be a function and $a\in X$. Then f is continuous at a if and only if*

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x\in X}\ \ d_X(a,x)<\delta\ \ \Rightarrow\ \ d_Y(f(x),b)<\varepsilon. \tag{3.36}$$

Proof: $\Longrightarrow$: Assume first that a is an isolated point in X, i.e.

$$\begin{aligned}&\exists_{\delta_o>0}\ \ B_{\delta_o}(a)\cap X=\{a\}\\ \Longleftrightarrow\ \ &\exists_{\delta_o>0}\ x\in B_{\delta_o>0}\ \ \Longleftrightarrow\ \ \exists_{\delta_o>0}\ d_X(x,a)<\delta_o.\end{aligned}$$

Therefore, we have

$$\exists_{\delta_o>0}\ \forall_{\varepsilon>0}\ \exists_{\delta=\delta_o>0}\ \forall_{x\in X}\ \ \overset{\text{i.e. } x=a}{d_X(a,x)<\delta}\ \ \Rightarrow\ \ d_Y(f(x),b)=0<\varepsilon.$$

One can easily notice that the implication $\Longleftarrow$ is trivially satisfied. □

Remark 3.56. Let us point out that it is very convenient to use the condition (3.36) (because it does not require consideration of two cases: a – isolated point in X, and a – a limit point in X) as a definition of the continuity of f at a.

Definition 3.57. We say that $f:X\to Y$ is *continuous* if and only if f is continuous at every point $a\in X$.

Consequently, by Theorem 3.55, we obtain immediately:

Theorem 3.58. *Let $f : X \to Y$ be a function. Then f is continuous if and only if*

$$\forall_{x\in X}\ \forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x'\in X}\ \ d_X(x,x') < \delta \ \ \Rightarrow \ \ d_Y(f(x), f(x')) < \varepsilon. \tag{3.37}$$

Remark 3.59. Suppose that a function $f : X \to Y$ is continuous. Then, by Theorem 3.58, for every $\varepsilon > 0$ and for $x \in X$ there exists $\delta > 0$, which clearly may depend on the choice of the point x, i.e. we can indicated this dependence by writing $\delta(x)$, such that for all $x' \in X$ we have

$$d_X(x,x') < \delta(x) \implies d_Y(f(x), f(x')) < \varepsilon.$$

In order to illustrate this situation, consider the following example: $f : (0,\infty) \to \mathbb{R}$, $f(t) = \frac{1}{t}$. Let $\varepsilon > 0$ be a fixed, but arbitrary number and $t > 0$ also be fixed. Then we can choose $\delta(t) := \min\left\{\frac{t}{2}, \frac{\varepsilon t^2}{4}\right\}$. Indeed, if $|t - t'| < \delta(t)$, then we have

$$\left|\frac{1}{t} - \frac{1}{t'}\right| = \frac{|t-t'|}{tt'} < \frac{4|t'-t|}{t^2} < \frac{4\delta(t)}{t^2} = \varepsilon.$$

So clearly, the function f is continuous, but it is also perfectly clear that $\delta(t) > 0$ depends on the point t.

The following concept of “stronger” continuity has an important place in analysis:

Definition 3.60. A function $f : X \to Y$ is said to be *uniformly continuous* if and only if

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x,x'\in X}\ \ d_X(x,x') < \delta \ \ \Rightarrow \ \ d_Y(f(x), f(x')) < \varepsilon. \tag{3.38}$$

Example 3.61. Consider two metric spaces (X, d_X) and (Y, d_Y) and let $F : X \to Y$ be a *Lipschitzian function*, i.e. there exists $L > 0$ such that

$$\forall_{x_1,x_2\in X}\ \ d_Y(F(x_1), F(x_2)) \le L d_X(x_1, x_2).$$

Then, the function F is uniformly continuous. Indeed, we have that

$$\begin{aligned}&\forall_{\varepsilon>0}\ \exists_{\delta:=\frac{\varepsilon}{L}}\ \forall_{x_1,x_2\in X}\quad d_X(x_1,x_2) < \delta \implies \\ &d_Y(F(x_1), F(x_2)) \le L d_X(x_1,x_2) < L\delta = L\frac{\varepsilon}{L} = \varepsilon.\end{aligned}$$

Proposition 3.62. *Let $f : X \to Y$ be a function and $a \in X$. Then f is continuous at a if and only if for every open set $U \subset Y$ such that $f(a) \in U$, there exists $\varepsilon > 0$ such that $B_\delta(a) \subset f^{-1}(U)$.*

Proof: $\implies$: Suppose that f is continuous at a, $U \subset Y$ an open set, and $f(a) \in U$. Then, since U is open $\exists_{\varepsilon>0}\ \ B_\varepsilon(f(a)) \subset U$. By Theorem 3.55

$$\begin{array}{ccc} \exists_{\delta>0}\ \ d_X(x,a) < \delta & \implies & d_Y(f(x), f(a)) < \varepsilon \\ \Updownarrow & & \Updownarrow \\ x \in B_\delta(a) & & f(x) \in B_\varepsilon(f(a)) \end{array}$$

Consequently, we obtain that

$$\begin{aligned}
&\exists_{\varepsilon>0}\,\exists_{\delta>0}\;\; x\in B_\delta(a) \;\;\Rightarrow\;\; f(x)\in B_\varepsilon(f(a))\\
\iff &\exists_{\varepsilon>0}\,\exists_{\delta>0}\;\; x\in B_\delta(a) \;\;\Rightarrow\;\; x\in f^{-1}(B_\varepsilon(f(a)) \;\;\wedge\;\; B_\varepsilon(f(a))\subset f^{-1}(U)\\
\iff &\exists_{\varepsilon>0}\,\exists_{\delta>0}\;\; B_\delta(a)\subset f^{-1}(U)
\end{aligned}$$

Conversely, $\Longleftarrow$: Assume that for every open set $U\subset Y$ such that $f(a)\in U$, there exists $\varepsilon>0$ such that $B_\delta(a)\subset f^{-1}(U)$. In particular, for every $\varepsilon>0$ (i.e. $\forall_{\varepsilon>0}$) the set $U=B_\varepsilon(f(a))$ is open, thus there exists $\delta>0$ (i.e. $\exists_{\delta>0}$) such that $B_\delta(a)\subset f^{-1}(B_\varepsilon(f(a))$, i.e.

$$\begin{aligned}
&\forall_{\varepsilon>0}\,\exists_{\delta>0}\;\; B_\delta(a)\subset f^{-1}(B_\varepsilon(f(a))\\
\iff &\forall_{\varepsilon>0}\,\exists_{\delta>0}\forall_{x\in X}\;\; x\in B_\delta(a) \;\;\Rightarrow\;\; x\in f^{-1}(B_\varepsilon(f(a))\\
\iff &\forall_{\varepsilon>0}\,\exists_{\delta>0}\forall_{x\in X}\;\; d(x,a)<\delta \;\;\Rightarrow\;\; f(x)\in B_\varepsilon(f(a)\\
\iff &\forall_{\varepsilon>0}\,\exists_{\delta>0}\forall_{x\in X}\;\; d(x,a)<\delta \;\;\Rightarrow\;\; d(f(x),f(a))<\varepsilon.
\end{aligned}$$

Clearly, the last condition is equivalent to the continuity of f at a. □

Theorem 3.63. *Let $f:X\to Y$ be a function. Then f is continuous if and only if for every open set $U\subset Y$ the set $f^{-1}(U)$ is open in X.*

Proof: $\Longrightarrow$: Assume that f is continuous and let $U\subset Y$ be an open set in Y. Then for every $a\in f^{-1}(U)$, by Proposition 3.62

$$\exists_{\delta>0}\;\; B_\delta(a)\subset f^{-1}(U)$$

which implies that $f^{-1}(U)$ is open.

$\Longleftarrow$: Conversely, assume that for every open $U\subset Y$ the set $f^{-1}(U)$ is open. Let $a\in X$ be an arbitrary point. Then for every $\varepsilon>0$ (i.e. $\forall_{\varepsilon>0}$) take $U=B_\varepsilon(f(a))$. Since, by assumption $f^{-1}(U)=f^{-1}(B_\varepsilon(f(a))$ is open, thus, in particular

$$a\in f^{-1}(U) \;\;\wedge\;\; \exists_{\delta>0}\;\; B_\delta(a)\subset f^{-1}(U).$$

In other words

$$\forall_{\varepsilon>0}\,\exists_{\delta>0}\;\; B_\delta(a)\subset f^{-1}(B_\varepsilon(f(a))$$

which is equivalent to

$$\begin{aligned}
&\forall_{\varepsilon>0}\,\exists_{\delta>0}\,\forall_{x\in X}\;\; x\in B_\delta(a) \;\;\Rightarrow\;\; x\in f^{-1}(B_\varepsilon(f(a))\\
\iff &\forall_{\varepsilon>0}\,\exists_{\delta>0}\,\forall_{x\in X}\;\; x\in B_\delta(a) \;\;\Rightarrow\;\; f(x_{\in}B_\varepsilon(f(a)\\
\iff &\forall_{\varepsilon>0}\,\exists_{\delta>0}\,\forall_{x\in X}\;\; d(x,a)<\delta \;\;\Rightarrow\;\; d(f(x),f(a))<\varepsilon.
\end{aligned}$$

Consequently, f is continuous at a, and since a was arbitrary, f is continuous. □

Corollary 3.64. *Let $f:X\to Y$ be a function. Then f is continuous if and only if for every closed set $C\subset Y$ the set $f^{-1}(C)$ is closed in X.*

Proof: Since $f : X \to Y$ is continuous (by Theorem 3.63) if and only if for every open set $U \subset Y$, the set $f^{-1}(U)$ is also open, which implies that for every closed set $C \subset Y$ the set $U := Y \setminus C$ is open and since $f^{-1}(U)$ is open, thus

$$f^{-1}(C) = f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$$

is closed.

Reciprocally, if for every closed set $C \subset Y$, the set $f^{-1}(C)$ is also closed, then for every open set $U \subset Y$, the set $D := Y \setminus U$ is closed, and thus $f^{-1}(D)$ is closed, which implies that

$$f^{-1}(U) = f^{-1}(Y \setminus D) = X \setminus f^{-1}(D)$$

is open. Theefore, f is continuous. □

Theorem 3.65. (Properties of Continuous Functions) *Let (X, d) be a metric space and $(V, \|\cdot\|)$ be a normed vector space. Assume that $f, g : X \to V$ and $h : X \to \mathbb{R}$ are continuous functions. Then*

(a) For every α, $\beta \in \mathbb{R}$, the function $\alpha f + \beta g : X \to V$, defined by

$$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x), \quad x \in X,$$

is continuous;

(b) the function $hf : X \to V$, defined by

$$(hf)(x) = h(x)f(x), \quad x \in X,$$

is continuous;

(c) If $h(x) \neq 0$ for all $x \in X$ then the function $\frac{1}{h} : X \to \mathbb{R}$, defined by

$$\left(\frac{1}{h}\right)(x) = \frac{1}{h(x)}, \quad x \in X,$$

is continuous.

Proof: (a): We need to prove that for every $a \in X$ the function $\alpha f + \beta g$ is continuous at a. We can assume that $|\alpha| + \beta| > 0$ (otherwise the statement is trivial). Since f and g are continuous, thus we have

$$\forall_{\varepsilon>0}\, \exists_{\delta_f>0}\, \forall_{x\in X}\;\; d(x,a) < \delta_f \;\;\Rightarrow\;\; \|f(x) - f(a)\| < \frac{\varepsilon}{|\alpha| + |\beta|}$$
$$\forall_{\varepsilon>0}\, \exists_{\delta_g>0}\, \forall_{x\in X}\;\; d(x,a) < \delta_g \;\;\Rightarrow\;\; \|g(x) - g(a)\| < \frac{\varepsilon}{|\alpha| + |\beta|}.$$

Therefore, if we choose $\delta := \min\{\delta_f, \delta_g\}$, then we have

$$\begin{aligned}
\forall_{\varepsilon>0}\, \exists_{\delta>0}\, \forall_{x\in X}\;\; d(x,a) < \delta \;\;\Rightarrow\;\; & \|(\alpha f(x) + \beta g(x)) - (\alpha f(a) + \beta g(x))\| \\
&= \|\alpha(f(x) - f(a)) + \beta(g(x) - g(a))\| \\
&\le \|\alpha(f(x) - f(a))\| + \|\beta(g(x) - g(a))\| \\
&\le |\alpha|\|f(x) - f(a)\| + |\beta|\|g(x) - g(a)\| \\
&< |\alpha|\frac{\varepsilon}{|\alpha| + |\beta|} + |\beta|\frac{\varepsilon}{|\alpha| + |\beta|} = \varepsilon.
\end{aligned}$$

(b): Since f and h are continuous, thus we have

$$\forall_{\varepsilon>0}\ \exists_{\delta_f>0}\ \forall_{x\in X}\ \ d(x,a)<\delta_f\ \ \Rightarrow\ \ \|f(x)-f(a)\|<\min\left\{1,\frac{\varepsilon}{|h(a)|+\|f(a)\|+1}\right\} \tag{3.39}$$

$$\forall_{\varepsilon>0}\ \exists_{\delta_h>0}\ \forall_{x\in X}\ \ d(x,a)<\delta_h\ \ \Rightarrow\ \ |h(x)-h(a)|<\min\left\{1,\frac{\varepsilon}{|h(a)|+\|f(a)\|+1}\right\}. \tag{3.40}$$

Therefore, if we choose $\delta:=\min\{\delta_f,\delta_h\}$, then (by (3.39)) we have that

$$d(x,a)<\delta\ \Longrightarrow\ |h(x)|-|h(a)|\le|h(x)-h(a)|<1\ \Leftrightarrow\ |h(x)<|h(a)|+1. \tag{3.41}$$

Consequently, we have

$$\begin{aligned}
\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x\in X}\ \ d(x,a)<\delta\ \ &\Rightarrow\ \|(h(x)f(x)-h(a)f(a)\|=\|h(x)(f(x)-f(a))+f(a)(h(x)-h(a))\|\\
&\le|h(x)|\|(f(x)-f(a)\|+\|f(a)\||h(x)-h(a)|\\
\text{by (3.41)}\quad&\le(|h(a)+1)\|f(x)-f(a)\|+\|f(a)\||h(x)-h(a)|\\
\text{by (3.39) and (3.40)}\quad&<(|h(a)+1)\frac{\varepsilon}{|h(a)|+\|f(a)\|+1}+\|f(a)\|\frac{\varepsilon}{|h(a)|+\|f(a)\|+1}\\
&=\frac{|h(a)|+\|f(a)\|+1}{|h(a)|+\|f(a)\|+1}\varepsilon=\varepsilon.
\end{aligned}$$

(c): Since and h are continuous and $h(x)>0$ thus we have

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x\in X}\ \ d(x,a)<\delta\ \ \Rightarrow\ \ |h(x)-h(a)|<\min\left\{\frac{|h(a)|}{2},\varepsilon\frac{h(a)^2}{2}\right\}. \tag{3.42}$$

Therefore, (by (3.42)) we have that

$$\begin{aligned}
\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x\in X}\ \ d(x,a)<\delta\ \ &\Rightarrow\ \left|\frac{1}{h(x)}-\frac{1}{h(a)}\right|=\left|\frac{h(x)-h(a)}{h(x)h(a)}\right|\\
\text{by (3.42)}\quad&\le\frac{2|h(x)-h(a)|}{h(a)^2}\\
\text{by (3.42)}\quad&<\varepsilon\frac{h(a)^2}{2}\cdot\frac{2}{h(a)^2}=\varepsilon.
\end{aligned}$$

□

Theorem 3.66. *Let $f:X\to Y$ and $g:Y\to Z$ be two continuous functions. Then $g\circ f:X\to Z$, defined $(g\circ f)(x)=g(f(x))$, $x\in X$, is continuous.*

Proof: We need to show, by Theorem 3.55, that for every open set $U\subset Z$ the set $(g\circ f)^{-1}(U)$ is open in X. Indeed, since g is continuous, again by Theorem 3.55, the set $V:=g^{-1}(U)$ is open in Y, and clearly, by continuity of f, the set $f^{-1}(g^{-1}(U))=f^{-1}(V)$ is open in X. Then we have

$$(g\circ f)^{-1}(U)=g^{-1}(f^{-1}(U))=g^{-1}(V),$$

and the conclusion follows. □

Example 3.67. Not every continuous function is uniformly continuous. Notice that $F : X \to Y$ is *not uniformly continuous*

$$\begin{aligned}
&\iff \sim \forall_{\varepsilon>0}\, \exists_{\delta>0}\, \forall_{x_1,x_2\in X}\ \ d_X(x_1,x_2)<\delta \ \Rightarrow\ d_Y(F(x_1),F(x_2))<\varepsilon\\
&\iff \exists_{\varepsilon>0}\, \forall_{\delta>0}\, \exists_{x_1,x_2\in X}\ \ \sim(\ d_X(x_1,x_2)<\delta \ \Rightarrow\ d_Y(F(x_1),F(x_2))<\varepsilon\ )\\
&\iff \exists_{\varepsilon>0}\, \forall_{\delta>0}\, \exists_{x_1,x_2\in X}\ \ d_X(x_1,x_2)<\delta \ \wedge\ d_Y(F(x_1),F(x_2))\geq\varepsilon.
\end{aligned}$$

Consider the function $f : \mathbb{R} \to \mathbb{R}$, given by $f(t) = t$. Clearly, the function f is uniformly continuous, therefore it is continuous. By Theorem 3.65, the function $g(t) := t^2 = f(t) \cdot f(t)$ is also continuous. Notice, that $g : \mathbb{R} \to \mathbb{R}$ is **not uniformly continuous**. Indeed, put $\varepsilon = 1$. Then for every $\delta > 0$, by Archimedes' axiom, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{\delta}$. Then, for $t_1 = n + \frac{\delta}{2}$ and $t_2 = n$, we have $|t_1 - t_2| = \frac{\delta}{2} < \delta$, and

$$|g(t_1)-g(t_2)| = \left(n+\frac{\delta}{2}\right)^2 - n^2 = n\delta + \frac{\delta^2}{4} > n\delta > 1.$$

Therefore, g is not uniformly continuous.

Example 3.68. Let (X, d) be a metric space and f, $g : X \to \mathbb{R}$ two continuous functions. The open intervals (α, β), $(-\infty, \beta)$ and (α, ∞) are open sets in $\mathbb{R}$. On the other hand, the closed intervals $[\alpha, \beta]$, $(-\infty, \beta]$ and $[\alpha, \infty)$ are closed sets in $\mathbb{R}$. Then clearly, the sets

$$\begin{aligned}
f^{-1}\Big((\alpha,\beta)\Big) &:= \{x\in X : \alpha < f(x) < \beta\},\\
g^{-1}\Big((\alpha,\infty)\Big) &:= \{x\in X : \alpha < g(x)\}
\end{aligned}$$

are open and the sets

$$\begin{aligned}
g^{-1}\Big([\alpha,\beta]\Big) &:= \{x\in X : \alpha \leq g(x) \leq \beta\},\\
f^{-1}\Big((\alpha,\infty)\Big) &:= \{x\in X : f(x) \leq \beta\}
\end{aligned}$$

are closed. Notice that the intersection of two open sets is open, thus the set

$$f^{-1}\Big((\alpha,\beta)\Big)\cap g^{-1}\Big((\alpha,\infty)\Big) = \{x\in X : \alpha < f(x) < \beta \ \ \wedge\ \ \alpha < g(x)\}$$

is open, and similarly, the intersection of two closed sets is closed, so the set

$$g^{-1}\Big([\alpha,\beta]\Big)\cap f^{-1}\Big((\alpha,\infty)\Big) = \{x\in X : \alpha \leq g(x) \leq \beta \ \ \wedge\ \ f(x) \leq \beta\}$$

is closed.

To be more specific, consider $X := \mathbb{R}^2$ and the sets

$$\begin{aligned}
A &:= \{(x,y) : 0 < x^2+y^2+2xy+x^3 < 4 \ \ \wedge\ \ \ x > y\}\\
B &:= \{(x,y) : 1 \leq \sin(xy) + \sqrt{1+y^2} + xy \leq 4 \ \ \wedge\ \ \ 1 \leq x+y\}.
\end{aligned}$$

Clearly, the set A is open (since the functions $f(x, y) = x^2 + y^2 + 2xy + x^3$ and $g(x, y) = x - y$ are continuous) and the set B is closed (since $f(x, y) = \sin(xy) + \sqrt{1 + y^2} + xy$ and $g(x, y) = x + y$ are also continuous).

Remark 3.69. (a) Notice that if (X, d_X) and (Y, d_Y) are two metric spaces, one can equip the product $X \times Y$ with the metric $\tilde{d} : (X \times Y) \times (X \times Y) \to \mathbb{R}$ defined by

$$\tilde{d}((x_1, y_1), (x_2, y_2)) := \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}, \quad (x_1, y_1),\ (x_2, y_2) \in X \times Y.$$

One can easily verify that the required properties of a metric are satisfied. The metric $\tilde{d}$ on the product $X \times Y$ is called the *product metric.*

(b) Suppose that (S, d) is a metric space. It is natural to ask the question if the distance function $d : X \times X \to \mathbb{R}$ is continuous? Indeed, the product space $S \times S$ is equipped with the product metric $\tilde{d}$. Then, we need to show that

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{(x_1,y_1),(x_2,y_2)\in S\times S}\quad \tilde{d}((x_1, y_1), (x_2, y_1)) < \delta \quad\Rightarrow\quad |d(x_1, y_1) - d(x_2, y_2)| < \varepsilon. \tag{3.43}$$

Since

$$d(x_1, y_1) \le d(x_1, x_2) + d(x_2, y_2) + d(y_1, y_2) \quad\Rightarrow\quad d(x_1, y_1) - d(x_2, y_2) \le d(x_1, x_2) + d(y_1, y_2)$$

By interchanging x_1, x_2 with y_1, y_2, we obtain

$$d(x_1, y_1) - d(x_2, y_2) \le d(x_1, x_2) + d(y_1, y_2) \le 2\tilde{d}((x_1, y_1), (x_2, y_2))$$

therefore, (3.43) is true for $\delta = \frac{\varepsilon}{2}$. Consequently, the metric $d : S \times S \to \mathbb{R}$ is a continuous function.

One of the easy consequences of the continuity of d, is the following fact: if $\{x_n\}$ and $\{y_n\}$ are two sequences in S such that

$$\lim_{n\to\infty} x_n = x_o \qquad\text{and}\qquad \lim_{n\to\infty} y_n = y_o,$$

then

$$\lim_{n\to\infty} d(x_n, y_n) = d(x_o, y_o).$$

Proposition 3.70. *Let (X, d) be metric space and $A \subset X$ a nonempty set. We define the distance of a point $x_o \in X$ from the set A by*

$$\operatorname{dist}(x_o, A) := \inf\{d(x_o, x) : x \in A\}. \tag{3.44}$$

Then we have

(a) $\operatorname{dist}(x_o, A) = 0 \quad\Longleftrightarrow\quad x_o \in \overline{A}$*;*
(b) $\forall_{x_o,y_o\in X}\quad |\operatorname{dist}(x_o, A) - \operatorname{dist}(y_o, A)| \le d(x_o, y_o)$*;*
(c) The function $\varphi_A : X \to \mathbb{R}$, defined by $\varphi_A(x) := \operatorname{dist}(x, A)$, $x \in X$, is Lipschitzian with constant 1 *(therefore it is uniformly continuous).*

Proof: (a): Suppose that $\operatorname{dist}(x_o, A) = 0$, i.e.

$$\inf\{d(x_o, x) : x \in A\} = 0 \quad\Longleftrightarrow\quad \begin{cases} \text{(i)} & \forall_{x\in A}\ \ d(x_o, x) \ge 0; \\ \text{(ii)} & \forall_{\varepsilon>0}\ \exists_{x\in X}\ \ d(x_o, x) < \varepsilon. \end{cases}$$

While the condition (i) is always true (by the definition of the metric), the condition (ii) can be written as

$$\forall_{x\in A}\ \ d(x_o,x)\geq 0 \quad\Leftrightarrow\quad \forall_{\varepsilon>0}\ \ B_\varepsilon(x_o)\cap A\neq\emptyset \quad\Leftrightarrow\quad x_o\in\overline{A}.$$

(b): Notice that we have by triangle inequalities

$$\forall_{x\in X}\quad d(x_o,x)\leq d(x_o,y_o)+d(y_o,x)\quad\text{and}\quad d(y_o,x)\leq d(x_o,y_o)+d(x_o,x) \tag{3.45}$$

therefore, by taking infimum of (3.45) with respect to $x\in A$, we obtain

$$\begin{aligned}\operatorname{dist}(x_o,A)&=\inf\{d(x_o,x):x\in A\}\leq\inf\{d(x_o,y_o)+d(y_ox):x\in A\}\\&=d(x_o,y_o)+\inf\{d(y_ox):x\in A\}=d(x_o,y_o)+\operatorname{dist}(y_o,A)\\ \operatorname{dist}(y_o,A)&=\inf\{d(x_o,x):x\in A\}\leq\inf\{d(x_o,y_o)+d(x_ox):x\in A\}\\&=d(x_o,y_o)+\inf\{d(x_ox):x\in A\}=d(x_o,y_o)+\operatorname{dist}(x_o,A)\end{aligned}$$

which implies

$$-d(x_o,y_o)\leq\operatorname{dist}(x_o,A)-\operatorname{dist}(y_o,A)\leq d(x_o,y_o)$$

and consequently

$$|\operatorname{dist}(x_o,A)-\operatorname{dist}(y_o,A)|\leq d(x_o,y_o).$$

(c) follows directly from (b). □

Corollary 3.71. *Let (X,d) be a metric space and A, B two closed, nonempty and disjoint sets in X. Then there exists a continuous function $\eta:X\to[0,1]$ such that*

$$\eta(x)=\begin{cases}0 & \textit{if}\ \ x\in A\\ 1 & \textit{if}\ \ x\in B.\end{cases}$$

Proof: By Proposition 3.70, the functions φ_A, $\varphi_B:X\to\mathbb{R}$, defined by

$$\varphi_A(x):=\operatorname{dist}(x,A)\quad\text{and}\quad\varphi_B(x):=\operatorname{dist}(x,B),\quad x\in X,$$

are continuous, and since A and B are closed and disjoint,

$$\forall_{x\in X}\quad\varphi_A(x)+\varphi_B(x)>0.$$

Therefore, we can define the function

$$\eta(x):=\frac{\varphi_A(x)}{\varphi_A*(x)+\varphi_B(x)},$$

which, as it can be easily verified, satisfies all the required properties. □

3.5 Compact Sets and Compact Spaces

Definition 3.72. Let (S,d) be a metric space and $A\subset S$.

(a) A family of subsets $\{U_\alpha\}_{\alpha\in\Lambda}$ of S is said to be a *cover* (or *covering*) of the set A if

$$A \subset \bigcup_{\alpha\in\Lambda} U_\alpha.$$

(b) We say that a cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of A is *open* (or that it is an *open cover* of A) if for every $\alpha \in \Lambda$ U_α is an open set.

(c) A family $\{V_\beta\}_{\beta\in B}$ is called a *subcover* of the cover $\{U_\alpha\}_{\alpha\in\Lambda}$, if
   (i) $\{V_\beta\}_{\beta\in B}$ is a cover of A,
   (ii) $\forall_{\beta\in B}\ \exists_{\alpha\in\Lambda}\ \ V_\beta = U_\alpha$.

(d) A cover composed of a finite number of sets is called a *finite cover*.

The following notion plays a fundamental role in mathematical analysis:

Definition 3.73. A set A is called *compact* if and only if for every open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of A there exists a finite subcover $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ of A.

Definition 3.74. A metric space (S, d) is called *compact*, if S is a compact set.

Proposition 3.75. *Let (S, d) be a metric space. Every compact set A in S is closed and bounded.*

Proof: We will show that if A is compact, then it is bounded. Indeed, assume for contradiction that $A \neq \overline{A}$, i.e. $\exists_{x_o\in\overline{A}\setminus A}$. Recall that

$$x_o \in \overline{A} \ \Leftrightarrow\ \forall_{\varepsilon>0}\ \exists_{x\in A}\ \ d(x, x_o) < \varepsilon.$$

In particular, we can take $\varepsilon = \frac{1}{n}$. Then $\exists_{x_n\in A}\ \ d(x_n, x) < \frac{1}{n}$, i.e. $\lim\limits_{n\to\infty} x_n = x_o$. Since $x_o \notin A$, for every $x \in A$ we have that $d(x, x_o) > 0$, thus we can take $\varepsilon_x = \frac{d(x,x_o)}{2}$ and define the open set $U_x := B_{\varepsilon_x}(x)$. It is clear that $\forall_{x\in A}\ \ x \in U_x$, thus

$$A \subset \bigcup_{x\in A} U_x,$$

thus $\{U_x\}_{x\in A}$ is an open cover of A. We will show that it is not possible do extract from $\{U_x\}_{x\in A}$ any finite subcover of A. Indeed, suppose for contradiction that $\{U_{x_1}, U_{x_2}, \dots, U_{x_n}\}$ is a finite collections of sets from $\{U_x\}_{x\in A}$. Since $U_{x_k} = B_{\varepsilon_{x_k}}(x_k)$, we can take

$$\varepsilon := \min\{\varepsilon_{x_1}, \varepsilon_{x_2}, \dots, \varepsilon_{x_n}\} > 0.$$

Since $x_o \in \overline{A}$, we have that

$$\exists_{x\in A}\ \ x \in B_\varepsilon(x_o) \cap A \iff d(x, x_o) < \varepsilon.$$

However, for all x_k, $k = 1, \dots, n$, we have

$$\begin{aligned} d(x, x_k) &\geq d(x_o, x_k) - d(x, x_o) = 2\varepsilon_{x_k} - d(x, x_o) \\ &> 2\varepsilon_{x_k} - \varepsilon > 2\varepsilon_{x_k} - \varepsilon_{x_k} \\ &= \varepsilon_{x_k} \geq \varepsilon, \end{aligned}$$

it follows that $x \notin B_{\varepsilon_{x_k}} = U_{x_k}$ for all $k = 1, 2, \dots, n$, which means that $x \notin \bigcup_{k=1}^{n} U_{x_k}$. Consequently, the family $\{U_{x_1}, U_{x_2}, \dots, U_{x_n}\}$ can not be a cover of A. That means, the cover $\{U_x\}_{x \in A}$ does not contain a finite subcover.

In order to prove that A is bounded, assume for contradiction that A is unbounded. Then we can construct a sequence $\{x_n\}_{n=1}^{\infty} \subset A$ satisfying

$$d(x_n, x_m) \geq 1 \quad \text{for} \;\; m \neq n. \tag{3.46}$$

For $x \in A$, we define the set $V_x := B_{\frac{1}{2}}(x)$. We will show that the open cover $\{V_x\}_{x \in A}$ does not contain a finite subcover. Suppose that $\{V_{y_1}, V_{y_2}, \dots, V_{y_N}\}$ is a finite collection of sets from the cover $\{V_x\}_{x \in A}$. We will show that it can not be a cover of A. Indeed, since the sequence $\{x_n\}$ is infinite, one of the sets $V_{y_1}, V_{y_2}, \dots, V_{y_N}$, say V_{y_k} , contains at least two points of this sequence (in fact there should be infinitely many elements of this sequence in V_{y_k}), say x_n and x_m. Then we have, since x_n, $x_m \in B_{\frac{1}{2}}(y_k)$, we have

$$d(x_n, x_m) \leq d(x_n, y_k) + d(x_m, y_k) < \frac{1}{2} + \frac{1}{2} = 1,$$

which contradicts (3.46). □

Definition 3.76. Let (S, d) be metric space. A set $A \subset S$ is said to be *totally bounded* if

$$\forall_{\varepsilon > 0} \; \exists_{\{a_1, \alpha_2, \dots, a_n\} \subset A} \;\; A \subset \bigcup_{k=1}^{n} B_\varepsilon(a_k). \tag{3.47}$$

The set $\{a_1, a_2, \dots, a_n\}$ is called an *ε-net in A*.

Proposition 3.77. *Let A be a totally bounded set in a metric space (S, d). Then A is bounded.*

Proof: Since A is totally bounded, then, by (3.47), for $\varepsilon = 1$, exists a finite subset $\{a_1, a_2, \dots, a_n\}$ of A such that

$$A \subset B_1(a_1) \cup B_1(a_2) \cup \dots \cup B_1(a_n). \tag{3.48}$$

Put $r = \max\{d(a_m, a_k) : m, k \in \{1, 2, \dots, n\}\} + 1$. Then, by (3.48), for every $a \in A$ there exists $k \in \{1, 2, \dots, n\}$, such that $a \in B_1(a_k)$. Consequently, we have

$$\begin{aligned} \forall_{a \in A} \; \exists_{k \in \{1,2,\dots,n\}} \;\; d(a_1, a) &\leq d(a_1, a_k) + d(a_k, a) \\ &< 1 + d(a_k, a) \leq 1 + \max\{d(a_m, a_k) : m, k \in \{1, 2, \dots, n\}\} \\ &= r, \end{aligned}$$

which implies that

$$\forall_{a \in A} \;\; a \in B_r(a_1),$$

i.e. $A \subset B_r(a_1)$, which means A is bounded. □

Lemma 3.78. *Let (S, d) be a metric space and $A \subset S$ a compact set. Then S is totally bounded.*

Proof: Let $\varepsilon > 0$ be an arbitrary number. For each $a \in A$ consider the set $U_a := B_\varepsilon(a)$. Since

$$A \subset \bigcup_{a\in A} U_a,$$

the family $\{U_a\}_{a\in A}$ is an open cover of A. By compactness of A, the cover $\{U_a\}_{a\in A}$ contains an open subcover $\{U_{a_1}, U_{a_2}, \dots, U_{a_n}\}$, i.e.

$$A \subset \bigcup_{k=1}^{n} U_{a_k} = \bigcup_{k=1}^{n} B_\varepsilon(a_k),$$

which implies that A is a totally bounded. □

Lemma 3.79. *Let (S, d) be a metric space and $\{x_n\}$ a sequence in S. Put*

$$C := \{x_n : n = 1, 2, 3, \dots\}.$$

If the sequence $\{x_n\}$ does not contain a convergent subsequence, then

(i) the set C infinite, and,
(ii) every point a in C is isolated,
(iii) the set C is closed.

Proof: Suppose that the set C is finite, say $C = \{a_1, a_2, \dots, a_N\}$. Then, clearly there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = a_k$ for some $k = 1, 2, \dots, N$. Since a constant sequence is convergent, it follows that $\{x_{n_k}\}$ is convergent subsequence of C.

Suppose now that $a \in S$ is a limit point of C, i.e.

$$\forall_{\varepsilon>0}\ \exists_n\ \ 0 < d(a, x_n) < \varepsilon. \tag{3.49}$$

Using the property (3.49), we will construct a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$. We put, $x_{n_1} := x_1$. Then, by (3.49), there exists x_{n_2} such that $0 < d(a, x_{n_1}) < \min\{d(a, x_{n_1}), \frac{1}{2})$. Suppose for induction, that we have constructed the points $\{x_{n_1}, x_{n_2}, \dots, x_{n_k}\}$ such that

$$n_1 < n_2 < \dots < n_k, \quad 0 < d(a, x_{n_l}) < \min\{d(a, x_1), d(a, x_2), \dots, d(a, x_{n_{l-1}}), \frac{1}{l}\}, \ \ l = 2, \dots, k.$$

Then, we can take

$$\varepsilon = \min\{d(a, x_1), d(a, x_2), \dots, d(a, x_{n_k}), \frac{1}{k+1}\},$$

and, by (3.49), tere exists $\{x_{k+1}\}$ such that $0 < d(a, x_{n_{k+1}}) < \varepsilon$. Clearly $n_{k+1} > n_k$, and since $d(a, x_{n_k}) < \frac{1}{k}$, it follows that $\lim_{k\to\infty} x_{n_k} = a$, so $\{x_{n_k}\}$ is a convergent subsequence of $\{x_n\}$. This implies that the set of limit points C' of C is empty, so C is closed. □

Proposition 3.80. *Let (S, d) be a metric space and A a compact set in S. Then every sequence $\{x_n\} \subset A$ contains a convergent subsequence $\{x_{n_k}\}$ to limit in A, i.e. $\lim_{k\to\infty} x_{n_k} \in A$.*

Proof: Suppose for contradiction that A contains a sequence $\{x_n\}$ without a convergent subsequence. Then by Lemma 3.79, the set $C := \{x_n : n = 1, 2, 3, \dots\}$ is closed and infinite, i.e. $C = \{a_1, a_2, a_3, \dots\}$. In addition, every point $a \in C$ is isolated, i.e.

$$\forall_{a_k \in C}\ \exists_{\varepsilon_k > 0}\ B_{\varepsilon_k}(a_k) \cap C = \{a_k\},$$

We put $U_0 := S \setminus C$, and for $k = 1, 2, 3, \dots$, $U_k := B_{\varepsilon_k}(a_k)$. Clearly, the family $\{U_0, U_1, U_2, \dots\}$ is an open cover of A, however, it does not contain any finite subcover. Indeed, consider $\{U_{n_1}, U_{n_2}, \dots, U_{n_N}\}$, Put $k := \max\{n_1, n_2, \dots, n_N\} + 1$. Then clearly

$$a_k \notin U_{n_1} \cup U_{n_2} \cup \dots \cup U_{n_N},$$

which means $\{U_{n_1}, U_{n_2}, \dots, U_{n_N}\}$ is not a cover. □

Lemma 3.81. *Let (S, d) be a metric space and $A \subset S$ a set satisfying the following condition: every sequence $\{x_n\}_{n=1}^{\infty} \subset A$ contains a convergent to an element in A subsequence. Then the set A is totally bounded.*

Proof: Suppose for contradiction that A is not totally bounded, i.e. (by applying the negation to (3.47))

$$\exists_{\varepsilon > 0}\ \forall_{\{y_1, y_2, \dots, y_n\} \subset A}\ \ A \setminus \bigcup_{k=1}^{n} B_\varepsilon(y_k) \neq \emptyset. \tag{3.50}$$

In order to show that this is impossible, we will construct (by applying the principle of mathematical induction) a sequence $\{x_n\} \subset A$ without any convergent subsequence.

Assume that $x_1 \in A$ is an arbitrary element. It follows from (3.50) that $A \setminus B_\varepsilon(x_1) \neq \emptyset$, thus there exists $x_2 \in A$ such that $d(x_1, x_2) \geq \varepsilon$. Assume for induction that we have constructed the elements x_1, x_2, $\dots$, $x_n \in A$, $n \leq 2$, which satisfy the condition

$$d(x_m, x_k) \geq \varepsilon \quad \text{for} \quad n \geq m > k \geq 1.$$

Notice that there exists $x_{n+1} \in A$ such that $d(x_{n+1}, x_k) \geq \varepsilon$ for all $k = 1, 2, \dots, n$. Indeed, since by (3.50)

$$A \setminus \bigcup_{k=1}^{n} B_\varepsilon(x_k) \neq \emptyset,$$

there exists $x_{n+1} \in A$ such that

$$x_{n+1} \notin \bigcup_{k=1}^{n} B_\varepsilon(x_k) \implies \forall_{k=1,2,\dots,n}\ \ d(x_{n+1}, x_k) \geq \varepsilon.$$

Therefore, by mathematical induction, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ satisfying the property

$$\forall_{k,m \in \mathbb{N}}\ \ k \neq m \quad \Rightarrow \quad d(x_k, x_m) \geq \varepsilon.$$

Therefore, the sequence $\{x_n\} \subset A$ doesn't contain any convergent subsequence, which is a contradiction with the assumption. Therefore, A is totally bounded. □

Lemma 3.82. *Let (S, d) be a metric space and $A \subset S$ a totally bounded set. Then A contains a countable set $C = \{a_1, a_2, a_3, \dots\}$ such that $\overline{C} = A$.*

Proof: Since A is totally bounded, therefore for every $n = 1, 2, 3, \dots,$ there exists a $\frac{1}{n}$-net $\{a_{1,n}, a_{2,n}, \dots, a_{N_n,n}\}$. Put

$$C := \{a_{k,n} : n = 1, 2, 3, \dots,\ 1 \le k \le N_k\} = \bigcup_{k=1}^{\infty} \{a_{1,n}, a_{2,n}, \dots, a_{N_n,n}\}.$$

As a union of countably many finite sets, the set C is countable. We will show that $\overline{C} = A$. Since $C \subset A$,it is clear that $\overline{C} = \overline{A} = A$. On the other hand, for every $a \in A$ and every $\varepsilon > 0$ there exists $n \in N$ (by Archimedes) such that $0 < \frac{1}{n} < \varepsilon$. On the other hand, since $\{a_{1,n}, a_{2,n}, \dots, a_{N_n,n}\}$ is a $\frac{1}{n}$-net in A, we have

$$A \in B_{\frac{1}{n}}(a_{1,n}) \cup B_{\frac{1}{n}}(a_{2,n}) \cup \dots \cup B_{\frac{1}{n}}(a_{N_n,n}),$$

which implies that there exists $a_{k,n} \in C$ such that $a_{k,n} \in C \cap B_{\frac{1}{n}}(a) \subset B_\varepsilon(a)$, i.e. $a \in \overline{C}$. □

Lemma 3.83. *Let (S, d) be a metric space and A a closed set of S such that there exists a countable subset $C = \{a_1, a_2, a_3, \dots\}$ of A satisfying $\overline{C} = A$. Then every open cover $\{U_\alpha\}_{\alpha \in \varLambda}$ of the set A contains a countable subcover $\{U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}, \dots\}$.*

Proof: Let $\{U_\alpha\}_{\alpha \in \varLambda}$ be an open cover of A. Let $\mathbb{Q}^+$ denotes the set of all positive rational numbers, which is countable. We define the family $\{B_r(c) : c \in C, r \in \mathbb{Q}^+\}$, which is indexed by the countable set $\mathbb{Q}^+ \times C$, thus it is composed of countably many sets. We put $\varUpsilon := \{(c, r) \in C \times \mathbb{Q}^+ : \exists_{\alpha \in \varLambda}\ B_r(c) \subset U_\alpha\}$. Since $\varUpsilon \subset \mathbb{Q}^+ \times C$, it is countable. We claim that $\{B_r(c)\}_{(r,c) \in \varUpsilon}$ is an open cover of A. Indeed, for every $a \in A$ there exists $\alpha \in \varLambda$ such that $a \in U_\alpha$. Since U_α is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(a) \subset U_\alpha$. On the other hand, if we take $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\varepsilon}{2}$, then, since $\overline{C} = A$, there exists $a_k \in C$ such that $d(a, a_k) < \frac{1}{n}$. On the other hand, if $d(x, a_k) < \frac{1}{n}$, then we have

$$d(a, x) \le d(a, a_k) + d(a_k, x) < \frac{1}{n} + \frac{1}{n} < \varepsilon,$$

thus $B_{\frac{1}{n}}(a_k) \subset B_\varepsilon(a) \subset U_\alpha$. But this means that $(\frac{1}{n}, a_k) \in \varUpsilon$ and $a \in B_{\frac{1}{n}}(a_k)$. Consequently, $\{B_r(c)\}_{(r,c) \in \varUpsilon}$ is an open cover which is countable. By the definition, for every $(r, c) \in \varUpsilon$ there exists $\alpha(r, c) \in \varLambda$ such that $B_r(c) \subset U_{\alpha(r,c)}$, thus

$$A \subset \bigcup_{(r,c) \in \varUpsilon} B_r(c) \subset \bigcup_{(r,c) \in \varUpsilon} U_{\alpha(r,c)},$$

thus the cover $\{U_{\alpha(r,c)}\}_{(r,c) \in \varUpsilon}$ is the required countable subcover of $\{U_\alpha\}_{\alpha \in \varLambda}$. □

Proposition 3.84. *Let (S, d) be a metric space and A be a subset in S such that every sequence $\{x_n\} \subset A$ has a convergent subsequence to a limit in A. Then A is compact.*

Proof: Clearly, the set A is closed (all limit points of A belong to A). By Lemma 3.81, A is totally bounded and consequently, by Lemma 3.82, it contains a countable set C such that $\overline{C} = A$.

Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover. By Lemma 3.83, $\{U_\alpha\}_{\alpha\in\Lambda}$ contains a countable subcover $\{U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}, \dots\}$. Assume for contradiction that the cover $\{U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}, \dots\}$ does not contain a finite subcover. Then we can put

$$W_n := \bigcup_{k=1}^{n} U_{\alpha_k}.$$

Clearly $\{W_n\}_{n=1}^{\infty}$ is also an open cover of A, and satisfying $W_n \subset W_{n+1}$, and also does not have a finite subcover. Consequently, for every n the set $A \setminus W_n$ is non-empty, thus we can choose $x_n \in A \setminus W_n$. By assumption, the sequence $\{x_n\}$ contains a convergent subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = x_o \in A$. Since $\{W_n\}$ is a cover, there exists W_{n_o} such that $x_o \in W_{n_o}$. The set W_{n_o} is open, thus

$$\exists_{\varepsilon>0}\ \forall_x\ \ d(x, x_o) < \varepsilon \ \ \Rightarrow \ \ x \in W_{n_o},$$

and on the other hand, by the convergence of $\{x_{n_k}\}$

$$\exists_{K>0}\ \forall_{k\geq K}\ \ d(x_{n_k}, x_o) < \varepsilon \ \ \Longrightarrow \ \ x_{n_k} \in W_{n_o}.$$

Choose $n_k \geq n_o$ with $k \geq K$, then we have $x_{n_k} \in (A \setminus W_{n_k}) \cap W_{n_o}$, but $W_{n_o} \subset W_{n_k}$, thus $(A \setminus W_{n_k}) \cap W_{n_o} = \emptyset$, and we get a contradiction. □

Theorem 3.85. (Characterization of Compact Sets) *Let (S, d) be a metric space and $A \subset S$. Then we have the following equivalent conditions*

(a) A is compact;
(b) A is complete and totally bounded;
(c) Every sequence in A has a convergent subsequence to limit in A.

Proof: (a) $\Leftrightarrow$ (c): The implication $\Rightarrow$ is given by Proposition 3.80, while $\Leftarrow$ follows from Proposition 3.84.
Since, by Lemma 3.78, we know that if A is compact, then it is totally bounded, thus we need to prove that it is also complete. However, we already know that (a) $\Leftrightarrow$ (c), thus if $\{x_n\}$ is a Cauchy sequence in A, then by (c), it contains a convergent subsequence, and by Lemma 3.41, it is convergent. Consequently A is complete.

Therefore, we only need to prove the implication (b) $\Rightarrow$ (c). For this purpose assume that A is totally bounded and complete, and let $\{x_n\}$ be an arbitrary sequence in A. For every natural number $n \in \mathbb{N}$, by assumption, there exists a finite $\frac{1}{n}$-net $\{B_{\frac{1}{n}}(a_{1,n}), \dots, B_{\frac{1}{n}}(a_{N_n,n})\}$. We begin with $n = 1$. It is clear that one of the balls $B_1(a_{k,1})$ of the cover $\{B_1(a_{1,1}), \dots, B_1(a_{N_1,1})\}$ of A (there are only finitely many of them) must contain infinitely many elements of the sequence $\{x_n\}$, i.e. for a certain $k_1 \in \{1, 2, \dots, N_1\}$, the set $I_1 := \{n : x_n \in B_1(a_{k_1 1})\}$ is infinite. Let chose $n_1 \in I_1$ and consider x_{n_1} to be the first element of our subsequence of $\{x_n\}$. Next, we can take $n = 2$, and since $\{B_{\frac{1}{2}}(a_{1,2}), \dots, B_{\frac{1}{2}}(a_{N_2,2})\}$ is a cover of A, which it is also a cover of $B_1(a_{k_1,1})$, so there exists a ball $B_{\frac{1}{2}}(a_{k_2,2})$, which contains infinitely many elements of of the set $\{x_n : n \in I_1\}$, i.e. there is an $k_2 \in \{1, 2, \dots, N_2\}$ such that the set $I_2 := \{n \in I_1 : x_n \in B_{\frac{1}{2}}(a_{k_2,2})\}$ is infinite. Then we chose $n_2 \in I_2$ such that $n_2 > n_1$ and consider x_{n_2} to be the second element of our subsequence. Next, by applying the mathematical induction, we assume that we have already constructed a sequence of balls

$$B_1(a_{k_1 1}) \supset B_{\frac{1}{2}}(a_{k_2,2}) \supset \cdots \supset B_{\frac{1}{m}}(a_{k_m,m})$$

such that the sets $I_l := \{n \in I_{l-1} : x_n \in B_{\frac{1}{l}}(a_{k_l,l})\}$, $l \geq 2$, are infinite, and we also have chosen elements x_{n_l}, $n_l \in I_l$ such that $n_{l-1} < n_l$.

Then, since $\{B_{\frac{1}{m+1}}(a_{1,m+1}), \dots, B_{\frac{1}{m+1}}(a_{N_{m+1},m+1})\}$ is a cover of A, which it is also a cover of $B_{\frac{1}{m}}(a_{k_m,m})$, so there exists a ball $B_{\frac{1}{m+1}}(a_{k,m+1})$, which contains infinitely many elements of of the set $\{x_n : n \in I_m\}$, i.e. there is an $k_{m+1} \in \{1, 2, \dots, N_{m+1}\}$ such that the set $I_{m+1} := \{n \in I_m : x_n \in B_{\frac{1}{m+1}}(a_{k_{m+1},m+1})$ is infinite. Then we chose $n_{m+1} \in I_{m+1}$ such that $n_{m+1} > n_m$ and consider $x_{n_{m+1}}$ to be the $m+1$-th element of our subsequence. Therefore, by the principle of mathematical induction, we have constructed a subsequence $\{x_{n_m}\}$ of $\{x_n\}$, which satisfies the property that

$$x_{n_l} \in B_{\frac{1}{m}}(a_{k_m,m}) \quad \text{for} \quad l \geq m, \quad \Rightarrow \quad \forall_{l_1,l_2 \geq m} \;\; d(x_{n_{l_1}}, x_{n_{l_2}}) < \frac{2}{m}.$$

Therefore, by Archimedes' axiom, the subsequence $\{x_{n_l}\}$ is Cauchy, i.e.

$$\forall_{\varepsilon>0} \, \exists_N \, \forall_{l_1,l_2 \geq N} \;\; d(x_{n_{l_1}}, x_{n_{l_2}}) < \varepsilon,$$

where $N \in \mathbb{N}$ is such that $\frac{2}{m} \leq \varepsilon$. Since A is complete, it follows that $\{x_{n_l}\}$ converge to a limit in A. $\square$

Example 3.86. Consider the space $S_h = \mathbb{R}^2$ to be equipped with the highway metric

$$d_h((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2, \\ |y_1| + |x_1 - x_2| + |y_2| & \text{if } x_1 \neq x_2. \end{cases}$$

Recall that (S_h, d_h) is a complete metric space (see Example 3.42. Suppose that $A \subset S_h$ is an arbitrary closed set and bounded set in $S_h$containing a sequence $\{(x^n, y^n)\}$ satisfying the condition

$$\exists_{r>0} \; \forall_{n,m \in \mathbb{N}} \quad n > m \quad \Rightarrow \quad x^n > x^m \quad \wedge \quad |y^n| \geq r. \tag{3.51}$$

Then the set A cannot be compact in (S_h, d_h). Indeed, one can easily verify that condition (3.51) implies

$$\forall_{n,m} \quad n \neq m \quad \Rightarrow \quad d_h((x^n, y^n), (x^m, y^m)) = |y^n| + |x^n - x^m| + |y^m| \geq 2r,$$

and consequently $\{(x^n, y^n)\} \subset A$ doesn't have a convergent subsequence.

Therefore, the following sets $A = \{(x, 1) : 0 \leq x \leq 1\}$, or $B = \{(x, y) : |x| + |y| \leq 1\}$ are closed, bounded but not compact in (S_h, d_h).

3.6 Compactness in Euclidean Space

Lemma 3.87. (Bolzano-Weierstarss Theorem) *Every bounded sequence in $\mathbb{R}^n$ contains a convergent subsequence.*

Proof: We will prove first that a bounded sequence $\{a_n\} \subset \mathbb{R}$ contains a convergent subsequence. Suppose that α and β are two real numbers such that $\alpha \leq a_n \leq \beta$ for all $n \in \mathbb{N}$. We will construct the subsequence a_{n_k} as follows. We put $\alpha_1 = \alpha$, $\beta_1 = \beta$ and we choose a_{n_1}, where $n_1 \geq 1$. We divide the interval $[\alpha_1, \beta_1]$ into two subintervals $[\alpha_1, \gamma_1]$ and $[\gamma_1, \beta_1]$, where $\gamma_1 = \frac{\alpha_1+\beta_1}{2}$. It is clear that one of these subintervals, which we denote by $[\alpha_2, \beta_2]$ must contain an infinite number of terms of the sequence $\{a_n\}$ and we can choose a term $a_{n_2} \in [\alpha_1, \beta_0]$ such that $n_2 > n_1$. The same construction can be applied to the interval $[\alpha_2, \beta_2]$, i.e. we can divide this interval into two subintervals $[\alpha_2, \gamma_2]$ and $[\gamma_2, \beta_2]$ of equal length (i.e. $\gamma_2 = \frac{\alpha_2+\beta_2}{2}$) and since $[\alpha_2, \beta_2]$ contains an infinite number of terms of the sequence $\{a_n\}$, one of these subintervals, which we denote by $[\alpha_3, \beta_3]$, it must contain also an infinite number of terms of $\{a_n\}$. Next, we can choose an element $a_{n_3} \in [\alpha_3, \beta_3]$ such that $n_3 > n_2$. By repeating this construction again and again, we can construct a sequence of subintervals $[\alpha_1, \beta_1] \supset [\alpha_2, \beta_2] \supset \cdots \supset [\alpha_k, \beta_k] \supset \dots$ and a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \in [\alpha_k, \beta_k]$. Since the length of the subinterval $[\alpha_k, \beta_k]$ is equal to $\frac{\beta-\alpha}{2^{k-1}}$, the subsequence $\{a_{n_k}\}$ is a Cauchy sequence. Indeed, for every $\varepsilon > 0$ there exist an $N >$ such that $\frac{\beta-\alpha}{2^{N-1}} < \varepsilon$ and therefore for all k, $m \geq N$ we have

$$|a_{n_k} - a_{n_m}| \leq \frac{\beta - \alpha}{2^{N-1}} < \varepsilon.$$

Consequently, by completeness of $\mathbb{R}$, the subsequence $\{a_{n_k}\}$ converges to a limit a such that $\alpha \leq a \leq \beta$.

Assume now that $\{x_n\}$ is a bounded sequence in $\mathbb{R}^m$, for $m > 1$, i.e.

$$\exists_{M>0}\ \forall_{n\in\mathbb{N}}\ \ \|x_n\| \leq M, \quad x_n = (x_n^1, x_2^n, \dots, x_n^m).$$

Notice, that the sequences $\{x_n^k\} \subset \mathbb{R}$, $k = 1, 2, \dots, m$, are bounded. Indeed, we have

$$\forall_{n\in\mathbb{N}}\ \ |x_n^k| \leq \|x_n\| \leq M, \quad k = 1, 2, \dots, m.$$

Then, by the first part of the proof, we have that the sequence $\{x_n^1\}$ contains a convergent subsequence $\{x_{n_{k_1}}^1\}$. Now, the sequence $\{x_n^2\}$ is also bounded, thus $\{x_{n_{k_1}}^2\}$ is bounded, so, by the same argument, we can extract from the subsequence $\{x_{n_{k_1}}^2\}$ another convergent subsequence $\{x_{n_{k_2}}^2\}$. By applying this argument repeatedly to $\{x_n^3\}$, $\dots$, $\{x_n^m\}$, we finally obtain a subsequences $\{x_{n_k}^l\}$, $l = 1, 2, \dots, m$, each of them being extracted from a convergent subsequence $\{x_{n_{k_l}}^l\}$, therefore each of them converges. Consequently, the obtain subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is convergent in $\mathbb{R}^m$.

□

Theorem 3.88. (Heine-Borel Theorem) *A set $A \subset \mathbb{R}^m$ is compact if and only if it is bounded and closed.*

Proof: If $A \subset \mathbb{R}^m$ is compact, then by Proposition 3.75, A is closed and bounded. Conversely, assume that A is bounded and closed, and let $\{x_n\}$ be a sequence in A. Since A is bounded, $\{x_n\}$ is also bounded, thus by Lemma 3.87, it contains a convergent subsequence $\{x_{n_k}\}$, i.e. $\lim\limits_{k\to\infty} x_{n_k} = a$. Since a is a limit point of A and A is closed, $a \in A$, and by Theorem 3.85(c), A is compact. □

Example 3.89. It should be pointed out that there exist metric spaces, where a bounded closed set is not compact. For example, if we take $S = (0, 1)$ equipped with the usual metric, one can see that S is bounded, closed (in S) but not compact. However this space is not complete. Another example of such a metric space (and this time it is a complete metric space) is the space $S = \mathbb{R}^2$ equipped with the highway metric d_h. Then the set $\{(x, 1) : x \in [0, 1]\}$ is closed and bounded but not compact (see Example 3.86).

Example 3.90. Consider the following set in $\mathbb{R}^2$:

$$A := \{(x, y) : 10 + 8|x| - 4|y| \geq x^2 + 2y^2\}.$$

Clearly, by the arguments explained in Example 3.68, the set A is closed. Is the set A compact? In order to be compact A needs to be bounded. Therefore, by Theorem 3.88, we need to check if the set A is bounded. Consider and arbitrary $(x, y) \in A$, i.e. the point (x, y) satisfies the condition

$$10 + 8|x| - 4|y| \geq x^2 + 2y^2 \iff 28 \geq (|x| - 4)^2 + 2(|y| + 1)^2$$

which clearly implies that

$$\sqrt{28} + 4 \geq |x| \quad \text{and} \quad \sqrt{28} + 1 \geq |y|,$$

therefore, the set A is bounded.

3.7 Continuity and Compactness

Proposition 3.91. *Let X and Y be two metric spaces, such that X is compact and $f : X \to Y$ a continuous map. Then $f(X)$ is a compact set of Y*

Proof: Suppose that $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover of $f(x)$. Then, by continuity of f, each set $f^{-1}(U_\alpha)$, $\alpha \in \Lambda$, is an open set in X, and

$$X \subset \bigcup_{\alpha\in\Lambda} f^{-1}(U_\alpha),$$

thus $\{f^{-1}(U_\alpha)\}_{\alpha\in\Lambda}$ is an open cover of X. Since X is compact, there exists a finite subcover $\{f^{-1}(U_{\alpha_1}), f^{-1}(U_{\alpha_2}), \dots, f^{-1}(U_{\alpha_n})\}$ of $\{f^{-1}(U_\alpha)\}_{\alpha\in\Lambda}$, i.e.

$$X \subset \bigcup_{k=1}^{n} f^{-1}(U_{\alpha_k}) \quad \Longrightarrow \quad f(X) \subset f\left(\bigcup_{k=1}^{n} f^{-1}(U_{\alpha_k})\right) = \bigcup_{k=1}^{n} U_{\alpha_k},$$

which means $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ is a finite subcover of $\{U_\alpha\}_{\alpha\in\Lambda}$. Consequently, $f(X)$ is compact. □

Theorem 3.92. (Weierstrass Theorem) *Let X be a compact metric space and $f : X \to \mathbb{R}$ a continuous function. Then there exist x_o, $y_o \in X$ such that*

$$f(x_o) = \min\{f(x) : x \in X\}, \quad \textit{and} \quad f(y_o) = \max\{f(x) : x \in X\}.$$

Proof: By Proposition 3.91, the set $f(X)$ is compact, thus it is closed and bounded. Therefore there exists (by LUB axiom) $\alpha = \sup\{f(x) : x \in X\}$. Since

$$\alpha = \sup\{f(x) : x \in X\} \quad \begin{cases} \forall_{x\in X} \ \ \alpha \geq f(x), \\ \forall_{\varepsilon>0} \ \exists_{x'\in X} \ \ \alpha - \varepsilon < f(x'). \end{cases}$$

Then for every $n \in \mathbb{N}$ there exists $x_n \in X$ such that $f(x_n) > \alpha - \frac{1}{n}$. By compactness of X, (Theorem 3.85(c)) $\{x_n\}$ contains a convergent subsequence $\{x_{n_k}\}$ such that $\lim\limits_{k\to\infty} x_{n_k} = x_o \in X$. Since for every $k \in \mathbb{N}$ we have

$$\alpha \geq f(x_{n_k}) > \alpha - \frac{1}{n_k},$$

it follows that $\lim_{k\to\infty} f(x_{n_k}) = \alpha$. On the other hand, by the continuity of $f(x)$ we get $\lim_{k\to\infty} f(x_{n_k}) = f(x_o)$, so $f(x_o) = \alpha$, and the conclusion follows. Similarly, one can show that there exists $y_o \in X$ such that $f(y_o) = \inf\{f(x) : x \in X\}$. $\square$

Theorem 3.93. (Cantor-Heine Theorem) *Let X be a compact metric space, Y a metric space and $f : X \to Y$ a continuous function. Then f is uniformly continuous.*

Proof: We need to show that f is uniformly continuous, i.e.

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x,x'\in X}\ \ d(x,x') < \delta \ \ \Rightarrow \ \ d(f(x), f(x')) < \varepsilon.$$

Assume for contradiction that the function $f(x)$ is not uniformly continuous on X, i.e.

$$\exists_{\varepsilon>0}\ \forall_{\delta>0}\ \exists_{x,x'\in X}\ \ d(x,x') < \delta \ \ \wedge \ \ d(f(x), f(x')) \geq \varepsilon. \tag{3.52}$$

Then by taking in (3.52) $\delta = \frac{1}{n}$, $n \in \mathbb{N}$, we construct two sequences $\{x_n\}$ and $\{x'_n\}$ in X

$$\forall_{n\in\mathbb{N}}\ \ d(x_n, x'_n) < \frac{1}{n} \ \ \wedge d(f(x_n), f(x'_n)) \geq \varepsilon. \tag{3.53}$$

Since X is compact, by Theorem 3.85, there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = x_o \in X$. Since $d(x_{n_k}, x'_{n_k}) < \frac{1}{n_k}$ it follows that

$$d(x_o, x'_{n_k}) \leq d(x_o, x_{n_k}) + d(x_{n_k}, x'_{n_k}) < d(x_o, x'_{n_k}) + \frac{1}{n_k}$$

thus $\lim_{k\to\infty} x'_{n_k} = x_o$. By continuity of f

$$\lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(x'_{n_k}) = f(x_o),$$

but, on the other hand

$$d(f(x_{n_k}), f(x'_{n_k})) \geq \varepsilon \ \ \Rightarrow \ \ \lim_{k\to\infty} d(f(x_{n_k}), f(x'_{n_k})) = d(f(x_o), f(x_o)) \geq \varepsilon,$$

which is a contradiction. Consequently, the function f is uniformly continuous. $\square$

Theorem 3.94. (Lebesgue Covering Lemma) *Let (X,d) be a compact metric space and $\{U_i\}_{i\in I}$ be an open cover of X. Then there exists a $\lambda > 0$ such that*

$$\forall_{x\in X}\ \ \exists_{i\in I}\ \ B_\lambda(x) \subset U_i.$$

Proof: Since X is compact, one can find a finite subcover $\{U_1, U_2, \dots, U_n\}$ of $\{U_i\}_{i\in I}$. Then define the functions $\varphi_k : X \to \mathbb{R}$ by

$$\varphi_k(x) = \operatorname{dist}(x, X \setminus U_k), \quad k = 1, 2, \dots, n.$$

Then, the functions are continuous and since $\varphi_k(x) = 0$ if and only if $x \in F \setminus U_k$, i.e. $x \notin U_k$, it follows that the function

$$\varphi(x) := \max\{\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)\}, \quad x \in X,$$

is continuous and $\varphi(x) > 0$ for all $x \in X$. Then, by Theorem 3.92, there exists $\lambda > 0$ such that

$$\lambda := \min\{\varphi(x) : x \in X\}.$$

One can easily verify that λ satisfies the above requirements. $\square$

Remark 3.95. One can use Theorem 3.94 to give a direct proof for Theorem 3.93. **Proof:** Indeed, assume that X is a compact metric space, Y a metric space and $f : X \to Y$ a continuous function. Then, we want to show that f is uniformly continuous. By assumption, since f is continuous at every $x \in X$,

$$\forall_{\varepsilon>0}\, \exists_{\delta_x>0}\, \forall_{x'\in X} \quad x' \in B_{\delta_x}(x) \quad \Rightarrow \quad d_Y(f(x), f(x')) < \frac{\varepsilon}{2}. \tag{3.54}$$

Since X is compact, by Theorem 3.94 applied to the cover $\{B_{\delta_x}(x)\}_{x\in X}$, there exists $\delta > 0$ such that

$$\forall_{x'\in X}\, \exists_{x''\in X} \;\; B_\delta(x') \subset B_{\delta_{x''}}(x'').$$

Therefore, for all x, $x' \in X$, if $d_X(x, x') < \delta$, i.e. $x \in B_\delta(x')$, there exists $x'' \in X$ such that $x \in B_\delta(x') \subset B_{\delta_{x''}}(x'')$. Therefore, by (3.54),

$$d_Y(f(x), f(x')) \le d_Y(f(x), f(x'')) + d_Y(f(x''), f(x')) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which implies

$$\forall_{\varepsilon>0}\, \exists_{\delta>0}\, \forall_{x,x'\in X} \quad d_X(x, x') < \delta \quad \Rightarrow \quad d_Y(f(x), f(x')) < \varepsilon,$$

f is uniformly continuous. $\square$

3.8 Connected and Path Connected Spaces

There are two formal definitions of connectedness of a metric space—the first one based on the idea that such a "connected" space consists of one piece meaning that it cannot be a union of two disjoint (nonempty) sets, and the second one that interprets connectedness as the possibility of moving continuously from any point to any other point.

Definition 3.96. A metric space (S, d) is said to be

(a) *connected* if S cannot be represented as the union of two nonempty disjoint open sets (or, equivalently, two nonempty disjoint closed sets);
(b) *path connected* if for any two points x_0, $x_1 \in S$ there exists a continuous function $\sigma : [0, 1] \to S$ such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. Such a function σ is called a *path* joining x_0 to x_1.

Definition 3.97. Let (S, d) be a metric space and $A \subset S$. The set A is said to be

(a) *connected* if (A, d_A) is connected space;
(b) *path connected* if (A, d_A) is a path-connected space.

Remark 3.98. Let (S, d) be a metric space and $A \subset S$. Notice that open sets in (A, d_A) are the sets $U \cap A$, where U is open in S. Therefore, a set A is *not connected* if there exist open sets U and V in S such that

$$U \cap V \cap A = \emptyset \quad \wedge \quad A \subset U \cup V \quad \wedge \quad A \cap V \neq \emptyset \quad \wedge \quad A \cap U \neq \emptyset.$$

Remark 3.99. One can easily notice that a set A of the space (S, d) is path-connected if for any two points x_0, $x_1 \in A$ there exists a path $\sigma : [0, 1] \to A$ joining x_0 and x_1.

Proposition 3.100. *Every interval* $[a, b]$ *in* $\mathbb{R}$ *(*$a < b$*) is connected.*

Proof: Assume for contradiction that $[a, b]$ is not connected. Then $[a, b]$ is a union of two non-empty disjoint open sets (in $[a, b]$) U and V, i.e.

$$[a, b] = U \cup V, \quad U \cap V = \emptyset, \quad u \neq \emptyset \neq V.$$

Notice that U and V are also closed in $[a, b]$. Since $a \notin V$, a is a lower bound of V. Put $\alpha = \inf(V)$. Since $a \leq \alpha \leq b$, we have that either $\alpha \in U$ or $\alpha \in V$. If $\alpha \in U$, then by openness of U, there is $\varepsilon > 0$ such that $(\alpha - \varepsilon, \alpha + \varepsilon) \cap [a, b] \subset U$, which implies α is not $\inf(A)$ (indeed: if $\alpha = \inf(A)$ then exists $y \in V$ such that $y < \alpha + \varepsilon$, which is impossible because such y belongs to U). Suppose therefore that $\alpha \in V$, but again by openness of V this would contradict the fact that $\alpha = \inf(V)$. Therefore, we obtain a contradiction (because $\alpha \in [a, b]$ but $\alpha \notin U$ and $\alpha \notin V$ while $[a, b] = U \cup V$).

□

Corollary 3.101. *If* $I \subset \mathbb{R}$ *is an interval (finite or infinite), then* I *is connected.*

Proof: Assume that I is an interval, an assume for contradiction that I is not connected. Then, there exist open sets U, $V \subset \mathbb{R}$ such that

$$I \subset U \cup V, \quad U \cap V \cap I = \emptyset, \;\; U \cap I \neq \emptyset, \;\; V \cap I \neq \emptyset.$$

Let $a \in U \cap I$, $b \in V \cap I$ and assume for definiteness that $a < b$. Then $[a, b] \subset I$ and

$$[a, b] \subset U \cup V, \quad U \cap V \cap [a, b] = \emptyset, \;\; a \in U \cap [a, b] \neq \emptyset, \;\; b \in V \cap [a, b] \neq \emptyset,$$

which implies that $[a, b]$ is not connected, which contradicts Proposition 3.100. □

Corollary 3.102. *The only connected sets in* $\mathbb{R}$ *are intervals.*

Proof: Suppose that $A \subset \mathbb{R}$ is a connected set. If a, $b \in A$, $a < b$, then $[a, b] \subset A$. Indeed, assume for contradiction that there is $c \in (a, b)$ such that $c \notin A$. Put $U = (-\infty, c)$, $V := (c, \infty)$, then $A \subset U \cup V$, $V \cap U = \emptyset$ and $a \in U \cap A$, $b \in V \cap A$. Therefore, we get a contradiction with the assumption that A is connected. Consequently,

$$A = \bigcup_{\substack{a,b \in A \\ a<b}} (a, b)$$

Put

$$\alpha := \begin{cases} \inf(A) & \text{if } A \text{ is bounded from below} \\ -\infty & \text{otherwise,} \end{cases}$$

and

$$\beta := \begin{cases} \sup(A) & \text{if } A \text{ is bounded from above} \\ \infty & \text{otherwise.} \end{cases}$$

Then, clearly $A \supset (\alpha, \beta)$ and

$$A = \begin{cases} (\alpha, \beta) & \text{if } \ \alpha,\, \beta \notin A \\ [\alpha, \beta) & \text{if } \ \alpha \in A,\ \beta \notin A \\ (\alpha, \beta] & \text{if } \ \alpha \notin A,\ \beta \in A \\ [\alpha, \beta] & \text{if } \ \alpha \in A,\ \beta \in A \end{cases}$$

which means that A is an interval. Since every interval, finite or infinite is connected, this proves that the only connected sets in $\mathbb{R}$ are intervals. □

Proposition 3.103. *A continuous image of a connected metric space is a connected space. More precisely, if (S, d) and (S', d') are two metric spaces such that (S, d) is connected and if there is a continuous surjective function $f : S \to S'$, then (S', d') is also connected.*

Proof: Suppose for the contradiction that (S', d') is not connected. Then we can represent S' as a union of two disjoint non-empty open sets U' and V'. By continuity of f the sets $U := f^{-1}(U')$ and $V := f^{-1}(V')$ are open, and since U' and V' are disjoint, U and V are also disjoint. Moreover, since f is surjective U and V are non-empty sets. Finally we have

$$S = f^{-1}(S') = f^{-1}(U' \cup V') = f^{-1}(U') \cup f^{-1}(V') = U \cup V,$$

therefore (S, d) is not connected what is a contradiction with the assumption. □

Theorem 3.104. (Intermediate Value Theorem) *Let (S, d) be a connected metric space and $f : S \to \mathbb{R}$ a continuous function. Suppose that $x_1,\ x_2 \in S$ are such that $f(x_1) < f(x_2)$. Then, for every $c \in (f(x_1), f(x_2))$ there exists $x \in S$ such that $f(x) = c$.*

Proof: Since S is connected and f is continuous, by Proposition 3.103, $f(S)$ is a connected set in $\mathbb{R}$, and thus by Corollary 3.101, $f(S)$ is an interval and $[f(x_1), f(x_2)] \subset f(S)$. Thus, $c \in f(S)$ whenever $f(x_1) < c < f(x_2)$. □

Proposition 3.105. *Let (S, d) be a path-connected metric space. Then (S, d) is connected.*

Proof: Suppose for the contradiction that (S, d) is not connected, then there exist two open, non-empty, disjoint sets U and V such that $U \cup V = S$. Since U and V are non-empty, there exists $x_0 \in U$ and $x_1 \in V$. By assumption, (S, d) is path-connected, thus there exists a continuous path $\sigma : [0, 1] \to S$ such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. However, by continuity of σ, $U' := \sigma^{-1}(U)$ and $V' := \sigma^{-1}(V)$ are two open, non-empty disjoint subsets of $[0, 1]$ such that $U' \cup V' = [0, 1]$, which implies that that $[0, 1]$ is not connected. But this is a contradiction with Proposition 3.100. □

3.9 Problems

1. A set A in a metric space (S, d) is called *bounded* if

$$\exists_{R>0}\, \exists_{x_o \in S}\;\; A \subset B_R(x_o).$$

Use the mathematical induction to show that if a set $A \subset S$ is *unbounded* (i.e. it is not bounded), then there exists a sequence $\{x_n\} \subset A$ such that $d(x_n, x_m) \geq 1$ for all $m \neq n$.

2. Let $(V, \|\cdot\|)$ be a normed vector space. Show that an open unit ball

$$B_1(0) := \{v \in V : \|v\| < 1\}$$

is a convex set, i.e.

$$\forall_{u,v \in B_1(0)}\, \forall_{t \in [0,1]}\;\; tu + (1-t)v \in B_1(0).$$

3. Consider the set $B \subset \mathbb{R}^2$ defined by

$$B := \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 < 4\}$$

Verify if the set B is open in the following metric spaces

(a) urban metric space (S_u, d_1);
(b) highway metric space (S_h, d_h);
(c) railway metric space (S_r, d_r).

4. Identify which of the following sets are open, which are closed, and which are neither, in the metric space $\mathbb{R}^2$ equipped with the Euclidean metric:

(a) $A = \{(x, y) : x^2 - 2x + y^2 = 0\} \cup \{(x, 0) : x \in [2, 3]\}$
(b) $B = \{(x, y) : y \geq x^2,\ 0 \leq y < 1\}$
(c) $C = \{(x, y) : x^2 - y^2 < 1,\ -1 < y < 1\}$

Find the interior, closure and boundary of the sets A, B, C (no proof is necessary)

5. Show that in a metric space (S, d) a closed ball is a closed set.

6. Suppose that (S, d) is a metric space and A, $B \subset S$ are such that $A \subset B$. Show that $\text{int}(A) \subset \text{int}(B)$ and $\overline{A} \subset \overline{B}$.

7. Give an example of two sets A and B in a metric space such that $(\text{int}(A \cup B) \neq \text{int}(A) \cup \text{int}(B)$. Use this example to show that the equality $\overline{C \cap D} = \overline{C} \cap \overline{D}$ does not hold in general.

8. Let (S_1, d_1) and (S_2, d_2) be two metric spaces. Show that each of the following functions d is a metric on $S = S_1 \times S_2$ (Hint: Knowing that d_1 and d_2 satisfies the three conditions of a metric, show that d also satisfies these conditions)

See the notes for the precise definition of these metric spaces

(a) $d((x_1,x_2),(y_1,y_2)) = \max\{d_1(x_1,y_1), d_2(x_2,y_2)\}$;
(b) $d((x_1,x_2),(y_1,y_2)) = \sqrt{(d_1(x_1,y_1))^2 + (d_2(x_2,y_2))^2}$;
(c) $d((x_1,x_2),(y_1,y_2)) = d_1(x_1,y_1) + d_2(x_2,y_2)$;

9. Let $S = \mathbb{R}^2$ and $d_h : S \times S \to \mathbb{R}$ be the *highway metric* given by

$$d_h((x_1,y_1),(x_2,y_2)) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2, \\ |y_1| + |x_1 - x_2| + |y_2| & \text{if } x_1 \neq x_2. \end{cases}$$

Check which of the following sets are open or closed in (S, d_h):

(a) $A = \{(x,y) : x^2 + y^2 < 1\}$;
(b) $B = \{(x,y) : y = 0 \ \wedge\ -1 \le x \le 1\}$;
(c) $C = \{(x,y) : x = 0 \ \wedge\ 0 < y < 1\}$;
(d) $D = \{(x,y) : -1 \le x \le 1 \ \wedge\ 1 < y < 2\}$.

HINT: Show that an open, with respect to the Euclidean metric d_2 on $\mathbb{R}^2$ (do not mix up this metric with the question 1, where this symbol was standing for something completely different), set U is also open with respect to the metric d_h. Use the definition of an open set for this question.

10. Let $S = \mathbb{R}^2$ and $d_2 : S \times S \to \mathbb{R}$ be the usual Euclidean metric on $\mathbb{R}^2$. Verify which of the sets A, B, C, and D listed in Problem 2, are open or closed in (S, d_2). In addition, find the boundaries of the sets A, B, C and D.

11. Use the definition of a limit and the properties of real numbers (including the Binomial Theorem) to show that

$$\forall_{a>0} \quad \lim_{n\to\infty} a^{\frac{1}{n}} = 1.$$

12. Let $\{\vec{x}_n\} \subset \mathbb{R}^m$ be a bounded sequence and $\{a_n\} \subset \mathbb{R}$ be a sequence convergent to zero. Show that the sequence $\{\vec{y}_n\}$ defined by $\vec{y}_n = a_n \vec{x}_n \in \mathbb{R}^m$ converges to $\vec{0} \in \mathbb{R}^m$.

13. Show that a convergent sequence in a metric space is a Cauchy sequence.

14. Let (X, d) be a metric space and $A \subset X$ a closed non-empty set. For a given point $x \in X$ we define the *distance from x to A* by the formula

$$d(x, A) := \inf\{d(x,a) : a \in A\}. \tag{3.55}$$

(a) Show that $x \in A \iff d(x, A) = 0$;
(b) Show that the function $\chi_A : X \to \mathbb{R}$ defined by $\chi_A(x) = d(x, A)$ is continuous on X.

15. Prove or give a counterexample to the following statement: *Let $f : X \to Y$ be a continuous map between two metric spaces. Then for every open set $U \subset X$ the set $f(U)$ is open in Y.*

16. Let (X, d) be a metric space and $f, g : X \to \mathbb{R}$ two continuous maps. Show that

$$\psi(x) := \max\{f(x), g(x)\} \quad \text{and} \quad \phi(x) := \min\{f(x), g(x)\},$$

are continuous on X.

17. Apply the definition of continuity to show that the function $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$ is continuous.

18. Give an example of a subspace X in $\mathbb{R}$ and a Cauchy sequence $\{x_n\}$ in X such that $\{x_n\}$ is not convergent. Verify the statement by proving it or giving a counterexample: *If $f : X \to \mathbb{R}$ is continuous, then for every Cauchy sequence $\{x_n\}$ in X the sequence $\{f(x_n)\}$ is Cauchy in $\mathbb{R}$.*

19. Let X and Y be two metric spaces. We say that a function $f : X \to Y$ is *Lipschitzian* with a constant $L \geq 0$ if

$$\forall_{x,y \in X} \;\; d(f(x), f(y)) \leq L\, d(x, y).$$

Show that every Lipschitzian function is uniformly continuous.

20. Check which of the following functions are uniformly continuous on the indicated sets:

(a) $f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$;
(b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{x}{|x|+1}$;
(c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$;
(d) $f : [-1, 1] \to \mathbb{R}$, $f(x) = x^2$.

21. Show that a function $f : X \to Y$ between two metric spaces is continuous if and only if for every closed set C in Y, the set $f^{-1}(C)$ is closed in X.

22. Which of the following sets in $\mathbb{R}^2$ (with the Euclidean metric) are compact (explain why!)

(a) $A := \{(x, y) : x^2 - y^2 \leq 1\}$
(b) $B := \left\{\left(\frac{1}{n}, 1\right) : n \in \mathbb{N}\right\} \cup \{(0, 1)\}$
(c) $C := \{(x, y) : 0 < x^2 + y^2 \leq 1\}$;
(d) $D := \{(x, y) : x^2 + y^4 \leq 1\}$.

23. Show that the union of two compact sets is compact.

24. Let (S, d) be a metric space. Show that

(a) the intersection of any number of compact sets in S is compact;
(b) the union of any finite number of compact sets is compact;
(c) the union of infinitely many compact sets may not be compact (even if it is bounded).

25. Let A be a compact set in a metric space (S, d). Show that

(a) for any closed set $B \subset S$ the intersection $A \cap B$ is compact;
(b) ∂A is compact;
(c) any finite set is compact.

26. Consider the metric space (S, d_h), where $S = \mathbb{R}^2$ and $d_h : S \times S \to \mathbb{R}$ is defined in Problem 2. Verify if in this metric space closed balls are always compact. Present examples and justify them.

27. Let $f : [a, b) \to \mathbb{R}$ be a bounded continuous function such that f is one-to-one. Show that the limit $\lim_{x\to b} f(x)$ exists.

28. Let $\{a_n\} \subset \mathbb{R}$ be an increasing bounded sequence. Show that

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$

SOLUTION: Since the sequence $\{a_n\}$ is increasing, we have $a_n < a_m$ for $n < m$, and $\alpha = \sup\{a_n : n \in \mathbb{N}\}$ implies that (i) $\forall_{n\in\mathbb{N}}\ a_n \le \alpha$, and (ii) $\forall_{\varepsilon>0}\exists_{m\in\mathbb{N}}\ \alpha - \varepsilon < a_m$. Since $0 \le \alpha - a_m < \alpha - a_n$ for $n \ge m$, consequently we get

$$\forall_{\varepsilon>0}\exists_{m\in\mathbb{N}}\forall_{n\ge m}\ \ 0 \le \alpha - \varepsilon = |\alpha - a_n| < \varepsilon.$$

Therefore,

$$\lim_{n\to\infty} a_n = \alpha = \sup\{a_n : n \in \mathbb{N}\}.$$

29. Sketch the following sets in $\mathbb{R}^2$ and decide whether they are connected or not. Justify your answer.

(a) $A := \{(x, y) : x^2 + y^2 < 1 \ \ \wedge\ \ |y| < |x|\}$;
(b) $B := \{(x, y) : xy > 1\}$;
(c) $C := \{(x, y) : \dfrac{x^2}{4} + \dfrac{y^2}{9} < 1 \ \ \wedge\ \ x^2 + y^2 > 1\}$.

30. Give examples of sets A and B in $\mathbb{R}^2$ which satisfy the following properties:

(a) The sets A and B are connected but $A \cup B$ is not connected.
(b) The sets A and B are connected but $A \cap B$ is not connected.
(c) The sets A and B are not connected but $A \cup B$ is connected.
(d) The sets A and B are not connected but $A \cap B$ is connected.
(e) The sets A and B are connected but $A \setminus B$ is not connected.

31. Give an example of a set in $\mathbb{R}^1$ with exactly four limit points.

32. Given $\mathbb{R}^2$ with the Euclidean metric. Show that the set

$$S = \Big\{(x, y) : 0 < x^2 + y^2 < 1\Big\}$$

is open. Describe the sets $S^{(0)}$, S', ∂S, S^c.

33. In $\mathbb{R}^2$ with the Euclidean metric, find an infinite collection of open sets $\{A_n\}$ such that $\bigcap_n A_n$ is the closed ball $\overline{B(0,1)}$.

34. In $\mathbb{R}^1$ with the Euclidean metric, give an example of a sequence $\{p_n\}$ with two subsequences converging to different limits. Give an example of a sequence which has infinitely many subsequences converging to different limits.

4

Banach spaces

4.1 Definition and Basic Properties of Normed Spaces. Definition of Banach Space.

Definition 4.1. Let V be a vector space. A function $\|\cdot\| : V \to \mathbb{R}$, i.e. $V \ni v \mapsto \|v\| \in \mathbb{R}$, is called a *norm* on V if

(N1) $\|v\| \geq 0$ for all $v \in V$;

(N2) $\|v\| = 0$ if and only if $v = \vec{0}$;

(N3) $\|\alpha v\| = |\alpha| \cdot \|v\|$ for all $v \in V$ and $\alpha \in \mathbb{R}$;

(N4) $\|v + w\| \leq \|v\| + \|w\|$ for all $v,\ w \in V$.

Definition 4.2. Let V be a vector space and $\|\cdot\| : V \to \mathbb{R}$ a norm on V. Then the pair $(V, \|\cdot\|)$ is called a *normed space*.

Remark 4.3. Notice that if $(V, \|\cdot\|)$ is a normed space that we can define on V a structure of a metric space (V, d), where the metric $d : V \times V \to \mathbb{R}$ is given by

$$d(v, w) = \|v - w\|, \qquad v,\ w \in V. \tag{4.1}$$

We will verify that the function d given by (4.1) satisfies the properties of a metric, i.e.

(M1) $d(v, w) \geq 0$ and $d(v, w) = 0 \quad \Leftrightarrow \quad v = w$;

(M2) $d(v, w) = d(w, v)$

(M3) $d(v, w) \leq d(v, u) + d(u, w)$

for all $v,\ w,\ u \in V$.

Indeed, we have
(M1): $d(v, w) = 0 \quad \Leftrightarrow \quad \|v - w\| = 0 \quad \Leftrightarrow \quad v - w = \vec{0} \quad \Leftrightarrow \quad v = w$.
(M2): $d(v, w) = \|v - w\| = \|(-1)(w - v)\| = |-1| \cdot \|w - v\| = \|w - v\| = d(w, v)$.
(M3): $d(v, u) = \|v - w\| = \|v - u + u - w\| \leq \|v - u\| + \|u - w\| = d(v, u) + d(u, w)$.

In this way, the topological notions such as *open* or *closed balls*, *open*, *closed* sets and etc., are defined in the normed space V.

More precisely, we have the following

Definition 4.4. Let V be a normed space. An *open ball* centered at v of radius $\varepsilon > 0$ is the set $B(v, \varepsilon) := \{w \in V : \|v - w\| < \varepsilon\}$, a *closed ball* centered at v of radius ε is the set $\overline{B}(v, \varepsilon) := \{w \in V : \|v - w\| \leq \varepsilon\}$.
A set $U \subset V$ is said to be *open* if

$$\forall_{u \in U} \, \exists_{\varepsilon > 0} \quad B(u, \varepsilon) \subset U,$$

a set $S \subset V$ is called *closed* if $S^c := V \setminus S$ is open, i.e. the set S contains all its limit points. For a given set $A \subset V$ we also define the sets

(i) A' — the set of *limit points* of A by

$$A' := \{u \in V : \forall_{\varepsilon > 0} \exists_{w \in A} \;\; 0 < \|u - w\| < \varepsilon\},$$

(ii) $\overline{A}$ — the *closure* of the set A by

$$\overline{A} := A \cup A'.$$

(iii)A^o — the *interior* of the set A by

$$A^o := \{u \in A : \exists_{\varepsilon > 0} \;\; B(u, \varepsilon) \subset A\},$$

(iv)∂A — the *boundary* of the set A by

$$\partial A := \overline{A} \setminus A^o,$$

or equivalently

$$\partial A := \{u \in V : \forall_{\varepsilon > 0} \;\; B(u, \varepsilon) \cap A \neq \emptyset \quad \wedge \quad B(u, \varepsilon) \cap A^c \neq \emptyset\}.$$

Definition 4.5. Let $\{u_n\}$ be a sequence in a normed space V. We say that the sequence $\{u_n\}$ *converges* to $u_o \in V$ (and we will call $\{u_n\}$ a *convergent sequence*) if and only if

$$\forall_{\varepsilon > 0} \, \exists_N \, \forall_{n \geq N} \;\; \|u_n - u_o\| < \varepsilon.$$

In such a case we will write $\lim_{n \to \infty} u_n = u_o$, or simply $\lim u_n = u_o$.

Remark 4.6. Notice that a sequence $\{u_n\}$ converges to $u_o \in V$ if and only if $\lim_{n \to \infty} \|u_n - u_o\| = 0$.

Definition 4.7. Let $\{u_n\}$ be a sequence in a normed space V. We say that the sequence $\{u_n\}$ is a *Cauchy sequence* if and only if

$$\forall_{\varepsilon > 0} \, \exists_N \, \forall_{n,m \geq N} \;\; \|u_n - u_m\| < \varepsilon.$$

Remark 4.8. Notice that every convergent sequence in V is a Cauchy sequence. However the converse statement is not true in general for an arbitrary normed space V.

Definition 4.9. A normed space V is said to be *complete* if every Cauchy sequence in V converges, i.e. if $\{u_n\}$ is a Cauchy sequence in V then there exists $u_o \in V$ such that $\lim u_n = u_o$. A complete normed space is called a *Banach space*.

Consider a normed space $(V, \|\cdot\|)$. Then by multiplying the norm $\|\cdot\|$ by a positve constant we can define another norm on V, i.e. for a given $\alpha > 0$ we put

$$\|u\|_\alpha := \alpha \cdot \|u\|, \qquad u \in V.$$

Obtained in this way function $\|\cdot\|_\alpha$ clearly satisfies all the properties (N1)—(N4) of a norm, which means that there are infinitely many norms which can be defined on V. What is important about all these normes $\|\cdot\|_\alpha$, is that the opens or closed sets in $(V, \|\cdot\|_\alpha)$ are exactly the same as in the normed space $(V, \|\cdot\|)$. Also, it is easy to notice that a sequence $\{u_n\}$ converges in $(V, \|\cdot\|)$ if and only if it converges in $(V, \|\cdot\|_\alpha)$ for $\alpha > 0$. These properties imply that the topological properties of the spaces $(V, \|\cdot\|_\alpha)$ for all $\alpha > 0$ are identical as the topological properties of the original space $(V, \|\cdot\|)$. In such a case we will simply say that the norms $\|\cdot\|$ and $\|\cdot\|_\alpha$ are *equivalent*. More precisely, we have the following definition:

Definition 4.10. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on the space V. We say that the norm $\|\cdot\|_2$ is *stronger* than the norm $\|\cdot\|_1$ (or, equivalently, that the norm $\|\cdot\|_1$ is *weaker* than the norm $\|\cdot\|_2$) if and only if for every sequence $\{u_n\}$ in V convergent to u_o with respect to the norm $\|\cdot\|_2$, we have that $\lim \|u_n - u_o\|_1 = 0$, i.e. the sequence $\{u_n\}$ also converges to u_o with respect to the norm $\|\cdot\|_1$.

Definition 4.11. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the space V are called equivaelnt, if $\|\cdot\|_1$ is stronger than $\|\cdot\|_2$ and $\|\cdot\|_2$ is stronger than $\|\cdot\|_1$.

Lemma 4.12. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on the space V. The norm $\|\cdot\|_2$ is stronger than the norm $\|\cdot\|_1$ if and only if*

$$\exists_{\alpha>0}\ \forall_{u\in V}\ \ \|u\|_1 \le \alpha \cdot \|u\|_2. \tag{4.2}$$

Proof: First, we will prove the implication $\Longleftarrow$: Suppose that the condition (4.2) is satisfied and let $\{u_n\}$ be a sequence convergent to u_o with respect to the norm $\|\cdot\|_2$, i.e. $\|u_n - u_o\|_2 \longrightarrow 0$ as $n \to \infty$. Since $\|u_n - u_o\|_1 \le \alpha \cdot \|u_n - u_o\|_2$, by squeeze property we obtain

$$0 \le \lim_{n\to\infty} \|u_n - u_o\|_1 \le \alpha \lim_{n\to\infty} \|u_n - u_o\|_2 = \alpha \cdot 0 = 0,$$

i.e. $\lim \|u_n - u_o\|_1 = 0$.

For the implication $\Longrightarrow$ we will use the prove by contradiction, i.e. we will assume that the norms $\|\cdot\|_2$ and $\|\cdot\|_1$ do not satisfy the condition (4.2), i.e. we have

$$\sim\ \exists_{\alpha>0}\ \forall_{u\in V}\ \ \|u\|_1 \le \alpha \cdot \|u\|_2 \quad \Longleftrightarrow \quad \forall_{\alpha>0}\ \exists_{u\in V}\ \ \|u\|_1 > \alpha \cdot \|u\|_2.$$

In particular, for $\alpha = n$, where $n \in \mathbb{N}$, there exists $x_n \in V$ such that

$$\|x_n\|_1 > n \cdot \|x_n\|_2,$$

which implies that

$$\frac{\|x_n\|_2}{\|x_n\|_1} < \frac{1}{n} \iff \left\| \frac{x_n}{\|x_n\|_1} \right\|_2 \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Consequently, the sequence $\{u_n\}$, where $u_n = \frac{x_n}{\|x_n\|_1}$ converges to $\vec{0}$ with respect to the norm $\|\cdot\|_2$, but on the other hand

$$\left\| \frac{x_n}{\|x_n\|_1} \right\|_1 = \frac{\|x_n\|_1}{\|x_n\|_1} = 1,$$

which implies that $u_n \not\to \vec{0}$ with respect to the norm $\|\cdot\|_1$. Consequently, the norm $\|\cdot\|_2$ can not be stronger that the norm $\|\cdot\|_1$. □

Corollary 4.13. *Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the space V are equivalent if and only if there exist $\alpha > 0$ and $\beta > 0$ such that*

$$\forall_{u \in V} \quad \|u\|_1 \le \alpha \cdot \|u\|_2, \quad \textit{and} \quad \|u\|_2 \le \beta \cdot \|u\|_1.$$

Example 4.14. Let us recall that we denote by $\mathbb{R}^n$ the n-dimensional Euclidean space. We have the following examples of norms in $\mathbb{R}^n$:

1. The most simple example of a norm is the norm *absolute value* on the space $\mathbb{R} = \mathbb{R}^1$, i.e. $\|x\| = |x|$ for $x \in \mathbb{R}$;
2. The *one-norm*
$$\|x\|_1 := \sum_{i=1}^n |x_i|, \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n;$$
3. The *Euclidean norm*
$$\|x\|_2 := \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}, \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n;$$
4. The *p-norm*, for $p > 1$, on $\mathbb{R}^n$ is defined by
$$\|x\|_p := \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}}, \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n; \tag{4.3}$$
5. The ∞-*norm* or *max-norm* on $\mathbb{R}^n$ is defined by
$$\|x\|_\infty := \max\{|x_i| : i = 1, 2, \dots, n\}, \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n;$$

Remark 4.15. The fact that the formula (4.3) defines the norm on $\mathbb{R}^n$ requires verification. We will show in several steps that $\|\cdot\|_p$ is indeed a norm. For $1 < p < \infty$; we will denote by p' the so-called *conjugate exponent of* p, which is defined as the number $p' > 1$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. We will need several inequalities.

Step 1: YOUNG INEQUALITY: For x, $y \in \mathbb{R}^n$, we have

$$x \bullet y \le \frac{1}{p}\|x\|_p^p + \frac{1}{p'}\|y\|_{p'}^{p'}. \tag{4.4}$$

This inequality follows immediately from the inequality

$$ab \le \frac{1}{p}\, a^p + \frac{1}{p'}\, b^{p'} \quad \forall\, a,\, b \ge 0$$

which is the consequence of the fact that the function ln is concave on $(0, \infty)$ (see section **??**) , so we have

$$\ln\left(\frac{1}{p}\, a^p + \frac{1}{p'}\, b^{p'}\right) \ge \frac{1}{p}\ \ln a^p + \frac{1}{p'}\ \ln b^{p'} = \ln ab.$$

Step 2: (Hölder Inequality) Let x and $y \in \mathbb{R}^n$ with $1 \le p \le \infty$. Then

$$x \bullet y \le \|x\|_p\, \|y\|_{p'}. \tag{4.5}$$

The conclusion is evident for $p = 1$ (with $p' = \infty$) and $p = \infty$ (with $p' = 1$). Assume therefore that $1 < p < \infty$. Since by Young inequality (with x replaced by tx, $t \in (0, \infty)$

$$x \bullet y \le \frac{t^{p-1}}{p}\|x\|_p^p + \frac{1}{tp'}\|y\|_{p'}^{p'} =: \varphi(t), \quad \forall_{t\in(0,\infty)}. \tag{4.6}$$

For fixed $x \ne 0$, $y \ne 0$ in $\mathbb{R}^n$, we find minimum of the function

$$\varphi(t) = \frac{t^{p-1}}{p}\|x\|_p^p + \frac{1}{tp'}\|y\|_{p'}^{p'},$$

by solving the equation $\varphi'(t) = 0$, i.e.

$$\varphi'(t) = \frac{p-1}{p} t^{p-2}\|x\|_p^p - \frac{1}{t^2p'}\|y\|_{p'}^{p'} = 0 \quad t = \frac{\|y\|_{p'}^{\frac{p'}{p}}}{\|x\|_p^p},$$

which, after substituting it to (4.6), gives the Hölder inequality.

Step 3: (Minkowsky Inequality), for all x, $y \in br^n$, we have

$$\|x+y\|_p \le \|x\|_p + \|y\|_p.$$

Indeed, by Hölder inequality we have

$$\begin{aligned}
(\|x+y\|_p^p &\le \sum_{k=1}^n |x_k + y_k|^p \\
&\le \sum_{k=1}^n (|x_k| + |y_k|)|x_k + y_k|^{p-1} \\
&\le \sum_{k=1}^n |x_k||x_k + y_k|^{p-1} + \sum_{k=1}^n |y_k||x_k + y_k|^{p-1} \\
&\le (\|x\|_p + \|y\|_p) \cdot \left(\sum_{k=1}^n |x_k + y_k|^{p'(p-1)}\right)^{\frac{1}{p'}} \\
&= (\|x\|_p + \|y\|_p) \cdot \left(\sum_{k=1}^n |x_k + y_k|^p\right)^{\frac{p-1}{p}} \\
&= (\|x\|_p + \|y\|_p)\|x+y\|_p^{p-1},
\end{aligned}$$

which implies the required inequality

4.2 Product of Normed Spaces

Let $(V, \|\cdot\|)$ and $(W, \|\cdot\|)$ be two normed spaces. Then it is possible to define the following norms (which are equivalent–see Problem 2 at the end of this chapter) on the cartesian product $V \times W$:

(a) $\|(v,w)\|_1 := \|v\| + \|w\|$, where $(v,w) \in V \times W$;
(b) $\|(v,w)\|_2 := \left(\|v\|^2 + \|w\|^2\right)^{\frac{1}{2}}$, where $(v,w) \in V \times W$;
(c) $\|(v,w)\|_p := (\|v\|^p + \|w\|^p)^{\frac{1}{p}}$, where $p > 1$ and $(v,w) \in V \times W$;
(c) $\|(v,w)\|_\infty := \max\{\|v\|, \|w\|\}$, where $(v,w) \in V \times W$;

Proposition 4.16. *Let V and W be two Banach spaces. Then the cartesian product $V \times W$, equipped with the norm $\|\cdot\|_2$ is also a Banach space.*

Proof: Let $\{(v_n, w_n)\}$ be a Cauchy sequence in the space $V \times W$, i.e. we have

$$\forall_{\varepsilon>0}\ \exists_N\ \ \forall_{n,m\geq N}\quad \|(v_n - v_m, w_n - w_m)\|_2 < \varepsilon. \tag{4.7}$$

Since

$$\|(v,w)\|_2 = \sqrt{\|v\|^2 + \|w\|^2} \geq \|v\|, \quad \text{and} \quad \|(v,w)\|_2 = \sqrt{\|v\|^2 + \|w\|^2} \geq \|w\|,$$

it follows from (4.7)

$$\forall_{\varepsilon>0}\ \exists_N\ \ \forall_{n,m\geq N}\quad \|v_n - v_m\| < \varepsilon,$$
$$\forall_{\varepsilon>0}\ \exists_N\ \ \forall_{n,m\geq N}\quad \|w_n - w_m\| < \varepsilon,$$

which means that that $\{v_n\}$ is a Cauchy sequence in V and $\{w_n\}$ is a Cauchy sequence in W. Since V and W are Banach spaces, we have

$$\lim v_n = v_o, \quad \lim w_n = w_o,$$

for some $v_o \in V$ and $w_o \in W$, which means

$$\forall_{\varepsilon>0}\ \exists_{N_1}\ \ \forall_{n\geq N_1}\quad \|v_n - v_o\| < \frac{\varepsilon}{\sqrt{2}},$$
$$\forall_{\varepsilon>0}\ \exists_{N_2}\ \ \forall_{n\geq N_2}\quad \|w_n - w_o\| < \frac{\varepsilon}{\sqrt{2}},$$

Put $N = \max\{N_1, N_2\}$ and notice that

$$\|(v,w)\|_2 \leq \sqrt{2}\max\{\|v\|, \|w\|\}.$$

Then we have

$$\forall_{\varepsilon>0}\ \exists_{N=\max\{N_1,N_2\}}\ \ \forall_{n\geq N}\quad \|(v_n - v_o, w_n - w_o)\|_2 < \sqrt{2}\max\{\|v_n - v_o\|, \|w_n - w_o\|\} < \sqrt{2}\frac{\varepsilon}{\sqrt{2}} = \varepsilon.$$

Consequently, the sequence $\{v_n, w_n\}$ converges to $\{v_o, w_o\}$ in $V \times W$, which completes the proof that $V \times W$ is a Banach space. $\square$

4.3 Examples of Normed and Banach Spaces

Let X be a set and $(V, \|\cdot\|)$ a normed space. A map $f : X \to V$ is called *bounded* if

$$\exists_{M>0}\ \forall_{x\in X}\quad \|f(x)\| < M,$$

i.e. the range $f(X)$ of the map f is contained in a ball of radius M centered at the origin.

We define the space

$$B(X,V) := \{f : X \to V \quad : \quad f \quad \text{is bounded}\ \}.$$

The space $B(X,V)$ is a vector space. Indeed, we have the following two natural operations defined on $B(X,V)$

$$(f_1 + f_2)(x) := f_1(x) + f_2(x), \quad (\alpha\, f)(x) := \alpha(f(x)),$$

where f_1, f_2, $f \in B(X,V)$ and $\alpha \in \mathbb{R}$.

We can also equip the space $B(X,V)$ with a norm, called the *uniform convergence* norm (see Problem 3), defined by

$$\|f\|_\infty := \sup\{\|f(x)\| : x \in X\}. \tag{4.8}$$

Notice that f is bounded, thus the set $\|f(X)\| := \{\|f(x)\| : x \in X\} \subset \mathbb{R}$ is bounded, so the supremum of $\|f(X\|$ always exists. The zero vector in the space $B(X,V)$ is the zero function $O : X \to V$, $O(x) = \vec{0} \in V$ for all $x \in X$.

Definition 4.17. A sequence $\{f_n\}$ of bounded maps $f_n : X \to V$, $n = 1, 2, \dots$, is said to be *uniformly convergent* to $f : X \to V$, if $\{f_n\}$ converges to f in the normed space $B(X,V)$, i.e. $f \in B(X,V)$ and

$$\forall_{\varepsilon>0}\ \exists_N\ \forall_{n\geq N}\ \forall_{x\in X}\quad \|f_n(x) - f(x)\| < \varepsilon.$$

Proposition 4.18. *Let X a set and V a Banach space. Then the vector space $B(X,V)$, equipped with the norm $\|\cdot\|_\infty$ given by (4.8), is a Banach space.*

Proof: The space $B(X,V)$ equipped with the norm $\|\cdot\|_\infty$ given by (4.8) is a normed space (see Problem 3 at the end of this chapter). Therefore, we need only to show that it is complete. Suppose that $\{f_n\}$, $f_n : X \to V$, is a Cauchy sequence in $B(X,V)$, i.e.

$$\forall_{\varepsilon>0}\ \exists_N\ \forall_{n,m\geq N}\quad \|f_n - f_m\|_\infty < \varepsilon. \tag{4.9}$$

In particular, it follows from (4.8) and (4.9) that for all $x \in X$ we have

$$\forall_{\varepsilon>0}\ \exists_N\ \forall_{n,m\geq N}\quad \|f_n(x) - f_m(x)\| < \varepsilon, \tag{4.10}$$

which means the sequence $\{f_n(x)\}$ is a Cauchy sequence in V. By assumption, V is a Banach space, i.e. it is complete, thus the sequence $\{f_n(x)\}$ converges. Let us denote its limit in V by $f(x)$. In this way we obtain that there is a mapping $f : X \to V$ satisfying

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for all} \quad x \in X.$$

By applying the limit (as $m \to \infty$) to the inequality (4.9), we obtain

$$\forall_{\varepsilon>0}\ \exists_N\ \forall_{n\geq N}\ \forall_{x\in X} \quad \lim_{m\to\infty} \|f_n(x) - f_m(x)\| = \|f_n(x) - \lim_{m\to\infty} f_m(x)\| = \|f_n(x) - f(x)\| \leq \varepsilon. \tag{4.11}$$

Now, we only need to prove that $f(X)$ is bounded. Let us fix $\varepsilon > 0$. Then it follows from the condition (4.11) that for some fixed $n \geq N$ we have

$$\forall_{x\in X}\ \|f_n(x) - f(x)\| \leq \varepsilon.$$

But, by the assumption $f_n(X)$ is bounded, i.e. there exists $M > 0$ such that

$$\forall_{x\in X}\ \ \|f_n(x)\| \leq M,$$

thus we have

$$\forall_{x\in X}\ \ \|f(x)\| \leq \|f(x) - f_n(x) + f_n(x)\| \leq \|f(x) - f_n(x)\| + \|f_n(x)\| \leq \varepsilon + M,$$

which means $f(X)$ is bounded. In this way, we showed that $f \in B(X,V)$ and $\|f_n - f\|_\infty \to 0$ as $n \to \infty$, i.e. the Cauchy sequence $\{f_n\}$ converges. Consequently, $B(X,V)$ is a Banach space. $\square$

Assume now that (X,d) is a metric space and define the subspace $BC(X,V)$ of the space $B(X,V)$ to be the set of all maps $f : X \to V$ which are continuous, i.e.

$$\forall_{x_o\in X}\ \forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x\in X}\ \ d(x_o, x) < \delta \implies \|f(x) - f(x_o)\| < \varepsilon,$$

We will show that, if V is a Banach space then $BC(X,V)$ is also a Banach space. For this purpose we need the following important result

Lemma 4.19. (Uniform Convergence Theorem) *Let (X,d) be a metric space and V a normed space. Suppose that $\{f_n\}$, $f_n : X \to V$, is a sequence of continuous functions uniformely convergent to a function $f : X \to V$, i.e.*

$$\forall_{\varepsilon>0}\ \exists_N\ \forall_{n\geq N}\ \forall_{x\in X}\ \ \|f_n(x) - f(x)\| < \varepsilon. \tag{4.12}$$

Then the function f is also continuous.

Proof: Let $x_o \in X$. We will show that the map f is continuous at x_o. First, by the condition (4.12) we have that

$$\forall_{\varepsilon>0}\ \exists_N\ \forall_{n\geq N}\ \forall_{x\in X}\ \ \|f_n(x) - f(x)\| < \frac{\varepsilon}{3}. \tag{4.13}$$

We can choose $n \geq N$ and consider the map $f_n : X \to V$. The map f_n is continuous at x_o, thus for a fixed $\varepsilon > 0$ we have

$$\exists_{\delta>0}\ \forall_{x\in X}\ \ d(x, x_o) < \delta \implies \|f_n(x) - f_n(x_o)\| < \frac{\varepsilon}{3}. \tag{4.14}$$

By applying (4.13) and (4.14) we obtain

$$
\begin{aligned}
\forall_{\varepsilon>0}\ \exists_N\ \forall_{n\ge N}\ \ d(x,x_o)<\delta \implies & \|f(x)-f(x_o)\| \\
&=\|f(x)-f_n(x)+f_n(x)-f_n(x_o)+f_n(x_o)-f(x_o)\| \\
&\le\|f(x)-f_n(x)\|+\|f_n(x)-f_n(x_o)\|+\|f_n(x_o)-f(x_o)\| \\
&<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.
\end{aligned}
$$

□

Proposition 4.20. *Let (X,d) be a metric space and V a Banach space. Then the space $BC(X,V)$ of all bounded continuous maps from X to V is a Banach space.*

Proof: Since we know that $BC(X,V)\subset B(X,V)$, and $B(X,V)$ is a Banach space, it is sufficient to show that for a Cauchy sequence $\{f_n\}$ in $B(X,V)$ of continuous maps, its limit f (which exists in $B(X,V)$, because $B(X,V)$ is a Banach space), is a continuous map. However, this is exactly the statement of Lemma 4.19 (the Uniform Convergence Theorem). □

Example 4.21. (a) Let us consider the space

$$C([0,1]) := \{f:[0,1]\to\mathbb{R}\ \ :\ \ f \quad \text{is continuous function}\}.$$

Then the space $C([0,1])$ equipped with the norm

$$\|f\|_\infty := \max\{|f(t)| : t\in[0,1]\},\qquad f\in C([0,1]),$$

is a Banach space. Indeed, it is sufficient to notice that this space coincide with $BC(X,V)$ with $X=[0,1]$ and $V=\mathbb{R}$.

(b) Let us introduce a different norm on the space $C([0,1])$. We define for $f\in C([0,1])$

$$\|f\|_1 := \int_0^1 |f(t)|dt. \tag{4.15}$$

It can be shown that the space $C([0,1])$ equipped with the norm $\|\cdot\|_1$ is a normed space which is **not a Banach space**. Indeed, we can show that it is not complete. Let us define a sequence $\{f_n\}$ in $C([0,1])$ by

$$f_n(t) := \begin{cases} 1 & \text{if } \ 0\le t\le \frac12-\frac1n, \\ -nt+\frac{n}{2} & \text{if } \ \frac12-\frac1n\le t\le\frac12, \\ 0 & \text{if } \ \frac12\le t\le 1, \end{cases}$$

where $t\in[0,1]$, $n=1,2,3,\dots$, (see Figure 4.21). We also define the function $f:[0,1]\to\mathbb{R}$ by

$$f(t)=\begin{cases} 1 & \text{if } \ 0\le t\le\frac12, \\ 0 & \text{if } \ \frac12<t\le 1. \end{cases}$$

Of course, the function f is discontinuous at $t=\frac12$, thus it is not in the space $C([0,1])$, however,

$$\|f_n-f\|_1 := \int_0^1 |f_n(t)-f(t)|dt = \int_{\frac12-\frac1n}^{\frac12}\left(-nt+\frac{n}{2}\right)dt = \frac{1}{2n} \longrightarrow 0 \qquad \text{as} \quad n\to\infty,$$

which means the Cauchy sequence $\{f_n\}$ **has no limit** in the space $C([0,1])$ with respect to the norm $\|\cdot\|_1$.

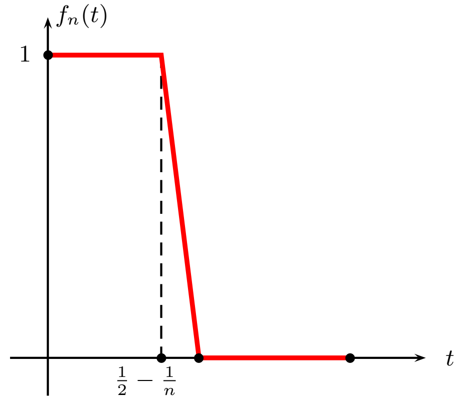


Fig. 4.1. Cauchy sequence $\{f_n\}$ in the space $C([0,1])$ equipped with the norm $\|\cdot\|_1$ (see (4.15)).

Example 4.22. Other examples of Banach spaces are the spaces of sequences. Let us put $X = \mathbb{N}$ and $V = \mathbb{R}$. Then the space $B(X,V) = B(\mathbb{N},\mathbb{R})$, which is commonly denoted by l^∞, is a Banach space. This space consists of all real bounded sequences $\{x_n\}$ such that

$$\|\{x_n\}\|_\infty := \sup\{|x_n| : n = 1, 2, \dots\} < \infty.$$

For $p \geq 1$ we can consider the space l^p of all real sequences $\{x_n\}$ such that

$$\|\{x_n\}\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} < \infty. \tag{4.16}$$

It can be verified that the function (4.16) satisfies the properties (N1)–(N4), i.e. it is a norm on the space l^p, and that the space l^p equipped with this norm is a Banach space.

4.4 Completion of Metric Spaces and Normed Spaces

Definition 4.23. Let (X,d) be a metric space. A *completion* of X is a metric space $(\widetilde{X}, wtd)$ together with a map $\iota : X \to \widetilde{X}$ satisfying the following properties:

(i) $(\widetilde{X}, \widetilde{d})$ is a complete metric space;
(ii) $\iota : X \to \widetilde{X}$ is an *isometry*, i.e. $\forall_{x,y\in X}\ d(x,y) = \widetilde{d}(\iota(x), \iota(y))$;
(iii) $\iota(X)$ is dense in $\widetilde{X}$, i.e. $\overline{\iota(X)} = \widetilde{X}$.

Theorem 4.24. (Completion of Metric Space) *Let (X,d) be a metric space. Then (X,d) has a completion $(\widetilde{X}, \widetilde{d})$.*

Proof: Put $\mathbb{E} := BC(X,\mathbb{R})$ equipped with the norm $\|\cdot\|_\infty$ given by (4.8). Then, by Proposition 4.20, $\mathbb{E}$ is a Banach space. Fix $x_o \in X$. For every $x \in X$ we define $f_x : X \to \mathbb{R}$ by

$$f_x(y) = d(x,y) - d(x_o,y), \quad y \in X.$$

since for y, $y' \in X$, we have

$$\begin{aligned}|f_x(y) - f_x(y')| &= |d(x,y) - d(x,y') + d(x_o,y') - d(x_o,y)| \\ &\le |d(x,y) - d(x,y')| + |d(x_o,y') - d(x_o,y)| \\ &\le d(y,y') + d(y',y) = 2d(y,y'),\end{aligned}$$

so f_x is Lipschitzian and therefore it is continuous. On the other hand, for every $y \in X$,

$$|f_x(y)| = |d(x,y) - d(x_o,y)| \le d(x,x_o), \quad \text{for all } \ y \in X,$$

thus f_x is also bounded. That means the function $\iota : X \to \mathbb{E}$ given by

$$\iota(x) := f_x : X \to \mathbb{R}, \quad x \in X,$$

is well defined. We will show that ι is an isometry. Indeed, for x, $x' \in X$ we have

$$\begin{aligned}\|f_x - f_{x'}\|_\infty &= \sup\{|f_x(y) - f_{x'}(y)| : y \in X\} \\ &= \sup\{|d(x,y) - d(x',y) + d(x_o,y) - d(x_o,y)| : y \in X\} \\ &= \sup\{|d(x,y) - d(x',y)| : y \in X\} \\ &\le d(x,x').\end{aligned}$$

Since for all $y \in X$

$$\|f_x - f_{x'}\|_\infty \ge |f_x(y) - f_{x'}(y)| = |d(x,y) - d(x',y)|,$$

thus for $y = x'$ we get

$$\|f_x - f_{x'}\|_\infty \ge d(x,x'),$$

which proves that

$$\|\iota(x) - \iota(x')\|_\infty = \|f_x - f_{x'}\|_\infty = d(x,x'), \quad \forall_{x,x' \in X},$$

i.e. $\iota : X \to \mathbb{E}$ is an isometry. Therefore, we can define $\widetilde{X} := \overline{\iota(X)} \subset \mathbb{E}$, equipped with the metric $\widetilde{d}(f,f') := \|f - f'\|_\infty$, $f, f' \in \widetilde{X}$. Since $\mathbb{E}$ is a Banach space and $\widetilde{X}$ is a closed set in $\mathbb{E}$, it follows that $\widetilde{X}$ is a complete metric space. □

Corollary 4.25. (Completion of Normed Spaces) *Let $(V, \|\cdot\|)$ be a normed space. Then there exists a Banach space $(V_o, \|\cdot\|_o)$, which is a completion of $(V, \|\cdot\|)$, i.e. there is a linear operator $\iota : V \to V_o$ such that*

(i) $\iota : V \to V_o$ is an isometry, i.e. $\forall_{v,u \in V}$ $\|v - u\| = \|\iota(v) - \iota(u)\|_o$;
(iii)$\iota(V)$ is dense linear subspace in V_o, i.e. $\overline{\iota(V)} = V_o$.

Proof: Let us recall the construction of $\iota : V \to V_o$ from the proof of Theorem 4.24: $\iota(v) := f_v : V \to \mathbb{R}$ is given by $\iota(v)(u) = f_v(u) := \|v - u\| - \|u\|$, $u \in V$. Since, V is a vector space, then on the image $\iota(V)$ one can also define the structure of a vector space (which is not necessarily a structure of a linear subspace in $\mathbb{E} := BC(V,\mathbb{R})$): for f, $f' \in \iota(V)$ we put $f + f' := \iota(v + v')$, and $\alpha f = \iota(\alpha v)$, where v, $v' \in V$ are such that $\iota(v) = f$ and $\iota(v') = f'$. Since that operations of vector addition and scalar multiplication are uniformly continuous, by using the result from Problem 5, they can be continuously extended on $V_o := \overline{\iota(V)}$, which implies that V_o admits a vector space structure with a norm $\|f\|_o := \widetilde{d}(f,0) = \|f\|_\infty$, $f \in V_o$. Clearly V_o equipped with the norm $\|\cdot\|_o$ is a Banach space. □

4.5 Bounded Linear Operators

Let us recall a definition of a linear operator between two vector spaces.

Definition 4.26. Let V and W be two vector spaces. A map $A : V \to W$ is called *linear* (or a *linear operator* from V to W) if it satisfies the following condition

$$A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 A(v_1) + \alpha_1 A(v_2), \quad v_1,\, v_2 \in V,\;\; \alpha_1,\, \alpha_2 \in \mathbb{R}.$$

In what follows in this subsection, we will assume that V and W are two normed spaces and we will consider the space $L(V, W)$ of all continuous linear operators from V to W.

Proposition 4.27. *Let V and W be two normed spaces and $A : V \to W$ a linear operator. Then the following conditions are equivalent*

(a) A is continuous;
(b) A is continuous at $v_o \in V$;
(c) A is continuous at 0;
(d) There exists a constant M such that

$$\forall_{v\in V} \;\; \|A(v)\| \le M \cdot \|v\|;$$

(e) The set $A(\overline{B}(0,1)) := \{A(v) : \|v\| \le 1\}$ is bounded in W;
(f) $\|A\| := \sup\{\|A(v)\| : \|v\| \le 1\} < \infty$.

Proof: Reacall that a linear operator $A : V \to W$ is continuous if for every $v_o \in V$ we have

$$\forall_{\varepsilon>0}\, \exists_{\delta>0}\, \forall_{v\in V} \;\; \|v - v_o\| < \delta \implies \|A(v) - A(v_o)\| < \varepsilon.$$

Since A is linear, $A(v) - A(v_o) = A(v - v_o)$, then by putting $u = v - v_o$ we obtain the following (simpler) condition for the continuity of the linear operator A

$$\forall_{\varepsilon>0}\, \exists_{\delta>0}\, \forall_{u\in V} \;\; \|u\| < \delta \implies \|A(u)\| < \varepsilon, \tag{4.17}$$

which implies that the equivalence between (a), (b) and (c).

In order to prove that the condition (4.17) implies (d), we notice that for every $v \in V$, $v \ne 0$, we have $u = \frac{v}{2\|v\|}\delta$ satisfies the condition $\|u\| = \frac{1}{2}\delta < \delta$, thus for all $v \in V$, $v \ne 0$, we have

$$\|A(u)\| < \varepsilon \iff \left\| A\left(\frac{v}{2\|v\|}\delta\right)\right\| < \varepsilon \iff \|A(v)\| < \frac{2\varepsilon}{\delta}\|v\|.$$

So it is possible to choose $M := \frac{2\varepsilon}{\delta}$. We will show that (d) imples (4.17). Assume that (d) is satisfied. Then

$$\forall_{\varepsilon>0}\, \exists_{\delta:=\frac{\varepsilon}{M}}\, \forall_{v\in V} \;\; \|v\| < \delta \implies \|A(v)\| \le M \cdot \|v\| < M \cdot \delta = M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

Of course, if (d) is satisfied, then we have that for every $v \in V$ with $\|v\| \le 1$ we have

$$\|A(v)\| \leq M \cdot \|v\| \leq M,$$

thus the set $A(\overline{B}(0,1))$ is bounded in W, so (e) is satisfied. On the other hand, if $A(B(0,1))$ is bounded in W, i.e.

$$\exists_{M>0} \, \forall_{v\in V} \quad \|v\| \leq 1 \implies \|A(v)\| \leq M,$$

then clearly we also have for $v \neq 0$

$$\left\| \frac{v}{\|v\|} \right\| = 1 \implies \left\| A\left(\frac{v}{\|v\|}\right) \right\| \leq M \iff \|A(v)\| \leq M\|v\|.$$

In this way we proved that (d) and (e) are equivalent. Notice that we have already proved that (d) implies (f). On the other hand, if (e) is satisfied, then

$$\forall_{v\in V} \quad v \neq 0 \implies \left\| A\left(\frac{v}{\|v\|}\right) \right\| \leq \|A\| \implies \|A(v)\| \leq \|A\|\|v\|,$$

thus by putting $M := \|A\|$ we get (d). □

Proposition 4.28. *Let V and W be two normed spaces. Then space $L(V,W)$ of continuous linear operators, equipped with the norm*

$$\|A\| := \sup\{\|A(v)\| : \|v\| \leq 1\}, \quad A \in L(V,W), \tag{4.18}$$

is an normed space. Moreover, if W is a Banach space, then $L(V,W)$ is also a Banach space.

Proof: The verification that the function $\|\cdot\|$ given by (4.18) satisfies the conditions (N1)–(N4) we leave as an exercise. We only need to show that $L(V,W)$ is complete when W is a Banach space. Let $\{A_n\}$ be a Cauchy sequence in $L(V,W)$. Notice that for every $A \in L(V,W)$ we can define a function $f : \overline{B}(0,1) \to W$, by $f(v) = A(v)$, $v \in \overline{B}(0,1)$. Then $\|A\| = \|f\|_\infty$, where $f \in BC(X,W)$, with $X = \overline{B}(0,1)$. Therefore, the sequence $f_n \in BC(X,W)$, $f_n(v) = A_n(v)$, $v \in X$, is a Cauchy sequence in $BC(X,W)$. Since by Proposition 4.20 the space $BC(X,W)$ is a Banch space, it follows that the sequence $\{f_n\}$ converges to continuous map f. It is easy to verify (by using the standard properties of limits) that f satisfies the linearity condition, thus, there is a linear coninuous operator $A : V \to W$ such that $f(v) = A(v)$ for $v \in X$. □

Proposition 4.29. *Let V, W and Y be Banach spaces and $A : V \to W$, $B : W \to Y$ two bounded linear operators. Then we have*

$$\|BA\| \leq \|B\|\|A\|.$$

Proof: Since A and B are continuous, the operator BA is bounded and $\|BA\| = \inf\{M > 0 : \forall_{v\in V} \; \|(BA)(v)\| \leq M\|v\|\}$. On the other hand, A and B are also bounded, thus for all $v \in V$ we have

$$\|(BA)(v)\| = \|B(A(v))\| \leq \|B\|\|A(v)\| \leq \|B\|\|A\|\|v\|,$$

and thus the number $M = \|B\|\|A\|$ satisfies the condition $\forall_{v\in V} \; \|(BA)(v)\| \leq M\|v\|$, so

$$\|BA\| \leq \|B\|\|A\|.$$

□

Remark 4.30. Since every continuous linear operator $A : V \to W$ satisfies the condition

$$\forall_{v \in V} \quad \|A(v)\| \le \|A\| \, \|v\|,$$

A is also called a *bounded* linear operator from V to W. In fact it is more common to say that the operator A is bounded rather than A is continuous.

Remark 4.31. In the case $V := \mathbb{R}^n$ and $W := \mathbb{R}^m$ it is clear that every linear operator $A : V \to W$ is bounded (continuous). However, this is not true for general normed spaces. Consider again the space $V := C([0,1])$ equipped with the norm $\|\cdot\|_1$ (see Example 4.21,(b)) and a linear operator $A : V \to \mathbb{R}$ defined by $A(f) := f(0)$, where $f : [0,1] \to \mathbb{R}$ is a continuous function. We claim that A is not a bounded operator. Indeed, define the sequence of functions $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(t) = \begin{cases} -n^2 t + n & \text{if } \ 0 \le t \le \frac{1}{n}, \\ 0 & \text{if } \ \frac{1}{n} < t \le 1. \end{cases}$$

The functions f_n are continuous and

$$\|f_n\|_1 = \int_0^1 |f_n(t)| dt = \int_0^{\frac{1}{n}} (-n^2 t + n) dt = \frac{1}{2},$$

thus $f_n \in X := \overline{B}(0,1) \subset C([0,1])$. However, $A(f_n) = f_n(0) = n \longrightarrow \infty$ as $n \to \infty$, which means the set $A(X)$ is unbounded in $\mathbb{R}$.

4.6 Invertibility of Linear Operators

Let V and W be Banach spaces. A bounded linear operator $A : V \to W$ is called *invertible* if it is one-to-one, onto, and its inverse $A^{-1} : W \to V$ is continuous. An invertible linear operator $A : V \to W$ is also called an *isomorphism.* The set of all invertible operators $A : V \to V$ will be denoted by $GL(V)$. Notice that $GL(V)$ is a subset of the Banach space $L(V) := L(V,V)$ equipped with the operator norm $\|A\| = \sup\{\|Av\| : \|v\| \le 1\}$.

Proposition 4.32. *Let V be a Banach space and $A : V \to V$ a linear operator. If $\|\mathrm{Id} - A\| < 1$ (where $\mathrm{Id} : V \to V$ denotes the identity operator), then A is invertible (i.e $A \in GL(V)$, and*

$$\|A^{-1}\| \le \frac{1}{1 - \|\mathrm{Id} - A\|}.$$

Proof: Put $S := \mathrm{Id} - A$, and let $\|S\| = \|\mathrm{Id} - A\| =: \rho < 1$. Let us consider the following sequence of linear operators $\{B_n\} : V \to V$, $n = 1, 2, \dots$, defined by

$$B_n := \sum_{k=0}^{n} S^k, \quad S^0 = \mathrm{Id}\,.$$

Notice that the sequence $B_n = \displaystyle\sum_{k=0}^{n} S^k$ is a Cauchy sequence. Indeed, for all $m,n \in \mathbb{N}$

$$\|B_{n+m} - B_n\| = \|\sum_{k=n+1}^{n+m} S^k\| \le \sum_{k=n+1}^{n+m} \|S\|^k$$
$$< \rho^{n+1}\frac{1}{1-\rho} \longrightarrow 0 \text{ as } n \to \infty,$$

and since the space $L(V)$ is complete, the sequence $\{B_n\}$ converges, i.e.

$$\lim_{n\to\infty} B_n =: B = \sum_{k=0}^{\infty} S^k,$$

is a well defined bounded linear operator. We will show that this operator is the inverse of A. Indeed,

$$AB = (\mathrm{Id} - S)\sum_{k=0}^{\infty} S^k = \sum_{k=0}^{\infty} S^k - \sum_{k=1}^{\infty} S^k = S^0 = \mathrm{Id}\,.$$

In addition, notice that

$$\|A^{-1}\| = \|B\| \le \sum_{k=0}^{\infty} \|S\|^k = \sum_{k=0}^{\infty} \rho^k = \frac{1}{1-\rho} = \frac{1}{1 - \|\mathrm{Id} - A\|}.$$

□

Corollary 4.33. *Let V be a Banach space and $S : V \to V$ a bounded linear operator such that $\|S\| < 1$. Then $\mathrm{Id} - S$ is invertible and*

$$(\mathrm{Id} - S)^{-1} = \sum_{k=0}^{\infty} S^k, \quad \|\mathrm{Id} - S\| \le \frac{1}{1 - \|S\|}.$$

Proposition 4.34. *Let V be a Banach space and $A \in GL(V)$. Then for every $B \in L(V)$ such that*

$$\|A - B\| < \frac{1}{\|A^{-1}\|}$$

we have that B is invertible (i.e. $B \in GL(V)$) and

$$\|B^{-1}\| \le \frac{\|A^{-1}\|}{1 - \|\mathrm{Id} - BA^{-1}\|}.$$

Proof: Since A is invertible, it is enough to show that $C = B \circ A^{-1}$ is invertible. Indeed, since $B = C \circ A$, if C is invertible then $A^{-1}C^{-1}$ is the inverse of B. From the inequality

$$\|\mathrm{Id} - C\| = \|\mathrm{Id} - BA^{-1}\| = \|(A - B)\circ A^{-1}\| \le \|A - B\|\|A^{-1}\| < 1$$

and Proposition 4.32, it follows that C is invertible. Since $B^{-1} = A^{-1}C^{-1}$, we get the norm estimate

$$\|B^{-1}\| \le \|A^{-1}\|\,\|C^{-1}\| \le \frac{\|A^{-1}\|}{1 - \|\mathrm{Id} - C\|} = \frac{\|A^{-1}\|}{1 - \|\mathrm{Id} - BA^{-1}\|}$$

□

Proposition 4.35. *Let V be a Banach space. Then the set of all invertible operators $GL(V)$ is open in $L(V)$.*

Proof: Let $A \in GL(V)$. By Proposition 4.34, we have that the ball $B\left(A, \frac{1}{2\|A^{-1}\|}\right)$ in the space $L(V)$ is contained in the set $GL(V,V)$, which implies that the set $GL(V)$ is open in $L(V)$. □

4.7 Problems

1. Show that all the norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_p$, $p>1$, and $\|\cdot\|_\infty$, on the space $\mathbb{R}^n$ are equivalent.

2. Let V and W be two normed spaces. Show that the norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_p$, $p>1$, and $\|\cdot\|_\infty$, on the space $V\times W$ given by

(a) $\|(v,w)\|_1 := \|v\|+\|w\|$, where $(v,w)\in V\times W$;
(b) $\|(v,w)\|_2 := \left(\|v\|^2+\|w\|^2\right)^{\frac{1}{2}}$, where $(v,w)\in V\times W$;
(c) $\|(v,w)\|_p := (\|v\|^p+\|w\|^p)^{\frac{1}{p}}$, where $p>1$ and $(v,w)\in V\times W$;
(c) $\|(v,w)\|_\infty := \max\{\|v\|,\|w\|\}$, where $(v,w)\in V\times W$;

are equivalent.

3. Verify that the function $\|\cdot\|_\infty : B(X,V)\to\mathbb{R}$, given by (4.8), satisfies the properties (N1)–(N4) of a norm.

4. Show that if $(\widetilde{X},\widetilde{d})$ and $(\widehat{X},\widehat{,}d)$ are two completions of a given metric space (X,d) then they are *isometric*, i.e. there exists a bijective function $f:\widetilde{X}\to\widehat{X}$ such that $\widehat{d}(f(x),f(y))=\widetilde{d}(x,y)$ for all $x,y\in\widetilde{X}$.

5. Let A be a dense subset in a metric space (X,d_X), (Y,d_Y) be a complete metric space and $f:A\to Y$ a uniformly continuous function. Show that

(a) If $\{x_n\}$ is a Cauchy sequence in A then $\{f(x_n)\}$ is a Cauchy sequence in Y;
(b) There exists a unique continuous function $\widetilde{f}:X\to Y$ such that $\widetilde{f}(x)=f(x)$ for all $x\in A$.

Part II

FUNCTIONS OF SINGLE VARIABLE: DIFFERENTIATION

5

Real Functions

5.1 Properties of Real Functions

We would like to give particular attention to the notion of an *inverse function*, in the case $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}$, is a one-to-one function (i.e. $f$is a bijection).

Let us recall the following definition.

Definition 5.1. We say that a real function : $X \to \mathbb{R}$, $X \subset \mathbb{R}$, is:

(i) *increasing* if $\forall_{x_1,x_2\in\mathbb{R}}\ \ x_1 < x_2 \ \ \Rightarrow \ \ f(x_1) < f(x_2)$;
(ii) *decreasing* if $\forall_{x_1,x_2\in\mathbb{R}}\ \ x_1 < x_2 \ \ \Rightarrow \ \ f(x_1) > f(x_2)$;
(iii)*non-decreasing* if $\forall_{x_1,x_2\in\mathbb{R}}\ \ x_1 < x_2 \ \ \Rightarrow \ \ f(x_1) \leq f(x_2)$;
(iv)*non-increasing* if $\forall_{x_1,x_2\in\mathbb{R}}\ \ x_1 < x_2 \ \ \Rightarrow \ \ f(x_1) \geq f(x_2)$;
(v) *monotonic* if it is non-decreasing or non-increasing;
(vi)*strictly monotonic* if it is increasing or decreasing.

It is clear that a strictly monotonic function is one-to-one.

Definition 5.2. Let $f : X \to \mathbb{R}$, $X \subset \mathbb{R}$, be a one-to-one function. Then its *inverse function* $f^{-1} : f(X) \to X$ is defined by

$$f^{-1}(y) = x \Longleftrightarrow f(x) = y, \tag{5.1}$$

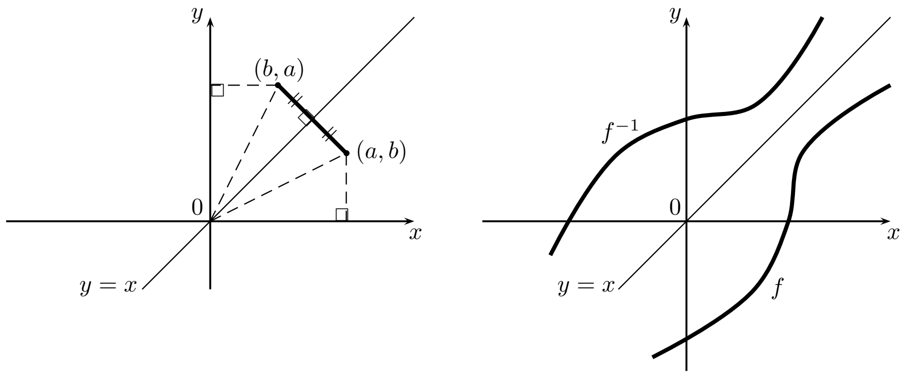
for any $y \in f(X)$, i.e. the domain of f^{-1} is $f(X)$.

Notice that the definition of the inverse function f^{-1} implies the following properties called the *cancelation rules*:

$$\begin{aligned} f^{-1}(f(x)) = x \quad &\text{for every} \quad x \in X; \\ f(f^{-1}(y) = y \quad &\text{for every} \quad y \in f(X). \end{aligned}$$

Notice that for an exponential function $f(x) = a^x$, $a > 0$, $a \neq 0$, the logarithmic function $g(y) = \log_a y$ is the inverse of f.

Since the definition of the inverse function is based on interchanging x and y, it suggest that the graph of f^{-1} can be obtained by reflecting the graph of f about the line $y = x$. Indeed, since $f(a) = b$ if and only if $f^{-1}(b) = a$, thus (a, b) belongs to the graph of f if and only if (b, a) belongs to the graph of f^{-1}.



The following result is known in Calculus as the *Intermediate Value Theorem* (or shortly IVT):

Theorem 5.3. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) \neq f(b)$. Then for every number α strictly between $f(a)$ and $f(b)$ there exists $c \in (a, b)$ such that $f(c) = \alpha$.*

Proof: Since $[a, b]$ is connected, the statement of Theorem 5.3 is a direct consequence of Theorem 3.104.

Nevertheless, we present below another proof, which could be regarded as a *numerical algorithm* for solving the equation $f(x) = \alpha$ for $x \in [a, b]$. Suppose, for definiteness that $f(a) < f(b)$ (the case $f(a) > f(b)$ can be considered in a similar way), and let α be a number such that $f(a) < \alpha < f(b)$. Define the function $\varphi(x) = f(x) - \alpha$. Then $\varphi(a) < 0$ and $\varphi(b) > 0$. If $\varphi(\frac{a+b}{2}) = 0$, then we put $c = \frac{a+b}{2}$ and the conclusion follows. Suppose that $\varphi(\frac{a+b}{2}) \neq 0$, then we consider the subinterval $[a_1, b_1]$ of $[a, b]$, where $a_1 = \frac{a+b}{2}$ and $b_1 = b$ if $\varphi(\frac{a+b}{2}) < 0$, or $a_1 = a$ and $b_1 = \frac{a+b}{2}$ if $\varphi(\frac{a+b}{2}) > 0$. Again, we have that $\varphi(a_1) < 0$ and $\varphi(b_1) > 0$, so we can repeat the above construction. We put $c = \frac{a_1+b_1}{2}$ if $\varphi(\frac{a_1+b_1}{2}) = 0$, or we consider the subinterval $[a_2, b_2]$, where $a_2 = \frac{a_1+b_1}{2}$ and $b_2 = b_1$ if $\varphi(\frac{a_1+b_1}{2}) < 0$, or $a_2 = a_1$ and $b_2 = \frac{a_1+b_1}{2}$ if $\varphi(\frac{a_1+b_1}{2}) > 0$. By induction, assume that we have constructed the interval $[a_n, b_n]$, such that $b_n - a_n = \frac{b-a}{2^n}$. Then, if $\varphi(\frac{a_n+b_n}{2}) \neq 0$, we define

$$\text{if} \quad \varphi(\frac{a_n+b_n}{2}) > 0 \quad \text{then} \quad \begin{cases} a_{n+1} = a_n \\ b_{n+1} = \frac{a_n+b_n}{2} \end{cases}$$

$$\text{if} \quad \varphi(\frac{a_n+b_n}{2}) < 0 \quad \text{then} \quad \begin{cases} a_{n+1} = \frac{a_n+b_n}{2} \\ b_{n+1} = b_n. \end{cases}$$

In this way, we obtained an infinite sequence of intervals

$$[a, b] \supset [a_1, b_1] \supset \cdots \supset [a_n, b_n] \supset \dots,$$

such that $\varphi(a_n) < 0$ and $\varphi(b_n) > 0$ for $k = 1, 2, \dots$. Since for $k,m \geq N$ we have

$$|a_m - a_k| \le \frac{b-a}{2^{N-1}}, \quad |b_m - b_k| \le \frac{b-a}{2^{N-1}} \quad \text{and} \quad b_n - a_n = \frac{b-a}{2^n},$$

it follows that $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences such that there exists $c \in [a,b]$ satisfying

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c.$$

On he other hand, by the continuity of $\varphi(x)$ we have that

$$0 \le \lim_{n\to\infty} \varphi(b_n) = \varphi(c) = \lim_{n\to\infty} \varphi(a_n) \le 0,$$

i.e. $\varphi(c) = 0$, and the conclusion follows. □

Theorem 5.4. *Let I denotes an interval of $\mathbb{R}$ (finite or infinite) and let $f : I \to \mathbb{R}$ be a continuous function. If $f(x)$ is a one-to-one function, then it is strictly monotone.*

Proof: We can assume, without loss of generality, that $I = [a,b]$ (notice that if a function $f(x)$ is strictly monotone on every finite closed subinterval of I, then it is strictly monotone). Suppose that $f(a) < f(b)$ (the proof for the other case $f(b) > f(a)$ is the same). We will show that $f(x)$ has to be an increasing function. Indeed, suppose that contrary to our claim, the function $f(x)$ is not increasing. Then there exist c, $d \in [a,b]$ such that $c < d$ and $f(c) > f(d)$. If $f(a) > f(c)$ then $f(d) < f(b)$. Put $\alpha_1 = \min(f(c), f(b))$. By Theorem 5.3, for every β such that $f(d) < \beta < \alpha_1$ there exist $x_1 \in (c,d)$ and $x_2 \in (d,b)$ such that $f(x_1) = f(x_2) = \beta$. But this is a contradiction with the assumption that $f(x)$ is one-to-one. Suppose therefore, that $f(a) < f(c)$. Let $\alpha_2 = \max(f(a), f(d))$. Again, by Theorem 5.3, for every β such that $\alpha_2 < \beta < f(c)$ there exist $x_1 \in (a,c)$ and $x_2 \in (c,d)$ such that $f(x_1) = f(x_2) = \beta$, and again we obtain the contradiction. Consequently, the function $f(x)$ must be increasing. □

In the case of functions of real variable, it is convenient to introduce the notions of the one-side limits:

Definition 5.5. Let $X \subset \mathbb{R}$ and $a \in X$. We say that the point a is a *left-hand limit point* of X if $\forall_{\varepsilon>0}\, \exists_{x\in X}\ \ 0 < a - x < \varepsilon$. Similarly, we say that the point a is a *right-hand limit point* of X if and only if $\forall_{\varepsilon>0}\, \exists_{x\in X}\ \ 0 < x - a < \varepsilon$.
Let $f : X \to \mathbb{R}$, where $A \subset \mathbb{R}$, be a function.

(i) Let a be a left-hand limit point of X. We say that the *left-hand limit* of f as x approaches a is $b \in \mathbb{R}$, and we write $\lim_{x\to a^-} f(x) = b$, if and only if

$$\forall_{\varepsilon>0}\, \exists_{\delta>0}\, \forall_{x\in X}\ \ 0 < a - x < \delta \ \Rightarrow\ |f(x) - b| < \varepsilon;$$

(ii) Let a be a right-hand limit point of X. We say that the *right-hand limit* of f as x approaches a is b, and we write $\lim_{x\to a^+} f(x) = b$, if and only if

$$\forall_{\varepsilon>0}\, \exists_{\delta>0}\, \forall_{x\in X}\ \ 0 < x - a < \delta \ \Rightarrow\ |f(x) - b| < \varepsilon;$$

Theorem 5.6. *Let $f : X \to \mathbb{R}$, $X \subset \mathbb{R}$, be a real function and a a limit point such that a is also a right-hand and left-hand limit point of X as well. Then we have that $\lim\limits_{x\to a} f(x) = b$ if and only if $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = b$.*

Proof: It is clear that if $\lim_{x\to a} f(x) = b$ then both limits $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are equal to b. Conversely, suppose that $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = b$ and assume that $\varepsilon > 0$ is an arbitrary number. Since $\lim_{x\to a^-} f(x) = b$, thus

$$\exists_{\delta^->0}\,\forall_{x\in X}\;\; 0 < a - x < \delta^- \;\;\Rightarrow\;\; |f(x) - b)| < \varepsilon. \tag{5.2}$$

On the other hand,

$$\exists_{\delta^+>0}\,\forall_{x\in X}\; 0 < x - a < \delta^+ \;\;\Rightarrow\;\; |f(x) - b| < \varepsilon. \tag{5.3}$$

Put $\delta = \min(\delta^+, \delta^-)$. Then (by (5.2) and (5.3))

$$\forall_{x\in X}\;\; 0 < |x - a| < \delta \;\;\Rightarrow\;\; |f(x) - b| < \varepsilon, \tag{5.4}$$

and it follows that $\lim_{x\to a} f(x) = b$. □

We will need the following important Lemma

Lemma 5.7. (Monotonic Convergence Theorem for Sequences) *Let $\{a_n\}$ be a bounded monotonic sequence (i.e. either nondecreasing or non-increasing) in $\mathbb{R}$. Then $\{a_n\}$ converges, i.e. there exists $\alpha \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = \alpha$.*

Proof: Assume for example that $\{a_n\}$ is a non-decreasing sequence, bounded from above (so clearly it is bounded). Then the set $\mathcal{S} = \{a_n : n \in \mathbb{N}\}$ is bounded and non-empty. It follows from the (LUB)-axiom that there exists $\alpha = \sup \mathcal{S}$. We claim that $\lim_{n\to\infty} a_n = \alpha$. Indeed, we have $\forall_n\; a_n \le \alpha$, and assume that $\varepsilon > 0$ is an arbitrary number. It follows from the definition of $\sup \mathcal{S}$ that $\alpha - \varepsilon$ is not an upper bound of $\mathcal{S}$, therefore there exists an index N such that $a_N > \alpha - \varepsilon$. On the other hand, since $\{a_n\}$ is non-decreasing, $\forall_{n\ge N}\;\; a_n \ge a_N$, so $a_n > \alpha - \varepsilon$. Consequently we have the following inequalities

$$\forall_{n\ge N}\;\; \alpha - \varepsilon < a_N \le a_n \le \alpha,$$

thus $-\varepsilon < a_n - \alpha \le 0 < \varepsilon$, so we proved that

$$\forall_{\varepsilon>0}\,\exists_{N>0}\,\forall_{n\ge N}\;\; |a_n - \alpha| < \varepsilon.$$

□

Lemma 5.8. *Let $f : (a, b) \to \mathbb{R}$ be a monotone function. Then for every $x_o \in (a, b)$ the one-side limits $\lim_{x\to x_o^+} f(x)$ and $\lim_{x\to x_o^-} f(x)$ exist.*

Proof: Assume for example that $f(x)$ is increasing and since for $a < \alpha < x_o < \beta < b$, we have $f(\alpha) \le f(x) \le f(\beta)$ for all $x \in [\alpha, \beta]$, so there exists $M > 0$ such that $|f(x)| < M$ for all $x \in [\alpha, \beta]$. Let $\{x_n\}$ be a sequence such that $x_1 < x_2 < \dots < x_n < \dots$ and $\lim_{n\to\infty} x_n = x_o$. Then the sequence $\{f(x_n)\}$ is non-decreasing and bounded (i.e. $|f(x_n)| < M$ for sufficiently large n), thus the limit (by Lemma 5.7) $\lim_{n\to\infty} f(x_n) = \gamma$ exists. We will show that $\lim_{x\to x_o^-} f(x) = \gamma$. Let $\varepsilon > 0$ be an arbitrary number. Since the sequence $\{f(x_n)\}$ converges to γ, there exists an N such that $\gamma - f(x_n) < \varepsilon$ for all $n \ge N$. Put $\delta = x_o - x_N$. Thus, if $0 < x_o - x < \delta$ then $x_o > x > x_N$, so $\gamma - f(x) \le \gamma - f(x_N) < \varepsilon$ and consequently $\lim_{x\to x_o^-} f(x) = \gamma$. The existence of the limit $\lim_{x\to x_o^+} f(x)$ can be proved in a similar way. □

Proposition 5.9. *Let $f : [a,b] \to \mathbb{R}$ be a continuous strictly monotone function and let $A = \min\{f(x) : x \in [a,b]\}$ and $B = \max\{f(x) : x \in [a,b]\}$. Then $f([a,b]) = [A,B]$ and the inverse function $f^{-1} : [A,B] \to \mathbb{R}$ is also strictly monotone and continuous.*

Proof: Assume for example that $f(x)$ is increasing. It follows from the cancellation equation that $f^{-1}(x)$ is also increasing. We need to show that $f^{-1}(x)$ is continuous. Since f is continuous, it maps connected sets, which by Corollary **??** are intervals, onto connected sets (intervals). Since f is strictly monotonic, it maps an open interval in $[a,b]$ onto an open interval in $[A,B]$. This implies that f is an open map, i.e. it maps open in $[a,b]$ sets onto open in $[A,B]$ sets. But is exactly what we need to show, that for any open set U in $[a,b]$, the set $(f^{-1})^{-1}(U) = f(U)$ is open, i.e. f^{-1} is continuous. □

As an immediate corollary of the previous results we obtain the following important Theorem:

Theorem 5.10. *Let I denote a finite or infinite interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a one-to-one continuous function. Then f^{-1} is also continuous.*

5.2 Elementary Functions

5.2.1 The Number e

In this section we will use limits to construct a new number e called the *Euler's number.* Leonard Euler (1707-1783) a Swiss mathematician, physicist and astronomer, was the first to take an interest in the number e and it was also him who introduced the symbol e.

We consider the following sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n, \quad N \in \mathbb{N}.$$

and will show that $\{a_n\}$ is an increasing and bounded sequence.

By the binomial formula

$$\begin{aligned}
a_n = \left(1+\frac{1}{n}\right)^n &= 1 + n\frac{1}{n} + \frac{n(n-1)}{1\cdot 2}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\frac{1}{n^3} + \\
&+ \dots + \frac{n(n-1)\dots(n-k+1)}{1\cdot 2\cdot\dots\cdot k}\cdot\frac{1}{n^k} + \dots + \frac{n(n-1)\dots(n-n+1)}{1\cdot 2\cdot\dots\cdot n}\cdot\frac{1}{n^n} \\
&= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots + \frac{1}{k!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{k-1}{n}\right) \\
&+ \dots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{n-1}{n}\right).
\end{aligned} \tag{5.5}$$

Notice that, a_{n+1} can be represented by a similar sum which differs from the sum representing a_n only by the fact that n is replaced by $n+1$ (which implies that each of the terms in this sum (5.5) will increase) and that there is one additional, the $n+2$-term in the sum. That means, that a_{n+1} is larger than a_n. Consequently we have

$$a_{n+1} > a_n, \quad \text{for all} \quad n = 1, 2, 3, \dots.$$

We will show now that the sequence $\{a_n\}$ is bounded from above. For this purpose notice that by dropping all the fractions standing in brackets in the formula (5.5) we obtain the following inequality:

$$a_n < 2 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.$$

Since $2^{n-1} < n!$ (why?), it follows that

$$a_n < 2 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}. \tag{5.6}$$

Put $x_n = 1 + \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}$. It is clear that $\{x_n\}$ is increasing and the properties of limits, $\lim_{n\to\infty} x_n = 1 + \frac{1}{1-\frac{1}{2}} = 3$. Therefore, it follows from (5.6) that $a_n < 3$ for all $n = 1, 2, 3, \dots$. Consequently, by Lemma 5.7, the sequence $\{a_n\}$ converges. We denote the limit of this sequence by the letter e, i.e.

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n. \tag{5.7}$$

Remark 5.11. Notice that for $n > 1$, by Bernoulli's inequality

$$\left(\frac{n^2}{n^2-1}\right)^n = \left(1 + \frac{1}{n^2-1}\right)^n \ge 1 + \frac{n}{n^2-1} > 1 + \frac{1}{n}.$$

Therefore, for $n \ge 1$, we have

$$\left(\frac{(n+1)^2}{n(n+2)}\right)^{n+1} > \frac{n+2}{n+1} \quad \iff \quad \left(\frac{n+1}{n}\right)^{n+1} > \left(\frac{n+1}{n+1}\right)^{n+2}.$$

The last inequality means that the sequence $c_n := \left(1 + \frac{1}{n}\right)^{n+1}$ is decreasing. On the other hand

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = e,$$

Thus, for all $n \ge 1$, we have the following inequalities

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}. \tag{5.8}$$

We are also going to prove that the number e is irrational but for this purpose we will need some additional properties of e. By using the formula (5.8) we can get an approximation of e, which is for example $e \approx 2.71828182859045$.

Definition 5.12. The logarithm to the base e is called the *natural logarithm* and we will denote it by $\ln \gamma = \log_e \gamma$.

Now, we will establish another formula, which can be used for the computations involving the number e. Since, by formula (5.5) we have

$$a_n > 2 + \tfrac{1}{2!}(1-\tfrac{1}{n}) + \tfrac{1}{3!}(1-\tfrac{1}{n})(1-\tfrac{2}{n}) + \dots + \tfrac{1}{k!}(1-\tfrac{1}{n})\dots(1-\tfrac{k-1}{n}),$$

for all $n \le k$, we obtain that the sequence $b_n = 2 + \dfrac{1}{2!} + \dfrac{1}{3!} + \dots \dfrac{1}{n!} = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}$, $n = 1, 2, \dots$, satisfies the inequalities

$$a_n \le b_n \le e \quad \text{for all} \quad n = 1, 2, 3, \dots$$

Therefore, by the Squeeze Property we also have

$$e = \lim_{n\to\infty} b_n = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!}.$$

Since the limit $\displaystyle\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!}$ exists , we can write the following formula for the number e

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

Notice that

$$\begin{aligned}
b_{n+m} - b_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots + \frac{1}{(n+m)!} \\
&= \frac{1}{(n+1)!}\left\{1 + \tfrac{1}{n+2} + \tfrac{1}{(n+2)(n+3)} + \dots + \tfrac{1}{(n+2)(n+3)\dots(n+m)}\right\} \\
&< \frac{1}{(n+1)!}\left\{1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \dots \frac{1}{(n+2)^{m-1}}\right\} \\
&= \frac{1}{(n+1)!}\left\{\frac{1-\left(\frac{1}{(n+2)^m}\right)}{1-\frac{1}{n+2}}\right\} \\
&< \frac{1}{(n+1)!}\frac{n+2}{n+1} \qquad\qquad \left(\text{since } \frac{1-\left(\frac{1}{(n+2)^m}\right)}{1-\frac{1}{n+2}} < \frac{1}{1-\frac{1}{n+2}}\right) \\
&< \frac{1}{n!}\frac{1}{n} \qquad\qquad \left(\text{since } \tfrac{n+2}{(n+1)^2} < \tfrac{1}{n}\right).
\end{aligned}$$

Consequently, we obtain that

$$0 < e - b_n < \frac{1}{n!n},$$

and we can use this formula to estimate the approximation error when using the sequence $\{b_n\}$ to approximate the value of e.

Assume that n is a fixed natural number and denote by A the value $(e-b)n!n$, i.e. we have

$$e = b_n + \frac{A}{n!n} = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \frac{A}{n!n}. \tag{5.9}$$

Since, by (5.8) we have that $0 < e - b_n < \frac{1}{n!n}$, it follows that $\frac{A}{n!n} < \frac{1}{n!n}$, thus $A < 1$.

Proposition 5.13. *The number e is irrational.*

Proof: Suppose that e is rational, i.e. we can represent e as a fraction $\frac{m}{n}$, where m and n are natural numbers. By (5.9), we should have

$$1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}+\frac{A}{n!n}=\frac{m}{n}. \tag{5.10}$$

Notice, that we took as the denominator of the fraction $\frac{m}{n}$ the integer n which is the subindex of the term b_n. By multiplying (5.10) by $n!$ we get

$$n!+n!+3\cdot 4\cdot\ldots n+\cdots+\frac{A}{n}=m(n-1)!. \tag{5.11}$$

It is clear that the number standing on the left hand side of (5.11) is not an integer, but the number standing on the right hand side of (5.11) is an integer, so we obtain a contradiction. □

Lemma 5.14. *Let n be a natural number. Then we have the following inequality*

$$\sqrt[n]{n!}\geq\frac{n+1}{e}. \tag{5.12}$$

Proof: We rewrite the inequality (5.12) as

$$n!\geq\left(\frac{n+1}{e}\right)^n \tag{5.13}$$

and we apply mathematical induction to prove (5.13). Notice that for $n=1$ we get $1\geq\frac{2}{e}$, so (5.13) is true for $n=1$. Assume now that (5.13) is true for a certain $n\geq 1$. Then by the induction assumption and (5.8) we have

$$\begin{aligned}(n+1)! = n!(n+1) &\geq \left(\frac{n+1}{e}\right)^n (n+1)\\ &= \left(\frac{n+2}{e}\right)^{n+1} e\left(\frac{n+1}{n+2}\right)^{n+1}\\ &= \left(\frac{n+2}{e}\right)^{n+1}\frac{e}{\left(1+\frac{1}{n+1}\right)^{n+1}}\geq\left(\frac{n+2}{e}\right)^{n+1}.,\end{aligned}$$

so (5.13) is true for $n+1$, and by mathematical induction, it is true for all $n\in\mathbb{N}$. □

5.2.2 Basic Elementary Functions

There is a large class of real functions, called *elementary functions*, which basically consists of all the functions that can be written down by a single analytic formula (notice that a function defined piecewise is using two or more analytic formulas, therefore, in general, such functions are not elementary). Among the elementary functions there are a few categories of functions that should be discussed separately.

I. Polynomials and Rational Functions.

A *polynomial* P is a function of the type

$$P(x) = a_0 + a_1x^1 + a_2x^2 + \cdots + a_nx^n, \quad a_n \neq 0$$

where a_0, a_1, a_2, $\ldots$, a_n are real constants. The number n is called the *degree* of the polynomial P and is denoted by $\deg P$. It is clear that the domain of a polynomial is $\mathbb{R}$. A *rational* function $R(x)$ is the quotient of two polynomials, i.e. $R(x) = \frac{P(x)}{Q(x)}$, where P and Q are two polynomials. There is an analogy between rational functions and fractions. For example, we say that a rational function $R = \frac{P}{Q}$ with $n = \deg P$ and $m = \deg Q$ is a *proper* rational fraction if $\frac{n}{m}$ is a proper fraction. The domain of $R = \frac{P}{Q}$ is the set of all real numbers x such that $Q(x) \neq 0$.

II. Power Functions.

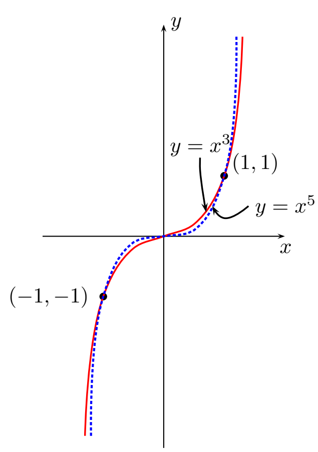
A *power* function is a function of the type

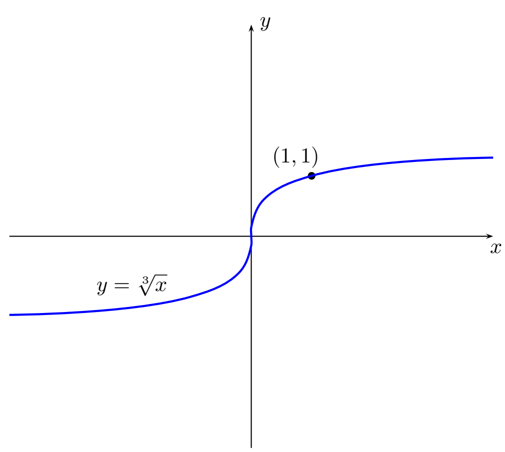
$$f(x) = x^\alpha, \text{ where } \alpha \in \mathbb{R}, \quad x > 0.$$

In the case where the exponent α is a rational number, i.e. $\alpha = \frac{m}{n}$, m, $n \in \mathbb{Z}$, $n \neq 0$, f can be written as a *root* function using the following expression

$$f(x) = x^{\frac{m}{n}} = \sqrt[n]{x^m}.$$

If n is an odd integer, the domain of f can be extended to negative numbers (including $x = 0$ if $m \geq 0$).



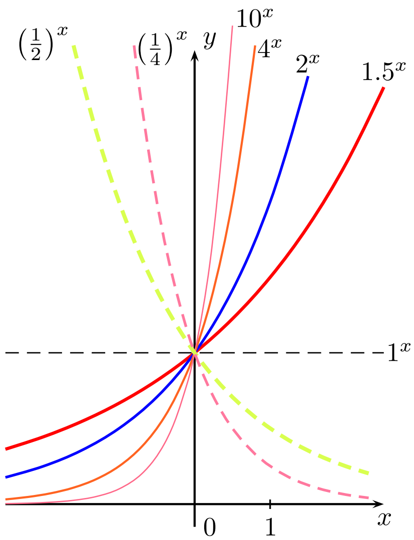


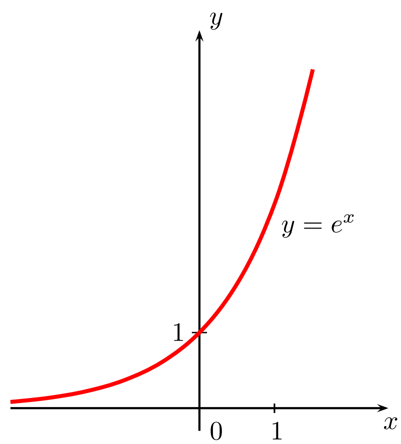
III. Exponential Functions.

An *exponential* function is a function of the type

$$f(x) = a^x,$$

where $a > 0$ is a constant. The exponential function e^x is also denoted by $\exp(x)$.



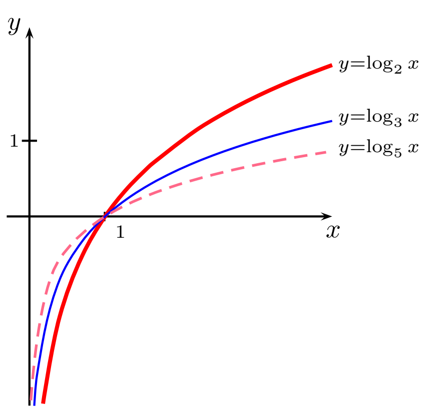


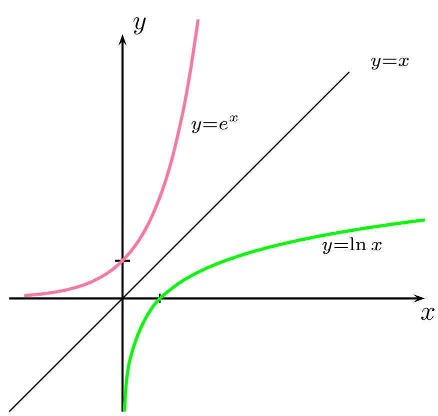
IV. Logarithmic Functions.

A *logarithmic* function is a function of the type

$$f(x) = \log_a x,$$

where $a > 0$, $a \neq 1$, and $x > 0$. The function $\ln x$ is simply called the *natural logarithm*.



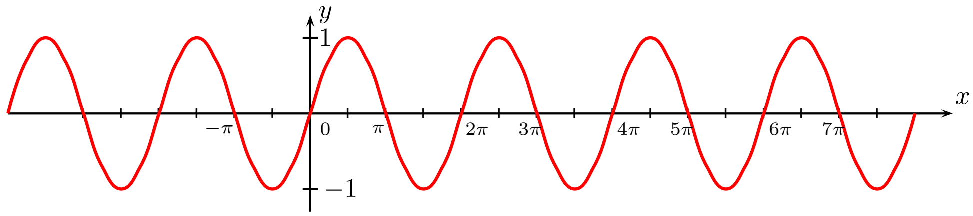


V. Trigonometric Functions.

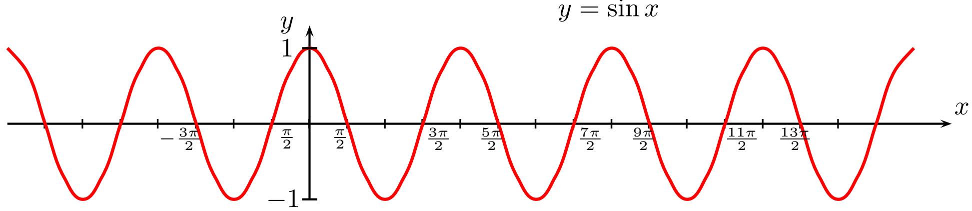
The functions

$$y = \sin x,\ y = \cos x,\ y = \tan x,\ y = \cot x,\ y = \sec x,\ y = \csc x,$$

are called *trigonometric* functions.



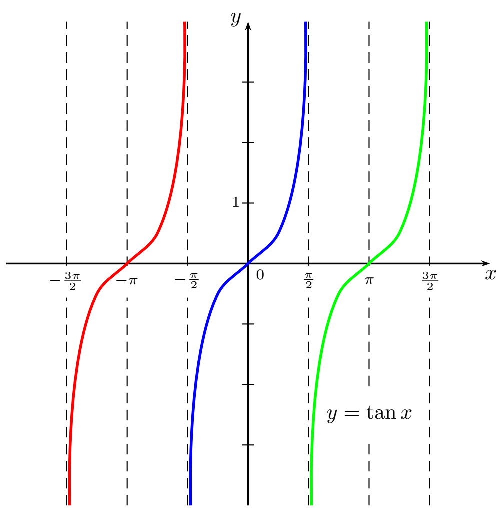
$y = \sin x$



$y = \cos x$

The following are the most important identities for trigonometric functions:

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$1 = \cos^2 x + \sin^2 x,$$
$$\cos 2x = \cos^2 x - \sin^2 x,$$
$$\sin 2x = 2 \sin x \cos x.$$

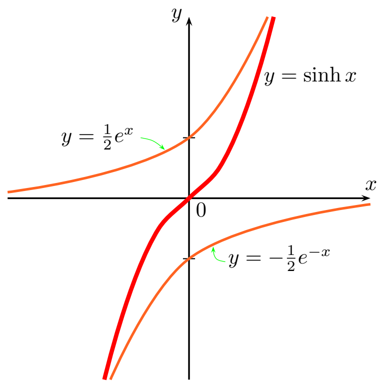


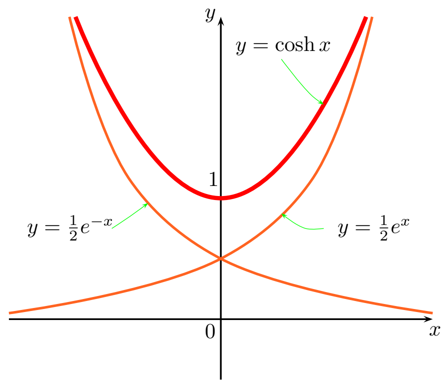
VI. Hyperbolic Functions.

The *hyperbolic* functions are defined by:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2},$$

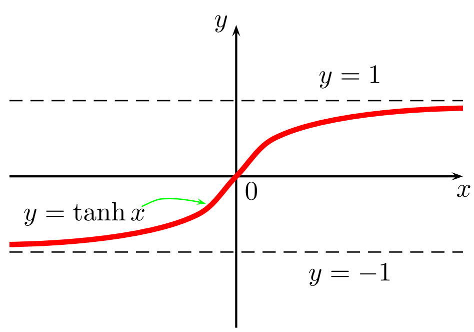
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$





Notice that the hyperbolic functions exhibit similar properties as trigonometric functions. For instance

$$\begin{aligned}
\cosh(x \pm y) &= \cosh x \cosh y \pm \sinh x \sinh y, \\
\sinh(x \pm y) &= \sinh x \cosh y \pm \cosh x \sinh y, \\
1 &= \cosh^2 x - \sinh^2 x, \\
\cosh 2x &= \cosh^2 x + \sinh^2 x, \\
\sinh 2x &= 2 \sinh x \cosh x.
\end{aligned}$$



The hyperbolic functions are quite popular among engineers. We owe to Johann Heinrich Lambert (1728-1777) the first systematic development of the theory of hyperbolic functions.

Let us introduce another category of elementary functions, called *inverse trigonometric* functions or *cyclometric* functions.

VII. Inverse Trigonometric Functions.

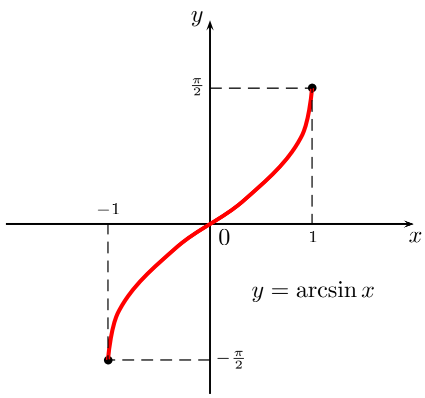
The inverse trigonometric functions are:

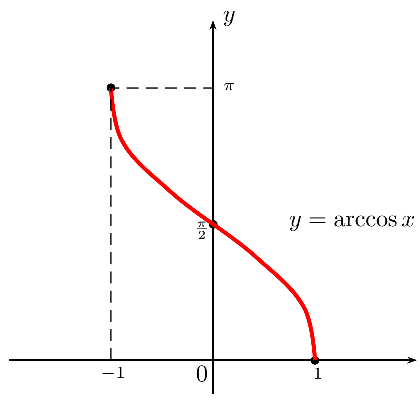
$$y = \arcsin x, \quad y = \arccos x, \quad y = \arctan x,$$

$$y = \operatorname{arccot} x, \quad y = \operatorname{arcsec} x, \quad y = \operatorname{arccsc} x.$$

We begin with the definition of $\arcsin x$. Clearly, $\sin x$ is not a one-to-one function (it is periodic of period 2π). In order to overcome this difficulty we need to restrict its domain so $\sin x$ becomes one-to-one. For this purpose we define $f : [-\frac{\pi}{2}; \frac{\pi}{2}] \to \mathbb{R}$ by $f(x) = \sin x$, i.e. we assume that the domain $Dom(f)$ is exactly the interval $[-\frac{\pi}{2}; \frac{\pi}{2}]$. Consequently, the function f is one-to-one and we can define $\arcsin x = f^{-1}(x)$, $x \in f([-\frac{\pi}{2}; \frac{\pi}{2}]) = [-1, 1]$. We also have the following cancelation equations:

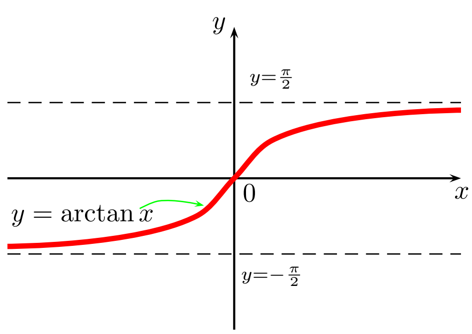
$$\begin{aligned} \arcsin(\sin x) &= x \quad \text{for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}, \\ \sin(\arcsin x) &= x \quad \text{for } -1 \le x \le 1. \end{aligned}$$





Similarly, we can define the function $\arccos x$. First, we restrict the domain of the function $\cos x$ to the set $[0, \pi]$, i.e we define the function $g(x) = \cos x$ for $x \in [0, \pi]$ and then we put $\arccos x = g^{-1}(x)$.

For $\arctan x$, we restrict the domain of $\tan x$ to the set $(-\frac{\pi}{2}, \frac{\pi}{2})$, i.e. we define $h(x) = \tan x$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and we put $\arctan x = h^{-1}(x)$.



For $\operatorname{arccot} x$, we define the function $r(x) = \cot x$, for $x \in (0, \pi)$, and we put $\operatorname{arccot} x = r^{-1}(x)$.

In some textbooks (for instance in 'standard' calculus books used across of the North America), the functions $\arcsin x$, $\arccos x$, $\arctan x$ and $\operatorname{arccot} x$ are denoted by $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ and $\cot^{-1} x$.

Definition 5.15. The class of *elementary functions* is composed of all the functions of type I-VII and all functions that can be obtained from these functions by repeatedly using addition, subtraction, multiplication, division and composition operations.

Intuitively, an elementary function is a real function that can be written down by a single formula. However, even some of the functions that are defined piecewise are also elementary.

Example 5.16. (a) Consider the function $f_1(x) = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$ Notice that $f_1(x)$ is an elementary function since $f_1(x) = \sqrt{x^2}$.

(b) Consider the function $f_2(x) = \begin{cases} 0 & \text{if } x < 0; \\ x & \text{if } x \geq 0. \end{cases}$ Since $f_1(x) = |x|$ is an elementary function, the function $f_2(x) = \frac{1}{2}\Big(x + |x|\Big)$ is also elementary;

(c) Let $f_3(x) = -x^2 f_2(-x) + x^4 f_2(x)$. Since $f_2(x)$ is an elementary function, the function $f_3(x)$ is also elementary function. However, we have

$$f_3(x) = \begin{cases} x^3 & \text{if } x < 0; \\ x^5 & \text{if } x \geq 0. \end{cases}$$

(d) The functions like $y = x^x$ or $y = x^{x^x}$ are also elementary, since $x^x = e^{x \ln x}$ and $x^{x^x} = e^{\ln x e^{x \ln x}}$.

Theorem 5.17. *All elementary functions are continuous.*

Proof: The class of elementary functions is composed of the functions of types I-VII and all the functions that can be obtained from these functions by repeatedly using addition, subtraction, multiplication, division and composition operations. Since we know that by applying the operations of addition, subtraction, multiplication, division and composition to continuous functions we will again obtain a continuous function, it is enough to show that every function of type I-VII is continuous.

Type I: In order to prove that every polynomial and consequently every rational function is continuous, it is enough to show that for every $n \in \mathbb{N}$ the function $f(x) = x^n$ is continuous. Let $x_o \in \mathbb{R}$ and assume that $\varepsilon > 0$ is an arbitrary number. We fix an $\alpha > 0$ such that $|x_o| < \alpha$. Let choose $\delta > 0$ such that $\delta < \min\left(\frac{\varepsilon}{n\alpha^{n-1}}, \alpha - |x_o|\right)$. Then for every x such that $|x - x_o| < \delta$ we have

$$\begin{aligned} |x^n - x^n| &= |x_o - x| \cdot |x_o^{n-1} + x_o^{n-2}x + \dots x_o x^{n-2} + x^{n-1}| \\ &\leq |x_o - x| \cdot |(|x_o|^{n-1} + |x_o|^{n-2}|x| + \dots + \dots |x_o||x|^{n-2} + |x|^{n-1}) \\ &\leq |x_o - x|(\alpha^{n-1} + \alpha^{n-2}\alpha + \dots + \alpha\alpha^{n-2} + \alpha^{n-1}) \\ &= |x_o - x|(n\alpha^{n-1}) < \delta \cdot (n\alpha^{n-1}) < \varepsilon. \end{aligned}$$

Therefore, the function $f(x) = x^n$ is continuous.

Type II: We will show that every power function $f(x) = x^\alpha$, where $x > 0$ and $\alpha \in \mathbb{R}$ is continuous. First, we assume that $\alpha > 0$. Notice that for every two numbers $0 < x < y$ and for $n > \alpha$ we have

$$\begin{aligned} |y^\alpha - x^\alpha| &= y^\alpha - x^\alpha = x^\alpha\left[\left(\frac{y}{x}\right)^\alpha - 1\right] \\ &\leq x^\alpha\left[\left(\frac{y}{x}\right)^n - 1\right] \\ &= x^\alpha\left(\frac{y}{x} - 1\right)\left(\left(\frac{y}{x}\right)^{n-1} + \left(\frac{y}{x}\right)^{n-2} + \dots + 1\right). \end{aligned}$$

Let $\varepsilon > 0$ be an arbitrary number. We fix a number $b > 1$ such that $b > x_o > \frac{1}{b}$. Notice that for any two numbers x and y such that $\frac{1}{b} < x < y, b$ we have $\frac{y}{x} < b^2$. and $0 < c < x_o$. Choose $\delta < \min(\frac{\varepsilon}{b^{\alpha+2n-1}n}, b - x_o, x_o - \frac{1}{b})$. Then, for every $x > 0$ such that $|x_o - x| < \delta$ we have

$$\begin{aligned} |x_o^\alpha - x^\alpha| &< b^\alpha \frac{|x_o - x|}{\frac{1}{b}} (b^{2(n-1)} + b^{(n-2)} + \cdots + 1) \\ &< nb^{\alpha+2n-1}\delta < \varepsilon \end{aligned}$$

and the continuity of $f(x) = x^\alpha$ follows. In the case of a function $g(x) = x^{-\alpha}$ we notice that $g(x) = \frac{1}{f(x)}$ thus $g(x)$ is also continuous.

Type III: Let $\alpha > 1$. We consider the function $f(x) = \alpha^x$. Assume that $x_o \in \mathbb{R}$. Notice that for $x < y$ such that $y - x < \frac{1}{n}$ for some natural number n. Then, the following inequality holds:

$$\begin{aligned} 0 < \alpha^y - \alpha^x &= \alpha^x(\alpha^{y-x} - 1) \\ &< \alpha^x(\alpha^{\frac{1}{n}} - 1). \end{aligned}$$

Put $1 + a = \alpha^{\frac{1}{n}}$, then by Bernoulli's inequality

$$\alpha = (\alpha^{\frac{1}{n}})^n = (1 + a)^n \geq 1 + na = 1 + n(\alpha^{\frac{1}{n}} - 1),$$

thus $\alpha^{\frac{1}{n}} - 1 \leq \frac{\alpha-1}{n}$ and consequently

$$0 < \alpha^y - \alpha^x = \alpha^x(\alpha^{y-x} - 1) < \alpha^x \frac{\alpha - 1}{n}.$$

Let $\varepsilon > 0$ be an arbitrary number. We fix a number b such that $b > x_o$. There exists a natural number n such that $n > \max(\frac{\alpha^b(\alpha-1)}{\varepsilon}, \frac{1}{b-x_o})$. Put $\delta = \frac{1}{n}$, and assume that $|x_o - x| < \delta$. Then we have:

$$|\alpha^{x_o} - \alpha^x| < \alpha^b \frac{\alpha - 1}{n} = \alpha^b(\alpha - 1)\delta < \varepsilon,$$

and therefore, the continuity of the function $f(x) = \alpha^x$ follows.

Suppose now that $0 < \beta < 1$, and let $g(x) = \beta^x$. Then $g(x) = \frac{1}{(\frac{1}{\beta})^x}$, where the function $f_1(x) = \left(\frac{1}{\beta}\right)^x$ is continuous (since $\frac{1}{\beta} > 1$), thus $g(x)$ is also continuous.

Type IV: Since the logarithmic function $f(x) = \log_\alpha x$ is the inverse of $g(x) = \alpha^x$, the continuity of $f(x)$ follows from Theorem 5.10.

Type V: Notice that for every x we have $|\sin x| \leq |x|$. Consequently, $\lim\limits_{x\to 0} \sin x = 0 = \sin 0$ and thus $\sin x$ is continuous at $x_o = 0$. On the other hand, since $\cos x = 1 - 2\sin^2 x$, it follows that $\cos x$ is also continuous at $x_o = 0$. We will show that $\sin x$ is continuous at every point x_o. Indeed, we have

$$\begin{aligned} \lim_{x\to x_o} \sin x &= \lim_{x\to x_o} (\sin(x - x_o)\cos x_o + \cos(x - x_o)\sin x_o) \\ &= \cos x_o \lim_{x\to x_o} \sin(x - x_o) + \sin x_o \lim_{x\to x_o} \cos(x - x_o) \\ &= \cos x_o \lim_{v\to 0} \sin v + \sin x_o \lim_{v\to 0} \cos v \\ &= \cos x_o \cdot 0 + \sin x_o \cdot 1 = \sin x_o, \end{aligned}$$

and consequently $\sin x$ is continuous. On the other hand, since $\cos x = \sin(\frac{\pi}{2} - x)$ it follows that $\cos x$ is also continuous. Therefore, the functions $\tan x$, $\cot c$, $\sec x$ and $\csc x$ are also continuous.

Type VI: All hyperbolic functions are continuous, since they are obtained by applying the operations of addition, subtraction and division to the exponential functions e^x and e^{-x}.

Type VII: The inverse trigonometric functions are continuous because the trigonometric functions are continuous. □

5.3 Problems

1. Show that the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \frac{1}{1+x^2}$ is uniformly continuous.
HINT: Show that

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{x,y\in\mathbb{R}}\ \ |x-y| < \delta \implies \left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right| < \varepsilon.$$

2. Let $b > a$ be two real numbers and $f : (a, b) \to \mathbb{R}$ be a uniformly continuous function. Show that the set of values of $f(x)$ is bounded, i.e. $\exists_{N>0}\forall_{x\in(a,b)}\ |f(x)| < N$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$. Show that the maximal value of $f(x)$ is 1, while there is no minimal value of $f(x)$. **HINT:** Show that $f(x) > 0$ and $\inf f(x) = 0$.

4. Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{3}\sin(2x + 1)$ is a contraction with constant $k = \frac{2}{3}$. Show that the equation $x = \frac{1}{3}\sin(2x + 1)$ has a unique solution.

5. Compute the following limits

(a) $\lim\limits_{x\to 1} \dfrac{\sqrt[m]{x} - 1}{\sqrt[n]{x} - 1}$, where m and n are natural numbers;
(b) $\lim\limits_{x\to -\infty} \sqrt{(x + a)(x + b)} + x$;
(c) $\lim\limits_{x\to\infty} (\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x})$;
(d) $\lim\limits_{x\to\infty} (\sqrt[3]{x^3 + x^2 + 1} - \sqrt[3]{x^3 - x^2 + 1})$;

6. Let $b > a$ and $f, g : [a, b] \to \mathbb{R}$ be two continuous functions. Show that the functions

$$\varphi(x) = \max\{f(x), g(x)\}$$

and

$$\psi(x) = \min\{f(x), g(x)\},$$

are also continuous on $[a, b]$.

7. Let $f : [a, \infty) \to \mathbb{R}$ be a continuous function which is neither bounded from above nor from below. Show that for every real number α there exist infinitely many numbers t_n, $n = 1, 2, \ldots$, in the interval $[a, \infty)$ such that $f(t_n) = \alpha$, i.e. the function $f(x)$ assumes every real value infinitely many times. Give an example of such a function.

8. Show that every function $f : (-a, a) \to \mathbb{R}$, where $a > 0$ can be represented as a sum of an even and an odd functions.

9. Let $f(x) = ax^2 + bx + c$. Show that

$$f(x+3) - 3f(x+2) + 3f(x+1) - f(x) = 0.$$

10. Let $f(x) = \frac{1}{2}\left(a^x + a^{-x}\right)$, where $a > 0$. Show that

$$f(x+y) + f(x-y) = 2f(x)f(y).$$

11. Find $f(x)$ if $f(x+1) = x^2 - 3x + 2$.

12. Find $f(x)$ if $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$ for $|x| \geq 2$.

13. Find $f(x)$ if $f\left(\frac{1}{x}\right) = x + \sqrt{1 + x^2}$ for $x > 0$.

14. Find $f(x)$ if $f\left(\frac{x}{1+x}\right) = x^2$.

6

Differentiable Functions of Real Variable

6.1 Derivative of a Function

Definition 6.1. Let $f : (a,b) \to \mathbb{R}$ be a function. Assume that that

$$\lim_{x\to x_o} \frac{f(x)-f(x_o)}{x-x_o}$$

exists. Then we put

$$f'(x_o) := \lim_{x\to x_o} \frac{f(x)-f(x_o)}{x-x_o} \tag{6.1}$$

and call it the *derivative* of the function f at the point x_o. We will also say that the function f is *differentiable* at x_o. If f is differentiable at every point in (a,b), we say that f is *differentiable*.

Motivated by historical development of calculus, the following symbols are also used to denote a derivative of f.

$\dfrac{dy}{dx}$	or	$\dfrac{df(x_o)}{dx}$	introduced by Gottfried Leibniz;
y'	or	$f'(x_o)$	introduced by Joseph Louis Lagrange;
Dy	or	$Df(x_o)$	introduced by Augustin-Louis Cauchy;
$\dot{y}$	or	$\dot{f}(x_o)$	introduced by Isaac Newton.

Remark 6.2. It is convenient to introduce additional notation. We put $\triangle f(x_o) = f(x) - f(x_o)$ and $\triangle x = x - x_o$, i.e. in simple words $\triangle f(x_o)$ denotes the "increment" of the function $f(x)$ corresponding to the "increment" $\triangle x = x - x_o$ of the independent variable x at the point x_o. Using this notation we can rewrite (6.1) in the following simplified form

$$f'(x_o) = \lim_{\triangle x\to 0} \frac{\triangle f(x_o)}{\triangle x}. \tag{6.2}$$

The one-sided limits

$$f'_+(x_o) := \lim_{\triangle x\to 0^+} \frac{\triangle f(x_o)}{\triangle x}, \qquad f'_-(x_o) := \lim_{\triangle x\to 0^-} \frac{\triangle f(x_o)}{\triangle x},$$

if they exist, are called the *right-hand* and *left-hand derivatives* of $f(x)$ at x_o, respectively.

Proposition 6.3. *Let $x_o \in (a,b)$ and $f : (a,b) \to \mathbb{R}$ be differentiable at the point x_o, i.e. the derivative $f'(x_o)$ exists. Then the function f is continuous at x_o.*

Proof: We have

$$\begin{aligned}\lim_{x\to x_o} f(x) &= \lim_{x\to x_o}\left[\frac{f(x)-f(x_o)}{x-x_o}(x-x_o)+f(x_o)\right]\\ &= f'(x_o)\cdot 0 + f(x_o) = f(x_o),\end{aligned}$$

so $\lim\limits_{x\to x_o} f(x) = f(x_o)$ and consequently the function f is continuous at x_o. □

We have the following basic properties of derivatives:

Theorem 6.4. *Let $f,\ g : (a,b) \to \mathbb{R}$ be two differentiable functions. Then we have*

(i) (Linearity Property) $\dfrac{d}{dx}[c_1 f(x) + c_2 g(x)] = c_1 f'(x) + c_2 g'(x)$, *where c_1 and c_2 are two constants;*

(ii) (Product Rule) $\dfrac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$*;*

(iii) (Quotient Rule) $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$, *where $g(x) \neq 0$.*

Proof: (i): By using the properties of the limit we have

$$\begin{aligned}\frac{d}{dx}[c_1 f(x)+c_2 g(x)] &= \lim_{\triangle x\to 0}\frac{c_1 f(x+\triangle x)+c_2 g(x+\triangle x)-c_1 f(x)-c_2 g(x)}{\triangle x}\\ &= c_1 \lim_{\triangle x\to 0}\frac{f(x+\triangle x)-f(x)}{\triangle x} + c_2 \lim_{\triangle x\to 0}\frac{g(x+\triangle x)-g(x)}{\triangle x}\\ &= c_1 f'(x) + c_2 g'(x).\end{aligned}$$

(ii): Similarly, we have

$$\begin{aligned}\frac{d}{dx}[f(x)g(x)] &= \lim_{\triangle x\to 0}\frac{f(x+\triangle x)g(x+\triangle x)-f(x)g(x)}{\triangle x}\\ &= \lim_{\triangle x\to 0}\frac{f(x+\triangle x)-f(x)}{\triangle x}g(x+\triangle x) + \lim_{\triangle x\to 0}\frac{g(x+\triangle x)-g(x)}{\triangle x}f(x).\end{aligned}$$

Since, in particular, the functions f and g have derivatives at x, by Proposition 6.3, they are continuous at x, so $\lim\limits_{\triangle x\to 0} g(x+\triangle x) = g(x)$, and therefore we obtain

$$\begin{aligned}\frac{d}{dx}[f(x)g(x)] &= \lim_{\triangle x\to 0}\frac{f(x+\triangle x)-f(x)}{\triangle x}\lim_{\triangle x\to 0} g(x+\triangle x) + \lim_{\triangle x\to 0}\frac{g(x+\triangle x)-g(x)}{\triangle x}f(x)\\ &= f'(x)g(x) + g'(x)f(x).\end{aligned}$$

(iii): Again we can use the properties of limits and the fact that f and g are continuous at x:

$$
\begin{aligned}
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \lim_{\triangle x\to 0}\frac{\frac{f(x+\triangle x)}{g(x+\triangle x)}-\frac{f(x)}{g(x)}}{\triangle x} = \lim_{\triangle x\to 0}\frac{\frac{f(x+\triangle x)g(x)-f(x)g(x+\triangle x)}{\triangle x}}{g(x+\triangle x)g(x)}\\
&= \lim_{\triangle x\to 0}\frac{\frac{f(x+\triangle x)-f(x)}{\triangle x}g(x)-\frac{g(x+\triangle x)-g(x)}{\triangle x}f(x)}{g(x+\triangle x)g(x)}\\
&= \frac{g(x)\lim_{\triangle x\to 0}\frac{f(x+\triangle x)-f(x)}{\triangle x}-f(x)\lim_{\triangle x\to 0}\frac{g(x+\triangle x)-g(x)}{\triangle x}}{\lim_{\triangle x\to 0}g(x+\triangle x)g(x)}\\
&= \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}.
\end{aligned}
$$

□

Theorem 6.5. (Chain Rule) *Suppose that $g:(c,d)\to\mathbb{R}$ and $f:(a,b)\to\mathbb{R}$ are two differentiable functions such that $f(a,b)\subset(c,d)$. Then the composite function $y=f(g(x))$ is differentiable and*

$$\frac{d}{dx}[g\circ f](x)=g'(f(x))\cdot f'(x).$$

Proof: We consider the functions $u=f(x)$ and $y=g(u)$, for which we will be using the following notation: $\triangle u=f(x)-f(x_o)$, $\triangle x=x-x_o$ and $\triangle y=g(f(x))-g(f(x_o))=g(u_o+\triangle u)-g(u_o)$. Since $g(u)$ has the derivative $g'(u_o)$ at u_o, we can write

$$\triangle y=g'(u_o)\triangle u+\alpha(u)\triangle u,$$

where $\alpha(u)=\frac{\triangle y}{\triangle u}-g'(u_o)$. It is clear that the existence of the derivative $g'(u_o)$ is equivalent to the property that $\lim\limits_{\triangle u\to 0}\alpha(u)=0$. Consequently, we can write

$$\frac{\triangle y}{\triangle x}=g'(u_o)\frac{\triangle u}{\triangle x}+\alpha(u)\frac{\triangle u}{\triangle x},$$

and by passing to the limit as $\triangle x\to 0$ we obtain

$$\lim_{\triangle x\to 0}\frac{\triangle y}{\triangle x}=g'(u_o)\lim_{\triangle x\to 0}\frac{\triangle u}{\triangle x}+\lim_{\triangle x\to 0}\alpha(u)\cdot\lim_{\triangle x\to 0}\frac{\triangle u}{\triangle x}.$$

By the continuity of $u=f(x)$ at x_o we have $\lim\limits_{\triangle x\to 0}\triangle u=0$. Thus $\lim\limits_{\triangle x\to 0}\alpha(u)=\lim\limits_{\triangle u\to 0}\alpha(u)=0$, so

$$y'_x=\lim_{\triangle x\to 0}\frac{\triangle y}{\triangle x}=g'(u_o)\cdot f'(x_o).$$

□

Proposition 6.6. (Inverse Function Rule) *Assume that $f:(a,b)\to\mathbb{R}$ is a continuous one-to-one function such that for $x_o\in(a,b)$ the derivative $f'(x_o)$ exists and is different from zero. Then the inverse function $f^{-1}(x)$ has a derivative at the point $y_o=f(x_o)$, which is equal to $\frac{1}{f'(x_o)}$, i.e.*

$$\frac{d}{dx}\left[f^{-1}(y_o)\right]=\frac{1}{f'(x_o)},\qquad \textit{where } y_o=f(x_o).$$

Proof: Since the function $y = f(x)$ is continuous, by Theorem 5.10, its inverse $x = f^{-1}(y)$ is also continuous. Consequently, $\lim\limits_{\triangle y \to 0} \triangle x = 0$, and we have

$$\begin{aligned}\frac{d}{dx}\left[f^{-1}(y_o)\right] &= \lim_{\triangle y\to 0}\frac{\triangle x}{\triangle y}\\ &= \frac{1}{\lim_{\triangle y\to 0}\frac{\triangle y}{\triangle x}}\\ &= \frac{1}{\lim_{\triangle x\to 0}\frac{\triangle y}{\triangle x}}\\ &= \frac{1}{f'(x_o)},\end{aligned}$$

where $\triangle y = f(x) - f(x_o)$ and $\triangle x = f^{-1}(y) - f^{-1}(y_o)$. $\square$

6.2 Fundamental Properties of Differentiable Functions

In this sections we collect the most important results describing the properties of differentiable functions. Throughout this section we assume that $f : (a, b) \to \mathbb{R}$ is a real function and that $x_o \in (a, b)$. For the function $y = f(x)$ we will denote $\triangle y = f(x) - f(x_o)$ and $\triangle x = x - x_o$. Then we have the following:

Definition 6.7. We say that the function $f : (a, b) \to \mathbb{R}$ has

(i) a *local maximum* at x_o if

$$\exists_{\delta>0}\ \forall_{x\in(a,b)}\ |x - x_o| < \delta \implies \triangle y \le 0,$$

which simply means that $\forall_{x\in(a,b)}\ |x - x_o| < \delta \ \Rightarrow\ f(x) \le f(x_o)$;

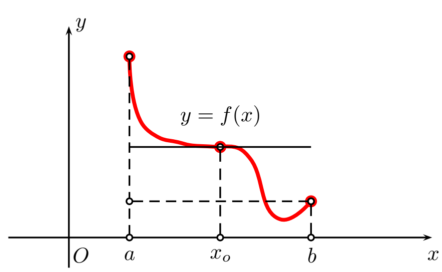
(ii) a *local minimum* at x_o if

$$\exists_{\delta>0}\ \forall_{x\in(a,b)}\ |x - x_o| < \delta \implies \triangle y \ge 0;$$

which simply means that $\forall_{x\in(a,b)}\ |x - x_o| < \delta \ \Rightarrow\ f(x) \ge f(x_o)$;

(iii) a *local extremum* at x_o if it has either a local minimum or local maximum at x_o.

A local maximum (resp. local minimum) is also called a *relative maximum* (resp. *relative minimum*).



$f(x)$ is increasing at x_o but is not increasing on $[a,b]$

We begin with the following important result:

Theorem 6.8. (FERMAT'S THEOREM) *Let $f : (a,b) \to \mathbb{R}$ be a function such that f has a local extremum at $x_o \in (a,b)$. If $f'(x_o)$ exists, then $f'(x_o) = 0$.*

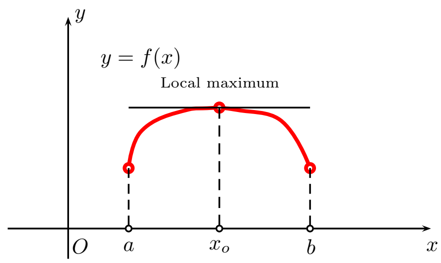
Proof: Suppose that f has a local maximum at x_o, i.e.

$$\exists_{\delta>0}\ \forall_{x\in(a,b)}\ |x - x_o| < \delta \implies f(x) - f(x_o) \le 0;$$

thus

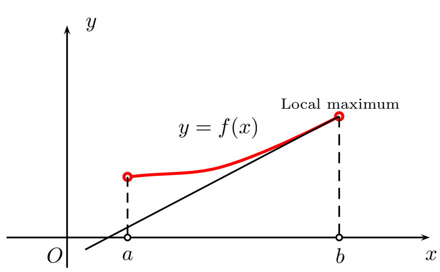
$$0 \ge \lim_{x\to x_o^+} \frac{f(x)-f(x_o)}{x-x_o} = \lim_{x\to x_o} \frac{f(x)-f(x_o)}{x-x_o} = \lim_{x\to x_o^-} \frac{f(x)-f(x_o)}{x-x_o} \ge 0,$$

which implies $f'(x_o) = 0$. □



Local maximum of a function $f(x)$

Remark 6.9. Notice that, in Theorem 6.8, it is an important assumption that x_o belongs to the interior of the interval (a,b). Indeed, in this case the fact that f is increasing (resp. decreasing) at x_o implies that the values $f(x)$ for $x < x_o$ are smaller (resp. greater) than $f(x_o)$, and the values $f(x)$ for $x > x_o$ are greater (resp. smaller) than $f(x_o)$, so it is impossible that $f(x_o)$ is a local extremum.



Local maximum at the endpoint of the interval $[a,b]$

The following result shows that a derivative of a differentiable function, which does not need to be continuous, satisfies the Intermediate Value Property:

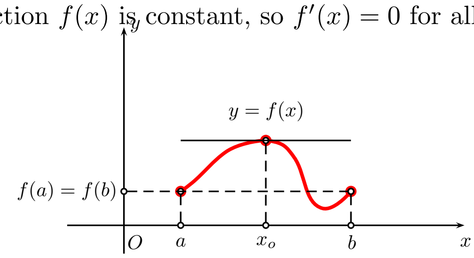
Theorem 6.10. (DARBOUX THEOREM) *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f'(x)$ exists for all $x \in [a,b]$ (i.e. we assume that at the endpoints $f'_+(a)$ and $f'_-(b)$ exist). Then, for every value α in between the numbers $f'_+(a)$ and $f'_-b)$ there exists $x_o \in (a,b)$ such that $f'(x_o) = \alpha$.*

Proof: First, we consider a special case of a differentiable function $\varphi : [a,b] \to \mathbb{R}$ where $\varphi'_+(a)$ and $\varphi'_-(b)$ have opposite signs, for example assume $\varphi'_+(a) > 0$ and $\varphi'_-(b) < 0$. We will show that there exists $x_o \in (a,b)$ such that $\varphi'(x_o) = 0$. By Proposition 6.3, since $\varphi : [a,b] \to \mathbb{R}$ is differentiable, it is continuous. Therefore, by Theorem 3.92, there exists $x_o \in [a,b]$ such that $\varphi(x_o) = \max_{x\in[a,b]} \varphi(x)$ (in the case $\varphi'_+(a) < 0$ and $\varphi'_-(b) > 0$ we should consider the point x_o such that $\varphi(x_o) = \min_{x\in[a,b]} \varphi(x)$). We claim that $x_o \neq a$ and $x_o \neq b$. Indeed, since $\varphi'_+(a) > 0$, there exists $\delta > 0$ such that $\frac{\triangle\varphi(a)}{\triangle x} > 0$ for $0 < x - a < \delta$ and thus $\varphi(x) > \varphi(a)$. Therefore, a is not a local maximum. Similarly, since $\varphi'_-(b) < 0$, $\varphi(b)$ is not a local maximum either. Consequently, $x_o \in (a,b)$ and, by Fermat's Theorem (Theorem 6.8), we obtain $\varphi'(x_o) = 0$.

Now, we are able to give the proof in the general case. Assume that $f'_+(a) > \alpha > f'_-(b)$. We consider the function $\varphi(x) = f(x) - \alpha x$. Then $\varphi'_+(a) = f'_+(a) - \alpha > 0$ and $\varphi'_-(b) = f'_-(b) - \alpha < 0$. Since the function $\varphi(x)$ satisfies the conditions of the previous case, there exists a point $x_o \in (a,b)$ such that $\varphi'(x_o) = 0$, i.e. $0 = f'(x_o) - \alpha$, so $f'(x_o) = \alpha$. □

Theorem 6.11. (Rolle's Theorem) *Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that f is differentiable in (a,b), i.e. the derivative $f'(x)$ exists for all $x \in (a,b)$. Assume that $f(a) = f(b)$, then there exists $x_o \in (a,b)$ such that $f'(x_o) = 0$.*

Proof: By Theorem 3.92, there exist $c \in [a,b]$ such $f(c) = \max\limits_{x\in[a,b]} f(x)$ and $d \in [a,b]$ such that $f(d) = \min\limits_{x\in[a,b]} f(x)$. If $f(c) \neq f(d)$, then at least one of the points c or d, which we denote by x_o, must be different from a and b, thus $x_o \in (a,b)$ and by Fermat's Theorem (Theorem 6.8), $f'(x_o) = 0$. If $f(c) = f(d)$ then the function $f(x)$ is constant, so $f'(x) = 0$ for all $x \in (a,b)$. □



Rolle's Theorem

As a consequence of Rolle's Theorem, we have the following important result, also called the *Mean Value Theorem*:

Theorem 6.12. (Lagrange Theorem) *Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that it is differentiable in (a,b), i.e. the derivative $f'(x)$ exists for all $x \in (a,b)$. Then there exists a point $x_o \in (a,b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(x_o).$$

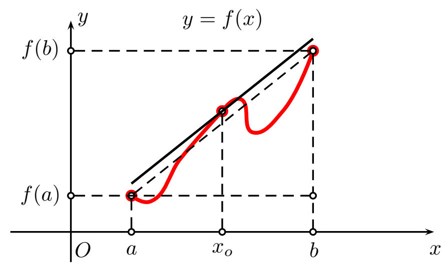
Proof: We consider the following auxiliary function $\varphi : [a,b] \to \mathbb{R}$ given by

$$\varphi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

It is clear that $\varphi(x)$ is continuous on $[a,b]$ and differentiable in (a,b). Moreover, $\varphi(a)=0$ and $\varphi(b)=0$, thus $\varphi(x)$ satisfies the assumptions of the Rolle's Theorem. Consequently, there exists $x_o\in(a,b)$ such that

$$0=\varphi'(x_o)=f'(x_o)-\frac{f(b)-f(a)}{b-a},$$

so $\frac{f(b)-f(a)}{b-a}=f'(x_o)$. □



Mean Value Theorem

Proposition 6.13. *Let $f:(a,b)\to\mathbb{R}$ be a differentiable function such that $f'(x)>0$ for all $x\in(a,b)$. Then f is increasing on (a,b). Similarly, if $f'(x)<0$ for all $x\in(a,b)$, then f is decreasing on (a,b).*

Proof: Assume that $f'(x)>0$ for all $x\in(a,b)$. Then, for all $x_1<x_2$, x_1, $x_2\in(a,b)$, by the Mean Value Theorem (Theorem 6.12), there exists $c\in(x_1,x_2)$ such that

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=f'(c)>0 \quad\Rightarrow\quad f(x_2)>f(x_2),$$

and the conclusion follows. □

Remark 6.14. Suppose that $f:(a,b)\to\mathbb{R}$ is a differentiable function such that $f'(x)>0$ for all $x\in(a,b)$ except a finite number of points $x_k\in(a,b)$, $k=1,2,\dots,n$, where $f'(x_k)=0$. Then f is increasing on the interval (a,b). Indeed, suppose that $a<x_1<x_2<\dots<x_n<b)$. Then by Proposition 6.13, the function f is increasing on the intervals (a,x_1), (x_k,x_{k+1}), $k=-1,2,\dots,n-1$, and (x_n,b). It is clear that, by Rolle's Theorem, it is also increasing on $(a,x_1]$, $[x_k,x_{k+1}]$, $k=-1,2,\dots,n-1$, and $[x_n,b)$, so it is increasing on (a,b).

The Mean Value Theorem can be generalized as follows:

Theorem 6.15. (Cauchy Theorem) *Suppose that $f,g:[a,b]\to\mathbb{R}$ are two continuous functions differentiable in (a,b), i.e. the derivatives $f'(x)$ and $g'(x)$ exist for all $x\in(a,b)$. Assume that $g'(x)\neq 0$ for all $x\in(a,b)$. Then there exists $x_o\in(a,b)$ such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(x_o)}{g'(x_o)}.$$

Proof: First, we will show that $g(b)-g(a)\neq 0$. Indeed, if $g(b)-g(a)=0$, then by Rolle's Theorem, there would exist $c\in(a,b)$ such that $g'(c)=0$, contrary to the assumption that $g'(x)\neq 0$ for all $x\in(a,b)$.

We consider the auxiliary function $\varphi:[a,b]\to\mathbb{R}$ given by

$$\varphi(x)=f(x)-f(a)-\frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)].$$

Notice that $\varphi:[a,b]\to\mathbb{R}$ is continuous, differentiable in (a,b), $\varphi(a)=0$ and $\varphi(b)=0$, thus it satisfies the assumptions of Rolle's Theorem. Thus, there exists $x_o\in(a,b)$ such that $\varphi'(x_o)=0$. On the other hand

$$\varphi'(x)=f'(x)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(x),$$

thus

$$f'(x_o)-\frac{f(b)-f(a)}{g(b)-g(a)}\cdot g'(x_o)=0$$

so

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(x_o)}{g'(x_o)}.$$

$\square$

Example 6.16. (a) Prove the inequality $\ln(1+x)<x$ for all $x>0$. Solution: We define the function $\varphi(x)=x-\ln(1+x)$. We have $\varphi(0)=0$ and $\varphi'(x)=1-\frac{1}{1+x}=\frac{x}{1+x}>0$ for $x>0$, thus by Proposition 6.13, $\varphi(x)$ is an increasing function. Consequently, $\varphi(x)>0$ for $x>0$ and the inequality $x>\ln(1+x)$ follows.

(b) Prob Use the Mean Value Theorem (Theorem 6.12) to show the following inequality

$$\frac{\alpha-\beta}{\cos^2\beta}\leq\tan\alpha-\tan\beta\leq\frac{\alpha-\beta}{\cos^2\alpha}.$$

Solution: Consider the function $f(x)=\tan x$. Then, for $\alpha>\beta$, by the Mean Value Theorem (Theorem 6.12), there exists $c\in(\beta,\alpha)$ such that

$$\frac{\tan\alpha-\tan\beta}{\alpha-\beta}=f'(c)=\frac{1}{\cos^2 c}.$$

If $0\leq\beta<\alpha<\frac{\pi}{2}$, then $\cos^2\beta>\cos^2 c>\cos^2\alpha$, i.e.

$$\frac{1}{\cos^2\beta}>\frac{1}{\cos^2 c}<\frac{1}{\cos^2\alpha},$$

so

$$\frac{1}{\cos^2\beta}<\frac{\tan\alpha-\tan\beta}{\alpha-\beta}<\frac{1}{\cos^2\alpha}$$

and therefore, for $0\leq\beta\leq\alpha<\frac{\pi}{2}$

$$\frac{\alpha-\beta}{\cos^2\beta}\leq\tan\alpha-\tan\beta\leq\frac{\alpha-\beta}{\cos^2\alpha}\tag{6.3}$$

If $0\leq\alpha<\beta<\frac{\pi}{2}$, then we have

$$\frac{1}{\cos^2\alpha} < \frac{\tan\alpha - \tan\beta}{\alpha-\beta} < \frac{1}{\cos^2\beta}$$

so

$$\frac{\alpha-\beta}{\cos^2\alpha} > \tan\alpha - \tan\beta > \frac{\alpha-\beta}{\cos^2\beta}.$$

That means (6.3) is true for all $\alpha,\ \beta \in [0, \frac{\pi}{2})$.

6.3 L'Hôpital's Rule

Theorem 6.17. (L'Hôpital's Rule) *Suppose that* $f, g : (a,b) \to \mathbb{R}$ *(resp.* $f, g : (d,a) \to \mathbb{R}$*), where* $d < a < b$*, are two differentiable functions such that*

(i) $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = 0$ *or* $\pm\infty$ *(resp.* $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^-} g(x) = 0$ *or* $\pm\infty$*);*
(ii)$g'(x) \neq 0$ *for all* $x \in (a,b)$ *(resp.* $x \in (d,a)$*);*
(iii)$\lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)}$ *(resp.* $\lim\limits_{x\to a^-} \dfrac{f'(x)}{g'(x)}$*) exists and is finite or equal to* $\pm\infty$*.*

Then, we have

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)} \tag{6.4}$$

$$\left(resp. \quad \lim_{x\to a^-} \frac{f(x)}{g(x)} = \lim_{x\to a^-} \frac{f'(x)}{g'(x)} \right) \tag{6.5}$$

Proof: We have to consider several cases. We begin with the case, where $f, g : (a,b) \to \mathbb{R}$ are differentiable functions such that $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = 0$. By putting $f(a) = g(a) = 0$ we can continuously extend the functions f and g to the interval $[a,b)$. By Cauchy's Theorem (Theorem 6.15) for every $x \in (a,b)$ there exists $c(x) \in (a,x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c(x))}{g'(c(x))}.$$

We can consider $c(x)$ as a function of x. Since $a < c(x) < x$, it follows from the squeeze property that $\lim\limits_{x\to a} c(x) = a$. Consequently, for $t = c(x)$ we obtain

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(c(x))}{g'(c(x))} = \lim_{t\to a^+} \frac{f'(t)}{g'(t)}.$$

Therefore

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

In the case, where $f, g : (d,a) \to \mathbb{R}$ and $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^-} g(x) = 0$, the proof of L'Hôpital's Rule is exactly 'parallel' and is left to the reader.

The case when $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = \infty$ is more complicated. Assume that the limit $\lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)} = \alpha$ is finite. Then,

$$\forall_{\varepsilon>0}\,\exists_{\theta>0}\,\forall_x\;\; 0<x-a<\theta \quad\Longrightarrow\quad \left|\frac{f'(x)}{g'(x)}-\alpha\right|<\frac{\varepsilon}{2}.$$

Put $x'=a+\theta$. Since $\lim\limits_{x\to a^+} g(x)=\infty$, we can assume that $g(x')>0$. Then, by Cauchy's Theorem (Theorem 6.15) there exists $c\in(x,x')$ such that

$$\frac{f(x)-f(x')}{g(x)-g(x')}=\frac{f'(c)}{g'(c)}.$$

Since $0<c-a<\theta$, we have

$$\left|\frac{f(x)-f(x')}{g(x)-g(x')}-\alpha\right|=\left|\frac{f'(c)}{g'(c)}-\alpha\right|<\frac{\varepsilon}{2}.$$

On the other hand

$$\begin{aligned}\frac{f(x)}{g(x)}-\alpha &= \frac{f(x)-\alpha g(x)}{g(x)}\\ &= \frac{f(x)+f(x')-f(x')-\alpha\big(g(x)+g(x')-g(x')\big)}{g(x)}\\ &= \frac{f(x')-\alpha g(x')}{g(x)}+\frac{f(x)-f(x')-\alpha\big(g(x)-g(x')\big)}{g(x)}\\ &= \frac{f(x')-\alpha g(x')}{g(x)}+\frac{g(x)-g(x')}{g(x)}\cdot\frac{f(x)-f(x')-\alpha\big(g(x)-g(x')\big)}{g(x)-g(x')}\\ &= \frac{f(x')-\alpha g(x')}{g(x)}+\left[1-\frac{g(x')}{g(x)}\right]\cdot\left[\frac{f(x)-f(x')}{g(x)-g(x')}-\alpha\right].\end{aligned}$$

Since the point $x'=a+\theta$ is assumed to be fixed and $\lim\limits_{x\to a^+} g(x)=\infty$, it follows that $\lim\limits_{x\to a^+}\dfrac{f(x')-\alpha g(x')}{g(x)}=0$ and $\lim\limits_{x\to a^+}\dfrac{g(x')}{g(x)}=0$. Therefore, there

$$\exists_{\delta>0}\;\; \delta<\theta\;\wedge\;\forall_x\quad 0<x-a<\delta \quad\Longrightarrow\quad \left|\frac{f(x')-\alpha g(x')}{g(x)}\right|<\frac{\varepsilon}{2}\quad\text{and}\quad 0<\frac{g(x')}{g(x)}<1.$$

Thus

$$\left|\frac{f(x)}{g(x)}-\alpha\right|\le\left|\frac{f(x')-\alpha g(x')}{g(x)}\right|+\left|\frac{f(x)-f(x')}{g(x)-g(x')}-\alpha\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Therefore, we have obtained

$$\forall_{\varepsilon>0}\,\exists_{\delta>0}\,\forall_x\quad 0<x-a<\delta\;\Longrightarrow\;\left|\frac{f(x)}{g(x)}-\alpha\right|<\varepsilon,$$

thus

$$\lim_{x\to a^+}\frac{f(x)}{g(x)}=\alpha=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

In the case when $\lim\limits_{x\to a^+}\dfrac{f'(x)}{g'(x)}=\infty$, instead of $\frac{f(x)}{g(x)}$, we consider the quotient function $\frac{g(x)}{f(x)}$. It follows from the assumptions that $\lim\limits_{x\to a^+}\dfrac{g'(x)}{f'(x)}=0^+$ and by the previous case we obtain $\lim\limits_{x\to a^+}\dfrac{g(x)}{f(x)}=0^+$, thus $\lim_{x\to a^+}\frac{f(x)}{g(x)}=\infty$.

The proof for the case when $f,g:(d,a)\to\mathbb{R}$ and $\lim\limits_{x\to a^-} f(x)=\lim\limits_{x\to a^-} g(x)=\infty$ is similar. $\square$

Corollary 6.18. (L'Hôpital's Rule at ∞) *Suppose that $f, g : (a, \infty) \to \mathbb{R}$ (resp. $f, g : (-\infty, b) \to \mathbb{R}$) are two differentiable functions such that*

(i) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$ *or* $\pm\infty$ *(resp.* $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} g(x) = 0$ *or* $\pm\infty$*);*
(ii)$g'(x) \neq 0$ *for all* $x \in (a, \infty)$ *(resp.* $x \in (-\infty, b)$*);*
(iii)$\lim_{x\to\infty} \dfrac{f'(x)}{g'(x)}$ *(resp.* $\lim_{x\to-\infty} \dfrac{f'(x)}{g'(x)}$*) exists and is finite or equal to* $\pm\infty$*.*

Then, we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

$$\left(\textit{resp.} \quad \lim_{x\to-\infty} \frac{f(x)}{g(x)} = \lim_{x\to-\infty} \frac{f'(x)}{g'(x)}.\right)$$

Proof: Consider the substitution $t = \frac{1}{x}$. Then we have

$$\begin{aligned}
\lim_{x\to\infty} \frac{f'(x)}{g'(x)} &= \lim_{t\to0^+} \frac{f'(\frac{1}{t})}{g'(\frac{1}{t})} = \lim_{t\to0^+} \frac{f'(\frac{1}{t})\cdot(-\frac{1}{t^2})}{g'(\frac{1}{t})\cdot(-\frac{1}{t^2})} = \lim_{t\to0^+} \frac{f(\frac{1}{t})}{g(\frac{1}{t})} \qquad \text{(by l'Hospital Rule)} \\
&= \lim_{x\to\infty} \frac{f(x)}{g(x)}.
\end{aligned}$$

$\square$

6.4 Higher Derivatives

Definition 6.19. Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. If the derivative $f' : (a, b) \to \mathbb{R}$ is also differentiable, we will say that f is *twice differentiable* and we will denote the derivative of f' by f'' and call *second derivative of f*. The *n-th derivative* $f^{(n)} : (a, b) \to \mathbb{R}$, $n = 1, 2, \dots$, is defined by induction

(i) $n = 0$ $f^{(0)} = f$;
(ii) If $f^{(n)}$ is defined, $n \geq 0$, then $f^{(n+1)} = \frac{d}{dx}\left(f^n\right)$.

If the function f has all derivatives up to the order n ($n \geq 1$), we will say that f is *n-times differentiable.* If in addition the derivative $f^{(n)} : (a, b) \to \mathbb{R}$ is continuous, we will say that f is *n-times continuously differentiable* or simply that f *is of class C^n*

The following are the properties of the n-derivative:

Theorem 6.20. *Suppose that $f, g : (a, b) \to \mathbb{R}$ are two n-differentiable functions, $n \geq 1$. Then for $1 \leq m \leq n$ we have*

(i) (Linearity Property)

$$\frac{d^m}{dx^m}\Big(\alpha f(x) + \beta g(x)\Big) = \alpha f^{(m)}(x) + \beta g^{(m)}(x),$$

where α, β are two constant;

(ii)(LEIBNITZ FORMULA)

$$\tfrac{d^m}{dx^m}\Big(f(x)g(x)\Big) = \sum_{k=0}^{m} \binom{m}{k} f^{(m-k)}(x)g^{(k)}(x),$$

where $f^{(0)}(x) = f(x)$ *and* $g^{(0)}(x) = g(x)$.

Proof: The proof of (i) follows immediately from the properties of the derivative,

We will prove (ii) by applying Mathematical Induction.

Step 1: It is clear that for $m = 1$ the Leibnitz Formula is exactly the Product Rule.

Step 2: Assume that for some n such that $n \geq 1$, we have

$$\frac{d^n}{dx^n}\left[f(x)g(x)\right] = \sum_{k=0}^{n} \binom{m}{k} f^{(n-k)}(x)g^{(k)}(x).$$

Then, we have

$$\begin{aligned}
&\frac{d^{n+1}}{dx^{n+1}}\left[f(x)g(x)\right] = \frac{d}{dx}\left[\frac{d^n}{dx^n}\left(f(x)g(x)\right)\right] = \frac{d}{dx}\left[\sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)g^{(k)}(x)\right] \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(m-k+1)}(x)g^{(k)}(x) + \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)g^{(k+1)}(x) \\
&= f^{(n+1)}(x)g(x) + \sum_{k=1}^{n} \binom{n}{k} f^{(n-k+1)}(x)g^{(k)}(x) + \sum_{k=1}^{n} \binom{n}{k-1} f^{(n-k+1)}(x)g^{(k)}(x) \\
&\quad + f(x)g^{(n+1)}(x) \\
&= f^{(n+1)}(x)g(x) + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1}\right] f^{(n-k)}(x)g^{(k)}(x) + f(x)g^{(n+1)}(x) \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(n+1-k)}(x)g^{(k)}(x).
\end{aligned}$$

Consequently, by Mathematical Induction, Leibnitz's formula is true for every $n \in \mathbb{N}$. □

6.5 Taylor Formula

Suppose that a function $f : (a, b) \to \mathbb{R}$ is n times differentiable . Then we can define the polynomial

$$T_n(x) := f(x_o) + \frac{f'(x_o)}{1!}(x - x_o) + \frac{f''(x_o)}{2!}(x - x_o)^2 + \cdots + \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n,$$

which is called the *Taylor Polynomial* of degree n of the function f. Clearly $T_n(x) \neq f(x)$ in general, however, we claim that $T_n(x)$ is an approximation of f. Therefore, it is important to estimate the difference

$$r_n(x) = f(x) - T_n(x) \tag{6.6}$$

which is the error of this approximation (also called *Taylor's remainder*). We will show that $\lim\limits_{x\to x_o} \dfrac{r_n(x)}{(x - x_o)^n} = 0$.

Lemma 6.21. *Let $r : (a,b) \to \mathbb{R}$ be $(n+1)$-differentiable function on (a,b). Assume that $x_o \in (a,b)$ is such that*

$$r(x_o) = r'(x_o) = r''(x_o) = \cdots = r^{(n)}(x_o) = 0.$$

Then

$$\lim_{x\to x_o} \frac{r(x)}{(x-x_o)^n} = 0. \tag{6.7}$$

Proof: We apply mathematical induction. For $n = 1$, if $r(x_o) = r'(x_o) = 0$, then $r(x) = o(x - x_o)$, i.e. $\lim\limits_{x\to x_o} \dfrac{r(x)}{x - x_o} = 0$. Indeed

$$\lim_{x\to x_o} \frac{r(x)}{x-x_o} = \lim_{x\to x_o} \frac{r(x) - r(x_o)}{x - x_o} = r'(x_o) = 0.$$

Assume therefore, that the statement of Lemma 6.21 is true for the natural number $n \geq 1$. Consider a function $r(x)$ which is $(n+2)$-differentiable and satisfies

$$r(x_o) = r'(x_o) = \cdots = r^{(n)}(x_o) = r^{(n+1)}(x_o) = 0.$$

Since $r_1(x) = r'(x)$ satisfies $r_1(x_o) = r_1'(x_o) = \cdots = r_1^{(n)}(x_o) = 0$, thus by the induction assumption $r'(x) = r_1(x) = o((x - x_o)^n)$. On the other hand, by the Lagrange Theorem (Theorem 6.12), there exists $c(x)$ in between x_o and x such that

$$r(x) = r(x) - r(x_o) = r'(c(x))(x - x_o).$$

Since $\lim\limits_{x\to x_o} c(x) = x_o$, it follows from the Substitution Rule that

$$\begin{aligned}\lim_{x\to x_o} \frac{r'(c(x))}{(c(x)-x_o)^n} &= \lim_{c(x)\to x_o} \frac{r'(c(x))}{(c(x)-x_o)^n} \\ &= \lim_{x\to x_o} \frac{r'(x)}{(x-x_o)^n} = 0,\end{aligned}$$

i.e.

$$r'(c(x)) = o((c(x) - x_o)^n).$$

On the other hand, since $|c(x) - x_o| < |x - x_o|$ we have

$$0 \leq \frac{|r'(c(x))|}{|x-x_o|^n} \leq \frac{|r'(c(x))|}{|c(x)-x_o|^n}.$$

thus

$$\lim_{x\to x_o} \frac{r'(c(x))}{(x-x_o)^n} =$$

and we obtain

$$\begin{aligned}\lim_{x\to x_o} \frac{r(x)}{(x-x_o)^{n+1}} &= \lim_{x\to x_o} \frac{r'(c(x))(x-x_o)}{(x-x_o)^{n+1}} \\ &= \lim_{x\to x_o} \frac{r'(c(x))}{(x-x_o)^n} = 0.\end{aligned}$$

Consequently, the conclusion of Lemma 6.21 follows from the Principle of Mathematical Induction.□

Now, we can state the following important result, which is an immediate consequence of Lemma 6.21:

Theorem 6.22. (Taylor's Theorem) *Let $f:(a,b)\to\mathbb{R}$ be an $(n+1)$-differentiable function on (a,b) and $x_o\in(a,b)$. Then we have*

$$f(x)=f(x_o)+\frac{f'(x_o)}{1!}(x-x_o)+\frac{f''(x_o)}{2!}(x-x_o)^2+\dots+\frac{f^{(n)}(x_o)}{n!}(x-x_o)^n+r_n(x) \tag{6.8}$$

where

$$\lim_{x\to x_o}\frac{r_n(x)}{(x-x_o)^n}=0.$$

The formula (6.8) is called *Taylor's Formula* for $f(x)$, where the remainder $r_n(x)$ is given in *Peano's form*. The following result is important for the error estimation of the approximation of $f(x)$ by Taylor's Polynomial $T_n(x)$:

Theorem 6.23. *Under the same assumptions as in Theorem 6.22, the remainder $r_n(x)$ for the Taylor's Formula (6.8) can be expressed in the following forms:*

(i) $r_n(x)=\frac{f^{(n+1)}(d)}{n!}(x-x_o)(x-d)^n$, *where d is a certain point in between x and x_o;*
(ii)(Lagrange Form) $r_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_o)^{n+1}$, *where c is a certain point in between x and x_o;*
(iii)(Cauchy Form) $r_n(x)=\frac{f^{(n+1)}(x_o+\theta(x-x_o))}{n!}(1-\theta)^n(x-x_o)^{n+1}$, *where θ is a certain number such that $0<\theta<1$.*

Proof: We have that

$$r_n(x)=f(x)-f(x_o)-\tfrac{f'(x_o)}{1!}(x-x_o)-\tfrac{f''(x_o)}{2!}(x-x_o)^2-\dots-\tfrac{f^{(n)}(x_o)}{n!}(x-x_o)^n.$$

We fix the number x and consider the following auxiliary function

$$\varphi(z)=f(x)-f(z)-\frac{f'(z)}{1!}(x-z)-\frac{f''(z)}{2!}(x-z)^2-\dots-\frac{f^{(n)}(z)}{n!}(x-z)^n,$$

where z belongs to the interval with the endpoints x and x_o. We can assume that $x_o<x$ (or $x<x_o$) in order to have a clear picture of the situation.

It follows from the assumptions that $\varphi(z)$ is differentiable and thus it is also continuous on $[x_o,x]$. Moreover

$$\varphi(x_o)=r_n(x),\quad\text{and}\quad\varphi(x)=0.$$

We have

$$\begin{aligned}\varphi'(z)=&-f'(z)-\left[\frac{f''(z)}{1!}(x-z)-f'(z)\right]-\left[\frac{f^{(3)}(z)}{2!}(x-z)^2-\frac{f''(z)}{1!}(x-z)\right]\\&-\left[\frac{f^{(4)}(z)}{3!}(x-z)^3-\frac{f^{(3)}(z)}{2!}(x-z)^2\right]-\dots\\&-\left[\frac{f^{(n+1)}(z)}{n!}(x-z)^n-\frac{f^{(n)}(z)}{(n-1)!}(x-z)^{n-1}\right]\\=&-\frac{f^{(n+1)}(z)}{n!}(x-z)^n.\end{aligned}$$

By applying the Mean Value Theorem, there exists a number d, in between x and x_o (i.e. because of our assumption we can write $x_o<d<x$) such that

$$\frac{r_n(x)}{x_o - x} = \frac{\varphi(x_o) - \varphi(x)}{x_o - x} = \varphi'(d)$$
$$= -\frac{f^{(n+1)}(d)}{n!}(x-d)^n,$$

i.e.

$$r_n(x) = \frac{f^{(n+1)}(d)}{n!}(x-d)^n(x-x_o).$$

Assume that $\psi(z)$ is a differentiable function on the open interval with the endpoints x_o and x (i.e. the interval (x_o, x)) such that $\psi'(z) \neq 0$ for all z. Then by Cauchy's Theorem (Theorem 6.15), there exists c in between x and x_o (i.e. $x_o < c < x$) such that

$$\frac{\varphi(x) - \varphi(x_o)}{\psi(x) - \psi(x_o)} = \frac{\varphi'(c)}{\psi'(c)}.$$

We can also write $c = x_o + \theta(x - x_o)$, where $0 < \theta < 1$. Thus, we obtain

$$r_n(x) = \frac{\psi(x) - \psi(x_o)}{\psi'(c)} \cdot \frac{f^{(n+1)}(c)}{n!}(x-c)^n.$$

Assume that the function $\psi(z)$ is given by $\psi(z) = (x-z)^p$, where $p > 0$. Then $\psi'(z) = -p(x-z)^{p-1} \neq 0$, so

$$r_n(x) = \frac{-(x-x_o)^p}{-p(x-c)^{p-1}} \cdot \frac{f^{(n+1)}(c)}{n!}(x-c)^n,$$

thus

$$r_n(x) = \frac{f^{(n+1)}(c)}{n!p}(x-c)^{n+1-p}(x-x_o)^p. \tag{6.9}$$

If we substitute $c = x_o + \theta(x - x_o)$ into (6.9), then $x - c = x - x_o - \theta(x - x_o) = (1-\theta)(x - x_o)$, and we obtain

$$r_n(x) = \frac{f^{(n+1)}(x_o + \theta(x - x_o))}{n!p} \cdot (1-\theta)^{n+1-p}(x-x_o)^{n+1}, \tag{6.10}$$

where $0 < \theta < 1$. Formula (6.10) is called the *Schlömilch-Roche* form of the remainder $r_n(x)$.

By taking $p = n+1$ we get from (6.9) the Lagrange form of $r_n(x)$:

$$r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_o)^{n+1}.$$

If we substitute $p = 1$, we obtain from (6.10) the Cauchy form of the remainder

$$r_n(x) = \frac{f^{(n+1)}(x_o + \theta(x - x_o))}{n!}(1-\theta)^n(x-x_o)^{n+1}.$$

□

Example 6.24. Consider the function $f(x) = \cos x$ for $x \in [0, a]$, where $a > 0$ is an arbitrary number. We will show that for every $\varepsilon > 0$ there exists a Taylor Polynomial

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$

such that

$$\forall_{x\in[0,a]}\ |\cos x - T_n(x)| < \varepsilon,$$

i.e. the function $\cos x$ can be *uniformly* approximated on the interval $[0,a]$ by the Taylor Polynomials $T_n(x)$. Indeed, since $\frac{d^m}{dx^m}(\cos x)|_{x=0} = \begin{cases} 0 & \text{if } m = 2k+1 \\ (-1)^k & \text{if } m = 2k, \end{cases}$ we have

$$T_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{x^{2k}}{(2k)!},$$

where the function $\lfloor x \rfloor$, called the *greatest integer function*, is defined by $f(x) =$ the greatest integer m such that $m \le x$. Then, by Taylor's Formula, for $x \in [0,a]$ we have that

$$\begin{aligned} |\cos x - T_n(x)| = |r_n(x)| &\le \left| \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right| \\ &\le \frac{|x|^{n+1}}{(n+1)!} \le \frac{a^{n+1}}{(n+1)!}. \end{aligned}$$

By the inequality $\sqrt[n]{n!} \ge \frac{n+1}{e}$ (see Lemma 5.14), we have

$$|\cos x - T_n(x)| \le \frac{a^{n+1}}{(n+1)!} \le \frac{e^{n+1}a^{n+1}}{(n+2)^{n+1}}.$$

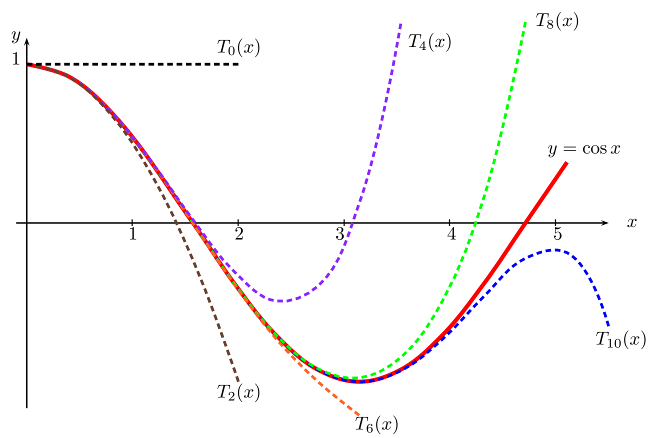
Therefore, for $n \ge ae-2$, we have that $\frac{ea}{n+2} \le 1$, hence

$$|\cos x - T_n(x)| \le \left(\frac{ea}{n+2}\right)^{n+1} \le \frac{ea}{n+2}$$

for all $x \in [0,a]$. Let $\varepsilon > 0$ be an arbitrary number, then by Archimedes axiom, there exists a natural number N such that $N \ge ae-2$ and $\frac{ea}{N+2} < \varepsilon$. Thus

$$|\cos x - T_N(x)| \le \frac{ea}{N+2} < \varepsilon \tag{6.11}$$

for all $x \in [0,a]$. In particular, the formula (6.11) can be considered as an algorithm for numerical computations of the function $f(x) = \cos x$.



6.6 Convex Functions

Definition 6.25. A function $f : [a, b] \to \mathbb{R}$ is called *convex* (or *concave upward*) if for any $x_1, x_2 \in [a, b]$ and $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ such that $\lambda_1 + \lambda_2 = 1$ we have

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2). \tag{6.12}$$

Similarly, a function $f : [a, b] \to \mathbb{R}$ is called *concave* (or *concave downward*) if for any $x_1, x_2 \in [a, b]$ and $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ such that $\lambda_1 + \lambda_2 = 1$ we have

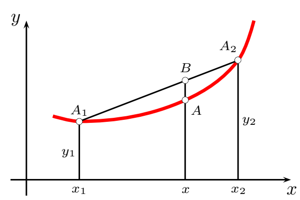
$$f(\lambda_1 x_1 + \lambda_2 x_2) \geq \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

In addition, we say that the function $f : I \to \mathbb{R}$ is *strictly convex* (resp. *strictly concave*) if for all $x_1 \neq x_2$ in I and $\lambda_1 > 0$, $\lambda_2 > 0$ such that $\lambda_1 + \lambda_2 = 1$ we have $f(\lambda_1 x_1 + \lambda_2 x_2) < \lambda_1 f(x_1) + \lambda_2 f(x_2)$ (resp. $f(\lambda_1 x_1 + \lambda_2 x_2) > \lambda_1 f(x_1) + \lambda_2 f(x_2)$).

The above definition of a convex function has a very simple meaning. For two points $x_1 < x_2$ every point x in between x_1 and x_2 can be expressed as a linear combination $x = \lambda_1 x_1 + \lambda_2 x_2$, where $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$. Indeed

$$\lambda_1 = \frac{x_2 - x}{x_2 - x_1}, \; \lambda_2 = \frac{x - x_1}{x_2 - x_1}.$$

Consider the points $A_1 = (x_1, f(x_1))$ and $A_2 = (x_2, f(x_2))$. Then the condition (6.12) implies that the arc of the graph of $f(x)$ between the points A_1 and A_2 must be located under the line segment joining A_1 and A_2 (they may touch but not cross). The following figure illustrates this situation.



Convex function.

The notion of a convex function was introduced by Johan L.W.V. Jensen. Notice that if $f(x)$ is convex, then $-f(x)$ is concave, i.e. the notions of convexity and concavity are 'dual'. Because of this duality it is sufficient to examine the properties of convex functions only. Similar properties for concave functions will follow.

Proposition 6.26. (Elementary Properties of Convex Functions)

(a) If $f(x)$ is convex and $c > 0$ then $y = cf(x)$ is also convex;
(b) If $f(x)$ and $g(x)$ are two convex functions, then $f(x) + g(x)$ is also convex;
(c) If $\varphi(x)$ is an increasing convex function and $f(x)$ is convex, then the composite function $\varphi(f(x))$ is also convex. We also have the following properties:

$\varphi(x)$	$u = f(x)$	$\varphi(f(x))$
convex decreasing	*concave*	*convex*
convave increasing	*concave*	*concave*
concave decreasing	*convex*	*concave*

(d) If $f(x)$ is one-to-one and $f^{-1}(x)$ denotes its inverse function, then we have

$f(x)$	$f^{-1}(x)$
convex increasing	*concave increasing*
convex decreasing	*convex decreasing*
concave decreasing	*concave decreasing*

(e) A non-constant convex function $f(x)$ defined on an open interval (a,b) can not attain its maximal value inside (a,b);

Proof: The proofs of (a) and (b) are very simple and are left as an exercise. In order to prove (c), we notice that since $\varphi(x)$ is increasing and $f(x)$ is convex, we have

$$\varphi(f(\lambda_1 x_1 + \lambda_2 x_2)) \le \varphi(\lambda_1 f(x_1) + \lambda_2 f(x_2)).$$

On the other hand $\varphi(x)$ is also convex, thus

$$\varphi(\lambda_1 f(x_1) + \lambda_2 f(x_2)) \le \lambda_1 \varphi(f(x_1)) + \lambda_2 \varphi(f(x_2)),$$

so

$$\varphi(f(\lambda_1 x_1 + \lambda_2 x_2)) \le \lambda_1 \varphi(f(x_1)) + \lambda_2 \varphi(f(x_2)),$$

and $\varphi(f(x))$ is convex.

(d): Assume that $f(x)$ is convex increasing and let $f(x_1) = y_1$ and $f(x_2) = y_2$. Then $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$. Since the inverse of an increasing function is also increasing, by applying f^{-1} to the inequality

$$f(\lambda_1 x_1 + \lambda_2 x_2) \le \lambda_1 f(x_1) + \lambda_2 f(x_2),$$

we obtain

$$\lambda_1 f^{-1}(y_1) + \lambda_2 f^{-1}(y_2) \le f^{-1}(\lambda_1 y_1 + \lambda_2 y_2),$$

and f^{-1} is also convex. The other cases are left as an exercise.

(e): Suppose that $f(x)$ attains its maximum at $x_o \in (a,b)$. Since the function $f(x)$ is non-constant, there exist points $x_1 < x_o < x_2$ such that either $f(x_1) < f(x_o)$ or $f(x_2) < f(x_o)$. Assume for example that $f(x_1) < f(x_o)$ and $f(x_2) \le f(x_o)$. Then $x_o = \lambda_1 x_1 + \lambda_2 x_2$, where

$$\lambda_1 = \frac{x_2 - x_o}{x_2 - x_1}, \; \lambda_2 = \frac{x_o - x_1}{x_2 - x_1},$$

and

$$f(x_o) = f(\lambda_1 x_1 + \lambda_2 x_2) > \lambda_1 f(x_1) + \lambda_2 f(x_2),$$

i.e. $f(x)$ can not be convex. $\square$

Proposition 6.27. *A function $f : [a,b] \to \mathbb{R}$ is convex if and only if for all x_1, x_2, $x \in I$ such that $x_1 < x < x_2$ we have*

$$(x_1 - x)f(x_1) + (x_1 - x_2)f(x) + (x - x_1)f(x_2) \geq 0, \tag{6.13}$$

or equivalently

$$\det \begin{pmatrix} 1 & x_1 & f(x_1) \\ 1 & x & f(x) \\ 1 & x_2 & f(x_2) \end{pmatrix} \geq 0. \tag{6.14}$$

Proof: It is clear that $f(x)$ is convex if and only if

$$f(x) \leq \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2),$$

so

$$(x_1 - x)f(x_1) + (x_1 - x_2)f(x) + (x - x_1)f(x_2) \geq 0.$$

□

Theorem 6.28. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. Then $f(x)$ is convex if and only if $f'(x)$ is non-decreasing, i.e.*

$$(f'(x_2) - f'(x_1))(x_2 - x_1) \geq 0, \quad \textit{for} \quad x_1,\, x_2 \in [a,b].$$

Moreover, if $f'(x)$ is increasing, then $f(x)$ is strictly convex.

Proof: Assume that the function $f(x)$ is convex. Then for $x_1 < x < x_2$, by (6.14),

$$\begin{aligned} 0 \leq \det \begin{pmatrix} 1 & x_1 & f(x_1) \\ 1 & x & f(x) \\ 1 & x_2 & f(x_2) \end{pmatrix} &= \det \begin{pmatrix} 0 & x_1 - x & f(x_1) - f(x) \\ 1 & x & f(x) \\ 0 & x_2 - x & f(x_2) - f(x) \end{pmatrix} \\ &= (x_2 - x)(f(x_1) - f(x)) - (x_1 - x)(f(x_2) - f(x)) \end{aligned}$$

thus

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}. \tag{6.15}$$

By passing to the limit $x \to x_1$ and $x \to x_2$, the (6.15) implies

$$f'(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{6.16}$$

and

$$f'(x_2) \geq \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

thus $f'(x_1) \leq f'(x_2)$, and consequently

$$(f'(x_2) - f'(x_1))(x_2 - x_1) \geq 0.$$

Conversely, assume that $f'(x)$ is non-decreasing. Let $x_1 < x < x_2$. Then by the Mean Value Theorem, there exist $\xi_1 \in (x_1, x)$ and $\xi_2 \in (x, x_2)$ such that

$$\frac{f(x)-f(x_1)}{x-x_1}=f'(\xi_1),\quad \frac{f(x_2)-f(x)}{x_2-x}=f'(\xi_2).$$

Since $f'(x)$ is non-decreasing, $f'(\xi_1)\le f'(\xi_2)$, so

$$\frac{f(x)-f(x_1)}{x-x_1}\le\frac{f(x_2)-f(x)}{x_2-x}$$

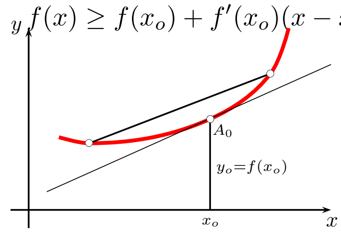
and the statement follows. □

Theorem 6.29. (Second Derivative Test for Convexity) *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f:[a,b]\to\mathbb{R}$ is continuous and the second derivative $f''(x)$ exists for all $x\in(a.b)$. Then the function $f(x)$ is convex in $[a,b]$ if and only if $f''(x)\ge 0$. Moreover, if $f''(x)>0$ for all $x\in(a,b)$, then $f(x)$ is strictly convex in $[a,b]$.*

Proof: By Proposition 6.13, the derivative $f'(x)$ is non-decreasing on $[a,b]$ if and only if $f''(x)\ge 0$ for all $x\in(a,b)$. Therefore, by Theorem 6.28, $f(x)$ is convex if and only if $f''(x)\ge 0$ for all $x\in(a,b)$. In the case where $f''(x)>0$ for all $x\in(a,b)$, Proposition 6.13 implies that $f'(x)$ is increasing, thus by Theorem 6.28 is strictly convex. □

Proposition 6.30. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function. Then $f(x)$ is convex if and only if any tangent line to the graph of $f(x)$ is located under the graph, i.e. for every x_o and $x\in[a,b]$ we have*

$$f(x)\ge f(x_o)+f'(x_o)(x-x_o). \tag{6.17}$$



Tangent to a convex function.

Proof: The inequality (6.17) is equivalent to the following two inequalities

$$f'(x_o)\begin{cases}\le\frac{f(x)-f(x_o)}{x-x_o} & \text{for } x>x_o,\\ \ge\frac{f(x)-f(x_o)}{x-x_o} & \text{for } x<x_o.\end{cases}$$

Consequently, for $x_1<x_2$, by taking $x=x_1$ and $x_o=x_2$ we obtain $f'(x_2)\ge\frac{f(x_1)-f(x_2)}{x_2-x_2}$, and by taking $x=x_2$ and $x_o=x_1$, we obtain $f'(x_1)\le\frac{f(x_1)-f(x_2)}{x_2-x_2}$, so $f'(x_2)\ge f'(x_1)$, and $f'(x)$ is non-decreasing. Consequently $f(x)$ is convex.

Conversely, suppose that $f(x)$ is convex, thus by Proposition 6.13, $f'(x)$ is increasing, thus by Lagrange Theorem, if $x>x_o$, then

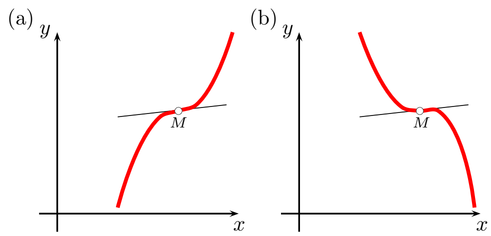
$$f'(x_o)\le f'(\xi_o)=\frac{f(x)-f(x_o)}{x-x_o}$$

for some $\xi_o\in(x_o,x)$, and if $x<x_o$, then

$$f'(x_o) \geq f'(\xi) = \frac{f(x) - f(x_o)}{x - x_o}$$

for some $\xi \in (x, x_o)$. Therefore, the inequality (6.17) follows. □

A point $M = (x_o, f(x_o))$ on the graph of a function $y = f(x)$ is called an *inflection point* if there is some $\delta > 0$ such that one of the restricted functions $f : (x_o - \delta, x_o] \to \mathbb{R}$ and $f : [x_o, x_o + \delta) \to \mathbb{R}$ is strictly convex while the other is strictly concave (see the figure below).



Inflection Points.

If the function $f(x)$ is twice differentiable, then $f(x)$ has an inflection point at $(x_o, f(x_o))$ if and only the first derivative $f'(x)$ is either decreasing on $(x_o - \delta, x_o]$ and increasing on $[x_o, x_o + \delta)$ (i.e. $f'(x)$ has a local minimum at x_o), or $f'(x)$ is increasing on $(x_o - \delta, x_o]$ and decreasing on $[x_o, x_o + \delta)$, (i.e. $f'(x)$ has local maximum at x_o). This is the case when the second derivative $f''(x)$ changes sign at x_o. On the other hand, by Fermat's Theorem, if $(x_o, f(x_o))$ is an inflection point, then $f''(x_o) = 0$. It is not difficult to see that this condition is a necessary but not sufficient condition for an inflection point. For example, the function $f(x) = x^4$ is strictly convex (i.e. it has no inflection point) but $f''(0) = 0$.

6.7 Properties of Convex Sets and Convex Functions

Consider the Euclidean space $\mathbb{R}^n$, $n \geq 1$. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is called *linear* if

$$\forall x, y \in \mathbb{R}^n \ \ \forall_{\alpha,\beta\in\mathbb{R}} \quad f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

A linear function $f : \mathbb{R}^n \to \mathbb{R}$ is also sometimes called a *linear functional* on $\mathbb{R}^n$.

Notice that for every linear function $f : \mathbb{R}^n \to \mathbb{R}$ there exists a unique vector $a \in \mathbb{R}^n$ such that

$$\forall_{x\in\mathbb{R}^n} \quad f(x) = x \bullet a = \sum_{k=1}^{n} x_x a_k,$$

where $a = (a_1, a_2, \dots, a_n)$ and $x = (x_1, x_2, \dots, x_n)$. Indeed, consider the standard basis $\{e_1, e_2, \dots, e_n\}$, $e_k = (0, 0, \dots, \underset{k\text{-th}}{1}, \dots, 0)$, $k = 1, \dots, n$, in $\mathbb{R}^n$. Then we have

$$\begin{aligned} f(x) &= f\left(\sum_{k=1}^{n} x_k e_k\right) = \sum_{k=1}^{n} f\left(\sum_{k=1}^{n} x_k f(e_k)\right) \\ &= \sum_{k=1}^{n} x_k a_k = x \bullet a, \end{aligned}$$

where $a = (a_1, a_2, \dots, a_n)$, $a_k = f(e_k)$, $k = 1, 2, \dots, n$).

6.7.1 Convex Sets and Hahn-Banach Theorem in $\mathbb{R}^n$

Definition 6.31. A set $C \subset \mathbb{R}^n$ is called *convex* if an only if

$$\forall_{x,y\in C}\ \ \forall_{t\in[0,1]}\quad tx + (1-t)y \in C. \tag{6.18}$$

One can easily prove that the convex sets satisfy the following properties:

Proposition 6.32. *(a) If $C \subset \mathbb{R}^n$ is a convex set, then $\overline{C}$ and $\operatorname{int}(C)$ are also convex sets.*
(b) Let $\{C_i\}_{i\in I}$ be an arbitrary collection of convex sets in $\mathbb{R}^n$. Then $\bigcap\limits_{i\in I} C_i$ is also a convex set in $\mathbb{R}^n$.
(c) Let $\{C_m\}_{m=1}^{\infty}$ be a sequence of convex sets in $\mathbb{R}^n$ such that $C_m \subset C_{m+1}$ for $m \geq 1$. Then $\bigcup\limits_{m=1}^{\infty} C_m$ is also a convex set in $\mathbb{R}^n$.

Definition 6.33. Let $\{x_1, x_2, \dots, x_m\} \subset \mathbb{R}^n$. A linear combination $\sum\limits_{k=1}^{m} \alpha_k x_k$ of the vectors $x_1, x_2, \dots, x_m$ is called a *convex combination* if $\sum\limits_{k=1}^{m} \alpha_k = 1$ and $\alpha_k \geq 1$ for all $k = 1, 2, \dots, m$.

Proposition 6.34. *A set $C \subset \mathbb{R}^n$ is convex if and only if for all vectors $x_1,\ x_2,\ \dots,\ x_m \in C$, $m \geq 2$, any convex combination $\sum\limits_{k=1}^{m} \alpha_k x_k$ of these vectors belongs to C, i.s.*

$$\forall_{\alpha_1,\alpha_2,\dots,\alpha_m\in[0,1]}\ \sum_{k=1}^{m} \alpha_k = 1 \implies \sum_{k=1}^{m} \alpha_k x_k \in C. \tag{6.19}$$

Proof: The implication '$\Longleftarrow$' is evident, because the condition (6.19) for $m = 2$, is equivalent to the condition (6.18).

We will prove the implication '$\Longrightarrow$'. Assume that C is a convex set. Clearly, the condition (6.19) is true for $m = 2$. We will apply the mathematical induction to show that (6.19) is true for any $m \geq 2$. Assume for induction that (6.19) is true for a certain $m \geq 2$ and consider a convex combination

$$\sum_{k=1}^{m+1} \widetilde{\alpha}_k x_k, \qquad \text{where}\quad x_1, x_2, \dots, x_m, x_{m+1} \in C,\quad \alpha_{m+1} \neq 1.$$

Then we have

$$\begin{aligned}\sum_{k=1}^{m+1} \widetilde{\alpha}_k x_k &= \sum_{k=1}^{m} \widetilde{\alpha}_k x_k + \widetilde{\alpha}_{m+1} x_{m+1} \\ &= (1-\alpha_{m+1}) \sum_{k=1}^{m} \frac{\widetilde{\alpha}_k}{1-\widetilde{\alpha}_{m+1}} x_k + \alpha_{m+1} x_{m+1}.\end{aligned}$$

Since

$$\sum_{k=1}^{m} \frac{\widetilde{\alpha}_k}{1-\widetilde{\alpha}_{m+1}} = \frac{1}{1-\widetilde{\alpha}_{m+1}} \sum_{k=1}^{m} \widetilde{\alpha}_k = \frac{1}{1-\widetilde{\alpha}_{m+1}}(1-\widetilde{\alpha}_{m+1}) = 1$$

thus $\displaystyle\sum_{k=1}^{m} \frac{\widetilde{\alpha}_k}{1-\widetilde{\alpha}_{m+1}} x_k$ is a convex combination of vectors x_1, x_2, ..., x_m, thus by the induction assumption $x := \displaystyle\sum_{k=1}^{m} \frac{\widetilde{\alpha}_k}{1-\widetilde{\alpha}_{m+1}} x_k \in C$. Put $t = \widetilde{\alpha}_{m+1}$, then we have

$$\begin{aligned} \sum_{k=1}^{m+1} \widetilde{\alpha}_k x_k &= (1-\alpha_{m+1}) \sum_{k=1}^{m} \frac{\widetilde{\alpha}_k}{1-\widetilde{\alpha}_{m+1}} x_k + \alpha_{m+1} x_{m+1} \\ &= (1-t)x + t x_{m+1} \in C \quad \text{by (6.18).} \end{aligned}$$

$\square$

Definition 6.35. Let $A \subset \mathbb{R}^n$ be a set. We define the *convex hull* $\operatorname{conv}(A)$ of A by

$$\operatorname{conv}(A) := \bigcap \{C : C \subset \mathbb{R}^n \quad \text{convex set such that} \quad A \subset C\}.$$

In other words (by Proposition 6.32(b)), $\operatorname{conv}(A)$ is the smallest convex set containing A.

Theorem 6.36. *Let A be a nonempty set in $\mathbb{R}^n$. Define for $m \geq 1$ the sets*

$$C_m(A) := \left\{ x : x = \sum_{k=1}^{m} \alpha_k x_k,\ \alpha_k \geq 0, \sum_{k=1}^{m} \alpha_k = 1,\ \text{for some} \quad x_1, x_2, \dots, x_m \in A \right\}$$

Then

$$\operatorname{conv}(A) = \sum_{m=1}^{\infty} C_m(A). \tag{6.20}$$

Proof: Put $\operatorname{conv}(A)$. Then by Proposition 6.34, $C_m(A) \subset \operatorname{conv}(A)$ for all $m \geq 1$, thus

$$\bigcup_{m=1}^{\infty} C_m(A) \subset \operatorname{conv}(A).$$

In order to prove that

$$\operatorname{conv}(A) \subset \bigcup_{m=1}^{\infty} C_m(A),$$

it is sufficient to show that the set $C := \displaystyle\bigcup_{m=1}^{\infty} C_m(A)$ is convex such that $A \subset C$. Then, by the definition of $\operatorname{conv}(A)$ (which is the smallest convex set containing A) we have $\operatorname{conv}(A) \subset C$. In order to show that C is convex, consider x, $y \in C$ and $t \in [0,1]$, i.e.there exist m, $m' \geq 1$ such that

$$x = \sum_{k=1}^{m} \alpha_k x_k, \quad y = \sum_{k'=1}^{m'} \beta_{k'} y_{k'}, \quad \{x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_{m'}\} \subset A,$$

where $\sum_{k=1}^{m} \alpha_k = 1$ and $\sum_{k'=1}^{m'} \beta_{k'} = 1$, $\alpha_k \geq 0$, $\beta_{k'} \geq 0$. Therefore, we have

$$(1-t)x + ty = \sum_{k=1}^{m}(1-t)\alpha_k x_k + \sum_{k'=1}^{m'} t\beta_{k'} y_{k'}. \tag{6.21}$$

Since

$$\sum_{k=1}^{m}(1-t)\alpha_k + \sum_{k'=1}^{m'} t\beta_{k'} = (1-t)\sum_{k=1}^{m}\alpha_k + t\sum_{k'=1}^{m'}\beta_{k'} = (1-t) + t = 1,$$

thus $\sum_{k=1}^{m}(1-t)\alpha_k x_k + \sum_{k'=1}^{m'} t\beta_{k'} y_{k'}$. is a convex combination of the vectors x_1, x_2, …, x_m, y_1, y_2, …, $y_{m'} \in A$, therefore it follows from (6.21) that $(1-t)x + ty \in C_{m+m'}(A) \subset C$. Therefore, C is convex. □

Theorem 6.37. (Caratheodory Theorem) *Let $A \subset \mathbb{R}^n$. Then*

$$\operatorname{conv}(A) = C_{n+1}(A).$$

Proof: We will show that for $m > n+1$, the convex combination

$$\sum_{i=1}^{m}\alpha_i x_i, \quad \{\alpha_i\} \in I_m, \ \ x_i \in A, \ \alpha_i > 0,$$

can be expressed as a convex combination

$$\sum_{i=1}^{m-1}\alpha_i' x_i', \quad \{\alpha_i'\} \in I_{m-1}, \ \ x_i' \in A.$$

Since $m-1 > n$, the vectors $x_m - x_1$, $x_{m-1} - x_1$, …, $x_2 - x_1$ are linearly depended, i.e. there exists the numbers $\beta_2, \beta_3, \dots, \beta_m$ (not all of them equal to zero) such that

$$\sum_{i=2}^{m}\beta_i(x_i - x_1) = 0.$$

Put $\beta_1 := -\sum_{i=2}^{m}\beta_i$, then we have

$$\sum_{i=1}^{m}\beta_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^{m}\beta_i = 0.$$

Then, for every $\varepsilon > 0$ we can write

$$\begin{aligned}\sum_{i=1}^{m}\alpha_i x_i &= \sum_{i=1}^{m}\alpha_i x_i - \varepsilon\sum_{i=1}^{m}\beta_i x_i \\ &= \sum_{i=1}^{m}(\alpha_i - \varepsilon\beta_i)x_i\end{aligned}$$

Notice that $\sum_{i=1}^{m}(\alpha_i - \varepsilon\beta_i) = 1$. Clearly,

$$\beta_{i_o} = \max\{\beta_i : i = 1, 2, \dots, m\} > 0$$

(since non of the numbers β_i are zero and $\sum_{i=1}^{m} \beta_i = 0$). Ten we can take $\varepsilon = \frac{\alpha_{i_o}}{\beta_{i_o}}$ and put $\alpha_i' := \alpha_i - \varepsilon\beta_i$. Clearly, $\alpha_i \geq 0$ and $\alpha_{i_o} = 0$. which implies that

$$\sum_{i=1,\, i\neq i_o}^{m} \alpha_i' x_i = \sum_{i=1}^{m} \alpha_i x_i$$

is required convex combination of $m-1$ vectors. □

Let us define for two nonempty sets A and B in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$ the sets

$$A + B := \{a + b : a \in A \text{ and } b \in B\} \quad \text{and} \quad \alpha A := \{\alpha x : x \in A\}.$$

Then we have the following proposition (which is easy to prove using the definition of a convex set, thus its proof is left to the reader):

Proposition 6.38. *Let C_1 and C_2 be two nonempty convex sets in $\mathbb{R}^n$ and α_1, $\alpha_2 \in \mathbb{R}$. Then the set $\alpha_1 C_1 + \alpha_2 C_2$ is convex.*

Definition 6.39. A *hyperplane* H in $\mathbb{R}^n$ is defined as

$$H := \{x \in \mathbb{R}^n : f(x) = \alpha\},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a non-zero linear function and $\alpha \in \mathbb{R}$ a fixed number. In such a case we will use the following notation $H = [f = \alpha]$. By applying the representation $f(x) = x \bullet a$, $x \in \mathbb{R}^n$ (for certain fixed non-zero vector $a \in \mathbb{R}^n$), then we have

$$H = \{x : x \bullet a = \alpha\}.$$

Definition 6.40. Let A and B be two nonempty sets in $\mathbb{R}^n$. We say that a hyperplane $H = [f = \alpha]$ *separates A from B* if

$$\begin{cases} \forall_{x\in A} & f(x) \leq \alpha \\ \forall_{x\in B} & f(x) \geq \alpha. \end{cases}$$

(See the Figure 6.1)

Our goal is to prove the following important theorem:

Theorem 6.41. (Hahn-Banach Theorem–geometric version) *Let A and B bet two non-empty disjoint convex sets in $\mathbb{R}^n$ such that A is an open set. Then there exists a hyperplane $H = [f = \alpha]$ which separates A from B.*

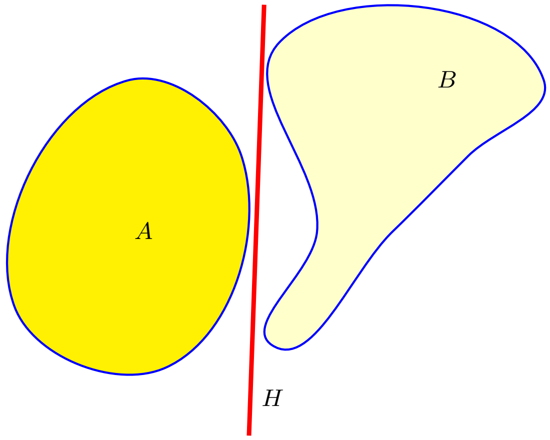


Fig. 6.1. Separation of two sets A and B by a hyperplane $H = [f = \alpha]$ in $\mathbb{R}^2$

Before we are able to prove Theorem 6.41, we need some additional results.

Lemma 6.42. *Let $C \subset \mathbb{R}^n$ be an open convex set such that $0 \in C$. Then the function* $p : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$\forall_{x\in\mathbb{R}^n} \quad p(x) := \inf\{\alpha > 0 : x \in \alpha C\}$$

satisfies the following properties:

(i) $\forall_{x\in\mathbb{R}^n}\ \forall_{\lambda>0} \quad p(\lambda x) = \lambda p(x)$;
(ii) $\forall_{x,y\in\mathbb{R}^n} \quad p(x+y) \le p(x) + p(y)$.
(iii) $\exists_{M>0}\ \forall_{x\in\mathbb{R}^n} \quad 0 \le p(x) \le M\|x\|$;
(iv) $C = \{x \in \mathbb{R}^n : p(x) < 1\} = p^{-1}([0,1))$;

Proof: (i): Let $\lambda > 0$ and $x \neq 0$. Then we have

$$\begin{aligned} p(\lambda x) &= \inf\{\alpha > 0 : \lambda x \in \alpha C\} \\ &= \inf\{\alpha > 0 : x \in \alpha\lambda^{-1} C\} \\ &= \lambda \inf\{\alpha' > 0 : x \in \alpha' C\} = \lambda p(x). \end{aligned}$$

where we used the substitution $\alpha' := \alpha\lambda^{-1}$.

(iii): Since C is open and $0 \in C$ there exists $r > 0$ such that $B_r(0) \subset C$, therefore

$$\forall_{x\in\mathbb{R}^n} \quad \|x\| < r \implies p(x) := \inf\{\alpha : x \in \alpha C\} \le 1.$$

. Therefore for $\varepsilon > 0$ and for every non-zero vector $x \in \mathbb{R}^n$ we have (by applying (a)) that

$$\left\| \frac{r}{\|x\| + \varepsilon} x \right\| = \frac{r\|x\|}{\|x\| + \varepsilon} < r \implies p\left(\frac{r}{\|x\| + \varepsilon} x\right) = \frac{rp(x)}{\|x\| + \varepsilon} \le 1$$

therefore

* Such a function $p : \mathbb{R}^n \to \mathbb{R}$ is called a *Minkowsky functional* associated with the convex set C

$$\forall_{\varepsilon>0} \quad p(x) \le \frac{1}{r}(\|x\|+\varepsilon) \quad \Longrightarrow \quad p(x) \le \frac{1}{r}\|x\|.$$

In order to prove (ii), consider x, $y \in \mathbb{R}^n$ and let $\varepsilon > 0$. By definition of the function p, $\frac{x}{p(x)+\varepsilon} \in C$ and $\frac{y}{p(y)+\varepsilon} \in C$. Thus, $\frac{tx}{p(x)+\varepsilon} + \frac{(1-t)y}{p(y)+\varepsilon} \in C$ for all $t \in [0,1]$. In particular for $t = \frac{p(x)+\varepsilon}{p(x)+p(y)+2\varepsilon}$ we obtain

$$\forall_{\varepsilon>0} \quad \frac{x+y}{p(x)+p(y)+2\varepsilon} \in C \quad \Longrightarrow \quad x+y \in (p(x)+p(y)+2\varepsilon)C,$$

thus

$$\forall_{\varepsilon>0} \;\; p(x+y) \le p(x)+p(y)+2\varepsilon \quad \Longrightarrow \quad p(x+y) \le p(x)+p(y).$$

(iv): Let $x \in C$ be such that $x \neq 0$. Since C is open there exists $r > 0$ such that $B_r(x) \subset C$. Then for every $\delta > 0$ such that $\delta < \frac{r}{\|x\|}$ we have

$$\|x-(1+\delta)x\| = \delta\|x\| < r \quad \Rightarrow \quad (1+\delta)x \in B_r(x) \subset C \quad \Rightarrow \quad x \in \frac{1}{1+\delta}C \quad \Rightarrow \quad p(x) \le \frac{1}{1+\delta} < 1$$

Thus $C \in \{x : p(x) < 1\}$. Conversely, if $p(x) < 1$, then there exists $0 < \alpha < 1$ such that $x \in \alpha C$. Since $\alpha^{-1}x \in C$ and C is convex, thus $x = \alpha(\alpha^{-1}x) + (1-\alpha)0 \in C$.

□

Lemma 6.43. (Hahn-Banach Theorem)] *Let $p : \mathbb{R}^n \to \mathbb{R}$ be a map satisfying*

$$\forall_{x\in\mathbb{R}^n}, \;\; \forall_{\lambda>0} \;\; p(\lambda x) = \lambda p(x) \tag{6.22}$$

$$\forall_{x,y\in\mathbb{R}^n} \;\; p(x+y) \le p(x)+p(y). \tag{6.23}$$

Suppose that $V \subset \mathbb{R}^n$ is a subspace and $g : V \to \mathbb{R}$ a linear function such that

$$\forall_{x\in V} \;\; g(x) \le p(x). \tag{6.24}$$

Then there exists a linear function $f : \mathbb{R}^n \to \mathbb{R}$ which extends g to $\mathbb{R}^n$, i.e.

$$\forall_{x\in V} \;\; g(x) = f(x) \quad \textit{and such that} \quad \forall_{x\in\mathbb{R}^n} \;\; f(x) \le p(x).$$

Proof: We will by applying the principle of mathematical induction that there exists a sequence of subspaces V_k, $k = 0, 1, \ldots, m$ ($m := n - \dim V$), of $\mathbb{R}^n$ and linear functions functions $g_k : V_k \to \mathbb{R}$ satisfying the properties

(a) $\dim V_k = \dim V + k$, $k = 0, 1, \ldots, m$,
(b) $V_0 = V$ and

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_m = \mathbb{R}^n,$$

(c) $g_0 = g : V \to \mathbb{R}$ and $\forall_{x\in\mathbb{R}^n} \;\; g_k(x) \le p(x)$.
(d) For $k, l \in \{0, 1, \ldots, m\}$ we have

$$\forall_{x\in V_i} \;\; g_k(x) = g_l(x), \quad \text{where} \quad i = \min\{k, l\}. \tag{6.25}$$

Assume for the induction (with respect that we have constructed the subspaces V_k and linear functions $g_k : V_k \to \mathbb{R}$ for $k = 0, 1, \dots, j$, where $0 \le j < m$, that satisfy the above conditions (a), (b), (c) and (d). We will construct the space V_{j+1} and the functional $g_{j+1} : V_{j+1} \to \mathbb{R}$ such that $\dim V_{j+1} = \dim V_j + 1$, $g_{j+1}(y) = g_j(y)$ for all $y \in V_j$ and $g_{j+1}(x) \le p(x)$ for all $x \in V_{j+1}$. Since $V_j \ne \mathbb{R}^n$, there exists $x_0 \notin V_j$. We put $V_{j+1} = \operatorname{span}\{V_j, x_0\}$, i.e. $V_{j+1} = \{x : x = y + tx_0$ where $y \in V_j$ and $t \in \mathbb{R}\}$. Then, we define $g_{j+1} : V_{j+1} \to \mathbb{R}$ by

$$g_{j+1}(y + tx_0) = g_j(y) + t\alpha,$$

where α is a constant to be determined later in such a way that

$$\forall_{x \in V_{j+1}}. \quad g_{j+1}(x) \le p(x). \tag{6.26}$$

The condition (6.26) can be written as

$$\forall_{y \in V_{j+1}} \forall_{t \in \mathbb{R}} \;\; g_j(y) + t\alpha \le p(y + tx_0)$$

should be satisfied. By (6.22), the above condition is equivalent to

$$\begin{cases} \forall_{y \in V_j} & g_j(y) + \alpha \le p(y + x_0) \\ \forall_{y \in V_j} & g_j(y) - \alpha \le p(y - x_0). \end{cases}$$

In other words, we need to choose α such that

$$\sup\{g_j(y) - p(y - x_0) : y \in V_j\} \le \alpha \le \inf\{p(y' + x_0) - g_j(y') : y' \in V_j\}. \tag{6.27}$$

Since

$$g_j(y') + g_j(y) = g_j(y' + y) \le p(y' + y) \le p(y' + x_0) + p(y - x_0)$$

we have

$$\forall_{y' \in V_j} \forall_{y \in V_j} \;\; g_j(y) - p(y - x_0) \le p(y' + x_0) - g_j(y')$$

therefore

$$a := \sup\{g_j(y) - p(y - x_0) : y \in V_j\} \le \inf\{p(y' + x_0) - g_j(y') : y' \in V_j\} =: b.$$

Therefore, if we choose $\alpha \in [a, b]$, then the condition (6.26) is satisfied and g_{j+1} has the required properties. Therefore, by the principle of mathematical induction, the extension $f := g_m$ exists. □

Lemma 6.44. *Let $\widetilde{C} \subset \mathbb{R}^n$ be an open non-empty convex and let $x_o \in \mathbb{R}^n$ such that $x_o \notin \widetilde{C}$. Then, there exists a linear function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) < f(x_o)$ for all $x \in \widetilde{C}$. In particular, the hyperplane with equation $[f = f(x_o)]$ separates $\{x_o\}$ from $\widetilde{C}$.*

Proof: Consider $y_o \in \widetilde{C}$ and define $C := \widetilde{C} - \{y_o\}$, $x_1 := x_o - y_o$. Clearly C is an open convex set such that $0 \in C$, therefore one can define the function

$$p(x) := \inf\{\alpha > 0 : x \in \alpha C\}.$$

By Lemma 6.42, satisfies the properties (6.22) and (6.23). Put $V := \operatorname{span}\{x_1\}$, and define $g : V \to \mathbb{R}$ by

$$\forall_{t\in\mathbb{R}} \;\; g(tx_1) = t.$$

Notice that

$$\forall_{x\in V} \;\; g(x) \le p(x);$$

Indeed, take $x = tx_0$ and consider two cases $t > 0$ and $t \le 0$): for $t > 0$, since $x_1 \notin C$ thus (by Lemma 6.42) $p(x_1) \ge 1$, thus

$$t = g(tx_1) = g(x) \le p(x) = p(tx_1) = tp(x_1).$$

By Hahn-Banach Theorem (cf. Lemma 6.43), there exists a linear function $f : \mathbb{R}^n \to \mathbb{R}$ extending g and such that

$$\forall_{x\in\mathbb{R}^n} \;\; f(x) \le p(x).$$

In particular we have $f(x_1) = 1$ and (by Lemma 6.42(iv)) $f(x) < 1$ for all $x \in C$.

Therefore, we also have $f(x_o) = f(x_1 + y_o) = f(x_1) + f(y_o) = 1 + f(y_o)$ and

$$\forall_{x\in\widetilde{C}} \;\; f(x) = f(x - y_o + y_o) = f(x - y_o) + f(y_o) < 1 + f(y_o).$$

□

Proof of Theorem 6.41: We put $C = A - B = \{x : x = a - b, a \in A, b \in B\}$. Notice that (by Proposition 6.38) C is convex and since

$$C = \bigcup_{y\in B} (A - \{y\})$$

it is also open. Moreover $0 \notin C$ (since $A \cap B = \emptyset$). By Lemma 6.44, there exists a linear function $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\forall_{z\in C} \;\; f(z) > 0.$$

Since $z \in C$ can be represented as $z = x - y$, where $x \in A$, $y \in B$, thus

$$\forall_{x\in A} \, \forall_{y\in B} \;\; f(x) < f(y).$$

We fix $\alpha \in \mathbb{R}$ such that

$$\sup\{f(x) : x \in A\} \le \alpha \le \inf\{f(y) : y \in B\},$$

and consequently the hyperplane $[f = \alpha]$ separates A from B. □

6.7.2 Convex Functions on $\mathbb{R}^n$

Definition 6.45. A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is called *convex* if and only if

$$\forall_{x,y\in\mathbb{R}^n} \;\; \forall_{t\in[0,1]} \;\; \varphi((1-t)x + ty) \le (1-t)\varphi(x) + t\varphi(y). \tag{6.28}$$

Proposition 6.46. *If a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is convex, then for every $\alpha \in \mathbb{R}$ the set*

$$\varphi^{-1}((-\infty, \alpha]) := \{x \in \mathbb{R}^n : \varphi(x) \le \alpha\}$$

is convex.

Proof: Assume x, $y \in \varphi^{-1}((-\infty, \alpha])$ and $t \in [0,1]$. Then we have

$$\varphi((1-t)x + ty) \le (1-t)\varphi(x) + t\varphi(y) \le (1-t)\alpha + t\alpha = \alpha,$$

thus $(1-t)x + ty \in \varphi^{-1}((-\infty, \alpha])$. Therefore, $\varphi^{-1}((-\infty, \alpha])$ is convex. $\square$

Definition 6.47. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a function. The set

$$\mathrm{epi}(\varphi) := \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : \varphi(x) \le \lambda\}$$

is called the *epigraph* of the function φ.

Proposition 6.48. *A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if* $\mathrm{epi}(\varphi)$ *is a convex set in* $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$.

Proof: Assume that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is convex and assume that (x, λ_1), $(y, \lambda_2) \in \mathrm{epi}(\varphi)$, i.e. $\varphi(x) \le \lambda_1$ and $\varphi(y) \le \lambda_2$. Then for $t \in [0,1]$, we have

$$\varphi((1-t)x + ty) \le (1-t)\varphi(x) + t\varphi(y) \le (1-t)\lambda_1 + t\lambda_2$$

thus

$$(1-t)(x, \lambda_1) + t(y, \lambda_2) = ((1-t)x + ty, (1-t)\lambda + t\lambda_2) \in \mathrm{epi}(\varphi).$$

Consequently, $\mathrm{epi}(\varphi)$ is convex.

Conversely, assume that $\mathrm{epi}(\varphi)$ is convex. Consider x and $y \in \mathbb{R}^n$. Since $(x, \varphi(x))$, $(y, \varphi(y)) \in \mathrm{epi}(\varphi)$, then for $t \in [0,1]$ we have

$$((1-t)x + ty, (1-t)\varphi(x) + t\varphi(y)) = (1-t)(x, \varphi(x)) + t(y, \varphi(y)) \in \mathrm{epi}(\varphi),$$

which means

$$\varphi((1-t)x + ty) \le (1-t)\varphi(x) + t\varphi(y),$$

thus φ is a convex function. $\square$

We leave the proof or the following result as an exercise:

Lemma 6.49. *A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is convex if an only if for any convex combination $\displaystyle\sum_{k=1}^{m} \alpha_k x_k$ of vectors x_1, $x_2, \dots$, x_m in $\mathbb{R}^n$ (i.e. $\displaystyle\sum_{k=1}^{m} \alpha_k = 1$, $\alpha_k \ge 0$) we have*

$$\varphi\left(\sum_{k=1}^{m} \alpha_k x_k\right) \le \sum_{k=1}^{m} \alpha_k \varphi(x_k).$$

Lemma 6.50. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $A \subset \mathbb{R}^n$ a nonempty set. If* $\sup\{\varphi(x) : x \in A\}$ *is finite then we have the following equality*

$$\sup\{\varphi(x) : x \in A\} = \sup\{\varphi(x) : x \in \mathrm{conv}(A)\}.$$

Proof: Put $a : \sup\{\varphi(x) : x \in A\}$ and $b := \sup\{\varphi(x) : x \in \operatorname{conv}(A)\}$. Since $A \subset \operatorname{conv}(A)$ we have that

$$\sup\{\varphi(x) : x \in A\} \leq \sup\{\varphi(x) : x \in \operatorname{conv}(A)\},$$

i.e.$a \leq b$. In order to show that $b \leq a$, consider an arbitrary $x \in \operatorname{conv}(A)$. By Theorem 6.36, there exists $x_1, x_2, \dots, x_m \in A$ such that

$$x = \sum_{k=1}^{m} \alpha_k x_k, \quad \text{for some } \alpha_k \geq 0 \quad \text{such that } \sum_{k=1}^{m} \alpha_k = 1.$$

Then, by Lemma 6.49

$$\varphi(x) = \varphi\left(\sum_{k=1}^{m} \alpha_k x_k\right) \leq \sum_{k=1}^{m} \alpha_k \varphi(x_k) \leq \sum_{k=1}^{m} \alpha_k a = a,$$

which implies that a is un upper bound of the set $\{\varphi(x) : x \in \operatorname{conv}(A)\}$. Therefore, $b \leq a$. □

Lemma 6.51. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $C := \operatorname{epi}(\varphi)$. Then $\operatorname{int}(C) \neq \emptyset$. More precisely, for every $x_o \in \mathbb{R}^n$ there exists λ such that $(x_o, \lambda) \in \operatorname{int}(C)$.*

Proof: Let $x_o \in \mathbb{R}^n$, $r > 0$. Consider the set $A := \{x_o \pm re_k : k = 1, 2, \dots, n\}$. Then, by Lemma 6.50

$$\lambda_o := \sup\{\varphi(x) : x \in \operatorname{conv}(A)\} = \max\{\varphi(x_o \pm re_k) : k = 1, 2, \dots, n\}.$$

Notice that $B_r(x_o) \subset \operatorname{conv}(A)$ and

$$B_r((x_o, \lambda_o + r) \subset \operatorname{epi}(\varphi) \subset \mathbb{R}^{n+1}.$$

Consequently, $(x_o, \lambda_o + r) \in \operatorname{int}(\operatorname{epi}(\varphi)) \neq \emptyset$. □

Lemma 6.52. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then for $x_o \in \mathbb{R}^n$ there exists a linear function $f : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\forall_{x \in \mathbb{R}^n} \quad \varphi(x) - \varphi(x_o) - f(x - x_o) \geq 0. \tag{6.29}$$

Proof: Put $C := \operatorname{int}(\operatorname{epi}(\varphi))$. Since C is open, nonempty and convex and $(x_o, \varphi(x_o)) \notin \operatorname{epi}(\varphi)$, by Theorem 6.41, there exist a non-zero linear function $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that

$$\begin{cases} \forall_{(x,\lambda) \in C} & \alpha < F(x, \lambda) \\ & \alpha = F(x_o, \varphi(x_o)). \end{cases}$$

Since F is continuous, it follows that (since points from ∂C are limit points of C and $\operatorname{epi}(\varphi) \subset \overline{C}$)

$$\begin{cases} \forall_{(x,\lambda) \in \operatorname{epi}(\varphi)} & \alpha \leq F(x, \lambda) \\ & \alpha = F(x_o, \varphi(x_o)). \end{cases}$$

The linear function F can be represented as

$$F(x, \lambda) = x \bullet a + \lambda \kappa, \quad x \in \mathbb{R}^n,$$

where $a \in \mathbb{R}^n$ and $\kappa \in \mathbb{R}$. Since

$$\forall_{(x,\lambda)\in \operatorname{epi}(\varphi)} \quad x_o \bullet a + \varphi(x_o)\kappa \le x \bullet a + \lambda\kappa,$$

by taking $x = x_o$ we obtain that $\kappa \ge 0$. Notice that κ cannot be zero. Indeed, if $\kappa = 0$ then we obtain that for every $x \in \mathbb{R}^n$ we have

$$x_o \bullet a \le x \bullet a \quad (x - x_o) \bullet a \le 0 \quad a = 0,$$

thus F is a zero function, which is a contradiction. Therefore, $\kappa > 0$ so we can define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \frac{1}{\kappa} a \bullet x$, $x \in \mathbb{R}^n$. Since $(x, \varphi(x)) \in \operatorname{epi}(\varphi)$, we have

$$\forall_{x\in\mathbb{R}^n} \quad f(x_o) + \varphi(x_o) \le f(x) + \varphi(x) \iff \forall_{x\in\mathbb{R}^n} \varphi(x) = f(x - x_o) - \varphi(x_o) \ge 0.$$

which concludes the proof. □

The goal of this subsection is the following important result:

Theorem 6.53. *Every convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuous.*

Proof: Assume that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is convex. Consider $x_o \in \mathbb{R}^n$ and assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a linear function (by Lemma 6.52) satisfying condition (6.29). Then the function $\widetilde{\varphi} : \mathbb{R}^n \to \mathbb{R}$, defined by

$$\forall_{x\in\mathbb{R}^n} \quad \widetilde{\varphi}(x) = \varphi(x) - \varphi(x_o) - f(x - x_o),$$

satisfies the conditions $\widetilde{\varphi}(x) \ge 0$ and $\widetilde{\varphi}(x_o) = 0$. Moreover, since f is continuous, φ is continuous if and only if $\widetilde{\varphi}$ is continuous. Therefore, in order to show that φ is continuous at x_o, it is sufficient to show that $\widetilde{\varphi}$ is continuous at x_o. By shifting the function $\widetilde{\varphi}$ we can always assume that $x_o = 0$. Consequently, we only need to show that the convex function $\widetilde{\varphi} : \mathbb{R}^n \to \mathbb{R}$ such that $\widetilde{\varphi}(x) \ge 0$, $\widetilde{\varphi}(0) = 0$, is continuous at zero. Put

$$C := \operatorname{conv}\{\pm e_k : k = 1, 2, \dots, n\}.$$

Since $0 \in C$, thus for every $r > 0$ we have (by Lemma 6.50)

$$\sup\{\widetilde{\varphi}(x) : x \in rC\} \le r \max\{\varphi(\pm e_k) : k = 1, 2, \dots, n\}.$$

Since $B_r(0) \subset rC$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\delta \max\{\widetilde{\varphi}(\pm e_k) : k = 1, 2, \dots, n\} < \varepsilon.$$

Therefore, we have

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x\in\mathbb{R}^n} \quad \|x\| < \delta \implies 0 \le \widetilde{\varphi}(x) < \varepsilon,$$

which implies that $\widetilde{\varphi}$ is continuous at zero. □

6.8 Problems

1. Discuss the differentiability at $x = 0$ of the function

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

2. Discuss the differentiability at $x = 0$ of the function

$$f(x) = \begin{cases} x^n \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

where n is an integer larger than 1. For what values of k, does the k-th derivative exist at $x = 0$?

3. Evaluate the following limit:

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}.$$

4. Suppose that f is defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

(a) Show that $f^{(n)}(0)$ exists for every positive integer n and has the value 0;

5. Let $1 > a > 0$ and $f(x) = (a^x + 1)^{\frac{1}{x}}$, show that the function f is decreasing for $x > 0$.

6. Use result in Problem 5 to prove the following inequality

$$(x^a + y^a)^{\frac{1}{a}} > (x^b + y^b)^{\frac{1}{b}}, \tag{6.30}$$

for all $x > 0$, $y > 0$ and $0 < a < b$.

7. Prove the following inequality for all $x > 0$

$$1 + 2 \ln x \leq x^2.$$

8. Compute the limit

$$\lim_{x \to 1} \frac{\ln(\cosh x)}{\ln(\cos x)}.$$

9. Prove the following theorem:

Theorem 6.54. *Let $f : (a, b) \to \mathbb{R}$ be a differentiable function which is unbounded. Then the derivative $f' : (a, b) \to \mathbb{R}$ is also unbounded.*

10. Give an example of a bounded differentiable function $f : (a,b) \to \mathbb{R}$ such that the derivative $f' : (a,b) \to \mathbb{R}$ is unbounded. What it has to do with the fact stated in the previous question?

11. Prove the following theorem:

Theorem 6.55. *Let $f : (a,b) \to \mathbb{R}$ be a differentiable function such that $f' : (a,b) \to \mathbb{R}$ is bounded. Then the function $f : (a,b) \to \mathbb{R}$ is uniformly continuous.*

12. Compute the derivative of the following functions:

(a) $f(x) = e^x + e^{e^x} + e^{e^{e^x}}$;
(b) $f(x) = \left(\frac{a}{b}\right)^x \left(\frac{b}{x}\right)^a \left(\frac{x}{a}\right)^b$, where $a > 0$ and $b > 0$;
(c) $f(x) = \ln \frac{b + a\cos x + \sqrt{b^2 - a^2}\sin x}{a + b\cos x}$, where $0 \le |a| < |b|$;

13. For what values of the integers n and m, where $m > 0$, the function

$$f(x) = \begin{cases} |x|^n \sin \frac{1}{|x|^m} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at $x = 0$.

14. Find $y^{(n)}$ for the following functions

(a) $y = \sin ax \cos bx$;
(b) $y = x^2 \sin ax$;
(c) $y = \cosh ax \cos bx$;

15. Explain why the functions $f(x) = x^2$ and $g(x) = x^3$ do not satisfy the assumptions of the Cauchy's Theorem (the conclusion of this theorem is also not true) on the interval $[-1, 1]$.

16. Prove the following inequalities

(a) $x - \frac{x^2}{2} < \ln(1+x) < x$ for all $x > 0$;
(b) $x - \frac{x^3}{6} < \sin x < x$ for all $x > 0$;
(c) $(x^a + y^b)^{\frac{1}{a}} > (x^b + y^b)^{\frac{1}{b}}$ for $x > 0$, $y > 0$ and $0 < a < b$;
(d) $x^a - 1 > a(x-1)$ for $a \ge 2$, $x > 1$;

17. Let $f(x)$ be a twice differentiable function in an interval $[a, \infty)$ such that (i) $f(a) = A > 0$; (ii) $f'(a) < 0$; (iii) $f''(x) \le 0$ for $x > a$. Show that the equation $f(x) = 0$ has exactly one root in the interval (a, ∞).

18. Use l'Hôpital's Rule to compute the following limits

(a) $\lim_{x\to 0} \frac{\cosh x - \cos x}{x^2}$;
(b) $\lim_{x\to 0} \frac{\arcsin 2x - 2\arcsin x}{x^3}$;
(c) $\lim_{x\to\infty} \frac{x^{\ln x}}{(\ln x)^x}$.

19. Show that

$$\int_0^a x^3 f(x^2)dx = \tfrac{1}{2}\int_0^{a^2} xf(x)\,dx,$$

where $a > 0$ and $f(x)$ is a continuous function on the interval $[0, a^2]$.

20. Consider the function $f : [-a, a] \to \mathbb{R}$, $a > 0$, defined by

$$f(x) = \cosh(3x), \quad x \in [a, b].$$

(a) Compute the n-th derivative of the function f.
(b) For a given $n \in \mathbb{N}$, write the Taylor polynomial $T_n(x)$ of f (centered at $x_o = 0$).
(c) For a given $\varepsilon > 0$, find $n \in \mathbb{N}$ such that

$$\sup\{|f(x) - T_n(x)| : x \in [-a, a]\} < \varepsilon.$$

Hint: Follow the same steps to those presented in Eample 6.24

21. Consider the function $f : [-1, 1] \to \mathbb{R}$, defined by

$$f(x) = \frac{1}{\sqrt{1 + \frac{1}{2}x}}, \quad x \in [-1, 1].$$

(a) Compute the n-th derivative of the function f.
(b) For a given $n \in \mathbb{N}$, write the Taylor polynomial $T_n(x)$ of f (centered at $x_o = 0$).
(c) For a given $\varepsilon > 0$, find $n \in \mathbb{N}$ such that

$$\sup\{|f(x) - T_n(x)| : x \in [-1, 1]\} < \varepsilon.$$

Hint: Review derivatives of the function $f(x) = (1+x)^\alpha$, the binomial series and facts related to its convergence.

22. Prove that if $C \subset \mathbb{R}^n$ is a convex set, then its closure $\overline{C}$ and its interior $\text{int}(C)$ are also convex sets.
Hint: Use Corollary **??** and the definition of the interior.

23. Prove that for an arbitrary collection $\{C_i\}_{i\in I}$ of convex sets in $\mathbb{R}^n$ their intersection $\bigcap_{i\in I} C_i$ is a convex set in $\mathbb{R}^n$.

24. Suppose that $\{C_m\}_{m=1}^\infty$ is a sequence of convex sets in $\mathbb{R}^n$ such that $C_m \subset C_{m+1}$ for $m \geq 1$. Show that $\bigcup_{m=1}^\infty C_m$ is also a convex set in $\mathbb{R}^n$.

Hint: Use the definition of generalized intersections and the definition of the convex set.

25. Suppose that $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary norm on $\mathbb{R}^n$ (see Definition 3.3 in subsection 3.1.2) and define

$$A := \{x \in \mathbb{R}^n : \|x\| \le 1\}$$

Show that A is convex.

26. Let U be a open set in $\mathbb{R}^n$. Show that convex hull $\operatorname{conv}(U)$ of U is also open.
Hint: Use Theorem 6.36 – show that the sets $C_m(U)$ are open

27. Give an example of a closed set $A \subset \mathbb{R}^2$ such that its convex hull $\operatorname{conv}(A)$ is not closed.

28. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $A \subset \mathbb{R}^n$ be a bounded set. Prove that $f(A)$ is a bounded set in $\mathbb{R}$.
Hint: Use Theorem 6.53

29. Show that if $A \subset \mathbb{R}^n$ is a compact set then $\operatorname{conv}(A)$ is also compact.
Use Caratheodory Theorem 6.37 (provide the proof of this theorem) and show that $C_{n+1}(A)$ is compact.

Part III

MULTIVARIABLE FUNCTIONS AND CALCULUS IN BANACH SPACES

7

Differential Calculus in Banach Spaces

7.1 Differentiation of Maps Between Normed Spaces

7.1.1 Notion of a Limit

Let V and W be two normed spaces, D a set in V and $F : D \to W$ a map. Suppose that x_o is a limit point of D (i.e. $x_o \in D'$). Then we have the following concept of a *limit* of F at the point x_o:

$$\lim_{x \to x_o} F(x) = y_o \iff \forall_{\varepsilon > 0}\ \exists_{\delta > 0}\ \forall_{x \in D}\ \ 0 < \|x - x_o\| < \delta \ \Rightarrow\ \|F(x) - y_o\| < \varepsilon.$$

Then we will sat that F has a limit $y_o \in W$ at x_o.

In the simple case of a real function $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^2$, of two variables (x, y), we have

$$\lim_{(x,y) \to (x_o,y_o)} f(x, y) = L \iff$$
$$\forall_{\varepsilon > 0}\ \exists_{\delta > 0}\ \forall_{(x,y) \in D}\ \ 0 < \sqrt{(x - x_o)^2 + (y - y_o)^2} < \delta \ \Rightarrow\ |f(x, y) - L| < \varepsilon.$$

The properties of limits are summarized in the following proposition (we leave the proof as an exercise):

Proposition 7.1. *Let V and W be two normed spaces, D a set in V such that x_o is a limit point of D, and $F, G, H : D \to W$ three maps. Suppose that the limits*

$$\lim_{x \to x_o} F(x) = y_1, \quad \lim_{x \to x_o} G(x) = y_2,$$

exist. Then we have the following properties:

(i) (Linearity Property) $\lim_{x \to x_o} (\alpha F(x) + \beta G(x)) = \alpha y_1 + \beta y_2$ *for all* $\alpha, \beta \in \mathbb{R}$*;*
(ii)(Squeeze Property*) If* $\|H(x)\| \le \|F(x)\|$ *and* $\lim_{x \to x_o} F(x) = 0$*, then we have*

$$\lim_{x \to x_o} H(x) = 0.$$

Let us illustrate, in a form of an example, the verification of the existence of a limit for a real function of two variables:

Example 7.2. (a) We will compute the limit $\lim_{(x,y)\to(0,0)} \dfrac{e^{x^2+y^2}-1}{x^2+y^2}$. Notice that by making a substitution $u = x^2+y^2$, we get an equivalent limit $\lim_{u\to 0} \dfrac{e^u-1}{u}$, which represents the derivative of the function $f(u) = e^u$ at $u = 0$, i.e. $\lim_{u\to 0} \dfrac{e^u-1}{u} = e^0 = 1$, and consequently

$$\lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}-1}{x^2+y^2} = 1.$$

In order to provide a formal argument (using the above definition of the limit) we first notice that

$$\lim_{u\to 0} \frac{e^u-1}{u} = 1 \iff \forall_{\varepsilon>0}\ \exists_{\delta_1>0}\ \forall_u\ 0<|u|<\delta_1 \ \Rightarrow\ \left|\frac{e^u-1}{u}-1\right| < \varepsilon,$$

thus we have

$$\forall_{\varepsilon>0}\ \exists_{\delta=\sqrt{\delta_1}>0}\ \forall_{(x,y)}\ \ 0<\sqrt{x^2+y^2}<\delta \ \Rightarrow\ \left|\frac{e^{x^2+y^2}-1}{x^2+y^2}-1\right| < \varepsilon.$$

(b) We will show that $\lim_{(x,y)\to(0,0)} \dfrac{x^3}{2x^2+y^4} = 0$. Indeed notice that we have the following inequalities

$$f(x,y) := \left|\frac{x^3}{2x^2+y^4}\right| \le \frac{|x|\left(x^2+\frac{y^4}{2}\right)}{2x^2+y^4} = \frac{|x|}{2} \le \frac{1}{2}\sqrt{x^2+y^2} =: g(x,y).$$

Since $\lim_{(x,y)\to(0,0)} g(x,y) = 0$, it follows by the Squeeze Property that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3}{2x^2+y^4} = 0.$$

(c) Assume that $p > q \ge 1$. We will show that $\lim_{(x,y)\to(0,0)} \dfrac{|x|^p+|y|^p}{|x|^q+|y|^q} = 0$. For this purpose we will need to establish the following inequality

$$\forall_{t\ge 0}\ \ 1+t^p \le (1+t^q)^{\frac{p}{q}}. \tag{7.1}$$

Put $\varphi(t) = (1+t^q)^{\frac{p}{q}} - 1 - t^p$. It is clear that $\varphi(0) = 0$ and since

$$\begin{aligned}
\varphi'(t) &= \frac{p}{q}(1+t^q)^{\frac{p}{q}-1} q t^{q-1} - p t^{p-1} \\
&= p\Big((1+t^q)^{\frac{p-q}{q}} t^{q-1} - t^{p-1}\Big) \\
&\ge p(t^{p-1} - t^{p-1}) = 0,
\end{aligned}$$

which means the function $\varphi(t)$ is increasing and consequently $\varphi(t) > 0$ for $t > 0$. Notice that the inequality (7.1) implies

$$|x|^p + |y|^p = |x|^p\left(1+\left|\frac{y}{x}\right|^p\right) \le |x|^p\left(1+\left|\frac{y}{x}\right|^q\right)^{\frac{p}{q}} = (|x|^q+|y|^q)^{\frac{p}{q}}.$$

By applying the last inequality we obtain

$$
\begin{aligned}
0 \leq & \frac{|x|^p + |y|^p}{|x|^q + |y|^q} \leq \frac{(|x|^q + |y|^q)^{\frac{p}{q}}}{|x|^q + |y|^q} \\
= & (|x|^q + |y|^q)^{\frac{p-q}{q}} =: g(x, y).
\end{aligned}
$$

Since $(x, y) \to (0, 0)$ and $\frac{p-q}{q} > 0$, it follows that $g(x, y) \to 0$, thus, by the Squeeze Property, $\lim_{(x,y)\to(0,0)} \frac{|x|^p + |y|^p}{|x|^q + |y|^q} = 0$.

Let us point out that the limit $\lim_{(x,y)\to(x_o,y_o)} f(x, y)$ is not equivalent to the iterated limits $\lim_{x\to x_o} \lim_{y)\to y_o} f(x, y)$. In fact we have that

$$
\lim_{(x,y)\to(x_o,y_o)} f(x, y) = L \implies \lim_{x\to x_o} \lim_{y\to y_o} f(x, y) = L,
$$

but the converse implication is not true.

Example 7.3. Notice that for every y we have $\lim_{x\to 0} \frac{xy^2}{x^2 + y^4} = 0$ and for every x we have $\lim_{x\to 0} \frac{xy^2}{x^2 + y^4} = 0$, which implies that

$$
\lim_{x\to 0} \lim_{y\to 0} \frac{xy^2}{x^2 + y^4} = \lim_{y\to 0} \lim_{x)\to 0} \frac{xy^2}{x^2 + y^4} = 0.
$$

However, the limit $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$ does not exist. Indeed, by choosing $(x, y) = (t^2, t)$ and passing to the limit as $t \to 0$ we get

$$
\lim_{(t^2,t)\to(0,0)} \frac{t^2 t^2}{t^4 + t^4} = \lim_{t\to 0} \frac{t^4}{2t^4} = \frac{1}{2} \neq 0.
$$

7.1.2 Definition of Derivative

We have the following definition

Definition 7.4. Let V and W be two normed spaces and U be an open set of V. Consider a map $F : U \to W$. We say that F is *differentiable* at $x_o \in U$ if there exists a bounded linear operator $A : V \to W$ satisfying the following condition

$$
\lim_{x\to x_o} \frac{[F(x) - F(x_o) - A(x - x_o)]}{\|x - x_o\|} = \lim_{h\to 0} \frac{[F(x_o + h) - F(x_o) - A(h)]}{\|h\|} = 0.
$$

Then the operator A is called the *derivative* of the map F at the point x_o and is denoted by $DF(x_o)$, i.e. $DF(x_o) := A$.

Remark 7.5. The above definition agrees with the usual definition of a derivatve for functions of single real variable. Indeed, for $f : U \to \mathbb{R}$, where U is open subset of $\mathbb{R}$, the derivative of f at x_o is a number $a \in \mathbb{R}$ such that

$$
\lim_{x\to x_o} \frac{[f(x) - f(x_o) - a(x - x_o)]}{|x - x_o|} = 0. \tag{7.2}
$$

We can consider the number a as an entry of an 1×1-matrix $[a]$, which represents a linear map $A : \mathbb{R} \to \mathbb{R}$. Then $Df(x_o) = A = [a]$ is exactly the operator $Df(x_o)(x) = ax$.

Remark 7.6. Let V and W be two Banach spaces, U be an open set of V, and $F : U \to W$ a differentiable map at $x_o \in U$. Then we can define

$$r(x_o; h) := F(x_o + h) - F(x_o) - DF(x_o)(h), \quad \text{for } \|h\| \text{ sufficiently small,}$$

or in other words

$$F(x_o + h) - F(x_o) = DF(x_o)(h) + r(x_o; h).$$

Then the property that F is differentiable at x_o is equivalent to

$$\lim_{h\to 0} \frac{r(x_o; h)}{\|h\|} = 0.$$

7.1.3 Properties of Derivatives

Theorem 7.7. *Let V and W be two Banach spaces and U be an open set of V. Assume that F, $G : U \to W$ are two differentiable maps at $x_o \in U$. Then the map $\alpha F + \beta G : U \to W$, α, $\beta \in \mathbb{R}$, defined by*

$$(\alpha F + \beta G)(x) = \alpha F(x) + \beta G(x), \quad x \in U,$$

is differentiable at x_o and

$$D(\alpha F + \beta G)(x_o) = \alpha DF(x_o) + \beta DG(x_o).$$

Proof: Let us put

$$r_1(x_o; h) := F(x_o + h) - F(x_o) - DF(x_o)(h),$$
$$r_2(x_o; h) := G(x_o + h) - G(x_o) - DG(x_o)(h).$$

Since F and G are differentiable at x_o we have

$$\lim_{h\to 0} \frac{r_1(x_o; h)}{\|h\|} = \lim_{h\to 0} \frac{r_2(x_o; h)}{\|h\|} = 0.$$

In order to prove the above statement it is sufficient to show that the function

$$r(x_o; h) := (\alpha F + \beta G)(x_o + h) - (\alpha F + \beta G)(x_o) - (\alpha DF(x_o) + \beta DG(x_o))(h),$$

satisfies the property

$$\lim_{h\to 0} \frac{r(x_o; h)}{\|h\|} = 0.$$

Indeed, we have

$$\begin{aligned}\lim_{h\to 0} \frac{r(x_o; h)}{\|h\|} &= \lim_{h\to 0} \frac{1}{\|h\|}\Big[(\alpha F + \beta G)(x_o + h) - (\alpha F + \beta G)(x_o) - (\alpha DF(x_o) + \beta DG(x_o))(h)\Big] \\ &= \alpha \lim_{h\to 0} \frac{F(x_o + h) - F(x_o) - DF(x_o)(h)}{\|h\|} + \beta \lim_{h\to 0} \frac{G(x_o + h) - G(x_o) - DG(x_o)(h)}{\|h\|} \\ &= \alpha \lim_{h\to 0} \frac{r_1(x_o, h)}{\|h\|} + \beta \lim_{h\to 0} \frac{r_2(x_o; h)}{\|h\|} = 0 + 0 = 0.\end{aligned}$$

□

Theorem 7.8. (Chain Rule) *Let V, W and Y be normed spaces, $\mathcal{U} \subset V$ and $\mathcal{V} \subset W$ two open sets, and $G : \mathcal{U} \to W$, $F : \mathcal{V} \to Y$ two maps such that $G(\mathcal{U}) \subset \mathcal{V}$. Assume that F is differentiable at $x_o \in \mathcal{U}$ and G is differentiable at $F(x_o) \in \mathcal{V}$. Then the map $F \circ G : \mathcal{U} \to Y$, defined by $(F \circ G)(x) := F(G(x))$, $x \in \mathcal{U}$, is differentiable at x_o and*

$$D(F \circ G)(x_o) = DF(G(x_o)) \circ DG(x_o) : V \to Y.$$

Proof: Let us put

$$\begin{aligned} r_1(x_o; h) &:= G(x_o + h) - G(x_o) - DG(x_o)(h), \\ r_2(G(x_o); k) &:= F(G(x_o) + k) - F(G(x_o)) - DF(G(x_o))(k), \end{aligned}$$

where we have

$$\lim_{h \to 0} \frac{r_1(x_o, h)}{\|h\|} = 0, \quad \text{and} \quad \lim_{k \to 0} \frac{r_2(G(x_o); k)}{\|k\|} = 0.$$

Let us consider

$$\begin{aligned} (F \circ G)(x_o + h) - (F \circ G)(x_o) &= F(G(x_o + h)) - F(G(x_o)) \\ &= F(G(x_o) + DG(x_o)(h) + r_1(x_o, h)) - F(G(x_o)) \\ &= DF(G(x_o))(DG(x_o)(h)) + DF(G(x_o))(r_1(x_o, h)) \\ &\quad + r_2\Big(G(x_o), DG(x_o)(h) + r_1(x_o, h)\Big) \\ &= DF(G(x_o)) \circ DG(x_o)(h) + R(x_o, h), \end{aligned}$$

where

$$R(x_o, h) := DF(G(x_o))(r_1(x_o; h)) + r_2\Big(G(x_o), DG(x_o)(h) + r_1(x_o; h)\Big).$$

We have

$$\begin{aligned} \frac{\|R(x_o; h)\|}{\|h\|} &= \frac{\Big\|DF(G(x_o))(r_1(x_o; h)) + r_2\Big(G(x_o), DG(x_o)(h) + r_1(x_o, h)\Big)\Big\|}{\|h\|} \\ &\le \frac{\|DF(G(x_o))(r_1(x_o; h))\|}{\|h\|} + \frac{\Big\|r_2\Big(G(x_o), DG(x_o)(h) + r_1(x_o; h)\Big)\Big\|}{\|h\|} \\ &= \frac{\|DF(G(x_o))(r_1(x_o; h))\|}{\|h\|} \\ &\quad + \frac{\Big\|r_2\Big(G(x_o), DG(x_o)(h) + r_1(x_o; h)\Big)\Big\| \cdot \|DG(x_o)(h) + r_1(x_o; h)\|}{\|DG(x_o)(h) + r_1(x_o; h)\| \cdot \|h\|} \\ &\le \|DF(G(x_o)) \circ DG(x_o)\| \cdot \frac{\|r_1(x_o; h)\|}{\|h\|} \\ &\quad + \frac{\Big\|r_2\Big(G(x_o), DG(x_o)(h) + r_1(x_o; h)\Big)\Big\|}{\|DG(x_o)(h) + r_1(x_o; h)\|} \left(\|DG(x_o)\| + \frac{\|r_1(x_o; h)\|}{\|h\|}\right), \end{aligned}$$

where we assume that $\|DG(x_o)(h) + r_1(x_o; h)\| \neq 0$. By passing to the limit $\|h\| \to 0$, we get $\|DG(x_o)(h) + r_1(x_o; h)\| \to 0$ and

$$\frac{\left\| r_2\Big(G(x_o), DG(x_o)(h) + r_1(x_o;h)\Big)\right\|}{\|DG(x_o)(h) + r_1(x_o;h)\|} \to 0.$$

Consequently, the conclusion follows. $\square$

7.1.4 Differentiation of Maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

In this subsection we consider the spaces $V := \mathbb{R}^n$ and $W := \mathbb{R}^m$. We assume that $\{e_1, \dots, e_n\}$ is a standard basis in $\mathbb{R}^n$, i.e.

$$e_k = (0, \dots, 0, \underset{k\text{-th}}{1}, 0, \dots, 0), \qquad k = 1, 2, \dots, n,$$

and similarly $\{\widetilde{e}_1, \dots, \widetilde{e}_m\}$ will denote the standard basis in $\mathbb{R}^m$. In other words, a vector $v = (x_1, \dots, x_n) \in \mathbb{R}^n$ can be represented as $v = \sum_{k=1}^{n} x_k e_k$. Suppose that $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator, then for every $v = (x_1, \dots, x_n)$ we have

$$A(v) = A\left(\sum_{k=1}^{n} x_k e_k\right) = \sum_{k=1}^{n} x_k A(e_k).$$

Since $A(e_k) \in \mathbb{R}^m$, then it can be expressed as linear combination of the basic vectors $\widetilde{e}_1, \dots, \widetilde{e}_m$, i.e.

$$A(e_k) = \sum_{j=1}^{m} a_{jk}\widetilde{e}_j, \qquad k = 1, 2, \dots, n,$$

thus

$$A(v) = \sum_{k=1}^{n} x_k A(e_k) = \sum_{k=1, j=1}^{n,m} a_{jk} x_k \widetilde{e}_j.$$

The last relation means that the operator A can be identified (with respect to the basis $\{e_1, \dots, e_k\}$ in $\mathbb{R}^n$, and the basis $\{\widetilde{e}_1, \dots, \widetilde{e}_m\}$ in $\mathbb{R}^m$) with the matrix $[a_{jk}]$, i.e.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

so

$$A(x) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Let $U \subset \mathbb{R}^n$ be an open set and consider a map $f : U \to \mathbb{R}^m$

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{bmatrix} = \sum_{j=1}^{m} f_j(x)\widetilde{e}_j, \qquad j = 1, 2, \dots, m.$$

Let $x_o \in U$ and assume that f is differentiable at x_o, such that $Df(x_o) = [a_{jk}]$. We will find the formula for computing the coefficients a_{jk} of the matrix $Df(x_o)$. By differentiability of f at x_o, we have

$$f(x_o + h) - f(x_o) = Df(x_o)(h) + r(x_o; h), \qquad h = (h_1, \dots, h_n) \in \mathbb{R}^n,$$

with

$$\lim_{h \to 0} \frac{r(x_o, h)}{\|h\|} = 0.$$

By linearity of $Df(x_o)$ we get

$$Df(x_o)(h) = \sum_{k=1, j=1}^{n,m} h_k a_{jk} \widetilde{e}_j,$$

and since $r(x_o; h) \in \mathbb{R}^m$, it can be expressed as

$$r(x_o; h) = \sum_{j=1}^{m} r_j(x_o; h) \widetilde{e}_j, \quad r_j(x_o; h) \in \mathbb{R}.$$

That means, we have

$$f(x_o + h) - f(x_o) = \sum_{j=1}^{m} \Big(f_j(x_o + h) - f_j(x_o) \Big) \widetilde{e}_j = \sum_{j=1}^{m} \left(\sum_{k=1}^{n} h_k a_{jk} + r_j(x_o; h) \right) \widetilde{e}_j.$$

In this way we obtain that for $j = 1, 2, \dots, m$, we have

$$f_j(x_o + h) - f_j(x_o) = \sum_{k=1}^{n} h_k a_{jk} + r_j(x_o; h),$$

where the function $r_j(x_o, h)$, $j = 1, \dots, m$, satisfies the property

$$\lim_{h \to 0} \frac{r_j(x_o; h)}{\|h\|} = 0, \tag{7.3}$$

which implies that each of the coordinate functions $f_j : U \to \mathbb{R}$ is differentiable and

$$Df_j(x_o)(h) = \sum_{k=1}^{n} h_k a_{jk}.$$

Suppose now that $h = (0, \dots, 0, \underset{k\text{-th}}{t}, 0, \dots, 0)$, $k = 1, 2, \dots, n$, then $Df_j(x_o)(h) = t a_{jk}$, and from the equation (7.3), with $h = t e_k$, we obtain

$$0 = \lim_{t \to 0} \frac{r_j(x_o; t e_k)}{t} = \lim_{t \to 0} \frac{f_j(x_o + t e_k) - f_j(x_o) - t a_{jk}}{t} = \lim_{t \to 0} \frac{f_j(x_o + t e_k) - f_j(x_o)}{t} - a_{jk},$$

which imples

$$a_{jk} = \lim_{t \to 0} \frac{f_j(x_o + t e_k) - f_j(x_o)}{t} = \frac{\partial f_j}{\partial x_k}(x_o).$$

In this way we have obtained that the derivative $Df(x_o)$ of the map f at the point x_o is the following matrix of partial derivatives:

$$Df(x_o) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_o) & \frac{\partial f_1}{\partial x_2}(x_o) & \dots & \frac{\partial f_1}{\partial x_n}(x_o) \\ \frac{\partial f_2}{\partial x_1}(x_o) & \frac{\partial f_2}{\partial x_2}(x_o) & \dots & \frac{\partial f_2}{\partial x_n}(x_o) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_o) & \frac{\partial f_m}{\partial x_2}(x_o) & \dots & \frac{\partial f_m}{\partial x_n}(x_o) \end{bmatrix}.$$

Theorem 7.9. *Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^m$,*

$$f(x) = (f_1(x), f_2(x), \dots, f_m(x))^T = \sum_{j=1}^{m} f_j(x)\widetilde{e}_j, \qquad j = 1, 2, \dots, m,$$

(where for a vector $v \in \mathbb{R}^m$ we denote by v^T the transpose vector of v) be a map such that the partial derivatives $\frac{\partial f_j}{\partial x_k}(x)$ exists in a neighborhood of x_o and are continuous for $k = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. Then the function f is differentiable at x_o and

$$Df(x_o) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_o) & \frac{\partial f_1}{\partial x_2}(x_o) & \dots & \frac{\partial f_1}{\partial x_n}(x_o) \\ \frac{\partial f_2}{\partial x_1}(x_o) & \frac{\partial f_2}{\partial x_2}(x_o) & \dots & \frac{\partial f_2}{\partial x_n}(x_o) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_o) & \frac{\partial f_m}{\partial x_2}(x_o) & \dots & \frac{\partial f_m}{\partial x_n}(x_o) \end{bmatrix}.$$

Proof: We can assume without loss of generality that $f : U \to \mathbb{R}$, i.e. f is a scalar-valued function. Put $x_o = (x_1, x_2, \dots, x_n)$ and

$$A(h) := \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(x_o)h_k, \qquad h = (h_1, h_2, \dots, h_n) \in \mathbb{R}^n.$$

Then we need to show that the function

$$R(x_o; h) := f(x_o + h) - f(x_o) - A(h), \quad h \in \mathbb{R}^n,$$

satisfies the property

$$\lim_{h\to 0} \frac{R(x_o; h)}{\|h\|} = 0. \tag{7.4}$$

Let us put

$$x_o^k := (x_1 + h_1, x_2 + h_2, \dots, x_{k-1} + h_{k-1}, x_k + h_k, x_{k+1}, \dots, x_n), \;\; x_o^0 := x_o, \quad k = 1, 2, \dots, n,$$

then we can write

$$f(x_o + h) - f(x_o) = \sum_{k=1}^{n} \Big(f(x_o^k) - f(x_o^{k-1}) \Big).$$

Since the partial derivatives of f exist in a neighborhood of x_o, we can apply the so-called Mean Value Theorem for scalar functions (with respect to the k-th variable) in order to obtain

$$\frac{f(x_o^k) - f(x_o^{k-1})}{h_k} = \frac{\partial f}{\partial x_k}(x_o^{k(c)}), \quad x_o^{k(c)} = (x_1 + h_1, x_2 + h_2, \dots, x_{k-1} + h_{k-1}, x_k + c(h_k), x_{k+1}, \dots, x_n),$$

where $c(h_k)$ is a number somewhere between 0 and h_k. Consequently, we obtain (by applying Cauchy-Schwarz inequality)

$$\begin{aligned}
\left|\frac{R(x_o;h)}{\|h\|}\right| &= \left|\frac{f(x_o+h)-f(x_o)-A(h)}{\|h\|}\right| \\
&= \left|\frac{\sum_{k=1}^n \Big(f(x_o^k)-f(x_o^{k-1})\Big) - \sum_{k=1}^n \frac{\partial f}{\partial x_k}(x_o)h_k}{\|h\|}\right| \\
&= \left|\frac{\sum_{k=1}^n \Big(\frac{\partial f}{\partial x_k}(x_o^{k(c)}) - \frac{\partial f}{\partial x_k}(x_o)\Big)h_k}{\|h\|}\right| \\
&\le \sqrt{\sum_{k=1}^n \Big(\frac{\partial f}{\partial x_k}(x_o^{k(c)}) - \frac{\partial f}{\partial x_k}(x_o)\Big)^2}\ \frac{\sqrt{\sum_{k=1}^n h_k^2}}{\|h\|} \\
&= \sqrt{\sum_{k=1}^n \Big(\frac{\partial f}{\partial x_k}(x_o^{k(c)}) - \frac{\partial f}{\partial x_k}(x_o)\Big)^2}.
\end{aligned}$$

By assumption, the partial derivatives of f are continuous at x_o, therefore

$$\lim_{h\to 0}\left(\frac{\partial f}{\partial x_k}(x_o^{k(c)}) - \frac{\partial f}{\partial x_k}(x_o)\right) = 0, \quad k=1,2,\dots,n,$$

and consequently, by squeeze theorem,

$$0 \le \lim_{h\to 0}\left|\frac{R(x_o;h)}{\|h\|}\right| \le \lim_{h\to 0}\sqrt{\sum_{k=1}^n \Big(\frac{\partial f}{\partial x_k}(x_o^{k(c)}) - \frac{\partial f}{\partial x_k}(x_o)\Big)^2} = 0,$$

which implies (7.4). □

Let us present several examples.

Example 7.10. Let us consider the following map

$$F(x,y,z) = (F_1(x,y,z), F_2(x,y,z))^T = \Big((\sin x)^{\ln y}, x^{y\arctan z}\Big).$$

We will compute the partial derivatives of F, establish if the map is differentiable and find its derivative. By applying the differentiation tules, we obtain

$$\frac{\partial F_1}{\partial x} = (\sin x)^{\ln y}\cdot\frac{\cos x}{\sin x}\ln y, \quad \frac{\partial F_1}{\partial y} = (\sin x)^{\ln y}\cdot\frac{\ln(\sin x)}{y}, \quad \frac{\partial F_1}{\partial z} = 0,$$

and

$$\frac{\partial F_2}{\partial x} = x^{y\arctan z}\cdot\frac{y\arctan z}{x}, \quad \frac{\partial F_2}{\partial y} = x^{y\arctan z}\cdot\arctan z\ln x, \quad \frac{\partial F_2}{\partial z} = x^{y\arctan z}\cdot\frac{y\ln x}{1+z^2}.$$

Since all the partial derivatives are continuous (they are given by formulas involving elementary functions) therefore it follows that at any point (x,y,z) belonging to the interior of the domain of definition of F, the map F is differentiable and we have

$$DF(x,y,z) = \begin{bmatrix} (\sin x)^{\ln y} \cdot \frac{\cos x}{\sin x} \ln y & (\sin x)^{\ln y} \cdot \frac{\ln(\sin x)}{y} & 0 \\ x^{y \arctan z} \cdot \frac{y \arctan z}{} & x^{y \arctan z} \cdot \arctan z \ln x & x^{y \arctan z} \cdot \frac{y \ln x}{1+z^2} \end{bmatrix}$$

Example 7.11. Let us illustrate some methods of finding derivatives of mappings defined on normed spaces. Let V and W be two normed spaces, U be an open set in V and $F : U \to W$ a differentiable map. Then, by the definition of the derivative $DF(x)$ we have that for $h \in V$ (with $\|h\|$ being sufficiently small) we have

$$F(x+h) - F(x) = DF(x)(h) + r(x;h), \quad \lim_{\|h\| \to 0} \frac{r(x,;h)}{\|h\|} = 0,$$

thus $DF(x)$ is the *linear approximation* of $F(x+h) - F(x)$. Therefore, in order to find $DF(x)$ one can consider the difference $F(x+h) - F(x)$ and "extract" from it a linear (with respect to h) operator A. If $A : V \to W$ is indeed the derivative $DF(x)$, then the condition

$$\lim_{\|h\| \to 0} \frac{F(x+h) - F(x) - A(h)}{\|h\|} = 0,$$

must be satisfied.

(a) Let us consider bounded linear operators A, B, $C : V \to V$, and define the map $F : L(V) \to L(V)$ by

$$F(X) = AX^2 + BX + C, \quad X \in L(V).$$

We will compute the derivative $DF(X)$. For this purpose we consider the difference

$$\begin{aligned} F(X+H) - F(X) &= \Big(A(X+H)^2 + B(X+H) + C\Big) - \Big(AX^2 + BX + C\Big) \\ &= AXH + AHX + BH + AH^2, \end{aligned}$$

where $H \in L(V)$. Notice that the expression $H \mapsto AXH + AHX + BH$ defines a linear operator $\mathcal{A} : L(V) to L(V)$ which may be the derivative of F at X. In order to prove that this is indeed the derivative of F at X, we need to consider

$$r(X;H) := F(X+H) - F(X) - \mathcal{A}(H) = AH^2.$$

Since

$$0 \le \lim_{\|H\| \to 0} \frac{\|r(X,;H)\|}{\|H\|} = \lim_{\|H\| \to 0} \frac{\|AH^2\|}{\|H\|} \le \lim_{\|H\| \to 0} \frac{\|A\| \|H\|^2}{\|H\|} = \lim_{\|H\| \to 0} \|A\| \|H\| = 0,$$

we obtain that

$$DF(X)(H) = AXH + AHX + BH, \quad H \in L(V).$$

By applying the same arguments, one can easily verify that we have for the maps

$$F_1(X) = X^2 A + XB + C, \quad F_2(X) = XAX + BX + C,$$

$$DF_1(X)(H) = XHA + HXA + HB, \quad DF_2(X)(H) = HAX + XAH + BH.$$

(b) Consider the Banach space $V := C([0,1])$ of continuous functions $\varphi : [0,1] \to \mathbb{R}$, equipped with the norm

$$\|\varphi\|_\infty := \sup\{|\varphi(t)| : t \in [0,1]\}, \quad \varphi \in V,$$

For $n \in \mathbb{N}$ we define the following map $F : V \to V$,

$$F(\varphi) = \varphi^n, \quad (\varphi^n)(t) = (\varphi(t))^n,\ t \in [0,1].$$

We will compute the derivative $DF(\varphi)$. Let us consider the difference $F(\varphi+h) - F(\varphi)$, where $h \in V$. By applying the binomial theorem, we get

$$\begin{aligned} F(\varphi+h) - F(\varphi) = (\varphi+h)^n - \varphi^n &= \sum_{k=0}^{n} \binom{n}{k} \varphi^k h^{n-k} - \varphi^n \\ &= \sum_{k=0}^{n-1} \binom{n}{k} \varphi^k h^{n-k} \\ &= n\varphi^{n-1}h + \sum_{k=0}^{n-2} \binom{n}{k} \varphi^k h^{n-k}. \end{aligned}$$

One can easily recognize that $n\varphi^{n-1}h$ is the "linear" part of $F(\varphi+h) - F(\varphi)$ (with respect to h). Put $A(h) = n\varphi^{n-1}h$, and

$$r(\varphi; h) := F(\varphi+h) - F(\varphi) - A(h) = \sum_{k=0}^{n-2} \binom{n}{k} \varphi^k h^{n-k}.$$

We will show that $\lim\limits_{\|h\|_\infty \to 0} \dfrac{r(\varphi; h)}{\|h\|_\infty} = 0$. Indeed, we have for all $t \in [0,1]$ and $h \in V$ such that $\|h\|_\infty < 1$

$$\begin{aligned} \frac{|r(\varphi;h)(t)|}{\|h\|_\infty} &= \frac{|F(\varphi+h)(t) - F(\varphi)(t) - A(h)(t)|}{\|h\|_\infty} = \frac{\left|\sum_{k=0}^{n-2} \binom{n}{k} \varphi^k(t) h^{n-k}(t)\right|}{\|h\|_\infty} \\ &\le \frac{\sum_{k=0}^{n-2} \binom{n}{k} |\varphi(t)|^k |h(t)|^{n-k}}{\|h\|_\infty} \le \frac{\sum_{k=0}^{n-2} \binom{n}{k} \|\varphi\|_\infty^k \|h\|_\infty^{n-k}}{\|h\|_\infty} \\ &\le \frac{\sum_{k=0}^{n-2} \binom{n}{k} \|\varphi\|_\infty^k \|h\|_\infty^{2}}{\|h\|_\infty} = \left(\sum_{k=0}^{n-2} \binom{n}{k} \|\varphi\|_\infty^k\right) \|h\|_\infty \end{aligned}$$

Consequently

$$\frac{\|r(\varphi;h)\|_\infty}{\|h\|_\infty} \le \left(\sum_{k=0}^{n-2} \binom{n}{k} \|\varphi\|_\infty^k\right) \|h\|_\infty$$

thus

$$\lim_{\|h\|_\infty \to 0} \frac{\|r(\varphi;h)\|_\infty}{\|h\|_\infty} = 0,$$

so $DF(\varphi)(h) = n\varphi^{n-1}h$, $h \in V$.

(c) Let us consider again the space $V = C([0,1])$ equipped with the norm $\|\cdot\|_\infty$. We define the map $F : V \to \mathbb{R}$ by

$$F(\varphi) = \int_0^1 \big[\varphi(t)\big]^3 dt, \quad \varphi \in C([0,1]).$$

We will compute $DF(\varphi)$. For this purpose we consider the the difference $F(\varphi+h) - F(\varphi)$, with $h \in V$, i.e.

$$\begin{aligned} F(\varphi+h) - F(\varphi) &= \int_0^1 \big[\varphi(t)+h(t)\big]^3 dt - \int_0^1 \big[\varphi(t)\big]^3 dt \\ &= 3\int_0^1 (\varphi(t))^2 h(t)dt + \int_0^1 \Big[3\varphi(t)(h(t))^2 + (h(t))^3\Big] dt. \end{aligned}$$

By extracting the linear part from $F(\varphi+h) - F(\varphi)$ we obtain that $DF(\varphi)(h) = 3\int_0^1 (\varphi(t))^2 h(t)dt$. Indeed, consider

$$r(\varphi; h) := \int_0^1 \Big[3\varphi(t)(h(t))^2 + (h(t))^3\Big] dt,$$

then we have for $h \in V$

$$\begin{aligned} 0 \le \frac{|r(\varphi;h)|}{\|h\|_\infty} &= \frac{\Big|\int_0^1 \Big[3\varphi(t)(h(t))^2 + (h(t))^3\Big] dt\Big|}{\|h\|_\infty} \\ &\le \frac{\int_0^1 \Big[3|\varphi(t)||h(t)|^2 + |h(t)|^3\Big] dt}{\|h\|_\infty} \le \frac{\int_0^1 \Big[3\|\varphi\|_\infty \|h\|_\infty^2 + \|h\|_\infty^3\Big] dt}{\|h\|_\infty} \\ &= 3\|\varphi\|_\infty \|h\|_\infty + \|h\|_\infty^2 \longrightarrow 0 \quad \text{as} \quad \|h\|_\infty \to 0. \end{aligned}$$

(d) Assume that V_1, V_2 and W are finite dimensional spaces. Recall that a map $B : V_1 \times V_2 \to W$ is said to be *bi-linear* (or a *bi-linear operator*) if it is linear with respect to each variable $v_i \in V_i$, $i = 1, 2$, separately, i.e. for all v_1, $v_1' \in V_1$, v_2, $v_2' \in V_2$ and α, $\beta \in \mathbb{R}$ we have

$$\begin{aligned} B(\alpha v_1 + \beta v_1', v_2) &= \alpha B(v_1, v_2) + \beta B(v_1', v_2), \\ B(v_1, \alpha v_2 + \beta v_2') &= \alpha B(v_1, v_2) + \beta B(v_1, v_2'). \end{aligned}$$

Assume that the space $V_1 \times V_2$ is equipped with the norm $\|(v_1, v_2)\| = \sqrt{\|v_1\|^2 + \|v_2\|^2}$ and put

$$\|B\| := \sup\{\|B(v_1, v_2)\| : \|v_1\| \le 1,\ \|v_2\| \le 1\}.$$

The number $\|B\|$ is called the *norm* of the bi-linear operator B and we have

$$\|B(v_1, v_2)\| \le \|B\|\|v_1\|\|v_2\|, \qquad \text{for all } v_1 \in V_1,\ v_2 \in V_2.$$

We will compute the derivative $DB(v)$ of the map B at $v = (v_1, v_2)$. We consider $h = (h_1, h_2) \in V_1 \times V_2$, and we will extract from the difference $B(v+h) - B(v)$ the "linear part." We have

$$\begin{aligned} B(v+h) - B(v) &= B(v_1 + h_1, v_2 + h_2) - B(v_1, v_2) \\ &= B(v_1, v_2) + B(v_1, h_2) + B(h_1, v_2) + B(h_1, h_2) - B(v_1, v_2) \\ &= B(v_1, h_2) + B(h_1, v_2) + B(h_1, h_2). \end{aligned}$$

We claim that $DB(v)(h) = B(v_1, h_2) + B(h_1, v_2)$. Indeed, consider

$$r(v; h) = B(v + h) - B(v) - DB(v)(h) = B(h_1, h_2).$$

Then we have (by applying the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$)

$$\begin{aligned}
0 \leq \lim_{\|h\| \to 0} \frac{\|r(v; h)\|}{\|h\|} &= \lim_{\|h\| \to 0} \frac{\|B(h_1, h_2\|}{\sqrt{\|h_1\|^2 + \|h_2\|^2}} \leq \lim_{\|h\| \to 0} \frac{\|B\|\|h_1\|\|h_2\|}{\sqrt{\|h_1\|^2 + \|h_2\|^2}} \\
&\leq \lim_{\|h\| \to 0} \frac{\|B\|\frac{1}{2}(\|h_1\|^2 + \|h_2\|^2)}{\sqrt{\|h_1\|^2 + \|h_2\|^2}} = \frac{1}{2}\|B\| \lim_{\|h\| \to 0} \frac{\|h\|^2}{\|h\|} = \frac{1}{2}\|B\| \lim_{\|h\| \to 0} \|h\| = 0.
\end{aligned}$$

(e) Let us consider, as a particular case of (d), the map $B : V \times V \to \mathbb{R}$ defined by $B(v_1, v_2) = v_1 \bullet v_2$, $v_1, v_2 \in V$, where V is an Euclidean space and $v_1 \bullet v_2$ stands for standard dot product in V. Clearly B is bi-linear, thus , by the formula obtained in (d) we obtain $DB(v_1, v_2)(h_1, h_2) = v_1 \bullet h_2 + h_1 \bullet v_2$.

(f) Let V be a Banach space. We consider a map $F : GL(V) \to L(V)$ defined by $F(X) = X^{-1}$. The map F is differentiable and we will show that $DF(X)(H) = -X^{-1}HX^{-1}$, for $H \in L(V)$. Let $X \in GL(V)$ be a fixed isomorphism. Then for $H \in L(V)$ such that $\|X^{-1}\|\|H\| = \rho < 1$ the operator $X + H$ is invertible. Indeed, we have $X + H = \Big(\mathrm{Id} - (-HX^{-1})\Big)X$, and since $\| - HX^{-1}\| \leq \|X^{-1}\|\|H\| < 1$, by Corollary **??**, the operator $\mathrm{Id} - (-HX^{-1})$ is invertible and

$$\Big(\mathrm{Id} - (-HX^{-1})\Big)^{-1} = \sum_{k=0}^{\infty}(-1)^k\Big(HX^{-1}\Big)^k.$$

Consequently

$$(X + H)^{-1} = X^{-1}\Big(\mathrm{Id} - (-HX^{-1})\Big)^{-1} = X^{-1}\sum_{k=0}^{\infty}(-1)^k\Big(HX^{-1}\Big)^k = \sum_{k=0}^{\infty}(-1)^k X^{-1}\Big(HX^{-1}\Big)^k.$$

Therefore, we have

$$F(X + H) - F(X) = \sum_{k=1}^{\infty}(-1)^k X^{-1}\Big(HX^{-1}\Big)^k = -X^{-1}HX^{-1} + \sum_{k=2}^{\infty}(-1)^k X^{-1}\Big(HX^{-1}\Big)^k.$$

Put

$$r(X; H) := F(X + H) - F(X) - (-X^{-1}HX^{-1}) = \sum_{k=2}^{\infty}(-1)^k X^{-1}\Big(HX^{-1}\Big)^k.$$

Then we have

$$\begin{aligned}
0 \leq \frac{\|r(X; H)\|}{\|H\|} &= \frac{\| \sum_{k=2}^{\infty}(-1)^k X^{-1}\Big(HX^{-1}\Big)^k\|}{\|H\|} \\
&\leq \frac{\|X^{-1}(HX^{-1})^2\| \sum_{k=0}^{\infty} \|\Big(HX^{-1}\|^k}{\|H\|} \\
&\leq \frac{\|X^{-1}\|^3\|H\|^2 \sum_{k=0}^{\infty} \rho^k}{\|H\|} \\
&\leq \|X^{-1}\|^3\|H\| \sum_{k=0}^{\infty} \rho^k = \|X^{-1}\|^3\|H\| \frac{1}{1 - \rho} \longrightarrow \quad \text{as} \quad \|H\| \to 0.
\end{aligned}$$

Of course an $n \times n$-matrix can be identified with a n^2-vector, thus the derivative $DF(X)$ can be represented in a matrix form (i.e. as an $n^2 \times n^2$-matrix with the entries being the partial derivatives of F). More precisely, let us denote by x_{ij}, $i, j \in \{1, 2, \dots, n\}$, denote the entries of the matrix X. Since $F(X)$ is an $n \times n$-matrix, we will denote it by $[F_{kl}(X)]$, $k, l \in \{1, 2, \dots, n\}$, i.e. $F_{kl}(X) \in \mathbb{R}$ is the kl-th entry of $F(X)$. In this way the map F can be represented as a map $F = [F_{kl}]$, and we can ask the following question: How to compute the partial derivatives $\dfrac{\partial F_{kl}}{\partial x_{ij}}(X)$? In order to answer this question, we denote by H_{ij} the $n \times n$-matrix $[h_{i'j'}]$, where $h_{i'j'} = 1$ if $i' = i$ and $j' = j$, and $h_{i'j'} = 0$ otherwise. In other words, the matrix H_{ij} is simply the standard ij-th basic vector, corresponding to the variable x_{ij}. Then, it is easy to notice, the partial derivative $\dfrac{\partial F_{kl}}{\partial x_{ij}}(X)$ is represented exactly by the kl-th entry of the the matrix $DF(X)(H_{ij}) = -X^{-1}H_{ij}X^{-1}$. As anb illustration, let us compute the partial derivatives of the map F in the case $n = 2$. Suppose that $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$. Then we have

$$X^{-1} = \frac{1}{\det(X)} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix}.$$

Applying this idea leads to

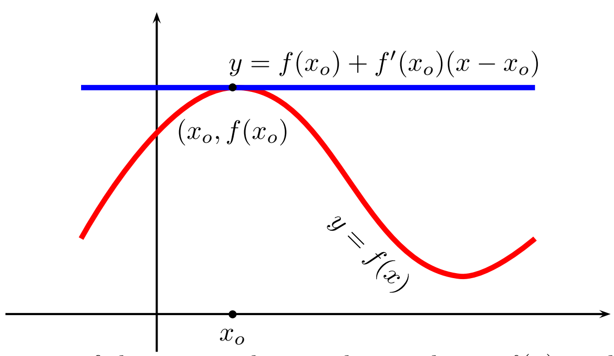
$$\begin{aligned}
-X^{-1}H_{11}X^{-1} &= \frac{-1}{\det^2(X)} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix} = \frac{-1}{\det^2(X)} \begin{bmatrix} x_{22}^2 & -x_{22}x_{12} \\ -x_{21}x_{22} & x_{21}x_{12} \end{bmatrix} \\
-X^{-1}H_{12}X^{-1} &= \frac{-1}{\det^2(X)} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix} = \frac{-1}{\det^2(X)} \begin{bmatrix} -x_{21}x_{22} & x_{22}x_{11} \\ x_{21}^2 & -x_{21}x_{11} \end{bmatrix} \\
-X^{-1}H_{21}X^{-1} &= \frac{-1}{\det^2(X)} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix} = \frac{-1}{\det^2(X)} \begin{bmatrix} -x_{22}x_{12} & x_{12}^2 \\ x_{22}x_{11} & -x_{11}x_{12} \end{bmatrix} \\
-X^{-1}H_{22}X^{-1} &= \frac{-1}{\det^2(X)} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix} = \frac{-1}{\det^2(X)} \begin{bmatrix} x_{21}x_{12} & -x_{12}x_{12} \\ -x_{21}x_{11} & x_1^2 \end{bmatrix}
\end{aligned}$$

In this way we obtain the following partial derivatives

$$
\begin{aligned}
\frac{\partial F_{11}}{\partial x_{11}}(X) &= \frac{-x_{22}^2}{(x_{11}x_{22}-x_{12}x_{21})^2}, & \frac{\partial F_{12}}{\partial x_{11}}(X) &= \frac{-x_{22}x_{12}}{(x_{11}x_{22}-x_{12}x_{21})^2}\\
\frac{\partial F_{21}}{\partial x_{11}}(X) &= \frac{x_{21}x_{22}}{(x_{11}x_{22}-x_{12}x_{21})^2}, & \frac{\partial F_{22}}{\partial x_{11}}(X) &= \frac{x_{21}x_{12}}{(x_{11}x_{22}-x_{12}x_{21})^2}\\
\frac{\partial F_{11}}{\partial x_{12}}(X) &= \frac{x_{21}x_{22}}{(x_{11}x_{22}-x_{12}x_{21})^2}, & \frac{\partial F_{12}}{\partial x_{12}}(X) &= \frac{-x_{22}x_{11}}{(x_{11}x_{22}-x_{12}x_{21})^2}\\
\frac{\partial F_{21}}{\partial x_{12}}(X) &= \frac{-x_{21}^2}{(x_{11}x_{22}-x_{12}x_{21})^2}, & \frac{\partial F_{22}}{\partial x_{12}}(X) &= \frac{-x_{21}x_{11}}{(x_{11}x_{22}-x_{12}x_{21})^2}\\
\frac{\partial F_{11}}{\partial x_{21}}(X) &= \frac{x_{22}x_{12}}{(x_{11}x_{22}-x_{12}x_{21})^2}, & \frac{\partial F_{12}}{\partial x_{21}}(X) &= \frac{-x_{12}^2}{(x_{11}x_{22}-x_{12}x_{21})^2}\\
\frac{\partial F_{21}}{\partial x_{21}}(X) &= \frac{-x_{22}x_{11}}{(x_{11}x_{22}-x_{12}x_{21})^2}, & \frac{\partial F_{22}}{\partial x_{21}}(X) &= \frac{x_{11}x_{12}}{(x_{11}x_{22}-x_{12}x_{21})^2}\\
\frac{\partial F_{11}}{\partial x_{22}}(X) &= \frac{-x_{21}x_{12}}{(x_{11}x_{22}-x_{12}x_{21})^2}, & \frac{\partial F_{12}}{\partial x_{22}}(X) &= \frac{x_{11}x_{12}}{(x_{11}x_{22}-x_{12}x_{21})^2}\\
\frac{\partial F_{21}}{\partial x_{22}}(X) &= \frac{x_{21}x_{11}}{(x_{11}x_{22}-x_{12}x_{21})^2}, & \frac{\partial F_{22}}{\partial x_{22}}(X) &= \frac{-x_{11}^2}{(x_{11}x_{22}-x_{12}x_{21})^2}
\end{aligned}
$$

7.1.5 Geometric Interpretation of Derivative

Let us recall the geometric interpretation of the notion of derivative in the case of a function of a single real variable. Suppose that $y = f(x)$ is differentiable at $x_o \in \mathbb{R}$, then the derivative $f'(x_o)$ (as a number) is simple the slope of the tangent line to the graph of f at the point $(x_o, f(x_o))$ (see the picture blow).



Then the equation of the tangent line to the graph $y = f(x)$ at the point $(x_o, f(x_o)$ is given by

$$y = f(x_o) + f'(x_o)(x - x_o),$$

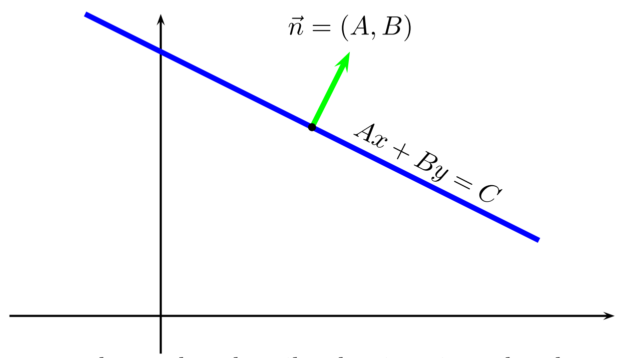
or if we use the notion of the derivative $Df(x_o)$ as a linear operator from $\mathbb{R}$ to $\mathbb{R}$ (i.e $DF(x_o)h = f'(x_o)h$, $h \in \mathbb{R}$), then the tangent equation of the tangent line becomes

$$y = f(x_o) + Df(x_o)(x - x_o). \tag{7.5}$$

The expression (7.5) represents (as a function) the so-called *linear approximation* of $y = f(x)$ near the point x_o. In general, an equation of a line in $\mathbb{R}^2$ can always be written in the form

$$Ax + By = C,$$

and it is clear that the vector $\vec{n} = (A, B)$ is the normal vector to this line (see the picture below).



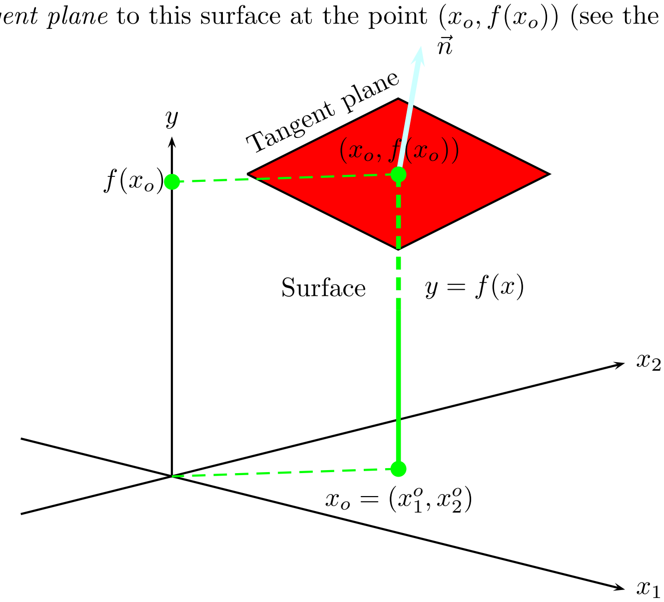
The same ideas, can be used to describe the situation related to a function $y = f(x)$ depending on two real variables $x = (x_1, x_2)$. Assume that the function f is differentiable at the point $x_o = (x_1^o, x_2^o)$ and let $Df(x_o)$ be its derivative, i.e. $Df(x_o) : \mathbb{R}^2 \to \mathbb{R}$ is a linear operator having the following matrix representation (with respect to the standard basis in $\mathbb{R}^2$)

$$Df(x_o) = \left[\frac{\partial f}{\partial x_1}(x_o), \frac{\partial f}{\partial x_2}(x_o)\right]$$

Then, according to the definition of the derivative, we have that the function

$$y = f(x_o) + Df(x_o)(x - x_o), \tag{7.6}$$

is an approximation of the function $f(x)$ near x_o, which is also called *linear approximation* of f. The graph of the function $y = f(x)$ is a surface in $\mathbb{R}^3$ and the graph of the linear approximation (7.6) is the *tangent plane* to this surface at the point $(x_o, f(x_o))$ (see the picture below).



The equation of the tangent plane to the surface $y = f(x)$ at the point x_o is exactly of the same type as in the case of single variable function, i.e.

$$y = f(x_o) + Df(x_o)(x - x_o), \tag{7.7}$$

which can be easily translated to the form

$$y = f(x_o) + \frac{\partial f}{\partial x_1}(x_o)(x_1 - x_1^o) + \frac{\partial f}{\partial x_2}(x_o)(x_2 - x_2^o). \tag{7.8}$$

Recall that an equation of a plane in $\mathbb{R}^3$ can be written as

$$Ax_1 + Bx_2 + Cy = D, \quad (x_1, x_2, y) \in \mathbb{R}^3,$$

and in our case we have

$$A = \frac{\partial f}{\partial x_1}(x_o), \quad B = \frac{\partial f}{\partial x_2}(x_o), \quad C = -1,$$
$$\frac{\partial f}{\partial x_1}(x_o)x_1^o + \frac{\partial f}{\partial x_2}(x_o)x_2^o - f(x_o).$$

Since the vector $\vec{n} = (A, B, C)$ is a normal vector to the plane $Ax_1 + Bx_2 + Cy = D$, it follows that the vector

$$\vec{n} = \left(\frac{\partial f}{\partial x_1}(x_o), \frac{\partial f}{\partial x_2}(x_o), -1\right)$$

is normal to the surface $y = f(x)$ at the point $(x_o, f(x_o))$. Let us also recall that the vector

$$\nabla f(x_o) = \left(\frac{\partial f}{\partial x_1}(x_o), \frac{\partial f}{\partial x_2}(x_o)\right),$$

is called the *gradient* of $f(x)$ at the point x_o. The gradient $\nabla f(x_o)$ should not be mixed up with the derivative $Df(x_o)$ (which is here the matrix representation of a linear map).

It is not hard to see that the same ideas can be worked out in the case of a function $y = f(x)$ depending on n-vectors $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. In this case we also have the *linear approximation* of the function $y = f(x)$ near the point x_o given by (7.6). The graph of the function $y = f(x)$ can be seen as a *hyper-surface* in the space $\mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}$ and we can also talk about the so-called *tangent hyperplane* to this surface. Of course, the graph of the linear approximation (7.6) is exactly the tangent hyperplane to $y = f(x)$ at x_o.

7.1.6 Chain Rule for Maps Between Euclidean Spaces

Let us discuss the Chain Rule formula for two maps between Euclidean spaces. Suppose that $V = \mathbb{R}^n$, $W = \mathbb{R}^m$ and $Y = \mathbb{R}^k$. Suppose that $\mathcal{U}$ is an open set in V and $\mathcal{V}$ an open set in W. Assume that $G : \mathcal{U} \to W$ is a map such that $G(\mathcal{U}) \subset \mathcal{V}$ and $F : \mathcal{V} \to Y$. Assume in addition that G is differentiable at $x_o \in \mathcal{U}$ and F is differentiable at $G(x_o)$. Then, by Theorem 7.8, the composition $H = F \circ G$, $H(x) = F(G(x))$, $x \in \mathcal{U}$, is differentiable at x_o and we have

$$DH(x_o) = DF(G(x_o)) \circ DG(x_o). \tag{7.9}$$

Put $G(x_o) =: y_o$. Let us use the following vector form of the maps H, F and G

$$H(x) = \begin{bmatrix} H_1(x) \\ H_2(x) \\ \vdots \\ H_k(x) \end{bmatrix}, \quad F(y) \begin{bmatrix} F_1(y) \\ F_2(y) \\ \vdots \\ F_k(y) \end{bmatrix}, \quad G(x) = \begin{bmatrix} G_1(x) \\ G_2(x) \\ \vdots \\ G_m(x) \end{bmatrix}.$$

By using the matrix representations of the derivatives $DH(x_o)$, $DF(y_o)$, and $DG(x_o)$, we can write the equality (7.9) as follows

$$\begin{bmatrix} \frac{\partial H_1}{\partial x_1}(x_o) & \frac{\partial H_1}{\partial x_2}(x_o) & \dots & \frac{\partial H_1}{\partial x_n}(x_o) \\ \frac{\partial H_2}{\partial x_1}(x_o) & \frac{\partial H_2}{\partial x_2}(x_o) & \dots & \frac{\partial H_2}{\partial x_n}(x_o) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial H_k}{\partial x_1}(x_o) & \frac{\partial H_k}{\partial x_2}(x_o) & \dots & \frac{\partial H_k}{\partial x_n}(x_o) \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\partial F_1}{\partial y_1}(y_o) & \frac{\partial F_1}{\partial y_2}(y_o) & \dots & \frac{\partial F_1}{\partial y_m}(y_o) \\ \frac{\partial F_2}{\partial y_1}(y_o) & \frac{\partial F_2}{\partial y_2}(y_o) & \dots & \frac{\partial F_2}{\partial y_m}(y_o) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_k}{\partial y_1}(y_o) & \frac{\partial F_k}{\partial y_2}(y_o) & \dots & \frac{\partial F_k}{\partial y_m}(y_o) \end{bmatrix} \circ \begin{bmatrix} \frac{\partial G_1}{\partial x_1}(x_o) & \frac{\partial G_1}{\partial x_2}(x_o) & \dots & \frac{\partial G_1}{\partial x_n}(x_o) \\ \frac{\partial G_2}{\partial x_1}(x_o) & \frac{\partial G_2}{\partial x_2}(x_o) & \dots & \frac{\partial G_2}{\partial x_n}(x_o) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial G_m}{\partial x_1}(x_o) & \frac{\partial G_m}{\partial x_2}(x_o) & \dots & \frac{\partial G_m}{\partial x_n}(x_o) \end{bmatrix}.$$

By comparing the entries of the above matrices we obtain the following explicit Chain Rule formula for the partial derivatives $\frac{\partial H_j}{\partial x_i}(x_o)$

$$\frac{\partial H_j}{\partial x_i}(x_o) = \sum_{l=1}^{m} \frac{\partial F_j}{\partial x_l}(y_o) \cdot \frac{\partial G_l}{\partial x_i}(x_o), \tag{7.10}$$

where $y_o = G(x_o)$.

Let us present some examples for application of the Chain Rule.

Example 7.12. (a) Let V be an Euclidean space. Define $F : V \to \mathbb{R}$ by $F(v) = \|v\|^2$, where $\|v\|^2 = v \bullet v$. We will compute the derivative $DF(v)$. Notice that F is a composition of two maps; the linear map $A : V \to V \times V$, $A(v) = (v, v)$, and bi-linear map $B : V \times V \to \mathbb{R}$, $B(v_1, v_2) = v_1 \bullet v_2$, i.e. $F(v) = (B \circ A)(v)$. Thus by the Chain Rule, we have

$$\begin{aligned} DF(v)(h) &= DB(v,v)\Big(DA(v)(h)\Big) = DB(v,v)(h,h) = B(v,h) + B(h,v) \\ &= v \bullet h + h \bullet v = 2v \bullet h. \end{aligned}$$

In particular, we have the following matrix representation $DF(v) = 2v$, which allows us to establish the partial derivatives $\frac{\partial F}{\partial x_k}(v) = 2x_k$, where $v = (x_1, x_2, \dots, x_n)$.

(b) For an Euclidean space we will compute the derivative of the map $G : V \to \mathbb{R}$ given by $G(v) = \|v\|$ at a point $v \neq 0$. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be defined by $f(t) = \sqrt{t}$. Then $G(v) = f(F(v))$, where $F : V \to \mathbb{R}$ was defined in (a), so

$$DG(v)(h) = Df(F(v))\Big(DF(v)(h)\Big) = Df(\|v]\|)\Big(2v \bullet h\Big) = -\frac{1}{2\sqrt{\|v\|^2}} \cdot 2v \bullet h = -\frac{v \bullet h}{\|v\|}.$$

Consequently, we obtain the following matrix representation $DG(v) = -\frac{v}{\|v\|}$, allowing us to compute the partial derivatives

$$\frac{\partial G}{\partial x_k}(v) = -\frac{x_k}{\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}}, \quad v = (x_1, x_2, \dots, x_n).$$

7.2 Mean Value Theorem

Let V and W be two Banach spaces, $\mathcal{U} \subset V$ an open set and $F : \mathcal{U} \to W$ a map. We say that the map F is of class C^1 (or C^1-*map* or *continuously differentiable*) if it is differentiable at every point $x \in \mathcal{U}$ and the map $DF : \mathcal{U} \to L(V, W)$, defined by $DF : x \mapsto DF(x) : V \to W$ is continuous.

Definition 7.13. Let V be a Banach space, x, $y \in V$. The subset $[x, y] \subset V$ defined by

$$[x, y] := \{tx + (1 - t)y : t \in [0, 1]\},$$

is called a *linear segment* between the points x and y.

Lemma 7.14. *Let U be an open subset of a Banach space V. Then*

$$\forall_{x\in U}\ \exists_{\delta>0}\ \forall_{h\in V}\ \ \|h\| < \delta \implies [x, x + h] \subset U.$$

Proof: Since U is open, for every $x \in U$ there exists $\delta > 0$ such that $B(x, \delta) \subset U$. In particular, if $\|h\| < \delta$ then $x + h \in B(x, \delta)$. On the other hand,

$[x, x+h] = \{x+th : t \in [0, 1]\}$, thus if $\|h\| < \delta$ then $\|th\| < \delta$ and consequently $[x, x+h] \subset B(x, \delta)$. $\square$

Theorem 7.15. (Mean Value Theorem for Functions) *Let U be an open subset in a Banach space V and $f : U \to \mathbb{R}$. If the function is differentiable on the linear segment $[x, x + h] \subset U$, then there exists $\theta \in (0, 1)$ such that*

$$f(x + h) - f(x) = Df(x + \theta h)(h). \tag{7.11}$$

Proof: Let us define the following scalar function $g : [0, 1] \to \mathbb{R}$ by

$$g(t) := f(x + th), \quad t \in [0, 1].$$

Then, by the assumption and the Chain Rule (Theorem 7.8) the function g is differentiable on the interval $(0, 1)$ and continuous on $[0, 1]$. Therefore, the Mean Value Theorem for scalar functions (Lagrange Formula) can be applied to g, i.e there exists $0 < \theta < 1$ such that

$$g(1) - g(0) = \frac{dg}{dt}(\theta),$$

where $g(1) = f(x + h)$ and $g(0) = f(x)$, and by the Chain Rule

$$\frac{dg}{dt}(\theta) = Df(x + \theta h)(h).$$

$\square$

Theorem 7.16. (Mean Value Theorem for Maps) *Let V and W be two Banach spaces, $\mathcal{U} \subset V$ and open set and $F : \mathcal{U} \to W$ a continuously differentiable map. Assume that $[x, a + h] \subset \mathcal{U}$. Then*

$$\|F(x + h) - F(x)\| \le M\|h\|,$$

where

$$M := \sup\{\|DF(x + \theta h)\| : 0 \le \theta \le 1\}.$$

Proof: We will present here the proof only when $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$. In the general case some additional information about the dual space W^* of W is needed, which was not covered by these notes.

For every $w \in W = \mathbb{R}^m$ we define

$$f_w : \mathcal{U} \to \mathbb{R}, \quad f_w(x) = F(x) \bullet w,$$

where $v \bullet w$ denotes the standard dot product of vectors v and w in $\mathbb{R}^m$. It is clear that the function f_w is differentiable (by Chain Rule) and

$$Df_w(x)h = (DF(x)(h)) \bullet w, \quad h \in \mathbb{R}^n.$$

By Theorem 7.15 (Mean Value Theorem for Functions), there exists $\theta \in (0,1)$ such that

$$f_w(x+h) - f_w(x) = Df_w(x+\theta h)(h).$$

Since

$$\|F(x+h) - F(x)\| = (F(x+h) - F(x)) \bullet \frac{F(x+h) - F(x)}{\|F(x+h) - F(x)\|}$$

it follows that

$$\|F(x+h) - F(x)\| \leq \sup\{(F(x+h) - F(x)) \bullet w : \|w\| \leq 1\}.$$

Therefore

$$\begin{aligned}\|F(x+h) - F(x)\| &\leq \sup\{\|Df_w(x+\theta h)(h)\| : \|w\| \leq 1,\ \theta\imath[0,1]\} \\ &\leq \|h\| \sup\{\|DF(x+\theta h)\| : \theta \in [0,1]\}.\end{aligned}$$

$\square$

7.3 Inverse Function Theorem

7.3.1 Invariance of Domain for Contractive Fields

Let X be a subset of a Banach space V. Given a map $F : X \to V$, the map $f : X \to V$, $f(x) := x - F(x)$ for $x \in X$, is called the *field associated with* F. The field f associated with contractive F is called a *contractive field.*

Theorem 7.17. (Invariance of Domain) *Let V be a Banach space, $\mathcal{U} \subset V$ an open set, and $F : \mathcal{U} \to V$ a contractive map with constant $\alpha < 1$. Let $f : \mathcal{U} \to V$ be the associated field, i.e. $f(x) = x - F(x)$. Then*

(a) $f : \mathcal{U} \to V$ is an open mapping, i.e. for every open subset $\mathcal{W} \subset \mathcal{U}$ the set $f(\mathcal{W})$ is open. In particular, the set $f(\mathcal{U})$ is open;
(b) $f : \mathcal{U} \to f(\mathcal{U})$ is a homeomorphism, i.e. f is one-to-one and onto, and the inverse map $f^{-1} : f(\mathcal{U}) \to \mathcal{U}$ is continuous.

Proof: In order to show that f is an open mapping it is sufficient to establish that for any $x_o \in \mathcal{U}$, if $B(x_o, r) \subset \mathcal{U}$, then $B(f(x_o), (1-\alpha)r) \subset f(B(x_o, r))$. For this purpose, choose $y_o \in B(f(x_o), (1-\alpha)r)$ and define $G : B(x_o, r) \to V$ by

$$G(y) = y_o + F(y), \quad y \in B(x_o, r).$$

Then G is contractive with constant $\alpha = L(F) < 1$ and

$$\|G(x_o) - x_o\| = \|y_o + F(x_o) - x_o\| = \|y_o - f(x_o)\| < (1-\alpha)r,$$

so, by Corollary **??**, there exists $z_o \in B(x_o, r)$ with $z_o = y_o + F(z_o)$, which means $f(z_o) = y_o$. Consequently, we have

$$B(f(x_o), (1-\alpha)r) \subset f(B(x_o, r)),$$

thus $f : \mathcal{U} \to V$ is an open mapping, and in particular $f(\mathcal{U})$ is open in V.

To prove (b), we observe that if $x, y \in \mathcal{U}$, then

$$\|f(x) - f(y)\| \geq \|x - y\| - \|F(x) - F(y)\| \geq (1-\alpha)\|x - y\|,$$

which implies that f is one-to-one. Since $f : \mathcal{U} \to f(\mathcal{U})$ is continuous, one-to-one and onto open mapping, its inverse is also continuous and consequently f is a homeomorphism. □

As a simple application of Invariance of Domain Theorem, we present an alternative proof of Proposition **??**

Corollary 7.18. *Let V be a Banach space. Then the set $GL(V)$ of all invertible linear operators is open in $L(V)$.*

Proof: Notice that the map $F(x) := (\mathrm{Id} - A)(x) = x - A(x)$ is contractive. Indeed,

$$\|F(x) - F(y)\| = \|(\mathrm{Id} - A)(x - y)\| \leq \|\mathrm{Id} - A\| \, \|x - y\|,$$

therefore, by the Invariance of Domain (cf. Theorem 7.17) the map $\mathrm{Id} - (\mathrm{Id} - A) = A$ is a homeomorphism. In particular, A is invertible. In addition we have

$$1 = \|AA^{-1}\| = \|A^{-1} - A^{-1}(\mathrm{Id} - A)\| \geq \|A^{-1}\| - \|A^{-1}\|\|\mathrm{Id} - A\|.$$

□

7.3.2 Inverse Function Theorem

Theorem 7.19. (Inverse Function Theorem) *Let V be a Banach space, $\mathcal{U} \subset V$ an open set and $f : \mathcal{U} \to V$ a continuously differentiable map (i.e. f is a C^1-map). Assume that for $x_o \in \mathcal{U}$ the derivative $Df(x_o)$ is invertible (i.e. $Df(x_o) \in GL(V, V)$). Then there exists a neighborhood $\mathcal{V}$ of x_o and a neighborhood $\mathcal{W}$ of $f(x_o)$ such that*

(a) $Df(x) : V \to V$ is invertible for all $x \in \mathcal{V}$;
(b) $f : \mathcal{V} \to \mathcal{W}$ is a homeomorphism;

(c) The inverse $g : \mathcal{W} \to \mathcal{V}$ is differentiable at each $y \in \mathcal{W}$ and

$$Dg(y) = [Df(G(y)]^{-1}.$$

Proof: Notice that the statement (a) follows from the fact that the set $GL(V,V)$ is open in $L(V,V)$ and the map Df is continuous (i.e. and inverse image of an open set under DF is open in V), i.e. there exists an open ball $B := B(x_o, r)$ such that for all $x \in B$ the derivative $Df(x)$ is invertible.

Let us consider the case when $x_o = 0$, $f(x_o) = 0$ and $Df(x_o) = Df(0) = \mathrm{Id}$. Define the map $F(x) := x - f(x)$. Notice that $DF(0) = \mathrm{Id} - Df(0) = \mathrm{Id} - \mathrm{Id} = 0$, and since F is continuously differentiable there exists an open ball (centered at 0) such that $\mathcal{V} \subset B$, $0 \in \mathcal{V}$, and

$$M := \sup\{\|DF(x)\| : x \in \mathcal{V}\} < \frac{1}{2}.$$

By Mean Value Theorem, we have

$$\|F(x_1) - F(x_2)\| \le M\|x_1 - x_2\| \le \frac{1}{2}\|x_1 - x_2\|, \quad \forall_{x_1, x_2 \in \mathcal{V}}.$$

Consequently, by Theorem 7.17 the map $f : \mathcal{V} \to V$ is a homeomorphism onto the open set $\mathcal{W} = f(\mathcal{V})$ containing $f(0) = 0$.

In order to prove (c), let $g : \mathcal{W} \to \mathcal{V}$ be the inverse of $f : \mathcal{V} \to \mathcal{W}$. For given pints $y, b \in \mathcal{W}$ we will write $g(b) = a$ and $g(y) = x$ and put $Df(a) = A$. By differentiability of f at a, we have

$$f(x) - f(a) = A(x-a) + r(a; x-a), \quad \lim_{x \to a} \frac{r(a; x-a)}{\|x-a\|} = 0. \tag{7.12}$$

By applying A^{-1} to the last equality and by substituting $f(x) = y$, $f(a) = b$, we get

$$A^{-1}(y-b) = g(y) - g(b) + A^{-1}(r(a; g(y) - g(a))),$$

so it is sufficient to prove that

$$\lim_{y \to b} \frac{\|A^{-1}(r(a; g(y) - g(a)))\|}{\|y-b\|} = 0.$$

Put

$$R := A^{-1}(r(a; g(y) - g(a))).$$

Notice that

$$\|x-a\| - \|f(x) - f(a)\| \le \|F(x) - F(a)\| \le \frac{1}{2}\|x-a\|, \quad \text{for all} \quad x,\, a \in \mathcal{V},$$

so

$$\|x-a\| \le 2\|f(x) - f(a)\| \iff \|g(y) - g(b)\| \le 2\|y-b\|.$$

Because $g : \mathcal{W} \to \mathcal{W}$ is one-to-one and onto, we have

$$\begin{aligned} R &\le \frac{\|A^{-1}\| \, \|r(a; g(y) - g(a))\|}{\|g(y) - g(b)\|} \cdot \frac{\|g(y) - g(b)\|}{\|y-b\|} \\ &\le 2\|A^{-1}\| \frac{\|r(a; g(y) - g(a))\|}{\|g(y) - g(b)\|} = 2\|A^{-1}\| \frac{\|r(a, x-a)\|}{\|x-a\|}. \end{aligned}$$

Therefore, by continuity of g we obtain that $\|y-b\| \to 0$ implies that $\|x-a\| \to 0$, therefore the conclusion follows from (7.12). $\square$

One can ask the following question: *Do we really need in the Inverse Function Theorem the assumption that the function f is continuously differentiable?* The following example provides the answer.

Example 7.20. Let us consider the function $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } \ x > 0 \\ 0 & \text{if } \ x \le 0. \end{cases}$$

The function φ is differentiable on $\mathbb{R}$, its derivative is continuous on the interval $(0,\infty)$ but it is not continuous at 0. Indeed, by direct computation $\varphi'(0) = 0$ and for $x > 0$, $\varphi'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$, and since $\lim x \to 0 \varphi'(x)$ does not exist, φ' is not continuous at zero. We define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = ax + \varphi(x), \quad x \in \mathbb{R}$$

where $a > 0$ is chosen to satisfy the condition $a < \frac{2}{\pi}$. We will show that the function f oscillates in every neighborhood of zero. More precisely, we define two sequences

$$x_n = \frac{1}{\frac{\pi}{2} + 2n\pi}, \quad y_n = \frac{1}{\frac{3\pi}{2} + 2n\pi}, \quad n = 1, 2, 3, \dots$$

Then we have $x_n > y_n > x_{n+1} > y_{n+1} > \dots$ and

$$\begin{aligned} f(x_n) &= \frac{a}{\frac{\pi}{2} + 2n\pi} + \frac{1}{(\frac{\pi}{2} + 2n\pi)^2} \sin(\frac{\pi}{2} + 2n\pi) = \frac{a}{\frac{\pi}{2} + 2n\pi} + \frac{1}{(\frac{\pi}{2} + 2n\pi)^2}, \\ f(y_n) &= \frac{a}{\frac{3\pi}{2} + 2n\pi} + \frac{1}{(\frac{3\pi}{2} + 2n\pi)^2} \sin(\frac{3\pi}{2} + 2n\pi) = \frac{a}{\frac{3\pi}{2} + 2n\pi} - \frac{1}{(\frac{3\pi}{2} + 2n\pi)^2}, \end{aligned}$$

It is clear that $f(x_n) > f(y_n)$. We claim that $f(y_n) < f(x_{n+1})$. Indeed, if $A > B > 0$ then we have the inequality $A^2 + B^2 > 2AB$, then by taking

$$A := \frac{1}{\frac{3\pi}{2} + 2n\pi}, \quad B := \frac{1}{\frac{5\pi}{2} + 2n\pi},$$

we obtain

$$\begin{aligned} \frac{1}{(\frac{3\pi}{2} + 2n\pi)^2} + \frac{1}{(\frac{5\pi}{2} + 2n\pi)^2} &> \frac{2}{(\frac{3\pi}{2} + 2n\pi)(\frac{5\pi}{2} + 2n\pi)} > \frac{a\pi}{(\frac{3\pi}{2} + 2n\pi)(\frac{5\pi}{2} + 2n\pi)} \\ &> \frac{a}{\frac{3\pi}{2} + 2n\pi} - \frac{a}{\frac{5\pi}{2} + 2n\pi}, \end{aligned}$$

which implies,

$$f(x_{n+1}) = \frac{a}{\frac{5\pi}{2} + 2n\pi} + \frac{1}{(\frac{5\pi}{2} + 2n\pi)^2} < \frac{a}{\frac{3\pi}{2} + 2n\pi} - \frac{1}{(\frac{3\pi}{2} + 2n\pi)^2} = f(y_n).$$

It follows from these inequalities that the function f can not be one-to-one in any neighborhood of 0.

7.3.3 Implicit Function Theorem

In this subsection we will consider only the case for finite dimensional spaces. Let $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$. Then $V \times W = \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$. We will write the elements of $V \times W$ in the form $x = (v, w)$, where $v \in V$ and $w \in W$. Let U be an open set of $V \times W$, and $F : U \to V$ be a differentiable map. Then for a fixed vector $v_o \in V$ (resp. $w_o \in W$) the map $F_{v_o}(w) = F(v_o, w)$ (resp. $F_{w_o}(v) = F(v, w_o)$) is differentiable map of the w-variable (resp. v-variable). The derivative $DF_{v_o}(w_o) : W \to V$ will be denoted as $D_w F(v_o, w_o)$ and the derivative $DF_{w_o}(v_o) : V \to V$ by $D_v F(v_o, w_o) : V \to V$, and we will call them W*-derivative* and V*-derivative* of F respectively. It is easy to notice that

$$DF_{v_o}(w_o)(w) = DF(v_o, w_o)(0, w), \quad \text{and} \quad D_w F(v_o, w_o)(v) = DF(v_o, w_o)(v, 0),$$

where $v \in V$ and $w \in W$.

Theorem 7.21. (Implicit Function Theorem) *Let $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, U be an open subset of $V \times W$, and $F : U \to V$ a continiously differentiable map such that for certain $x_o = (v_o, w_o) \in V \times W$ we have*

(i) $F(v_o, w_o) = 0$;
(ii)$A := D_v F(v_o, w_o) : V \to V$ is an isomorphism.

Then there exist open sets $\mathcal{U} \subset U \subset V \times W$ and $\mathcal{V} \subset W$ satisfying the following property: to every $w \in \mathcal{V}$ corresponds a unique $v \in V$ such that $(v, w) \in \mathcal{U}$ and

$$F(v, w) = 0. \tag{7.13}$$

Moreover, if this v is defined to be $g(w)$, then $g : \mathcal{V} \to V$ is a continuously differentiable map such that $g(w_o) = v_o$,

$$F(g(w), w) = 0, \quad w \in \mathcal{W}, \tag{7.14}$$

and

$$Dg(w) = -\big[D_v F(g(v), w)\big]^{-1} D_w F(g(v), w), \quad w \in \mathcal{V}. \tag{7.15}$$

Proof: Define $G : U \to V \times W$ by $G(v, w) = (F(v, w), w)$ for $(v, w) \in V \times W$. The map G is continuously differentiable. Let us compute its derivative $DG(v_o, w_o)$. By assumption, F is differentiable at (v_o, w_o), thus

$$\begin{aligned} F(v_o + h, w_o + k) - F(v_o, w_o) &= DF(v_o, w_o)(h, k) + r((v_o, w_o), (h, k)) \\ &= D_w F(v_o, w_o)(h) + D_v F(v_o, w_o)(k) + r((v_o, w_o), (h, k)), \end{aligned}$$

where $(h, k) \in V \times W$. Thus

$$\begin{aligned} G(v_o + h, w_o + k) - G(v_o, w_o) &= \Big(DF(v_o, w_o)(h, k), k\Big) + (r((v_o, w_o), (h, k)), 0) \\ &= \Big(D_v F(v_o, w_o)(h) + D_w F(v_o, w_o)(k), k\Big) + (r((v_o, w_o), (h, k)), 0), \end{aligned}$$

i.e.

$$DG(v_o, w_o)(h, k) = \Big(DF(v_o, w_o)(h, k), k\Big) = \Big(D_v F(v_o, w_o)(h) + D_w F(v_o, w_o)(k), k\Big).$$

In matrix form, the derivative $DG(v_o, w_o)$ can be written as follows

$$DG(v_o, w_o) = \begin{bmatrix} D_v F(v_o, w_o) & D_w F(v_o, w_o) \\ & \\ 0 & \mathrm{Id} \end{bmatrix}$$

Since $DG(v_o, w_o)(h, k) = (0, 0)$ if and only if $k = 0$ and $D_v F(v_o, w_o)(h) = 0$, thus $h =$ and we obtain that $DG(v_o, w_o)$ is an isomorphism. Therefore we can apply the Invers Function Theorem (cf. Theorem 7.19). In particular, there exist open sets $\mathcal{W}$, $\mathcal{V} \subset V \times W$ such that $G : \mathcal{V} \to \mathcal{W}$ is an one-to-one and onto map, the inverse map $G^{-1} : \mathcal{W} \to \mathcal{V}$ is differentiable, and $DG^{-1}(x, y) = \Big[DG(v, w)\Big]^{-1}$ for $(x, y) = G(v, w)$. Since $G(v, w) = (F(v, w), w)$, it follows that $G^{-1}(x, w) = (f(x, w), w)$. Moreover, since $F(v, w) = 0$, $(v, w) \in \mathcal{U}$, if and only if $G(v, w) = (0, w)$, it follows that the slution set for the equation $F(v, w) = 0$, $(v, w) \in \mathcal{U}$, is exactly the set $\{G^{-1}(0, w) : w \in \mathcal{W}\}$. In particular, we can define the map $g(w) = f(0, w)$, $w \in \mathcal{W}$ and, of course, it satisfies the requirement $F(g(w), w) = 0$, $w \in \mathcal{W}$. In order to compute the derivative of g, we apply the Chain Rule to the equation $F(g(w), w) = 0$. Then we get for $(h, k) \in V \times W$

$$0 = DF(g(w), w) \circ \Big[Dg(w) + \mathrm{Id}\Big] = D_v F(g(w), w) \circ Dg(w) + D_w F(g(w), w)$$

thus

$$D_v F(g(w), w) \circ Dg(w) = -D_w F(g(w), w) \;\Leftrightarrow\; Dg(w) = -\Big[D_v F(g(w), w)\Big]^{-1} \circ D_w F(g(w), w).$$

□

7.4 Derivatives of Higher Order

7.4.1 Multilinear Maps

Definition 7.22. Let V_1, V_2, $\dots$, V_n, and W ($n \geq 2$) be vector spaces. We say that a map $B : V_1 \times V_2 \times \dots V_n \to W$ is *n-linear* (or *multilinear of order n*), if it is linear with respect to each of the variables $v_k \in V_k$ separately, i.e. for $k = 1, 2, \dots, n$, we have

$$\begin{aligned} &B(v_1, \dots, v_{k-1}, \alpha v_k + \beta w_k, v_{k+1}, \dots, v_n) \\ &= \alpha B(v_1, \dots, v_{k-1}, v_k, v_{k+1}, \dots, v_n) + \beta B(v_1, \dots, v_{k-1}, w_k, v_{k+1}, \dots, v_n) \end{aligned}$$

for all $v_k \in V_k$, $k = 1, \dots, n$ and α, $\beta \in \mathbb{R}$.
In particular, if $n = 2$, the map $B : V_1 \times V_2 \to W$ is called *bi-linear*.

Notice that the set of all n-linear maps $B : V_1 \times V_2 \times \dots V_n \to W$ form a vector space. Indeed, it is easy to check that if B_1, $B_2 : V_1 \times V_2 \times \dots V_n \to W$ are two n-linear maps, then $\alpha B_1 + \beta B_2 : V_1 \times V_2 \times \dots V_n \to W$, for α, $\beta \in \mathbb{R}$, is also an n-linear map.

Suppose that $B : V_1 \times V_2 \times \dots V_n \to W$ is an n-linear map. Then for $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \dots V_n$, and α_1, α_2, $\dots$, $\alpha_n \in \mathbb{R}$, we have

$$B(\alpha_1 v_1, \alpha_2 v_2, \dots, \alpha_n v_n) = \alpha_1 \alpha_2 \dots \alpha_n B(v_1, v_2, \dots, v_n). \tag{7.16}$$

Assume that V_1, V_2, ..., V_n and W are Banach spaces. Put $v = (v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \dots \times V_n$. We equip the product space $V := V_1 \times V_2 \times \dots \times V_n$ with the norm

$$\|v\| = \|(v_1, v_2, \dots, v_n)\| = \sqrt{\|v_1\|^2 + \|v_2\|^2 + \dots + \|v_n\|^2}.$$

Then the space V is also also a Banach space.

We will denote by $L(V_1, V_2, \dots, V_n; W)$ the space of all *continuous* n-linear maps $B : V_1 \times \dots V_n \to W$. We wioll show that the space $L(V_1, V_2, \dots, V_n; W)$ can be equipped with a norm, which will make her a Banach space.

Theorem 7.23. *Let V_1, V_2, ..., V_n and W, be Banach spaces. Then an n-linear map $B : V_1 \times \dots \times V_n \to W$ is continuous if and only if there exists a constant $M > 0$ such that*

$$\|B(v_1, v_2, \dots, v_n)\| \le M \|v_1\| \|v_2\| \dots \|v_n\| \tag{7.17}$$

for all $v_k \in V_k$, $k = 1, 2, \dots, n$.

Proof: For simplicity and clarification of the underlying ideas, we will present first the proof for the case of a bi-linear map $B : V_1 \times V_2 \to W$ (i.e. $n = 2$) and later we will prove the above statement for the general case of an arbitrary $n \ge 2$.

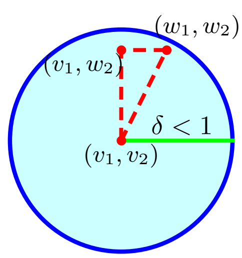
Assume therefore that $B : V_1 \times V_2 \to W$ is a bi-linear map satisfying the condition

$$\|B(v_1, v_2)\| \le M \|v_1\| \|v_2\|. \tag{7.18}$$

We claim that B is continuous. Indeed, for every $v = (v_1, v_2) \in V_1 \times V_2 =: V$ and every $\varepsilon > 0$ we choose $\delta := \min\left\{1, \frac{\varepsilon}{M(\|v\|+1)}\right\}$. Then, if $\|v - w\| < \delta$, where $w = (w_1, w_2) \in V_1 \times V_2$, we have

$$\begin{aligned}
\|B(v) - B(w)\| &= \|B(v_1, v_2) - B(w_1, w_2)\| \\
&= \|B(v_1, v_2) - B(w_1, v_2) + B(w_1, v_2) - B(w_1, w_2)\| \\
&\le \|B(v_1, v_2) - B(w_1, v_2)\| + \|B(w_1, v_2) - B(w_1, w_2)\| \\
&= \|B(v_1 - w_1, v_2)\| + \|B(w_1, v_2 - w_2)\| \\
&\le M\Big(\|v_2\| \|v_1 - w_1\| + \|w_1\| \|v_2 - w_2\|\Big) \\
&\le M\sqrt{\|w_1\|^2 + \|v_2\|^2}\sqrt{\|v_1 - w_1\|^2 + \|v_2 - w_2\|^2} \quad \text{by Cauchy Schwarz.}
\end{aligned}$$

Since $\|v - w\| < \delta \le 1$, we have that $\sqrt{\|w_1\|^2 + \|v_2\|^2} = \|(w_1, v_2)\| \le \|(v_1, v_2)\| + 1 = \|v\| + 1$ (see the picture below).



Therefore, we obtain

$$\begin{aligned}\forall_{\varepsilon>0}\ \exists_{\delta=\min\left\{1,\frac{\varepsilon}{M(\|v\|+1)}\right\}}\ \forall_{w\in V}\quad \|v-w\|<\delta \quad\Rightarrow\quad \|B(v)-B(w)\| &\le M(\|v\|+1)\|v-w\| \\ &< \delta M(\|v\|+1) \\ &\le \frac{\varepsilon}{M(\|v\|+1)}M(\|v\|+1) \\ &= \varepsilon.\end{aligned}$$

Let us assume now that $B: V_1\times V_2\to W$ is a continuous bi-linear map. Then, in particular, it is continuous at $0=(0,0)\in V_1\times V_2$, and for a fixed $\varepsilon>0$ there exists $\delta>0$ such that

$$\forall_{w\in V}\quad \|w\|<\delta \implies \|B(w)\|<\varepsilon, \tag{7.19}$$

where $w=(w_1,w_2)$. Then for every $v=(v_1,v_2)\in V_1\times V_2$, such that $v_1\neq 0\neq v_2$ (otherwise $B(v)=0$), the vector $w=(w_1,w_2)$ defined by

$$(w_1,w_2)=\left(\frac{\delta}{2\|v_1\|}v_1,\frac{\delta}{2\|v_2\|}v_2\right),$$

satisfies the condition

$$\|w\|=\sqrt{\left(\frac{\delta}{2\|v_1\|}\right)^2\|v_1\|^2+\left(\frac{\delta}{2\|v_2\|}\right)^2\|v_2\|^2}=\frac{\delta\sqrt{2}}{2}=\frac{\delta}{\sqrt{2}}<\delta,$$

thus we have $\|B(w)\|<\varepsilon$. Since

$$v=(v_1,v_2)=\left(\frac{2\|v_1\|}{\delta}w_1,\frac{2\|v_2\|}{\delta}w_2\right),$$

we obtain

$$\|B(v_1,v_2)\|=\frac{2\|v_1\|}{\delta}\cdot\frac{2\|v_2\|}{\delta}\cdot\|B(w_1,w_2)\|=M\,\|v_1\|\,\|v_2\|,$$

where

$$M:=\frac{4\varepsilon}{\delta^2},$$

which imples that B satisfies the condition (7.18).

The prove in general case follows exactly the same ideas. Assume that the condition (7.17) is satisfied. Consider $v=(v_1,v_2,\dots,v_n)$ and $w=(w_1,w_2,\dots,w_n)$. Then we have

$$\begin{aligned}B(v)-B(w)&=B(v_1,v_2,\dots,v_n)-B(w_1,w_2,\dots,w_n)\\ &=\sum_{k=1}^n\Big(B(w_1,\dots,w_{k-1},v_k,v_{k+1},\dots,v_n)-B(w_1,\dots,w_{k-1},w_k,v_{k+1},\dots,v_n)\Big)\\ &=\sum_{k=1}^n B(w_1,\dots,w_{k-1},v_k-w_k,v_{k+1},\dots,v_n).\end{aligned}$$

Thus (by applying Cauchy-Schwarz inequality)

$$\begin{aligned}
\|B(v)-B(w)\| &\le \sum_{k=1}^{n} \|B(w_1,\dots,w_{k-1},v_k-w_k,v_{k+1},\dots,v_n)\| \\
&\le M\sum_{k=1}^{n} \|w_1\|\dots\|w_{k-1}\|\|v_k-w_k\|\|v_{k+1}\|\dots\|v_n\| \\
&\le M\sqrt{\sum_{k=1}^{n} \|w_1\|^2\dots\|w_{k-1}\|^2\|v_{k+1}\|^2\dots\|v_n\|^2}\ \|v-w\|.
\end{aligned}$$

Put

$$C := \sqrt{\sum_{k=1}^{n}(\|v_1\|+1)^2\dots(\|v_{k-1}\|+1)^2(\|v_{k+1}\|+1)^2\dots(\|v_n\|+1)^2}.$$

Let us fix $v \in V_1 \times \dots \times V_n$. Then for every $\varepsilon > 0$ there exists $\delta = \min\left\{1, \frac{\varepsilon}{CM}\right\}$ such that for all $w \in V_1 \times \dots \times V_n$ we have

$$\begin{aligned}
\|v-w\| < \delta \implies \|B(v)-B(w)\| &\le M\sqrt{\sum_{k=1}^{n} \|w_1\|^2\dots\|w_{k-1}\|^2\|v_{k+1}\|^2\dots\|v_n\|^2}\ \|v-w\| \\
&\le MC\|v-w\| \\
&< MC\delta \le MC\frac{\varepsilon}{MC} \\
&= \varepsilon.
\end{aligned}$$

In order to show that for a continuous n-linear map $B : V_1 \times \dots V_n \to W$ there exists a constant $M > 0$ satisfying (7.17), we fix $\varepsilon > 0$, and by continuity of B at 0, we have that there exists $\delta > 0$ such that

$$\forall_{w\in V} \quad \|w\| < \delta \implies \|B(w)\| < \varepsilon.$$

Let $v = (v_1, v_2, \dots, v_n)$ be an arbitrary vector such that $v_k \ne 0$ for all $k = 1, 2, \dots, n$ (otherwise we have $B(v) = 0$). Define the vector w by

$$w = (w_1, w_2, \dots, w_n) = \left(\frac{\delta v_1}{n\|v_1\|}, \frac{\delta v_2}{n\|v_2\|}, \dots, \frac{\delta v_n}{n\|v_n\|}\right)$$

. Then we have

$$\|w\| = \sqrt{\sum_{k=1}^{n}\left(\frac{\delta}{n\|v_k\|}\right)^2 \|v_k\|^2} = \frac{\delta}{n}\sqrt{n} = \frac{\delta}{\sqrt{n}} < \delta,$$

thus $\|B(w)\| < \varepsilon$. On the other hand we have

$$v = (v_1, v_2, \dots, v_n) = \left(\frac{n\|v_1\|}{\delta}w_1, \frac{n\|v_2\|}{\delta}w_2, \dots, \frac{n\|v_n\|}{\delta}w_n\right),$$

and consequently

$$\begin{aligned}\|B(v_1, v_2, \dots, v_n)\| &= \left\| B\left(\frac{n\|v_1\|}{\delta} w_1, \frac{n\|v_2\|}{\delta} w_2, \dots, \frac{n\|v_n\|}{\delta} w_n\right) \right\| \\ &= \frac{n^n}{\delta^n} \|B(w)\| \, \|v_1\| \, \|v_2\| \, \dots \, \|v_n\| \\ &\le \frac{n^n \varepsilon}{\delta^n} \, \|v_1\| \, \|v_2\| \, \dots \, \|v_n\| \\ &= M \, \|v_1\| \, \|v_2\| \, \dots \, \|v_n\|,\end{aligned}$$

where

$$M := \frac{n^n \varepsilon}{\delta^n}.$$

□

For an n-linear continuous map $B : V_1 \times \dots \times V_n \to W$ we define the *norm* $\|B\|$ of B by

$$\|B\| := \inf\{M > 0 : \forall_{v=(v_1,\dots,v_n)} \; \|B(v_1, \dots, v_n)\} \le M \, \|v_1\| \dots \|v_n\|\}.$$

By applying the same idea as in the case of linear operators, it is easy to see that

$$\|B\| = \sup\{\|B(v_1, v_2, \dots, v_n)\| : \|v_1\| \le 1, \; \|v_2\| \le 1, \; \dots \; , \|v_n\| \le 1\}. \tag{7.20}$$

Proposition 7.24. *Let V_1, V_2, ..., V_n, and W be Banach spaces. Then the space $L(V_1, V_2, \dots, V_n; W)$ of continuous n-linear maps $B : V_1 \times V_2 \times \dots \times V_n \to W$, equipped with the norm $\|B\|$ given by (7.20), is a Banach space.*

Proof: One can apply the same idea as in the oproof of Proposition **??**. □

Theorem 7.25. *Let V_1, V_2 and W be Banach spaces. Then for every $A \in L(V_1, L(V_2, W))$, the map $B : V_1 \times V_2 \to W$ defined by*

$$B(v_1, v_2) = A(v_1)(v_2), \quad \text{where } A(v_1) \in L(V_2, W), \; v_1 \in V_1, \; v_2 \in V_2, \tag{7.21}$$

is continuous bilinear map such that $\|A\| = \|B\|$. In other words, the space $L(V_1, L(V_2, W))$ is isometrically isomorphic to the space $L(V_1, V_2; W)$.

Proof: We will show that the map $\Phi : L(V_1, L(V_2, W)) \to L(V_1, V_2; W)$, defined by

$$\Phi(A)(v_1, v_2) = A(v_1)(v_2), \quad v_1 \in V_1, \; v_2 \in V_2,$$

is a well defined one-to-one and onto linear operator. Indeed, it is clear that

$$\Phi(\alpha A_1 + \beta A_2) = \alpha\Phi(A_1) + \beta\Phi(A_2), \quad A_1, \; A_2 \in L(V_1, L(V_2, W)), \; \alpha, \; \beta \in \mathbb{R}.$$

The operator Φ is onto. Indeed, let $B \in L(V_1, V_2; W)$. Then the map $A : V_1 \to L(V_2, W)$, i.e. $A(v_1) : V_2 \to W$ for $v_1 \in V_1$, defined by

$$A(v_1)(v_2) = B(v_1, v_2), \quad v_2 \in V_2,$$

is a continuous linear operator. Indeed, we have that for all $v_1 \in V_1$, $\|v_1\| \leq 1$ we have

$$\|A(v_1)\| = \sup\{\|A(v_1)(v_2)\| : \|v_2\| \leq 1\} = \sup\{\|B(v_1, v_2)\| : \|v_2\| \leq 1\},$$

thus

$$\|A\| = \sup\{\|A(v_1)\| : \|v_1\| \leq 1\} = \sup\{\|B(v_1, v_2)\| : \|v_1\| \leq 1,\ \|v_2\| \leq 1\} = \|B\|.$$

It is easy to notice that $\Phi(A) = B$, thus Φ is surjective. In order to show that Φ is one-to-one, suppose that $\Phi(A) = 0$. Then for all $v_1 \in V_1$ the operator $A(v_1)$ is zero, i.e. $A(v_1)(v_2) = \Phi(A)(v_1, v_2) = 0$ and consequently $A = 0$. □

By applying an inductive argument, one can prove the following result:

Theorem 7.26. *Let V_1, V_2, ..., V_n, and W be Banach spaces. Then the Banach space $L(V_1, V_2, \dots, V_n; W)$ is isometrically isomprphic to the space $L(V_1, L(V_2, \dots, L(V_n, W) \dots))$.*

Let us discuss the case of finite dimensional spaces. Suppose that $V_1 = \mathbb{R}^{n_1}$, $V_2 = \mathbb{R}^{n_2}$, and $W = \mathbb{R}$, and consider a bi-linear map $B : V_1 \times V_2 \to \mathbb{R}$. We denote by $\{e^1_1, e^1_2, \dots, e^1_{n_1}\}$ the standard basis in V_1, and by $\{e^2_1, e^2_2, \dots, e^2_{n_2}\}$ the standard basis in V_2. Then for $v_1 = \sum_{k=1}^{n_1} \alpha^1_k e^1_k$ and $v_2 = \sum_{k=1}^{n_2} \alpha^2_k e^2_k$ we have

$$B(v_1, v_2) = B\left(\sum_{k=1}^{n_1} \alpha^1_k e^1_k, \sum_{j=1}^{n_2} \alpha^2_j e^2_j\right) = \sum_{k=1,j=1}^{n_1,n_2} \alpha^1_k \alpha^2_j B(e^1_k, e^2_j).$$

Denote by $A = [a_{k,j}]$ the $n_1 \times n_2$-matrix given by

$$a_{k,j} = B(e^1_k, e^2_j).$$

If we write

$$v_1 = \begin{bmatrix} \alpha^1_1 \\ \alpha^1_2 \\ \vdots \\ \alpha^1_{n_1} \end{bmatrix}, \quad v_2 = \begin{bmatrix} \alpha^2_1 \\ \alpha^2_2 \\ \vdots \\ \alpha^2_{n_2} \end{bmatrix},$$

then the bi-linear map B acts on (v_1, v_2) as the following matrix multiplication

$$B(v_1, v_2) = v_1^T [a_{k,j}] v_2 = [\alpha^1_1, \alpha^1_2, \dots, \alpha^1_{n_1}] \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n_2} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n_2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_1,1} & a_{n_1,2} & \dots & a_{n_1,n_2} \end{bmatrix} \begin{bmatrix} \alpha^2_1 \\ \alpha^2_2 \\ \vdots \\ \alpha^2_{n_2} \end{bmatrix}$$

In the case of an n-linear map $B : V_1 \times V_2 \times \dots \times V_n \to \mathbb{R}$, we suppose that $\{e^k_1, e^k_2, \dots, e^k_{n_k}\}$ denotes the standard basis in V_k, $k = 1, 2, \dots, n$. Then we can associate with the map B a *generalized matrix* $[a_{i_1,i_2,\dots,i_n}]$, defined by

$$a_{i_1,i_2,\dots,i_n} = B(e^1_{i_1}, e^2_{i_2}, \dots, e^n_{i_n}), \quad i_k = 1, 2, \dots, n_k, \ \ k = 1, 2, \dots, n.$$

Then, for $v_k = (\alpha^k_1, \alpha^k_2, \dots, \alpha^k_{n_k})^T$, $k = 1, 2, \dots, n$, we have

$$B(v_1, v_2, \dots, v_n) = \sum_{i_1=1,\dots,i_n=1}^{n_1,\dots,n_n} a_{i_1,\dots,i_n} \alpha^1_{i_1} \dots \alpha^n_{i_n}.$$

In the case we are dealing with an n-linear map $B : V_1 \times V_2 \times \cdots \times V_n \to \mathbb{R}^m$, the map B can be represented as $B(v) = (B_1(v), B_2(v), \dots, B_m(v))$, $v = (v_1, v_2, \dots, v_n)$, where $B_k : V_1 \times V_2 \times \cdots \times V_n \to \mathbb{R}$, $k = 1, 2, \dots, m$ are bi-linear (scalar-valued) maps. That means each or the maps B_k, $k = 1, 2, \dots, n$, can be represented by the formula

$$B_k(v_1, v_2, \dots, v_n) = \sum_{i_1=1,\dots,i_n=1}^{n_1,\dots,n_n} a^k_{i_1,\dots,i_n} \alpha^1_{i_1} \dots \alpha^n_{i_n},$$

where

$$a^k_{i_1,i_2,\dots,i_n} = B_k(e^1_{i_1}, e^2_{i_2}, \dots, e^n_{i_n}), \quad i_j = 1, 2, \dots, n_j, \;\; j = 1, 2, \dots, n.$$

7.4.2 Higher Derivatives

Definition 7.27. Let V and W be two Banach spaces and U an open set of V. A map $F : U \to W$ is said to be *twice differentiable* if

(i) F is differentiable, i.e. the map $DF : U \to L(V, W)$ is well defined, and
(ii) the map DF is differentiable, i.e. for every $v_o \in U$, there exists a bounded operator $D(DF)(v_o)) : V \to L(V; W)$ such that

$$\lim_{h \to v_o} \frac{DF(v_o + h) - DF(v_o) - D(DF)(v_o)(h)}{\|h\|} = 0.$$

Since the operator $D(DF)(v_o) \in L(V, L(V, W))$ can be identified with a bilinear map $D^2F(v_o) : V \to V \to W$ (i.e. $D^2F(v_o) \in L(V, V; W)$), defined by

$$D^2F(v_o)(v_1, v_2) = D(DF)(v_o)(v_1)(v_2),$$

it we will call the bi-linear operator $D^2F(v_o)$ the *second derivative* of F at v_o.

Theorem 7.28. (Clairaut's Theorem) *Let V and W be two Banach spaces and $U \subset V$ an open set. Suppose that a map $F : U \to W$ is twice differentiable. Then for every $v_o \in U$, the second derivative $D^2F(v_o) : V \times V \to W$ is a symmetric bi-linear map, i.e.*

$$D^2F(v_o)(v, w) = D^2F(v_o)(w, v), \quad \text{for all} \quad v,\, w \in V.$$

Proof: We will present here the proof only in the case $W = \mathbb{R}^m$. Assume first that $W = \mathbb{R}$, i.e. the function F is scalar-valued. Let v, $w \in V$ be two fixed vectors. We define the following auxiliary function $\varphi : (-1, 1) \to \mathbb{R}$ by

$$\varphi(t) = F(v_o + tv + w) - F(v_o + tv), \quad t \in [-1, 1].$$

It is clear that the function φ is differentiable (by Chain Rule) and

$$\varphi'(t) = DF(v_o + tv + w)v - DF(v_o + tv)v, \tag{7.22}$$

where by $\varphi'(t)$ we denote the vector in W such that $D\varphi(t)s = \varphi'(t) \cdot s$. By Mean Value Theorem,

$$\varphi(1) - \varphi(0) = \varphi'(\theta), \quad \text{for some} \quad \theta \in (0,1),$$

thus

$$|\varphi(1) - \varphi(0) - \varphi'(0)| \le \sup\{|\varphi'(\theta) - \varphi'(0)| : \theta \in [-1,1]\}. \tag{7.23}$$

By the definition of the second derivative, we have

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{h\in V} \;\; 0 < \|h\| < 2\delta \implies \frac{\|r(v_o;h)\|}{\|h\|} < \varepsilon,$$

where

$$r(v_o;h) := DF(v_o + h) - DF(v_o) - D(DF)(v_o)(h).$$

In particular, if $\|v\| < \delta$ and $\|w\| < \delta$, then we have

$$\begin{aligned}\|DF(v_o + tv + w)(v) - DF(v_o)(v) - (D(DF)(v_o)(tv + w))(v)\| &\le \varepsilon\|v\|(\|(tv + w)\| \\ &\le \varepsilon\|v\|(\|v\| + \|w\|).\end{aligned} \tag{7.24}$$

Thus, by (7.22) and (7.24) we get for $\theta \in [-1,1]$

$$\begin{aligned}|g'(\theta) - (D(DF)(v_o)(v))(w)\| &= |DF(v_o + tv + w)(v) - DF(v_o + tv)(v) - (D(DF)(v_o)(v))(w)| \\ &\le \varepsilon\|v\|(\|v\| + \|w\|).\end{aligned} \tag{7.25}$$

On the other hand, by(7.23) and (7.25)

$$\begin{aligned}\|\varphi(1) - \varphi(0) - D(DF)(v_o)(v)\| &= \|\varphi(1) - \varphi(0) - \varphi'(0)\| + \|\varphi'(0) - D(DF)(v_o)(v)\} \\ &\le \sup\{|\varphi'(\theta) - \varphi'(0)| : \theta \in [-1,1]\} + \varepsilon\|v\|(\|v\| + \|w\|) \\ &\le \sup\{|\varphi'(\theta) - (D(DF)(v_o)(v))(w)\| : \theta \in [-1,1]\} \\ &\quad + \|(D(DF)(v_o)(v))(w) - \varphi'(0)\| + \varepsilon\|v\|(\|v\| + \|w\|) \\ &\le 3\varepsilon\|v\|(\|v\| + \|w\|).\end{aligned}$$

which leads to

$$|\varphi(1) - \varphi(0) - D((DF)(v_o)(v))(w)| \le 3\varepsilon\|v\|(\|v\| + \|w\|). \tag{7.26}$$

But, the expression

$$\varphi(1) - \varphi(0) = F(v_o + tv + w) - F(v_o + v) - F(v + w) + F(v_o)$$

is symmetric with respect v and w (i.e. interchanging v and w leads to exactly the same expression), thus by interchanging v with w we obtain

$$|\varphi(1) - \varphi(0) - (D(DF)(v_o)(w))(v)| \le 3\varepsilon\|w\|(\|v\| + \|w\|), \quad \text{for} \;\; \|v\| < \delta, \; \|w\| < \delta. \tag{7.27}$$

Therefore, by (7.26) and (7.27), we obtain

$$\begin{aligned}|(D(DF)(v_o)(v))(w)-(D(DF)(v_o)(w))(v)| &\le |\varphi(1)-\varphi(0)-D((DF)(v_o)(v))(w)|\\ &\quad+|\varphi(1)-\varphi(0)-(D(DF)(v_o)(w))(v)|\\ &\le 3\varepsilon\|v\|(\|v\|+\|w\|)+3\varepsilon\|w\|(\|v\|+\|w\|)\\ &=3\varepsilon(\|v\|+\|w\|)^2,\end{aligned}$$

for all v, $w \in V$ such that $\|v\| < \delta$ and $\|w\| < \delta$. However, for arbitrary vectors v, $w \in V$, we can take $b > \max\left\{\frac{\|v\|}{\delta}, \frac{\|w\|}{\delta}\right\}$, thus $\|b^{-1}v\| < \delta$ and $\|b^{-1}w\| < \delta$. Consequently, we have

$$\begin{aligned}|(D(DF(v_o)(v))(w)-(D(DF(v_o)(w))(v)| &= \left|\left(D(DF(v_o)\left(\frac{v}{b}\right)\right)\left(\frac{w}{b}\right)-\left(D(DF(v_o)\left(\frac{w}{b}\right)\right)\left(\frac{v}{b}\right)\right|b^2\\ &\le 3\varepsilon\left(\frac{\|v\|}{b}+\frac{\|w\|}{b}\right)^2\cdot b^2\\ &=3\varepsilon(\|v\|+\|w\|)^2.\end{aligned}$$

In this way we obtain that for every $\varepsilon > 0$ we have

$$|D^2F(v_o)(v,w)-D^2F(w,v)| = |(D(DF)(v_o)(v))(w)-(D(DF)(v_o)(w))(v)| \le \varepsilon(\|v\|+\|w\|)^2,$$

which clearly implies that

$$D^2F(v_o)(v,w) = D^2F(v_o)(w,v).$$

□

Part IV

FUNCTIONS OF SINGLE VARIABLE: INTEGRATION

8

Riemann Integration

8.1 Riemann Integral and Conditions for Integrability

Let $[a, b]$ be a finite interval. We consider a function $f : [a, b] \to \mathbb{R}$. A finite sequence of points $\{x_k\}_{k=0}^n$, such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k < \cdots < x_{n-1} < x_n = b$$

is called a *partition* of the interval $[a, b]$ of $[a, b]$), and we will denote it by the letter P, i.e. $P = \{x_k\}_{k=0}^n$. To be more precise, we should say that $P = \{x_k\}_{k=0}^n$ is a *partition of* $[a, b]$ *from* a *to* b. Indeed, we can also consider a *partition* $\{t_j\}$ of the interval $[a, b]$ *from* b *to* a, which is a decreasing sequence of points

$$a = t_m < t_{m-1} < \cdots < t_m < t_{m-1} < \cdots < t_1 < t_0 = b.$$

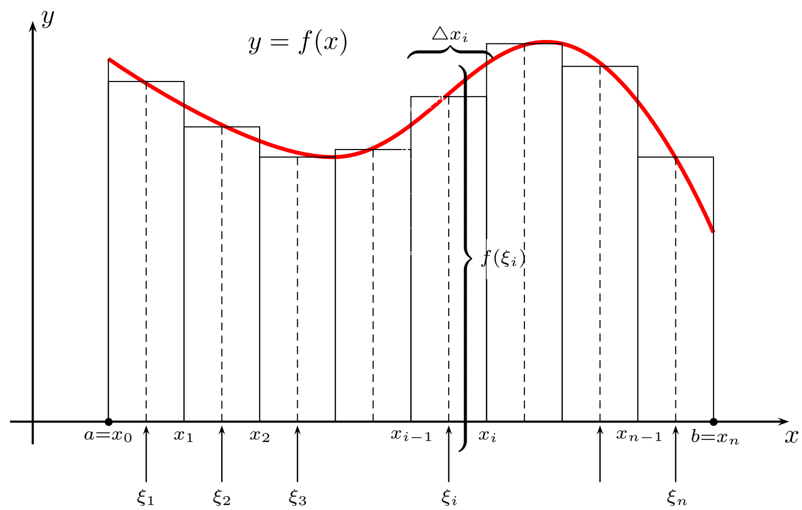
For a given partition $P = \{x_k\}$ we will also use the notation

$$\triangle x_k = x_k - x_{k-1}, \qquad \|P\| := \max\{| \triangle x_k| : k = 1, 2, \dots, n\}.$$

The number $\|P\|$ is called the *diameter* of the partition P. Suppose that for every $k = 1, \dots, n$ we choose a number $\xi_k \in [x_{k-1}, x_k]$. Then the partition P together with $\xi := \{\xi_k\}_{k=1}^n$, is called a *tagged partition* and will be denoted by (P, ξ). We define for a tagged partition (P, ξ)

$$\sigma(f, P, \xi) := \sum_{k=1}^{n} f(\xi_k) \triangle x_k, \tag{8.1}$$

and we will call $\sigma(f,P,\xi)$ the *Riemann sum* for f (on $[a,b]$) *associated with* (P,ξ). Notice that in the case where $f($ is a positive function representing a curve in $\mathbb{R}^2$, the Riemann sum can be considered as an approximation of the area in between the curve and the x-axis (see the figure below).



Definition 8.1. We say that the Riemann sums $\sigma(f,P,\xi)$ (given by (8.1)) have a limit as $\|P\| \to 0$ if there exists a real number I such that

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{(P,\xi)}\ \ \|P\| < \delta \implies |I - \sigma(f,P,\xi)| < \varepsilon. \tag{8.2}$$

Notice that for the tagged partition (P,ξ) in (8.2), the points $\xi_k \in [x_{k-1}, x_k]$ can be chosen arbitrarily (i.e. the number I does not depend on the choice of the points ξ_k). The limit I will be denoted by $I = \displaystyle\int_a^b f(x)dx$, i.e.

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(\xi_k) \triangle x_k \tag{8.3}$$

and we will call it the *definite integral* (or the *Riemann integral*) of f from a to b . The function f is called the *integrand* and the numbers a and b are called the *lower* and the *upper limits* of the integration, respectively.

Proposition 8.2. *Let $f : [a,b] \to \mathbb{R}$ be a function such that the Riemann integral $\displaystyle\int_a^b f(x)dx$ exists. Then we have*

$$\begin{cases} \forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{P=\{x_k\}}\ \forall_{Q=\{y_m\}}\ \forall_{\xi_k\in[x_{k-1},x_k]}\ \forall_{\zeta_m\in[y_{m-1},y_m]} \\ \|P\| < \delta\ \wedge\ \|Q\| < \delta\ \Rightarrow\ \left|\sum_m f(\zeta_m) \triangle y_m - \sum_k f(\xi_k) \triangle x_k\right| < \varepsilon. \end{cases} \tag{8.4}$$

Proof: By (8.2) we have

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{P=\{x_k\}}\ \forall_{\xi_k\in[x_{k-1},x_k]}\ \ \|P\|<\delta \implies \left|I-\sum_k f(\xi_k)\triangle x_k\right|<\frac{\varepsilon}{2} \tag{8.5}$$

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{Q=\{y_m\}}\ \forall_{\zeta_m\in[y_{m-1},y_m]}\ \ \|Q\|<\delta \implies \left|I-\sum_m f(\zeta_m)\triangle y_m\right|<\frac{\varepsilon}{2} \tag{8.6}$$

thus by combining (8.5) and (8.6) and using the inequality

$$\left|\sum_m f(\zeta_m)\triangle y_m-\sum_k f(\xi_k)\triangle x_k\right|\le\left|I-\sum_m f(\zeta_m)\triangle y_m\right|+\left|I-\sum_k f(\xi_k)\triangle x_k\right|$$

we obtain (8.4). □

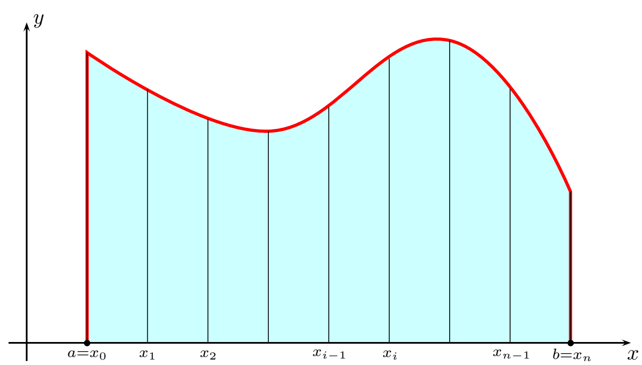
Remark 8.3. Notice that in Definition 8.1 we do not need to assume that $a<b$. In the case $a>b$ we assume that the considered partition $P=\{x_k\}$ is from a to b, so all the numbers $\triangle x_k$ are negative. It is clear, that by taking a partition $Q=\{t_k\}$ from b to a defined by $t_k=x_{n-k}$, we obtain that

$$\sum_{k=1}^n f(\xi_k)\triangle x_k=-\sum_{k=1}^n f(\xi_{n-k})\triangle t_{n-k},$$

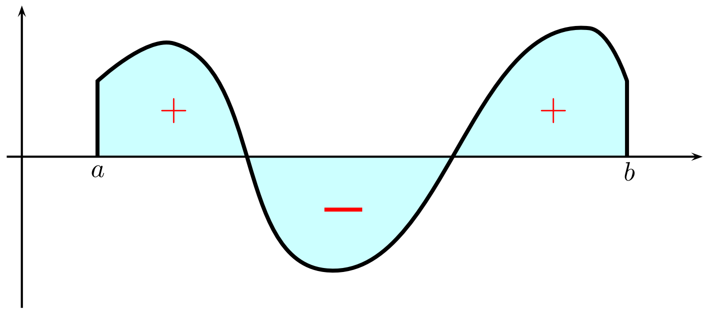
so the integral $\int_b^a f(x)dx$ also exists and we have

$$\int_a^b f(x)dx=-\int_b^a f(x)dx.$$

Remark 8.4. Notice that if $f:[a,b]\to\mathbb{R}$ is a positive function, then the integral $\displaystyle\int_a^b f(x)dx$ can be interpreted as the area of the region bounded by the curve $y=f(x)$, the x-axis and the lines $x=a$ and $x=b$ (see the picture below).



However, in the case the function f is not positive on $[a,b]$, the integral $\displaystyle\int_a^b f(x)dx$ represents the difference between the area of this region above the x-axis and the area of the portion of this region under the x-axis.



The above definition of the integral was introduced by the German mathematician Bernhard Riemann(1826-1866). The symbol $\int$ was introduced by Isaac Newton.

Our main goal in this section is to find sufficient conditions for the function $f : [a,b] \to \mathbb{R}$ that would imply the Riemann integral $\displaystyle\int_a^b f(x)dx$ exists.

Definition 8.5. Let $f : [a,b] \to \mathbb{R}$ be a function. We say that f is *integrable* on $[a,b]$ if the integral $\displaystyle\int_a^b f(x)dx$ exists. Otherwise, we say that f is *not integrable* on $[a,b]$.

Proposition 8.6. *If the function $f : [a,b] \to \mathbb{R}$ is unbounded, then it is not integrable on $[a,b]$.*

Proof: Suppose, for example, that the function f is unbounded from above, i.e.

$$\forall_{M>0} \ \exists_{t\in[a,b]} \ f(t) > M.$$

Then

$$\forall_{n\in\mathbb{N}} \ \exists_{t_n\in[a,b]} \ f(t_n) > n. \tag{8.7}$$

Since $[a,b]$ is compact, we can choose a convergent subsequence $\{t_{n_k}\}$ of $\{t_n\}$ to a limit $t_o \in [a,b]$. In order to show that the integral $\displaystyle\int_a^b f(x)dx$ does not exist, by negating (8.4), we need to show that

$$\exists_{\varepsilon>0} \ \forall_{\delta>0} \ \exists_{P,Q} \ \ \|P\| < \delta, \ \|Q\| < \delta \quad \text{and} \quad |\sigma(f,P,\xi) - \sigma(f,Q,\zeta)| \ge \varepsilon.$$

Put $\varepsilon = 1$ and let $\delta > 0$ be an arbitrary number. We choose the both partitions P and Q to be the same, namely $P = Q = \{x_k\}_{k=0}^m$, where $x_k = a + k\frac{b-a}{m}$. We choose m to be such that $m > \frac{b-a}{\delta}$. Notice that $|\triangle x_k| = \frac{b-a}{m}$, thus $\|P\| = \|Q\| = \frac{b-a}{m}$. There exists a natural number $k \le m$ such that $t_o \in [x_{k-1}, x_k]$ and the interval $[x_{k-1}, x_k]$ contains (by convergence of $\{t_{n_k}\}$) infinitely many elements of the sequence $\{t_{n_k}\}$. We put for $l \ne k$ $\xi_l = \zeta_l = x_k$, and for $l = k$ we put $\xi_k = t_o$ and $\zeta_k = t_{n_l}$, where k, is chosen to be such that $f(t_{n_l}) > f(t_o) + \frac{m}{b-a}$. Then we have

$$\left|\sum_{s=1}^m f(\xi_s) \triangle x_s - \sum_{s=1}^m f(\zeta_s) \triangle x_s\right| = |f(z_o) - f(t_{n_l})| \cdot \frac{b-a}{m} > \frac{m}{b-a} \cdot \frac{b-a}{m} = 1.$$

and the conclusion follows. $\square$

8.2 Lower and Upper Darboux Integrals

A fundamental question related to the notion of the definite integral is: *What are the integrable functions on a closed interval* $[a,b]$*?* It is clear from Proposition 8.6, that only a bounded function can be integrable. Assume therefore, that $f : [a,b] \to \mathbb{R}$ is a bounded function, i.e. there are constants M and m such that

$$\forall_{x\in[a,b]} \quad m \le f(x) \le M.$$

We consider a partition P given by

$$a = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k < \cdots < x_{n-1} < x_n = b.$$

We put

$$m_k := \inf\{f(x) : x \in [x_{k-1}, x_k]\}, \quad \text{and} \quad M_k := \sup\{f(x) : x \in [x_{k-1}, x_k]\},$$

and we define the following sums:

$$s(f,P) := \sum_{k=1}^{n} m_k \triangle x_k, \quad S(f,P) := \sum_{k=1}^{n} M_k \triangle x_k.$$

The sum $s(f,P)$ is called the *lower integral sum* and $S(f,P)$ is called the *upper integral sum* or simply *Darboux sums* for the partition P. Notice that for a continuous function $f(x)$ (by Weierstrass Theorem, see Theorem 3.92, there exist $t_k,\ t'_k \in [x_{k-1}, x_k]$ such that $f(t_k) = m_k$ and $f(t'_k) = M_k$ for all k, so both sums $s(f,P)$ and $S(f,P)$ are Riemann sums, but this is not true in general for non-continuous functions. However, it follows from the definition of the Riemann sum and the inequality

$$m_k \le f(\xi_k) \le M_k, \qquad \text{for} \ \ \xi_k \in [x_{k-1}, x_k],$$

that

$$s(f,P) \le \sigma(f,P,\xi) = \sum_{k=1}^{n} f(\xi_k) \triangle x_k \le S(f,P).$$

That means the sum $s(f,P)$ is a lower bound and $S(f,P)$ is an upper bound for all Riemann sums $\sigma(f,P,\xi)$ taken for the partition P. On the other hand, for every $\varepsilon > 0$ and for every $k = 1, 2 \ldots, n$, we can find $\xi_k, \xi'_k \in [x_{k-1}, x_k]$ such that $f(\xi_k) - m_k < \frac{\varepsilon}{(b-a)}$ and $M_k - f(\xi'_k) < \frac{\varepsilon}{(b-a)}$. Then for the Riemann sum $\sigma(f,P,\xi) = \sum_{k=1}^{m} f(\xi_k) \triangle x_k$ we have

$$\sigma(f,P,\xi) - s(f,P) = \sum_{k=1}^{n} (f(\xi_k) - m_k) \triangle x_k < \tfrac{\varepsilon}{(b-a)} \sum_{k=1}^{n} \triangle x_k = \tfrac{\varepsilon}{(b-a)}(b-a) = \varepsilon,$$

and for the Riemann sum $\sigma(f,P,\xi) = \sum_{k=1}^{m} f(\xi'_k) \triangle x_k$ we have

$$S(f,P) - \sigma(f,P,\xi') = \sum_{k=1}^{n} (M_k - f(\xi'_k)) \triangle x_k < \tfrac{\varepsilon}{(b-a)} \sum_{k=1}^{n} \triangle x_k = \tfrac{\varepsilon}{(b-a)}(b-a) = \varepsilon.$$

Consequently $s(f,P)$ is equal to the infimum and $S(f,P)$ is equal to the supremum of the all Riemann sums $\sigma(f,P,\xi)$, i.e.

$$s(f,P) = \inf\{\sigma(f,P,\xi) : \sigma(f,P,\xi) = \sum_{k=1}^{m} f(\xi_k) \bigtriangleup x_k, \quad \xi_k \in [x_{k-1}, x_k], \;\; k = 1,2,\dots,m\},$$
$$S(f,P) = \sup\{\sigma(f,P,\xi) : \sigma(f,P,\xi') = \sum_{k=1}^{m} f(\xi'_k) \bigtriangleup x_k, \quad \xi'_k \in [x_{k-1}, x_k], \;\; k = 1,2,\dots,m\}.$$

It is convenient to introduce the following definition:

Definition 8.7. Let P and Q be two partition of the interval $[a,b]$, i.e.

$$P : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$
$$Q : a = y_0 < y_1 < y_2 < \cdots < y_{m-1} < y_m = b.$$

We will say that the partition Q is *finer or equal* than the partition P if

$$\{x_0, x_1, x_2, \dots, x_{n-1}, x_n\} \subset \{y_0, y_1, \dots, y_{m-1}, y_m\},$$

i.e. the partition Q contains all the partition points of P and possibly additional points. In such a case we will write $P \leq Q$. Notice that this relation '$\leq$' is a partial order on the set of all partitions.

Proposition 8.8. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function and P and Q two partitions of $[a,b]$ such that $P \leq Q$. Then, we have*

$$s(f,P) \leq s(f,Q) \leq S(f,Q) \leq S(f,P). \tag{8.8}$$

Proof: Suppose first that Q is a partition obtained from the partition $P = \{x_k\}$ by adding one extra point x' to P, i.e.

$$a = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x' < x_k < \cdots < x_{n-1} < x_n = b.$$

Let $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$, $M''_k = \sup\{f(x) : x \in [x', x_k]\}$ and $M'_k = \sup\{f(x) : x \in [x_{k-1}, x']\}$. Since $M_k \geq M'_k$ and $M_k \geq M''_k$ we have that

$$M_k \bigtriangleup x_k \geq M'_k(x' - x_{k-1}) + M''_k(x_k - x'),$$

so $S(f,P) \geq S(f,Q)$. Similarly, we can show that $s(f,P) \leq s(f,Q)$. Therefore, by adding one or more extra points to P, we obtained a new partition Q such that

$$s(f,P) \leq s(f,Q) \leq S(f,Q) \leq S(f,P).$$

In the general case, when $P < Q$, the partition Q can be obtained from the partition P by adding a finite number of points, therefore by applying the principle of mathematical induction, the statement follows. $\square$

Proposition can be generalized into the following

Proposition 8.9. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function with $M = \sup_{x\in[a,\, b]} f(x)$ and $m = \inf_{x\in[a,\, b]} f(x)$. Suppose that P and Q are two partitions of $[a, b]$ where Q is obtained by adding n_0 more points to P. That is, $P \subset Q$ and the cardinality of the set $Q \setminus P$ is n_0. Then we have*

$$S(f,\, P) \geq S(f,\, Q) \geq S(f,\, P) - n_0(M-m)\|P\|$$
$$s(f,\, P) \leq s(f,\, Q) \leq s(f,\, P) + n_0(M-m)\|P\|.$$

Proof: Let $\{x_i^*\}_{i=1}^{n_0}$ be the n_0 points added to P and $Q = P \cup \{x_i^*\}_{i=1}^{n_0}$. Let $\{Q_i\}_{i=1}^{n_0}$ be partitions obtained as follows: $Q_1 = P \cup \{x_1^*\}$, $Q_2 = Q_1 \cup \{x_2^*\}$, $Q_3 = Q_2 \cup \{x_3^*\} \cdots$, $Q_{n_0} = Q_{n_0-1} \cup x_{n_0}^*$. It is clear that $\|P\| \geq \max\{\|Q_1\|, \|Q_2\|, \cdots, \|Q_{n_0}\|\}$.

Now we first consider partitions $P = \{a = x_0,\, x_1,\, x_2,\, \cdots, x_n = b\}$ and Q_1. Assume $x_1^* \in [x_{k-1},\, x_k]$ and let

$$M_k = \sup_{x\in[x_{k-1},\, x_k]} f(x),\ M_k' = \sup_{x\in[x_{k-1},\, x_1^*]} f(x),\ M_k'' = \sup_{x\in[x_1^*,\, x_k]} f(x).$$

Then we have $0 \leq M_k - M_k'' \leq M - m$ and $0 \leq M_k - M_k' \leq M - m$. It follows that

$$\begin{aligned} S(f,\, P) - S(f,\, Q_1) &= M_k\delta x_k - (M_k'(x_1^* - x_{k-1}) + M_k''(x_k - x_1^*)) \\ &= M_k(x_k - x_1^* + x_1^* - x_{k-1}) - (M_k'(x_1^* - x_{k-1}) + M_k''(x_k - x_1^*)) \\ &= (M_k - M_k'')(x_k - x_1^*) + (M_k - M_k')(x_1^* - x_{k-1}) \\ &\geq 0, \end{aligned} \tag{8.9}$$

and that

$$\begin{aligned} S(f,\, P) - S(f,\, Q_1) &= (M_k - M_k'')(x_k - x_1^*) + (M_k - M_k')(x_1^* - x_{k-1}) \\ &\leq (M-m)(x_k - x_1^*) + (M-m)(x_1^* - x_{k-1}) \\ &\leq (M-m)(x_k - x_{k-1}) \\ &\leq (M-m)\|P\|. \end{aligned} \tag{8.10}$$

Then by (8.9) and (8.10) we have

$$0 \leq S(f,\, P) - S(f,\, Q_1) \leq (M-m)\|P\|. \tag{8.11}$$

By the same token for (8.11) we have

$$\begin{array}{lcccc} 0 & \leq S(f,\, Q_1) - S(f,\, Q_2) & \leq (M-m)\|Q_1\| & \leq (M-m)\|P\| \\ 0 & \leq S(f,\, Q_2) - S(f,\, Q_3) & \leq (M-m)\|Q_2\| & \leq (M-m)\|P\| \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \leq S(f,\, Q_{n_0-1}) - S(f,\, Q_{n_0}) & \leq (M-m)\|Q_{n_0-1}\| & \leq (M-m)\|P\|. \end{array} \tag{8.12}$$

Adding similar terms in the inequalities (8.11) and (8.12) we have

$$0 \leq S(f,\, P) - S(f,\, Q) \leq n_0(M-m)\|P\|,$$

which leads to

$$S(f,\, P) \geq S(f,\, Q) \geq S(f,\, P) - n_0(M-m)\|P\|.$$

Similarly we can show that

$$s(f,\, P) \leq s(f,\, Q) \leq s(f,\, P) + n_0(M-m)\|P\|.$$

□

Notice that $P_0 : a = x_1 < x_2 = b$ is the smallest partition of $[a,b]$, and $S(f,P_0) = M(b-a)$, $s(f,P_0) = m(b-a)$, where

$$M = \sup\{f(x) : x \in [a,b]\} \quad \text{and} \quad m = \inf\{f(x) : x \in [a,b]\}$$

Therefore, by Proposition 8.8, all the lower sums $s(f,P)$ are bounded from above by $M(b-a)$ and all the upper sums $S(f,P)$ are bounded from below by $m(b-a)$. We can introduce the following definition:

Definition 8.10. The numbers

$$s = \underline{\int_a^b} f(x)dx := \sup\left\{\sum_{k=1}^{n} m_k \triangle x_k : P = \{x_k\}\right\},$$

and

$$S = \overline{\int_a^b} f(x)dx := \inf\left\{\sum_{k=1}^{n} M_k \triangle x_k : P = \{x_k\}\right\},$$

where the supremum and infimum are taken over all partitions P of $[a,b]$, are called the *lower Darboux integral* and the *upper Darboux integral* respectively.

Now we are in the position to prove the following Darboux Theorem:

Theorem 8.11. *Let $f : [a,\, b] \to \mathbb{R}$ be a bounded function. Then we have*

$$S = \lim_{\|P\|\to 0} S(f,\, P),$$
$$s = \lim_{\|P\|\to 0} s(f,\, P).$$

Proof: We only prove the first limit. The second one can be proved similarly. For every $\epsilon > 0$, we want to find $\delta > 0$ such that for every partition P with $\|P\| < \delta$ we have

$$0 \leq S(f,\, P) - S < \epsilon, \tag{8.13}$$

where the nonnegativity of $S(f,\, P) - S$ is clear since we have $S = \inf_P S(f,\, P)$. $S = \inf_P S(f,\, P)$ also implies that there exists partition P' such that

$$S(f,\, P') - S < \frac{\epsilon}{2}. \tag{8.14}$$

In order to relate $S(f,\, P')$ with $S(f,\, P)$, we consider the partition $Q = P \cup P'$ and assume that Q has n_0 more points than P. Then by Proposition 8.9 we have

$$S(f,\,P) \geq S(f,\,Q) \geq S(f,\,P) - n_0(M-m)\|P\|,$$
$$S(f,\,P') \geq S(f,\,Q).$$

which lead to $S(f,\,P') \geq S(f,\,P) - n_0(M-m)\|P\|$ and hence to

$$S(f,\,P) - S(f,\,P') \leq n_0(M-m)\|P\|. \tag{8.15}$$

Choose $\delta = \frac{\epsilon}{2n_0(M-m)}$. Then by (8.15) for every P with $\|P\| < \delta$ we have

$$S(f,\,P) - S(f,\,P') \leq n_0(M-m)\|P\| < \frac{\epsilon}{2}. \tag{8.16}$$

By (8.14) and (8.16) we have

$$S(f,\,P) - S = S(f,\,P) - S(f,\,P') + S(f,\,P') - S < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{8.17}$$

Therefore (8.13) is proved. □

Theorem 8.12. *A bounded function $f : [a,b] \to \mathbb{R}$ is integrable if and only if*

$$\lim_{\|P\|\to 0} (S(f,P) - s(f,P)) = 0. \tag{8.18}$$

Proof: Suppose $f(x)$ is integrable on $[a,b]$, i.e. there exists a number I such that

$$\lim_{\|P\|\to 0} \sigma(f,P,\xi) = I \iff \left[\forall_{\varepsilon>0}\, \exists_{\delta>0} \forall_P \quad \|P\| < \delta \quad \Longrightarrow \quad |\sigma(f,P,\xi) - I| < \tfrac{\varepsilon}{2}\right], \tag{8.19}$$

where $\sigma(f,P,\xi) = \sum_{k=1}^n f(\xi_k)\triangle x_k$ and the points $\xi_k \in [x_{k-1}, x_k]$ are chosen arbitrarily. In particular, it follows from (8.2) that

$$I - \frac{\varepsilon}{2} < \sigma(f,P,\xi) < I + \frac{\varepsilon}{2}.$$

On the other hand, $s(f,P)$ is the supremum of numbers $\sigma(f,P,\xi)$ (with all possible values of ξ_k), so we have

$$I - \frac{\varepsilon}{2} \leq s(f,P) \leq I + \frac{\varepsilon}{2}.$$

Similarly, $S(f,P)$ is the infimum of numbers $\sigma(f,P,\xi)$, thus

$$I - \frac{\varepsilon}{2} \leq S(f,P) \leq I + \frac{\varepsilon}{2},$$

and since $s(f,P) \leq S(f,P)$, we obtain $S(f,P) - s(f,P) \leq \varepsilon$.

Conversely, suppose that we have

$$\lim_{\|P\|\to 0} \Big(S(f,P) - s(f,P)\Big) = 0.$$

Therefore, by Theorem 8.11 we have $I = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$ and for every partition P

$$s(f,P) \leq I \leq S(f,P), \qquad s(f,P) \leq \sigma(f,P,\xi) \leq S(f,P) \tag{8.20}$$

We have by the assumption

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{P=\{x_k\}}\quad \|P\|<\delta \implies |S(P)-s(P)|<\varepsilon.$$

By (8.20) we have that I, $\sigma(f,P,\xi)\in[s(f,P),S(f,P)]$, thus $|I-\sigma(f,P,\xi)|\le\varepsilon$, thus

$$\lim_{\|P\|\to 0}\sigma(f,P,\xi)=I$$

and the conclusion follows. $\square$

The following corollaries are immediate consequences of Theorems 8.11 and 8.12.

Corollary 8.13. *A bounded function $f:[a,b]\to\mathbb{R}$ is integrable if and only if $S=s$.*

Corollary 8.14. *A bounded function $f:[a,b]\to\mathbb{R}$ is integrable if and only if for every $\epsilon>0$, there exists a partition P such that $S(f,\,P)-s(f,\,P)\le\epsilon$.*

Theorem 8.15. *Every continuous function $f:[a,b]\to\mathbb{R}$ is integrable on $[a,b]$.*

Proof: Let $\varepsilon>0$ be an arbitrary number. By Theorem 3.93, the function $f:[a,b]\to\mathbb{R}$ is uniformly continuous on $[a,b]$, therefore

$$\exists_{\delta>0}\ \forall_{t,s\in[a,b]}\quad |s-t|<\delta \implies |f(s)-f(t)|<\frac{\varepsilon}{(b-a)}.$$

Let $P=\{x_k\}$ be a partition such that $\|P\|<\delta$, then by Weierstrass Theorem (Theorem 3.92) we have that for every $k=1,\dots,n$, there exist ξ_k, $\xi_k'\in[x_{k-1},x_k]$ such that $m_k=\inf\{f(x):x\in[x_{k-1},x_k]\}$ and $M_k=\sup\{f(x):x\in[x_{k-1},x_k]\}$. Since $|\xi_k-\xi_k'|<\delta$ we have that $M_k-m_k<\frac{\varepsilon}{(b-a)}$, and thus

$$\begin{aligned}
S(f,P)-s(f,P)&=\sum_{k=1}^n M_k\bigtriangleup x_k-\sum_{k=1}^n m_k\bigtriangleup x_k\\
&=\sum_{k=1}^n (M_k-m_k)\bigtriangleup x_k<\sum_{k=1}^n\frac{\varepsilon}{(b-a)}\bigtriangleup x_k\\
&=\frac{\varepsilon}{(b-a)}\sum_{k=1}^n\bigtriangleup x_k=\frac{\varepsilon}{(b-a)}(b-a)=\varepsilon.
\end{aligned}$$

Consequently, for all partitions P such that $\|P\|<\delta$ we have

$$|S(f,P)-s(f,P)|<\varepsilon,$$

so $\lim\limits_{\|P\|\to 0}\Big(S(f,P)-s(f,P)\Big)=0$, and the integral $\int_a^b f(x)dx$ exists. $\square$

Theorem 8.15 can be generalized. In fact we have the following result

Theorem 8.16. *Let $f:[a,b]\to\mathbb{R}$ be a bounded function such that it has only a finite number of discontinuity points in $[a,b]$. Then $f(x)$ is integrable on $[a,b]$.*

Proof: Suppose that $m \le f(x) \le M$ for all $x \in [a,b]$ and let $\varepsilon > 0$ be an arbitrary number. Assume that the discontinuity points of $f(x)$ are $t_1 < t_2 < \cdots < t_r$. For each of the points t_j we can find an open interval $(t_j - \rho_j, t_j + \rho_j)$ such that $\sum_{j=1}^r 2\rho_j < \frac{\varepsilon}{4(M-m)}$, i.e. the total length of all these intervals is smaller than $\frac{\varepsilon}{4(M-m)}$. On the other hand, the set $K := [a,b] \setminus \bigcup_{j=1}^r (t_j - \rho_j, t_j + \rho_j)$ is a finite union of closed intervals and $f(x)$ is continuous on each of these intervals, so, by Theorem 3.92, f is uniformly continuous on K. Therefore

$$\exists_{\delta>0}\ \forall_{s,t\in K} \quad |s-t| < \delta \implies \quad |f(s) - f(t)| < \frac{\varepsilon}{2(b-a)}.$$

Let $P = \{x_k\}$ be an arbitrary partition of $[a,b]$ such that $\|P\| < \min\{\delta, \rho_j, j = 1,\dots,r\}$. There are exactly two types of subintervals $[x_{k-1}, x_k]$:

(1) Type I if $[x_{k-1}, x_k] \subset K$, and;
(2) Type II otherwise, i.e $[x_{k-1}, x_k]$ intersects one of the subintervals $(t_j - \rho_j, t_j + \rho_j)$.

We can write

$$s(f,P) = \sum_{k=1}^n m_k \bigtriangleup x_k = \sum_{\mathrm{I}} m_k \bigtriangleup x_k + \sum_{\mathrm{II}} m_k \bigtriangleup x_k,$$

and

$$S(f,P) = \sum_{k=1}^n M_k \bigtriangleup x_k = \sum_{\mathrm{I}} M_k \bigtriangleup x_k + \sum_{\mathrm{II}} M_k \bigtriangleup x_k,$$

where we denote by $\sum_{\mathrm{I}}$ and $\sum_{\mathrm{II}}$ the summation over intervals of the type I and II respectively. Then we have

$$\begin{aligned}
S(f,P) - s(f,P) &= \sum_{\mathrm{I}} \Big(M_k - m_k\Big) \bigtriangleup x_k + \sum_{\mathrm{II}} \Big(M_k - m_k\Big) \bigtriangleup x_k \\
&< \sum_{\mathrm{I}} \frac{\varepsilon}{2(b-a)} \bigtriangleup x_k + \sum_{\mathrm{II}} (M-m) \bigtriangleup x_k \\
&< \frac{\varepsilon}{2(b-a)}(b-a) + (M-m)\sum_{\mathrm{II}} \bigtriangleup x_k \\
&< \frac{\varepsilon}{2} + (M-m)\cdot 2\frac{\varepsilon}{4(M-m)} \\
&= \varepsilon.
\end{aligned}$$

Consequently, $\lim_{\|P\|\to 0} \big(S(P) - s(P)\big) = 0$ and $f(x)$ is integrable on $[a,b]$. □

Theorem 8.17. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic bounded function, then $f(x)$ is integrable on $[a,b]$.*

Proof: Let $\varepsilon > 0$ be an arbitrary number and let $P = \{x_k\}$ be a partition such that $\|P\| < \frac{\varepsilon}{f(b)-f(a)}$. Assume for example that the function f is non-decreasing, i.e. for all $x \in [x_{k-1}, x_k]$ we have

$$f(x_{k-1}) \le f(x) \le f(x_k)$$

thus

$$m_k = f(x_{k-1}), \quad M_k = f(x_k).$$

Therefore

$$\begin{aligned}
S(f,P) - s(f,P) &= \sum_{k=1}^{n} M_k \triangle x_k - \sum_{k=1}^{n} m_k \triangle x_k \\
&= \sum_{k=1}^{n} \big(f(x_k) - f(x_{k-1})\big) \triangle x_k \\
&\le \sum_{k=1}^{n} \big(f(x_k) - f(x_{k-1})\big)\delta \\
&= \delta \sum_{k=1}^{n} \big(f(x_k) - f(x_{k-1}) = \delta \cdot \big(f(b) - f(a)\big) \\
&< \frac{\varepsilon}{f(b) - f(a)} \big(f(b) - f(a)\big) = \varepsilon.
\end{aligned}$$

Consequently, $\lim\limits_{\|P\|\to 0} \Big(S(P) - s(P)\Big) = 0$ and the function f is integrable on $[a,b]$. □

We have identified the following classes of integrable functions on an interval $[a,b]$

(I) Bounded functions with a finite number of discontinuities in $[a,b]$;
(II) Bounded monotonic functions.

Example 8.18. We consider the function $f : [0,1] \to \mathbb{R}$, called the *Dirichlet function* defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number,} \\ 0 & \text{if } x \text{ is irrational number.} \end{cases}$$

Let P be an arbitrary partition of $[0,1]$. Notice that for every $k = 1, 2, \dots, n$, $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} = 0$ and $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} = 1$. Therefore,

$$s(f,P) = \sum_{k=1}^{n} m_k \triangle x_k = 0, \qquad \text{and} \qquad S(f,P) = \sum_{k=1}^{n} M_k \triangle x_k = \sum_{k1}^{n} \triangle x_k = [a,b],$$

so $\lim\limits_{\|P\|\to 0} \Big(S(P) - s(P)\Big) = [a,b]$ and the function f is not integrable on $[0,1]$. Notice that f has infinitely many discontinuity points, namely every point $x_o \in [0,1]$ is a discontinuity point of f.

Proposition 8.19. *Let $f,\ g : [a,b] \to \mathbb{R}$ be two integrable functions on $[a,b]$. Then*

(i) The functions $|f(x)|$ and $kf(x)$, for $k \in \mathbb{R}$, are integrable on $[a,b]$;
(ii) The functions $f(x) - g(x)$, $f(x) + g(x)$ and $f(x)g(x)$ are also integrable on $[a,b]$,
(iii) The function $f(x)$ is integrable on every subinterval $[\alpha, \beta]$ of $[a,b]$. Moreover, if $f(x)$ is integrable on subintervals $[c_{k-1}, c_k]$ such that $a = c_0 < c_1 < \cdots < c_m = b$, then $f(x)$ is also integrable on $[a,b]$,
(iv) Every function $h(x)$ such that $h(x) \neq f(x)$ for only a finite number of points x in $[a,b]$, is also integrable on $[a,b]$.

Proof: For a given partition P and a function $F : [a, b] \to \mathbb{R}$ we will denote by $S(F, P)$ and $s(F, P)$ respectively the upper and lower sums of F with respect to the partition P.
(i): Since for every t, $s \in [x_{k-1}, x_k]$

$$\Big| |f(s)| - |f(t)| \Big| \le |f(s) - f(t)|,$$

we have that

$$\sup\{|f(x)| : x \in [x_{k-1}, x_k]\} - \inf\{|f(x)| : x \in [x_{k-1}, x_k]\} \le M_k - m_k,$$
$$0 \le \lim_{\|P\|\to 0} \Big(S(|f|, P) - s(|f|, P) \le \lim_{\|P\|\to 0} \Big(S(f, P) - s(f, P)\Big) = 0,$$

so the function $|f(x)|$ is integrable.

Similarly, we have that

$$S(kf, P) - s(kf, P) = k\Big(S(f, P) - s(f, P)\Big) \to 0 \quad \text{as} \quad \|P\| \to 0,$$

so $kf(x)$ is also integrable.

(ii): We have that

$$\sup\{f(x) + g(x) : x \in [x_{k-1}, x_k]\} \le \sup\{f(x) : x \in [x_{k-1}, x_k]\} + \sup\{g(x) : x \in [x_{k-1}, x_k]\}$$

and

$$\inf\{f(x) : x \in [x_{k-1}, x_k]\} + \inf\{g(x) : x \in [x_{k-1}, x_k]\} \le \inf\{f(x) + g(x) : x \in [x_{k-1}, x_k]\}$$

hence

$$\begin{aligned} S(f+g, P) - s(f+g, P) &\le \Big(S(f, P) + S(g, P)\Big) - \Big(s(f, P) + s(g, P)\Big) \\ &= \Big(S(f, P) - s(f, P)\Big) + \Big(S(g, P) - s(g, P)\Big) \\ &\longrightarrow 0 \quad \text{as } \|P\| \to 0. \end{aligned}$$

Since $f(x)$ and $g(x)$ are bounded, there exists a constant M such that $f(x)$, $g(x) \le M$ for all $x \in [a, b]$. For s, $t \in [x_{k-1}, x_k]$ we have

$$f(s)g(s) - f(t)g(t) = \big(f(s) - f(t)\big)g(s) + \big(g(s) - g(t)\big)f(t),$$

thus

$$\sup\{f(s)g(s)\} - \inf\{f(t)g(t)\} \le \big(\sup\{f(s)\} - \inf\{f(t)\}M + \big(\sup\{g(t)\} - \inf\{g(t)\}\big)M.$$

Consequently,

$$S(fg, P) - s(fg, P) \le \Big(S(f, P) - s(f, P)\Big)M + \Big(S(g, P) - s(g, P)\Big)M \longmapsto 0 \quad \text{as } \|P\| \to 0,$$

hence $f(x)g(x)$ is integrable.

(iii): If $f(x)$ is integrable on $[a, b]$ then for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|P\| < \delta \quad \Longrightarrow S(f,P) - s(f,P) < \varepsilon.$$

Let Q be a partition of $[\alpha,\beta] \subset [a,b]$ such that $\|Q\| < \delta$. We can extend the partition Q to a partition $\tilde{Q}$ of $[a,b]$ such that $\left\|\tilde{Q}\right\| < \delta$, and thus we have

$$S(f,Q) - s(f,Q) \le S(f,\tilde{Q}) - s(f,\tilde{Q}) < \varepsilon,$$

so the function $f(x)$ is integrable on $[\alpha,\beta]$.

Suppose that $a < c < b$ and that the function $f(x)$ is integrable on the intervals $[a,c]$ and $[c,b]$. Let $\varepsilon > 0$ be an arbitrary number. Since $f(x)$ is integrable on $[a,c]$ and $[c,d]$ there exists $\delta > 0$ such that for every partition P_1 of $[a,c]$ and every partition P_2 of $[c,b]$, such that $\|P_1\| < \delta$ and $\|P_2\| < \delta$, we have

$$S(P_1,f) - s(P_2,f) < \frac{\varepsilon}{3}, \qquad S(P_2,f) - s(P_2,f) < \frac{\varepsilon}{3}.$$

Let $\eta = \min\{\delta, \frac{\varepsilon}{3(M-m)}\}$, where $m \le f(x) \le M$ for $x \in [a,b]$. Let P be an arbitrary partition of $[a,b]$ and assume that $c \in [x_{l-1}, x_l]$. Then we have

$$\begin{aligned} S(f,P) - s(f,P) &= \sum_{k=0}^{l-1} \big(M_k - m_k\big) \triangle x_k + \big(M_l - m_l\big) \triangle x_l + \sum_{k=l+1}^{n} \big(M_k - m_k\big) \triangle x_k \\ &< \frac{\varepsilon}{3} + (M-m)\frac{\varepsilon}{3(M-m)} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

and therefore $\lim_{\|P\|\to 0} \Big(S(f,P) - s(f,P)\Big) = 0$, i.e. the function $f(x)$ is integrable on $[a,b]$.

(iv): The idea of the proof is similar to the proof of the case (iii). In this case we changed the value of the function $f(x)$ at the point c where $a < c < b$ we can apply the same construction as above, where m and M are respectively a lower and upper bound for both the function $f(x)$ and the new value at c. The rest of the proof is without change. □

Example 8.20. (a) We will use the definition of the definite integral to compute $I = \int_0^a x^2 dx$, where $a > 0$ is an arbitrary number. Since the function $f(x) = x^2$ is continuous, it is integrable and we have that

$$\int_0^a x^2 dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(\xi_k) \triangle x_k.$$

We can choose the partition P to be the partition of $[0,a]$ obtained by dividing $[0,a]$ into n subintervals of equal length, i.e. $P = \{x_k\}$, where $x_k = k\frac{a}{n}$. We can also choose $\xi_k \in [x_{k-1}, x_k]$ to be x_k, i.e. $\xi_k = \frac{ka}{n}$. Then we have

$$\int_0^a x^2 dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{ka}{n}\right)^2 \cdot \frac{a}{n} = \lim_{n\to\infty} a^3 \cdot \frac{1^2 + 2^2 + 3^2 + \cdots + n^2}{n^3}.$$

Since

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

so it follows

$$\int_0^a x^2 dx = \int_0^a x^2 dx = a^3 \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}a^3.$$

(b) We will compute the integral $\int_a^b x^\mu dx$, where μ is an arbitrary real number and $b > a > 0$. Again, by continuity of $f(x) = x^\mu$ we have that

$$\int_a^b x^\mu dx = \lim_{\|P\|\to 0} \sum_{k=1}^n f(\xi_k) \triangle x_k.$$

Let $q = q_n = \sqrt[n]{\frac{b}{a}}$. We define the partition $P_n = \{x_n\}$ by $x_k = aq^k$, $k = 0, 1, 2, \dots, n$. Since $\lim_{n\to\infty} q_n = \lim_{n\to\infty} \sqrt[n]{\frac{b}{a}} = 1$, it follows that

$$0 \le \lim_{n\to\infty} (x_k - x_{k-1}) = \lim_{n\to\infty} (aq_n^k - aq_n^{k-1}) \le \lim_{n\to\infty} b(q_n - 1) = 0,$$

it follows that $\lim_{n\to\infty} \|P_n\| = 0$. Choose $\xi_k = aq^{k-1}$, then we have

$$\sigma(f, P_n, \xi) = \sum_{k=1}^n (aq^{k-1})^\mu (aq^k - aq^{k-1}) = a^{\mu+1}(q-1) \sum_{k=1}^n (q^{\mu+1})^{k-1}$$

Suppose that $\mu \ne -1$, then we have

$$\sigma(f, P_n, \xi) = a^{\mu+1}(q-1) \frac{\left(\frac{b}{a}\right)^{\mu+1} - 1}{q^{\mu+1} - 1} = (b^{\mu+1} - a^{\mu+1}) \frac{q-1}{q^{\mu+1} - 1}.$$

Consequently,

$$\int_a^b x^\mu dx = (b^{\mu+1} - a^{\mu+1}) \lim_{n\to\infty} \frac{q-1}{q^{\mu+1} - 1} = \frac{b^{\mu+1} - a^{\mu+1}}{\mu + 1}.$$

In the case $\mu = -1$ we have

$$\sigma(f, P_n, \xi) = n(q_n - 1) = n\left(\sqrt[n]{\frac{b}{a}} - 1\right),$$

hence

$$\int_a^b \frac{dx}{x} = \lim_{n\to\infty} n\left(\sqrt[n]{\frac{b}{a}} - 1\right) = \ln b - \ln a.$$

(c) We will compute $\int_a^b \sin x dx$. We consider the partition P_n consisting of points x_k obtained by dividing the interval $[a, b]$ into n equal parts of length $h = \frac{b-a}{n}$, i.e. $x_k = a + kh$, $k = 0, 1, 2, \dots, n$. We also assume that $\xi_k = x_k = a + kh$. We have the following Riemann sum associated with the partition P_n

$$\sigma(f, P_n, \xi) = h \sum_{k=1}^n \sin(a + kh).$$

On the other hand

$$\begin{aligned}
\sum_{k=1}^n \sin(a + kh) &= \frac{1}{2\sin\frac{1}{2}h} \sum_{k=1}^n 2\sin(a + kh)\sin\tfrac{1}{2}h \\
&= \frac{1}{2\sin\frac{1}{2}h} \sum_{k=1}^n \left[\cos\left(a + \left(k - \tfrac{1}{2}\right)h\right) - \cos\left(a + \left(k + \tfrac{1}{2}\right)h\right)\right] \\
&= \frac{\cos\left(a + \frac{1}{2}h\right) - \cos\left(a + \left(n + \frac{1}{2}\right)h\right)}{2\sin\frac{1}{2}h}.
\end{aligned}$$

Consequently

$$\sigma(f, P_n, \xi) = \frac{\frac{1}{2}h}{\sin\frac{1}{2}h}\left[\cos\left(a+\tfrac{1}{2}h\right) - \cos\left(b+\tfrac{1}{2}h\right)\right].$$

Since $h \to 0$ as $n \to \infty$ we have

$$p\int_a^b \sin x dx = \lim_{h\to 0}\frac{\frac{1}{2}h}{\sin\frac{1}{2}h}\left[\cos\left(a+\tfrac{1}{2}h\right) - \cos\left(b+\tfrac{1}{2}h\right)\right] = \cos a - \cos b.$$

8.3 Properties of Definite Integrals and the Fundamental Theorem of Calculus

We have the following basic properties of the definite integral:

Proposition 8.21. *(i) Let $a < c < b$ (respectively $a < b < c$) and $f(x)$ be an integrable on the interval $[a, b]$ (respectively on $[a, c]$). Then*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

(ii) Let $f,\ g : [a, b] \to \mathbb{R}$ be two integrable functions and $\alpha,\ \beta \in \mathbb{R}$. Then

$$\int_a^b \Big(\alpha f(x) + \beta g(x)\Big)dx = \alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx.$$

(iii) Let $f : [a, b] \to \mathbb{R}$ be a non-negative integrable function. Then

$$\int_a^b f(x)dx \geq 0.$$

(iv) Let $f : [a, b] \to \mathbb{R}$ be an integrable function such that $m \leq f(x) \leq M$ for all $x \in [a, b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Proof: (i): Assume that $a < c < b$. By Proposition 8.19, the function $f(x)$ is integrable on $[a, c]$ and $[c, b]$. We consider partitions $P = \{x_k\}$ of the interval $[a, b]$ such that c is one of the partition points, i.e.

$$a + x_0 < x_1 < \cdots < x_{l-1} < x_l = c < x_{l+1} < \cdots < x_{n-1} < x_n = b,$$

and let $\xi_k \in [x_{k-1}, x_k]$. Then we have

$$\sum_{k=1}^{n} f(\xi_k) \bigtriangleup x_k = \sum_{k=1}^{l} f(\xi_k) \bigtriangleup x_k + \sum_{k=l+1}^{n} f(\xi_k) \bigtriangleup x_k,$$

and by passing to the limit as $\|P\| \to 0$ we obtain

$$\begin{aligned}\int_a^b f(x)dx &= \lim_{\|P\|\to 0}\sum_{k=1}^n f(\xi_k)\triangle x_k\\ &= \lim_{\|P\|\to 0}\sum_{k=1}^l f(\xi_k)\triangle x_k + \lim_{\|P\|\to 0}\sum_{k=l+1}^n f(\xi_k)\triangle x_k\\ &= \int_a^c f(x)dx + \int_c^b f(x)dx.\end{aligned}$$

(ii): Let $P = \{x_k\}$ be a partition of $[a, b]$ and $\xi_k \in [x_{k-1}, x_k]$. Then we have

$$\begin{aligned}\int_a^b \big(\alpha f(x) + \beta g(x)\big)dx &= \lim_{\|P\|\to 0}\sum_{k=1}^n \big(\alpha f(\xi_k) + \beta g(\xi_k)\big)\triangle x_k\\ &= \alpha\lim_{\|P\|\to 0}\sum_{k=1}^n f(\xi_k)\triangle x_k + \beta\lim_{\|P\|\to 0}\sum_{k=1}^n g(\xi_k)\triangle x_k\\ &= \alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx.\end{aligned}$$

(iii): If $f(x) \geq 0$ for $x \in [a, b]$ then for every partition $P = \{x_k\}$ we have

$$\sigma(f, P, \xi) = \sum_{k=1}^n f(\xi_k)\triangle x_k \geq 0,$$

so

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0}\sum_{k=1}^n f(\xi_k)\triangle x_k \geq 0.$$

(iv): If $m \leq f(x) \leq M$ for all $x \in [a, b]$ then for every partition $P = \{x_k\}$ of $[a, b]$ we have

$$m(b-a) = \sum_{k=1}^n m\triangle x_k \leq \sum_{k=1}^n f(\xi_k)\triangle x_k \leq \sum_{k=1}^n M\triangle x_k = M(b-a),$$

thus

$$m(b-a) \leq \int_a^b f(x)dx = \lim_{\|P\|\to 0}\sum_{k=1}^n f(\xi_k)\triangle x_k \leq M(b-a).$$

□

Corollary 8.22. *Let $f,\ g : [a, b] \to \mathbb{R}$ be two integrable on $[a, b]$ functions such that $f(x) \leq g(x)$ for all $x \in [a, b]$. Then*

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Proof: Since $g(x) - f(x) \geq 0$, it follows from Proposition 8.21 that

$$\int_a^b g(x)dx - \int_a^b f(x)dx = \int_a^b (g(x) - f(x))dx \geq 0.$$

□

Corollary 8.23. *Let* $f : [a,b] \to \mathbb{R}$ *be an integrable function on* $[a,b]$. *Then*

$$\left| \int_a^b f(x)dx \right| \le \int_a^b |f(x)|dx.$$

Proof: Since

$$-|f(x)| \le f(x) \le |f(x)|, \qquad \text{for all} \quad x \in [a,b],$$

we have

$$-\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx,$$

so

$$\left| \int_a^b f(x)dx \right| \le \int_a^b |f(x)|dx.$$

□

Theorem 8.24. (Mean Value Theorem) *Let* $f : [a,b] \to \mathbb{R}$ *be an integrable function on* $[a,b]$ *such that* $m \le f(x) \le M$ *for all* $x \in [a,b]$. *Then there exists* $\mu \in [m,M]$ *such that*

$$\int_a^b f(x)dx = \mu(b-a).$$

Proof: By Proposition 8.21(iv) we have that

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a),$$

thus

$$m \le \frac{1}{b-a} \int_a^b f(x)dx \le M,$$

so $\mu = \frac{1}{b-a} \int_a^b f(x)dx \in [m,M]$. □

Corollary 8.25. *Let* $f : [a,b] \to \mathbb{R}$ *be a continuous function. Then there exists* $c \in [a,b]$ *such that*

$$\int_a^b f(x)dx = f(c)(b-a).$$

Proof: Since $f(x)$ is continuous, by Weierstrass' Theorem (Theorem 3.92) it follows that there exists $t,\ t' \in [a,b]$ such that for all $x \in [a,b]$

$$f(t) = \inf\{f(x) : x \in [a,b]\} \le f(x) \le \sup\{f(x) : x \in [a,b]\} = f(t'),$$

thus by the Intermediate Value Theorem (Theorem 5.3) there exists c between t and t' such that $f(c) = \mu \in [f(t), f(t')] = [m,M]$, where $\mu = \frac{1}{b-a} \int_a^b f(x)dx$. □

Theorem 8.26. (Generalization of the Mean Value Theorem) *Let* $f,\ g : [a,b] \to \mathbb{R}$ *be two integrable functions on* $[a,b]$ *such that*

(i) $m \le f(x) \le M$ *for all* $x \in [a, b]$*;*
(ii) either $g(x) \ge 0$ *for all* $x \in [a, b]$ *or* $g(x) \le 0$ *for all* $x \in [a, b]$*.*

Then there exists $\mu \in [m, M]$ *such that*

$$\int_a^b f(x)g(x)dx = \mu \int_a^b g(x)dx. \tag{8.21}$$

Proof: Suppose that $g(x) \ge 0$ for $x \in [a, b]$. We have

$$mg(x) \le f(x)g(x) \le Mg(x), \qquad \text{for } x \in [a, b],$$

then, by Corollary 8.25,

$$m \int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M \int_a^b g(x)dx. \tag{8.22}$$

Since $g(x) \ge 0$, by Proposition 8.21(iii),

$$\int_a^b g(x)dx \ge 0.$$

If $\int_a^b g(x)dx = 0$ then by (8.22) $\int_a^b f(x)g(x)dx = 0$ and the equality (8.21) is trivially satisfied for every number $\mu \in [m, M]$. Suppose therefore, that $\int_a^b g(x)dx > 0$. Then the number

$$\mu = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx},$$

satisfies the required properties. □

Theorem 8.27. *Let* $f : [a, b] \to \mathbb{R}$ *be integrable on* $[a, b]$*. We define the function*

$$F(x) = \int_a^x f(t)dt.$$

Then we have

(i) The function $F : [a, b] \to \mathbb{R}$ *is uniformly continuous;*
(ii) If $f(x)$ *is continuous at* $x_o \in [a, b]$*, then the function* $F(x)$ *is differentiable at* x_o *and*

$$F'(x_o) = f(x_o).$$

Proof: (i) The function $f(x)$ is integrable, therefore it is bounded, i.e. there exists M such that

$$|f(x)| \le M, \qquad \text{for all} \quad x \in [a, b].$$

We will show that the function $F(x)$ is uniformly continuous on $[a, b]$. Indeed, for every $\varepsilon > 0$ we can choose $\delta \le \frac{\varepsilon}{M}$, then for all x, $x' \in [a, b]$ such that $x < x'$ and $|x' - x| < \delta$ we have

$$
\begin{aligned}
|F(x') - F(x)| &= \left| \int_a^{x'} f(x)dx - \int_a^x f(x)dx \right| \\
&= \left| \int_x^{x'} f(t)dt \right| \le \int_x^{x'} |f(t)|dt \\
&\le M(x'-x) < M\delta \le M\frac{\varepsilon}{M} = \varepsilon.
\end{aligned}
$$

Since $\delta > 0$ does not depend on the point x, it follows that the function $F(x)$ is uniformly continuous.

(ii): Assume that the function $f(x)$ is continuous at $x_o \in [a,b]$, i.e.

$$\forall_{\varepsilon>0} \ \exists_{\delta>0} \ \forall_{x\in[a,b]} \quad |x - x_o| < \delta \quad \Longrightarrow \quad |f(x) - f(x_o)| < \varepsilon.$$

Then, for $|h| < \delta$ we have

$$
\begin{aligned}
\left| \tfrac{1}{h}\Big(F(x_o+h) - F(x_o)\Big) - f(x_o) \right| &= \left| \tfrac{1}{h} \left(\int_{x_o}^{x_o+h} f(x)dx - f(x_o)h \right) \right| \\
&= \left| \tfrac{1}{h} \int_{x_o}^{x_o+h} \Big(f(x) - f(x_o)\Big)dx \right| \\
&\le \tfrac{1}{|h|}|h|\varepsilon = \varepsilon.
\end{aligned}
$$

Therefore,

$$\lim_{h\to 0} \frac{F(x_o+h) - F(x_o)}{h} = f(x_o).$$

□

Corollary 8.28. (The Fundamental Theorem of Calculus – Newton-Leibnitz Formula) *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then the function*

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of $f(x)$, i.e.

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Corollary 8.29. *Let $f : [a,b] \to \mathbb{R}$ be continuous and $\Phi : [a,b] \to \mathbb{R}$ be its antiderivative on $[a,b]$. Then*

$$\int_a^b f(x)dx = \Phi(b) - \Phi(a).$$

Proof: Since $F(x) = \int_a^x f(x)dx$ is an antiderivative of $f(x)$, it follows by Proposition 5.1.2, there exists a constant C such that $F(x) = \Phi(x) + C$. Therefore,

$$\int_a^b f(x)dx = F(b) = F(b) - F(a) = \Phi(b) + C - (\Phi(a) + C) = \Phi(b) - \Phi(a).$$

□

Theorem 8.30. (The Second Fundamental Theorem of Calculus) *Let $F : [a,b] \to \mathbb{R}$ be a continuous function, differentiable on (a,b) and let $f : [a,b] \to \mathbb{R}$ be an integrable function on $[a,b]$ such that $f(x) = F'(x)$ for all $x \in (a,b)$ except for a finite number of points. Then we have*

$$F(x) = F(a) + \int_a^x f(t)dt$$

for all $x \in [a,b]$.

Proof: Let $x \in (a,b]$ and let $P = \{x_k\}$ be an arbitrary partition of $[a,x]$ containing all points t such that $f(t) \neq F'(t)$. Then we have

$$F(x) - F(a) = \sum_{k=1}^{n} [F(x_k) - F(x_{k-1})]. \tag{8.23}$$

Since F is continuous on $[x_{k-1}, x_k]$ and differentiable on (x_{k-1}, x_k), there exists $\xi_k \in (x_{k-1}, x_k)$ such that

$$F'(\xi_k) = \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}},$$

so it follows from (**??**) that

$$F(x) - F(a) = \sum_{k=1}^{n} F'(\xi_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} f(\xi_k) \triangle x_k. \tag{8.24}$$

Since we assumed that $f(x)$ is integrable on $[a,b]$ there exists the limit

$$\int_a^x f(t)dt = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(\xi_k) \triangle x_k,$$

which does not depend on the choice of the points ξ_k. Consequently, by (8.24)

$$F(x) - F(a) = \int_a^x f(t)dt.$$

□

For a given function $\Phi : [a,b] \to \mathbb{R}$ we will denote

$$\Phi(x)\big|_a^b = \Phi(b) - \Phi(a).$$

Consequently, if $\Phi : [a,b] \to \mathbb{R}$ is an antiderivative of $f : [a,b] \to \mathbb{R}$ then, by Corollary 8.29, we can write

$$\int_a^b f(x)dx = \Phi(x)\big|_a^b .$$

Corollary 8.31. *(i)* (Integration by Parts Rule) *Let $f, g : [a,b] \to \mathbb{R}$ be two differentiable functions such that their derivatives $f'(x)$ and $g'(x)$ are also continuous on $[a,b]$. Then we have*

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\big|_a^b - \int_a^b f'(x)g(x)dx;$$

(ii) *Let $f : g : [a,b] \to \mathbb{R}$ be two functions such that their derivatives up to the order $n+1$ exist and are continuous on $[a,b]$. Then we have*

$$\int_a^b f(x)g^{(n+1)}(x)dx = f(x)g^{(n)}(x)\Big|_a^b - f'(x)g^{(n-1)}(x)\Big|_a^b - \dots$$
$$+ (-1)^n f^{(n)}(x)g(x)\Big|_a^b + (-1)^{n+1}\int_a^b f^{(n+1)}(x)g(x)dx.$$

Example 8.32. We will compute the integral

$$J_m = \int_0^{\frac{\pi}{2}} \sin^m x dx.$$

By applying the Integration by Parts Rule we obtain

$$\begin{aligned} J_m &= \int_0^{\frac{\pi}{2}} \sin^{m-1} x \tfrac{d}{dx}(-\cos x)dx \\ &= -\sin^{m-1} x \cos x\big|_0^{\frac{\pi}{2}} + (m-1)\int_0^{\frac{\pi}{2}} \sin^{m-2} x \cos^2 x dx \\ &= (m-1)\int_0^{\frac{\pi}{2}} \sin^{m-2} x(1-\sin^2 x)dx \\ &= (m-1)\int_0^{\frac{\pi}{2}} \sin^{m-2} x dx - (m-1)\int_0^{\frac{\pi}{2}} \sin^m x dx \\ &= (m-1)J_{m-2} - (m-1)J_m, \end{aligned}$$

thus

$$J_m = \frac{m-1}{m} J_{m-2}.$$

Since

$$J_0 = \frac{\pi}{2}, \qquad J_1 = -\cos x\big|_0^{\frac{\pi}{2}} = 1,$$

hence, for $m = 2n$ we have

$$\begin{aligned} J_{2n} &= \int_0^{\frac{\pi}{2}} \sin^{2n} x dx = \frac{(2n-1)(2n-3)\cdot \dots \cdot 3\cdot 1}{2n(2n-2)\cdot \dots \cdot 4 \cdot 2} \cdot \frac{\pi}{2} \\ &= \frac{(2n)!}{\left(2^n n!\right)^2} \cdot \frac{\pi}{2}. \end{aligned}$$

and for $m = 2n+1$ we have

$$\begin{aligned} J_{2n+1} &= \int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx = \frac{2n(2n-2)\cdot \dots \cdot 4\cdot 2}{(2n+1)(2n-1)\cdot \dots \cdot 3 \cdot 1} \\ &= \frac{\left(2^n n!\right)^2}{(2n+1)!}. \end{aligned}$$

Example 8.33. In this example we will derive a formula for computing approximations of the number π. For $0 < x < \frac{\pi}{2}$ we have the following inequalities

$$\sin^{2n+1} x < \sin^{2n} x < \sin^{2n-1} x. \tag{8.25}$$

By integrating (8.25) from 0 to $\frac{\pi}{2}$ we obtain

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx < \int_0^{\frac{\pi}{2}} \sin^{2n} x dx < \int_0^{\frac{\pi}{2}} \sin^{2n-1} x dx.$$

By applying the formulas derived in Example 8.32, we have

$$\frac{(2^n n!)^2}{(2n+1)!} < \frac{(2n)!}{(2^n n!)^2} \cdot \frac{\pi}{2} < \frac{(2^{n-1}(n-1)!)^2}{(2n-1)!},$$

so

$$a_n := \frac{1}{2n+1} \cdot \left[\frac{(2^n n!)^2}{(2n)!}\right]^2 < \frac{\pi}{2} < \left[\frac{(2^n n!)^2}{(2n)!}\right]^2 \cdot \frac{1}{2n} =: b_n. \tag{8.26}$$

The first inequality in (8.26) implies that

$$0 < b_n - a_n = \left[\frac{(2^n n!)^2}{(2n)!}\right]^2 \cdot \frac{1}{(2n+1)2n} < \frac{\pi}{2} \cdot \frac{1}{2n}.$$

Since $\lim\limits_{n\to\infty} (b_n - a_n) = 0$ it follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{\pi}{2},$$

in particular, we obtain the following *Wallis Formula*

$$\frac{\pi}{2} = \lim_{n\to\infty} \left[\frac{(2^n n!)^2}{(2n)!}\right]^2 \cdot \frac{1}{2n}, \tag{8.27}$$

and we have the following ‘error’ estimation

$$\left| \frac{\pi}{2} - \left[\frac{(2^n n!)^2}{(2n)!}\right]^2 \cdot \frac{1}{2n} \right| < \frac{\pi}{4n}.$$

Formula (8.27) can be used to compute approximations of the number π, but it is a quite inefficient way to do it.

8.4 Improper Integrals

Notice that the definition of the definite integral $\int_a^b f(x)dx$ required that the function $f(x)$ be bounded on the closed interval $[a, b]$. In this section we extend the notion of the definite integral to the case of functions defined on infinite intervals or unbounded functions. This extension is called the *improper integral* (we may also call it the *improper integral of the first type*).

Definition 8.34. Let $f : [a, \infty) \to \mathbb{R}$ be a function which is integrable on every closed subinterval $[a, A] \subset [a, \infty)$. If the (finite or infinite) limit

$$\lim_{A\to\infty} \int_a^A f(x)dx$$

exists we will denote it by

$$\int_a^\infty f(x)dx = \lim_{A\to\infty}\int_a^A f(x)dx$$

and we will call it an *improper integral* from a to ∞. If $\int_a^\infty f(x)dx$ is finite we will say that this improper integral *converges* or simply that $\int_a^\infty f(x)dx$ is convergent. Otherwise we will say that $\int_a^\infty f(x)dx$ *diverges* or that it is a *divergent* improper integral.

Sometimes, for convenience we will use in place of the term 'improper integral' simply the term 'integral' but it should not be mixed up with the definite integral.

Example 8.35. Let $a > 0$. We will find for what values of λ the improper integral

$$I = \int_a^\infty \frac{dx}{x^\lambda}$$

converges. We have

$$\begin{aligned}
\int_a^A \frac{dx}{x^\lambda} &= \lim_{A\to\infty}\int_a^A \frac{dx}{x^\lambda}\\
&= \begin{cases} \frac{1}{1-\lambda}x^{1-\lambda}\Big|_a^A, & \text{if } \lambda \neq 1\\ \ln x|_a^A, & \text{if } \lambda = 1,\end{cases}\\
&= \begin{cases} \frac{1}{1-\lambda}\left(A^{1-\lambda}-a^{1-\lambda}\right) & \text{if } \lambda \neq 1,\\ \ln A - \ln a & \text{if } \lambda = 1,\end{cases}
\end{aligned}$$

thus

$$\begin{aligned}
\int_a^\infty \frac{dx}{x^\lambda} &= \begin{cases} \frac{1}{1-\lambda}\lim_{A\to\infty}\left(A^{1-\lambda}-a^{1-\lambda}\right) & \text{for } \lambda \neq 1,\\ \lim_{A\to\infty}\ln A - \ln a & \text{for } \lambda = 1,\end{cases}\\
&= \begin{cases} \infty & \text{for } \lambda < 1,\\ \infty & \text{for } \lambda = 1,\\ \frac{1}{\lambda-1}a^{1-\lambda} & \text{for } \lambda > 1.\end{cases}
\end{aligned}$$

This means if $\lambda > 1$ the integral $\int_a^\infty \frac{dx}{x^\lambda}$ converges to $\frac{1}{\lambda-1}a^{1-\lambda}$, i.e.

$$\int_a^\infty \frac{dx}{x^\lambda} = \frac{1}{\lambda-1}a^{1-\lambda}.$$

Similarly we can define the improper integral on the infinite interval $(-\infty, a]$:

Definition 8.36. Let $f : (-\infty, a] \to \mathbb{R}$ be a function which is integrable on every closed subinterval $[A, a] \subset (-\infty, a]$. If the (finite or infinite) limit

$$\lim_{A\to-\infty}\int_A^a f(x)dx$$

exists we will denote it by

$$\int_{-\infty}^{a} f(x)dx = \lim_{A\to-\infty}\int_{A}^{a} f(x)dx$$

and we will call it an *improper integral* from $-\infty$ to a. If $\int_{-\infty}^{a} f(x)dx$ is finite we will say that this improper integral *converges* or simply that $\int_{-\infty}^{a} f(x)dx$ is *convergent*. Otherwise we will say that $\int_{-\infty}^{a} f(x)dx$ *diverges* or that it is a *divergent* improper integral.

Definition 8.37. Let $f : (-\infty,\infty) \to \mathbb{R}$ be a function which is integrable on every finite interval $[A,B]$. If for some $a \in (\infty,\infty)$ there exist convergent improper integrals

$$\int_{-\infty}^{a} f(x)dx, \qquad \text{and} \qquad \int_{a}^{\infty} f(x)dx,$$

we say that the improper integral

$$\int_{-\infty}^{\infty} f(x)dx := \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$$

converges.

Notice that the definition of the convergent improper integral $\int_{-\infty}^{\infty} f(x)dx$ does not depend on the choice of the point a.

Example 8.38. (a) We will compute the integral

$$\int_0^{\infty} e^{-ax}\sin bx\,dx,$$

where $a>0$. An antiderivative of the function $y=e^{-ax}\sin bx$ was evaluated in Example 5.2.5. We have that

$$\int e^{-ax}\sin bx\,dx = -e^{-ax}\frac{a\sin bx + b\cos bx}{a^2+b^2} + C,$$

thus

$$\begin{aligned}\int_0^{\infty} e^{-ax}\sin bx\,dx &= -\lim_{A\to\infty} e^{-ax}\frac{a\sin bx + b\cos bx}{a^2+b^2}\Big|_0^A\\ &= \frac{b}{a^2+b^2} - \lim_{A\to\infty}\frac{a\sin(bA)+b\cos(bA)}{a^2+b^2}e^{-aA}\\ &= \frac{b}{a^2+b^2}.\end{aligned}$$

Similarly, we can show that

$$\int_0^{\infty} e^{-ax}\cos bx\,dx = e^{-ax}\frac{b\sin bx - a\cos bx}{a^2+b^2}\Big|_0^{\infty} = \frac{a}{a^2+b^2}.$$

(b) We will compute the integral $\int_0^{\infty}\frac{dx}{1+x^4}$. One can compute that

$$\int\frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}}\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1} + \frac{1}{2\sqrt{2}}\arctan\frac{x\sqrt{2}}{x\sqrt{2}+1} + \frac{1}{2\sqrt{2}}\arctan(x\sqrt{2}-1) + C.$$

Thus

$$\int_0^\infty \frac{dx}{1+x^4} = \left(\tfrac{1}{4\sqrt{2}} \ln \tfrac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1} + \tfrac{1}{2\sqrt{2}}\arctan \tfrac{x\sqrt{2}}{x\sqrt{2}+1} + \tfrac{1}{2\sqrt{2}}\arctan\left(x\sqrt{2}-1\right)\right)\Big|_0^\infty$$
$$= \frac{\pi}{2\sqrt{2}}.$$

Proposition 8.39. *Let $f : [a,\infty) \to \mathbb{R}$ be a function such that*

(i) $f(x) \geq 0$ for all $x \in [a,\infty)$;
(ii) $f(x)$ is integrable on $[a,b]$ for all $a < b$.

Then the improper integral

$$\int_a^\infty f(x)dx$$

converges if and only if there exists a real number $L \geq 0$ such that

$$\int_a^A f(x)dx \leq L$$

for all $A > a$.

Proof: Since the function $f(x)$ is non-negative, the function

$$\varphi(t) = \int_a^t f(x)dx$$

is non-decreasing, thus there exists the limit

$$\int_0^\infty f(x)dx = \lim_{t\to\infty} \varphi(t).$$

It is clear that if $\varphi(t) \leq L$ for all $t > a$ then $\int_0^\infty f(x)dx \leq L$, thus it converges.
Conversely, if the integral $\int_0^\infty f(x)dx$ converges, then

$$\varphi(t) \leq L = \int_0^\infty f(x)dx$$

for all $t > a$. □

Corollary 8.40. *Let $f : [a,\infty) \to \mathbb{R}$ be a function such that*

(i) $f(x) \geq 0$ for all $x \in [a,\infty)$;
(ii) $f(x)$ is integrable on $[a,b]$ for all $a < b$.

If the numbers $\int_a^A f(x)dx$ are not bounded from above, then

$$\int_a^\infty f(x)dx = \infty.$$

Theorem 8.41. (Comparison Test) *Let f, $g : [a,\infty) \to \mathbb{R}$ be two functions such that*

(i) $f(x)$ *and* $g(x)$ *are integrable on intervals* $[a, A]$*, for* $a < A$*;*
(ii) There exists a number $M > a$ *such that*

$$\forall_{x \geq M} \quad 0 \leq f(x) \leq g(x);$$

Then

(a) If $\int_a^\infty g(x)dx$ *converges then* $\int_a^\infty f(x)dx$ *also converges;*
(b) If $\int_a^\infty f(x)dx$ *diverges then* $\int_a^\infty g(x)dx$ *also diverges.*

Proof: (a): Let $\varphi(t) = \int_M^t g(x)dx$. Since $\int_M^\infty g(x)dx$ converges, there exists $T := \lim_{t\to\infty} \varphi(t)$ such that for all $A > M$ we have

$$\int_M^A f(x)dx \leq \int_M^A g(x)dx = \varphi(A) \leq T,$$

thus, by Proposition 8.39, the integral $\int_M^\infty f(x)dx$ converges, therefore $\int_a^\infty f(x)dx$ also converges.

(b): Assume that $\int_a^\infty f(x)dx = \infty$, then we have that

$$\int_M^A f(x)dx \leq \int_M^A g(x)dx,$$

thus the numbers $\int_M^A g(x)dx$ are not bounded from above, thus $\int_M^\infty g(x)dx$ diverges to ∞, so $\int_a^\infty g(x)dx$ also diverges to ∞. □

Theorem 8.42. (LIMIT COMPARISON TEST) *Let* f*,* $g : [a, \infty) \to \mathbb{R}$ *be two non-negative functions such that*

(i) $f(x)$ *and* $g(x)$ *are integrable on intervals* $[a, A]$*, for* $a < A$*;*
(ii) There exists $\infty \geq K \geq 0$ *such that*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = K;$$

Then

(a) If $\infty > K > 0$ *then* $\int_a^\infty g(x)dx$ *converges if and only if* $\int_a^\infty f(x)dx$ *converges;*
(b) If $K = 0$ *then the convergence of* $\int_a^\infty g(x)dx$ *implies the convergence of* $\int_a^\infty f(x)dx$*;*
(c) If $K = \infty$ *then the divergence of* $\int_a^\infty g(x)dx$ *implies the divergence of* $\int_a^\infty f(x)dx$*.*

Proof: (a): Suppose that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = K$, where K is a finite number. Let $\varepsilon > 0$ be such that $0 < K - \varepsilon$. Then, by assumption (ii), there exists $M > a$ such that

$$\forall_{x \geq M} \quad K - \varepsilon < \frac{f(x)}{g(x)} < K + \varepsilon.$$

Consequently,

$$\forall_{x \geq M} \quad (K - \varepsilon)g(x) < f(x) \quad \text{and} \quad f(x) < (K + \varepsilon)g(x).$$

This implies, by Theorem 6.4.8, that if $\int_M^\infty f(x)dx$ converges then $\int_M^\infty (K - \varepsilon)g(x)dx = (K - \varepsilon)\int_M^\infty g(x)dx$ also converges, and similarly, if $\int_M^\infty (K + \varepsilon)g(x)dx = (K + \varepsilon)\int_M^\infty g(x)dx$ converges then $\int_M^\infty f(x)dx$ also converges.Therefore, $\int_a^\infty f(x)$ is convergent if and only if $\int_a^\infty g(x)dx$ converges.

(b): Assume that $K = 0$ and let $\varepsilon > 0$. Then there exists $M > a$ such that

$$\forall_{x\geq M} \quad \frac{f(x)}{g(x)} < \varepsilon,$$

hence

$$\forall_{x\geq M} \quad f(x) < \varepsilon g(x).$$

Using a similar method as above, we can show that if $\int_a^\infty g(x) < \infty$ then $\int_a^\infty f(x)dx < \infty$.

(c): Assume that $K = \infty$ and let $R > 0$. Then there exists $M > a$ such that

$$\forall_{x\geq M} \quad \frac{f(x)}{g(x)} > R,$$

hence

$$\forall_{x\geq M} \quad f(x) > Rg(x).$$

Consequently, by Theorem 6.4.8, if $\int_a^\infty f(x)dx = \infty$ then $\int_a^\infty g(x)dx = \infty$. □

Theorem 8.43. (Bolzano-Cauchy Criterion) *Let $f : [a,\infty) \to \mathbb{R}$ be a function which is integrable on every closed interval $[a, A]$, $A > a$. Then the improper integral*

$$\int_a^\infty f(x)dx$$

converges if and only if

$$\forall_{\varepsilon>0} \ \exists_{R>a} \ \forall_{A,A'>R} \quad \left|\int_a^{A'} f(x)dx - \int_a^A f(x)dx\right| = \left|\int_{A'}^A f(x)dx\right| < \varepsilon. \tag{8.28}$$

Proof: We define the function

$$\Phi(t) = \int_a^t f(x)dx, \qquad t > a.$$

It is clear that $\int_a^\infty f(x)dx$ converges if and only if $\lim_{t\to\infty} \Phi(t) = \alpha$ exists and is finite.

Assume that $\lim_{t\to\infty} \Phi(t) = \alpha$, where α is a finite number. Then we have

$$\forall_{\varepsilon>0} \ \exists_{R>a} \ \forall_{t>R} \quad |\Phi(t) - \alpha| < \frac{\varepsilon}{2}.$$

If A and A' are two numbers such that A, $A' > R$ then we have

$$|\Phi(A) - \alpha| < \frac{\varepsilon}{2}, \qquad \text{and} \qquad |\Phi(A') - \alpha| < \frac{\varepsilon}{2},$$

thus, by triangle inequality

$$|\Phi(A) - \Phi(A')| \leq |\Phi(A) - \alpha| + |\Phi(A') - \alpha| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so the condition (8.28) is satisfied.

Assume now, that condition (8.28) holds. Then for every sequence $\{t_n\}$, $t_n > a$ and $t_n \to \infty$ as $n \to \infty$, we have that the sequence $\Phi(t_n)$ is a Cauchy sequence. Indeed, for every $\varepsilon > 0$ there exists $R > a$ such that for all t, $t' > R$ we have

$$|\Phi(t) - \Phi(t')| < \varepsilon.$$

On the other hand, there exists an $N \in \mathbb{N}$ such that for all n, $m \geq N$ we have t_n, $t_m > R$, thus

$$|\Phi(t_n) - \Phi(t_m)| < \varepsilon.$$

By the completeness of real numbers, the sequence $\{\Phi(t_n)\}$ converges to a finite limit α. It is easy to see that if $\{t'_n\}$ is another sequence such that $t'_n \to \infty$ then $\lim_{n\to\infty} \Phi(t'_n) = \alpha$. Therefore, we have

$$\lim_{t\to\infty} \Phi(t) = \alpha,$$

so the integral $\int_a^\infty f(x)dx$ converges. □

Corollary 8.44. *Let $f : [a, \infty) \to \mathbb{R}$ be a function such that $f(x)$ is integrable on every closed interval $[a, A]$, $A > a$. If the integral $\int_a^\infty |f(x)|dx$ converges, then $\int_a^\infty f(x)dx$ also converges.*

Proof: By Bolzano-Cauchy Criterion, we have that

$$\forall_{\varepsilon>0} \ \exists_{R>a} \ \forall_{A>A'>R} \quad \int_{A'}^{A} |f(x)|dx < \varepsilon.$$

since

$$\left| \int_{A'}^{A} f(x)dx \right| \leq \int_{A'}^{A} |f(x)|dx$$

we also obtain

$$\forall_{\varepsilon>0} \ \exists_{R>a} \ \forall_{A>A'>R} \quad \left| \int_{A'}^{A} f(x)dx \right| \leq \int_{A'}^{A} |f(x)|dx < \varepsilon,$$

thus, by Bolzano-Cauchy Criterion, the integral $\int_a^\infty f(x)dx$ converges. □

Definition 8.45. Let $f : [a, \infty) \to \mathbb{R}$ be a function such that $f(x)$ is integrable on every closed interval $[a, A]$, $A > a$. We say that the improper integral $\int_a^\infty f(x)dx$ is *absolutely convergent* if $\int_a^\infty |f(x)|dx$ converges. If the improper integral $\int_a^\infty f(x)dx$ converges but $\int_a^\infty |f(x)|dx$ diverges, we will say that $\int_a^\infty f(x)dx$ is *conditionally convergent.*

Another type of improper integral (which we may call the *second type*) is related to functions that are unbounded on finite intervals. Suppose that $f : [a, b) \to \mathbb{R}$ is a function integrable on every closed subinterval $[a, A] \subset [a, b)$, in particular this implies that $f(x)$ is bounded on every such subinterval $[a, A]$, but we also assume that the function $f(x)$ is unbounded on the whole interval $[a, b)$. In this case we will say that b is a *singular point* of $f(x)$. As an example, we can consider the function $f : [0, 1) \to \mathbb{R}$ given by

$$f(x) = \frac{1}{(1-x)^3}.$$

We have the following definition:

Definition 8.46. Let $f : [a, b) \to \mathbb{R}$ be a function integrable on every closed subinterval $[a, A] \subset [a, b)$. If the limit (finite or infinite) $\lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)dx$ exists, we denote it by

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)dx.$$

We will say that $\int_a^b f(x)dx$ *converges* if $\lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)dx$ is finite. Otherwise we say that $\int_a^b f(x)dx$ *diverges.*

A similar definition can be introduced for an improper integral of an unbounded function $f : (a, b] \to \mathbb{R}$, i.e.

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b} f(x)dx.$$

If the function $f : [a, b] \to \mathbb{R}$ is unbounded near a point $c \in (a, b)$, i.e. c is a singular point of $f(x)$, then we can define an improper integral of this type of function as the sum of two improper integrals, i.e.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Example 8.47. We will compute the improper integral

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}.$$

The function $f(x) = \frac{1}{\sqrt{1-x^2}}$ is unbounded near the point $x = 1$, i.e. 1 is a singular point of $f(x)$. Therefore we have the following improper integral

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{1-x^2}} &= \lim_{\varepsilon \to 0^+} \int_0^{1-\varepsilon} \frac{dx}{\sqrt{1-x^2}} \\ &= \lim_{\varepsilon \to 0^+} \arcsin x|_0^{1-\varepsilon} = \lim_{\varepsilon \to 0^+} \arcsin (1-\varepsilon) \\ &= \frac{\pi}{2}.\end{aligned}$$

Exactly the same conditions for the convergence of improper integrals of the second type hold as in the case of improper integrals of the first type. Since the proofs are similar we will not repeat them.

Proposition 8.48. *Let $f : [a, b) \to \mathbb{R}$ be a function such that*

(i) $f(x) \geq 0$ for all $x \in [a, b)$;
(ii)$f(x)$ is integrable on $[a, M] \subset [a, b)$.

Then the improper integral $\int_a^b f(x)dx$ converges if and only if there exists a real number $L \geq 0$ such that

$$\int_a^{b-\varepsilon} f(x)dx \leq L$$

for all $0 < \varepsilon < b - a$.

Corollary 8.49. *Let $f : [a, b) \to \mathbb{R}$ be a function such that*

(i) $f(x) \geq 0$ for all $x \in [a, b)$;
(ii) $f(x)$ is integrable on $[a, A] \subset [a, b)$.

If the numbers $\int_a^{b-\varepsilon} f(x)dx$ are not bounded from above, then

$$\int_a^b f(x)dx = \infty.$$

Theorem 8.50. *(*Comparison Test*) Let $f,\ g : [a, b) \to \mathbb{R}$ be two functions such that*

(i) $f(x)$ and $g(x)$ are integrable on intervals $[a, A] \subset [a, b)$;
(ii) There exists a number M such that $a < M < b$ and

$$\forall_{x \in [M,b)} \quad 0 \leq f(x) \leq g(x);$$

Then

(a) If $\int_a^b g(x)dx$ converges then $\int_a^b f(x)dx$ also converges;
(b) If $\int_a^b f(x)dx$ diverges then $\int_a^b g(x)dx$ also diverges.

Theorem 8.51. (Limit Comparison Test) *Let $f,\ g : [a, b) \to \mathbb{R}$ be two functions such that*

(i) $f(x)$ and $g(x)$ are integrable on intervals $[a, A] \subset [a, b)$;
(ii) There exists $\infty \geq K \geq 0$ such that

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = K;$$

Then

(a) If $\infty > K > 0$ then $\int_a^b g(x)dx$ converges if and only if $\int_a^b f(x)dx$ converges;
(b) If $K = 0$ then the convergence of $\int_a^b g(x)dx$ implies the convergence of $\int_a^b f(x)dx$;
(c) If $K = \infty$ then the divergence of $\int_a^b f(x)dx$ implies the divergence of $\int_a^b g(x)dx$.

Theorem 8.52. (Bolzano-Cauchy Criterion) *Let $f : [a, b) \to \mathbb{R}$ be a function which is integrable on every closed interval $[a, A] \subset [a, b)$. Then the improper integral*

$$\int_a^b f(x)dx$$

converges if and only if

$$\forall_{\delta>0} \ \ \exists_{b-a>\varepsilon>0} \ \ \forall_{A,A' \in (b-\varepsilon,b)} \quad \left| \int_a^{A'} f(x)dx - \int_a^A f(x)dx \right| = \left| \int_{A'}^A f(x)dx \right| < \delta.$$

Corollary 8.53. *Let $f : [a, b) \to \mathbb{R}$ be a function such that $|f(x)|$ is integrable on every closed interval $[a, A] \subset [a, b)$. If the integral $\int_a^b |f(x)|dx$ converges, then $\int_a^b f(x)dx$ also converges.*

Definition 8.54. Let $f : [a, b) \to \mathbb{R}$ be a function such that $|f(x)|$ is integrable on every closed interval $[a, A] \subset [a, b)$. We say that the improper integral $\int_a^b f(x)dx$ is *absolutely convergent* if $\int_a^b |f(x)|dx$ converges. If the improper integral $\int_a^b f(x)dx$ converges but $\int_a^b |f(x)|dx$ diverges, we will say that $\int_a^b f(x)dx$ is *conditionally convergent*.

Theorem 8.55. *(*CHANGE OF VARIABLE RULE FOR IMPROPER INTEGRALS*) Consider an interval $[a, b)$, where b is either infinite or finite. Let $f : [a, b) \to \mathbb{R}$ be a continuous function and $\varphi : [\alpha, \beta) \to [a, b)$ be an increasing differentiable function such that*

(i) $\varphi'(t)$ is continuous on $[\alpha, \beta)$;
(ii)$\varphi(\alpha) = a$ and $\lim_{t\to\beta^-} \varphi(t) = b$.

Then we have the following equality of the improper integrals

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt.$$

Moreover, if either of these improper integrals converge then the other also converges.

Proof: We have

$$\begin{aligned}\int_a^b f(x)dx &= \lim_{T\to b^-} \int_a^T f(x)dx \\ &= \lim_{T\to b^-} \int_\alpha^{\varphi^{-1}(T)} f(\varphi(t))\varphi'(t)dt \\ &= \lim_{T'\to \beta^-} \int_\alpha^{T'} f(\varphi(t))\varphi'(t)dt \\ &= \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt.\end{aligned}$$

□

Theorem 8.56. (INTEGRATION BY PARTS RULE FOR IMPROPER INTEGRALS) *Consider an interval $[a, b)$, where b is either infinite or finite. Let f, $g : [a, b) \to \mathbb{R}$ be two functions having continuous derivatives on $[a, b)$. Then we have*

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)dx, \tag{8.29}$$

where

$$f(x)g(x)\Big|_a^b = \lim_{t\to b^-} f(t)g(t) - f(a)g(a),$$

where we assume that any two of the three limits"Notice that an improper integral is a limit. in (8.29) exist and are finite, so the third limit also exists and is finite.*

"

Proof: Assume that two of the three limits in (8.29) exist. Then we have

$$\begin{aligned}\int_a^b f(x)g'(x)dx &= \lim_{T\to b^-}\int_a^T f(x)g'(x)dx\\ &= \lim_{T\to b^-}\left(f(x)g(x)\big|_a^T - \int_a^T f'(x)g(x)dx\right)\\ &= \lim_{T\to b^-} f(x)g(x)\big|_a^T - \lim_{T\to b^-}\int_a^T f'(x)g(x)dx\\ &= f(x)g(x)\big|_a^b - \int_a^b f'(x)g(x)dx.\end{aligned}$$

□

The rest of this section is devoted to examples.

Example 8.57. Test the following improper integrals for convergence

(a) $I=\int_0^\infty \frac{dx}{1+2x^2+3x^4}$;
(b) $J=\int_0^\infty \frac{dx}{x+\sin^2 x}$.

We have:

(a): Since the function $f(x)=\frac{1}{1+2x^2+3x^4}$ satisfies the inequality

$$f(x)=\frac{1}{1+2x^2+3x^4}\le\frac{1}{x^4}=g(x),$$

and the improper integral

$$\int_0^\infty\frac{dx}{1+x^4}$$

converges (see Example 8.38), by the Comparison Test (Theorem 8.41), the integral I also converges.

(b): Let $f(x)=\frac{1}{x+\sin^2 x}$ and $g(x)=\frac{1}{x}$. Then we have

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x}{x+\sin^2 x}=1.$$

Since the improper integral $\int_0^\infty\frac{dx}{x}$ diverges (see Example 8.38), by the Limit Comparison Test (Theorem 8.42), we have that the integral J also diverges.

Example 8.58. We will evaluate the improper integral

$$\int_0^\infty\frac{dx}{(1+x^2)^n},$$

where n is a natural number. We apply the substitution $x=\tan t$, where $0\le t<\frac{\pi}{2}$. Then by the Change of Variable Rule we obtain

$$\int_0^\infty\frac{dx}{(1+x^2)^n}=\int_0^{\frac{\pi}{2}}\frac{1}{\sec^{2n}t}\cdot\sec^2 t\,dt=\int_0^{\frac{\pi}{2}}\cos^{2n-2}t\,dt.$$

Since

$$\int_0^{\frac{\pi}{2}} \cos^{2n-2} t\, dt = \int_0^{\frac{\pi}{2}} \sin^{2n-2} t\, dt,$$

by the result in Example 8.32, we obtain that

$$\int_0^{\infty} \frac{dx}{(1+x^2)^n} = \frac{(2n-2)!}{(2^{n-1}(n-1)!} \frac{\pi}{2}.$$

Now, as examples, we will consider several important improper integrals.

Example 8.59. (Euler Integral) We will apply changes of variable to compute the following improper integral

$$J = \int_0^{\frac{\pi}{2}} \ln \sin x\, dx.$$

We begin with $x = 2t$, so we obtain

$$\begin{aligned} J &= 2\int_0^{\frac{\pi}{4}} \ln \sin 2t\, dt \\ &= 2\int_0^{\frac{\pi}{4}} \ln(2 \sin t \cos t) dt \\ &= 2\left[\int_0^{\frac{\pi}{4}} \ln 2\, dt + \int_0^{\frac{\pi}{4}} \ln \sin t\, dt + \int_0^{\frac{\pi}{4}} \ln \cos t\, dt\right] \\ &= \frac{1}{2}\pi \ln 2 + 2\int_0^{\frac{\pi}{4}} \ln \sin t\, dt + 2\int_0^{\frac{\pi}{4}} \ln \cos t\, dt. \end{aligned}$$

Next, in the integral $\int_0^{\frac{\pi}{4}} \ln \cos t\, dt$ we substitute $t = \frac{\pi}{2} - u$, so we obtain

$$\int_0^{\frac{\pi}{4}} \ln \cos t\, dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln \sin u\, du,$$

hence we have

$$J = \frac{1}{2}\pi \ln 2 + 2\int_0^{\frac{\pi}{4}} \ln \sin t\, dt + 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln \sin t\, dt = \frac{1}{2}\pi \ln 2 + 2\int_0^{\frac{\pi}{2}} \ln \sin x\, dx,$$

this means we have the following relation

$$J = \frac{1}{2}\pi \ln 2 + 2J,$$

thus

$$J = -\frac{1}{2}\pi \ln 2.$$

Example 8.60. (Euler-Poisson Integral) We will compute the integral

$$I = \int_0^{\infty} e^{-x^2} dx.$$

It is easy to show that

$$e^t > 1 + t, \qquad \text{for all} \quad t \neq 0,$$

thus

$$(1+t)e^{-t} < 1, \qquad \text{for} \quad t \neq 0,$$

hence by substituting $t = \pm x^2$ we obtain

$$(1-x^2)e^{x^2} < 1 \qquad \text{and} \qquad (1+x^2)e^{-x^2} < 1.$$

Consequently, we obtain

$$1 - x^2 < e^{-x^2} < \frac{1}{1+x^2} \qquad \text{for} \quad x > 0. \tag{8.30}$$

We can apply the substitution $x = \sqrt{n}u$ to obtain

$$I = \int_0^\infty e^{-x^2}dx = \sqrt{n}\int_0^\infty e^{-nu^2}du.$$

Therefore, we have

$$\int_0^\infty e^{-nx^2}dx = \frac{1}{\sqrt{n}}I.$$

By taking the nth powers of the inequality (8.30) we also obtain

$$(1-x^2)^n < e^{-nx^2} \qquad \text{for} \quad 0 < x < 1,$$

and

$$e^{-nx^2} < \frac{1}{(1+x^2)^n} \qquad \text{for} \quad x > 0.$$

Consequently, we have the following inequalities

$$\int_0^1 (1-x^2)^n dx < \int_0^1 e^{-nx^2}dx < \int_0^\infty e^{-nx^2}dx < \int_0^\infty \frac{dx}{(1+x^2)^n}.$$

By Example 8.32 , we obtain

$$\int_0^1 (1-x^2)^n dx = \int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx = \frac{(2^n n!)^2}{(2n+1)!},$$

and again by Example 8.32 and 8.58

$$\int_0^\infty \frac{dx}{(1+x^2)^n} = \frac{(2n-2)!}{(2^{n-1}(n-1)!)^2}\frac{\pi}{2}.$$

Consequently, we have

$$\sqrt{n}\frac{(2^n n!)^2}{(2n+1)!} < I < \sqrt{n}\frac{(2n-2)!}{(2^{n-1}(n-1)!)^2}\frac{\pi}{2}. \tag{8.31}$$

By taking the square of the inequality (8.31) we obtain

$$\frac{1}{2}\cdot\frac{1}{2n}\cdot\left[\frac{(2^n n!)^2}{(2n)!}\right]^2 < I^2 < \frac{\left(\frac{\pi}{2}\right)^2}{\frac{2(n-1)}{n}\cdot\frac{1}{2(n-1)}\left[\frac{(2^{n-1}(n-1)!)^2}{(2(n-1))!}\right]^2}. \tag{8.32}$$

By the Wallis Formula (8.27), the left and the right hand sides of the inequality (8.32) tend to the same limit, namely $\frac{\pi}{4}$, therefore we have

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$$

Example 8.61. (Gamma Function) Let $a > 0$. We define $\Gamma(a)$ as the following improper integral

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx. \tag{8.33}$$

We claim that the integral(8.33) is convergent for every $a > 0$. Indeed, we have

$$\lim_{x\to\infty} \frac{x^{a-1}e^{-x}}{e^{-\frac{x}{2}}} = \lim_{x\to\infty} \frac{x^{a-1}}{e^{\frac{x}{2}}} = 0,$$

thus since $\int_1^\infty e^{-\frac{x}{2}} dx$ converges, by Theorem 8.41, the improper integral

$$\int_1^\infty x^{a-1} e^{-x} dx$$

converges. On the other hand, if $a < 1$ the integral $\int_0^1 x^{a-1}e^{-x}dx$ is also improper and we have

$$x^{a-1}e^{-x} < x^{a-1} \qquad \text{for} \quad 0 < x < 1.$$

Since, we have

$$\begin{aligned}\int_0^1 x^{a-1} dx &= \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 x^{a-1} dx \\ &= \lim_{\varepsilon\to 0^+} \left.\frac{x^a}{a}\right|_\varepsilon^1 = \frac{1}{a},\end{aligned}$$

i.e. the integral $\int_0^1 x^{a-1}dx$ converges, therefore by Theorem 8.50 , the improper integral $\int_0^1 x^{a-1}e^{-x}dx$ also converges. Consequently, the function $\Gamma : (0,\infty) \to \mathbb{R}$ is well defined. The function Gamma is one of the most important functions of mathematical analysis, after the elementary functions. By applying the substitution $x = \ln\frac{1}{t}$ we obtain another form of the Gamma Function:

$$\Gamma(a) = \int_0^\infty \left(\ln\tfrac{1}{t}\right)^{a-1} dt.$$

Notice that

$$\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x}|_0^\infty = 1.$$

On the other hand, for every number $a > 0$, by integration by parts, we obtain

$$a\Gamma(a) = a\int_0^\infty x^{a-1}e^{-x}dx = x^a e^{-x}\big|_0^\infty + \int_0^\infty x^a e^{-x} dx = \int_0^\infty x^a e^{-x} dx = \Gamma(a+1).$$

In particular, for every natural number n we have

$$\Gamma(n+1) = n\Gamma(n).$$

Since $\Gamma(1) = 1$, we also have $\Gamma(2) = 1 \cdot 1 = 1$, $\Gamma(3) = 2 \cdot 1$, and in general $\Gamma(n+1) = n!$. It can be shown (using additional results on the differentiation of integrals depending on parameter) that Γ is a differentiable function (so it is also continuous). We also have (by applying the substitution $x = t^2$, that

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx = \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2},$$

where the last integral was evaluated in Example 8.60.

8.5 Problems

1. Give an example of a function $f : [a, b] \to \mathbb{R}$ which is bounded but not integrable. Include a formal argument showing that your function is indeed not integrable.

2. If f is increasing on the interval $I = \{x : a \le x \le b\}$, show that f is integrable. *Hint:* Use the formula

$$S(f, P) - s(f, P) = \sum_{k=1}^{n} \big(f(x_k) - f(x_{k-1})\big)\triangle x_k.$$

Replace each $\triangle x_k$ by the longest subinterval getting an inequality, and then observe that the terms "telescope." (Notice that we are using here slightly different notation than the textbook.) Support your claims by referring to appropriate theorems (which should be stated).

3. Give an example of a function f defined on $I = \{x : 0 \le x \le 1\}$ such that $|f|$ is integrable but f is not. (Include a formal argument showing that your function f is indeed not integrable.)

4. Use the previous problem to prove the following statement:

Theorem 8.62. *Let $f : [a, b] \to \mathbb{R}$ be an integrable function, then $|f| : [a, b] \to \mathbb{R}$ is also integrable and we have*

$$\left| \int_a^b f(x)dx \right| \le \int_a^b |f(x)|dx.$$

5. Suppose that $f,\, g : [a, b] \to \mathbb{R}$ are integrable.

(a) Show that $f^2,\, g^2,\, fg : [a, b] \to \mathbb{R}$ are integrable;
Hint: Consider first the case where f and g are non-negative, and show that

$$M_k(fg) \le M_k(f)M_k(g), \quad m_k(fg) \ge m_k(f)m_k(g),$$

then observe

$$\begin{aligned} M_k(fg) - m_k(fg) &\le M_k(f)M_k(g) - m_k(f)m_k(g) \\ &= M_k(f)\Big(M_k(g) - m_k(g)\Big) + m_k(g)\Big(M_k(f) - m_k(f)\Big) \\ &\le M(f)\Big(M_k(g) - m_k(g)\Big) + M(g)\Big(M_k(f) - m_k(f)\Big), \end{aligned}$$

where

$$M(f) = \sup\{f(x) : x \in [a, b]\}, \quad M(g) = \sup\{g(x) : x \in [a, b]\},$$

and use the appropriate results to show that

$$\lim_{\|P\| \to 0} \Big(s_+(fg, P) - s_-(fg, P)\Big) = 0.$$

Next, consider the general case, where f and g are not necessarily non-negative, and use the representation (see Problem 4 and appropriate theorems)

$$fg = (f_+ - f_-)(g_+ - g_-) = f_+g_+ - f_+g_- - f_-g_+ + f_-g_-,$$

to conclude that fg is integrable.

(b) Define the "dot-product" for the functions f and g by

$$f \bullet g := \int_a^b f(x)g(x)dx.$$

Show that the product $f \bullet g$ satisfies the following conditions
(i) $f \bullet f \geq 0$;
(ii) $f \bullet g = g \bullet f$;
(iii) $f \bullet (g_1 + g_2) = f \bullet g_1 + f \bullet g_2$;

6. Use result in Problem 5 to prove the following Cauchy-Schwarz inequality:

$$\left[\int_a^b f(x)g(x)dx\right]^2 \leq \left[\int_a^b f^2(x)dx\right]\left[\int_a^b g^2(x)dx\right],$$

where $f,\ g : [a, b] \to \mathbb{R}$ are integrable functions.
Hint: Use the dot product $f \bullet g$, which was defined in Problem 6, in the same way it was used in the proof of the Cauchy-Schwarz inequality for sequences.

7. Show that if $f : [a, b] \to \mathbb{R}$ is a positive function, which is integrable on $[a, b]$, then for every $n \in \mathbb{N}$ the function $f^n : [a, b] \to \mathbb{R}$ is also integrable. Is this result true for functions which are not necessarily positive? *Hint:* Use the following inequality

$$M_k^n - m_k^n = (M_k - m_k)(M_k^{n-1} + M_k^{n-2}m_k + \cdots + M_k m_k^{n-2} + m_k^{n-2}) \leq nM^{n-1}(M_k - m_k),$$

where

$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}, \quad m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\},$$
$$M = \sup\{f(x) : x \in [a, b]\}.$$

8. Prove the following theorem:

Theorem 8.63. *Let $f : [a, b] \to \mathbb{R}$ be a continuous non-negative function such that*

$$\int_a^b f(x)dx = 0,$$

then $f(x) = 0$ for all $x \in [a, b]$.

9. For a continuous function $f : [a, b] \to \mathbb{R}$ we define the *norm* of f by

$$\|f\| = \sqrt{f \bullet f}.$$

Show that this norm $\|\cdot\|$ satisfies the following properties:

(i) $\|f\| \geq 0$ and $\|f\| = 0 \iff f(x) = 0$ for all $x \in [a, b]$ (we assume here that f is continuous);
(ii) $\|\alpha f\| = |\alpha|\,\|f\|$, for $\alpha \in \mathbb{R}$;

(iii) $\|f+g\| \leq \|f\| + \|g\|$.

10. We define the function $F : [0,\infty) \to \mathbb{R}$ by the formula

$$F(x) = \int_0^x \Big||t-2|-1\Big| dt.$$

Find the explicit formula for $F(x)$.

11. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that we have the following equality

$$\int_0^{\pi} x\, f(\sin x) dx = \pi \int_0^{\frac{\pi}{2}} f(\sin x) dx.$$

12. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that

(a) $\int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx$;
(b) $\int_0^{\pi} x f(\sin x) dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx$.

13. Let $f(x)$ be a continuous function in the interval $[0,1]$. Show that

$$\int_0^{\pi} x f(\sin x) dx = \pi \int_0^{\frac{\pi}{2}} f(\sin x) dx.$$

14. Show that

$$\int_0^1 x^m (1-x)^n dx = \int_0^1 x^n (1-x)^m dx.$$

15. Let $f(x)$ be a continuously differentiable function on the interval $[a,b]$ such that $f(a) = 0$.Prove that

$$\sup_{x\in[a,b]} |f(x)| \leq \sqrt{(b-a) \int_a^b \left[f'(x)\right]^2 dx}.$$

16. Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function such that there exists the limit $f(\infty) = \lim_{x\to\infty} f(x)$. Show that

$$\int_0^{\infty} \frac{f(ax) - f(bx)}{x} dx = \Big(f(0) - f(\infty)\Big) \ln \tfrac{b}{a},$$

where $a > 0$ and $b > 0$.

17. Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function such that for every $A > 0$ the integral $\int_A^{\infty} \frac{f(x)}{x} dx$ converges. Show that

$$\int_0^{\infty} \frac{f(ax) - f(bx)}{x} dx = f(0) \ln \tfrac{b}{a},$$

where $a > 0$ and $b > 0$.

18. Compute the following improper integrals:

(a) $\int_1^\infty \frac{dx}{x^4}$;
(b) $\int_0^\infty xe^{-x^2}\,dx$;
(c) $\int_0^\infty e^{-x}\sin x\,dx$.

19. Check which of the following improper integrals converge

(a) $\int_0^\infty \sqrt{x}e^{-x}dx$,
(b) $\int_{e^2}^\infty \frac{dx}{x\ln\ln x}$.

20. Evaluate the following improper integrals

(a) $\int_1^3 \frac{x\,dx}{\sqrt{x-1}}$,
(b) $\int_0^1 x\ln x\,dx$,
(c) $\int_{-1}^0 \frac{e^{\frac{1}{x}}}{x^3}\,dx$.

21. Check which of the following improper integrals converge:

(a) $\int_0^1 \frac{\sqrt{x}}{\sqrt{1-x^4}}dx$;
(b) $\int_0^1 \frac{dx}{e^{\sqrt{x}}-1}$.

22. Consider the so-called Dirichlet function $f:\mathbb{R}\to\mathbb{R}$, defined by

$$f(x) = \begin{cases} 0 & \text{if } \; x \text{ is irrational} \\ 1 & \text{if } \; x \text{ is rational.} \end{cases}$$

Show that for any interval $[a,b]$ $(a<b)$ the function f is not Riemann integrable on $[a,b]$.

23. Consider the so-called Riemann function $f:\mathbb{R}\to\mathbb{R}$, defined by

$$f(x) = \begin{cases} 0 & \text{if } \; x \text{ is irrational or } x=0 \\ \frac{1}{n} & \text{if } \; x=\frac{m}{n} \text{ where the fraction } \frac{m}{n} \text{ is irreducible and } n\geq 1. \end{cases}$$

Show that the function f is Riemann integrable on $[0,1]$. (Actually f is integrable on every $[a,b]$).
Hint: Show that for given $\varepsilon>0$ there exists only finitely many $x\in(0,1]$ such that $x=\frac{m}{n}$ and $\frac{1}{n}\geq\frac{\varepsilon}{2(b-a)}$. Use this fact to find $\delta>0$ such that for every partition P satisfying $\|P\|<\delta$ we have $S(f,P)-s(f,P)<\varepsilon$.

24. Show that the function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{x}-\left\lfloor\frac{1}{x}\right\rfloor & \text{if } \; 0<x<1 \\ 0 & \text{if } \; x=0, \end{cases}$$

is Riemann integrable on $[0,1]$.

9

Stieltjes Integrals

9.1 Definition of Riemann-Stieltjes Integral

Recall that in a definition of a Riemann integral of a bounded function $f : [a, b] \to \mathbb{R}$ we used the Riemann sums of the type

$$\sigma(f, P) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}), \tag{9.1}$$

where P is a partition of the interval $[a, b]$. i.e.

$$P : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

and $\xi_k \in [x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$. In statistics, probability and other applied areas, it is often required to to modify the formula (9.1) to include the weighted (by means of a certain function $g : [a, b] \to \mathbb{R}$) change of f on the partition subintervals. This simply means, that factors $(x_k - x_{k-1})$ in (9.1) are replaced by $(g(x_k) - g(x_{k-1}))$, i.e.

$$\sigma(f, P, \xi, g) = \sum_{k=1}^{n} f(\xi_k)(g(x_k) - g(x_{k-1})) =: \sum_{k=1}^{n} f(\xi_k)\triangle g(x_k), \tag{9.2}$$

where $\triangle g(x_k) := g(x_k) - g(x_{k-1})$ and $\xi_k \in [x_{k-1}, x_k]$. Then, similarly to the definition of the Darboux upper sum and Darboux lower sum, assuming that g is increasing, we can define

$$S(f, P, g) = \sup_{\xi} \sigma(f, P, \xi, g), \; s(f, P, g) = \inf_{\xi} \sigma(f, P, \xi, g).$$

If $\sup_P s(f, P, g) = \inf_P s(f, P, g) = I$, we can define the *Darboux–Stieltjes integral* of the function $f(x)$ with respect to $g(x)$ on $[a, b]$ by:

$$\int_a^b f(x) dg(x) = I. \tag{9.3}$$

If $\lim_{|P| \to 0} = \sigma(f, P, \xi, g) = I$, we can define the *Riemann–Stieltjes integral* of the function $f(x)$ with respect to $g(x)$ on $[a, b]$ by:

$$\int_a^b f(x) dg(x) = I. \tag{9.4}$$

If the Riemann–Stieltjes integral (9.4) exists, we say that the function $f(x)$ is *integrable on* $[a,b]$ *with respect to* $g(x)$.

Notice that the Riemann integral of $f(x)$ on $[a,b]$ is a special case of a Riemann-Stieltjes integral with $g(x)=x$.

In this chapter we will establish the conditions for the existence of the Riemann–Stieltjes integral of $f(x)$ on $[a,b]$ with respect to $g(x)$ and describe the techniques for its computations.

9.2 Functions of Bounded Variation

This section is devoted to the so-called *functions of bounded variation*—the concept that was introduced by C. Jordan* and which will be used to establish the condition for existence of Riemann–Stieltjes integral. The functions with bounded variation are also used in studying other important concepts of analysis.

Assume that $g:[a,b]\to\mathbb{R}$ is a function and denote by P a partition $\{x_k\}_{k=0}^n$ of $[a,b]$, i.e.

$$P: a=x_0<x_1<\dots x_{k-1}<x_k<\dots<x_{n-1}<x_n=b.$$

Then we can define for g and P the number

$$v(g,P):=\sum_{k=1}^n |g(x_k)-g(x_{k-1})|. \tag{9.5}$$

Definition 9.1. A function $g:[a,b]\to\mathbb{R}$ is said to be of *bounded variation* (on the interval $[a,b]$) if the set ,

$$\Big\{v(g,P): P=\{x_k\}_{k=0}^n\} \ \text{ a partition of } [a,b]\Big\}$$

is bounded from above. Then we can define the number

$$\operatorname*{Var}_{[a,b]}(g):=\sup\Big\{v(g,P): P=\{x_k\}_{k=0}^n\} \ \text{ a partition of } [a,b]\Big\}. \tag{9.6}$$

The number $\operatorname*{Var}_{[a,b]}(g)$ is called the *total variation* of $g(x)$ on $[a,b]$. Sometimes, when it is clear on which interval the total variation of g is considered, we will simply write $\operatorname{Var}(g)$.

Example 9.2. Let $g:[a,b]\to\mathbb{R}$ be a monotonic function. Then $g(x)$ is of bounded variation. Indeed, if g is non-decreasing, i.e. $g(x)\le g(y)$ for $x<y$, then we have for any partition P of $[a,b]$

$$v(g,P)=\sum_{k=1}^n |g(x_k)-g(x_{k-1})|=\sum_{k=1}^n (g(x_k-g(x_{k-1}))=g(b)-g(a),$$

or if g is non-increasing , i.e. $g(x)\ge g(y)$ for $x<y$, then

* C. Jordan was a French mathematicians (January 5, 1838 –January 22, 1922) well-known for his work in algebra (Jordan form of a matrix) and analysis (e.g. Jordan curve)

$$v(g,P)=\sum_{k=1}^{n}|g(x_k)-g(x_{k-1})|=\left|\sum_{k=1}^{n}(g(x_k)-g(x_{k-1}))\right|=|g(b)-g(a)|,$$

therefore, for any monotonic function $g:[a,b]$ we have

$$v(g,P)\le|g(b)-g(a)|\quad\text{ for any partition } P \text{ of } [a,b].$$

Since $\{v(g,P): P \text{ a partition of } [a,b]\}=|g(b)-g(a)|$ for any partition P, g is of bounded variation and ,

$$\operatorname*{Var}_{[a,b]}(g)=|g(b)-g(a)|. \tag{9.7}$$

Example 9.3. As an example of a continuous function $g:[a,b]\to\mathbb{R}$ that is not of bounded variation, consider $g:[0,1]\to\mathbb{R}$ defined by

$$g(x)=\begin{cases} x\cos\left(\frac{\pi}{2x}\right) & \text{for } \ 0<x\le 1\\ 0 & \text{for } \ x=0.\end{cases}$$

Then, consider the partition

$$P_n: 0<\frac{1}{2n}<\frac{1}{2n-1}<\dots<\frac{1}{3}<\frac{1}{2}<1.$$

One can easily compute that

$$v(g,P_n)=1+\frac{1}{2}+\dots+\frac{1}{n}=:H_n,$$

and since the limit

$$\lim_{n\to\infty}H_n=\sum_{k=1}^{\infty}\frac{1}{k}=\infty,$$

it follows that the sequence $\{v(g,P_n)\}$ diverges to infinity, and consequently the function $g(x)$ is not of bounded variation.,

Proposition 9.4. *Let $g:[a,b]\to\mathbb{R}$ be a Lipschitzian function, i.e. there exists $L>0$ such that*

$$\forall_{x,y\in[a,b]}\quad |g(x)-g(y)|\le L|x-y|,$$

then $g(x)$ is of bounded variation and

$$\operatorname*{Var}_{[a,b]}(g)\le L(b-a).$$

Proof: For any partition $P=\{x_k\}_{k=0}^{n}$ we have

$$v(g,P)=\sum_{k=1}^{n}|g(x_k)-g(x_{k-1})|\le\sum_{k=1}^{n}L(x_k-x_{k-1})=L(b-a),$$

which implies that $L(b-a)$ is un upper bound for the set $\{v(g,P): P \text{ a partition of } [a,b]\}$, thus

$$\operatorname*{Var}_{[a,b]}(g)\le L(b-a).$$

□

Corollary 9.5. *Let $g : [a, b] \to \mathbb{R}$ be a continuous function such that g is differentiable on (a, b) and*

$$\exists_{L>0} \ \forall_{x\in(a,b)} \quad |g'(x)| < L, \tag{9.8}$$

i.e. $g'(x)$ is bounded on (a, b). Then $g : [a, b] \to \mathbb{R}$ is a function of bounded variation.,

Proof: Since the condition (9.8) implies (by Lagrange Theorem 6.12) that g is Lipschitzian on $[a, b]$, the conclusion follows from Proposition 9.4. □

Lemma 9.6. *Let $g : [a, b] \to \mathbb{R}$ be a bounded function. Then for two partitions P and Q of $[a, b]$, such that $P \leq Q$ we have*

$$v(g, P) \leq v(g, Q).$$

Proof: Let P be a partition of $[a, b]$ given by $P : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and assume that Q is a partition obtained from the partition $P = \{x_k\}$ by adding one extra point x' to P, i.e.

$$a = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x' < x_k < \cdots < x_{n-1} < x_n = b.$$

Then, since

$$|g(x_k) - g(x_{k-1})| = |(g(x_k) - g(x')) + (g(x') - g(x_{k-1}))| \leq |(g(x_k) - g(x'))| + |(g(x') - g(x_{k-1}))|,$$

thus

$$v(g, P) \leq v(g, Q).$$

The result in general case, when $P < Q$, can be obtained by applying mathematical induction. □

Theorem 9.7. (Properties of Functions of Bounded Variation) ,

(a) Every function $g : [a, b] \to \mathbb{R}$ of bounded variation is bounded.

(b) If $g_1, g_2 : [a, b] \to \mathbb{R}$ are two functions of bounded variation, then the linear combination $\alpha g_1 + \beta g_2 : [a, b] \to \mathbb{R}$ ($\alpha, \beta \in \mathbb{R}$) of g_1 and g_2 is also a function of bounded variation and

$$\operatorname*{Var}_{[a,b]}(\alpha g_1 + \beta g_2) \leq |\alpha| \operatorname*{Var}_{[a,b]}(g_1) + |\beta| \operatorname*{Var}_{[a,b]}(g_2).$$

(c) If $g_1, g_2 : [a, b] \to \mathbb{R}$ are two functions of bounded variation, then the function $h : [a, b] \to \mathbb{R}$, defined by $h(x) = g_1(x)g_2(x)$, $x \in [a, b]$, is also of bounded variation. Moreover, if

$$\forall_{x\in[a,b]} \quad |g_1(x)| \leq M \quad \text{and} \quad |g_2(x)| \leq N$$

for some $M, N > 0$, then we have

$$\operatorname*{Var}_{[a,b]}(h) \leq N \operatorname*{Var}_{[a,b]}(g_1) + M \operatorname*{Var}_{[a,b]}(g_2).$$

(d) If $g, h : [a, b] \to \mathbb{R}$ are two functions of bounded variation such that

$$\exists_{\varepsilon>0} \ \forall_{x\in[a,b]} \quad |h(x)| \geq \varepsilon,$$

then the quotient $\frac{g(x)}{h(x)}$ is also a function of bounded variation. Moreover,

$$\operatorname*{Var}_{[a,b]}\left(\frac{1}{h}\right) \leq \frac{1}{\varepsilon^2} \operatorname*{Var}_{[a,b]}(h).$$

(e) A function $g : [a, b] \to \mathbb{R}$ is of bounded variation if and only if for every $c \in (a, b)$ the restrictions of g to $[a, c]$ and $[c, b]$ (i.e. $g; [a, c] \to \mathbb{R}$ and $g : [c, b] \to \mathbb{R}$) are of bounded variation. Moreover, in such a case

$$\operatorname*{Var}_{[a,b]}(g) = \operatorname*{Var}_{[a,c]}(g) + \operatorname*{Var}_{[c,b]}(g). \tag{9.9}$$

(f) If $g : [a, b] \to \mathbb{R}$ is a function of bounded variation, then $m : [a, b] \to \mathbb{R}$ defined by

$$m(x) = \operatorname*{Var}_{[a,x]}(g)$$

is a bounded non-decreasing function.

Proof: (a): Let $x \in (a, b]$ be an arbitrary point, and consider the partition $P : a = x_0 < x = x_1 \leq b$. By the definition of the total variation, we have

$$v(g, P) := |g(x) - g(a)| + |g(b) - g(x)| \leq \operatorname*{Var}_{[a,b]}(g),$$

thus, by applying the inequality $|g(x)| - |g(a)| \leq |g(x) - g(a)|$, we get

$$|g(x)| \leq |g(x) - g(a)| + |g(a)| \leq |g(a)| + \operatorname*{Var}_{[a,b]}(g) := M.$$

Since, the point $x \in (a, b]$ was chosen to be arbitrary, we obtain

$$\forall_{x\in[a,b]} \ \ |g(x)| \leq M,$$

which implies that $g(x)$ is bounded.

(b): Put $h(x) := \alpha g_1(x) + \beta g_2(x)$, $x \in [a, b]$. Consider a partition $P = \{x_k\}_{k=0}^n$. Since

$$\begin{aligned} |h(x_k) - h(x_{k-1})| &= |\alpha(g_1(x_k - g_1(x_{k-1})) + \beta(g_2(x_k) - g_2(x_{k-1}))| \\ &\leq |\alpha||g_1(x_k) - g_1(x_{k-1})| + |\beta||g_2(x_k) - g_2(x_{k-1})|, \end{aligned}$$

therefore, we have

$$\begin{aligned} v(h, P) &= \sum_{k=1}^n |h(x_k - h(x_{k-1})| \\ &\leq \sum_{k=1}^n \Big(|\alpha||g_1(x_k) - g_1(x_{k-1})| + |\beta||g_2(x_k) - g_2(x_{k-1})|\Big) \\ &= |\alpha| \sum_{k=1}^n |g_1(x_k) - g_1(x_{k-1})| + |\beta| \sum_{k=1}^n |g_2(x_k) - g_2(x_{k-1})| \\ &\leq |\alpha| \operatorname*{Var}_{[a,b]}(g_1) + |\beta| \operatorname*{Var}_{[a,b]}(g_2). \end{aligned}$$

Therefore, h is of bounded variation and

$$\operatorname*{Var}_{[a,b]}(h) \leq |\alpha| \operatorname*{Var}_{[a,b]}(g_1) + |\beta| \operatorname*{Var}_{[a,b]}(g_2)$$

(c): Let $P = \{x_k\}_{k=0}^n$ be an arbitrary partition of $[a, b]$, then we have

$$\begin{aligned}|h(x_k) - h(x_{k-1})| &= |g_1(x_k)||g_2(x_k) - g_2(x_{k-1})| + |g_2(x_{k-1})||g_1(x_k) - g_1(x_{k-1})| \\ &\le M|g_2(x_k) - g_2(x_{k-1})| + N|g_1(x_k) - g_1(x_{k-1})|,\end{aligned}$$

therefore we have

$$\begin{aligned}v(h, P) &= \sum_{k=1}^{n} |h(x_k) - h(x_{k-1})| \\ &\le M \sum_{k=1}^{n} |g_2(x_k) - g_2(x_{k-1})| + N \sum_{k=1}^{n} |g_1(x_k) - g_1(x_{k-1})| \\ &\le M \operatorname*{Var}_{[a,b]}(g_2) + N \operatorname*{Var}_{[a,b]}(g_1),\end{aligned}$$

which implies

$$\operatorname*{Var}_{[a,b]}(h) \le N \operatorname*{Var}_{[a,b]}(g_1) + M \operatorname*{Var}_{[a,b]}(g_2).$$

(d): By the property (c), it is sufficient to show that the function $k : [a, b] \to \mathbb{R}$, $k(x) = \frac{1}{h(x)}$, $x \in [a, b]$ is of bounded variation. Let $P = \{x_k\}_{k=0}^{n}$ be an arbitrary partition of $[a, b]$, then we have

$$\begin{aligned}|k(x_k) - k(x_{k-1})| &= \left| \frac{1}{h(x_k)} - \frac{1}{h(x_{k-1})} \right| \\ &= \frac{|h(x_k) - h(x_{k-1})|}{|h(x_k)||h(x_{k-1})|} \le \frac{1}{\varepsilon^2} |h(x_k) - h(x_{k-1})|,\end{aligned}$$

thus

$$\begin{aligned}v(k, P) &= \sum_{k=1}^{n} |k(x_k) - k(x_{k-1})| \le \frac{1}{\varepsilon^2} \sum_{k=1}^{n} |h(x_k) - h(x_{k-1})| \\ &\le \frac{1}{\varepsilon^2} \operatorname*{Var}_{[a,b]}(h).\end{aligned}$$

(e): Assume that $g : [a, b] \to \mathbb{R}$ is a function of bounded variation in $[a, b]$. Consider the two partitions ,

$$\begin{aligned}P_1 &: a = y_0 < y_1 < y_2 < \cdots < y_{m-1} < y_m = c \quad \text{(a partition of } [a, c] \\ P_2 &: c = y_0 < y_1 < y_2 < \cdots < y_{m-1} < y_m = c \quad \text{(a partition of } [a, c].\end{aligned}$$

Let's combine these two partitions into a partition P of $[a, b]$, i.e.

$$P : a = y_0 < y_1 < y_2 < \cdots < y_{m-1} < y_m = c = y_0 < y_1 < y_2 < \cdots < y_{m-1} < y_m = c.$$

The obtained partition P will be denoted by $P =: P_1 \cup P_2$. Next consider the following sums:

$$v_1(g, P_1) := \sum_{k=1}^{m} |g(y_k) - g(y_{k-1})| \quad \text{and} \quad v_2(g, P_2) := \sum_{l=1}^{n} |g(z_l) - g(z_{l-1})|.$$

Since

$$v_1(g, P_1) + v_2(g, P_2) = v(g, P) \le \operatorname*{Var}_{[a,b]}(g),$$

therefore

$$\operatorname*{Var}_{[a,c]}(g) + \operatorname*{Var}_{[c,b]}(g) \leq \operatorname*{Var}_{[a,b]}(g). \tag{9.10}$$

Assume now that the function $g(x)$ is of bounded variation in each of the intervals $[a, c]$ and $[c, b]$, and consider an arbitrary partition P of $[a, b[$. If the point c doesn't belong to the partition P, we can replace it by a finer partition P' with the point c added it the partition P. Since by Lemma 9.6, $v(g, P) \leq v(g, P')$, we can assume that the point c belongs to the partition P. Then we have that P can be represented as $P = P_1 \cup P_2$, where P_1 is a partition of $[a, c]$ and P_2 is a partition of $[c, b]$. Consequently, we have

$$v(g, P) = v_1(g, P_1) + v_2(g, P_2) \leq \operatorname*{Var}_{[a,c]}(g) + \operatorname*{Var}_{[c,b]}(g),$$

which implies

$$\operatorname*{Var}_{[a,b]}(g) \leq \operatorname*{Var}_{[a,c]}(g) + \operatorname*{Var}_{[c,b]}(g), \tag{9.11}$$

By combining (9.10) with (9.11), we obtain the equality (9.9).

(f): Since by (e), we have for $a \leq x' < x'' \leq b$

$$\operatorname*{Var}_{[a,x'']}(g) = \operatorname*{Var}_{[a,x']}(g) + \operatorname*{Var}_{[x',x'']}(g),$$

thus

$$m(x'') - m(x') = \operatorname*{Var}_{[x',x'']}(g) \geq 0.$$

$\square$

Definition 9.8. , Let $g : [a, b] \to \mathbb{R}$ be a function of bounded variation. A non-decreasing function $\mathfrak{m} : [a, b] \to \mathbb{R}$ is called a *majorant* of g if

$$\forall_{x',x'' \in [a,b]} \quad x' < x'' \implies |g(x') - g(x'')| \leq \mathfrak{m}(x'') - \mathfrak{m}(x').$$

Lemma 9.9. *A function $g : [a, b] \to \mathbb{R}$ is of bounded variation if and only if g has a majorant.*

Proof: Assume that $g : [a, b] \to \mathbb{R}$ is a function of bounded variation. Then (by Theorem 9.7(f))

$$m(x) = \operatorname*{Var}_{[a,x]}(g), \quad x \in [a, b],$$

is a majorant of g, i.e. we have

$$|g(x') - g(x'')| \leq m(x'') - m(x') = \operatorname*{Var}_{[x',x'']}(g).$$

Conversely, suppose that $\mathfrak{m} : [a, b] \to \mathbb{R}$ is a majorant of g, and consider a partition $P = \{x_k\}_{k=0}^n$. Then we have

$$v(g, P) = \sum_{k=1}^n |g(x_k) - g(x_{k-1})| \leq \sum_{k=1}^n (\mathfrak{m}(x_k) - \mathfrak{m}(x_{k-1})) = \mathfrak{m}(b) - \mathfrak{m}(a),$$

which implies that g is of bounded variation. $\square$

Theorem 9.10. *A function $g : [a,b] \to \mathbb{R}$ is of bounded variation if and only if g can be represented as a difference of two non-decreasing functions g_1, $g_2 : [a,b] \to \mathbb{R}$, i.e. $g(x) = g_1(x) - g_2(x)$, $x \in [a,b]$.*

Proof: Suppose that $g : [a,b] \to \mathbb{R}$ is a function of bounded variation. Then by Lemma 9.9, g has a majorant $\mathfrak{m} : [a,b] \to \mathbb{R}$. Let us put $g_1(x) := \mathfrak{m}(x)$, $x \in [a,b]$. We will show that the function $g_2 : [a,b] \to \mathbb{R}$, defined by $g_2(x) = \mathfrak{m}(x) - g(x)$, $x \in [a,b]$, is also non-decreasing, i.e. for x', $x'' \in [a,b]$, if $x' < x''$ then $g_2(x'') - g_2(x') \geq 0$. Indeed, since

$$g(x'') - g(x') \leq |g(x'') - g(x')| \leq \mathfrak{m}(x'') - \mathfrak{m}(x')$$

we have

$$\begin{aligned} g_2(x'') - g_2(x') &= (\mathfrak{m}(x'') - g(x'')) - (\mathfrak{m}(x') - g(x')) \\ &= (\mathfrak{m}(x'') - \mathfrak{m}(x')) - (g(x'') - g(x')) \geq 0. \end{aligned}$$

Conversely, if $g(x) = g_1(x) - g_2(x)$, where g_1 and g_2 are two non-decreasing functions, the statement follows from Theorem 9.7(b) and Example 9.2. □

Theorem 9.11. *Let $g : [a,b] \to \mathbb{R}$ be a function of bounded variation. If the function g is continuous at $x_o \in [a,b]$ then the majorant*

$$m(x) := \operatorname*{Var}_{[a,x]}(g), \quad x \in [a,b]$$

is also continuous at x_o.

Proof: Assume that $x_o < b$. We will prove that the function $m(x)$ is continuous from from the right at x_o, i.e. we have

$$\lim_{x \to x_o^+} m(x) = m(x_o) \quad \Leftrightarrow \quad \forall_{\varepsilon>0}\, \exists_{\delta>0}\, \forall_{x\in[a,b]}\ \ 0 \leq x - x_o < \delta \ \Rightarrow\ m(x) - m(x_o) < \varepsilon.$$

Since, by Theorem 9.7 (e), if $x > x_o$, then

$$m(x) - m(x_o) = \operatorname*{Var}_{[a,x]}(g) - \operatorname*{Var}_{[a,x_o]}(g) = \operatorname*{Var}_{[x_o,x]}(g).$$

Assume that $\varepsilon > 0$ is a fixed number. Since $g(x)$ is continuous at x_o, there exists $\delta > 0$ such that

$$0 < x - x_o < \delta \quad \Rightarrow \quad |g(x) - g(x_o)| < \frac{\varepsilon}{2}.$$

On the other hand, from the definition of $\operatorname*{Var}_{[x_o,b]}(g)$, there exists a partition $P : x_o < x_1 < \cdots < x_n = b$, such that

$$v(g,P) = \sum_{k=1}^{n} |g(x_k) - g(x_{k-1})| > \operatorname*{Var}_{[x_o,b]}(g) - \frac{\varepsilon}{2}.$$

By Lemma 9.6, we can assume that $x_1 - x_o < \delta$, therefore $|g(x_1) - g(x_o)| < \frac{\varepsilon}{2}$, and consequently,

$$\begin{aligned}\operatorname*{Var}_{[x_o,b]}(g) &< \frac{\varepsilon}{2} + \sum_{k=1}^{n} |g(x_k) - g(x_{k-1})| \\ &= \frac{\varepsilon}{2} + |g(x_1) - g(x_o)| + \sum_{k=2}^{n} |g(x_k) - g(x_{k-1})| \\ &< \varepsilon + \sum_{k=2}^{n} |g(x_k) - g(x_{k-1})| \\ &\le \varepsilon + \operatorname*{Var}_{[x_1,b]}(g),\end{aligned}$$

therefore (for $x = x_1$)

$$\operatorname*{Var}_{[x_o,x]}(g) = \operatorname*{Var}_{[x_o,b]}(g) - \operatorname*{Var}_{[x_1,b]}(g) < \varepsilon.$$

On the other hand, if $x_o > a$, by applying the same idea, we can show that the function $m(x)$ is continuous from the left, i.e.

$$\lim_{x\to x_o^-} m(x) = m(x_o) \quad \Leftrightarrow \quad \forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x\in[a,b]}\ \ 0 \le x_o - x < \delta\ \Rightarrow\ m(x_o) - m(x) < \varepsilon.$$

Therefore, $m(x)$ is continuous at x_o. □

Corollary 9.12. *Let $g : [a, b] \to \mathbb{R}$ be a continuous function of bounded variation. Then, g can be represented as a difference of two non-decreasing continuous functions g_1, $g_2 : [a, b] \to \mathbb{R}$, i.e. $g(x) = g_1(x) - g_2(x)$, $x \in [a, b]$.*

Proposition 9.13. *, Let $g : [a, b] \to \mathbb{R}$ be a continuous function of bounded variation. Then*

$$\operatorname*{Var}_{[a,b]}(g) = \lim_{\|P\|\to 0} v(g, P) \ \stackrel{def}{\Longleftrightarrow}\ \forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_P\ \ \|P\| < \delta \ \Rightarrow\ \operatorname*{Var}_{[a,b]}(g) - v(g, P) < \varepsilon. \tag{9.12}$$

Proof: Let $\varepsilon > 0$. We need to find $\delta > 0$ such that for all P such that $\|P\| < \delta$ we have

$$\alpha := \operatorname*{Var}_{[a,b]}(g) - \varepsilon < v(g, P). \tag{9.13}$$

Since

$$\operatorname*{Var}_{[a,b]}(g) = \sup \Big\{ v(g, P) : P = \{x_k\}_{k=0}^{n}\ \text{ a partition of } [a, b] \Big\}.$$

by the definition of supremum, there exists a partition $P^* : a = x_0^* < x_1^* < x_2^* < \cdots < x_{m-1}^* < x_m^* = b$ such that

$$\alpha < v(P^*, g).$$

On the other hand, since the function g is continuous and $[a, b]$ is compact, by Cantor-Heine Theorem 3.93, it is uniformly continuous, i.e.

$$\forall_{\eta>0}\ \exists_{\delta>0}\ \forall_{x',x''\in[a,b]}\ \ |x' - x''| < \delta \ \Rightarrow\ |g(x') - g(x'')| < \eta.$$

Therefore, for $\eta := \frac{v(g,P^*)-\alpha}{4m} > 0$ there exists $\delta > 0$ such that whenever $|x' - x''| < \delta$ we have

$$|g(x') - g(x'')| < \eta = \frac{v(g, P^*) - \alpha}{2m}. \tag{9.14}$$

Suppose that P is an arbitrary partition such that $\|P\| < \delta$. Put $Q := P \cup P^*$. Then Q has at most m more points than P. Let $\{Q_i\}_{i=1}^m$ be the sequence of partitions obtained by letting $Q_0 = P$, $\{Q_i\} = Q_{i-1} \cup \{x_i^*\}$, $i = 1, 2, \cdots m$. Then we have

$$\begin{aligned}
0 &\le v(g, Q_1) - v(g, P) < \frac{v(g, P^*) - \alpha}{2m} \\
0 &\le v(g, Q_2) - v(g, Q_2) < \frac{v(g, P^*) - \alpha}{2m} \\
\cdots \quad & \cdots \quad \cdots \\
0 &\le v(g, Q_m) - v(g, Q_{m-1}) < \frac{v(g, P^*) - \alpha}{2m}
\end{aligned}$$

By adding the above inequalities we obtain

$$0 \le v(g, Q) - v(g, P) < \frac{v(g, P^*) - \alpha}{2}.$$

Consequently (since $Q \ge P^*$), by Lemma 9.6, we have

$$v(g, P) > v(g, Q) - \frac{v(g, P^*) - \alpha}{2} \ge v(g, P^*) - \frac{v(g, P^*) - \alpha}{2} = \frac{v(g, P^*) + \alpha}{2} > \alpha.$$

□

9.3 Conditions Necessary for the Existence of the Stieltjes Integrals

The Darboux-Stieltjes integral is a direct generalization of the Riemann integral.

Consider two bounded functions $f, g : [a, b] \to \mathbb{R}$ and let $P : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ be a partition of $[a, b]$. From each $k = 1, 2, \ldots, k$, we choose $\xi_k \in [x_{k-1}, x_k]$, and we put

$$\triangle g(x_k) := g(x_k) - g(x_{k-1}), \quad k = 1, 2, \ldots, n.$$

The sum

$$\sigma(f,\ P,\ \xi,\ g) := \sum_{k=1}^{n} f(\xi_k) \triangle g(x_k)$$

is called the *Riemann-Stieltjes sum* of f relative to g for the partition P.

Let us consider first the case where the function $g : [a, b] \to \mathbb{R}$ is non-decreasing.

Following the same ideas from Chapter 5, we introduce the notion of *lower* and *upper Darboux-Stieltjes sum*: for a partition $P = \{x_k\}_{k=0}^n$ of $[a, b]$ we put

$$\begin{aligned}
s(f, P, g) &:= \sum_{k=1}^{n} m_k \triangle g(x_k), \quad \text{where} \quad m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}, \ \ k = 1, 2, \ldots, n; \\
S(f, P, g) &:= \sum_{k=1}^{n} M_k \triangle g(x_k), \quad \text{where} \quad M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}, \ \ k = 1, 2, \ldots, n
\end{aligned}$$

One can easily prove that for all partitions P and Q satisfying $P \le Q$ we have the following inequality:

$$s(f, P, g) \le s(f, Q, g) \le S(f, Q, g) \le S(f, P, g).$$

Therefore, we can define the *lower* and *upper Darboux-Stieltjes integrals*:

$$s = \underline{\int_a^b} f(x)dg(x) := \sup\left\{s(f, P, g) : P \text{ partition of } [a, b]\right\},$$
$$S = \overline{\int_a^b} f(x)dg(x) := \inf\left\{S(f, P, g) : P \text{ partition of } [a, b]\right\}.$$

Theorem 9.14. *Let f, $g : [a, b] \to \mathbb{R}$ be bounded functions and g is increasing, we have*

$$s \le S.$$

Proof: Let P, Q be arbitrary partitions of $[a, b]$, then we have

$$s(f,\, P,\, g) \le s(f,\, P \cup Q,\, g) \le S(f,\, P \cup Q,\, g) \le S(f,\, Q,\, g).$$

It follows that

$$\sup_P s(f,\, P,\, g) \le \inf_Q S(f,\, Q,\, g).$$

That is $s \le S$. □

Definition 9.15. Let f, $g : [a, b] \to \mathbb{R}$ be bounded functions and g is increasing. If

$$\underline{\int_a^b} f(x)dg(x) = \overline{\int_a^b} f(x)dg(x) = I$$

then we say that the Darboux-Stieltjes integral of f (relative to g) exists and

$$\int_a^b f(x)dg(x) = \underline{\int_a^b} f(x)dg(x) = \overline{\int_a^b} f(x)dg(x).$$

I is called the *Darboux-Stieltjes integral* of f *relative to* g on $[a, b]$, and it is denoted by $\displaystyle\int_a^b f(x)dg(x)$. Moreover, if the Riemann-Stieltjes integral of f *relative to* g on $[a, b]$ exists, we will say that f is *Riemann-Stieltjes (R-S) integrable on* $[a, b]$ *relative to* g and denote $f \in \mathcal{RS}(g)$.

Remark 9.16. Under the definition for R-S integrability with $\lim_{\|P\|\to 0} \sigma(f,\, P,\, \xi,\, g) = I$, $s = S$ is not sufficient to imply $\lim_{\|P\|\to 0} \sigma(f,\, P,\, \xi,\, g) = I$. The definition of R-S integrability is generalized in many textbooks using $s = S$ instead of $\lim_{\|P\|\to 0} \sigma(f,\, P,\, \xi,\, g) = I$. In this notes we still adopt the definition with $s = S$. Let us explain through an example the situation that we have $s = S$ but $\lim_{\|P\|\to 0} \sigma(f,\, P,\, \xi,\, g)$ does not exist.

Let $f = g : [-1,\, 1] \to \mathbb{R}$ be functions defined by

$$f(x) = \begin{cases} \frac{x}{|x|} & \text{if} \quad x \ne 0, \\ 1 & \text{if} \quad x = 0. \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{x}{|x|} & \text{if} \quad x \ne 0, \\ -1 & \text{if} \quad x = 0. \end{cases}$$

Let P be a partition of $[-1, 1]$. We have

$$S(f, P, g) = \begin{cases} 2 & \text{if} \quad 0 \in P \\ 2 & \text{if} \quad 0 \notin P, \end{cases}$$

and

$$s(f, P, g) = \begin{cases} -2 & \text{if} \quad 0 \in P \\ 2 & \text{if} \quad 0 \notin P. \end{cases}$$

Clearly $s = S$ but $\lim_{\|P\| \to 0} \sigma(f, P, \xi, g)$ does not exist since $\lim_{\|P\| \to 0} s(f, P, g)$ does not exist.

Theorem 9.17. *Let $f, g : [a, b] \to \mathbb{R}$ be two bounded function such that g is a non-decreasing function. The Darboux-Stieltjes integral of f relative to g on $[a, b]$ exists if and only if for every $\epsilon > 0$, there exists a partition P such that*

$$S(f, P, g) - s(f, P, g) < \epsilon.$$

Proof: By Theorem 9.14, we have for every partition P,

$$s(f, P, g) \le s \le S \le S(f, p, g).$$

Then for every $\epsilon > 0$, there exists a partition P such that

$$S - s \le S(f, P, g) - s(f, P, g) < \epsilon,$$

which means $S = s$. f is D-S integrable.

Conversely, if f is D-S integrable, then $S = s = I$. For every $\epsilon > 0$, there exist partitions P_1 and P_2 such that

$$S(f, P_1, g) \le I + \frac{\epsilon}{2}$$
$$s(f, P_2, g) \ge I - \frac{\epsilon}{2}.$$

Then we have

$$S(f, P_1 \cup P_2, g) - I \le S(f, P_1, g) - I \le \frac{\epsilon}{2}$$
$$I - s(f, P_1 \cup P_2, g) \le I - s(f, P_2, g) \le \frac{\epsilon}{2}.$$

Adding the above two inequalities we have

$$S(f, P_1 \cup P_2, g) - s(f, P_1 \cup P_2, g) \le \epsilon.$$

That is, for every $\epsilon > 0$, there exists partition $P_1 \cup P_2$ such that $S(f, P_1 \cup P_2, g) - s(f, P_1 \cup P_2, g) \le \epsilon$.
□

Now we discuss some typical D-S integrable functions.

Theorem 9.18. *Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions and g is increasing. If f is continuous, then f is D-S integrable with respect to g.*

Proof: We want to show that for every $\epsilon > 0$, there exists a partition P such that

$$S(f,\, P,\, g) - s(f,\, P,\, g) = \sum_{k=1}^{n}(M_k - m_k)(g(x_k) - g(x_{k-1})) \leq \epsilon.$$

Note that if g is a constant function, the statement is trivial. So we assume that $g(b) > g(a)$ and let $\eta = \frac{\epsilon}{g(b)-g(a)}$. Since f is continuous on $[a,\, b]$, it is uniformly continuous and there exists $\delta > 0$ so that for every $t_1,\, t_2 \in [a,\, b]$ with $|t_1 - t_2| \leq \delta$ we have

$$|f(t_1) - f(t_2)| \leq \eta.$$

Let P be an arbitrary partition with $\|P\| \leq \delta$. Then for every $k = 1,\, 2,\, \cdots$, we have

$$M_k - m_k \leq \eta.$$

Since f is continuous, there exist $\xi_k,\, \bar{\xi}_k \in [x_{k-1},\, x_k]$ such that

$$M_k - m_k = f(\xi_k) - f(\xi_{k-1}) \leq \eta.$$

Then we have

$$\begin{aligned} S(f,\, P,\, g) - s(f,\, P,\, g) =& \sum_{k=1}^{n}(M_k - m_k)(g(x_k) - g(x_{k-1})) \\ \leq& \sum_{k=1}^{n}\eta(g(x_k) - g(x_{k-1})) \\ \leq& \eta(g(b) - g(a)) = \epsilon. \end{aligned}$$

By Theorem 9.41, f is D-S integrable with respect to g. □

Similarly we can show that

Theorem 9.19. *Let $f,\, g : [a,b] \to \mathbb{R}$ be monotonic functions. If g is continuous, then f is D-S integrable with respect to g.*

Theorem 9.20. *Let $f,\, g_1,\, g_2 : [a,b] \to \mathbb{R}$ be bounded functions and g_1 and g_2 are non-decreasing functions. If f is D-S integrable with respect to g_1 and g_2, respectively, then f is D-S integrable with respect to $g_1 + g_2$ and*

$$\int_a^b f(x)\mathrm{d}g_1(x) + \int_a^b f(x)\mathrm{d}g_2(x) = \int_a^b f(x)\mathrm{d}(g_1(x) + g_2(x)).$$

Proof: For every $\epsilon > 0$, there exist partitions P_1 and P_2 such that

$$\begin{aligned} S(f,\, P_1,\, g_1) - s(f,\, P_1,\, g_1) < \frac{\epsilon}{2}, \\ S(f,\, P_2,\, g_2) - s(f,\, P_2,\, g_2) < \frac{\epsilon}{2}. \end{aligned}$$

Let $P = P_1 \cup P_2$, we have $P_1 \leq P$, $P_2 \leq P$ and

$$S(f,\, P,\, g_1) - s(f,\, P,\, g_1) < \frac{\epsilon}{2},$$
$$S(f,\, P,\, g_2) - s(f,\, P,\, g_2) < \frac{\epsilon}{2}.$$

Note that $\Delta(g_1+g_2)(x_k) = \Delta(g_1)(x_k) + \Delta g_2(x_k)$. We have

$$S(f,\, P,\, g_1+g_2) - s(f,\, P,\, g_1+g_2) = S(f,\, P_1,\, g_1) - s(f,\, P_1,\, g_1) + S(f,\, P_2,\, g_2) - s(f,\, P_2,\, g_2) \leq \epsilon.$$

That is, f is R-S integrable with respect to g_1+g_2. Moreover, we have

$$\begin{aligned} s(f,\, P,\, g_1+g_2) =& s(f,\, P,\, g_1) + s(f,\, P,\, g_2) \\ \leq& \int_a^b f(x)\mathrm{d}g_1(x) + \int_a^b f(x)\mathrm{d}g_2(x) \\ \leq& S(f,\, P,\, g_1) + S(f,\, P,\, g_2) \\ \leq& S(f,\, P,\, g_1+g_2). \end{aligned}$$

Since f is R-S integrable with respect to g_1+g_2, we have

$$\int_a^b f(x)\mathrm{d}g_1(x) + \int_a^b f(x)\mathrm{d}g_2(x) = \int_a^b f(x)\mathrm{d}(g_1(x)+g_2(x)).$$

□

Theorem 9.21. *Let $f,\ g, h : [a,b] \to \mathbb{R}$ be bounded functions and h is increasing, $c \in \mathbb{R}$. If f and g are D-S integrable with respect to h, respectively, then $f+g$ and cf are D-S integrable with respect to h and*

i) $\displaystyle\int_a^b (f(x)+g(x))\mathrm{d}h(x) = \int_a^b f(x)\mathrm{d}h(x) + \int_a^b g(x)\mathrm{d}h(x);$

ii) $\displaystyle\int_a^b cf(x)\mathrm{d}h(x) = c\int_a^b f(x)\mathrm{d}h(x).$

Proof: We only prove (i) as (ii) is trivial. Let $P = \{x_k\}_{k=0}^n$ be an arbitrary partition of $[a,\, b]$ and denote

$$M_k^f = \sup_{x\in[x_{k-1},\, x_k]} f(x),\ m_k^f = \inf_{x\in[x_{k-1},\, x_k]} f(x).$$

Then we have

$$M_k^{f+g} \leq M_k^f + M_k^g, \quad m_k^{f+g} \geq m_k^f + m_k^g,\ k = 1,\, 2,\, \cdots,\, n,$$

and

$$S(f+g,\, P,\, h) \leq S(f,\, P,\, h) + S(g,\, P,\, h), \tag{9.15}$$
$$s(f+g,\, P,\, h) \geq s(f,\, P,\, h) + s(g,\, P,\, h). \tag{9.16}$$

For every $\epsilon > 0$, there exist P_f, P_g such that

$$S(f,\, P_f,\, h) \leq \int_a^b f(x)\mathrm{d}h(x) + \frac{\epsilon}{2},$$
$$S(f,\, P_g,\, h) \leq \int_a^b g(x)\mathrm{d}h(x) + \frac{\epsilon}{2}.$$

It follows that

$$S(f,\, P_f \cup P_g,\, h) + S(g,\, P_f \cup P_g,\, h) \leq S(f,\, P_f,\, h) + S(g,\, P_g,\, h) \leq \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}h(x) + \epsilon. \tag{9.17}$$

By (9.15 and (9.17), we have

$$S(f+g,\, P_f \cup P_g,\, h) \leq S(f,\, P_f \cup P_g,\, h) + S(g,\, P_f \cup P_g,\, h) \leq \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}h(x) + \epsilon,$$

which leads to

$$\inf_P S(f+g,\, P,\, h) \leq \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x.$$

Similarly, we have

$$\sup_P s(f+g,\, P,\, h) \geq \int_a^b f(x)\mathrm{d}h(x) + \int_a^b g(x)\mathrm{d}h(x).$$

Then by Theorem 9.14, we have

$$\sup_P s(f+g,\, P,\, h) = \int_a^b f(x)\mathrm{d}h(x) + \int_a^b g(x)\mathrm{d}h(x) = \sup_P s(f+g,\, P,\, h).$$

and hence

$$\int_a^b f(x)\mathrm{d}h(x) + \int_a^b g(x)\mathrm{d}h(x) = \int_a^b (f(x)+g(x))\mathrm{d}h(x).$$

□

Theorem 9.22. *Let a, b and c be constants in $\mathbb{R}$ with $c \in [a,\, b]$. Let $f,\, g : [a, b] \to \mathbb{R}$ be bounded functions and g is increasing. Then f is D-S integrable with respect to g on $[a,\, b]$ if and only if f is D-S integrable with respect to g on $[a,\, c]$ and on $[c,\, b]$, respectively and*

$$\int_a^b f(x)\mathrm{d}g(x) = \int_a^c f(x)\mathrm{d}g(x) + \int_c^b f(x)\mathrm{d}g(x).$$

Proof: Suppose that f is D-S integrable with respect to g on $[a,\, b]$. For every $\epsilon > 0$, there exists a partition P such that $S(f,\, P,\, g) - s(f,\, P,\, g) \leq \epsilon$. Let $P^* = P \cup \{c\}$ we have $P^* \geq P$ and hence

$$S(f,\, P^*,\, g) - s(f,\, P^*,\, g) \leq S(f,\, P,\, g) - s(f,\, P,\, g) \leq \epsilon.$$

Let $P_1 = P^* \cap [a,\, c]$ and $P_2 = P^* \cap [c,\, b]$. Then P_1 and P_2 are partitions of $[a,\, c]$ and $[c,\, b]$, respectively. Then we have $P^* = P_1 \cup P_2$ and

$$S(f,\, P^*,\, g) - s(f,\, P^*,\, g) = S(f,\, P_1,\, g) - s(f,\, P_1,\, g) + S(f,\, P_2,\, g) - s(f,\, P_2,\, g) \leq \epsilon.$$

Note that $S(f,\, P_1,\, g) - s(f,\, P_1,\, g)$ and $S(f,\, P_2,\, g) - s(f,\, P_2,\, g)$ are both nonnegative hence we have

$$S(f,\, P_1,\, g) - s(f,\, P_1,\, g) \leq \epsilon$$
$$S(f,\, P_2,\, g) - s(f,\, P_2,\, g) \leq \epsilon.$$

f is D-S integrable with respect to g on $[a,\, c]$ and on $[c,\, b]$, respectively.

Conversely, if f is D-S integrable with respect to g on $[a,\, c]$ and on $[c,\, b]$, respectively, then for every $\epsilon > 0$ there exist partitions P_1 and P_2 such that

$$\begin{aligned} S(f,\, P_1,\, g) - s(f,\, P_1,\, g) &< \frac{\epsilon}{2}, \\ S(f,\, P_2,\, g) - s(f,\, P_2,\, g) &< \frac{\epsilon}{2}. \end{aligned}$$

Let $P = P_1 \cup P_2$. Then P is a partition of $[a,\, b]$. We have

$$S(f,\, P,\, g) - s(f,\, P,\, g) = S(f,\, P_1,\, g) - s(f,\, P_1,\, g) + S(f,\, P_2,\, g) - s(f,\, P_2,\, g) < \epsilon.$$

That is, f is R-S integrable with respect to g on $[a,\, b]$. $\square$

Remark 9.23. In the previous theorem, with the Darboux-Stieltjes integral definition of $s = S$, the existence of the integral $\displaystyle\int_a^b f(x)dg(x)$ is equivalent to the existence of the integrals $\displaystyle\int_a^c f(x)dg(x)$ and $\displaystyle\int_c^b f(x)dg(x)$. However, if we use the Darboux-Stieltjes integral definition of $\lim_{\|P\|\to 0} \sigma(f,\, P,\, \xi,\, g) = I$, the existence of $\displaystyle\int_a^c f(x)dg(x)$ and $\displaystyle\int_c^b f(x)dg(x)$ doesn't imply the existence of $\displaystyle\int_a^b f(x)dg(x)$. Indeed, consider the following example:

$$f(x) = \begin{cases} -1 & \text{for} \quad -1 \le x < 0 \\ 1 & \text{for} \quad 0 \le x \le 1, \end{cases} \quad g(x) = \begin{cases} -1 & \text{for} \quad -1 \le x \le 0 \\ 1 & \text{for} \quad 0 < x \le 1, \end{cases}$$

It is easy to verify that the D-S integrals under the limit definition $\displaystyle\int_{-1}^0 f(x)dg(x)$ and $\displaystyle\int_0^1 f(x)dg(x)$ exist with $\displaystyle\int_{-1}^0 f(x)dg(x) = 0$ and $\displaystyle\int_0^1 f(x)dg(x) = 2$. On the other hand, the D-S integrals under the limit definition $\displaystyle\int_{-1}^1 f(x)dg(x)$ doesn't exist. Indeed, by taking any partition $P = \{x_k\}_{k=1}^n$ of $[-1, 1]$, which doesn't contain the point 0, i.e. there exists i such that $x_{i-1} < 0 < x_i$. Then, consider the sum

$$\sigma(f, P,\, \xi,\, g) = \sum_{k=1}^{n} f(\xi_k) \triangle g(x_k) = f(\xi_i)(g(x_i) - g(x_{i-1})) = f(\xi_i).$$

Depending on $\xi_i < 0$ or $\xi_i > 0$, we obtain that $\sigma(f, P, g) = -2$ or $\sigma(f, P, g) = 2$, respectively. That is, under the limit definition of $R - S$ integral, f is not integrable with respect to g on $[-1,\, 1]$. But by Remark 9.16, it is integrable under the $s = S$ definition.

Theorem 9.24. *Let $f,\, g : [a, b] \to \mathbb{R}$ be bounded functions. Suppose that f has finitely many discontinuities, and g is increasing and continuous at every discontinuity of f. Then f is D-S integrable with respect to g.*

Proof: Let $D = \{c_1,\, c_2,\, \cdots,\, c_m\} \subseteq [a,\, b]$ be the set of all discontinuities of f with $c_1 < c_2 < \cdots < c_m$. Since g is continuous on the finite set D, for every $\epsilon > 0$, there exists a sequence of disjoint sets $\{[u_i,\, v_i]\}_{i=1}^m$ such that $c_i \in [u_i,\, v_i]$, $i = 1,\, 2,\, \cdots,\, m$ and

$$\sum_{i=1}^{m}[g(v_i) - g(u_i)] \le \epsilon.$$

Let K be defined as follows:

$$K = [a,\, b] \setminus (\cup_{i=1}^{m}(u_i,\, v_i) \cup (D \cap \{a,\, b\}))$$

Then K is closed and hence f is uniformly continuous on K. There exists $\delta > 0$ so that for every $s,\, t \in K$ with $|s - t| \le \delta$ we have

$$|f(s) - f(t)| \le \epsilon.$$

Let P be a partition such that $D \subset P$ with $\|P\| \le \max\{\delta,\, |u_i - v_i| : i = 1,\, 2,\, \cdots,\, m\}$ and $(P \setminus D) \cap (\cup_{i=1}^{m}(u_i,\, v_i)) = \emptyset$. Then we have

$$\begin{aligned}
&S(f,\, P,\, g) - s(f,\, P,\, g)\\
&= \sum_{k=1}^{m}(M_k - m_k)(g(x_k) - g(x_{k-1}))\\
&= \sum_{x_k,\, x_{k-1} \in D}(M_k - m_k)(g(x_k) - g(x_{k-1})) + \sum_{x_k,\, x_{k-1} \in P \setminus D}(M_k - m_k)(g(x_k) - g(x_{k-1}))\\
&\le (M - m)\epsilon + \epsilon(g(b) - g(a))\\
&= (M - m + g(b) - g(a))\epsilon,
\end{aligned}$$

where M, m are the supremum and infimum of f on $[a,\, b]$, respectively, and $M_k = \sup_{x \in [x_{k-1},\, x_k]} f(x)$, $m_i = \inf_{x \in [x_{k-1},\, x_k]} f(x)$, $k = 1,\, 2,\, \cdots$. By Theorem 9.41, f is D-S integrable with respect to g. □

9.4 Darboux-Stieltjes Integral with respect to Functions of Bounded Variation

Now we are in the position to consider a more general case that g is of bounded variation on $[a, b]$, when it can be represented as a difference $g(x) = g_1(x) - g_2(x)$, where $g_1,\, g_2 : [a, b] \to \mathbb{R}$ are two non-decreasing functions (see Corollary 9.12). Note that

$$\begin{aligned}
S(f, P, g) - s(f, P, g) &= \sum_{k=1}^{n} \omega_k \triangle g(x_k)\\
&= \sum_{k=1}^{n} \omega_k \triangle g_1(x_k) - \sum_{k=1}^{n} \omega_k \triangle g_2(x_k)\\
&=: (S(f,\, P,\, g_1) - s(f,\, P,\, g_1)) - (S(f,\, P,\, g_2) - s(f,\, P,\, g_2)).
\end{aligned}$$

It seems feasible to define the Darboux-Stieltjes integral $\int_a^b f(x)dg(x)$ as

$$\int_a^b f(x)\mathrm{d}g(x) := \int_a^b f(x)\mathrm{d}g_1(x) - \int_a^b f(x)\mathrm{d}g_2(x). \tag{9.18}$$

As the representation $g_1 - g_2$ of g is clearly not unique, we need to show that if $g = g_1 - g_2 = g_1' - g_2'$ where g_1, g_2, g_1' and g_2' are increasing functions on $[a, b]$ and f is D-S integrable with respect to each of them, then

$$\int_a^b f(x)\mathrm{d}g_1x - \int_a^b f(x)\mathrm{d}g_2x = \int_a^b f(x)\mathrm{d}g_1'(x) - \int_a^b f(x)\mathrm{d}g_2'(x),$$

which is true by Theorem 9.20. In the following we discuss D-S integral of functions with respect to functions of bounded variations in the sense of (9.37).

Theorem 9.25. *Assume that $f : [a, b] \to \mathbb{R}$ is a Riemann integrable function on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is a Lipschitzian function, i.e.*

$$\exists_{L>0}\, \forall_{x',x''\in[a,b]} \quad |g(x') - g(x'')| \le L|x' - x''|.$$

Then, the Darboux-Stieltjes integral $\displaystyle\int_a^b f(x)dg(x)$ exists.

Proof: Let us first consider the case when $g : [a, b] \to \mathbb{R}$ is a non-decreasing function. For a partition $P = \{x_k\}_{k=1}^n$, we denote

$$\omega_k := M_k - m_k, \quad k = 1, 2, \dots, n,$$

where

$$M_k := \sup\{f(x) : x \in [x_k, x_{k-1}]\}, \quad m_k := \inf\{f(x) : x \in [x_k, x_{k-1}]\}.$$

Then, (since g is Lipschitzian) we have

$$\sum_{k=1}^n \omega_k \triangle g(x_k) \le L \sum_{k=1}^n \omega_k \triangle x_k.$$

By the assumption, f is Riemann integrable, thus

$$\lim_{\|P\|\to 0} \sum_{k=1}^n \omega_k \triangle x_k = 0,$$

and therefore, for every $\epsilon > 0$, there exists P such that

$$S(f,\, P,\, g) - s(f,\, P,\, g) = \sum_{k=1}^n \omega_k \triangle g(x_k) \le \epsilon.$$

That is, the Darboux-Stieltjes integral $\displaystyle\int_a^b f(x)dg(x)$ exists.

Assume now that g is an arbitrary Lipschitzian function (with constant L) of bounded variation. Then, we can write $g(x) = g_1(x) - g_2(x)$, $x \in [a, b]$, where $g_1(x) := Lx$, $g_2 := Lx - g(x)$. We claim that g_1, $g_2 : [a, b] \to \mathbb{R}$ are non-decreasing. Indeed, g_1 is increasing (since $L > 0$), and for all $a \le x < x' \le b$ we have

$$g_2(x') - g_2(x) = L(x' - x) - (g(x') - g(x)) \ge 0.$$

In addition g_2 is Lipschitzian (with respect to the constant $2L$). Then,

$$|g_2(x') - g_2(x)| \le L(x' - x) + |g(x') - g(x)| \le 2L|x - x'|.$$

Therefore, by the same argument for the proof of the first part, the Darboux-Stieltjes integral $\displaystyle\int_a^b f(x)dg(x)$ exists. $\square$

Theorem 9.26. *Assume that $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is a function such that it can be represented as*

$$g(x) = c + \int_a^x \varphi(t)dt \tag{9.19}$$

where c is a constant, and $\varphi(t)$ is Riemann integrable on $[a, b]$. Then, the Darboux-Stieltjes integral $\int_a^b f(x)dg(x)$ exists and

$$\int_a^b f(x)dg(x) = \int_a^b f(x)\varphi(x)dx,$$

where the second integral is the Riemann integral.

Proof: By the assumption, the function $\varphi(x)$ is Riemann integrable, thus it has to be bounded, i.e. there exists $L > 0$ such that $|\varphi(x)| \le L$ fo all $x \in [a, b]$. Thus, for all x, $x' \in [a, b]$, $x > x'$, we have

$$|g(x) - g(x')| = |\int_{x'}^x \varphi(t)dt| \le L|x - x'|,$$

which means that g is a Lipschitzian function, therefore the Darboux-Stieltjes $\int_a^b f(x)dg(x)$ exists.

Without loss of generality, we can assume that the function φ is non-negative. Indeed, we can write $\varphi(t) = \varphi_1(t) - \varphi_2(t)$, where

$$\varphi_1(t) = \frac{|\varphi(t)| + \varphi(t)}{2}, \quad \varphi_2(t) = \frac{|\varphi(t)| - \varphi(t)}{2,}$$

where φ_1 and φ_2 are non-negative and Lipschitzian. This implies that $g(x) = g_1(x) - g_2(x)$, where

$$g_1(x) = c + \int_a^x \varphi_1(t)dt, \quad g_2(x) = \int_a^x \varphi_2(t)dt.$$

Since,

$$\int_a^b f(x)dg(x) = \int_a^b f(x)dg_1(x) - \int_a^b f(x)dg_2(x),$$

we can consider the integrals $\int_a^b f(x)dg_1(x)$ and $\int_a^b f(x)dg_2(x)$ separately.

Thus, if $\varphi(t)$ is non-negative function, then for every partition $P = \{x_k\}_{k=1}^n$, we have

$$S(f, P, g) = \sum_{k=1}^n M_k(g(x_k) - g(x_{k-1})) = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} M_k\varphi(x)dx,$$
$$s(f, P, g) = \sum_{k=1}^n m_k(g(x_k) - g(x_{k-1})) = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} m_k\varphi(x)dx,$$

where $M_k = \sup_{x\in[x_{k-1},\, x_k]} f(x)$ and $m_k = \inf_{x\in[x_{k-1},\, x_k]} f(x)$. On the other hand,

$$\int_a^b f(x)\varphi(x)dx = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x)\varphi(x)dx,$$

thus
$$S(f, P, g) - \int_a^b f(x)\varphi(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} (M_k - f(x))\varphi(x)dx.$$
Notice that for all $k = 1, 2, \ldots, n$, $x_{k-1} \le x \le x_k$, $k = 1, 2, \ldots, n$, we have
$$M_k - f(x) < \omega_k = M_k - m_k,$$
therefore,
$$\begin{aligned} S(f, P, g) - \int_a^b f(x)\varphi(t)dx &\le \sum_{k=1}^{n} \omega_k \int_{x_{k-1}}^{x_k} \varphi(x)dx \\ &= \sum_{k=1}^{n} \omega_k \triangle g(x_k) \\ &= S(f, P, g) - s(f, P, g). \end{aligned}$$
Since f is D-S integrable with respect to g, for every $\epsilon > 0$, there exists P_1 such that
$$S(f, P_1, g) - \int_a^b f(x)\varphi(t)dx \le S(f, P_1, g) - s(f, P_1, g) \le \epsilon,$$
which leads to
$$\inf_P S(f,\ P,\ g) \le \int_a^b f(x)\varphi(t)dx.$$
By the same token, there exists P_2 such that
$$\int_a^b f(x)\varphi(t)dx - s(f, P_2, g) \le S(f, P_2, g) - s(f, P_2, g) \le \epsilon,$$
which leads to
$$\sup_P s(f,\ P,\ g) \ge \int_a^b f(x)\varphi(t)dx.$$
Consequently, we have
$$\inf_P S(f,\ P,\ g) \le \int_a^b f(x)\varphi(t)dx \le \sup_P S(f,\ P,\ g).$$
It follows that
$$\inf_P S(f,\ P,\ g) = \int_a^b f(x)\varphi(t)dx = \sup_P S(f,\ P,\ g),$$
Since f is R-S integrable with respect to g implies that $\inf_P S(f,\ P,\ g) = \sup_P S(f,\ P,\ g)$. □

Remark 9.27. Let $g : [a, b] \to \mathbb{R}$ be a continuous function such that the derivative $g'(x)$ exists for all $x \in [a, b]$ except a finite number of points $y_1,\ y_2,\ \ldots,\ y_N \in [a, b]$. We can define
$$\varphi(x) := \begin{cases} g'(x) & \text{if } \ x \ne y_l,\ l = 1, 2, \ldots, N \\ 0 & \text{otherwise.} \end{cases}$$
Then, if the function $|\varepsilon(x)|$ is integrable on $[a, b]$, then by Theorem 8.30, we have
$$g(x) = g(a) + \int_a^x \varphi(t)dt.$$

From Remark 9.46 and Theorem 9.45, we obtain:

Corollary 9.28. *Assume that $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ and the function $g : [a,b] \to \mathbb{R}$ is differentiable in the interval $[a,b]$ except a finite number of points, such that $|g'(x)|$ is integrable on $[a,b]$. Then*

$$\int_a^b f(x)dg(x) = \int_a^b f(x)g'(x)dx.$$

9.5 Properties of the Darboux–Stieltjes Integral

The following properties of Darboux-Stieltjes integral follow directly from the definition:

Proposition 9.29. *Let $g,\ g_1,\ g_2 : [a,b] \to \mathbb{R}$ be functions of bounded variation. Then we have the following properties:*

(P1) $\displaystyle\int_a^b dg(x) = g(b) - g(a);$

(P2) $\displaystyle\int_a^b (f_1(x) \pm f_2(x))dg(x) = \int_a^b f_1(x)dg(x) \pm \int_a^b f_2(x)dg(x);$

(P3) $\displaystyle\int_a^b f(x)d(g_1(x) \pm g_2(x)) = \int_a^b f(x)dg_1(x) \pm \int_a^b f(x)dg_2(x);$

(P4) $\displaystyle\int_a^b \alpha f(x)dg(x) = \alpha \int_a^b f(x)dg(x);$

(P5) $\displaystyle\int_a^b f(x)d(\beta g(x)) = \beta \int_a^b f(x)dg(x)$

where we assume in (P2)–(P5) the existence of all the specified integrals and $\alpha,\ \beta \in \mathbb{R}$. Moreover, under the assumption that $a < c < b$, we have

(P6) $\displaystyle\int_a^b f(x)dg(x) = \int_a^c f(x)dg(x) + \int_c^b f(x)dg(x),$

where we also assume that all the related integrals exist.

Proof: All these properties can be obtain directly from the definition of the Darboux-Stieltjes integral. For the property (P6), it is need to add the point c to the considered partition P of $[a,b]$. □

Proposition 9.30. *Let $f,\ g : [a,b] \to \mathbb{R}$ be two bounded functions such that both function have the same discontinuity point $c \in [a,b]$ and are discontinuous from the same side of c. Assume that g is increasing. Then the integral $\displaystyle\int_a^b f(x)dg(x)$ does not exist.*

Proof: We have two cases to be considered separately:

(i) the both functions g and f are not continuous from the right at c;
(ii) the both functions g and f are not continuous from the left at c.

We first consider the situation (i). By way of contradiction, we assume that f is R-S integrable with respect to g. Choose $\epsilon_f > 0$ so that for every $\delta_f > 0$, there exists x_f such that

$$0 < x_f - c < \delta_f \quad \text{and} \quad |f(x_f) - f(c)| \geq \sqrt{\epsilon_f}. \tag{9.20}$$

Choose $\epsilon_g > 0$ so that for every $\delta_g > 0$, there exists x_g such that

$$0 < x_g - c < \delta_g \quad \text{and} \quad |g(x_g) - g(c)| \geq \sqrt{\epsilon_g}. \tag{9.21}$$

Let $\epsilon = \min\{\epsilon_f, \epsilon_g\}$. Since f is D-S integrable with respect to g, there exists a partition P of $[a, b]$ such that

$$S(f, P, g) - s(f, P, g) \leq \epsilon.$$

Let $P^* = P \cup \{c\}$. Then we have

$$S(f, P^*, g) - s(f, P^*, g) \leq \epsilon. \tag{9.22}$$

For the partition P^*, there exists an interval $[x_{i-1}, x_i]$ such that $x_{i-1} < c < x_i$. Choose $\delta^* = \min\{\delta_f, \delta_g, c - x_{i-1}, x_i - c\}$. Then by (9.20) and by (9.21), there exist x_f^* and x_g^* such that

$$0 < x_f^* - c < \delta^* \quad \text{and} \quad |f(x_f^*) - f(c)| \geq \sqrt{\epsilon_f} \geq \sqrt{\epsilon} \tag{9.23}$$

$$0 < x_g^* - c < \delta^* \quad \text{and} \quad |g(x_g^*) - g(c)| \geq \sqrt{\epsilon_g} \geq \sqrt{\epsilon}. \tag{9.24}$$

Let $M_i^- = \sup_{x\in[x_{i-1}, c]} f(x)$, $M_i^+ = \sup_{x\in[c, x_i]} f(x)$, $m_i^- = \inf_{x\in[x_{i-1}, c]} f(x)$ and $m_i^+ = \inf_{x\in[c, x_i]} f(x)$. Then we have

$$\begin{aligned} S(f, P^*, g) - s(f, P^*, g) &= \sum_{k=1}^{i-1}(M_k - m_k)\Delta g(x_k) + (M_i^- - m_i^-)(g(c) - g(x_{i-1})) \\ &\quad + (M_i^+ - m_i^+)(g(x_i) - g(c)) + \sum_{k=i+1}^{n} (M_k - m_k)\Delta g(x_k) \\ &\geq (M_i^- - m_i^-)(g(c) - g(x_{i-1})) + (M_i^+ - m_i^+)(g(x_i) - g(c)) \\ &\geq (M_i^+ - m_i^+)(g(x_i) - g(c)). \end{aligned}$$

Since f and g are discontinuous from the right side of c, then by (9.23) we have

$$S(f, P^*, g) - s(f, P^*, g) \geq (M_i^+ - m_i^+)(g(x_i) - g(c)) \geq \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon,$$

which is a contradiction to (9.22). Similar contradiction can be obtained if f and g are discontinuous from the left side of c. $\square$

Theorem 9.31. (Integration by Parts Formula) *Let f, $g : [a, b] \to \mathbb{R}$ be two functions of bounded variation. Then*

a) $\int_a^b f(x)dg(x)$ if and only if $\int_a^b g(x)df(x)$ exists;
b) If $\int_a^b f(x)dg(x)$ exists, then

$$\int_a^b f(x)dg(x) = f(x)g(x)\Big|_a^b - \int_a^b g(x)df(x). \tag{9.25}$$

Proof: Since $f,\ g : [a, b] \to \mathbb{R}$ are functions of bounded variation we assume that both $f,\ g : [a, b] \to \mathbb{R}$ are increasing without loss of generality. Consider a partition $P = \{x_k\}_{k=1}^{n}$ with a sequence $\xi_k \in [x_{k-1}, x_k]$, i.e.

$$a = x_0 \le \xi_1 \le x_1 \le \xi_2 \le \cdots \le x_{k-1} \le \xi_k \le x_k \le \xi_{k+1} \le x_{k+1} \le \cdots \le x_{n-1} \le \xi_n \le x_n = b.$$

Then, we have

$$\begin{aligned}
S(f, P, g) &= \sum_{k=1}^{n} f(x_k)(g(x_k) - g(x_{k-1})) && (9.26)\\
&= \sum_{k=1}^{n} f(x_k)g(x_k) - \sum_{k=1}^{n} f(x_k)g(x_{k-1})\\
&= f(b)g(b) + \sum_{k=1}^{n-1} f(x_k)g(x_k) - \sum_{k=2}^{n} f(x_k)g(x_{k-1}) - f(x_1)g(a)\\
&= f(b)g(b) + \sum_{k=1}^{n-1} f(x_k)g(x_k) - \sum_{k=1}^{n-1} f(x_{k+1})g(x_k) - f(x_1)g(a)\\
&= f(b)g(b) - f(a)g(a) + f(a)g(a) - f(x_1)g(a) + \sum_{k=1}^{n-1} g(x_k)(f(x_{k+1}) - f(x_k))\\
&= f(x)g(x)\Big|_a^b - \left(g(a)(f(x_1) - f(a)) + \sum_{k=1}^{n-1} g(x_k)\Big(f(x_{k+1}) - f(x_k)\Big)\right). && (9.27)
\end{aligned}$$

That is,

$$S(f,\ P,\ g) = f(x)g(x)\Big|_a^b - s(g,\ P,\ f). \tag{9.28}$$

Similarly we have

$$s(f,\ P,\ g) = f(x)g(x)\Big|_a^b - S(g,\ P,\ f). \tag{9.29}$$

Therefore

$$S(f,\ P,\ g) - s(f,\ P,\ g) = S(g,\ P,\ f) - s(g,\ P,\ f), \tag{9.30}$$

which means that for every $\epsilon > 0$, $S(f,\ P,\ g) - s(f,\ P,\ g) \le \epsilon$ for some P if and only if $S(g,\ P,\ f) - s(g,\ P,\ f) \le \epsilon$. That is, $\int_a^b f(x)dg(x)$ exists if and only if $\int_a^b g(x)df(x)$ exists.

If $\int_a^b f(x)dg(x)$ exists, then by (9.28) and (9.29) we have

$$\int_a^b f(x)\mathrm{d}g(x) = f(x)g(x)\Big|_a^b - \int_a^b g(x)\mathrm{d}f(x).$$

□

One can deduce from the proof of Theorem 9.48 the following property:

Corollary 9.32. *If the function $f : [a, b] \to \mathbb{R}$ is Darboux-Stieltjes integrable with respect to g on $[a, b]$, then the function $g : [a, b] \to \mathbb{R}$ is also Darboux-Stieltjes integrable with respect to f on $[a, b]$.*

9.6 Computations of the Darboux–Stieltjes Integral

Example 9.33. Consider a function $\rho : \mathbb{R} \to \mathbb{R}$ defined by

$$\rho(x) := \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases} \tag{9.31}$$

The function ρ has a jump discontinuity at 0 and we have

$$\rho(0^+) - \rho(0) = 1, \quad \text{where} \quad \rho(0^+) := \lim_{x\to 0^+} \rho(x).$$

Let $f : [a,b] \to \mathbb{R}$ be a continuous function at $c \in [a,b)$. Put $g(x) := \rho(x-c)$, $x \in [a,b]$. We will compute the integral $\int_a^b f(x)dg(x) = \int_a^b f(x)d\rho(x-c)$. Consider a partition $P = \{x_k\}_{k=0}^n$ such that $c \in [x_{i-1}, x_i)$. Then $\triangle g(x_i) = \triangle\rho(x_i - c) = 1$ and for $k \neq i$, $\triangle g(x_k) = \triangle\rho(x_k - c) = 0$. Therefore, we have

$$\sigma(f,P,g) = \sum_{k=1}^n f(\xi_k)\triangle\rho(x_k - c) = f(\xi_i), \quad \xi_i \in [x_{i-1}, x_i].$$

Since

$$\int_a^b f(x)d\rho(x-c) = \lim_{\|P\|\to 0} \sigma(f,P,g) = \lim_{\|P\|\to 0} f(\xi_i) = f(c).$$

Notice that for $c = b$ we have $\int_a^b f(x)d\rho(x-c) = 0$. Similarly, for $a < c \leq b$ we have

$$\int_a^b f(x)\rho(c-x) = -f(c).$$

Theorem 9.34. *Assume that $f : [a,b] \to \mathbb{R}$ is a continuous function and $g : [a,b] \to \mathbb{R}$ is a differentiable on $[a,b]$ function, except for a finite number of points in $[a,b]$, say*

$$a = c_0 < c_1 < \cdots < c_k < \cdots < c_m = b,$$

at which g may have jump discontinuities, i.e the limits

$$\lim_{x\to c_k^+} g(x) = g(c_k^+) \quad \text{and} \quad \lim_{x\to c_k^-} g(x) = g(c_k^-),$$

exist. Assume also that $g'(x)$ is bounded on $[a.b]$. Then the Darboux-Stieltjes integral of f on $[a,b]$ relative to g exists and

$$\int_a^b f(x)dg(x) = \begin{cases} f(a)\big(g(a^+) - g(a)\big) + \displaystyle\int_a^b f(x)g'(x)dx \\ + \displaystyle\sum_{k=1}^{m-1} f(c_k)\big(g(c_k^+) - g(c_k^-)\big) + f(b)\big(g(b) - g(b^-)\big). \end{cases} \tag{9.32}$$

Proof: Put

$$\alpha_k^+ := g(c_k^+) - g(c_k), \quad k = 0,1,2,\dots,m-1,$$
$$\alpha_k^- := g(c_k) - g(c_k^-), \quad k = 1,2,\dots,m-1,m,$$

Notice that $g(c_k^+) - g(c_k^-) = \alpha_k + +\alpha_k^-$ for $k = 1, 2, \dots, m-1$.

Let us define the following auxiliary function $g_1 : [a, b] \to \mathbb{R}$, by

$$g_1(x) = \sum_{k=0}^{m-1} \alpha_k^+ \rho(x - c_k) - \sum_{k=1}^{m} \alpha_k^- \rho(c_k - x), \quad x \in [a, b], \tag{9.33}$$

where $\rho : \mathbb{R} \to \mathbb{R}$ was defined by (9.31). Notice that g_1 has exactly the same discontinuity points as g. We will prove that the function $g_2 : [a, b] \to \mathbb{R}$, defined by

$$g_2(x) = g(x) - g_1(x), \quad x \in [a, b],$$

is continuous. Clearly, for $x \neq c_k$, $k = 0, 1, \dots, m]$, the functions g_2 is continuous at x (because both functions g and g_1 are continuous at x). We will show that the function g_2 is continuous from the right at all points c_k. Indeed, we have that all the summands in (9.33) are continuous from the right at c_k, except the summand $\alpha_k^+ \rho(x - c_k)$, therefore, we need to compute the limit

$$\lim_{x \to c_k^+} (g(x) - \alpha_k^+ \rho(x - c_k)) = g(c_k^+) - \alpha_k^+ = g(c_k).$$

On the other hand, all the summands in (9.33) are continuous from the left at c_k, except $\alpha_k^- \rho(c_k - x)$, therefore we need to compute the limit

$$\lim_{x \to c_k^-} (g(x) + \alpha_k^- \rho(c_k - x)) = g(c_k^-) + \alpha_k^- = g(c_k),$$

which implies g_2 is continuous at c_k (at $c_0 = a$ and $c_m = b$ we only need check the continuity from the right and from the left respectively). Moreover, notice that $g_2'(x) = g'(x)$ whenever $g'(x)$ exists. Thus, by Theorem 9.45, we have

$$\int_a^b f(x) dg_2(x) = \int_a^b f(x) g_2'(x) dx = \int_a^b f(x) g'(x) dx.$$

Since, by the properties of the Darboux-Stieltjes integral (see Proposition 9.29

$$\begin{aligned}
\int_a^b f(x) dg_1(x) &= \sum_{k=0}^{m-1} \alpha_k^+ \int_a^b f(x) d\rho(x - c_k) - \sum_{k=1}^{m} \alpha_k^- \int_a^b f(x) d\rho(c_k - x) \\
&= \sum_{k=0}^{m-1} \alpha_k^+ f(c_k) + \sum_{k=1}^{m} \alpha_k^- f(c_k) \\
&= f(a)(g(a^+) - g(a)) + \sum_{k=1}^{m-1} f(c_k)(g(c_k^+) - g(c_k^-)) + f(b)(g(b) - g(b^-)).
\end{aligned}$$

Therefore, by Proposition 9.29, we obtain

$$\int_a^b f(x) dg(x) = \int_a^b f(x) dg_1(x) + \int_a^b f(x) dg_2(x),$$

and the conclusion follows. □

Example 9.35. We can apply Theorem 9.45 to compute the following Darboux-Stieltjes integrals:

(a) $\displaystyle\int_0^2 x^2 d\ln(1+x)$;

(b) $\displaystyle\int_0^{\frac{\pi}{2}} xd\sin(x)$;

(c) $\displaystyle\int_{-1}^1 xd\arctan(x)$.

Indeed, for (a):

$$\int_0^2 x^2 d\ln(1+x) = \int_0^2 \frac{x^2 dx}{1+x} = \left(\frac{1}{2}x^2 - x + \ln(1+x)\right)\Big|_0^2 = \ln 3.$$

For (b):

$$\int_0^{\frac{\pi}{2}} xd\sin(x) = \int_0^{\frac{\pi}{2}} x\cos(x)dx = (x\sin + \cos x)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1.$$

For (c):

$$\int_{-1}^1 xd\arctan(x) = \int_{-1}^1 \frac{xdx}{1+x^2} = \frac{1}{2}\ln(1+x^2)\Big|_{-1}^1 = \frac{1}{2}(\ln 2 - \ln 2) = 0.$$

Example 9.36. We will apply Theorem 9.34 to evaluate the following Darboux-Stieltjes integrals.

(a) $\displaystyle\int_{-1}^3 xdg(x)$, where $g(x) = \begin{cases} 0 & \text{for } x = -1, \\ 1 & \text{for } -1 < x < 2, ; \\ -1 & \text{for } 2 < x \le 3. \end{cases}$

(b) $\displaystyle\int_0^2 x^2 dg(x)$ where $g(x) = \begin{cases} 0 & \text{for } 0 \le x < \frac{1}{2}, \\ 0 & \text{for } \frac{1}{2} \le x < \frac{3}{2}, \\ 2 & \text{for } x = \frac{3}{2}, \\ -2 & \text{for } \frac{3}{2} < x \le 2. \end{cases};$

For (a): We have $g(-1^+) - g(-1) = 1$, $g(2^+) - g(2^-) = -2$, $g'(x) = 0$, thus we have

$$\int_{-1}^3 xdg(x) = (-1)\cdot 1 + 2\cdot(-2) = -5.$$

For (b): We have $g(\frac{1}{2}^+) - g(\frac{1}{2}^-) = 1$, $g(\frac{3}{2}^+) - g(\frac{3}{2}^-) = -2$ (the value $g(\frac{3}{2})$ has no impact on this interval). Thus

$$\int_0^2 x^2 dg(x) = \left(\frac{1}{2}\right)^2 \cdot 1 + \left(\frac{3}{2}\right)^2 \cdot (-2) = -\frac{17}{4}.$$

Example 9.37. Consider $g : [-2, 2] \to \mathbb{R}$ given by

$$g(x) = \begin{cases} x+2 & \text{for } -2 \le x \le -1 \\ 2 & \text{for } -1 < x < 0, \\ x^2 + 3 & \text{for } 0 \le x \le 2. \end{cases}.$$

We will apply Theorem 9.34 to evaluate the following Darboux-Stieltjes integrals.

(a) $\displaystyle\int_{-2}^2 xdg(x)$;

(b) $\displaystyle\int_{-2}^{2} x^2 dg(x)$;

(c) $\displaystyle\int_{-2}^{2} (x^3+1) dg(x)$;

Since

$$g'(x) = \begin{cases} 1 & \text{for } -2 \le x < -1, \\ 0 & \text{for } -1 < x < 0, \\ 2x & \text{for } 0 \le x \le 2. \end{cases}$$

we have, for (a):

$$\int_{-2}^{2} x dg(x) = \int_{-2}^{-1} x\, dx + 2\int_{0}^{2} x^2\, dx + (-1)\cdot 1 + 0 \cdot 1 = \frac{17}{6}.$$

For (b):

$$\int_{-2}^{2} x^2 dg(x) = \int_{-2}^{-1} x^2\, dx + 2\int_{0}^{2} x^3\, dx + 1\cdot 1 + 0 \cdot 1 = \frac{34}{3}.$$

For (c):

$$\int_{-2}^{2} (x^3+1) dg(x) = \int_{-2}^{-1} (x^3+1)\, dx + 2\int_{0}^{2} (x^4+x)\, dx + 0\cdot 1 + 1 \cdot 1 = \frac{301}{20}.$$

9.7 Geometric Interpretation of the Darboux-Stieltjes Integral

Assume that the function $f : [a,b] \to \mathbb{R}$ is continuous and $g : [a,b] \to \mathbb{R}$ is a non-decreasing function (having discontinuities). We will find out what is a geometrical interpretation of the Riemann-Stieltjes integral

$$\int_a^b f(x)dg(x) = \int_a^b f(x)g'(x)dx$$

The parametric equations

$$\begin{cases} u = g(x), \\ v = f(x), \end{cases}, \quad \text{where} \quad x \in [a,b],$$

defines a certain curve C in the plane (u,v). This curve has discontinuities, if the function $g(x)$ has discontinuities. Assume for simplicity that g is differentiable except at the points c_1 and $c_2 \in (a,b)$, $c_1 < c_2$, where it has two jump-discontinuities. Thus the curve C jumps from the point $A_1^- := (g(c_1^-), f(c_1))$ to the point $A_1^+ := (g(c_1^+), f(c_1))$, and from the point t $A_2^- := (g(c_2^-), f(c_2))$ to the point $A_2^+ := (g(c_2^+), f(c_2))$ (see Figure 9.1).

By applying Theorem 9.34, we have that

$$\int_a^b f(x)dg(x) = \int_a^b f(x)g'(x)dx + f(c_1)\big(g(c_1^+) - g(c_1^-)\big) + f(c_2)\big(g(c_2^+) - g(c_2^-)\big).$$

Since the integrals $\displaystyle\int_a^{c_1} f(x)g'(x)dx$, $\displaystyle\int_{c_1}^{c_2} f(x)g'(x)dx$ and $\displaystyle\int_b^{c_s} f(x)g'(x)dx$, represent the area under the curve C and above the intervals $[g(a), g(c_1^-)]$, $[g(c_1^+), g(c_2^-)]$ and $[g(c_2^+), g(b)]$, thus this integral represent the area under the curve C (see Figure 9.1).

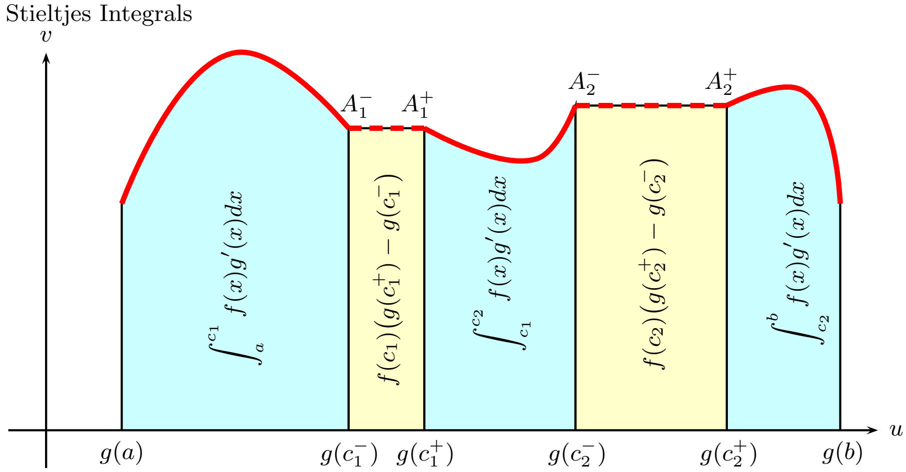


Fig. 9.1. The area under the curve C represents the Riemann-Stieltjes integral of f on $[a, b]$ relative to g.

9.8 Existence of the Riemann–Stieltjes Integral in the classical sense

There are different definitions for Riemann–Stieltjes Integral in the literature. In this section we discuss Riemann-Stieltjes integral in the classical sense.

Consider two bounded functions $f,\, g : [a, b] \to \mathbb{R}$ and let $P : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ be a partition of $[a, b]$. From each $k = 1, 2, \ldots, k$, we choose $\xi_k \in [x_{k-1}, x_k]$, and we put

$$\triangle g(x_k) := g(x_k) - g(x_{k-1}), \quad k = 1, 2, \ldots, n.$$

The sum

$$\sigma(f, P, g) := \sum_{k=1}^{n} f(\xi_k)\triangle g(x_k)$$

is called the *Riemann-Stieltjes sum* of f relative to g for the partition P.

Definition 9.38. For two bounded functions $f,\, g : [a, b] \to \mathbb{R}$, if the following limit exist (which is finite)

$$\lim_{\|P\|\to 0} \sigma(f, P, g) = I \quad \overset{\text{def}}{\Longleftrightarrow}$$

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{P=\{x_k\}_{k=0}^{n}}\ \forall_{\xi_k\in[x_{k-1},x_k]} \quad \|P\| < \delta \quad \Rightarrow \quad \left|\sum_{k=1}^{n} f(\xi_k)\triangle g(x_k) - I\right| < \varepsilon$$

then I is called the *Riemann-Stieltjes integral* of f *relative to* g on $[a, b]$ in the classical sense, and it is denoted by $\displaystyle\int_a^b f(x)dg(x)$. Moreover, if the Riemann-Stieltjes integral of f *relative to* g on $[a, b]$ exists, we will say that f is *Riemann-Stieltjes integrable on* $[a, b]$ *relative to* g in the classical sense.

Let us consider first the case where the function $g : [a, b] \to \mathbb{R}$ is non-decreasing.

Following the same ideas from Chapter 5, we introduce the notion of *lower* and *upper Darboux-Stieltjes sum*: for a partition $P = \{x_k\}_{k=0}^{n}$ of $[a, b]$ we put

$$s(f,P,g) := \sum_{k=1}^{n} m_k \triangle g(x_k), \qquad \text{where} \quad m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}, \;\; k = 1, 2, \dots, n;$$
$$S(f,P,g) := \sum_{k=1}^{n} M_k \triangle g(x_k), \qquad \text{where} \quad M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}, \;\; k = 1, 2, \dots, n$$

One can easily prove that for all partitions P and Q satisfying $P \le Q$ we have the following inequality:

$$s(f,P,g) \le s(f,Q,g) \le S(f,Q,g) \le S(f,P,g).$$

Moreover, we have

Theorem 9.39. *Let $f,\, g : [a,b] \to \mathbb{R}$ be bounded functions and $g : [a,b] \to \mathbb{R}$ is increasing. For every partitions P and Q, we have*

$$s(f,\, P,\, g) \le S(f,\, Q,\, g).$$

Proof: We have

$$s(f,\, P,\, g) \le s(f,\, P \cup Q,\, g) \le S(f,\, P \cup Q,\, g) \le S(f,\, Q,\, g).$$

□

Therefore, we can define the *lower* and *upper Darboux-Stieltjes integrals*:

$$\underline{\int_a^b} f(x)dg(x) := \sup\{s(f,P,g) : P \text{ partition of } [a,b]\}$$
$$\overline{\int_a^b} f(x)dg(x) := \inf\{S(f,P,g) : P \text{ partition of } [a,b]\}$$

Theorem 9.40. *Let $f,\, g : [a,b] \to \mathbb{R}$ be bounded functions and $g : [a,b] \to \mathbb{R}$ is increasing. If*

$$\lim_{\|P\| \to 0} \Big(S(f,P,g) - s(f,P,g) \Big) = 0,$$

then $\lim_{\|P\|\to 0} S(f,P,g)$ and $\lim_{\|P\|\to 0} s(f,P,g)$ exist.

Proof: It is sufficient to show that $\lim_{\|P\|\to 0} S(f,P,g)$ exists. Suppose not. Then there exists $\epsilon_0 > 0$ such that for every $\delta_0 > 0$, there exist partitions P and Q with $\|P\| \le \delta_0$, $\|Q\| \le \delta_0$ and

$$S(f,\, P,\, g) - S(f,\, Q,\, g) > \epsilon_0.$$

It follows that

$$\sup_{\|P\| \le \delta_0} S(f,\, P,\, g) - \inf_{\|Q\| \le \delta_0} S(f,\, Q,\, g)) > \epsilon_0. \tag{9.34}$$

On the other hand, since we have $\lim_{\|P\|\to 0} \Big(S(f,P,g) - s(f,P,g) \Big) = 0$, there exists $\delta^* > 0$ such that $\|P^*\| \le \delta^*$ implies that

$$S(f,\, P^*,\, g) \leq s(f,\, P^*,\, g) + \epsilon_0. \tag{9.35}$$

Take $\delta = \min\{\delta_0,\, \delta^*\}$. From (9.35), we have

$$\begin{aligned}\sup_{\|P\|\leq\delta}(S(f,\, P,\, g) - \inf_{\|P\|\leq\delta} S(f,\, P,\, g)) \leq & \sup_{\|P\|\leq\delta} s(f,\, P,\, g) + \epsilon_0 - \inf_{\|P\|\leq\delta} S(f,\, P,\, g)) \\ =&\epsilon_0 - \left(\inf_{\|P\|\leq\delta} S(f,\, P,\, g) - \sup_{\|P\|\leq\delta} s(f,\, P,\, g)\right) \\ \leq&\epsilon_0, \end{aligned} \tag{9.36}$$

where the last inequality follows from Theorem 9.39. This is a contradiction to (9.34). □

Theorem 9.41. *Let $f,\, g : [a,b] \to \mathbb{R}$ be two bounded function such that g is a non-decreasing function. The Riemann-Stieltjes integral of f relative to g on $[a,b]$ exists if and only if*

$$\lim_{\|P\|\to 0}\Big(S(f,P,g) - s(f,P,g)\Big) = 0.$$

Proof: Suppose that the Riemann-Stieltjes integral I of f relative to g on $[a,b]$ exists. Then for every $\epsilon > 0$, there exists $\delta > 0$ such that $\|P\| \leq \delta$ implies

$$|\sigma(f,\, P,\, \xi,\, g) - I| \leq \epsilon.$$

It follows that

$$\begin{aligned}|S(f,\, P,\, g) - I| &\leq \epsilon,\\ |s(f,\, P,\, g) - I| &\leq \epsilon,\end{aligned}$$

which lead to

$$|S(f,\, P,\, g) - s(f,\, P,\, g)| \leq |S(f,\, P,\, g) - I| + |s(f,\, P,\, g) - I| \leq 2\epsilon.$$

That is, $\lim_{\|P\|\to 0}(S(f,P,g) - s(f,P,g)) = 0$.

Conversely, if $\lim_{\|P\|\to 0}(S(f,P,g) - s(f,P,g)) = 0$, then by Theorem 9.40, there exists $I \in \mathbb{R}$ such that $\lim_{\|P\|\to 0} S(f,P,g) = \lim_{\|P\|\to 0} s(f,P,g) = I$. Applying the Sandwich Theorem to

$$s(f,\, P,\, g) \leq \sigma(f,\, P,\, \xi,\, g) \leq S(f,\, P,\, g)$$

we have $\lim_{\|P\|\to 0} \sigma(f,\, P,\, \xi,\, g) = I$. □

By Theorems 9.40 and 9.41, we have

Corollary 9.42. *Let $f,\, g : [a,b] \to \mathbb{R}$ be two bounded function such that g is a non-decreasing function. If the Riemann-Stieltjes integral of f relative to g on $[a,b]$ in the classical sense exists then we have*

$$s = S.$$

Now we are in the position to consider a more general case that g is of bounded variation on $[a,b]$, when it can be represented as a difference $g(x) = g_1(x) - g_2(x)$, where $g_1,\, g_2 : [a,b] \to \mathbb{R}$ are two non-decreasing functions (see Corollary 9.12). Note that

$$
\begin{aligned}
S(f,P,g)-s(f,P,g) &= \sum_{k=1}^{n}\omega_k \triangle g(x_k)\\
&= \sum_{k=1}^{n}\omega_k \triangle g_1(x_k) - \sum_{k=1}^{n}\omega_k \triangle g_2(x_k)\\
&=: (S(f,\, P,\, g_1)-s(f,\, P,\, g_1)) - (S(f,\, P,\, g_2)-s(f,\, P,\, g_2)).
\end{aligned}
$$

It seems feasible to define the Riemann-Stieltjes integral $\displaystyle\int_a^b f(x)dg(x)$ as

$$
\int_a^b f(x)\mathrm{d}g(x) := \int_a^b f(x)\mathrm{d}g_1(x) - \int_a^b f(x)\mathrm{d}g_2(x). \tag{9.37}
$$

As the representation g_1-g_2 of g is clearly not unique, we need to show that if $g=g_1-g_2=g_1'-g_2'$ where g_1, g_2, g_1' and g_2' are increasing functions on $[a,\, b]$, then

$$
\int_a^b f(x)\mathrm{d}g_1 x - \int_a^b f(x)\mathrm{d}g_2 x = \int_a^b f(x)\mathrm{d}g_1'(x) - \int_a^b f(x)\mathrm{d}g_2'(x),
$$

which is true by Theorem 9.20. In the following we discuss R-S integral of functions with respect to functions of bounded variations in the sense of (9.37).

Theorem 9.43. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function and $g:[a,b]\to\mathbb{R}$ a function of bounded variation. Then the Riemann-Stieltjes integral $\displaystyle\int_a^b f(x)dg(x)$ exists.*

Proof: Let us consider first the case when $g:[a,b]\to\mathbb{R}$ is a non-decreasing function. Therefore, by Theorem 9.41, we need to show that

$$
\lim_{\|P\|\to 0}\Big(S(f,P,g)-s(f,P,g)\Big)=0.
$$

Notice that for a partition $P=\{x_k\}_{k=1}^n$, we have

$$
S(f,P,g)-s(f,P,g)=\sum_{k=1}^{n}(M_k-m_k)\triangle g(x_k).
$$

Put

$$
\omega_k := M_k - m_k, \quad k=1,2,\dots,n,
$$

We need to show that

$$
\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{P=\{x_k\}_{k=1}^n}\quad \|P\|<\delta \ \Rightarrow\ \sum_{k=1}^{n}\omega_k\triangle(x_k)<\varepsilon.
$$

Since $f:[a,b]\to\mathbb{R}$ is continuous and $[a,b]$ is compact, by Heine-Cantor Theorem, f is uniformly continuous, therefore

$$
\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x',x''\in[a,b]}\quad |x'-x''|<\delta \ \Rightarrow\ |f(x')-f(x'')|<\frac{\varepsilon}{g(b)-g(a)},
$$

and consequently, for every partition $P = \{x_k\}_{k=1}^n$ such that $\|P\| \leq \delta$, we have for all $k = 1, 2, \ldots, n$, there exist x', $x'' \in [x_k, x_{k-1}]$ such that

$$\begin{aligned}
\omega_k &= \sup\{f(x) : x \in [x_k, x_{k-}]\} - \inf\{f(x) : x \in [x_k, x_{k-}]\} \\
&= \max\{f(x) : x \in [x_k, x_{k-}]\} - \min\{f(x) : x \in [x_k, x_{k-}]\} \\
&= f(x') - f(x'') < \frac{\varepsilon}{g(b) - g(a)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{k=1}^n \omega_k \triangle(x_k) &< \sum_{k=1}^n \frac{\varepsilon}{g(b) - g(a)} \triangle g(x_k) \\
&= \frac{\varepsilon}{g(b) - g(a)} \sum_{k=1}^n \triangle g(x_k) \\
&= \frac{\varepsilon}{g(b) - g(a)} (g(b) - g(a)) = \varepsilon.
\end{aligned}$$

In the general case, if g is of bounded variation on $[a, b]$, then it can be represented as a difference $g(x) = g_1(x) - g_2(x)$, where g_1, $g_2 : [a, b] \to \mathbb{R}$ are two non-decreasing functions (see Corollary 9.12). Since,

$$\begin{aligned}
\sigma(f, P, g) &= \sum_{k=1}^n f(\xi_k) \triangle g(x_k) \\
&= \sum_{k=1}^n f(\xi_k) \triangle g_1(x_k) - \sum_{k=1}^n f(\xi_k) \triangle g_2(x_k) \\
&=: \sigma(f, P, g_1) - \sigma(f, P, g_2).
\end{aligned}$$

Since each of the sums $\sigma(f, P, g_1)$ and $\sigma(f, P, g_2)$ tends to $\displaystyle\int_a^b f(x)dg_1(x)$ and $\displaystyle\int_a^b f(x)dg_2(x)$ for $\|P\| \to 0$, respectively, we also have that

$$\lim_{\|P\| \to 0} \sigma(f, P, g) = \int_a^b f(x)dg_1(x) + \int_a^b f(x)dg_2(x),$$

so the Riemann-Stieltjes integral $\displaystyle\int_a^b f(x)dg(x)$ exists. □

Theorem 9.44. *Assume that $f : [a, b] \to \mathbb{R}$ is a Riemann integrable function on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is a Lipschitzian function, i.e.*

$$\exists_{L>0}\ \forall_{x', x'' \in [a,b]} \quad |g(x') - g(x'')| \leq L|x' - x''|.$$

Then, the Riemann-Stieltjes integral $\displaystyle\int_a^b f(x)dg(x)$ exists.

Proof: Let us first consider the case when $g : [a, b] \to \mathbb{R}$ is a non-decreasing function. For a partition $P = \{x_k\}_{k=1}^n$, we denote

$$\omega_k := M_k - m_k, \quad k = 1, 2, \dots, n,$$

where

$$M_k := \sup\{f(x) : x \in [x_k, x_{k-1}]\}, \quad m_k := \inf\{f(x) : x \in [x_k, x_{k-1}]\}.$$

Then, (since g is Lipschitzian) we have

$$\sum_{k=1}^{n} \omega_k \triangle g(x_k) \leq L \sum_{k=1}^{n} \omega_k \triangle x_k.$$

By the assumption, f is Riemann integrable, thus

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} \omega_k \triangle x_k = 0,$$

and therefore,

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} \omega_k \triangle g(x_k) = 0,$$

so, the Riemann-Stieltjes integral $\displaystyle\int_a^b f(x)dg(x)$ exists.

Assume now that g is an arbitrary Lipschitzian function (with constant L) of bounded variation. Then, we can write $g(x) = g_1(x) - g_2(x)$, $x \in [a, b]$, where $g_1(x) := Lx$, $g_2 := Lx - g(x)$. We claim that g_1, $g_2 : [a, b] \to \mathbb{R}$ are non-decreasing. Indeed, g_1 is increasing (since $L > 0$), and for all $a \leq x < x' \leq b$ we have

$$g_2(x') - g_2(x) = L(x' - x) - (g(x') - g(x)) \geq 0.$$

In addition g_2 is Lipschitzian (with respect to the constant $2L$). Then,

$$|g_2(x') - g_2(x)| \leq L(x' - x) + |g(x') - g(x)| \leq 2L|x - x'|.$$

Therefore, by the same argument as in the proof of Theorem 9.43, the Riemann-Stieltjes integral $\displaystyle\int_a^b f(x)dg(x)$ exists. □

Theorem 9.45. *Assume that $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is a function such that it can be represented as*

$$g(x) = c + \int_a^x \varphi(t)dt \tag{9.38}$$

where c is a constant, and $|\varphi(t)|$ is Riemann integrable on $[a, b]$. Then, the Riemann-Stieltjes integral $\displaystyle\int_a^b f(x)dg(x)$ exists and

$$\int_a^b f(x)dg(x) = \int_a^b f(x)\varphi(x)dx,$$

where the second integral is the Riemann integral.

Proof: By the assumption, the function $\varphi(x)$ is Riemann integrable, thus it has to be bounded, i.e. there exists $L > 0$ such that $|\varphi(x)| \le L$ fo all $x \in [a,b]$. Thus, for all x, $x' \in [a,b]$, $x > x'$, we have

$$|g(x) - g(x')| = |\int_{x'}^{x} \varphi(t)dt| \le L|x - x'|,$$

which means that g is a Lipschitzian function, therefore the Riemann-Stieltjes $\int_a^b f(x)dg(x)$ exists.

Without loss of generality, we can assume that the function φ is non-negative. Indeed, we can write $\varphi(t) = \varphi_1(t) - \varphi_2(t)$, where

$$\varphi_1(t) = \frac{|\varphi(t)| + \varphi(t)}{2}, \quad \varphi_2(t) = \frac{|\varphi(t)| - \varphi(t)}{2,}$$

where φ_1 and φ_2 are non-negative and Lipschitzian. This implies that $g(x) = g_1(x) - g_2(x)$, where

$$g_1(x) = c + \int_a^x \varphi_1(t)dt, \quad g_2(x) = \int_a^x \varphi_2(t)dt.$$

Since,

$$\int_a^b f(x)dg(x) = \int_a^b f(x)dg_1(x) - \int_a^b f(x)dg_2(x),$$

we can consider the integrals $\int_a^b f(x)dg_1(x)$ and $\int_a^b f(x)dg_2(x)$ separately.

Thus, if $\varphi(t)$ is non-negative function, then for every partition $P = \{x_k\}_{k=1}^n$, we have

$$\sigma(f,P,g) = \sum_{k=1}^{n} f(\xi_k)(g(x_k) - g(x_{k-1})) = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(\xi_k)\varphi(x)dx.$$

On the other hand,

$$\int_a^b f(x)\varphi(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)\varphi(x)dx,$$

thus

$$\sigma(f,P,g) - \int_a^b f(x)\varphi(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} (f(\xi_k) - f(x))\varphi(x)dx.$$

Notice that for all $k = 1,2,\dots,n$, $x_{k-1} \le x \le x_k$, $k = 1,2,\dots,n$, we have

$$|f(\xi_k) - f(x)| < \omega_k = M_k - m_k,$$

therefore,

$$\begin{aligned}\left|\sigma(f,P,g) - \int_a^b f(x)\varphi(t)dx\right| &\le \sum_{k=1}^{n} \omega_k \int_{x_{k-1}}^{x_k} \varphi(x)xdx \\ &= \sum_{k=1}^{n} \omega_k \triangle g(x_k) \\ &= S(f,P,g) - s(f,P,g) \overset{\|P\|\to 0}{\longrightarrow} 0.\end{aligned}$$

Consequently,

$$\lim_{\|P\|\to 0} \sigma(f,P,g) = \int_a^b f(x)\varphi(x)dx,$$

what completes the proof. □

Remark 9.46. Let $g:[a,b]\to\mathbb{R}$ be a continuous function such that the derivative $g'(x)$ exists for all $x\in[a,b]$ except a finite number of points $y_1, y_2, \dots, y_N \in [a,b]$. We can define

$$\varphi(x) := \begin{cases} g'(x) & \text{if } x\neq y_l,\ l=1,2,\dots,N \\ 0 & \text{otherwise.} \end{cases}$$

Then, if the function $|\varepsilon(x)|$ is integrable on $[a,b]$, then by Theorem 8.30, we have

$$g(x) = g(a) + \int_a^x \varphi(t)dt.$$

From Remark 9.46 and Theorem 9.45, we obtain:

Corollary 9.47. *Assume that $f:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$ and the function $g:[a,b]\to\mathbb{R}$ is differentiable in the interval $[a,b]$ except a finite number of points, such that $|g'(x)|$ is integrable on $[a,b]$. Then*

$$\int_a^b f(x)dg(x) = \int_a^b f(x)g'(x)dx.$$

Theorem 9.48. (Integration by Parts Formula) *Let f, $g:[a,b]\to\mathbb{R}$ be two functions such that the integrals $\int_a^b f(x)dg(x)$ and $\int_a^b g(x)df(x)$ exist. Then we have*

$$\int_a^b f(x)dg(x) = f(x)g(x)\Big|_a^b - \int_a^b g(x)df(x). \tag{9.39}$$

Proof: Assume that both integrals $\int_a^b f(x)dg(x)$ and $\int_a^b g(x)df(x)$ exist, and consider a partition $P=\{x_k\}_{k=1}^n$ with a sequence $\xi_k\in[x_{k-1},x_k]$, i.e.

$$a=x_0\le\xi_1\le x_1\le\xi_2\le\cdots\le x_{k-1}\le\xi_k\le x_k\le\xi_{k+1}\le x_{k+1}\le\cdots\le x_{n-1}\le\xi_n\le x_n=b.$$

Then, we have

$$\begin{aligned}
\sigma(f,P,g) &= \sum_{k=1}^n f(\xi_k)(g(x_k)-g(x_{k-1})) = \sum_{k=1}^n f(\xi_k)g(x_k) - \sum_{k=1}^n f(\xi_k)g(x_{k-1}) \\
&= -\left(f(\xi_1)g(a) - \sum_{k=1}^{n-1} f(\xi_k)g(x_k) + \sum_{k=2}^{n} f(\xi_k)g(x_{k-1}) - f(\xi_n)g(b)\right) \\
&= -\left(f(\xi_1)g(a) + \sum_{k=1}^{n-1} g(x_k)\Big(f(\xi_{k+1})-f(\xi_k)\Big) - f(\xi_n)g(b)\right).
\end{aligned} \tag{9.40}$$

Next, we add and subtract

$$f(x)g(x)\Big|_a^b = f(b)g(b) - f(a)g(a)$$

to (9.40),

$$\sigma(f,P,g) = f(x)g(x)\Big|_a^b - \Big(g(a)(f(\xi_1) - f(a)) \\ + \sum_{k=1}^{n-1} g(x_k)\Big(f(\xi_{k+1}) - f(\xi_k)\Big) - g(b)(f(b) - f(\xi_n))\Big)$$

Put $\xi_0 = a$ and $\xi_{n+1} = b$, and denote by Q the partition

$$a = \xi_0 \le \xi_1 < \xi_2 < \dots < \xi_k < \xi_{k+1} < \dots \xi_n \le \xi_{n+1} = b,$$

where $x_k \in [\xi_k, \xi_{k+1}]$. Then we get

$$\sigma(f,P,g) = f(x)g(x)\Big|_a^b - \sigma(g,Q,f).$$

Notice that, if $\|P\| < \delta$, then $\|Q\| < 2\delta$, therefore as $\|P\| \to 0$, then $\|Q\| \to 0$, therefore (by the assumption that the following limits exist

$$\lim_{\|P\|\to 0} \sigma(f,P,g) = \int_a^b f(x)dg(x), \quad \text{and} \quad \lim_{\|Q\|\to 0} \sigma(g,Q,f) = \int_a^b g(x)df(x),$$

we obtain (9.39). □

9.9 Problems

1. Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = x^4 - 4x^3 + 4x^2 + 1, \quad x \in \mathbb{R}.$$

Compute $\operatorname*{Var}_{[0,4]}(g)$.

2. Given the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(x) = x - \lfloor x \rfloor, \quad x \in \mathbb{R}.$$

For given $a < b$, compute $\operatorname*{Var}_{[a,b]}(g)$.

3. Let $g : [a,b] \to \mathbb{R}$ be a function of bounded variation. Show that

$$|g(x)| \le |g(a)| + \operatorname*{Var}_{[a,b]}(g).$$

4. Let $g : [a,b] \to \mathbb{R}$ be a function of bounded variation. Show that

$$\operatorname*{Var}_{[a,b]}(|g|) \le \operatorname*{Var}_{[a,b]}(g).$$

Give an example of a function g such that $\operatorname*{Var}_{[a,b]}(|g|) < \operatorname*{Var}_{[a,b]}(g)$.

5. Let $f, g : [a,b] \to \mathbb{R}$ be two functions of bounded variations. Show that

$$\operatorname*{Var}_{[a,b]}(f+g) \le \operatorname*{Var}_{[a,b]}(f) + \operatorname*{Var}_{[a,b]}(g).$$

Give an example of two functions f and g such that $\operatorname*{Var}_{[a,b]}(f+g) < \operatorname*{Var}_{[a,b]}(f) + \operatorname*{Var}_{[a,b]}(g)$.

6. Evaluate the following Riemann-Stieltjes integrals:

(a): $\displaystyle\int_{-1}^{1} x dg(x)$, where $g(x) = e^{|x|}$ for $x \in [-1,1]$.

(b): $\displaystyle\int_{-1}^{1} f(x) dg(x)$, where $f,\ g : [-1,1] \to \mathbb{R}$ are defined by

$$f(x(= \begin{cases} x^2 & \text{if } \ -1 \le x \le 0 \\ x^3 & \text{if } \quad 0 < x \le 1, \end{cases} \quad g(x) = \begin{cases} 1 & \text{if } \quad x = -1 \\ 2x^2 & \text{if } \ -1 < x < 1 \\ -1 & \text{if } \quad x = 1. \end{cases}$$

7. Let $g(x) = \begin{cases} 0 & \text{for } x \in [0,1] \\ 1 & \text{for } x \in (1,2]. \end{cases}$.

Show that $\displaystyle\int_0^2 g dg$ does not exist.

8. Let $\lfloor x \rfloor$ denote the largest integer less than or equal to the number x. Find the value of

$$\int_0^a (x^2+1) d(\lfloor x \rfloor).$$

where $a > 2$ is an arbitrary non-integer number.

9. (a): Let $g : [0,a] \to \mathbb{R}$ be the function $g(x) = x^\alpha$, where $\alpha \in (0,1)$. Show that g is of bounded variation but it is **not Lipschitzian**. Compute $\operatorname*{Var}_{[0,a]}(g)$

(b): Define $g : [0,\pi] \to \mathbb{R}$ by $g(x) = \sin x$. Find the explicit formula for the function $m : [0,\pi] \to \mathbb{R}$ defined by

$$m(x) := \operatorname*{Var}_{[0,x]}(g), \quad x \in [0,\pi].$$

10. Compute $\operatorname*{Var}_{[a,b]}(f)$ for the following functions $f : [a,b] \to \mathbb{R}$

(a) $f(t) = \begin{cases} 1 & \text{for } \ t = a; \\ 2 & \text{for } a < t < b; \\ 3 & \text{fort } t = b, \end{cases}$

(b) $f(t) = e^{t^2}$, where $a < 0 < b$.

11. Let $a < c < b$ be given numbers, $f : [a,b] \to \mathbb{R}$ a continuous function, and $g : [a,b] \to \mathbb{R}$ be defined by

$$g(t) = \begin{cases} \alpha & \text{for } \quad t \le c \\ \beta & \text{for } t > c, \end{cases}, \quad \text{where } \ \alpha \ne \beta.$$

Compute the Riemann-Stieltjes integral

$$\int_a^b f(t)dg(t).$$

12. Compute:

(a) $V_0^4(f)$ for $f(t) = \begin{cases} t^2 & \text{for } \ 0 \le t \le 1; \\ 4 & \text{for } 1 < t \le 2; \\ 4 - t & \text{for } 2 < t \le 4. \end{cases}$

(b) $V_0^{\sqrt{\pi}}(f)$ for $f(t) = \sin t^2$.

13. Consider the function $f : [0,2] \to \mathbb{R}$, $f(t) = e^{t^2}$ and the function $g : [0,2] \to \mathbb{R}$ be defined by

$$g(t) = \begin{cases} t^2 & \text{for } \ 0 \le t \le 1 \\ 2 & \text{for } 1 < t \le 2, \end{cases}.$$

Compute the Riemann-Stieltjes integral

$$\int_0^2 f(t)dg(t).$$

14. Compute $\operatorname*{Var}_{[0,\pi]}(f)$ for $f(x) = \sin^2(nx)$, where $n \in \mathbb{N}$.

15. Given

$$f(x) = \begin{cases} \alpha & \text{if } 0 \le x < 1 \\ \beta & \text{if } 1 \le x < 2 \\ \gamma & \text{if } 2 \le x \le 3. \end{cases}$$

Find $\operatorname*{Var}_{[0,3]}(f)$.

10

Lebesgue Integration

10.1 Introduction

In the Chapter on Riemann integral, we firstly defined Riemann integral as an extension to the concept of area and then studied its properties and applications. Unfortunately we cannot follow the same procedure to introduce Lebesgue integral, for that we need some motivation to develop Lebesgue integral and that the development will be much more complicated than establishing Riemann integral.

Let's recall that the Dirichlet function $D : [0,\, 1] \to \mathbb{R}$

$$D(x) = \begin{cases} 1 & \text{if } x \in [0,\, 1] \cap \mathbb{Q}, \\ 0 & \text{if } x \in [0,\, 1] \setminus \mathbb{Q}, \end{cases}$$

where $\mathbb{Q}$ is the set of all rational numbers, is not Riemann integrable, because for every partition $P = \{x_0 = 0,\, x_1,\, x_2,\, \cdots,\, x_n = 1\}$ of $[0,\, 1]$ we have the difference of Darboux sums

$$\begin{aligned} S(f,\, P) - s(f,\, P) &= \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} (x_k - x_{k-1}) \\ &= 1, \end{aligned} \tag{10.1}$$

where $M_k = \sup_{x \in [x_{k-1},\, x_k]} D(x)$ and $m_k = \inf_{x \in [x_{k-1},\, x_k]} D(x)$, and clearly $S(f,\, P) - s(f,\, P)$ is not tending to zero as $\|P\| \to 0$.

In addition to the above example, certainly there are many other scenarios which exhibit the limitations of Riemann integral. A natural question is that can we design a new integral such that the Diriclet function becomes integrable and is still an extension to area? The answer is affirmative. Let us investigate the equalities at (10.1). The non-integrability of the Dirichlet function is that the amplitude $M_k - m_k$ is constantly equal to 1 on every interval $[x_{k-1},\, x_k]$ and leads to nonzero limit of the difference of Darboux sums as $\|P\| \to 0$. Based on this observation, we want to partition the interval $[0,\, 1]$ in such a way that the amplitude of D on each part of the partition is controllable, instead of letting the amplitude be passively determined by the partition on $[0,\, 1]$. A straight forward method is that we actively partition the range of D so that this partition induces a partition in the domain $[0, 1]$. Then the amplitude of D is fully controllable, leaving the induced partition of the domain $[0, 1]$ for further analysis. This further analysis will be complicated and is part of the answer to the question why we cannot develop Lebesgue integral in the same manner for Riemann integral.

The range of the Diriclet function D is simple enough to contain only two values $\{0, 1\}$. We regard $P := \{0, 1\}$ as a partition of the range of D and then the induced partition of the domain $[0, 1]$ is

$$\begin{aligned}[0, 1] =& \{x : D(x) = 0, x \in [0, 1]\} \cup \{x : D(x) = 1, x \in [0, 1]\} \\ =& \{x : x \in [0, 1] \setminus \mathbb{Q}\} \cup \{x : x \in [0, 1] \cap \mathbb{Q}\} \\ =& ([0, 1] \setminus \mathbb{Q}) \cup ([0, 1] \cap \mathbb{Q}),\end{aligned}$$

where $[0, 1]$ is partitioned into two parts $[0, 1] \setminus \mathbb{Q}$ and $[0, 1] \cap \mathbb{Q}$. It is clear that on each part of $[0, 1]$, the amplitude of D is small (zero). Then the area under the graph of D is

$$\begin{aligned}A =& 0 \cdot \text{Length of } [0, 1] \setminus \mathbb{Q} + 1 \cdot \text{Length of } [0, 1] \cap \mathbb{Q} \\ =& 1 \cdot \text{Length of } [0, 1] \cap \mathbb{Q}.\end{aligned}$$

where the "Length of $[0, 1] \cap \mathbb{Q}$" requires careful measurement. We can notice that the region under the graph of D is partition into two "rectangles" with broken bases since $[0, 1] \setminus \mathbb{Q}$ and $[0, 1] \cap \mathbb{Q}$ are not intervals anymore.

Before we turn to the measurement of $[0, 1] \setminus \mathbb{Q}$, let us consider a more general function $f : [a, b]] \to \mathbb{R}$ which satisfies

$$m \leq f(x) \leq M$$

for every $x \in [a, b]$, where m and M are constants. In order to control the amplitude of f such that f does not change much on each part of certain partition of $[a, b]$. We partition the super set $[m, M]$ for the range of f. That is, let $P = \{m = y_0, y_1, y_2, \cdots, y_n = M\}$ be a partition of $[m, M]$. Then the partition P on the range of f induces a partition in the domain $[a, b]$ as follows:

$$[a, b] = \cup_{k=1}^{n} E_k,$$

where

$$E_k = \{x : x \in [a, b], y_{k-1} \leq f(x) < y_k\}.$$

For every $k \in \{1, 2, \cdots\}$, choose $\xi_k \in [y_{k-1}, y_k]$. Then

$$A_n = \sum_{k=1}^{n} \xi_i \cdot (\text{Length of } E_k)$$

is an approximation of the area of the region under the graph of f, provided that "Length of E_k", $k \in \{1, 2, \cdots\}$, are well-defined. Moreover, if the limit

$$\lim_{\|P\| \to 0} A_n = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \xi_i \cdot (\text{Length of } E_k)$$

exists, we obtain the area of the region under the graph of f and call it the "new" integral of f.

It turns out that we need to extend our definition of "length" for intervals to more complicated sets in $\mathbb{R}$ such as $[0, 1] \cap \mathbb{Q}$. If the "Length of $[0, 1] \cap Q$" and "Length of $[0, 1] \setminus \mathbb{Q}$" are well-defined, the Dirichlet function D will become integrable! In the following sections, we will illustrate how to extend the definition of "length" for families of sets in $\mathbb{R}$ for a better integral than Riemann integral in the sense that more functions are integrable under the new integral.

10.2 Outer Measure

In $\mathbb{R}$ space we can define the length of an interval I with endpoints a and b with $a \le b$ (i.e., $I = (a, b)$, or $[a, b)$, or $(a, b]$, or $[a, b]$) as follows:

$$|I| := b - a.$$

More generally, for an interval $I \subset \mathbb{R}^d$ with

$$I := \{x = (x_1, x_2, \cdots, x_d) : x_i \in I_i \subset \mathbb{R}, i = 1, 2, \cdots, d\},$$

where each I_i is an interval in $\mathbb{R}$, the length is defined to be

$$|I| := \Pi_{i=1}^{d} |I_i|.$$

For convenience of discussion, we consider subsets in $\mathbb{R}$ only. But the discussion can be extended to $\mathbb{R}^d$, $d \in \mathbb{N}$ with a little effort.

It has been noted that the definition of length does not work for more complicated sets which are not intervals. The most ambitious effort is to find a function $\mu : 2^{\mathbb{R}} \to [0, +\infty]$, where we use $[0, +\infty]$ to mean the function value can be extended to $+\infty$, such that

1) $\mu(E)$ is defined for every $E \in 2^{\mathbb{R}}$;
2) For an interval I, $\mu(I) = |I|$;
3) If $\{E_i\}_{i=1}^{+\infty}$ is a sequence of disjoint sets, then

$$\mu\left(\bigcup_{i=1}^{+\infty} E_i\right) = \sum_{i=1}^{+\infty} \mu(E_i);$$

4) μ is translation invariant. That is, for every $y \in \mathbb{R}$ and $E \subseteq \mathbb{R}$, we have

$$\mu(E + y) = \mu(E).$$

Unfortunately it is impossible to find such a (set) function μ which satisfies all of the above four conditions. Later we will show that we can a derive contradiction if such an μ existed. One compromise is not to define μ for all subsets of $\mathbb{R}$ but only for a certain class of subsets $\mathfrak{M} \subseteq 2^{\mathbb{R}}$. Namely, we attempt to define $\mu : \mathfrak{M} \to [0, +\infty]$ which satisfies the conditions 1)– 4). An immediate question is how to find an $\mathfrak{M}$? One method to answer this question is to use the so-called outer measure which we will define soon. The idea of using outer measure is to define a function $\mu^* : 2^{\mathbb{R}} \to [0, +\infty]$ for every set in $2^{\mathbb{R}}$ which satisfies 1), 2), 4) and a weakened version of 3):

3') If $\{E_i\}_{i=1}^{+\infty}$ is a sequence of disjoint sets, then

$$\mu^*\left(\bigcup_{i=1}^{+\infty} E_i\right) \le \sum_{i=1}^{+\infty} \mu^*(E_i),$$

which is called "subadditivity". It turns out there exists such an outer measure, which serves as an machinery to define a family of sets $\mathfrak{M} \subset 2^{\mathbb{R}}$ such that the restriction $\mu^*|_{\mathfrak{M}}$ satisfies the conditions 1), 2), 3) and 4). The set function $\mu := \mu^*|_{\mathfrak{M}} : \mathfrak{M} \to [0, +\infty]$ is the extended notion of "Length" which we are seeking for and we call it measure. Every set in $\mathfrak{M}$ is then called measurable in the sense of the measure μ.

Definition 10.1. The outer measure of $A \subset \mathbb{R}$ is defined to be

$$\mu^*(A) = \inf\left\{\sum_{k=1}^{+\infty} |I_k| : A \subset \bigcup_{k=1}^{+\infty} I_k, \text{ where } I_k \text{ is an interval in } \mathbb{R}\right\}.$$

It is clear that μ^* is defined for every set in $2^{\mathbb{R}}$ and satisfies 1), 2) and 4). To verify 3'), we have

Proposition 10.2. *For any sequence of subsets $\{A_n\}_{n=1}^{\infty}$ of $\mathbb{R}$ we have*

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu^*(A_n). \tag{10.2}$$

Proof: Let $\varepsilon > 0$. For every $n \in \mathbb{N}$, since (by definition of infimum)

$$\mu^*(A_n) = \inf\left\{\sum_{k=1}^{\infty} \mu(I_k^n) : A_n \subset \bigcup_{k=1}^{\infty} I_k^n\right\},$$

we can find the sequence of intervals $\{I_k^n\}_{k=1}^{\infty}$ such that

$$A_n \subset \bigcup_{k=1}^{\infty} I_k^n, \quad \text{and} \quad \sum_{k=1}^{\infty} \mu(I_k^n) \le \mu^*(A_n) + \frac{\varepsilon}{2^n}.$$

Since the collection $\{I_k^n\}_{k,l=1}^{\infty}$ is countable and $\displaystyle\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{k,l}^{\infty} I_k^n$, we obtain

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{k,n=1}^{\infty} \mu(I_k^n) = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \mu(I_k^n)\right) \le \sum_{n=1}^{\infty}\left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon.$$

Since the inequality

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon,$$

is true for every $\varepsilon > 0$, therefore

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu^*(A_n).$$

□

For finite union of sets, we have

Proposition 10.3. *For every $A \subseteq \mathbb{R}$ and $B \subset \mathbb{R}$, we have $\mu^*(A \cup B) \le \mu^*(A) + \mu^*(B)$.*

Proof: Let $\{I_i\}_{i=1}^{+\infty}$ and $\{J_i\}_{i=1}^{+\infty}$ be sequences of intervals such that $A \subset \cup_{i=1}^{+\infty} J_i$ and $B \subset \cup_{i=1}^{+\infty} I_i$, then we have $A \cup B \subset (\cup_{i=1}^{+\infty} I_i) \cup (\cup_{i=1}^{+\infty} J_i)$ and hence $\mu^*(A \cup B) \le \sum_{i=1}^{+\infty} |I_i| + \sum_{i=1}^{+\infty} |J_i|$. Then by definition of outer measure we have $\mu^*(A \cup B) \le \mu^*(A) + \mu^*(B)$. □

Another consequence of outer measure is monotonicity property

Proposition 10.4. *For every $A \subseteq B \subset \mathbb{R}$, we have $\mu^*(A) \le \mu^*(B)$.*

Proof: Let $\{I_i\}_{i=1}^{+\infty}$ be a sequence of intervals such that $B \subset \cup_{i=1}^{+\infty} I_i$, then we have $A \subset \cup_{i=1}^{+\infty} I_i$ and hence $\mu^*(A) \leq \sum_{i=1}^{+\infty} |I_i|$. Then by definition of outer measure we have $\mu^*(A) \leq \mu^*(B)$. □

In order to find the class $\mathfrak{M}$ of measurable sets, we want to borrow some idea for Riemann integrability. Recall that a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if the Darboux lower and upper sums have the same limit as the diameter of the partition approaches zero. Namely, the outer approximation of the area has the same limit of the inner approximation. Noticing that the outer measure of a set A is exactly the outer approximation of the length. It is natural to think of defining an inner measure μ_* which is the inner approximation of the length, such that $\mu^*(A) = \mu_*(A)$ implies measurability of A. However, it happens that it is not convenient to define the inner measure $\mu_*(A)$ directly as

$$\sup \left\{ \sum_{k=1}^{+\infty} |I_k| : A \supset \bigcup_{k=1}^{+\infty} I_k, \text{ where } I_k \text{ is an interval in } \mathbb{R} \right\}. \tag{10.3}$$

Let us illustrate the reason by the following example.

Example 10.5. The sets $[0, 1] \cap \mathbb{Q}$ and $[0, 1] \setminus \mathbb{Q}$ have empty interior sets because of the density of rational numbers in $[0, 1]$. Therefore they both do not contain any intervals. Suppose such an inner measure (10.3) is defined, then we have

$$\mu_*([0, 1] \cap \mathbb{Q}) = 0, \ \mu_*([0, 1] \setminus \mathbb{Q}) = 0.$$

If $\mu^*(A) = \mu_*(A)$ implies measurability of A and denote the measure by $\mu(A) = \mu^*(A)$, then at least one of the sets $[0, 1] \cap \mathbb{Q}$ and $[0, 1] \setminus \mathbb{Q}$ becomes not measurable. Otherwise, by 1) and 3) we have

$$\begin{aligned} \mu([0, 1]) &= \mu([0, 1] \cap Q) + \mu([0, 1] \setminus \mathbb{Q}) \\ &= \mu_*([0, 1] \cap \mathbb{Q}) + \mu_*([0, 1] \setminus \mathbb{Q}) \\ &= 0. \end{aligned}$$

This is a contradiction to 2) which requires the measure of an interval to be its length. This at least shows that we still cannot discuss the integrability of the Dirichlet function.

It turns out that we can define inner measure in an indirect manner. If we only consider measure on 2^{I_0} where $I_0 \subset \mathbb{R}$ is a bounded interval, then we can define the inner measure of $A \subset I_0$ as follows:

Definition 10.6. Let $I_0 = [a, b]$ be a bounded interval of $\mathbb{R}$. The inner measure of every subset A of I_0 is defined by

$$\mu_*(A) = |I_0| - \mu^*(I_0 \setminus A). \tag{10.4}$$

Namely we approximate the length of A by means of the outer measure of its compliment. By the subadditivity of outer measure, we know that $\mu^*(I_0) = |I_0| \leq \mu^*(A) + \mu^*(I_0 \setminus A)$ which implies that $|I_0| \leq \mu^*(A) - \mu^*(I_0 \setminus A) \leq \mu^*(A)$. That is, we have

Proposition 10.7. *For every set $A \subset I_0 = [a, b]$ we have*

$$\mu_*(A) \leq \mu^*(A).$$

With the definition of inner measure at (10.4), let us return to the discussion of the sets $[0,\,1]\cap\mathbb{Q}$ and $[0,\,1]\setminus\mathbb{Q}$ under the assumption that $\mu^*(A)=\mu_*(A)$ implies measurability of the set A.

Example 10.8. The set $[0,\,1]\cap\mathbb{Q}$ is countable which can be written as $\{r_i\}_{i=1}^{+\infty}$. Clearly each single point set $\{r_i\}$, $i=1,\,2,\,\cdots$, is measurable with measure zero and hence by Propositions 10.4 and 10.7, we have

$$0\le\mu_*([0,\,1]\cap\mathbb{Q})\le\mu^*([0,\,1]\cap\mathbb{Q})\le\sum_{i=1}^{+\infty}\mu(\{r_i\})=0.$$

It follows that $\mu^*([0,\,1]\cap\mathbb{Q})=\mu_*([0,\,1]\cap\mathbb{Q})=0$, which means that the set $[0,\,1]\cap\mathbb{Q}$ is measurable.

Next we consider $[0,\,1]\setminus\mathbb{Q}$. We have $\mu_*([0,\,1]\setminus\mathbb{Q})=1-\mu^*([0,\,1]\cap\mathbb{Q})=1-0=1$. Moreover, by Propositions 10.4 and 10.7 we have

$$1=\mu^*([0,\,1])\ge\mu^*([0,\,1]\setminus\mathbb{Q})\ge\mu_*([0,\,1]\setminus\mathbb{Q})=1.$$

Therefore we have $\mu^*([0,\,1]\setminus\mathbb{Q})\ge\mu_*([0,\,1]\setminus\mathbb{Q})=1$ which means $[0,\,1]\setminus\mathbb{Q}$ is also measurable. This shows that the Dirichlet function is integrable in the sense of the new definition of measure.

Definition 10.9. A set $A\subset I_0=[a,\,b]$ is Lebesgue measurable if

$$\mu^*(A)=\mu_*(A).$$

Let $\mathfrak{M}_0$ be the family of Lebesgue measurable sets in I_0. We define Lebesgue measure $\mu:\mathfrak{M}_0\to[0,\,+\infty)$ by

$$\mu(A)=\mu^*(A),\ A\in\mathfrak{M}_0.$$

A conspicuous condition in the definition of inner measure is that we only consider inner measure for family of subsets of bounded set $I_0=[a,\,b]$. Certainly we could extend inner measure to $2^{\mathbb{R}}$ by writing $\mathbb{R}$ into a countable union of bounded sets. In the following we introduce the Caratheodory condition for measurability without using inner measure and show that in the case of bounded set $I_0=[a,\,b]$, it is equivalent to Definition 10.9.

Proposition 10.10. *Let $A\subset I_0$. The following conditions are equivalent*

(a) $\mu^*(A)=\mu_*(A)$*;*
(b) (Caratheodory Condition) $\mu^*(B)=\mu^*(B\cap A)+\mu^*(B\setminus A)$ *for all* $B\subset I_0$*.*

Proof: Notice that the condition (b) implies (a). Indeed, take $B=I_0$, then we get

$$|I_0|=\mu^*(I_0)=\mu^*(I_0\cap A)+\mu^*(I_0\setminus A)=\mu^*(A)+\mu^*(I_0\setminus A),$$

which is equivalent to $\mu_*(A)=\mu^*(A)$.

The proof that (a) implies (b) will be done in three steps:
Step 1: Consider first the case $A=J$, where $J\subset I_0$ is an interval. Then it is clear that for any interval I we have

$$|I|=|I\cap J|+|I\setminus J|.$$

Suppose therefore that for an $\varepsilon>0$, $\{I_k\}_{k=1}^{\infty}$ is a cover of B such that

$$\sum_{k=1}^{\infty} |I_k| < \mu^*(B) + \varepsilon.$$

Then,

$$\begin{aligned}
\mu^*(B \cap J) + \mu^*(B \setminus J) &\le \sum_{k=1}^{\infty} |I_k \cap J| + \sum_{k=1}^{\infty} |I_k \setminus J| \\
&= \sum_{k=1}^{\infty} \Big(|I_k \cap J| + |I_k \setminus J| \Big) = \sum_{k=1}^{\infty} |I_k| < \mu(B) + \varepsilon.
\end{aligned}$$

Since, $\varepsilon > 0$ is an arbitrary number, it follows that $\mu^*(B \cap J) + \mu^*(B \setminus J) \le \mu^*(B)$, and consequently, by the sub-additivity property of the outer measure (cf. Proposition 10.2), we have

$$\mu^*(B \cap J) + \mu^*(B \setminus J) = \mu^*(B) \quad \text{for all } \ B \subset I_0. \tag{10.5}$$

Step 2: Assume now that $A \subset I_0$ satisfies the condition (a), which is equivalent to

$$|I_0| = \mu^*(A) + \mu^*(I_0 \setminus A).$$

In addition, by (10.5), we have

$$\mu^*(A \cap J) + \mu^*(A \setminus J) = \mu^*(A)$$

and

$$\mu^*((I_0 \setminus A) \cap J) + \mu^*((I_0 \setminus A) \setminus J) = \mu^*(I_0 \setminus A).$$

Therefore, we have

$$\begin{aligned}
\mu^*(I_0) &= \mu^*(A) + \mu^*(I_0 \setminus A) \\
&= \mu^*(A \cap J) + \mu^*(A \setminus J) + \mu^*((I_0 \setminus A) \cap J) + \mu^*((I_0 \setminus A) \setminus J) \\
&= \Big(\mu^*(A \cap J) + \mu^*((I_0 \setminus A) \cap J) \Big) + \Big(\mu * (A \setminus J) + \mu^*((I_0 \setminus A) \setminus J) \Big) \\
&= \Big(\mu^*(J \cap A) + \mu^*(J \setminus A) \Big) + \Big(\mu^*((I_0 \setminus J) \cap A) + \mu * ((I_0 \setminus J) \setminus A) \Big) \\
&\ge \mu^*(J) + \mu^*(I_0 \setminus J) = |J| + |I_0 \setminus J| = |I_0|.
\end{aligned}$$

However, the last inequality has to be an equality, thus

$$\begin{aligned}
\mu^*(A \cap J) + \mu^*((I_0 \setminus A) \cap J) &= |J| \\
\mu^*((I_0 \setminus J) \cap A) + \mu^*((I_0 \setminus J) \setminus A) &= |I_0 \setminus J|.
\end{aligned} \tag{10.6}$$

Step 3: Now we can show that the condition (a) implies (b). Suppose that $A \subset I_0$ satisfies (a), $B \subset I_0$ be an arbitrary set and let $\varepsilon > 0$. Then there exists a cover $\{I_k\}_{k=1}^{\infty}$ of B such that

$$\sum_{k=1}^{\infty} |I_k| < \mu^*(B) + \varepsilon.$$

Then we have, by (10.6), that for every $k \in \mathbb{N}$, $|I_k| = \mu^*(A \cap I_k) + \mu^*(I_k \setminus A)$, therefore

$$\mu^*(B\cap A)+\mu^*(B\setminus A)\le \mu^*\left(\bigcap_{k=1}^{\infty} I_k\cap A\right)+\mu^*\left(\bigcap_{k=1}^{\infty} I_k\setminus A\right)$$
$$\le \sum_{k=1}^{\infty}(\mu^*(I_k\cap A))+\mu^*(I_k\setminus A))=\sum_{k=1}^{\infty}|I_k|\le \mu^*(B)+\varepsilon$$

Since $\varepsilon>0$ was chosen arbitrary, we obtain that

$$\mu^*(B\cap A)+\mu^*(B\setminus A)=\mu^*(B).$$

□

Definition 10.11. A set $A\subset\mathbb{R}$ is Lebesgue measurable if for every $B\subset\mathbb{R}$, we have

$$\mu^*(B)=\mu^*(B\cap A)+\mu^*(B\setminus A).$$

Let $\mathfrak{M}$ be the family of Lebesgue integrable sets in $\mathbb{R}$. We define Lebesgue measure $\mu:\mathfrak{M}\to[0,+\infty]$ by

$$\mu(A)=\mu^*(A),\ A\in\mathfrak{M}.$$

We notice that the Carathéodory condition for measurability does not involve inner measure and can deal with unbounded sets in $\mathbb{R}$, even though it is not intuitively obvious.

Let us bear in mind that it still remains to show that the measure defined at Definition **??** is indeed a measure satisfying conditions 1), 2), 3) and 4) on the family of measurable sets $\mathfrak{M}$, while 1) and 2) are clearly satisfied. We undertake this task to verify 3) and 4) during the discussion of the next section. Let us close this section by the following property of outer measure.

Proposition 10.12. *Let $E\subset\mathbb{R}$ be a set. Then we have*

$$\mu^*(E)=\inf\{\mu^*(G): G \text{ is open with } G\supset E\}.$$

Proof: For every open set $G\supset E$, by Propositions 10.4, we have $\mu^*(G)\ge\mu^*(E)$. If $\mu^*(E)=+\infty$, then we are done. If $\mu^*(E)<+\infty$, for every $\epsilon>0$, there exists a sequence of intervals $\{I_i\}_{i=1}^{+\infty}$ such that $\sum_{i=1}^{+\infty} I_i\supset E$ and

$$\sum_{i=1}^{+\infty}|I_i|<\mu^*(E)+\frac{\epsilon}{2}. \tag{10.7}$$

For every $i\in\mathbb{N}$, there exists an open set $J_i\supset I_i$ such that

$$|J_i|<|I_i|+\frac{\epsilon}{2^{i+1}}. \tag{10.8}$$

Then we have $\cup_{i=1}^{+\infty}J_i\supset\cup_{i=1}^{+\infty}I_i$ and

$$\sum_{i=1}^{+\infty}|J_i|<\sum_{i=1}^{+\infty}|I_i|+\frac{\epsilon}{2}. \tag{10.9}$$

Put $G=\cup_{i=1}^{+\infty}I_i$. Then G is open with $G\supset E$ and $\mu^*(G)\ge\mu^*(E)$. Moreover, by (10.7), (10.8) and (10.9) we have

$$\begin{aligned}\mu^*(G) &\le \sum_{i=1}^{+\infty}|J_i| \\ &< \sum_{i=1}^{+\infty}|I_i| + \frac{\epsilon}{2} \\ &< \mu^*(E) + \epsilon.\end{aligned}$$

Then we have $\mu^*(E) = \inf\{\mu^*(G) : G \text{ is open with } G \supset E\}$. □

10.3 Measurable Sets

We are now in the position to discuss properties of measurable sets including the countable additivity of Lebesgue measure. We will also investigate how to describe the family of Lebesgue measurable sets in $\mathbb{R}$.

Proposition 10.13. *For every $E \subset \mathbb{R}$, if $\mu^*(E) = 0$, then E is measurable.*

Proof: For every $A \subset \mathbb{R}$, by subadditivity of outer measure (Proposition 10.3), we have

$$\mu^*(A) \le \mu^*(A \cap E) + \mu^*(A \cap E^c),$$

where E^c stands for its compliment. We have only to show that $\mu^*(A) \ge \mu^*(A \cap E) + \mu^*(A \cap E^c)$. Indeed, we have $E \supset A \cap E$ and $A \supset A \cap E^c$ which by Proposition 10.4 lead to

$$\begin{aligned}0 = \mu^*(E) &\ge \mu^*(A \cap E) \\ \mu^*(A) &\ge \mu^*(A \cap E^c),\end{aligned}$$

and hence to $\mu^*(A) \ge \mu^*(A \cap E) + \mu^*(A \cap E^c)$. □

The following two propositions concern necessary and sufficient condition for measurable sets.

Proposition 10.14. *For every $E \subset \mathbb{R}$, E is measurable if and only if E^c is measurable.*

Proof: Since $E = (E^c)^c$, the conclusion follows by the Carathéodory condition for the definition of measurable set. □

Proposition 10.15. *For every $E \subset \mathbb{R}$, E is measurable if and only if for every $A \subseteq E$ and $B \subseteq E^c$ we have $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.*

Proof: Suppose E is measurable. Then for every $T \subset \mathbb{R}$, we have $\mu^*(T) = \mu^*(T \cap E) + \mu^*(T \cap E^c)$. Take $T = A \cup B$. Then we have

$$\begin{aligned}T \cap E =& (A \cup B) \cap E = A, \\ T \cap E^c =& (A \cup B) \cap E^c = B.\end{aligned}$$

Therefore, we have $\mu^*(A \cup B) = \mu^*(T) = \mu^*(T \cap E) + \mu^*(T \cap E^c) = \mu^*(A) + \mu^*(B)$.

Now we assume that if for every $A \subseteq E$ and $B \subseteq E^c$ we have $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$. Then for every $T \subset \mathbb{R}$, let $A = T \cap E$ and $B = T \cap E^c$. Then we have $A \cup B = T$ and

$$\mu^*(T) = \mu^*(A \cup B) = \mu^*(A) + \mu^*(B) = \mu^*(T \cap E) + \mu^*(T \cap E^c),$$

which means $\mu^*(T) = \mu^*(T \cap E) + \mu^*(T \cap E^c)$ and E is measurable. □

To prove the additivity of Lebesgue measure, we need

Proposition 10.16. *If E_1 and E_2 are measurable then $E_1 \cup E_2$ is also measurable.*

Proof: We only have to show that for every $A \subset \mathbb{R}$ we have

$$\mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1 \cup E_2)^c) \leq \mu^*(A).$$

Since E_1 is measurable, we have

$$\mu^*(A \cap E_1) + \mu^*(A \cap E_1^c) = \mu^*(A). \tag{10.10}$$

In order to relate $A \cap E_1^c \cap E_2^c$, we consider $A \cap E_1^c$ and E_2. E_2 is measurable implies that

$$\mu^*(A \cap E_1^c) = \mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap E_1^c \cap E_2^c). \tag{10.11}$$

Then by (10.10) and (10.11) we have

$$\mu^*(A \cap E_1) + \mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap E_1^c \cap E_2^c) = \mu^*(A). \tag{10.12}$$

It breaks down to show that

$$\mu^*(A \cap (E_1 \cup E_2)) \leq \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c \cap E_2)$$

which is indeed true since we have

$$A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2) = (A \cap E_1) \cup (A \cap E_1^c \cap E_2).$$

□

An immediate consequence is of Proposition 10.16 is that the union of finite number of measurable sets is measurable. Moreover, we have

Corollary 10.17. *Let E_1 and E_2 be measurable sets and $E_1 \cap E_2 = \emptyset$. Then for every $T \subset \mathbb{R}$ we have*

$$\mu^*(T \cap (E_1 \cup E_2)) = \mu^*(T \cap E_1) + \mu^*(T \cap E_2).$$

Proof: Since E_1, E_2 are measurable, by Proposition 10.10, $E_1 \cup E_2$ is measurable. Note that $T \cap E_1 \subseteq E_1$, $T \cap E_2 \subseteq E_2 \subseteq E_1^c$. Then by Proposition 10.15, we have

$$\mu^*(T \cap (E_1 \cup E_2)) = \mu^*((T \cap E_1) \cup (T \cap E_2)) = \mu^*(T \cap E_1) + \mu^*(T \cap E_2).$$

□

Choose $T = E_1 \cup E_2$ in Corollary 10.17, we have

Corollary 10.18. *Let E_1 and E_2 be measurable sets and $E_1 \cap E_2 = \emptyset$. Then $E_1 \cup E_2$ is measurable and*

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2).$$

Now we are ready to prove the following additivity of Lebesgue Measure.

Theorem 10.19. *Let $\{E_i\}_{i=1}^{+\infty}$ be a sequence of measurable sets and $E_i \cap E_j = \emptyset$ with $i \neq j$. Then $\cup_{i=1}^{+\infty} E_i$ is measurable and $\mu(\cup_{i=1}^{+\infty} E_i) = \sum_{i=1}^{+\infty} \mu(E_i)$*

Proof: By Proposition 10.10, we know that for every $k \in \mathbb{N}$, $\cup_{i=1}^{k} E_i$ is measurable. For every $T \subset \mathbb{R}$, by Corollary 10.18, we have

$$\mu^*\left(T \cap \left(\cup_{i=1}^{k} E_i\right)\right) = \sum_{i=1}^{k} \mu^*(T \cap E_i).$$

Therefore we have

$$\begin{aligned} \mu^*(T) &= \mu^*\left(T \cap \left(\cup_{i=1}^{k} E_i\right)\right) + \mu^*(T \cap \left(\cup_{i=1}^{k} E_i\right)^c) \\ &= \sum_{i=1}^{k} \mu^*(T \cap E_i) + \mu^*(T \cap \left(\cup_{i=1}^{k} E_i\right)^c) \\ &\geq \sum_{i=1}^{k} \mu^*(T \cap E_i) + \mu^*(T \cap (\cup_{i=1}^{\infty} E_i)^c). \end{aligned} \tag{10.13}$$

Let $k \to +\infty$ in (10.13) we have

$$\mu^*(T) \geq \sum_{i=1}^{+\infty} \mu^*(T \cap E_i) + \mu^*(T \cap (\cup_{i=1}^{\infty} E_i)^c) \tag{10.14}$$

$$\geq \mu^*(T \cap \left(\cup_{i=1}^{+\infty} E_i\right)) + \mu^*(T \cap (\cup_{i=1}^{\infty} E_i)^c). \tag{10.15}$$

Note that $T = \left(T \cap \left(\cup_{i=1}^{+\infty} E_i\right)\right) \cup \left(T \cap \left(\cup_{i=1}^{+\infty} E_i\right)^c\right)$. It follows that

$$\mu^*(T) \leq \mu^*(T \cap \left(\cup_{i=1}^{+\infty} E_i\right)) + \mu^*(T \cap (\cup_{i=1}^{\infty} E_i)^c). \tag{10.16}$$

Then by (10.15) and by (10.16) we have

$$\mu^*(T) = \mu^*(T \cap \left(\cup_{i=1}^{+\infty} E_i\right)) + \mu^*(T \cap (\cup_{i=1}^{\infty} E_i)^c), \tag{10.17}$$

which means $\cup_{i=1}^{+\infty} E_i$ is measurable.

Let $T = \cup_{i=1}^{+\infty} E_i$. Then we have $T \cap (\cup_{i=1}^{+\infty} E_i)^c = \emptyset$, $T \cap E_i = E_i$, $i = 1, 2, \cdots$. By (10.14) we obtain that

$$\mu(\cup_{i=1}^{+\infty} E_i) \geq \sum_{i=1}^{+\infty} \mu(E_i).$$

By subadditivity of outer measure (Proposition 10.2), we also have

$$\mu(\cup_{i=1}^{+\infty} E_i) \leq \sum_{i=1}^{+\infty} \mu(E_i).$$

Then we have $\mu(\cup_{i=1}^{+\infty} E_i) = \sum_{i=1}^{+\infty} \mu(E_i)$. $\square$

Theorem 10.20. *Let $E \subset \mathbb{R}$ be a measurable set $x_0 \in \mathbb{R}$. Then $x_0 + E := \{x_0 + x : x \in E\}$ is also measurable and $\mu(x_0 + E) = \mu(E)$.*

Proof: For every interval $I \subset \mathbb{R}$, $x_0 + I$ is also an interval with length $|x_0 + I| = |I|$. Therefore, we have

$$\begin{aligned}
\mu^*(E) &= \inf\{\sum_{i=1}^{+\infty} |I_i| : E \subset \cup_{i=1}^{+\infty} |I_i|\} \\
&= \inf\{\sum_{i=1}^{+\infty} |x_0 + I_i| : E \subset \cup_{i=1}^{+\infty} |I_i|\} \\
&= \inf\{\sum_{i=1}^{+\infty} |x_0 + I_i| : x_0 + E \subset \cup_{i=1}^{+\infty} x_0 + I_i\} \\
&= \mu^*(x_0 + E).
\end{aligned}$$

Since $(x_0 + E)^c = x_0 + E^c$ we have $\mu^*((x_0 + E)^c) = \mu^*(E^c)$. Then for every $B \subset \mathbb{R}$, we want to show that $\mu^*(B) = \mu^*(B \cap (x_0 + E)) + \mu^*(B \cap (x_0 + E)^c)$. Choose $T \subset \mathbb{R}$ such that $B = x_0 + T$. Since E is measurable, we have

$$\mu^*(T) = \mu^*(T \cap E) + \mu^*(T \cap E^c). \tag{10.18}$$

By (**??**) and (10.18), we obtain

$$\begin{aligned}
\mu^*(B) = \mu^*(x_0 + T) = \mu^*(T) & \\
&= \mu^*(T \cap E) + \mu^*(T \cap E^c) \\
&= \mu^*(x_0 + T \cap E) + \mu^*(x_0 + T \cap E^c) \\
&= \mu^*((x_0 + T) \cap (x_0 + E)) + \mu^*((x_0 + T) \cap (x_0 + E)^c) \\
&= \mu^*(B \cap (x_0 + E)) + \mu^*(B \cap (x_0 + E)^c).
\end{aligned} \tag{10.19}$$

It follows that $x_0 + E$ is measurable and $\mu(x_0 + E) = \mu(E)$. ☐ With Theorems 10.19 and 10.20 we complete the verification that Lebesgue measure defined at Definition 10.11 is indeed a measure satisfying the conditions 1), 2), 3) and 4).

Definition 10.21. Let X be a set and $\mathfrak{N} \subset 2^X$ a family of subsets. We say that $\mathfrak{N}$ is a *σ-algebra* in X if

(σ_1) $X \in \mathfrak{N}$,

(σ_2) if $A \in \mathfrak{N}$ then $A^c \in \mathfrak{N}$ (here we denote by $A^c := X \setminus A$ the complement of A),

(σ_3) If $\{A_n\}_{n=1}^{\infty} \subset \mathfrak{N}$ then $\displaystyle\bigcup_{n=1}^{\infty} A_n \in \mathfrak{N}$.

Then the pair $(X, \mathfrak{N})$ is called a *measurable space* (or just space with a measure) and $A \in \mathfrak{N}$ i s called a *measurable set*.

Propositions 10.14 and Theorem 10.19 actually have proved that the family of Lebesgue measurable sets is a σ-algebra. We leave the discussion of abstract measure theory to more advanced course. Let us we finish this section with the following example and the theorem about representation of open sets in $\mathbb{R}$:

Example 10.22. Put $I_0 = [-2,2]$. Consider the following equivalence relation for numbers $x, y \in [0,1]$: $x \sim y \iff x - y \in \mathbb{Q}$. Indeed, the relation '$\sim$' is an equivalent relation (it is reflexive: $x - x = 0 \in \mathbb{Q}$, symmetric: $x - y \in \mathbb{Q}$ iff $y - x \in \mathbb{Q}$, and transitive: if $x - y \in \mathbb{Q}$ and $y - z \in \mathbb{Q}$, then $(x-y) + (y-z) = x - z \in \mathbb{Q}$). Consequently, the set $[-1,1]$ can be represented as a disjoint union of the equivalence classes $[x] := \{x' \in [-1,1] : x \sim x'\}$. By the Axiom of Choice, we can choose from each of these disjoint classes exactly one element and create a set A composed of these elements. In other words, we have

$$[-1,1] = \bigcup_{a\in A}[a], \quad \text{where} \quad [a]\cap[b] = \emptyset \quad \text{for} \quad a \neq b.$$

Therefore, for every element $x \in [-1,1]$ there exists a class $[a]$, $a \in A$ such that $x \in [a]$, which means there exists $r \in \mathbb{Q}\cap[-1,1]$ such that $x = r + a$, i.e.

$$[-1,1] \subset \bigcup_{r\in[-1,1]\cap\mathbb{Q}} (r + A) \subset [-2,2], \quad \text{and} \quad r + A \cap r' + A = \emptyset$$

where r', $r \in [-1,1]\cap\mathbb{Q}$ and $r \neq r'$. We claim that the set A is not measurable. Indeed, assume for contradiction that A is measurable, then by Theorem 10.20, the sets $r + A$ are also measurable and we have

$$2 = \mu([-1,1]) \leq \mu\left(\bigcup_{r\in[-1,1]\cap\mathbb{Q}} (r + A)\right) = \sum_{r\in[-1,1]\cap\mathbb{Q}} \mu(r + A) = \sum_{k=1}^{\infty}\mu(A) \leq \mu([-2,2]). \tag{10.20}$$

But (10.20) implies a contradiction. If $\mu(A) = 0$ then it implies $2 = 0$, and if $\mu(A) > 0$ then it implies $\sum_{k=1}^{\infty}\mu(A) = \infty \leq 4$.

Theorem 10.23. *Let U be an open set in $\mathbb{R}$, then there exists a sequence $\{I_i\}_{i\in\mathbb{N}}$, $I_i = (a_i, b_i)$ of disjoint open intervals (i.e $(a_i,b_i)\cap(a_j,b_j) = \emptyset$ for $i \neq j$), $a_i \leq b_i$ (i.e. we do not exclude a possibility that $a_i = b_i$, i.e. $(a_i,b_i) = \emptyset$, and/or $a_i = -\infty$ and $b_i = \infty$) such that*

$$U = \bigcup_{i=1}^{\infty} I_i. \tag{10.21}$$

Proof: Consider a nonempty open set $U \subset \mathbb{R}$. We introduce the following relation '$\approx$' in U:

$$\forall_{x,\,y\in U} \quad x \approx y \quad \overset{\text{def}}{\Leftrightarrow} \quad [a,b] \subset U, \;\; a := \min\{x,y\}, \; b := \max\{x,y\}.$$

In other words for $x \leq y$ we have $x \approx y$ if and only if $[x,y] \subset U$ and if $x \geq y$, then $x \approx y$ if $[y,x] \subset U$. We will show that that '$\approx$' is an equivalence relation in U. indeed, we have

- (reflexivity) $\forall_{x\in U}, \quad x \approx x$, since $[x,x] = \{x\} \subset U$;

- (symmetry) $\forall_{x,y\in U} \quad x \approx y \quad \Leftrightarrow \quad y \approx y$;
- (transitivity) $\forall_{x,y,z\in U} \quad x \approx y$ and $y \approx z$ then $x \approx z$. Indeed, if the intervals I_1 with the endpoints x and y and the interval I_2 with the endpoints y and z are contained in U then the interval $I := I_1 \cup I_2$ is also contained in U. Since the interval with the endpoints x and z is a subinterval of I, it is also contained in U.

Notice that, if two intervals $[a,b]$ and $[c,d]$ are such that $[a,b] \cap [c,d] \neq \emptyset$ then $[a,b] \cup [c,d]$ is also an interval $[\alpha,\beta]$, where $\alpha := \min\{a,c\}$ and $\beta := \max\{b,d\}$.

We will show that for every $x \in U$, the equivalence class

$$[x] := \{y \in U : x \approx y\}$$

is an open interval. Indeed, if the set $[x]$ is bounded from above (resp. from below) we put $b := \sup\{y : y \in [x]\}$ (resp. $a := \inf\{y : y \in [x]\}$), otherwise we put $b = \infty$ (resp. $a = -\infty$). We claim that $[x] = (a,b)$. In order to show that $[x] \subset (a,b)$ we notice that if a is finite then $a \notin [x]$. Indeed, if $a \in [x]$ then $a \in U$. Since U is open, there is an $\varepsilon > 0$ such that $(a - 2\varepsilon, a + 2\varepsilon) \subset U$. Since $a - \varepsilon \approx a + \varepsilon$ and (by definition of infimum) there is $x' \in [x]$ such that $x' < a + \varepsilon$ and $x' \approx x$, we obtain (by transitivity) that $a - \varepsilon \approx x$, i.e. $a - \varepsilon \in [x]$, which contradicts the fact that a is the infimum of $[x]$. Similarly, we show that if b is finite then $b \notin [x]$. Since a is a lower bound of $[x]$ and b an upper bound of $[x]$, which do not belong to $[x]$ thus $[x] \subset (a,b)$. Let us show that $(a,b) \subset [x]$. Indeed, suppose for some $x' \in (a,b)$ that $x' < x$. Take $\varepsilon := x' - a$ and by the definition of the infimum there exists $y \in [x]$ such that $a + \varepsilon > y$, i.e. $x' < y$ and $x' \in [y,x]$, it follows that $x' \approx x$, thus $x' \in [x]$. For the case $x' > x$ we apply similar argument based on the definition of supremum.

Since

$$U = \bigcup_{x\in U} [x], \quad [x] \cap [y] \neq \emptyset \quad \Rightarrow \quad [x] = [y] \tag{10.22}$$

meaning that this a disjoint union. Since each of the equivalence classes $[x]$ is an open interval containing a rational number, by countability of $\mathbb{Q}$ there may be only countably many such intervals. That means we can index them by a finite or infinite sequence of natural numbers. If there are only finitely many such disjoint open intervals in the union (10.22), we can extend this collection to an infinite (countable) collection by adding empty intervals (a,a). □

10.4 Lebesgue Integration

In this section we will consider the integral on $[a,b]$. We denote by $\mathfrak{M}$ the σ-algebra of all Lebesgue measurable sets in $[a,b]$ and by $\mu : \mathfrak{M} \to \mathbb{R}_+$ the Lebesgue measure on $[a,b]$. We will also denote by $\mathcal{T}$ the collection of all open sets in $[a,b]$.

10.4.1 Lebesgue Measurable Functions

Definition 10.24. A function $f : [a,b] \to \mathbb{R}$ is said to be *Lebesgue measurable* (or simply *measurable*) if for every open set $U \subset \mathbb{R}$ the set $f^{-1}(U)$ is measurable, i.e.

$$\forall_{U\in\mathcal{T}} \quad f^{-1}(U) \in \mathfrak{M}.$$

Proposition 10.25. *Let $f : [a,b] \to \mathbb{R}$ be a function. The following conditions are equivalent*

(a) f *is measurable;*
(b) $\forall_{\alpha<\beta} \quad f^{-1}(\alpha,\beta) \in \mathfrak{M}$*;*
(c) $\forall_{\alpha\in\mathbb{R}} \quad f^{-1}(-\infty,\alpha) \in \mathfrak{M}$*;*
(d) $\forall_{\alpha\in\mathbb{R}} \quad f^{-1}[\alpha,\infty) \in \mathfrak{M}$*;*
(e) $\forall_{\alpha\in\mathbb{R}} \quad f^{-1}(\alpha,\infty) \in \mathfrak{M}$*;*
(f) $\forall_{\alpha\in\mathbb{R}} \quad f^{-1}(-\infty,\alpha] \in \mathfrak{M}$*;*

Proof: In order to show that (a) $\Leftrightarrow$ (b), it is sufficient to notice that (a) $\Rightarrow$ (b), and that every open set $U \subset \mathbb{R}$ is a disjoint union of open intervals (see Theorem 10.23), i.e.

$$U = \bigcup_{k=1}^{\infty} (\alpha_k, \beta_k).$$

Then, if $f^{-1}(\alpha_k,\beta_k) \in \mathfrak{M}$, for every $k \in \mathbb{N}$, by Theorem 10.19,

$$f^{-1}(U) = f^{-1}\left(\bigcup_{k=1}^{\infty} (\alpha_k,\beta_k)\right) = \bigcup_{k=1}^{\infty} f^{-1}(\alpha_k,\beta_k) \in \mathfrak{M}.$$

On the other hand, knowing that $f^{-1}(-\infty,\alpha) \in \mathfrak{M}$ for all α, then again, since $\mathfrak{M}$ is a σ-algebra, thus

$$f^{-1}[\alpha,\infty) = \Big(\mathbb{R} \setminus f^{-1}(-\infty,\alpha)\Big)^c \in \mathfrak{M},$$

which means that (c) $\Rightarrow$ (d). Then we have

$$f^{-1}(\alpha,\infty) = f^{-1}\left(\bigcup_{k=1}^{\infty}\left[\alpha - \frac{1}{k}, \infty\right)\right) = \bigcup_{k=1}^{\infty} f^{-1}\left[\alpha - \frac{1}{k}, \infty\right) \in \mathfrak{M},$$

thus (d) $\Rightarrow$ (e). Then again, Therefore, if f satisfies (e), then clearly it also satisfies (f) (by taking a complement of $(-\infty,\alpha]$), and by applying similar arguments as above, we can also show that $f^{-1}(-\infty,\alpha] \in \mathfrak{M}$ and $f^{-1}(-\infty,\beta) \in \mathfrak{M}$ (for α, $\beta \in \mathbb{R}$), thus

$$f^{-1}(\alpha,\beta) = f^{-1}\Big((\alpha,\infty) \cap (-\infty,\beta)\Big) = f^{-1}(\alpha,\infty) \cap f^{-1}(-\infty,\beta) \in \mathfrak{M},$$

so (e) $\Rightarrow$ (b). $\square$

Proposition 10.26. *We have the following properties:*

(a) If f, $g : [a,b] \to \mathbb{R}$ *are measurable functions and* α, $\beta \in \mathbb{R}$, *then* $\alpha f + \beta g : [a,b] \to \mathbb{R}$ *is a measurable function;*
(b) If $f : [a,b] \to \mathbb{R}$ *is measurable, then* $|f| : [a,b] \to \mathbb{R}$ *is measurable;*
(c) If $f : [a,b] \to \mathbb{R}$ *is measurable, then* $f_+ := \frac{1}{2}(|f| + f)$ *and* $f_- := \frac{1}{2}(|f| - f)$ *are measurable;*
(d) If f, $g : [a,b] \to \mathbb{R}$ *are measurable functions then* $fg : [a,b] \to \mathbb{R}$ *is a measurable function;*
(e) If $f : [a,b] \to \mathbb{R}$ *is a measurable function such that* $f(x) \neq 0$ *for* $x \in [a,b]$, *then* $\frac{1}{f} : [a,b] \to \mathbb{R}$ *is a measurable function;*

(f) *Let* $f_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, *be a sequence of measurable functions such that* $f : [a,b] \to \mathbb{R}$ *satisfies the condition*

$$f(x) := \sup\{f_n(x) : n \in \mathbb{N}\}, \quad x \in [a,b].$$

Then, f *is a measurable function.*

(g) *Let* $f_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, *be a sequence of measurable functions such that* $f : [a,b] \to \mathbb{R}$ *satisfies the condition*

$$f(x) := \inf\{f_n(x) : n \in \mathbb{N}\}, \quad x \in [a,b].$$

Then, f *is a measurable function.*

Proof: (a): One can easily verify that if f is measurable then αf ($\alpha \in \mathbb{R}$) is measurable, therefore we only need to show that if f, g are measurable then $f+g$ is also measurable. Notice that for every $\alpha \in \mathbb{R}$ we have

$$\{x \in [a,b] : f(x) + g(x) < \alpha\} = \bigcup_{\substack{p+q<\alpha \\ p,q\in\mathbb{Q}}} \{x \in [a,b] : f(x) < p\} \cap \{x \in [a,b] : g(x) < q\}.$$

Clearly, for $p + q < \alpha$,

$$\{x \in [a,b] : f(x) < p\} \cap \{x \in [a,b] : g(x) < q\} \subset \{x \in [a,b] : f(x) + g(x) < \alpha\},$$

thus

$$\{x \in [a,b] : f(x) + g(x) < \alpha\} \supset \bigcup_{\substack{p+q<\alpha \\ p,q\in\mathbb{Q}}} \{x \in [a,b] : f(x) < p\} \cap \{x \in [a,b] : g(x) < q\}.$$

Suppose therefore that $f(x) + g(x) < \alpha$. Put $2\varepsilon := \alpha - (f(x) + g(x)) > 0$. Then, there exist rational numbers p and q such that $p \in (f(x), f(x)+\varepsilon)$ and $(g(x), g(x)+\varepsilon)$. Then we have that $x \in \{x' \in [a,b] : f(x') < p\} \cap \{x' \in [a,b] : g(x') < q\}$, which implies

$$\{x' \in [a,b] : f(x') + g(x') < \alpha\} \subset \bigcup_{\substack{p+q<\alpha \\ p,q\in\mathbb{Q}}} \{x' \in [a,b] : f(x') < p\} \cap \{x' \in [a,b] : g(x') < q\}.$$

Therefore, (since the above union is countable) we have

$$f^{-1}(-\infty, \alpha) = \bigcup_{\substack{p+q<\alpha \\ p,q\in\mathbb{Q}}} f^{-1}(-\infty, p) \cap f^{-1}(-\infty, q) \in \mathfrak{M}.$$

Consequently, $f + g$ is measurable.

We leave the proofs of (b) and (c) as exercises.

In order to prove (d), assume that $f(x)$, $g(x) \geq 0$ for all $x \in [a,b]$. If $\alpha \leq 0$ then the set $\{x \in [a,b] : f(x)g(x) \leq 0\}$ is empty, thus it is measurable. Suppose therefore that $\alpha > 0$, then

$$\{x \in [a,b] : f(x)g(x) < \alpha\} \supset \bigcup_{\substack{pq<\alpha \\ p,q\in\mathbb{Q}_+}} \{x \in [a,b] : f(x) < p\} \cap \{x \in [a,b] : g(x) < q\},$$

where $\mathbb{Q}_+$ denote the set of all positive rational numbers. Then clearly,

$$(fg)^{-1}(-\infty,\alpha) = \bigcup_{\substack{pq<\alpha\\ p,q\in\mathbb{Q}_+}} f^{-1}(-\infty,p)\cap f^{-1}(-\infty,q)\in\mathfrak{M}.$$

In order to prove the general case, we write $f = f_+ - f_-$ and $g = g_+ - g_-$, then since $f_+(x)$, $f_-(x)$, $g_+(x)$, $g_-(x)\geq 0$ and

$$f(x)g(x) = f_+(x)g_+(x) + f_-(x)g_-(x) - (f_-(x)g_+(x) + f_+(x)g_-(x)),\quad x\in[a,b],$$

the conclusion follows from (a).

(e): Notice that we have for $\alpha > 0$

$$\begin{aligned}\left\{x : \frac{1}{f(x)} < \alpha\right\} &= \left\{x : \frac{1}{f(x)} < \alpha \text{ and } f(x)>0\right\}\cup\left\{x : \frac{1}{f(x)} < \alpha \text{ and } f(x)<0\right\}\\ &= \left\{x : \frac{1}{\alpha} < f(x) \text{ and } f(x)>0\right\}\cup\left\{x : \frac{1}{\alpha} > f(x) \text{ and } f(x)<0\right\}\\ &= \left((f^{-1}\left(\frac{1}{\alpha},\infty\right)\cap f^{-1}(0,\infty)\right)\right)\cup\left((f^{-1}\left(-\infty,\frac{1}{\alpha}\right)\cap f^{-1}(-\infty,0)\right)\right)\in\mathfrak{M}.\end{aligned}$$

The proof in the case when $\alpha\leq 0$ is left as an exercise.

(f): Notice that for $\alpha\in\mathbb{R}$ we have

$$\{x\in[a,b] : f(x)<\alpha\} = \bigcup_{\substack{p<\alpha\\ p\in\mathbb{Q}}}\bigcap_{k=1}^{\infty}\{x\in[a,b] : f_k(x)<p\}. \tag{10.23}$$

Indeed, if $p<\alpha$, then clearly, if $x\in[a,b]$ is such that $f_k(x)<p$, then $f(x) = \sup\{f_k(x) : k\in\mathbb{N}\}\leq p<\alpha$, thus

$$\{x\in[a,b] : f(x)<\alpha\} \supset \bigcup_{\substack{p<\alpha\\ p\in\mathbb{Q}}}\bigcap_{k=1}^{\infty}\{x\in[a,b] : f_k(x)<p\}.$$

On the other hand, if $f(x)<\alpha$ then for every $p\in\mathbb{Q}$ such that $f(x)<p<\alpha$, we have $f_k(x)\leq f(x)<p$ fo all $k\in\mathbb{N}$, thus we have

$$\{x\in[a,b] : f(x)<\alpha\} \subset \bigcup_{\substack{p<\alpha\\ p\in\mathbb{Q}}}\bigcap_{k=1}^{\infty}\{x\in[a,b] : f_k(x)<p\}.$$

Therefore, (10.23) implies

$$f^{-1}(-\infty,\alpha) = \bigcup_{\substack{p<\alpha\\ p\in\mathbb{Q}}}\bigcap_{k=1}^{\infty} f_k^{-1}(-\infty,p)\in\mathfrak{M}.$$

The proof of (g) is left as an exercise. $\square$

Definition 10.27. A function $s : [a, b] \to \mathbb{R}$ is called *simple*, if it is measurable and has only a finite number of values $\{r_1, \dots, r_n\} \subset \mathbb{R}$. Then, clearly it has a unique representation

$$s(x) = \sum_{k=1}^{N} \chi_{E_k}(x), \quad E_k = f^{-1}(\{r_k\}), \ \ k = 1, \dots, N.$$

Proposition 10.28. *Let $f : [a, b] \to [0, \infty)$ be a measurable function. Then there exists a sequence of simple functions $s_n : [a, b] \to [0, \infty)$ such that*

(i) for every $x \in [a, b]$ we have

$$0 \le s_1(x) \le s_2(x) \le \dots \le s_n(x) \le \dots,$$

(ii)for every $x \in [a, b]$

$$\lim_{n\to\infty} s_n(x) = f(x).$$

Proof: For every $n = 1, 2, \dots,$ and $1 \le i \le n2^n$ define the sets

$$E_{n,i} := f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right), \quad F_n := f^{-1}([n, \infty]).$$

Then the required sequence can be defined as

$$s_n(x) := \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}}(x) - n\chi_{F_n}(x).$$

□

Definition 10.29. Let $E \subset [a, b]$ be a measurable set and $s : [a, b] \to \mathbb{R}$ a simple function, i.e s can be represented as $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, where $A_i \cap A_j = \emptyset$ for $i \neq j$. Then the *Lebesgue integral* of f on $E \subset [a, b]$ with respect to μ is defined by the formula

$$\int_E f(x) d\mu(x) := \sum_{i=1}^{n} \alpha_i \mu(E \cap A_i).$$

Definition 10.30. Let $f : [a, b] \to [0, \infty)$ be a measurable function. The *integral* of f (which can be equal to ∞) on a measurable set E with respect to μ is defined by

$$\int_E f(x) d\mu(x) := \sup\left\{\int_E s d\mu : \forall_{x\in X} \ \ 0 \le s(x) \le f(x)\right\}.$$

If $\int_E f d\mu < \infty$, then f is said to be *summable* on E.

For a given function $f : [a, b] \to \mathbb{R}$ we put

$$f^+(x) := \max\{f(x), 0\}, \quad f^-(x) := \max\{-f(x), 0\}, \ \ x \in [a, b].$$

If f is measurable and f^+, f^- are summable on $[a, b]$, then, we define the Lebesgue integral of f on $[a, b]$ by

$$\int_a^b f(x)\,d\mu(x) := \int_a^b f^+(x)\,d\mu(x) - \int_a^b f^-(x)\,d\mu(x).$$

Moreover, if $E \subset [a,b]$ is a measurable set, then we define

$$\int_E f(x)d\mu(x) := \int_a^b \chi_E(x)f(x)d\mu(x).$$

10.4.2 Properties of Lebesgue Integral

Theorem 10.31. (Monotone Convergence Theorem) *Let $f_n : [a,b] \to \mathbb{R}$, $n = 1, 2, \dots,$ be a sequence of measurable functions such that*

(i) for all $x \in [a,b]$ we have $0 \le f_1(x) \le f_2(x) \le \dots \le f_n(x) \le \dots$
(ii) $\lim\limits_{n\to\infty} f_n(x) = f(x)$

Then f is measurable. Moreover, if $\displaystyle\int_a^b f(x)d\mu(x) < \infty$, then

$$\lim_{n\to\infty} \int_a^b f_n(x)d\mu(x) = \int_a^b f(x)d\mu(x).$$

Proof: The property that f is measurable can be obtained directly from the definition of measurability. Notice that $\int_a^b f_n(x)d\mu(x) \le \int_a^b f_{n+1}(x)d\mu(x)$ thus it is a convergent sequence to an $\alpha \in [0,\infty)$. Since $\int_a^b f_n(x)d\mu(x) \le \int_a^b f(x)d\mu(x)$, we have $\alpha \le \int_a^b f(x)d\mu(x)$. Consider an arbitrary simple function s such that $0 \le s(x) \le f(x)$ (for all $x \in [a,b]$) and let $0 < c < 1$. Put

$$E_n := \{x : f_n(x) \ge cs(x)\}.$$

Then

$$E_1 \subset E_2 \subset \dots \subset E_n \subset \dots, \quad [a,b] = \bigcup_n E_n.$$

Clearly,

$$\int_a^b f_n(x)\,d\mu(x) \ge \int_{E_n} f_n(x)\,d\mu(x) \ge c\int_{E_n} s(x)\,d\mu(x),$$

thus $\alpha \ge c\int_a^b s(x)\,d\mu(x)$ for all $0 < c < 1$. Consequently, $\alpha \ge \int_a^b s(x)\,d\mu(x)$, which implies that $\alpha \ge \int_a^b f(x)\,d\mu(x)$. □

Corollary 10.32. *Let $f_n : [a,b] \to [0,\infty]$, $n = 1, 2, \dots,$ be a sequence of measurable functions and*

$$f(x) := \sum_{n=1}^{\infty} f_n(x).$$

Then

$$\int_a^b f(x)\,d\mu = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,d\mu(x).$$

Proposition 10.33. (Properties of Lebesgue Integral) *Let f, $g : [a,b] \to \mathbb{R}$ be a summable on $[a,b]$ functions and $E \subset [a,b]$ a measurable set. Then we have the following properties*

(a) For α, $\beta \in \mathbb{R}$,

$$\int_E (\alpha f(x) + \beta g(x))d\mu(x) = \alpha \int_E f(x)d\mu(x) + \beta \int_E g(x)d\mu(x).$$

(b) If $g(x) \le f(x)$ for all $x \in E \subset [a,b]$, then

$$\int_E g(x)d\mu(x) \le \int_E f(x)d\mu(x).$$

(c) If $E = \bigcup_{k=1}^{n} E_k$, where E_k are measurable sets such that $E_k \cap E_{k'} = \emptyset$ for $k \neq k'$, then

$$\int_E f(x)d\mu(x) = \sum_{k=1}^{n} \int_{E_k} f(x)d\mu(x).$$

(d) $\int_E d\mu(x) = \mu(E)$.

(e) $\left| \int_E f(x)d\mu(x) \right| \le \int_E |f(x)|d\mu(x)$.

(f) If $\mu(E) = 0$ then $\int_E f(x)d\mu(x) = 0$.

Proof: (a): It is sufficient to show this property assuming that f and g are nonnegative on E, i.e $f(x) \ge 0$ and $g(x) \ge 0$, $x \in E$. On the other hand, since

$$\int_E f(x)d\mu(x) = \int_a^b \chi_E(x) f(x)d\mu(x),$$

we only need to consider the case $E = [a,b]$ and $\alpha, \beta > 0$. Next, by applying the Monotone Convergence Theorem 10.31 to the sequences of simple functions s_n^1, $s_n^2 : [a,b] \to \mathbb{R}$, such that $s_n^1(x) \le f(x)$, $s_n^2(x) \le g(x)$ and $\lim_{n\to\infty} s_n^1(x) = f(x)$, $\lim_{n\to\infty} s_n^2(x) = g(x)$ for all $x \in [a,b]$, we obtain

$$\begin{aligned}
\alpha \int_a^b f(x)d\mu(x) + \beta \int_a^b g(x)d\mu(x) &= \alpha \lim_{n\to\infty} \int_a^b s_n^1(x)d\mu(x) + \beta \lim_{n\to\infty} \int_a^b s_n^2(x)d\mu(x) \\
&= \lim_{n\to\infty} \left(\alpha \int_a^b s_n^1(x)d\mu(x) + \beta \int_a^b s_n^2(x)d\mu(x) \right) \\
&= \lim_{n\to\infty} \int_a^b (\alpha s_n^1(x)d\mu(x) + \beta \int_a^b s_n^2(x))d\mu(x) \\
&= \int_a^b (\alpha f(x) + \beta g(x))d\mu(x).
\end{aligned}$$

The proof of (b) we leave as an exercise.

In order to prove (c), it is sufficient to consider the case $E = E_1 \cup E_2$, where E_1 and E_2 are two disjoint measurable sets. Then notice that for every simple function $s : [a,b] \to \mathbb{R}$, $s_1(x) := \chi_{E_1}(x)s(x)$

and $s_2(x) := \chi_{E_2}(x)s(x)$, $x \in [a,b]$ are simple and $\chi_E(x)s(x) = s_1(x)+s_2(x)$. Therefore, the conclusion follows again from the Monotone Convergence Theorem 10.31.

(d): Follows trivially from the definition of the Lebesgue integral.

(e):

Notice that for a measurable function $f : [a,b] \to \mathbb{R}$, if $\int_a^b f\,d\mu \geq 0$ then we have

$$\left|\int_a^b f(x)\,d\mu(x)\right| = \int_a^b f(x)\,d\mu(x) = \int_a^b f^+(x)\,d\mu(x) - \int_a^b f^-(x)\,d\mu(x)$$
$$\leq \int_a^b f^+(x)\,d\mu(x) + \int_a^b f^-(x)\,d\mu(x) = \int_a^b |f|\,d\mu.$$

If $\int_a^b f(x)\,d\mu(x) < 0$, the same relations applied to $-f$ lead to the inequality

$$\left|\int_a^b f(x)\,d\mu\right| \leq \int_a^b |f(x)|\,d\mu(x),$$

and consequently, we also obtain

$$\left|\int_E f(x)\,d\mu(x)\right| = \left|\int_a^b \chi_E(x)f(x)\,d\mu(x)\right| \leq \int_a^b \chi_E(x)|f(x)|\,d\mu(x) = \int_E |f(x)|\,d\mu(x).$$

The proof of (f) is trivial. □

10.4.3 Riemann Integrable Functions and Lebesgue Integrable Functions

Consider the interval $[a,b]$, $a < b$. For $\delta > 0$ and $x_o \in [a,b]$ we define the set $D_\delta(x_o) \subset [a,b]$ by

$$D_\delta(x_o) := \{x \in [a,b] : |x - x_o| \leq \delta\}.$$

Let $f : [a,b] \to \mathbb{R}$ be a bounded function. We put for $x_o \in [a,b]$ and $\delta > 0$

$$\omega(f,\delta)(x_o) := \sup\{f(x) : x \in D_\delta(x_o)\} - \inf\{f(x) : x \in D_\delta(x_o)\}.$$

Notice that, if $0 < \delta_1 < \delta_2$, then

$$0 \leq \omega(f,\delta_1)(x_o) \leq \omega(f,\delta_2)(x_o)$$

therefore, there exists the limit

$$\omega(f)(x_o) := \lim_{\delta\to 0^+} \omega(f,\delta)(x_o).$$

The number $\omega(f)(x_o)$ is sometimes called the *local variation* of f at x_o.

Proposition 10.34. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function and $x_o \in [a,b]$. The function f is continuous at x_o if and only if $\omega(f)(x_o) = 0$.*

Proof: Suppose that f is continuous at x_o, then

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\forall_{x\in[a,b]}\quad |x-x_o|\le\delta\ \ \Rightarrow\ \ |f(x)-f(x_o)|<\varepsilon.$$

Since $x'\in D_\delta(x_o)\ \ \Leftrightarrow\ \ x'\in[a,b]$ and $|x'-x_o|<\delta$, it follows that

$$\forall_{x',x''\in D_\delta(x_o)}\ \ f(x')-f(x'')\le|f(x')-f(x'')|<\varepsilon\ \implies\ \omega(f,\delta)(x_o)\le\varepsilon,$$

which implies that

$$\forall_{\varepsilon>0}\ \ \exists_{\delta>0}\ \ 0\le\omega(f)(x_o)\le\omega(f,\delta)(x_o)\le\varepsilon\quad\implies\quad\omega(f)(x_o)=0.$$

Conversely, suppose that $\omega(f)(x_o)=0$, then, by the definition,

$$\forall_{\varepsilon>0}\ \ \exists_{\delta>0}\ \ \omega(f,\delta)(x_o)=\sup\{f(x):x\in D_\delta(x_o)\}-\inf\{f(x):x\in D_\delta(x_o)<\varepsilon.$$

Since for all x', $x''\in D_\delta(x_o)$,

$$|f(x')-f(x'')|\le\sup\{f(x):x\in D_\delta(x_o)\}-\inf\{f(x):x\in D_\delta(x_o)<\varepsilon,$$

it follows that

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\forall_{x\in[a,b]}\quad |x-x_o|\le\delta\ \ \Rightarrow\ \ |f(x)-f(x_o)|<\varepsilon.$$

□

Lemma 10.35. *Let $f:[a,b]\to\mathbb{R}$ be a bounded function. Then for every $\lambda>0$ the set*

$$E_\lambda:=\{x\in[a,b]:\omega(f)(x)\ge\lambda\}$$

is closed in $[a,b]$.

Proof: It is sufficient to show that if $\{x_n\}\subset E_\lambda$ and $\lim\limits_{n\to\infty}x_n=x_o$ then $x_o\in E_\lambda$. Assume that $\delta>0$ is an arbitrary number.

$$\exists_N\ \forall_{n\ge N}\ \ x_n\in[a,b]\subset B_\delta(x_o)=\operatorname{int}(D_\delta(x_o))\subset D_\delta(x_o).$$

Since x_n belongs to the interior of $D_\delta(x_o)$, there exists σ such that $\delta>\sigma>0$ and $D_\sigma(x_n)\subset D_\delta(x_o)$. Consequently, we have

$$\lambda\le\omega(f)(x_n)\le\omega(f,\sigma)(x_n)\le\omega(f,\delta)(x_o),$$

which implies

$$\omega(f)(x_o)=\lim_{\delta\to0^+}\omega(f,\delta)(x_o)\ge\lambda\quad\implies\quad x_o\in E_\lambda.$$

□

Lemma 10.36. *Let $f:[a,b]\to\mathbb{R}$ be a bounded function, $\lambda>0$ and U an open set containing*

$$E_\lambda:=\{x\in[a,b]:\omega(f)(x)\ge\lambda\}.$$

Then, there exists $\delta>0$ such that

$$\forall_x\quad x\in[a,b]\setminus U\ \ \implies\ \ \omega(f,\delta)<2\lambda.$$

Proof: By Lemma 10.35, the set E_λ is compact, $E_\lambda \subset U$, thus there exists a compact subset $F \subset [a,b]$ such that $F \cap E_\lambda = \emptyset$ and $U \cup F = [a,b]$. Then, for every $x \in F$ there exists an interval $(x-\delta_x, x+\delta_x) \cap [a,b]$, $\delta_x > 0$, such that

$$\sup\{f(x') : x' \in D_{\delta_x}(x)\} - \inf\{f(x') : x' \in D_{\delta_x}(x) < 2\lambda.$$

Since F is compact and $\{\mathrm{int}(D_{\delta_x}(x))\}_{x\in F}$ is an open cover of F, there exists a finite subcover

$$\{\mathrm{int}(D_{\delta_1}(x_1)),\ \mathrm{int}(D_{\delta_2}(x_2)), \dots, \mathrm{int}(D_{\delta_n}(x_n))\}.$$

Then, by Theorem 3.94, there exists $\delta > 0$ satisfying the property

$$\forall_{x\in F}\ \exists_{k\in\{1,2,\dots,n\}} \quad D_\delta(x) \subset \mathrm{int}(D_{\delta_k}(x_k)).$$

Therefore, the conclusion follows. □

Theorem 10.37. (Lebesgue Theorem) *A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if the set $E = \omega\{x \in [a,b] : \omega(f)(x) > 0\}$ of all discontinuities of f has measure zero, i.e.*

$$\mu(\{x : \omega(f)(x) > 0\}) = 0.$$

Proof: Assume that $f : [a,b] \to \mathbb{R}$ is a bounded function (i.e. there exists $M > 0$ such that $|f(x)| \le M$ for all $x \in [a,b]$), such that the set of its discontinuity points E is of measure zero (i.e. $\mu(E) = 0$). Assume that $\varepsilon > 0$ is an arbitrary number. We will show that there exists $\delta > 0$ such that for any partition $P = \{x_k\}_{k=1}^n$ such that $\|P\| < \delta$, we have

$$\begin{aligned} S(f,P) - s(f,P) &= \sum_{k=1}^n (M_k - m_k)(x_k - x_{k-1}) < \varepsilon, \\ M_k &= \sup\{f(x) : x \in [x_{k-1}, x_k]\}, \\ m_k &= \inf\{f(x) : x \in [x_{k-1}, x_k]\}. \end{aligned}$$

Indeed, choose $\lambda > 0$ such that $4\lambda(b-a) < \varepsilon$. Since $E_\lambda \subset E$ and $\mu(E) = 0$, thus $\mu(E_\lambda) = 0$. By Theorem **??** (i), there exists an open in $[a,b]$ set U such that $E_\lambda \subset U$ and $\mu(U) < \frac{\varepsilon}{4M}$. Since the set U, by Theorem 10.23, is a union of pairwise disjoint open intervals, by compactness of E_λ (see Lemma 10.35), we can assume that U is composed of a finite number disjoint open intervals and assume the ∂U is the set of their endpoints. Moreover, by Lemma 10.36 there exists $\delta > 0$ such that for all $x \in [a,b] \setminus U$ we have $\omega(f,\delta)(x) < 2\lambda$. Suppose that $P = \{x_k\}_{k=1}^n$ is a partition of $[a,b]$ such that $\|P\| < \delta$. We can assume without loss of generality that $\partial U \subset P$. Let us divide the subintervals $[x_{k-1}, x_k]$ into two groups:

(i) The group (I) composed of those subintervals $[x_{k-1}, x_k]$ such that $(x_{k-1}, x_k) \subset U$, and
(ii The group (II) composed of those subintervals $[x_{k-1}, x_k)$ such that $(x_{k-1}, x_k) \cap U = \emptyset$.

Then, since for the subintervals $[x_{k-1}, x_k]$ of the type (II) we have $M_k - m_k \le 2\lambda$, we have

$$\begin{aligned}
S(f,P)-s(f,P) &= \sum_{k=1}^{n}(M_k-m_k)(x_k-x_{k-1})\\
&= \sum_{(I)}(M_k-m_k)(x_k-x_{k-1})+\sum_{(II)}(M_k-m_k)(x_k-x_{k-1})\\
&= 2M\sum_{(I)}(x_k-x_{k-1})+2\lambda\sum_{(I)}(x_k-x_{k-1})\\
&\le 2M\mu(U)+2\lambda(b-a)<2M\frac{\varepsilon}{4M}+2\frac{\varepsilon}{4}(b-a)=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

Conversely, assume that a bounded function $f:[a,b]\to\mathbb{R}$ is Riemann integrable. Consider an arbitrary number $\lambda>0$. Then, the integrability of f on $[a,b]$ implies that

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{P=\{x_k\}_{k=1}^n}\quad \|P\|<\delta \implies S(f,P)-s(f,P)<\varepsilon\lambda. \tag{10.24}$$

Assume that for a fixed $\varepsilon>0$, a partition $P=\{x_k\}_{k=1}^n$ satisfies the condition (10.24). Then we can divide the subintervals $[x_{k-1},x_k]$ into two groups

(i) The group (I) composed of those subintervals $[x_{k-1},x_k]$ such that $(x_{k-1},x_k)\cap E_\lambda\neq\emptyset$, and
(ii) The group (II) composed of all the remaining subintervals.

Put

$$U:=\bigcup_{[x_{k-1},x_k]\in(I)}(x_{k-1},x_k).$$

Then, clearly $E_\lambda\subset U$. Notice that if $x\in(x_{k-1},x_k)\cap E_\lambda$, then there exists $\sigma>0$ such that $D_\sigma(x)\subset[x_{k-1},x_k]$. and therefore

$$M_k-m_k\ge\sup\{f(x'):x'\in D_\sigma(x)\}-\inf\{f(x'):x'\in D_\sigma(x)=\omega(f,\sigma)(x)\ge\omega(f)(x)\ge\lambda.$$

Consequently,

$$\begin{aligned}
\lambda\varepsilon>S(f,P)-s(f,P) &= \sum_{k=1}^{n}(M_k-m_k)(x_k-x_{k-1})\\
&= \sum_{(I)}(M_k-m_k)(x_k-x_{k-1})+\sum_{(II)}(M_k-m_k)(x_k-x_{k-1})\\
&\ge \sum_{(I)}(M_k-m_k)(x_k-x_{k-1})\ge\lambda\sum_{(I)}(x_k-x_{k-1})\\
&= \lambda\mu(U)\ge\lambda\mu(E_\lambda).
\end{aligned}$$

Therefore, for every $\varepsilon>0$, we have that $\mu(E_\lambda)<\varepsilon$, i.e. $\mu(E_\lambda)=0$. On the other hand,

$$E=\bigcup_{k\in\mathbb{N}}E_{\frac{1}{k}}\implies \mu(E)\le\sum_{k=1}^{\infty}\mu(E_{\frac{1}{k}})=\sum_{k=1}^{\infty}0=0.$$

$\square$

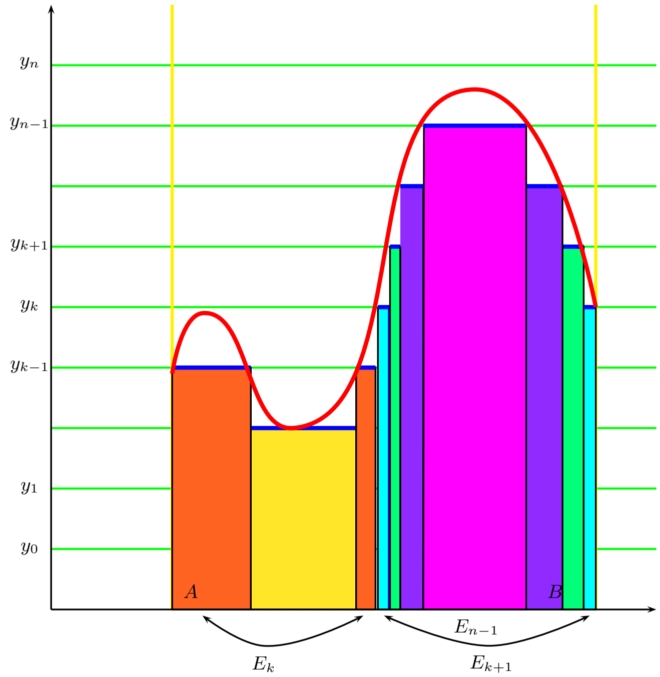


Fig. 10.1. Geometric Interpretation of Lebesgue Integral

Remark 10.38. Suppose that $f : [a,b] \to \mathbb{R}$ is a non-negative bounded measurable function such that $\int_a^b f(x)d\mu(x) < \infty$, i.e. f is summable on $[a,b]$. Suppose that $A \le f(x) \le B$ for all $x \in [a,b]$. Consider an arbitrary partition $P = \{y_k\}_{k=1}^n$ of the interval $[A,B]$ and then define two step functions s, $S : [a,b] \to \mathbb{R}$ by the formula

$$s(x) = \sum_{k=1}^n y_{k-1}\chi_{E_k}(x),$$
$$s(x) = \sum_{k=1}^n y_k\chi_{E_k}(x),$$

where $E_k = f^{-1}([y_{k-1}, y_k))$, $k = 1, 2, \dots, n$. Clearly, by the definition,

$$\forall_{x\in[a,b]} \quad s(x) \le f(x) \le S(x).$$

Since s and S are simple function, therefore we obtain

$$\sum_{k=1}^n y_{k-1}\mu(E_k) = \int_a^b s(x)d\mu(x) \le \int_a^b f(x)d\mu(x) \le \int_a^b S(x)d\mu(x) = \sum_{k=1}^n y_k\mu(E_k).$$

Then, since

$$\int_a^b S(x)d\mu(x) - \int_a^b s(x)d\mu(x) = \sum_{k=1}^n (y_k - y_{k-1})\mu(E_k) \le \|P\| \sum_{k=1}^n \mu(E_k) = \|P\|(b-a),$$

it follows that

$$\int_a^b f(x)d\mu(x) = \lim_{\|P\|\to 0} \sum_{k=1}^n y_{k-1}\mu(E_k) = \lim_{\|P\|\to 0} \sum_{k=1}^n y_k\mu(E_k).$$

Remark 10.39. In order to compare the definition of the Lebesgue integral with Riemann integral, assume that a non-negative measurable function $f : [a,b] \to \mathbb{R}$ is bounded and $f([a,b]) \subset [A,B]$, where $[A,B] \subset \mathbb{R}$ is an interval. The approximation of the function f by step functions $s : [a,b] \to \mathbb{R}$ such that $0 \le s(x) \le f(x)$, can be achieved by considering partitions $P = \{y_k\}_{k=1}^n$ of the interval $[A,B]$ and then defining $s(x)$ by the formula

$$s(x) = \sum_{k=1}^n y_{k-1}\mu(E_k), \quad E_k = f^{-1}([y_{k-1}, y_k)), \quad k = 1, 2, \dots, n.$$

Then, the area under the curve $y = f(x)$, $a \le x \le b$ can be approximated by the value

$$\int_a^b s(x)d\mu = \sum_{k=1}^n y_{k-1}\mu(E_k).$$

By taking supremum of the integrals $\displaystyle\int_a^b s(x)d\mu$, over these approximations $s(x)$, we get the value of the Lebesgue integral $\int_a^b f(x)d\mu$. See Figure 10.1.

Theorem 10.40. *Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function. Then f is Lebesgue integrable and*

$$\int_a^b f(x)d\mu(x) = \int_a^b f(x)dx.$$

Proof: Assume that $f : [a,b] \to \mathbb{R}$ is Riemann integrable (thus it is bounded) and we have that the functions $f^+(x) := \max\{f(x), 0\}$, $f^-(x) := \max\{-f(x), 0\}$, $x \in [a,b]$ are also integrable and

$$\int_a^b f(x)dx = \int_a^b f^+(x)dx - \int_a^b f^-(x)dx.$$

Thus, it is enough to show that

$$\int_a^b f(x)d\mu(x) = \int_a^b f(x)dx$$

for a Riemann integrable function $f : [a,b] \to [0,\infty)$.

Notice that if f is Riemann integrable, than, by Theorem 10.37, it is continuous outside a set of measure zero, i.e. the set E of all discontinuity points of f has the measure zero ($\mu(E) = 0$). Put $X := [a,b] \setminus E$. Since $f|_X : X \to \mathbb{R}$ is continuous, for every open set U in $\mathbb{R}$, $(f|_X)^{-1}(U) = f^{-1}(U) \cap X$ is open in X. Then there exists an open set V in $[a,b]$ such that $f^{-1}(U) \cap X = V \cap X$. Since E is of measure zero, by completeness of μ, E and $f^{-1}(U) \cap E$ are measurable. Therefore X is also measurable. Since the open set V is measurable, thus the intersection $V \cap X$ is also measurable, and therefore the set

$$f^{-1}(U) = (f^{-1}(U) \cap X) \cup (f^{-1}(U) \cap E),$$

is also measurable. This implies that $f^{-1}(U)$ is a measurable set for every open set U in $\mathbb{R}$, thus f s measurable. Let $P = \{x_k\}_{k=1}^{\infty}$ be an arbitrary partition of $[a,b]$. Denote by $E_k := [x_{k-1}, x_k)$, for $k = 1, \dots, n-1$, and $E_n := [x_{n-1}, b]$, and consider the step functions

$$S_P(x) := \sum_{k=1}^{n} M_k \chi_{E_k}(x), \quad \text{and} \quad s_P(x) := \sum_{k=1}^{n} m_k \chi_{E_k}(x),$$

where

$$M_k := \sup\{f(x) : x \in [x_{k-1}, x_k]\}, \quad m_k := \inf\{f(x) : x \in [x_{k-1}, x_k]\}.$$

Then, clearly

$$\forall_{x \in [a,b]} \ s_P(x) \le f(x) \le S_P(x),$$

and therefore

$$\begin{aligned} s(f,P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \int_a^b s_P(x) d\mu(x) \qquad &\le \int_a^b f(x) d\mu(x) \le \int_a^b S_P(x) d\mu(x) \\ = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = S(f,P). \end{aligned}$$

Since, by assumption $\lim_{\|P\| \to 0} \Big(S(f,P) - s(f,P) \Big) = 0$ and $\lim_{\|P\| \to 0} s(f,P) = \int_a^b f(x) dx$, it follows from the Squeeze Property, tat

$$\int_a^b f(x) d\mu(x) = \int_a^b f(x) dx.$$

□

Example 10.41. The the so-called *Dirichlet function* $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad x \text{ is irrational} \\ 1 & \text{if} \quad x \text{ is rational.} \end{cases}$$

is not Riemann integrable on any interval $[a,b]$, $a < b$. Indeed, notice that for all $x \in \mathbb{R}$ and $\delta > 0$ we have

$$\omega(f,\delta)(x) = \sup\{f(x) : x \in D_\delta(x)\} - \inf\{f(x) : x \in D_\delta(x)\} = 1 - 0 = 1,$$

which implies that $\omega(f)(x) = 1$ for all $x \in \mathbb{R}$. Consequently,

$$E_1 := \{x \in [a,b] : \omega(f)(x) \ge 1\} = [a,b],$$

so the function f is discontinuous at every point $x \in [a,b]$, and by Lebesgue Theorem **??**, f is not Riemann integrable. However, notice that $f : [a,b] \to \mathbb{R}$ is measurable. Indeed, the set $A := \mathbb{Q} \cap [a,b]$ is countable, i.e. $A = \bigcup_{k=1}^{\infty} \{x_n\}$, and since $\mu(\{x_k\}) = 0$ for every $k \in \mathbb{N}$, we have $\mu(A) = 0$. By completeness of the Lebesgue measure, A is measurable. Therefore, the set $B := [a,b] \setminus A$ is also measurable. Notice that

$$\{x \in [a,b] : f(x) \geq \alpha\} = \begin{cases} \emptyset & \text{if } \alpha > 1 \\ A & \text{if } 0 < \alpha \leq 1 \\ [a,b] & \text{if } \alpha \leq 1. \end{cases}$$

thus the function f is measurable. On the other hand, f is a simple function, because it can be written as $f(x) = \chi_A(x)$ and therefore

$$\int_a^b f(x)d\mu(x) = \int_a^b \chi_A(x)d\mu(x) = \int_A d\mu(x) = \mu(A) = 0.$$

Example 10.42. Consider the so-called *Riemann function* $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 0 & \text{if } \; x \text{ is irrational or } x = 0 \\ \frac{1}{n} & \text{if } \; x = \frac{m}{n} \text{ where the fraction } \frac{m}{n} \text{ is irreducible and } n \geq 1. \end{cases}$$

We will show that for every $a < b$, the function f is Riemann integrable on $[a,b]$. We claim that fora given $\varepsilon > 0$ there exists only finitely many rational $x \in [a,b]$ such that $x = \frac{m}{n}$ (with the fraction $\frac{m}{n}$ is irreducible and $n \geq 1$) and satisfying $\frac{1}{n} \geq \varepsilon$. Indeed, for any $\varepsilon > 0$ there exists only finitely many rational numbers $r = \frac{m}{n}$ in $[a,b]$ such that $f(r) \geq \varepsilon$, i.e. $\frac{m}{n}$ is an irreducible fraction and $\frac{1}{n} \geq \varepsilon$. Indeed, there are only finitely many natural numbers n such that $\frac{1}{\eta} \geq n$ and, for each such natural number n there exists only finitely many natural numbers k such that $\frac{k}{n} \in [a,b]$ (but not every such a fraction $\frac{k}{n}$ has to be irreducible), thus this statement is true.

Let x_o be an irrational number in $[a,b]$. Assume that $\varepsilon > 0$ is a fixed and r_1, r_2, $\dots$, r_m is the finite set of all rational numbers satisfying the property $f(r_j) \geq \varepsilon$. We can find $\delta > 0$ such that there

$$\delta := \frac{1}{2} \min\{|x_o - r_1|, |x_o - r_2|, \dots, |x_o - r_m|\}.$$

Therefore, for every $x \in D_\delta(x_o)$, we have that $f(x) = 0$ if x is irrational, or $f(x) = \frac{1}{n} < \varepsilon$ (for some natural number n) if x is rational. Consequently,

$$\forall_{\varepsilon>0} \, \exists_{\delta>0} \quad \omega(f,\delta)(x_o) = \sup\{f(x) : x \in D_\delta\} - \inf\{f(x) : x \in D_\delta\} \leq \varepsilon,$$

which implies that

$$\omega(f)(x_o) = \lim_{\delta \to 0} \omega(f,\delta)(x_o) = 0.$$

for every irrational number $x_o \in [a,b]$. Consequently, the set of all discontinuity points E of f has to be contained in $\mathbb{Q} \cap [a,b]$. Since $0 \leq \mu(E) \leq \mu(\mathbb{Q} \cap [a,b]) = 0$, it follows from Theorem **??** that f is Riemann integrable.

Consider an unbounded function $f : [a,b] \to \mathbb{R}$, which is Riemann integrable on every closed subinterval $[a,A] \subset [a,b)$. Of course f is not Riemann integrable on $[a,b]$, however, then the limit $\displaystyle\lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)dx$ (if it exists and is finite) exists, then it is denote by

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)dx.$$

and is called (convergent) *improper* (at b) *integral* of f on $[a,b]$ (see Definition 8.46). A similar definition can be introduced for an improper integral of an unbounded function $f : [a,b] \to \mathbb{R}$ for which the Riemann integrals $\displaystyle\int_{a+\varepsilon}^{b} f(x)dx$ exist. More precisely, the limit (if it is finite)

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^{b} f(x)dx,$$

is also called (convergent) *improper* (at a) *integral* of f on $[a,b]$. Moreover, if $c \in (a,b)$ is such that f is unbounded around c, i.e. one of the integrals $\displaystyle\int_a^b f(x)dx$ or $\displaystyle\int_c^b f(x)dx$ is improper, then the integral

$$\int_a^b f(x)dx := \int_a^c f(x)dx + \int_c^b f(x)dx$$

is also an improper (Riemann) integral.

Proposition 10.43. *Assume that $f : [a,b] \to \mathbb{R}$ is a non-negative unbounded (near a or b) function such that the improper Riemann integral $\displaystyle\int_{a+\varepsilon}^{b} f(x)dx$ converges. Then f is measurable and Lebesgue integrable of $[a,b]$ and we have*

$$\int_a^b f(x)d\mu(x) = \int_a^b f(x)dx.$$

Proof: Assume for definiteness that we are dealing here with an improper integral at b, i.e.

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^{b-\frac{1}{n}} f(x)dx.$$

Put for $n \in \mathbb{N}$

$$f_n(x) := \chi_{E_n}(x)f(x), \quad \text{where } E_n = \left[a, b-\frac{1}{n}\right], \quad x \in [a,b].$$

Since $f(x) \ge 0$, we have that

$$f_1(x) \le f_2(x) \le \cdots \le f_n(x) \le \cdots \le f(x), \quad \text{for all } x \in [a,b],$$

and $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in [a,b)$. Then, by the Monotone Convergence Theorem 10.31 and Theorem 10.40, we have that $f(x)$ is measurable and

$$\begin{aligned}
\int_a^b f(x)d\mu(x) &= \lim_{n\to\infty} \int_a^b f_n(x)d\mu(x) = \lim_{n\to\infty} \int_a^b \chi_{E_n}(x)f(x)d\mu(x) \\
&= \lim_{n\to\infty} \int_a^{b-\frac{1}{n}} f(x)d\mu(x) = \lim_{n\to\infty} \int_a^{b-\frac{1}{n}} f(x)dx \\
&= \int_a^b f(x)dx
\end{aligned}$$

□

10.4.4 Space $L^1([a,b],\mu)$

We introduce the space $L^1\{[a,b];\mu\}$ (which is a prototype for the space $L^1([a,b],\mu)$ that will be introduced later) by

$$L^1\{[a,b];\mu\} := \{f : [a,b] \to \mathbb{R} : f \text{ is measurable and } \int_a^b |f(x)|\, d\mu(x) < \infty\}.$$

Recall that for a sequence of real numbers $\{a_n\}$ we have the following notions

$$\liminf_{n\to\infty} a_n := \sup_n \inf_{k\ge n} a_n, \quad \limsup_{n\to\infty} a_n := \inf_n \sup_{k\ge n} a_n.$$

Theorem 10.44. (Fatou Lemma) *Let $f_n : [a,b] \to [0,\infty)$, $n = 1, 2, \dots,$ be a sequence of measurable functions. Then*

$$\int_a^b \liminf_{n\to\infty} f_n(x)\, d\mu(x) \le \liminf_{n\to\infty} \int_a^b f_n(x)\, d\mu(x).$$

Proof: Put $g_n(x) := \inf_{i\ge n} f_i(x)$, $x \in [,ab]$. Then $g_n(x) \le f_n(x)$, $x \in [a,b]$ and

$$\int_a^b g_n(x) d\mu(x) \le \int_a^b f_n(x)\, d\mu(x).$$

Since $\lim_{n\to\infty} g_n(x) = \liminf_{n\to\infty} f_n(x)$, we obtain

$$\int_a^b \liminf_{n\to\infty} f_n(x)\, d\mu(x) \le \liminf_{n\to\infty} \int_a^b f_n(x)\, d\mu(x).$$

□

Theorem 10.45. (Dominated Convergence Theorem) *Let $\{f_n\}_{n=1}^\infty \subset L^1\{[a,b]; d\mu\}$ be a sequence such that*

(i) $f(x) := \lim_{n\to\infty} f_n(x)$ for all $x \in [a,b]$;
(ii) there exists $g \in L^1\{[a,b]; d\mu\}$ such that

$$\forall_{n=1,2,3,\dots} \forall_{x\in[a,b]} \;\; f_n(x) \le g(x).$$

Then

1. $f \in L^1\{[a,b]; d\mu\}$;
2. $\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)| d\mu(x) = 0$;
3. $\lim_{n\to\infty} \int_a^b f_n(x) d\mu(x) = \int_a^b f(x) d\mu(x)$.

Proof: (i) Since $|f(x)| \le g(x)$, for $x \in [a,b]$, thus $f \in L^1\{[a,b];\mu\}$. (ii) Since $|f_n(x) - f(x)| \le 2g(x)$, thus $2g(x) - |f_n(x) - f(x)| \ge 0$ for $x \in [a,b]$, and by Fatou Lemma

$$\begin{aligned}2\int_a^b g(x)\,d\mu(x) &\le \liminf_{n\to\infty} \int_a^b (2g(x) - |f_n(x) - f(x)|)d\mu(x)\\ &= \int_a^b 2g(x)d\mu(x) - \limsup_{n\to\infty} \int_a^b |f_n(x) - f(x)|d\mu(x),\end{aligned}$$

we get

$$\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|d\mu = \limsup_{n\to\infty} \int_a^b |f_n(x) - f(x)|d\mu(x) = 0.$$

Finally

$$\left|\int_a^b f_n(x) - \int_a^b f(x)d\mu\right| = \left|\int_a^b (f_n(x) - f(x))d\mu(x)\right| \le \int_a^b |f_n(x) - f(x)|d\mu(x),$$

thus $\lim_{n\to\infty} \int_a^b f_n(x)d\mu(x) = \int_a^b f(x)d\mu(x)$. □

Definition 10.46. We say that two measurable functions $f, g : [a,b] \to \mathbb{R}$ are *equal almost everywhere* (in abbreviation we will write $f = g$ *a.e.*), if

$$\mu(\{x \in [a,b] : f(x) \ne g(x)\}) = 0.$$

We can define the equivalence relation in $L^1\{[a,b];\mu\}$ as follows: for $f, g \in L^1\{[a,b];\mu\}$, $f \sim g$ if $f = g$ a.e., and we define the space $L^1([a,b];\mu)$ by

$$L^1([a,b];\mu) := L^1\{[a,b];\mu\}/\sim .$$

One can easily verify that $L^1([a,b];\mu)$ is a vector space.

Definition 10.47. Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of functions and $f : [a,b] \to \mathbb{R}$. We ay that $\{f_n\}$ converges to f *almost everywhere* (in abbreviation *a.e.*) if there exists a set $E \subset [a,b]$ of measure zero such that

$$\forall_{x\in[a,b]\setminus E} \quad \lim_{n\to\infty} f_n(x) = f(x).$$

In such a case we will write $f_n \to f$ a.e., or $f_n \xrightarrow{\text{a.e.}} f$.

Proposition 10.48. *Let $\{f_n\}_{n=1}^\infty \subset L^1\{[a,b];\mu\}$ be a sequence such that $\displaystyle\sum_{n=1}^\infty \int_a^b |f_n(x)|d\mu(x) < \infty$. Then there exists $f \in L^1\{[a,b],\mu\}$ such that*

(i) $f(x) = \displaystyle\sum_{n=1}^\infty f_n(x)$ *a.e.,*

(ii) $\displaystyle\int_a^b f(x)\,d\mu(x) = \sum_{n=1}^\infty \int_a^b f_n(x)\,d\mu(x).$

Proof: Assume that $\sum_{k=1}^{n} |f_k(x)| \to f(x)$ as $n \to \infty$ for $x \in S$ with $\mu([a,b] \setminus S) = 0$. Put $\varphi(x) := \sum_{n=1}^{\infty} |f_n(x)|$ for $x \in S$ and $\varphi(x) = 0$ for $x \in S^c$. Then $\varphi \in L^1\{[a,b],\mu\}$. Since $f(x) \le \varphi(x)$ a.e. thus by Theorem 10.31 (applied on S) we obtain that $f \in L^1\{[a,b];\mu\}$. Put $g_n(x) := \sum_{k=1}^{n} f_k(x)$. Since $|g_n(x)| \le \varphi(x)$, by Theorem 10.45, $\int_a^b g_n(x) d\mu(x) \to \int_a^b f(x) d\mu(x)$. □

Proposition 10.49. *We have*

(i) Let $E \in \mathfrak{M}$ and $f : [a,b] \to [0,\infty)$ be a measurable function such that $\int_E f(x) d\mu(x) = 0$. Then $f(x) = 0$ a.e. on E;

(ii) Let $f \in L^1\{[a,b];\mu)$ be such that $\int_E f(x)\, d\mu(x) = 0$ for all $E \in \mathfrak{M}$. Then $f(x) = 0$ a.e. on $[a,b]$;

(iii) Let $f \in L^1\{[a,b];\mu)$ be such that $\left|\int_a^b f\, d\mu\right| = \int_a^b |f(x)| d\mu(x)$ for all $E \in \mathfrak{M}$.. Then there exists a constant $\alpha \in \mathbb{R}$ such that $|f| = \alpha f$.

10.5 Problems

1. Assume that $I_0 := [a,b]$ is a fixed interval. Use the "sub-additivity property" of the outer measure to prove that for A, $B \subset I_0$ we have

$$|\mu^*(A) - \mu^*(B)| \le \mu^*(A \triangle B),$$

where

$$A \triangle B := (A \setminus B) \cup (B \setminus A)$$

denotes the symmetric difference of sets A and B.

2. Assume that $I_0 := [a,b]$ is a fixed interval. Show that if the set $A \subset I_0$ satisfies the property $\mu^*(A) = 0$ then A is measurable and $\mu(A) = 0$.

3. Assume that $I_0 := [a,b]$ is a fixed interval. Show that for every $A \subset I_0$ such that $\mu^*(A) = \mu(I_0)$, we have

$$\overline{A} = I_0.$$

4. Assume that $I_0 := [a,b]$ is a fixed interval. Let $K \subset I_0$ be a compact set. Show that for every $\varepsilon > 0$ there exists an open set U such that

$$K \subset U \quad \text{and} \quad \mu(U) - \mu(K) < \varepsilon.$$

5. Use the additivity property of the Lebesgue measure μ to show that for every open set $U \subset I_0$ (here $I_0 = [a,b]$ is a fixed interval) there exists a cover $\{I_k\}_{k=1}^{\infty}$ of U by open intervals $I_k = (\alpha_k, \beta_k) \cap I_0$ such that

$$\mu(U) = \sum_{k=1}^{\infty} \mu(I_k).$$

6. (a) Let $I_0 = [a, b]$ be a fixed interval. Consider an open set U in I_0 such that for some $\lambda \in \mathbb{R}$ the set $\lambda U := \{\lambda x : x \in U\}$ is contained in I_0. Use the result in Problem 1 to show that

$$\mu(\lambda U) = |\lambda|\mu(U).$$

(b) Assume that $K \subset I_0$ is a compact set such that $\lambda K \subset I_0$ for some $\lambda \in \mathbb{R}$. Show that

$$\mu(\lambda K) = |\lambda|\mu(K).$$

7. Consider a set $A \subset I_0$ (here $I_0 = [a, b]$ is a fixed interval) such that for some $\lambda \in \mathbb{R}$, we have

$$\lambda A := \{\lambda x : x \in A\} \subset I_0.$$

Show that

(a) $\mu^*(\lambda A) = |\lambda|\mu^*(A)$.
(b $\mu_*(\lambda A) = |\lambda|\mu_*(A)$.
(c) if A is measurable, then λA is measurable and $\mu(A) = |\lambda|\mu(A)$.

8. Let $f : [a, b] \to \mathbb{R}$ be a measurable function. Show that

(a) the function $f_+ : [a, b] \to \mathbb{R}$, defined by $f_+(x) := \max\{0, f(x)\}$, $x \in [a, b]$, is measurable
(b) the function $f_- : [a, b] \to \mathbb{R}$, defined by $f_-(x) := f_+(x) - f(x)$, $x \in [a, b]$, is measurable
(c) the function $|f| : [a, b] \to \mathbb{R}$, defined by $|f|(x) := |f(x)|$, $x \in [a, b]$, is measurable

9. Let $f_n : [a, b] \to \mathbb{R}$, $n = 1, 2, \dots,$, be a sequence of measurable functions. Suppose there is a function $f : [a, b] \to \mathbb{R}$ such that

$$f(x) := \inf\{f_n(x) : n \in \mathbb{N}\}.$$

Show that $f : [a, b] \to \mathbb{R}$ is measurable.

10. Assume that $\{A_k\}_{k=1}^{\infty}$ is a family of measurable sets in $I_0 = [a, b]$. Show that

$$\lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right).$$

11. Let

$$E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots \subset I_0 =: [a, b],$$

be a sequence of measurable sets and let $f : [a, b] \to \mathbb{R}$ be a summable function. Show that

$$\lim_{n\to\infty} \int_{E_n} f(x)d\mu(x) = \int_E f(x)d\mu(x) \quad \text{where} \quad E = \bigcup_{k=1}^{\infty} E_k.$$

12. Consider a function $s : [a, b] \to \mathbb{R}$ with finite number of values, i.e.

$$s([a,b]) = \{c_1, c_2, \dots, c_N\}.$$

Show that s is measurable if and only if for all $k = 1, 2, \dots, N$, the set $A_k := s^{-1}(\{c_k\})$ is measurable.

13. For the following functions $f : [-1,1] \to \mathbb{R}$ compute the Lebesgue integrals

$$\int_{-1}^{1} f(x) d\mu(x).$$

(a): $f(x) = \begin{cases} 1 & \text{if } x \in [-1,1] \cap \mathbb{Q} \\ x^2 & \text{if } x \in [-1,1] \setminus \mathbb{Q} \end{cases}$

(b): $f(x) = \begin{cases} 1 & \text{if } x \in [-1,0] \\ \frac{1}{n+2} & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right],\ n \in \mathbb{N}. \end{cases}$

14. Let $A \subset [a,b]$ be a measurable set and assume that K is a compact set and U is an open set in $[a,b]$ satisfying the properties

(i) $K \subset A \subset U$;
(ii) $\mu(U) - \mu(K) < \varepsilon$ for some $\varepsilon > 0$.

Define two functions d_K, $d_{U^c} : [a,b] \to \mathbb{R}$ by

$$d_K(x) := \inf\{|x-k| : k \in K\} \quad \text{and} \quad d_{U^c}(x) := \inf\{|x-v| : v \in U^c\}$$

(a): Show that

$$\forall_{x,x' \in [a,b]}\ |d_K(x) - d_K(x')| \le |x-x'| \quad \text{and} \quad |d_{U^c}(x) - d_{U^c}(x')| \le |x-x'|$$

(b): Show that

$$\forall_{x \in [a,b]}\ d_K(x) + d_{U^c}(x) > 0.$$

(c): Show that the function $f : [a,b] \to \mathbb{R}$ defined by $f(x) = \frac{d_{U^c}(x)}{d_K(x) + d_{U^c}(x)}$, $x \in [a,b]$ is continuous and satisfies the inequality

$$\int_a^b |f(x) - \chi_A(x)| d\mu(x) < \varepsilon.$$

where $\chi_A : [a,b] \to \mathbb{R}$ is the characteristic function of the set A.

11

Infinite Series and Power Series

11.1 Infinite Series

Let $\{a_n\}$ be a sequence of real or complex numbers. We consider the following sequence of *partial sums*:

$$\begin{aligned} s_1 &= a_1; \\ s_2 &= a_1 + a_2; \\ \vdots\ &\ \ \vdots \\ s_n &= a_1 + a_2 + \cdots + a_n. \end{aligned}$$

If the limit $s = \lim_{n\to\infty} s_n$ exists and is finite, then it can be treated as the infinite sum

$$a_1 + a_2 + \cdots + a_n + \dots,$$

which is called an *infinite series* (or just *series*) and is denoted by the symbol $\sum_{n=1}^{\infty} a_n$. In this case we also say that the series $\sum_{n=1}^{\infty} a_n$ *converges* to s, or simply call it is *convergent*. The number s is called the *sum* of the series $\sum_{n=1}^{\infty} a_n$ and we will write

$$s = \sum_{n=1}^{\infty} a_n.$$

The numbers a_n are called the *terms* of the infinite series. If the limit $\lim_{n\to\infty} s_n$ does not exist, or if it is infinite, we assume that the symbol $\sum_{n=1}^{\infty} a_n$ represents the sequence of the partial sums $\{s_n\}$ and we will call the series $\sum_{n=1}^{\infty} a_n$ *divergent*.

Example 11.1. (a) Let q be a real or complex number such that $|q| < 1$. The series

$$\sum_{n=1}^{\infty} aq^{n-1} = a + aq + aq^2 + \cdots + aq^{n-1} + \dots$$

is called the *geometric series*. We have

$$s_n = a + aq + \cdots + aq^{n-1} = \frac{a - aq^n}{1 - q}.$$

Since $|q| < 1$, thus

$$\lim_{n\to\infty} |q|^n = \lim_{n\to\infty} e^{n \ln |q|} = e^{-\infty} = 0,$$

so $\lim_{n\to\infty} q^n = 0$ and consequently,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a - aq^n}{1-q} = \frac{a}{1-q}.$$

Hence

$$\sum_{n=1}^{\infty} aq^{n-1} = \frac{a}{1-q}.$$

(b) We consider the series

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right) = \sum_{n=1}^{\infty} \left[\ln(n+1) - \ln n\right].$$

Since

$$\begin{aligned} s_n &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln n - \ln(n-1)) + (\ln(n+1) - \ln n) \\ &= \ln(n+1) \longrightarrow \infty \quad \text{as} \quad n \to \infty, \end{aligned}$$

the series $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$ diverges.

(c) Suppose α is an arbitrary number which is not a negative integer, i.e. $\alpha \neq -1, -2, -3, \ldots$. By applying the partial fraction decomposition we obtain

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)} = \sum_{n=1}^{\infty} \left[\frac{1}{\alpha+n} - \frac{1}{\alpha+n+1}\right].$$

It is clear that

$$\begin{aligned} s_n &= \left(\frac{1}{\alpha+1} - \frac{1}{\alpha+2}\right) + \left(\frac{1}{\alpha+2} - \frac{1}{\alpha+3}\right) + \cdots + \left(\frac{1}{\alpha+n} - \frac{1}{\alpha+n+1}\right) \\ &= \frac{1}{\alpha+1} - \frac{1}{\alpha+n+1} \longrightarrow \frac{1}{\alpha+1} \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)} = \frac{1}{\alpha+1}.$$

Similarly, we have

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)(\alpha+n+2)} = \sum_{n=1}^{\infty} \frac{1}{2}\left[\frac{1}{(\alpha+n)(\alpha+n+1)} - \frac{1}{(\alpha+n+1)(\alpha+n+2)}\right].$$

Thus

$$\begin{aligned} s_n &= \frac{1}{2}\left[\left(\frac{1}{(\alpha+1)(\alpha+2)} - \frac{1}{(\alpha+2)(\alpha+3)}\right) + \cdots + \left(\frac{1}{(\alpha+n)(\alpha+n+1)} - \frac{1}{(\alpha+n+1)(\alpha+n+2)}\right)\right] \\ &= \frac{1}{2}\left[\frac{1}{(\alpha+1)(\alpha+2)} - \frac{1}{(\alpha+n+1)(\alpha+n+2)}\right] \longrightarrow \frac{1}{2(\alpha+1)(\alpha+2)} \quad \text{as} \quad n \to \infty. \end{aligned}$$

Consequently,

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)(\alpha+n+2)} = \frac{1}{2(\alpha+1)(\alpha+2)}.$$

In general, we can consider the series

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)\dots(\alpha+n+p)}, \tag{11.1}$$

where p is a natural number. The series (11.1) is called a *telescopic series*. Notice that

$$\frac{1}{(\alpha+n)(\alpha+n+1)\dots(\alpha+n+p)} = \frac{1}{p}\Bigg[\frac{1}{(\alpha+n)(\alpha+n+1)\dots(\alpha+n-1+p)} - \frac{1}{(\alpha+n+1)(\alpha+n+2)\dots(\alpha+n+}$$

therefore,

$$\begin{aligned} s_n &= \frac{1}{p}\left[\frac{1}{(\alpha+1)(\alpha+2)\dots(\alpha+p+1)} - \frac{1}{(\alpha+n+1)(\alpha+n+2)\dots(\alpha+n+p)}\right] \\ &\longrightarrow \frac{1}{p(\alpha+1)(\alpha+2)\dots(\alpha+p+1)} \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence (11.1) converges and

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)\dots(\alpha+n+p)} = \frac{1}{p(\alpha+1)(\alpha+2)\dots(\alpha+p+1)}.$$

(d) We consider the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}. \tag{11.2}$$

Notice that we have the following inequality

$$1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} \ge \sqrt{n},$$

for $n \ge 1$. Indeed, it is clear that the inequality (11.2) is true for $n = 1$, thus we can assume that it is also valid for a certain natural number n. Since

$$\begin{aligned} 1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} &\ge \sqrt{n} + \frac{1}{\sqrt{n+1}} \\ &= \frac{\sqrt{n}\sqrt{n+1}}{\sqrt{n+1}} \ge \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}, \end{aligned}$$

the inequality (11.2) is true by the Principle of Mathematical Induction.
Since $s_n \ge \sqrt{n}$, it follows that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sqrt{n} = \infty$, and consequently the series (11.2) diverges, or $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$.

(e) Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n}. \tag{11.3}$$

Since the partial sums

$$s_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$$

for $n = 2^k$ satisfy the inequalities

$$\begin{aligned}
s_2 &= 1 + \frac{1}{2}\\
s_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}\\
s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)\\
&> 1 + \frac{2}{2} + \frac{4}{8} = 1 + \frac{3}{2}\\
\vdots \quad & \vdots\\
s_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right)\\
&> 1 + \frac{k}{2},
\end{aligned}$$

we have

$$\lim_{k\to\infty} s_{2^k} = \lim_{k\to\infty}\left(1 + \frac{k}{2}\right) = \infty.$$

The sequence $\{s_n\}$ is increasing, thus we obtain that $\lim_{n\to\infty} s_n = \infty$ and

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

i.e. this series is divergent. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the *harmonic series*, which is a particular case of a series, also called the *harmonic* series or the *p-series*, of the type

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \tag{11.4}$$

where p is an arbitrary real number. Assume that $p < 1$, then we have

$$s_n = 1 + \frac{1}{2^p} + \cdots + \frac{1}{n^p} \geq 1 + \frac{1}{2} + \cdots + \frac{1}{n} \longrightarrow \infty \quad \text{as} \quad n \to \infty,$$

therefore, the series (11.4) diverges for $p \leq 1$. Suppose therefore, that $p > 1$. Since

$$\frac{1}{(2^{k-1}+1)^p} + \cdots + \frac{1}{(2^k)^p} \leq \frac{1}{(2^{k-1})^p} + \cdots + \frac{1}{(2^{k-1})^p} = \frac{2^{k-1}}{(2^{k-1})^p} = \frac{1}{(2^{k-1})^{p-1}}.$$

Then for $n = 2^k$ we have

$$\begin{aligned}
s_{2^k} &= 1 + \frac{1}{2^p} + \left(\frac{1}{3^p} + \frac{1}{4^p}\right) + \left(\frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p}\right) + \ldots\\
&+ \left(\frac{1}{(2^{k-1}+1)^p} + \cdots + \frac{1}{(2^k)^p}\right)\\
&\leq \left(1 + \frac{1}{2^p}\right) + \frac{1}{2^{p-1}} + \cdots + \frac{1}{(2^{p-1})^{k-1}}\\
&= 1 + \frac{1}{2^p} + \frac{1 - \frac{1}{(2^{p-1})^k}}{1 - \frac{1}{2^{p-1}}}\\
&\leq 1 + \frac{1}{2^p} + \frac{2^{p-1}}{2^{p-1} - 1} =: M.
\end{aligned}$$

Since the sequence $\{s_n\}$ is increasing and $s_{2^k} \le M$, it follows that $s_n \le M$ for all n, therefore it converges*. In summary, the series (11.4) converges if and only if $p > 1$.

(f) It was proved in subsection 4.2.1 that we

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

The basic properties of the infinite series can be summarized in the following result

Proposition 11.2. *Let* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be two convergent series. Then*

(i) the series $\sum_{n=1}^{\infty}(a_n + b_n)$ *is also convergent and*

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n;$$

(ii)the series $\sum_{n=1}^{\infty} \alpha a_n$ *is also convergent, where* α *is an arbitrary number, and*

$$\sum_{n=1}^{\infty} \alpha a_n = \alpha \sum_{n=1}^{\infty} a_n.$$

Proof: Since the partial sum s_n of the series $\sum_{n=1}^{\infty}(a_n + b_n)$ is the sum of the partial sum s'_n of the series $\sum_{n=1}^{\infty} a_n$ and the partial sum s''_n of the series $\sum_{n=1}^{\infty} b_n$,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} s'_n + \lim_{n\to\infty} s''_n,$$

and the result follows. Similarly, the partial sum $\tilde{s}_n$ of the series $\sum_{n=1}^{\infty} \alpha a_n$ is $\alpha s'_n$, thus

$$\lim_{n\to\infty} \tilde{s}_n = \lim_{n\to\infty} \alpha s'_n = \alpha \lim_{n\to\infty} s'_n.$$

□

Remark 11.3. Notice that the convergence of the series $\sum_{n=1}^{\infty}(a_n + b_n)$ does not in general imply that

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

Indeed, the telescopic series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges, but since $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left[\frac{1}{n} + \frac{-1}{n+1}\right] \ne \sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{-1}{n+1}$$

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* The function

$$\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p},$$

is the famous *Riemann's zeta function*, which plays an important role in number theory. This function can be extended to a function defined for complex numbers. The *Riemann Hypothesis*, which remains the last of the acclaimed unsolved old problems, describes the location of the zeros of the function $\zeta(p)$.

Theorem 11.4. (Cauchy Criterion) *An infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if*

$$\forall_{\varepsilon>0}\exists_N\forall_{n\geq N}\forall_{m\geq n} \qquad \left|\sum_{k=n}^{m} a_k\right| < \varepsilon. \tag{11.5}$$

Proof: Condition (11.5)) states simply that the sequence $\{s_n\}$ of partial sums are a Cauchy sequence, thus the conclusion follows from the completeness of $\mathbb{R}$. □

Corollary 11.5. (Necessary Condition for Convergence) *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Then $\lim_{n\to\infty} a_n = 0$.*

Proof: By assumption, there exists a number s such that

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \big(s_{n-1} + a_n\big) = s + \lim_{n\to\infty} a_n,$$

thus

$$\lim_{n\to\infty} a_n = s - s = 0.$$

□

Corollary 11.6. (Divergence Test) *Let $\{a_n\}$ be a sequence such that the limit $\lim_{n\to\infty} a_n$ does not exists, or $\lim_{n\to\infty} a_n \neq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ diverges.*

Example 11.7. Suppose that q and a are two real or complex numbers such that $|q| \geq 1$ and $a \neq 0$. Since $|aq^{n-1}| \geq |a|$ for all $n \in \mathbb{N}$, it follows that the limit of aq^{n-1}, as $n \to \infty$, can not be equal to 0. Consequently, if $|q| \geq 1$, the geometric series $\sum_{n=1}^{\infty} aq^{n-1}$ diverges.

Remark 11.8. Notice that the condition $\lim_{n\to\infty} a_n = 0$ does not necessarily imply that the series $\sum_{n=1}^{\infty} a_n$ converges. For example, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but still $\lim_{n\to\infty} \frac{1}{n} = 0$. Other divergent series $\sum_{n=1}^{\infty} a_n$, satisfying the condition $\lim_{n\to\infty} a_n = 0$, are $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \ln\big(1 + \frac{1}{n}\big)$.

We should emphasize that an infinite series $\sum_{n=1}^{\infty} a_n$, where a_n are real numbers, can be viewed as an improper integral. Indeed, suppose that $f : [0, \infty) \to \mathbb{R}$ is defined by

$$f(x) = a_n \quad \text{if} \quad x \in [n-1, n),$$

then

$$\sum_{n=1}^{\infty} a_n = \int_0^{\infty} f(x)dx.$$

As in the case of improper integrals, infinite series with positive terms deserve special consideration.

Proposition 11.9. *Let $\{a_n\}$ be such that $a_n > 0$ for all $n = 1, 2, \ldots$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if*

$$\exists_{M>0}\forall_{n\in\mathbb{N}} \quad s_n \leq M.$$

Proof: Since $a_n > 0$, the sequence of partial sums $\{s_n\}$ is increasing and consequently, by Lemma 5.7, it converges if and only if it is bounded from above. $\square$

Theorem 11.10. (Comparison Test) *Suppose $\{a_n\}$ and $\{b_n\}$ are two sequences such that*

$$\exists_{N>0}\forall_{n\geq N}\quad 0 \leq a_n \leq b_n.$$

Then,

(i) if the series $\sum_{n=1}^{\infty} b_n$ converges then the series $\sum_{n=1}^{\infty} a_n$ also converges;
(ii) if the series $\sum_{n=1}^{\infty} a_n$ diverges then the series $\sum_{n=1}^{\infty} b_n$ also diverges.

Proof: Put

$$A_k = a_N + a_{N+1} + \cdots + a_{N+k}$$
$$B_k = b_N + b_{N+1} + \cdots + b_{N+k}.$$

We denote

$$s_n = \sum_{k=1}^{n} a_k, \qquad \text{and} \qquad S_n = \sum_{k=1}^{n} b_k.$$

Then for $n > N$ we have

$$s_n = s_N + A_{n-N}, \qquad \text{and} \qquad S_n = S_N + B_{n-N},$$

thus s_n converges if and only if A_n converges, and S_n converges if and only if B_n converges. Since A_n and B_n are two increasing sequences such that

$$A_n \leq B_n$$

if B_n converges then it is bounded and consequently A_n is also bounded, thus it is convergent. On the other hand, if $\lim_{n\to\infty} A_n = \infty$ then $\lim_{n\to\infty} B_n = \infty$, so the divergence of $\sum_{n=1}^{\infty} a_n$ implies the divergence of $\sum_{n=1}^{\infty} b_n$. $\square$

Theorem 11.11. (Limit Comparison Test) *Let $\{a_n\}$ and $\{b_n\}$ be two positive sequences such that there exists the limit*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = K, \qquad (0 \le K \le \infty).$$

(i) If $0 < K < \infty$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} b_n$ converges;
(ii) If $K = 0$, then the convergence of the series $\sum_{n=1}^{\infty} b_n$ implies the convergence of the series $\sum_{n=1}^{\infty} a_n$;
(iii) If $K = \infty$, then the divergence of the series $\sum_{n=1}^{\infty} b_n$ implies the divergence of the series $\sum_{n=1}^{\infty} a_n$.

Proof: Suppose that $K < \infty$, then

$$\forall_{\varepsilon>0}\exists_N\forall_{n\ge N} \quad K-\varepsilon < \frac{a_n}{b_n} < K+\varepsilon,$$

thus

$$\forall_{\varepsilon>0}\exists_N\forall_{n\ge N} \quad (K-\varepsilon)b_n < a_n < (K+\varepsilon)b_n.$$

Consequently, by the Comparison Test (Theorem 8.1.10), if the series $\sum_{n=1}^{\infty}(K+\varepsilon)b_n = (K+\varepsilon)\sum_{n=1}^{\infty} b_n$ converges, the series $\sum_{n=1}^{\infty} a_n$ also converges. If $K > 0$, then we can choose $\varepsilon > 0$ such that $K - \varepsilon > 0$, so

$$\forall_{n\ge N} \quad 0 < (K-\varepsilon)b_n < a_n,$$

by the Comparison Test, the convergence of the series $\sum_{n=1}^{\infty} a_n$ implies the convergence of the series $\sum_{n=1}^{\infty} b_n$.

Assume now that $K = \infty$, then

$$\forall_{M>0}\exists_N\forall_{n\ge N} \quad \frac{a_n}{b_n} > M,$$

so

$$\forall_{n\ge N} \quad a_n > Mb_n > 0.$$

Therefore, the divergence of the series $\sum_{n=1}^{\infty} b_n$ implies the divergence of $\sum_{n=1}^{\infty} a_n$. □

Theorem 11.12. (Ratio Comparison Test) *Suppose $\{a_n\}$ and $\{b_n\}$ are two positive sequences such that*

$$\exists_{N>0}\forall_{n\ge N} \quad \frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n}$$

Then,

(i) if the series $\sum_{n=1}^{\infty} b_n$ converges then the series $\sum_{n=1}^{\infty} a_n$ also converges;
(ii) if the series $\sum_{n=1}^{\infty} a_n$ diverges then the series $\sum_{n=1}^{\infty} b_n$ also diverges.

Proof: Assume for simplicity, that $\frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n}$ for all n. Since we have

$$\frac{a_2}{a_1} \le \frac{b_2}{b_1}, \quad \frac{a_3}{a_2} \le \frac{b_3}{b_2}, \quad \dots, \quad \frac{a_n}{a_{n-1}} \le \frac{b_n}{b_{n-1}},$$

by cross multiplication, we obtain

$$\frac{a_1}{b_1} \ge \frac{a_2}{b_2} \ge \dots \ge \frac{a_{n-1}}{b_{n-1}} \ge \frac{a_n}{b_n},$$

thus for all n

$$a_n \le \frac{a_1}{b_1} b_n.$$

Consequently the result follows from the Comparison Test (cf. Theorem 11.10). □

Example 11.13. (a) Let a be a real number. If $a \le 1$ then the limit $\lim_{n\to\infty} \frac{1}{1+a^n}$ either does not exist or it is different than 0, thus by the Divergence Test (Corollary 11.6), the series

$$\sum_{n=1}^{\infty} \frac{1}{1+a^n},$$

diverges. Assume therefore that $a > 1$. Since

$$\frac{1}{1+a^n} < \left(\frac{1}{a}\right)^n,$$

and the geometric series

$$\sum_{n=1}^{\infty} \left(\frac{1}{a}\right)^n = \frac{1}{a-1}$$

converges, by the Comparison Test (Theorem 11.10), the series $\sum_{n=1}^{\infty} \frac{1}{1+a^n}$ also converges.

(b) Consider the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}.$$

Since

$$\begin{aligned} a_n := \frac{(n!)^2}{(2n)!} &= \frac{n! \cdot n!}{2 \cdot 4 \cdot \ldots (2n-2)(2n) \cdot 1 \cdot 3 \cdot \cdots \cdot (2n-3)(2n-1)} \\ &= \frac{n! \cdot n!}{2^n \cdot n! \cdot 1 \cdot 3 \cdot \cdots \cdot (2n-3)(2n-1)} \\ &= \frac{1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n}{2^n \cdot 1 \cdot 3 \cdot \cdots \cdot (2n-3)(2n-1)} < \frac{1}{2^n} := b_n, \end{aligned}$$

and the geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges, by the Comparison Test, the series $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ also converges.

(c) Let $\alpha \in (0, 3\pi)$. We consider the following infinite series

$$\sum_{n=1}^{\infty} 2^n \sin\left(\frac{\alpha}{3^n}\right). \tag{11.6}$$

Since

$$2^n \sin\left(\frac{\alpha}{3^n}\right) < \alpha \left(\frac{2}{3}\right)^n,$$

the convergence of the series (11.6) follows again from the Comparison Test (Theorem 11.10):

(d) By the Limit Comparison Test (Theorem 11.11), we conclude that the series

$$\sum_{n=1}^{\infty} \frac{1}{(\ln n)^\alpha},$$

where $\alpha > 0$, is divergent. Indeed,

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{1}{(\ln n)^\alpha}}{\frac{1}{n}} &= \lim_{n\to\infty} \frac{n}{(\ln n)^\alpha} \\ &= \lim_{n\to\infty} \frac{n}{(\ln n)^{\alpha-[\alpha]-1}} \qquad \text{by applying repeatedly L'Hospital Rule} \\ &= \lim_{n\to\infty} n(\ln n)^{[\alpha]+1-\alpha} = \infty, \end{aligned}$$

where $[\alpha]$ denotes the largest integer that is less than or equal to α.

(e) The series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} \tag{11.7}$$

converges. Indeed, since for $n > 3$

$$\begin{aligned}\frac{n!}{n^n} &= \frac{1 \cdot 2 \cdot \ldots (n-1)n}{n \cdot n \cdot \cdots \cdot n \cdot n} \\ &< \frac{2}{n^2},\end{aligned}$$

it follows from the Comparison Test that ((11.7) converges.

(f) The series

$$\sum_{n=1}^{\infty} \frac{1}{(\ln n)^{\ln n}} \tag{11.8}$$

converges. Indeed, we have that

$$(\ln n)^{\ln n} = e^{\ln\left((\ln n)^{\ln n}\right)} = \left(e^{\ln n}\right)^{\ln \ln n} = n^{\ln \ln n},$$

thus

$$\begin{aligned}\lim_{n\to\infty} \frac{\frac{1}{(\ln n)^{\ln n}}}{\frac{1}{n^2}} &= \lim_{n\to\infty} \frac{n^2}{n^{\ln \ln n}} \\ &= \lim_{n\to\infty} \frac{1}{n^{\ln \ln n - 2}} = \frac{1}{\infty} = 0,\end{aligned}$$

thus the convergence of the series (11.8) follows from the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. In a similar way, by comparing it with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, we show that the series

$$\sum_{n=1}^{\infty} \frac{1}{(\ln \ln n)^{\ln n}}$$

converges*.

- (g) We consider the series

$$\sum_{n=1}^{\infty} \frac{1}{(\ln n)^{\ln \ln n}}. \tag{11.9}$$

Notice that for every $n \in \mathbb{N}$ we have

$$(\ln n)^{\ln \ln n} = e^{(\ln \ln n)^2} < e^{\ln n} = n,$$

thus

$$\frac{1}{(\ln n)^{\ln \ln n}} > \frac{1}{n}$$

and, by the Comparison Test and the fact that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series (11.9) diverges.

* Notice that $(\ln \ln n)^{\ln n} = n^{\ln \ln \ln n}$ and $\lim_{n\to\infty}(\ln \ln \ln n - 2) = \infty$.

(h) The series

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$$

is divergent. Indeed, since

$$\lim_{n\to\infty} \frac{\frac{1}{n\sqrt[n]{n}}}{\frac{1}{n}} = \lim_{n\to\infty} n^{-\frac{1}{n}} = 1,$$

the conclusion follows from the Limit Comparison Test and the fact that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(i) Let α be an arbitrary real number. We consider the series

$$\sum_{n=1}^{\infty} \left(1 - \cos\frac{\alpha}{n}\right). \tag{11.10}$$

Since

$$\lim_{n\to\infty} \frac{1-\cos\left(\frac{\alpha}{n}\right)}{\frac{1}{n^2}} = \frac{\alpha^2}{2},$$

the series (11.10) converges by the Limit Comparison Test (Theorem 11.11 and the fact that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(j) Finally, we consider the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln\frac{n+1}{n}\right). \tag{11.11}$$

Since, for $x \in (-1,\infty)$ and $x \neq 0$ we have $\ln(1+x) < x$, thus

$$\ln\frac{n+1}{n} = \ln\left(1+\frac{1}{n}\right) < \frac{1}{n},$$

and

$$\ln\frac{n+1}{n} = -\ln\frac{n}{n+1} = -\ln\left(1-\frac{1}{n+1}\right) > \frac{1}{n+1},$$

so

$$0 < \frac{1}{n} - \ln\frac{n+1}{n} < \frac{1}{n} - \frac{1}{n+1} = \frac{1}{a(n+1)} < \frac{1}{n^2},$$

therefore, the series (11.11) converges by the Comparison Test (Theorem 11.10).

The idea of using the comparison of the geometric series $\sum_{n=1}^{\infty} q^n$ with an infinite series $\sum_{n=1}^{\infty} a_n$ with positive terms leads to the following tests:

Theorem 11.14. (Cauchy Convergence Criterion) *Let $\{a_n\}$ be a sequence of positive real numbers. If there exists a number $1 > q > 0$ such that*

$$\exists_N \forall_{n\geq N} \quad \sqrt[n]{a_n} \leq q, \tag{11.12}$$

then the series $\sum_{n=1}^{\infty} a_n$ converges. If

$$\forall_N \exists_{n\geq N} \quad \sqrt[n]{a_n} \geq 1, \tag{11.13}$$

then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof: Since the inequality (11.12) implies that for $n \geq N$ we have

$$0 < a_n < q^n,$$

thus, by the Comparison Test (Theorem 11.10) and the fact that the geometric series $\sum_{n=1}^{\infty} q^n$ converges, the infinite series $\sum_{n=1}^{\infty} a_n$ converges.

On the other hand, the inequality (11.13) implies that the sequence $\{a_n\}$ can not converge to zero, thus the series $\sum_{n=1}^{\infty} a_n$ diverges. □

Corollary 11.15. (Root Test) *Assume that $\{a_n\}$ is a sequence of positive real numbers such that there exists the limit*

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lambda.$$

(i) If $\lambda < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges;
(ii) If $\lambda > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof: Assume that $\lambda < 1$ and let $\varepsilon > 0$ be such that $q := \lambda + \varepsilon < 1$. Then

$$\exists_N \forall_{n\geq N} \quad \sqrt[n]{a_n} < \lambda + \varepsilon = q < 1,$$

and (i) follows from the Cauchy Convergence Criterion (Theorem 11.14).

If $\lim_{n\to\infty} \sqrt[n]{a_n} = \lambda > 1$, then for sufficiently large n $\sqrt[n]{a_n} \geq 1$ and again the conclusion (ii) follows from Theorem 8.1.14. □

Theorem 11.16. (D'Alembert Convergence Criterion) *Let $\{a_n\}$ be a sequence of positive real numbers. If there exists a number $1 > q > 0$ such that*

$$\exists_N \forall_{n\geq N} \quad \frac{a_{n+1}}{a_n} \leq q, \tag{11.14}$$

then the series $\sum_{n=1}^{\infty} a_n$ converges. If

$$\exists_N \forall_{n\geq N} \quad \frac{a_{n+1}}{a_n} \geq 1, \tag{11.15}$$

then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof: Notice, that the condition (11.14) implies that

$$\begin{aligned} a_{N+1} &< q a_N, \\ a_{N+2} &< q a_{N+1} < q^2 a_N, \\ \vdots &\ \vdots \\ a_{N+k} &< q^k a_N, \end{aligned}$$

consequently, for $n > N$ we have that

$$0 < a_n < q^{n-N} a_N,$$

and since the geometric series $\sum_{n=1}^{\infty} q^n$ converges, the convergence of $\sum_{n=1}^{\infty} a_n$ follows from the Comparison Test (Theorem 11.10).

In the case where condition (11.15) is satisfied, we obtain, in a similar way, that for $n > N$

$$a_n \geq a_N,$$

thus by the Divergence Test, the series $\sum_{n=1}^{\infty} a_n$ diverges. □

Corollary 11.17. *(Ratio Test) Assume that $\{a_n\}$ is a sequence of positive real numbers such that the limit*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda$$

exists.

(i) If $\lambda < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges;
(ii) If $\lambda > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

By using the idea of comparing the series $\sum_{n=1}^{\infty} a_n$ with the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

we obtain another criterion, called the *Raabe Convergence Criterion*.

Theorem 11.18. *(Raabe Convergence Criterion) Let $\{a_n\}$ be a sequence of positive real numbers. If there exists a number $p > 1$ such that*

$$\exists_N \forall_{n\geq N} \quad n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq p, \tag{11.16}$$

then the series $\sum_{n=1}^{\infty} a_n$ converges. If

$$\exists_N \forall_{n\geq N} \quad n\left(\frac{a_n}{a_{n+1}} - 1\right) < 1, \tag{11.17}$$

then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof: Assume that condition (11.16) is satisfied and let s be a real number such that $p > s > 1$. Then we have

$$\lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^s - 1}{\frac{1}{n}} = s,$$

thus for sufficiently large n we also have

$$\frac{\left(1+\frac{1}{n}\right)^s - 1}{\frac{1}{n}} < p,$$

so

$$\left(1+\frac{1}{n}\right)^s < 1 + \frac{p}{n},$$

and consequently

$$\frac{a_n}{a_{n+1}} > \left(1+\frac{1}{n}\right)^s. \tag{11.18}$$

The inequality (11.18) can be written as

$$\frac{a_{n+1}}{a_n} < \left(\frac{n}{n+1}\right)^s = \frac{\frac{1}{(n+1)^s}}{\frac{1}{n^s}}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges (we assumed that $s > 1$), thus by the Ratio Test (Corollary 11.17), the series $\sum_{n=1}^{\infty} a_n$ also converges.

If for sufficiently large n

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \leq 1,$$

then

$$\frac{a_{n+1}}{a_n} \geq \frac{n}{n+1} = \frac{\frac{1}{n+1}}{\frac{1}{n}},$$

and again the conclusion follows from the Ratio Test. □

Corollary 11.19. (Raabe Test) *Assume that $\{a_n\}$ is a sequence of positive real numbers such that the limit*

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \rho$$

exists.

(i) If $\rho > 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges;
(ii) If $\rho < 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example 11.20. (a) The series

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n},$$

converges by Root Test (Corollary 11.15). Indeed, we have

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 < 1.$$

(b) Let $x > 0$ be a certain real number. Then the series

$$\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n$$

converges by the Root Test (Corollary 11.15). Indeed

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{x}{n}\right)^n} = \lim_{n\to\infty} \frac{x}{n} = 0 < 1.$$

(c) Let $x > 0$ be a real number. The series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges by the Ratio Test (Corollary 11.17). Indeed, we have

$$\lim_{n\to\infty} \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \lim_{n\to\infty} \frac{x}{n} = 0 < 1.$$

(d) Let $x > 0$ be a real number. We consider the series

$$\sum_{n=1} \frac{n!x^n}{n^n}. \tag{11.19}$$

By applying the Ratio Test (Corollary 11.17)we obtain that

$$\lim_{n\to\infty} \frac{\frac{(n+1)!x^{n+1}}{(n+1)^{n+1}}}{\frac{n!x^n}{n^n}} = \lim_{n\to\infty} \frac{x}{\left(1+\frac{1}{n}\right)^n} = \frac{x}{e}.$$

Therefore, if $x < e$, then the series (11.19) converges, and if $x > e$, the series ((11.19) diverges. In the case $x = e$ the Ratio Test (Corollary 11.17) is inconclusive. However, the series $\sum_{n=1}^{\infty} \frac{n!e^n}{n^n}$ satisfies

$$\frac{\frac{(n+1)!e^{n+1}}{(n+1)^{n+1}}}{\frac{n!e^n}{n^n}} = \frac{e}{\left(1+\frac{1}{n}\right)^n} > 1,$$

thus the series

$$\sum_{n=1}^{\infty} \frac{n!e^n}{n^n}$$

diverges by the D'Alambert Criterion (Theorem 11.16).

(e) Let $x > 0$ be a real number. We apply the Ratio Test to the series

$$\sum_{n=1}^{\infty} \frac{(nx)^n}{n!}. \tag{11.20}$$

Since

$$\lim_{n\to\infty} \frac{\frac{((n+1)x)^{n+1}}{(n+1)!}}{\frac{(nx)^n}{n!}} = x \lim_{n\to\infty} (\left(1+\frac{1}{n}\right)^n = xe.$$

The series (11.20) converges if $x < \frac{1}{e}$ and it diverges if $x > \frac{1}{e}$. If $x = \frac{1}{e}$ the Ratio Test is inconclusive. Suppose that $x = \frac{1}{e}$, i.e. we consider the series

$$\sum_{n=1}^{\infty} \frac{n^n}{n!e^n}. \tag{11.21}$$

We apply the Raabe Test (Corollary 11.19)

$$\lim_{n\to\infty} n\left(\frac{\frac{n^n}{n!e^n}}{\frac{(n+1)^{n+1}}{(n+1)!e^{n+1}}} - 1\right) = \lim_{n\to\infty} n\left(\frac{e}{\left(1+\frac{1}{n}\right)^n} - 1\right). \tag{11.22}$$

In order to compute the limit (11.22) we apply the l'Hospital's Rule

$$\begin{aligned}\lim_{x\to 0}\frac{1}{x}\left(\frac{e}{(1+x)^{\frac{1}{x}}}-1\right) &= \lim_{x\to 0}\frac{e}{(1+x)^{\frac{1}{x}}}\left(\frac{-\frac{x}{1+x}+\ln(1+x)}{x^2}\right)\\ &= \lim_{x\to 0}\frac{(1+x)\ln(1+x)-x}{x^2(1+x)}\\ &= \lim_{x\to 0}\frac{-1+\ln(1+x)+1}{2x^2+3x^2}\\ &= \lim_{x\to 0}\frac{\frac{1}{1+x}}{2+6x}=\frac{1}{2}.\end{aligned}$$

Since the limit (11.22) is equal to $\frac{1}{2}$, by the Raabe Test (Corollary 11.19) , the series (11.21) diverges.

Theorem 11.21. (MACLAURIN-CAUCHY INTEGRAL TEST) *Let $\{a_n\}$ be a sequence of positive numbers and $f:[0,\infty)\to\mathbb{R}$ a positive function such that*

(i) $a_n=f(n)$ for $n=1, 2, \dots$;
(ii) There exists $N>0$ such that the function $f(x)$ is non-increasing on the interval $[N,\infty)$.

Then the series

$$\sum_{n=1}^{\infty}a_n$$

converges if and only if the improper integral

$$\int_0^{\infty}f(x)dx$$

converges.

Proof: Assume for simplicity that $f(x)$ is non-increasing on the whole interval $[0,\infty)$. Then we have for every $n_o\in\mathbb{N}$

$$\int_{n_o}^{n_o+1}f(x)dx\ge a_{n_o+1},$$

thus

$$\sum_{n=1}^{n_o}a_n\le\int_0^{n_o}f(x)dx,$$

thus the convergence of the improper integral $\int_0^{\infty}f(x)dx$ implies, by Proposition 11.9 , that the series $\sum_{n=1}^{\infty}a_n$ converges (see the picture below).

On the other hand, since

$$\int_{n_o}^{n_o+1}f(x)dx\le a_{n_o},$$

for all $n_o\ge 1$ we have

$$\sum_{n=1}^{\infty}a_n\ge\int_1^{\infty}f(x)dx,$$

therefore, if the series $\sum_{n=1}^{\infty}a_n$ diverges, the improper integral also diverges (see the picture above).
□

Example 11.22. (a) The series
$$\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln \ln n}$$
is divergent by the Integral Test. Indeed,
$$\int_3^x \frac{dt}{t \ln t \ln \ln t} = \ln \ln \ln x - \ln \ln \ln 3 \longrightarrow \infty \quad \text{as} \quad x \to \infty.$$
(b) We consider the series
$$\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln \ln n)^{p+1}}, \tag{11.23}$$
where $p > 0$. Since
$$\int_3^x \frac{dt}{t \ln t (\ln \ln t)^{p+1}} = -\frac{1}{p(\ln \ln x)^p} + \frac{1}{p(\ln \ln 3)^p} \longrightarrow \frac{1}{p(\ln \ln 3)^p} \quad \text{as} \quad x \to \infty,$$
by the Integral Test, the series (11.23) converges.

11.2 Absolute and Conditional Convergence of Infinite Series

In the previous section we developed several tests to determine whether an infinite series with positive terms is convergent or not. It is possible to apply these tests to determine the convergence or divergence of an infinite series with arbitrary real or complex terms.

Theorem 11.23. *Let $\{a_n\}$ be a sequence of real or complex numbers. If the series*
$$\sum_{n=1}^{\infty} |a_n|$$
converges, then the series
$$\sum_{n=1}^{\infty} a_n$$
also converges.

Proof: Since $\sum_{n=1}^{\infty} |a_n|$ converges, we have
$$\forall_{\varepsilon>0} \exists_N \forall_{m>n\geq N} \quad \sum_{k=n}^{m} |a_k| < \varepsilon.$$
Since,
$$\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k|,$$
it follows that
$$\forall_{\varepsilon>0} \exists_N \forall_{m>n\geq N} \quad \left| \sum_{k=n}^{m} a_k \right| < \varepsilon,$$
so the series $\sum_{n=1}^{\infty} a_n$ is also convergent. □

Definition 11.24. Let $\{a_n\}$ be a sequence of real or complex numbers. We say that the series $\sum_{n=1}^{\infty} a_n$ *converges absolutely*, or that it is *absolutely convergent*, if the series $\sum_{n=1}^{\infty} |a_n|$ converges. If a series $\sum_{n=1}^{\infty} a_n$ converges but the series $\sum_{n=1}^{\infty} |a_n|$ diverges, then we say that the series $\sum_{n=1}^{\infty} a_n$ *converges conditionally.*

Corollary 11.25. (RATIO TEST) *Let $\{a_n\}$ be a sequence of non-zero numbers such that the limit*

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lambda$$

exists.

(i) If $\lambda < 1$ then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely;
(ii) If $\lambda > 1$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof: Notice that if $\lambda \neq 1$ then the series $\sum_{n=1}^{\infty} |a_n|$ can be compared to the geometric series $\sum_{n=1}^{\infty} q^n$, where $q < 1$ if $\lambda < 1$ and $q = 1$ when $\lambda > 1$. Since in the second case $\lim_{n\to\infty} |a_n| \neq 0$, the divergence of the series $\sum_{n=1}^{\infty} a_n$ follows from the Divergence Test. □

Example 11.26. We define the *hypergeometric* series

$$H(x, \alpha, \beta, \gamma) := 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\dots(\alpha+n-1)\cdot\beta(\beta+1)\dots(\beta+n-1)}{n!\cdot\gamma(\gamma+1)\dots(\gamma+n-1)} x^n, \tag{11.24}$$

where α, β and γ are non-zero real numbers not equal to any negative integer, and x is any number. In this case the terms of the series

$$a_n = \frac{\alpha(\alpha+1)\dots(\alpha+n-1)\cdot\beta(\beta+1)\dots(\beta+n-1)}{n!\cdot\gamma(\gamma+1)\dots(\gamma+n-1)} x^n$$

can be negative. We can apply the Ratio Test to the series $\sum_{n=1}^{\infty} |a_n|$, i.e. we compute

$$\begin{aligned}\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{\left|\frac{\alpha(\alpha+1)\dots(\alpha+n-1)(\alpha+n)\cdot\beta(\beta+1)\dots(\beta+n-1)(\beta+n)}{(n+1)!\cdot\gamma(\gamma+1)\dots(\gamma+n-1)(\gamma+n)} x^{n+1}\right|}{\left|\frac{\alpha(\alpha+1)\dots(\alpha+n-1)\cdot\beta(\beta+1)\dots(\beta+n-1)}{n!\cdot\gamma(\gamma+1)\dots(\gamma+n-1)} x^n\right|} \\ &= \lim_{n\to\infty} \frac{|\alpha+n||\beta+n|}{(n+1)|\gamma+n|}|x| = |x|.\end{aligned}$$

Then, by the Ratio Test the series (11.24) converges for $|x| < 1$ and it diverges for $|x| > 1$.

For $x = 1$ we have

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}.$$

Since

$$\frac{1}{1+\frac{a}{n}} = 1 - \frac{a}{n} + \frac{a^2}{1+\frac{a}{n}}\cdot\frac{1}{n^2},$$

it follows that

$$\frac{a_n}{a_{n+1}} = 1 + \frac{\gamma-\alpha-\beta+1}{n} + \frac{\theta_n}{n^2},$$

where θ_n is a bounded sequence. If $\gamma-\alpha-\beta+1<0$, then for n sufficiently large $\frac{\gamma-\alpha-\beta+1}{n}+\frac{\theta_n}{n^2}<0$, thus

$$\frac{a_{n+1}}{a_n}\geq 1,$$

and consequently the series $\sum_{n=1}^{\infty} a_n$ diverges. However, in order to examine the convergence or divergence of this series in a general case, one needs a more powerful convergence test.

Corollary 11.27. (Root Test) *Let $\{a_n\}$ be a sequence of non-zero numbers such that the limit*

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lambda$$

exists.

(i) If $\lambda<1$ then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely;
(ii) If $\lambda>1$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Definition 11.28. Let $\{c_n\}$ be a sequence of positive numbers. The series

$$\sum_{n=1}(-1)^{n+1}c_n,$$

is called an *alternating series.*

Theorem 11.29. (Leibnitz Theorem – Alternating Series Test) *Let $\{c_n\}$ be a sequence of positive numbers and $n_o\in\mathbb{N}$ such that*

(i) $c_{n+1}<c_n$ for all $n\geq n_o$;
(ii) $\lim_{n\to\infty} c_n=0$.

Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}c_n$ converges and for $n\geq n_o$

$$\left|\sum_{k=1}^{\infty}(-1)^{k+1}c_k-\sum_{k=1}^{n}(-1)^{k}c_k\right|<c_{n+1}.$$

Proof: The $2k$-th partial sum s_{2k} of the series $\sum_{n=1}^{\infty}(-1)^n c_n$, where $k\geq 1$, can be written in the form

$$s_{2k}=(c_1-c_2)+(c_3-c_4)+\cdots+(c_{2k-1}-c_{2k}).$$

Since $c_n-c_{n+1}>0$, we have

$$s_2<s_4<\cdots<s_{2k}<s_{2k+2}.$$

Moreover,

$$s_{2k}=c_1-(c_2-c_3)-\cdots-(c_{2k-2}-c_{2k-1})-c_{2k},$$

thus

$$s_{2k}<c_1.$$

Consequently, the sequence $\{s_{2k}\}$ is increasing and bounded from above. By Lemma 5.7, it converges, i.e. there exists

$$s_* = \lim_{k\to\infty} s_{2k}.$$

On the other hand, the $2k+1$-partial sum s_{2k+1}, $k \geq 1$, can be written as

$$s_{2k+1} = c_1 - (c_2 - c_3) - \cdots - (c_{2k} - c_{2k+1}),$$

thus

$$s_1 > s_3 > \cdots > s_{2k-1} > s_{2k+1}.$$

Since

$$s_{2k+1} = (c_1 - c_2) + (c_3 - c_4) + \cdots + (c_{2k-1} - c_{2k}) + c_{2k+1},$$

it follows that

$$s_{2k+1} > s_{2k},$$

so the sequence $\{s_{2k+1}\}$ is a bounded from below decreasing sequence. By Lemma 5.7, , there exists

$$s^* = \lim_{k\to\infty} s_{2k+1}.$$

Since $s_{2k+1} - s_{2k} = c_{2k+1}$ it follows that

$$s^* - s_* = \lim_{k\to\infty} s_{2k+1} - \lim_{k\to\infty} s_{2k} = \lim_{k\to\infty} c_{2k+1} = 0,$$

therefore, all the partial sums (i.e. even and odd) converge to the limit $s = s_* = s^*$ such that

$$s_{2l} < s < s_{2s+1},$$

for $l, s \in \mathbb{N}$. Since $|s_{2k} - s_{2k-1}| = c_{2k}$, we have

$$\begin{aligned}\left|\sum_{k=1}^{n}(-1)^{k+1}c_k - \sum_{k=1}^{\infty}(-1)^{k+1}c_k\right| &= |s_n - s| \\ &< |s_n - s_{n+1}| = c_{n+1},\end{aligned}$$

and the conclusion follows. □

Example 11.30. By the Alternating Series Test (Theorem 11.29), the following infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \ln^p n}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot \ln n \cdot (\ln \ln n)^p},$$

where $p > 0$, converge. Moreover, if $p > 1$, these series converge absolutely and if $p \leq 1$ they converge conditionally.

Theorem 11.31. (ABEL'S TEST) *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers such that:*

(i) the series $\sum_{n=1}^{\infty} b_n$ converges,
(ii) the sequence $\{a_n\}$ is monotonic and bounded.

Then the series

$$\sum_{n=1}^{\infty} a_n b_n$$

also converges.

Proof: Assume that $\varepsilon > 0$ is an arbitrary number. The sequence $\{a_n\}$ is bounded, thus

$$\exists_{L>0} \forall_{n\in\mathbb{N}} \quad |a_n| < L.$$

On the other hand, the series $\sum_{n=1}^{\infty} b_n$ converges, therefore

$$\exists_N \forall_{m>n\geq N} \left| \sum_{k=n+1}^{m} b_k \right| < \frac{\varepsilon}{3L}. \tag{11.25}$$

Suppose that $n \geq N$. For $m > n$ we denote by

$$\sigma_m = b_{n+1} + \cdots + b_m$$

the partial sum of the series $\sum_{k=n+1}^{\infty} b_k$. Notice, that by (11.25)

$$|\sigma_m| < \frac{\varepsilon}{3L} \tag{11.26}$$

for all $m > n$.

We will show that, for $m > n \geq N$ we have

$$|s_m - s_n| < \varepsilon,$$

where $s_n = \sum_{k=1}^{n} a_k b_k$. Indeed, we have

$$\begin{aligned}
s_m - s_n &= a_{n+1}b_{n+1} + a_{n+2}b_{n+2} + \cdots + a_m b_m \\
&= a_{n+1}\sigma_{n+1} + a_{n+2}(\sigma_{n+2} - \sigma_{n+1}) + \cdots + a_m(\sigma_m - \sigma_{m-1}) \\
&= \sigma_{n+1}(a_{n+1} - a_{n+2}) + \sigma_{n+2}(a_{n+2} - a_{n+3}) + \cdots + \sigma_m(a_{m-1} - a_m) + a_m\sigma_m \\
&= \sum_{k=n+1}^{m-1} (a_k - a_{k+1})\sigma_k + a_m\sigma_m.
\end{aligned}$$

Therefore

$$|s_m - s_n| < \sum_{k=n+1}^{m} |a_k - a_{k+1}| \frac{\varepsilon}{3L} + |a_m| \frac{\varepsilon}{3L}.$$

Since the sequence $\{a_n\}$ is monotonic, all the terms $(a_k - a_{k+1})$ have the same sign, therefore for all k, either $|a_k - a_{k+1}| = a_k - a_{k+1}$ (if $\{a_n\}$ is decreasing), or $|a_k - a_{k+1}| = -(a_k - a_{k+1})$ (if $\{a_n\}$ is increasing). Hence

$$\sum_{k=n+1}^{m} |a_k - a_{k+1}| = |a_{n+1} - a_m|,$$

thus

$$\begin{aligned}
|s_m - s_n| &< (|a_{n+1} - a_m| + |a_m|) \frac{\varepsilon}{3L} \\
&\leq (|a_{n+1}| + 2|a_m|) \frac{\varepsilon}{3L} \leq 3L \frac{\varepsilon}{3L} = \varepsilon,
\end{aligned}$$

and the conclusion of the Theorem 11.31 follows. $\square$

Theorem 11.32. (Dirichlet's Test) *Assume that the partial sums of the series*

$$\sum_{n=1}^{\infty} b_n$$

are bounded, i.e.

$$\exists_{M>0}\forall_{n\in\mathbb{N}} \quad |b_1 + b_2 + \cdots + b_n| \le M,$$

and let $\{a_n\}$ *be a monotonic sequence such that*

$$\lim_{n\to\infty} a_n = 0.$$

Then the series

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

Proof: Assume that $\varepsilon > 0$ is an arbitrary number. The sequence $\{a_n\}$ converges to zero, thus

$$\exists_{N>0}\forall_{n\ge N} \quad |a_n| < \frac{\varepsilon}{6M}.$$

On the other hand, the partial sums of the series $\sum_{n=1}^{\infty} b_n$ are bounded, therefore

$$\forall_{m>n\ge N} \left|\sum_{k=n+1}^{m} b_k\right| \le \left|\sum_{k=1}^{n} b_k\right| + \left|\sum_{k=1}^{m} b_k\right| \le 2M. \tag{11.27}$$

Suppose that $n \ge N$. For $m > n$ we denote by

$$\sigma_m = b_{n+1} + \cdots + b_m$$

the partial sum of the series $\sum_{k=n+1}^{\infty} b_k$. Notice, that by (11.27)

$$|\sigma_m| < 2M$$

for all $m > n$.

We will show that, for $m > n \ge N$ we have

$$|s_m - s_n| < \varepsilon,$$

where $s_n = \sum_{k=1}^{n} a_k b_k$. We show, in the same way as in the proof of Theorem 11.31 that

$$s_m - s_n = \sum_{k=n+1}^{m-1} (a_k - a_{k+1})\sigma_k + a_m\sigma_m.$$

Therefore

$$|s_m - s_n| \le 2M(|a_{n+1}| + 2|a_m|) < 6M\frac{\varepsilon}{6M} = \varepsilon,$$

and the conclusion follows. □

Example 11.33. An application of complex numbers leads to the following formulas

$$\sum_{k=1}^{n} \sin kx = \frac{\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}, \qquad \sum_{k=1}^{n} \cos kx = \frac{\sin\left(n+\frac{1}{2}\right)x - \sin\frac{x}{2}}{2\sin\frac{x}{2}}.$$

Indeed, consider the complex number $\gamma = e^{ix} = \cos x + i\sin x$. By de Moivre's formula, we have that

$$\sum_{k=1}^{n} \cos kx = \operatorname{Re}\left(\sum_{k=1}^{n} \gamma^k\right), \qquad \sum_{k=1}^{n} \sin kx = \operatorname{Im}\left(\sum_{k=1}^{n} \gamma^k\right).$$

Therefore,

$$\begin{aligned}
\sum_{k=1}^{n} \gamma^k &= \gamma \frac{1-\gamma^n}{1-\gamma} \\
&= \frac{e^{ix} - e^{i(n+1)x}}{1-e^{ix}} \\
&= \frac{e^{ix} - 1 - e^{i(n+1)x} + e^{inx}}{2(1-\cos x)} \\
&= \frac{\cos nx - \cos(n+1)x - (1-\cos x) + i(\sin x - \sin(n+1)x + \sin nx)}{4\sin^2\frac{x}{2}} \\
&= \frac{\left(2\sin\left(n+\frac{1}{2}\right)x\sin\frac{x}{2} - 2\sin^2\frac{x}{2}\right) + i\left(2\sin\frac{x}{2}\cos\frac{x}{2} - 2\sin\frac{x}{2}\cos\left(n+\frac{1}{2}\right)x\right)}{4\sin^2\frac{x}{2}} \\
&= \frac{\sin\left(n+\frac{1}{2}\right)x - \sin\frac{x}{2}}{2\sin\frac{x}{2}} + i\frac{\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}.
\end{aligned}$$

Consequently, we obtain that the partial sums of the series

$$\sum_{n=1}^{\infty} \sin nx$$

are bounded. Since the sequence $\left\{\frac{1}{n}\right\}$ converges monotonically to zero, by Dirichlet Test (Theorem 11.32), the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

converges.
Similarly, by applying the Dirichlet Test (Theorem 11.32), the series

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) \frac{\sin nx}{n}$$

converges.

Theorem 11.34. (Associativity of Infinite Series) *Assume that the series*

$$\sum_{n=1}^{\infty} a_n \tag{11.28}$$

converges. Then the series $\sum_{n=1}^{\infty} a_n^*$ obtained from the sums of terms of the series (11.28)

$$\underbrace{(a_1 + \dots + a_{n_1})}_{a_1^*} + \underbrace{(a_{n_1+1} + \dots + a_{n_2})}_{a_2^*} + \dots + \underbrace{(a_{n_{k-1}+1} + \dots + a_k)}_{a_k^*} + \dots,$$

also converges to the same sum.

Proof: Notice, that the partial sums s_n^* of the series $\sum_{n=1}^{\infty} a_n^*$ form a subsequence of the sequence of the partial sums $\{s_n\}$ of the series $\sum_{n=1}^{\infty} a_n$ and consequently, the limit of the subsequence $\{s_n^*\}$ is equal to the limit $\lim_{n\to\infty} s_n$. $\square$

Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series. We denote by $\{p_n\}$ the subsequence of $\{a_n\}$ composed of all positive values of a_n, and by $\{q_n\}$ the subsequence composed of all values of $-a_n$ for a_n negative. By the assumption, there exists s such that $\sum_{n=1}^{\infty} |a_n| = s$. Since every partial sum of the series $\sum_{n=1}^{\infty} p_n$ or $\sum_{n=1}^{\infty} q_n$ is a part of a certain partial sum s_n of the series $\sum_{n=1}^{\infty} |a_n|$ and $s_n \le s$ for all $n \ge 1$, it follows that the partial sums σ_n^+ of $\sum_{n=1}^{\infty} p_n$ and the partial sums σ_n^- of $\sum_{n=1}^{\infty} q_n$, are bounded by s. By Proposition 11.9, the series $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ converge. On the other hand, if the partial sum σ_n of $\sum_{n=1}^{\infty} a_n$ contains $k(n)$ positive terms and $l(n)$ negative terms, then

$$\sigma_n = \sigma_{k(n)}^+ - \sigma_{l(n)}^-. \tag{11.29}$$

If the sequence $\{a_n\}$ contains infinitely many positive and negative terms, then $k(n) \to \infty$ and $l(n) \to \infty$ as $n \to \infty$. Therefore, by passing in (11.29) to the limit as $n \to \infty$, we obtain that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n.$$

Consequently we have the following

Proposition 11.35. *Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers. Then the sum of $\sum_{n=1}^{\infty} a_n$ is equal to the difference of the sum of the series $\sum_{n=1}^{\infty} p_n$, composed of the all terms a_n with $a_n > 0$, and the series $\sum_{n=1}^{\infty} q_n$, composed of the terms $-a_n$ with $a_n < 0$, i.e.*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n.$$

The next result states that for an absolutely convergent series its sum does not depend on the order of its terms.

Theorem 11.36. (Commutativity Property) *Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers. Then any series $\sum_{k=1}^{\infty} a_{n(k)}$, obtained from $\sum_{n=1}^{\infty} a_n$ by rearrangement of its terms, converges to the same limit.*

Proof: We assume first that the sequence $\{a_n\}$ is positive and let $s = \sum_{n=1}^{\infty} a_n$. Since any partial sum s_n' of the rearranged series $\sum_{k=1}^{\infty} a_{n(k)}$ is a part of a certain partial sum s_N of $\sum_{n=1}^{\infty} a_n$, thus we have

$$s_n' \le s_N \le s,$$

thus the partial sums s_n' are bounded by s and, by Proposition 11.9, the series $\sum_{k=1}^{\infty} a_{n(k)}$ converges, so

$$\sum_{k=1}^{\infty} a_{n(k)} \le \sum_{n=1}^{\infty} a_n.$$

On the other hand, the series $\sum_{n=1}^{\infty} a_n$ can be obtained from $\sum_{k=1}^{\infty} a_{n(k)}$ by rearranging its terms, therefore exactly the same argument proves that

$$\sum_{n=1}^{\infty} a_n \le \sum_{k=1}^{\infty} a_{n(k)},$$

so

$$\sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} a_{n(k)}.$$

Assume now that the a_n are arbitrary real numbers. Since the series $\sum_{n=1}^{\infty} a_n$ can be written as the difference of two series $\sum_{n=1}^{\infty} p_n$ (composed of all terms a_n with $a_n > 0$) and the series $\sum_{n=1}^{\infty} q_n$ (composed of terms $-a_n$ with $a_n < 0$), where $p_n > 0$ and $q_n > 0$, and similarly, the series $\sum_{k=1}^{\infty} a_{n(k)}$ can be written as the difference of two series $\sum_{k=1}^{\infty} p_{n(k)}$ (composed of all terms $a_{n(k)}$ with $a_{n(k)} > 0$) and the series $\sum_{k=1}^{\infty} q_{n(k)}$ (composed of terms $-a_{n(k)}$ with $a_{n(k)} < 0$), where $p_{n(k)} > 0$ ad $q_{n(k)} > 0$, we have that*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n = \sum_{k=1}^{\infty} p_{n(k)} - \sum_{k=1}^{\infty} q_{n(k)} = \sum_{k=1}^{\infty} a_{n(k)}.$$

$\square$

The above property of the absolutely convergent series is not satisfied by conditionally convergent series.

Theorem 11.37. (Riemann's Theorem) *Let $\{a_n\}$ be a sequence of real numbers such that the series $\sum_{n=1}^{\infty} a_n$ converges conditionally and let α denote an arbitrary real number or $\pm\infty$. Then it is possible to rearrange the terms of the series $\sum_{n=1}^{\infty} a_n$ in such a way that the obtained new series $\sum_{k=1}^{\infty} a_{n(k)}$ satisfies*

$$\sum_{k=1}^{\infty} a_{n(k)} = \alpha.$$

Proof: Since the series $\sum_{n=1}^{\infty} a_n$ converges conditionally, the series $\sum_{n=1}^{\infty} p_n$, composed of all the terms a_n with $a_n > 0$, and the series $\sum_{n=1}^{\infty} q_n$, composed of the terms $-a_n$ with $a_n < 0$, diverge, i.e.

$$\sum_{n=1}^{\infty} p_n = \infty, \qquad \sum_{n=1}^{\infty} q_n = \infty.$$

Suppose that α is a finite number. We construct the series $\sum_{k=1}^{\infty} a_{n(k)}$ as follows. The first terms of this series are the first k_1 terms of the series $\sum_{n=1}^{\infty} p_n$ such that

$$p_1 + p_2 + \dots p_{k_1} > \alpha.$$

Next, we subtract the terms q_n of the series $\sum_{n=1}^{\infty} q_n$ until the obtained partial sum satisfies

$$p_1 + p_2 + \dots p_{k_1} - q_1 - q_2 - \dots - q_{m_1} < \alpha.$$

* the series $\sum_{k=1}^{\infty} p_{n(k)}$ is obtained from $\sum_{n=1}^{\infty} p_n$ by rearrangement of its terms, and the series $\sum_{k=1}^{\infty} q_{n(k)}$ is also obtained from $\sum_{n=1}^{\infty} q_n$ by rearrangement of terms. Thus, by the first part of the proof $\sum_{k=1}^{\infty} p_{n(k)} = \sum_{n=1}^{\infty} p_n$ and $\sum_{k=1}^{\infty} q_{n(k)} = \sum_{n=1}^{\infty} q_n$.

Next, we continue adding the terms of the series $\sum_{n=1}^{\infty} p_n$ until we obtain

$$p_1 + p_2 + \dots p_{k_1} - q_1 - q_2 - \dots - q_{m_1} + p_{k_1+1} + \dots + p_{k_2} > \alpha.$$

By continuing this process, we construct the series

$$p_1 + p_2 + \dots p_{k_1} - q_1 - q_2 - \dots - q_{m_1} + \dots + p_{k_r+1} + \dots + p_{k_{r+1}} - q_{m_r+1} - \dots - q_{m_{r+1}} + \dots.$$

Since the series $\sum_{n=1}^{\infty} a_n$ converges, it follows that $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = 0$. Therefore, the obtained series converges to α.

If $\alpha = \infty$, we construct the series $\sum_{k=1}^{\infty} a_{n(k)}$ in a similar way, except we request that at the step consisting of adding the terms p_k to the series, the obtained value of the partial sum should be greater than α_n (instead α) for some sequence $\alpha_n \to \infty$ as $n \to \infty$. □

Definition 11.38. The *product* of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ is the infinite series $\sum_{n=1}^{\infty} c_n$ where

$$c_n = \sum_{k=1}^{n} a_k b_{n-k+1} = a_1 b_n + a_2 b_{n-1} + \dots + a_{n-1} b_2 + a_n b_1 \quad \text{for } n \geq 1. \tag{11.30}$$

The following result justify the above definition of the product of two infinite series.

Theorem 11.39. *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two absolutely convergent series. Then the series $\sum_{n=1}^{\infty} c_n$, where c_n is given by (11.30), converges absolutely and*

$$\sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} c_n.$$

Proof: We put

$$s_n = a_1 + \dots + a_n, \qquad t_n = b_1 + \dots + b_n, \quad \text{and} \quad u_n = c_1 + \dots + c_n,$$

where

$$c_n = a_1 b_n + a_2 b_{n-1} + \dots + a_{n-1} b_2 + a_n b_1.$$

Notice that

$$u_n = a_1 t_n + a_2 t_{n-1} + \dots + a_n t_1.$$

Then

$$\begin{aligned} s_n t_n - u_n &= a_1 t_n + a_2 t_n + \dots + a_n t_n - u_n \\ &= a_2(t_n - t_{n-1}) + a_2(t_n - t_{n-2}) + \dots + a_n(t_n - t_1). \end{aligned} \tag{11.31}$$

Since the series $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} |a_n|$ converge

$$\exists_{M>0} \forall_{k\in\mathbb{N}} \quad |t_k| < M, \quad |a_1| + |a_2| + \dots + |a_k| < M. \tag{11.32}$$

Moreover,

$$\forall_{\varepsilon>0}\exists_N\forall_{n>m>N}\quad |t_n-t_m|<\frac{\varepsilon}{3M}\quad\text{and}\quad |a_{k+1}|+|a_{k+2}|+\cdots+|a_n|<\frac{\varepsilon}{3M}. \tag{11.33}$$

Assume that $n>2N$, then by (11.31)

$$\begin{aligned}|s_nt_n-u_n| &\le (|a_2||t_n-t_{n-1}|+\cdots+|a_N||t_n-t_{n-N+1}|)\\ &\quad+(|a_{N+1}||t_n-t_{n-N}|+\cdots+|a_n||t_n-t_1|)\,.\end{aligned}$$

Since $n-N+1>N$ and $|t_n-t_k|\le|t_n|+|t_k|<2M$, by (11.32) and (11.33) , we obtain

$$|s_nt_n-u_n|\le(|a_2|+\cdots+|a_N|)\frac{\varepsilon}{3M}+(|a_{N+1}|+\cdots+|a_n|)\cdot 2M<M\frac{\varepsilon}{3M}+\frac{\varepsilon}{3M}\cdot 2M=\varepsilon.$$

Consequently,

$$\lim_{n\to\infty}s_n\cdot\lim_{n\to\infty}t_n=\lim_{n\to\infty}u_n,$$

i.e.

$$\sum_{n=1}^{\infty}a_n\cdot\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}c_n.$$

□

Example 11.40. (a) We will show that

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}\cdot\sum_{n=0}^{\infty}\frac{y^n}{n!}=\sum_{n=0}^{\infty}\frac{(x+y)^n}{n!},$$

where x and y are two real numbers. Since for any real number $t\neq 0$, we have

$$\lim_{n\to\infty}\frac{\left|\frac{t^{n+1}}{(n+1)!}\right|}{\left|\frac{t^n}{n!}\right|}=\lim_{n\to\infty}\frac{|t|}{n+1}=0,$$

by the Ratio Test, the both series

$$\sum_{n=0}^{\infty}\frac{x^n}{n!},\quad\text{and}\quad\sum_{n=0}^{\infty}\frac{y^n}{n!}$$

are absolutely convergent. By Theorem 8.2.17 and Proposition 3.1.7,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{x^n}{n!}\cdot\sum_{n=0}^{\infty}\frac{y^n}{n!}&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{x^k}{k!}\cdot\frac{y^{n-k}}{(n-k)!}\right)\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}x^ky^{n-k}\right)\\&=\sum_{n=0}^{\infty}\frac{1}{n!}(x+y)^n.\end{aligned}$$

(b) We know that for $|x|<1$ we have

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n.$$

Therefore, by Theorem 8.2.17,

$$\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)x^n.$$

11.3 Power Series

. A *power series* in $(x-a)$ centered at a is a series of the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n, \tag{11.34}$$

where x is a variable and $(x-a)^0 = 1$. The numbers a_n are called the *coefficients* of the power series 11.34. In general, we can assume that the number a, coefficients a_n and the variable x are all complex numbers. The set of all values x for which the power series 11.34 converges is called its *region of convergence*. It is clear that the sum of the power series defines a function on the region of convergence. We will show that the region of convergence of 11.34 (in the complex plane $\mathbb{C}$) is either the whole plane, the single point a or the disk centered at a of a certain radius R, which is called the *radius of convergence*.

Lemma 11.41. *Let z_o be a complex number such that the series*

$$\sum_{n=0}^{\infty} a_n(z_o-a)^n \tag{11.35}$$

converges. Then for every $x \in \mathbb{C}$ such that $|x-a| < |z_o-a|$ the power series

$$\sum_{n=0}^{\infty} a_n(x-a)^n$$

converges absolutely.

Proof: Let x be such that $|x-a| < |z_o-a|$. Since the series (8.3.2) converges, the sequence $|a_n(z_o-a)^n|$ is bounded, i.e.

$$\exists_{M>0}\forall_{n\geq 0} \quad |a_n||z_o-a|^n \leq M.$$

Let $q = \frac{|x-a|}{|z_o-a|}$. Since $|q|<1$, the geometric series

$$\sum_{n=0}^{\infty} Mq^n$$

converges. Since for all n

$$|a_n||x-a|^n = |a_n||z_o-a|^n \frac{|x-a|^n}{|z_o-a|^n} \leq Mq^n,$$

by the Comparison Test, the series

$$\sum_{n=0}^{\infty} |a_n(x-a)^n|$$

converges. $\square$

Corollary 11.42. *Let $\mathcal{D}$ denotes the region of convergence of the power series*

$$\sum_{n=0}^{\infty} a_n(x-a)^n.$$

If $z_o \in \mathcal{D}$ then the open disk

$$B(a,r) := \{z \in \mathbb{C} : |z-a| < r\},$$

where $r = |z-o-a|$, is contained in $\mathcal{D}$.

Definition 11.43. Let $\mathcal{D}$ denotes the region of convergence of the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$. The number

$$R = \sup\{r : B(a,r) \subset \mathcal{D}\}$$

is called the *radius of convergence* of the series $\sum_{n=0}^{\infty} a_n(x-a)^n$.

It is clear that if R is the radius of convergence of 11.34 then for every x such that $|x-a| < R$ the series 11.34 converges, and for every x such that $|x-a| > R$ the series 11.34 diverges. This means that $\mathcal{D}$ is the disk centered at a of radius R. However, we have to state clearly that on the circle $\{z : |z-a| = R\}$ the series (11.34 may or may not be convergent.

In what follows we will restrict our consideration to the power series 11.34 where a_n are real coefficients and x is also real. In this case the region of convergence of 11.34 contains the interval $(a-R, a+R)$ and if $x \in (-\infty, a-R) \cup (a+R, \infty)$, then the series 11.34 diverges. We will call the interval $(a-R, a+R)$ the *interval of convergence* of the power series 11.34 .

Theorem 11.44. (Cauchy-Hadamard Theorem) *Let $\{a_n\}$ be a sequence of real numbers. We define*

$$\rho = \inf_{n\ge 0} \sup_{k\ge n} \sqrt[k]{|a_k|}.$$

Then the radius of convergence R of the power series $\sum_{n=0}^{\infty} a_n x^n$ is

$$R = \frac{1}{\rho},$$

where we adopt the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Proof: For an arbitrary sequence $\{c_n\}$, the value

$$\inf_{n\ge 0} \sup_{k\ge n} c_k$$

is called the *upper limit* of the sequence $\{c_n\}$ and it is denoted by

$$\overline{\lim_{n\to\infty}} c_n.$$

We consider the following cases:

Case 1: $\rho = 0$.

Since the sequence $\{\sqrt[n]{|a_n|}\}$ is non-negative, it follows that the sequence $\{\sqrt[n]{|a_n|}\}$ converges and

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0.$$

Therefore, for any $x \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n(x-a)^n|} = |x-a| \lim_{n\to\infty} \sqrt[n]{|a_n|} = 0,$$

thus by the Root Test, the series 11.34 converges for any x, i.e. the radius of convergence of 11.34 is equal ∞.

Case 2: $\rho = \infty$.

Since $\rho = \overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, there exists a subsequence $\{a_{n_k}\}$ such that

$$\lim_{k\to\infty} \sqrt[n_k]{|a_{n_k}|} = \infty.$$

Consequently, if $x \neq a$, then

$$\exists_N \forall_{k\geq N} \quad \sqrt[n_k]{|a_{n_k}|} > \frac{1}{|x-a|},$$

and consequently,

$$\exists_N \forall_{k\geq N} \quad |a_{n_k}(x-a)^{n_k}| > 1. \tag{11.36}$$

Since the inequality 11.36 implies that the sequence $a_n(x-a)^n$ does not converge to zero, the series 11.34 diverges for all $x \neq a$. This implies that $R = 0$.

Case 3: ρ is a finite positive number.

Assume that $|x-a| < \frac{1}{\rho}$. Then, there exists an $\varepsilon > 0$ such that

$$|x-a| < \frac{1}{\rho+\varepsilon}.$$

By the definition of the number ρ

$$\exists_{N_\varepsilon} \forall_{n>N_\varepsilon} \quad \sqrt[n]{|a_n|} < \rho + \varepsilon.$$

Consequently,

$$\forall_{n>N_\varepsilon} \sqrt[n]{|a_n||x-a|^n} = |x-a| \sqrt[n]{|a_n|} < |x-a|(\rho+\varepsilon) < 1,$$

and consequently, by the Cauchy Convergence Criterion, the series 11.34 converges.

Assume now that $|x-a| > \frac{1}{\rho}$. Then there exists $\varepsilon > 0$ such that

$$|x-a| > \frac{1}{\rho-\varepsilon} > 0.$$

Again, by the definition of the number ρ we have

$$\forall_{m\in\mathbb{N}} \exists_{n>m} \quad \sqrt[n]{|a_n|} > \rho - \varepsilon.$$

In particular, there exist arbitrarily large integers n such that

$$\sqrt[n]{|a_n|} > \rho - \varepsilon.$$

Therefore, for every such n

$$\sqrt[n]{|a_n(x-a)^n|} = |x-a|\sqrt[n]{|a_n|} > |x-a|(\rho-\varepsilon) > 1,$$

thus, by the Cauchy Convergence Criterion, the series 11.34 diverges. Consequently, the conclusion follows. □

Corollary 11.45. *Let $\{a_n\}$ be a sequence of real numbers.*

(i) If the limit (finite or infinite)

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho,$$

exists , then the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n(x-a)^n, \tag{11.37}$$

is equal $R = \frac{1}{\rho}$;

(ii) If the numbers a_n are different from zero for n sufficiently large and the limit

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \tau$$

exists (finite or infinite), then the radius of convergence R of the power series 11.37 is equal to τ.

Example 11.46. (a) Let k be a positive integer. We will find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n. \tag{11.38}$$

Let $a_n = \frac{(n!)^k}{(kn)!}$, then the radius of convergence R of (11.38) is equal to

$$\begin{aligned} R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} &= \lim_{n\to\infty} \frac{(n!)^k \cdot (k(n+1))!}{(kn)! \cdot ((n+1)!)^k} \\ &= \lim_{n\to\infty} \frac{(kn+1)(kn+2)\cdot \dots \cdot (kn+k)}{(n+1)^k} = k^k. \end{aligned}$$

(b) We apply Corollary ????(i) to compute the radius of convergence R of the power series

$$\sum_{n=0}^{\infty} \sqrt{n}(3x+2)^n = \sum_{n=0}^{\infty} \sqrt{n}3^n \left(x + \frac{2}{3}\right)^n.$$

We have $a_n = \sqrt{n}3^n$ and we compute

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} n^{\frac{1}{n}} \cdot 3 = 3.$$

Consequently, the radius of convergence is $R = \frac{1}{\rho} = \frac{1}{3}$.

(c) We consider the following power series

$$\sum_{n=0}^{\infty}(-1)^n \frac{n!}{n^n} x^{2n+1}. \tag{11.39}$$

In order to compute the radius of convergence of 11.39 , we put

$$b_n = (-1)^n \frac{n!}{n^n} x^{2n+1}.$$

The series (11.39 converges if

$$\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} < 1.$$

Since

$$\begin{aligned}
\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} &= \lim_{n\to\infty} \frac{(n+1)! \cdot n^n}{(n+1)^{n+1} \cdot n!} |x|^2 \\
&= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} |x|^2 = \frac{|x|^2}{e},
\end{aligned}$$

the series 11.38 converges if

$$\frac{|x|^2}{e} < 1 \quad \Longleftrightarrow \quad |x| < \sqrt{e}.$$

On the other hand, by the same argument, 11.39 diverges if $|x| > \sqrt{e}$, thus the radius of convergence R is equal $\sqrt{e}$.

Assume that the power series

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

has a non-zero radius of convergence R. Then for every $x \in (a-R, a+R)$ we can define

$$F(x) = \sum_{n=0}^{\infty} a_n (x-a)^n. \tag{11.40}$$

We would like to know what the properties of the function $F(x)$ are. Is it continuous or differentiable and how do we compute its derivative? In order to answer these questions we need some additional results.

Lemma 11.47. *Let R be the radius of convergence of the power series*

$$\sum_{n=0}^{\infty} a_n (x-a)^n.$$

Then

$$\forall_{0<r<R} \forall_{\varepsilon>0} \exists_{N\in\mathbb{N}} \forall_x \quad |x-a| \le r \implies \sum_{n=N}^{\infty} |a_n (x-a)^n| < \varepsilon.$$

Proof: Notice that, since $0 < r < R$, the series

$$\sum_{n=0}^{\infty} |a_n| r^n$$

converges. In particular, we have

$$\forall_{\varepsilon>0} \exists_{N\in\mathbb{N}} \quad \sum_{n=N}^{\infty} |a_n| r^n < \varepsilon$$

and consequently, for $|x-a| \leq r$ we have

$$\sum_{n=N}^{\infty} |a_n(x-a)^n| \leq \sum_{n=N}^{\infty} |a_n| r^n < \varepsilon,$$

hence the conclusion follows. □

Lemma 11.48. *Let $\{a_n\}$ be a sequence of real numbers. Then the power series*

$$\sum_{n=0}^{\infty} a_n(x-a)^n, \quad \textit{and} \quad \sum_{n=1}^{\infty} a_n n(x-a)^{n-1},$$

have the same radius of convergence.

Proof: Let R denote the radius of convergence of $\sum_{n=0}^{\infty} a_n(x-a)^n$ and R' the radius of convergence of the series $\sum_{n=1}^{\infty} a_n n(x-a)^{n-1}$. Since the series $\sum_{n=1}^{\infty} a_n n(x-a)^n$ converges if and only if the series $\sum_{n=1}^{\infty} a_n n(x-a)^{n-1}$ converges, we have that they have the same radius of convergence. By the Cauchy-Hadamard Theorem, $R = \frac{1}{\rho}$ and $R' = \frac{1}{\rho'}$, where

$$\rho = \inf_{n\geq 0} \sup_{k\geq n} \sqrt[k]{|a_k|} \quad \text{and} \quad \rho' = \inf_{n\geq 0} \sup_{k\geq n} \sqrt[k]{|a_k|} \cdot \sqrt[k]{k}.$$

Consequently, we get

$$\rho \leq \rho' \tag{11.41}$$

On the other hand, since $\lim_{n\to\infty} \sqrt[n]{n} = 1$, for every arbitrary $\varepsilon > 0$ there exists $N > 0$ such that

$$\forall_{k\geq N} \sqrt[k]{k} < 1 + \varepsilon,$$

thus it follows

$$\forall_{n\geq N} \quad \rho' \leq \sup_{k\geq n} \sqrt[k]{|a_k|} \cdot \sqrt[k]{k} \leq (1+\varepsilon) \sup_{k\geq n} \sqrt[k]{|a_k|}.$$

Since the sequence $b_n := \sup k \geq n \sqrt[k]{|a_k|}$ is non-increasing, we have that

$$\rho = \inf_{n\geq 0} b_n = \inf_{n\geq N} b_n,$$

and consequently

$$\rho' \leq (1+\varepsilon)\rho.$$

Since $\varepsilon > 0$ is an arbitrary positive number, it follows that $\rho' \leq \rho$, by (8.3.8), which implies that $\rho = \rho'$. Therefore, $R = R'$. □

Theorem 11.49. *Let $\{a_n\}$ be a sequence of real numbers such that the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ has a non-zero radius of convergence R. Then the function $F:(a-R,a+R)\to\mathbb{R}$, defined by*

$$F(x)=\sum_{n=0}^{\infty} a_n(x-a)^n,$$

is differentiable on $(a-R,a+R)$ and

$$F'(x)=\sum_{n=1}^{\infty} a_n n(x-a)^{n-1}.$$

Proof: Suppose that $x_o\in(a-R,a+R)$. Then, there exists $0<r<R$ such that

$$|x_o-a|<r.$$

Let $\varepsilon>0$ be an arbitrary number. By Lemma 8.3.8,

$$\exists_{N>0}\forall_{m>n\geq N}\ \ |x-a|<r \implies \sum_{k=n+1}^{m} |a_k k(x-a)^{k-1}|<\frac{\varepsilon}{2}.$$

Let us fix two numbers m and n such that $m>n\geq N$ and denote by

$$U(x)=\sum_{k=n+1}^{m} a_k(x-a)^k.$$

Then

$$U'(x)=\sum_{k=n+1}^{m} a_k k(x-a)^{k-1}.$$

Therefore, we have

$$\frac{1}{x-x_o}\left(\sum_{k=n+1}^{m} a_k(x-a)^k-\sum_{k=n+1}^{m} a_k(x_o-a)^k\right)=\frac{U(x)-U(x_o)}{x-x_o}.$$

By the Lagrange Theorem, there exists a point c between x and x_o such that

$$\frac{U(x)-U(x_o)}{x-x_o}=U'(c).$$

Since $|c-a|<r$, we have

$$U'(c)<\sum_{k=n+1}^{m} |a_k k(c-a)^{k-1}|<\frac{\varepsilon}{2},$$

and consequently

$$\left|\frac{1}{x-x_o}\left(\sum_{k=n+1}^{m} a_k(x-a)^k-\sum_{k=n+1}^{m} a_k(x_o-a)^k\right)\right|<\frac{\varepsilon}{2}. \tag{11.42}$$

Since the inequality (11.42)) is true for every m such that $m>n$, and by Lemma ??? the series $\sum_{n=1}^{\infty} a_n n(x-a)^{n-1}$ has the same radius of convergence as the series $\sum_{n=0}^{\infty} a_n(x-a)^n$, it follows

$$\left|\frac{1}{x-x_o}\left(\sum_{k=n+1}^{\infty} a_k(x-a)^k - \sum_{k=n+1}^{\infty} a_k(x_o-a)^k\right)\right| \le \frac{\varepsilon}{2}. \tag{11.43}$$

Since the function $x \mapsto \sum_{k=0}^{n} a_k(x-a)^k$ is differentiable, there exists $\delta > 0$ such that $|x_o - a| + \delta < r$ and

$$|x - x_o| < \delta \implies \left|\frac{\sum_{k=0}^{n} a_k(x-a)^k - \sum_{k=1}^{n} a_k(x_o-a)^k}{x-x_o} - \sum_{k=1}^{n} a_k k(x_o-a)^k\right| < \frac{\varepsilon}{2}. \tag{11.44}$$

Consequently, by (11.43) and (11.44), we obtain that if $|x - x_o| < \delta$ then

$$\begin{aligned}
&\left|\frac{F(x)-F(x_o)}{x-x_o} - \sum_{k=1}^{\infty} a_k k(x_o-a)^k\right| \\
&= \left|\frac{\sum_{k=0}^{\infty} a_k(x-a)^k - \sum_{k=1}^{\infty} a_k(x_o-a)^k}{x-x_o} - \sum_{k=1}^{\infty} a_k k(x_o-a)^k\right| \\
&\le \left|\frac{\sum_{k=0}^{n} a_k(x-a)^k - \sum_{k=1}^{n} a_k(x_o-a)^k}{x-x_o} - \sum_{k=1}^{n} a_k k(x_o-a)^k\right| \\
&+ \left|\frac{\sum_{k=n+1}^{\infty} a_k(x-a)^k - \sum_{k=n+1}^{\infty} a_k(x_o-a)^k}{x-x_o} - \sum_{k=n+1}^{\infty} a_k k(x_o-a)^k\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This means

$$\lim_{x \to x_o} \frac{F(x)-F(x_o)}{x-x_o} = \sum_{k=1}^{\infty} a_k k(x_o-a)^k.$$

□

Remark 11.50. Theorem ??? simply states that every power series defines a differentiable functions on its interval of convergence and that its derivative can be obtained by differentiating the power series term by term.

Theorem 11.51. *Let $\{a_n\}$ be a sequence of real numbers such that the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ has a non-zero radius of convergence R. Then for any two numbers α, $\beta \in (a-R, a+R)$ we have that*

$$\begin{aligned}
\int_{\alpha}^{\beta} \sum_{k=0}^{\infty} a_n(x-a)^n &= \sum_{k=0}^{\infty} a_n \frac{(a-x)^{n+1}}{n+1}\bigg|_{\alpha}^{\beta} \\
&= \sum_{k=0}^{\infty} a_n \frac{(a-\beta)^{n+1}}{n+1} - \sum_{k=0}^{\infty} a_n \frac{(a-\alpha)^{n+1}}{n+1}.
\end{aligned}$$

Proof: It follows from Theorem ???, that the function

$$\Phi(x) = \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1},$$

is an antiderivative of the function

$$F(x) = \sum_{n=0}^{\infty} a_n (x-a)^n.$$

Consequently, the statement follows from the Fundamental Theorem of Calculus. □

Taylor's Theorem (Theorem ????) shows that if a function $f(x)$ is infinitely differentiable in a neighborhood of a point a, i.e. it has derivatives of all orders, then one can associate with $f(x)$ its *Taylor series* at a, which is the following power series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{11.45}$$

Since the Taylor series (11.45) defines a function $F(x)$ on its interval of convergence, and this function, by Theorem ????, is infinitely differentiable, it is interesting to consider whether $f(x) = F(x)$. The answer to this question is given in the following Theorem, which is an immediate consequence of Theorem ????.

Theorem 11.52. *Let $f(x)$ be a function infinitely differentiable in a neighborhood of a. Then the Taylor series (11.45) converges to $f(x)$ if and only if*

$$\lim_{n\to\infty} r_n(x) = 0,$$

where $r_n(x)$ denotes the remainder of the n-th Taylor polynomial of $f(x)$, i.e.

$$r_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Definition 11.53. A function $f(x)$ is said to be *analytic* at a if it can be represented in a neighborhood of a as a power series. We say that $f(x)$ is *analytic* if it is analytic at every point of its domain of definition.

Notice that every analytic function is infinitely differentiable, but not every infinitely differentiable function is analytic.

Example 11.54. Consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

The function $f(x)$ is infinitely differentiable at every point $x \neq 0$. We will show that it is also infinitely differentiable at $x = 0$. Indeed, by l'Hôpital's Rule, we have

$$f'(0) = \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x} = \lim_{x\to 0} \frac{\frac{1}{x}}{e^{\frac{1}{x^2}}} = \lim_{x\to 0} \frac{-\frac{1}{x^2}}{e^{\frac{1}{x^2}}\frac{-2}{x^3}} = 0.$$

Notice that a similar argument yields

$$\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = 0,$$

where n is an arbitrary natural number. On the other hand, we have

$$f'(x) = \frac{(-2)e^{-\frac{1}{x^2}}}{x^3}. \tag{11.46}$$

We claim, that for all $n \in \mathbb{N}$ there exists a polynomial $p_n(x)$ and a positive integer k_n such that

$$f^{(n)}(x) = \frac{p_n(x)e^{-\frac{1}{x^2}}}{x^{k_n}}. \tag{11.47}$$

Indeed, (11.46) shows that the formula (11.14) is valid for $n = 1$. Assume therefore, that (11.47) is true for some $n \geq 1$. Then we have

$$\begin{aligned} f^{(n+1)}(x) &= \frac{d}{dx} f^{(n)}(x) = \frac{d}{dx}\left(\frac{p_n(x)e^{-\frac{1}{x^2}}}{x^{k_n}}\right) \\ &= \frac{p_n'(x)e^{-\frac{1}{x^2}}}{x^{k_n}} + p_n(x)\frac{e^{-\frac{1}{x^2}}\left(\frac{-2}{x^3} - k_n x^{k_n-1}\right)}{x^{2k_n}} \\ &= \frac{\Big(p_n'(x)x^{k_n+3} + p_n(x)(-2 - k_n x^{k_n+1})\Big)e^{-\frac{1}{x^2}}}{x^{2k_n+3}} \\ &= \frac{p_{n+1}e^{-\frac{1}{x^2}}}{x^{k_{n+1}}}, \end{aligned}$$

where

$$p_{n+1}(x) = p_n'(x)x^{k_n+3} + p_n(x)(-2 - k_n x^{k_n+1}),$$

and

$$k_{n+1} = 2k_n + 3.$$

By applying the Principle of Mathematical Induction, we show that $f^{(n)}(0) = 0$ for all $n \geq 1$. Indeed, for $n = 1$ this statement is true. Assume that $f^{(n)}(0) = 0$ for some $n \geq 1$. Then we have that

$$f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x)}{x} = \lim_{x\to 0} \frac{p_n(x)e^{-\frac{1}{x^2}}}{x^{k_n+1}} = p_n(0) \cdot 0 = 0,$$

which concludes the induction argument. Thus, all the derivatives $f^{(n)}(0)$ are zero, the Taylor series of the function $f(x)$ is the zeros series, i.e.

$$F(x) = \sum_{n=0}^{\infty} \frac{0}{n!} x^n \equiv 0.$$

Since the function $f(x)$ is not constant, we get $F(x) \neq f(x)$, i.e. the function $f(x)$ is not analytic.

It follows immediately from the basic properties of the power series that a sum or a product of two analytic functions is an analytic function. In fact, it can be proved that the quotient or composition of two analytic functions is also an analytic function.

Example 11.55. (a) We consider the function $f(x) = e^x$. Since $f^{(n)}(x) = e^x$, the function $f(x)$ has the Taylor Series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. In order to show that $f(x)$ is analytic, we consider the remainder

$$r_n(x) = \frac{f^{(n+1)}(c(x))}{(n+1)!} x^{n+1} = \frac{e^{c(x)}}{(n+1)!} x^{n+1},$$

where $c(x)$ is a point between 0 and x (which depends on x and n). Since, by the Ratio Test, the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges, it follows that

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0,$$

so

$$\begin{aligned} 0 \le \lim_{n\to\infty} |r_n(x)| &= \lim_{n\to\infty} \frac{e^{c(x)}}{(n+1)!} |x|^{n+1} \\ &\le e^{|x|} \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = e^{|x|} \cdot 0 = 0. \end{aligned}$$

Consequently, the exponential function $f(x) = e^x$ is analytic and we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(b) Notice that for the function $f(x) = \sin x$ the remainder $r_n(x)$ satisfies

$$\lim_{n\to\infty} |r_n(x)| = \lim_{n\to\infty} \frac{|\sin(c(x) + \frac{n+1}{2}\pi)|}{(n+1)!} |x|^{n+1} = 0,$$

Thus the function $\sin x$ is analytic and

$$\sin x = \sum_{n=0}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n!} x^n = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

Similarly, we show

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

and

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \quad \text{and} \quad \cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}.$$

(c) By applying a simple substitution ($-x$ in place of x) we obtain that

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}.$$

(d) We have for $|x| < 1$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

thus

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

By Theorem ?????, we have that for $|x| < 1$

$$\arctan x = \int_0^x \frac{dt}{1+t^2} = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^{2n} \right) dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

This power series can be used for the computation of the number π. Notice that

$$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy},$$

thus if $\frac{x+y}{1-xy} = 1$, then

$$\frac{\pi}{4} = \arctan x + \arctan y = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1} + y^{2n+1}}{2n+1}.$$

For example, we can take $x = \frac{1}{2}$ and $y = \frac{1}{3}$, so we get

$$\pi = 4 \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2^{2n+1}(n+1)} + \frac{1}{5^{2n+1}(n+1)} \right). \tag{11.48}$$

The series (11.48) is alternating and its general term

$$a_n = \left(\frac{1}{2^{2n+1}(n+1)} + \frac{1}{5^{2n+1}(n+1)} \right)$$

is decreasing monotonically to zero, so it is true, by the Alternating Series Test, that we have the following error estimation of the approximation of the number π:

$$\left| \pi - 4 \sum_{n=0}^{N} (-1)^n \left(\frac{1}{2^{2n+1}(n+1)} + \frac{1}{5^{2n+1}(n+1)} \right) \right| < 4 \left(\frac{1}{2^{2N+3}(N+2)} + \frac{1}{5^{2N+3}(N+2)} \right).$$

(e) By integrating term by term the power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

we obtain the power series representation of the function $\ln(1+x)$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}.$$

In particular, for $x = 1$, we obtain the alternating series

$$\ln 2 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}.$$

Example 11.56. We consider the function $f(x) = (1+x)^\alpha$, where $x > -1$ and α is not a positive integer. Then we have that

$$f^{(n)}(x) = \alpha(\alpha-1)\dots(\alpha-n+1)(1+x)^{\alpha-n},$$

consequently the Taylor series of the function $f(x)$ is the following power series

$$\sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n.$$

We introduce the notation

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \quad (n \geq 1) \quad \text{and} \quad \binom{\alpha}{0} = 1.$$

By applying the Ratio Test, we obtain that the radius of convergence R of this series is equal to 1. Indeed, we have

$$R = \lim_{n\to\infty} \left| \frac{\binom{\alpha}{n}}{\binom{\alpha}{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{\alpha - n} \right| = 1.$$

In particular, for $|x| < 1$ we have that the series

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

converges absolutely, by the necessary condition for convergence for an infinite series, we have

$$\lim_{n\to\infty} \binom{\alpha}{n} x^n = 0.$$

We will show that the function $f(x)$ is analytic. For this purpose we will use the Cauchy form of the Taylor remainder

$$r_n(x) = \frac{\alpha(\alpha-1)\dots(\alpha-n)(1+\theta x)^{\alpha-n-1}}{1\cdot 2\cdot \dots \cdot n}(1-\theta)^n x^{n+1},$$

where $\theta \in (0,1)$. Since the remainder $r_n(x)$ can be written as

$$\begin{aligned} r_n(x) &= \left[\frac{(\alpha-1)(\alpha-2)\dots\big((\alpha-1)-n+1\big)}{1\cdot 2\cdot \dots \cdot n}\right] \cdot \left[\alpha x(1+\theta x)^{\alpha-1}\right] \left(\frac{1-\theta}{1-\theta x}\right)^n \\ &= \left[\binom{\alpha-1}{n} x^n\right] \cdot \left[\alpha x(1+\theta x)^{\alpha-1}\right] \left(\frac{1-\theta}{1-\theta x}\right)^n \end{aligned}$$

and the power series

$$\sum_{n=0}^{\infty} \binom{\alpha-1}{n} x^n$$

converges, we have that

$$\lim_{n\to\infty} \binom{\alpha-1}{n} x^n = 0.$$

Since

$$|\alpha x|(1-|x|)^{\alpha-1} \leq \left|\alpha x(1+\theta x)^{\alpha-1}\right| \leq |\alpha x|(1+|x|)^{\alpha-1},$$

it follows that

$$\left[\alpha x(1+\theta x)^{\alpha-1}\right]\left(\frac{1-\theta}{1-\theta x}\right)^n$$

is bounded, hence

$$\lim_{n\to\infty} r_n(x) = 0.$$

Therefore, the function $f(x) = (1+x)^\alpha$ is analytic and

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n. \tag{11.49}$$

The formula (11.49) is called Newton's *Binomial Formula.*

Example 11.57. By applying the Binomial Formula we obtain

$$\begin{aligned}
\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\dots\left(-\frac{1}{2}-n+1\right)}{n!} x^n \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1\cdot 3\cdot 5\cdot \dots \cdot (2n-1)}{2^n \cdot n!} x^n \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{(2^n\cdot n!)^2} x^n.
\end{aligned}$$

By substitution, we obtain

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n\cdot n!)^2} x^{2n},$$

so, by integration term by term, we get

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n\cdot n!)^2\,((2n+1)} x^{2n+1}.$$

11.4 Problems

1. Show that the following series converge and find their sums

(a) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$;
(b) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$;
(c) $\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)}$;
(d) $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$;
(e) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$;

(f) $\sum_{n=1}^{\infty} \frac{3^n+2^n}{6^n}$;
(g) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2(2n+1)^2}$;
(h) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$;
(i) $\sum_{n=1}^{\infty} \frac{2n-1}{2^n}$;
(j) $\sum_{n=1}^{\infty} \frac{n^2}{(n+1)(n+2)(n+3)(n+4)}$.

2. Apply the Divergence Test to show divergence of the following series

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$;
(b) $\sum_{n=1}^{\infty} 2^{(-1)^n n}$;
(c) $\sum_{n=1}^{\infty} \cos\left(\sin \frac{1}{n}\right)$;
(d) $\sum_{n=1}^{\infty} n(\sqrt{n^2+1} - \sqrt{n^2-1})$.

3. Apply Comparison Tests to check which of the following series converge;

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n\sqrt{n+1}}}$;
(b) $\sum_{n=1}^{\infty} \frac{1}{n}\sqrt{\sin \frac{1}{n}}$;
(c) $\sum_{n=1}^{\infty} \frac{1}{n}\left(\sqrt{n+1} - \sqrt{n}\right)$;
(d) $\sum_{n=1}^{\infty} \frac{1}{n^2}\left(\sqrt{n^2+n\sqrt{n}} - \sqrt{n^2-n\sqrt{n}}\right)$;
(e) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$;
(f) $\sum_{n=1}^{\infty} \frac{1}{n}\ln\left(1+\frac{1}{n}\right)$;
(g) $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$;
(h) $\sum_{n=1}^{\infty} \sqrt{\ln \frac{n^3+1}{n^3}}$;
(i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\ln\left(\frac{\sqrt{n}+1}{\sqrt{n}}\right)$;
(j) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+\sqrt{n}}-\sqrt{n^2-\sqrt{n}}}{\sqrt{n}}$;
(k) $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}$;
(l) $\sum_{n=1}^{\infty} \frac{1}{n}e^{-1/n}$;
(m) $\sum_{n=1}^{\infty} \sin \frac{1}{n}$;
(n) $\sum_{n=1}^{\infty} \tan^2 \frac{1}{n}$;
(o) $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}\sin \frac{1}{n}$, where $\alpha > 0$;
(p) $\sum_{n=1}^{\infty} \left(\cos \frac{1}{n+1} - \cos \frac{1}{n}\right)$;
(q) $\sum_{n=1}^{\infty} \sin\left(\tan \frac{1}{n}\right)$;
(r) $\sum_{n=1}^{\infty} \left(\cos \frac{1}{n} + \sin \frac{1}{n}\right)$;
(s) $\sum_{n=1}^{\infty} \tan \frac{\pi}{4^n}$;
(t) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n(n+1)(n+2)(n+3)}}$;
(u) $\sum_{n=1}^{\infty} \frac{\sin x^n}{n^2}$;
(v) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln \frac{n+1}{n}\right)$.

4. Apply the Ratio Test to check which of the following series are convergent

(a) $\sum_{n=1}^{\infty} \frac{10^n}{n!}$;
(b) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$;

(c) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$;
(d) $\sum_{n=1}^{\infty} \frac{(n!)^2}{2^{n^2}}$;
(e) $\sum_{n=1}^{\infty} \frac{n^5}{2^n+3^n}$;
(f) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2 e^n}$;
(g) $\sum_{n=2}^{\infty} \frac{1}{2^n \ln(n!)}$;
(h) $\sum_{n=1}^{\infty} \frac{a^n n!}{n^n}$, where a is a positive constant such that $a \neq e$.

5. Apply the Root Test to check which of the following series converge

(a) $\sum_{n=1}^{\infty} \left(\arctan(n^2+1)\right)^n$;
(b) $\sum_{n=1}^{\infty} \frac{n^2}{\left(2+\frac{1}{n}\right)^n}$;
(c) $\sum_{n=1}^{\infty} \frac{n^{n+\frac{1}{n}}}{\left(2n+\frac{1}{n}\right)^n}$;
(d) $\sum_{n=1}^{\infty} \frac{1}{\ln^n(n+1)}$;
(e) $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$;
(f) $\sum_{n=1}^{\infty} \frac{\left(\frac{n+1}{n}\right)^{n^2}}{3^n}$;
(g) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} 2^n$;
(h) $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{n^2}}$.

6. Apply the Integral Test to check which of the following series converge

(a) $\sum_{n=2}^{\infty} \frac{1}{n \ln^{1+s} n}$, where $s > 0$;
(b) $\sum_{n=1}^{\infty} \frac{1}{n \ln n \cdot \ln \ln n}$;
(c) $\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln^2(n+1)}$;
(d) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$;
(e) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \ln\left(\frac{n+1}{n-1}\right)$.

7. Determine which of the following alternating series converge

(a) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+100}{3n+1}\right)^n$;
(b) $\sum_{n=1}^{\infty} \frac{\ln^{100} n}{n} \sin\frac{(2n+1)\pi}{2}$;
(c) $\sum_{n=1}^{\infty} (-1)^n \frac{\sin^2 n}{n}$;
(d) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+100}$;
(e) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n-\ln n}$;
(f) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+(-1)^n}$;
- (g) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}$;
- (h) $\sum_{n=2}^{\infty} \frac{1}{\ln^2 n} \cos \pi n^2$.

8. Determine whether the following series converge or diverge

(a) $\sum_{n=2}^{\infty} \frac{\sqrt{n+2}-\sqrt{n-2}}{n^\alpha}$;
(b) $\sum_{n=1}^{\infty} \left(\sqrt{n+a} - \sqrt[4]{n^2+n+b}\right)$;

(c) $\sum_{n=1}^{\infty}\left(\cot\frac{n\pi}{4n-2}-\sin\frac{n\pi}{2n+1}\right)$;
(d) $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\sqrt{\ln\frac{n+1}{n}}\right)$;
(e) $\sum_{n=1}^{\infty}\frac{\ln(n!)}{n^{\alpha}}$;
(f) $\sum_{n=1}^{\infty}n^2e^{-\sqrt{n}}$;
• (g) $\sum_{n=1}^{\infty}\left(n^{\frac{1}{n^2+1}}-1\right)$;
• (h) $\sum_{n=1}^{\infty}\frac{a\ln n+b}{e^{c\ln n+b}}$;
(i) $\sum_{n=1}^{\infty}\frac{1}{\ln^2\left(\sin\frac{1}{n}\right)}$;
(j) $\sum_{n=1}^{\infty}\left(\cos\frac{a}{n}\right)^{n^3}$;

9. Determine whether the following series converge absolutely, converge conditionally or diverge:

(a) $\sum n=1^{\infty}\frac{(-1)^{n-1}}{b^{p+\frac{1}{n}}}$;
(b) $\sum_{n=2}^{\infty}\ln\left[1+\frac{(-1)^n}{n^p}\right]$;
(c) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n\sin^{2n}x}{n}$;
(d) $\sum_{n=2}^{\infty}\frac{(-1)^n}{\left(n+(-1)^n\right)^p}$;
(e) $\sum_{n=1}^{\infty}(-1)^{\frac{n(n-1)}{2}}\frac{n^{100}}{2^n}$;
(f) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n\sqrt[2]{n}}$;
(g) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\left(\sqrt{n}+(-1)^{n-1}\right)^p}$;
• (h) $\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{12}}{\ln n}$;
(i) $\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{4}}{n^p+\sin\frac{n\pi}{4}}$;
(j) $\sum_{n=1}^{\infty}(-1)^n\frac{n-1}{n+1}\frac{1}{\sqrt[100]{n}}$;
(k) $\sum_{n=1}^{\infty}\frac{(-1)^{[\ln]}}{n}$, where $[x]$ denotes the greatest integer m satisfying $m\le x$;
(l) $\sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}{2\cdot4\cdot6\cdot\dots\cdot(2n)}\right]^p$;
(m) $\sum_{n=1}^{\infty}\frac{\sin n\cdot\sin n^2}{n}$;
(n) $\sum_{n=1}^{\infty}\sin n^2$.

10. Find the interval of convergence of the following power series

(a) $\sum_{n=0}^{\infty}10^nx^n$;
(b) $\sum_{n=1}^{\infty}\frac{x^n}{n10^{n-1}}$;
(c) $\sum_{n=0}^{\infty}n!x^n$;
(d) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)(2n-1)!}$;
(e) $\sum_{n=1}^{\infty}(n-1)3^{n-1}\cdot x^{n-1}$;
(f) $\sum_{n=1}^{\infty}\frac{(nx)^n}{n!}$;
(g) $\sum_{n=1}^{\infty}\frac{\ln(n+1)}{n+1}x^{n+1}$;
(h) $\sum_{n=1}^{\infty}\left[\left(\frac{n+1}{n}\right)^n\cdot x\right]^n$.

11. Find the Maclaurin series of the following functions

(a) $f(x)=x^2e^x$;

(b) $f(x) = e^x \sin x$;
(c) $f(x) = \sin 3x$;
(d) $f(x) = \ln(1 + e^x)$;
(e) $f(x) = e^{-x^2}$;
(f) $f(x) = \sin \frac{x}{2}$;
(g) $f(x) = \sin^2 x$;
(h) $f(x) = (1 + x)e^{-x}$;
(i) $f(x) = e^x \sin x$;
(j) $f(x) = (1 - x)^2 \cosh \sqrt{x}$;
(k) $f(x) = \frac{\ln(1+x)}{1+x}$;
(l) $f(x) = \ln^2(1 - x)$;
(m) $f(x) = (\arctan x)^2$;
(n) $f(x) = e^x \cos x$;
(o) $f(x) = \left(\frac{\arcsin x}{x}\right)^2$.

12. Suppose the function $\sec x$ is represented by the following power series

$$\sec x = \sum_{n=0}^{\infty} \frac{E_n}{(2n)!} x^{2n}.$$

Find the recurrence relations for the coefficients E_n (which are called the *Euler's numbers*).

13. Find the Maclaurin series of the function

$$f(x) = \frac{1}{\sqrt{1 - 2tx + x^2}}, \qquad \text{where} \quad |x| < 1.$$

14. Find the Maclaurin series of the following functions

(a) $f(x) = \int_0^x e^{-t^2} dt$;
(b) $f(x) = \int_0^x \frac{\sin t}{t} dt$;
(c) $f(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}}$;
(d) $f(x) = \int_0^x \frac{\arctan t}{t} dt$.

15. Use the equality

$$\frac{\pi}{6} = \arctan \frac{1}{2}$$

to compute the number π with error at most 0.0001.

16. Use the equality

$$\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3}$$

to compute the number π with error at most 0.0001.

17. Use the equality

$$\frac{\pi}{4} = 4\arctan \frac{1}{5} - \arctan \frac{1}{239}$$

to compute the number π with error at most 10^{-9}.

18. Use the equality

$$\ln(n+1) = \ln n + 2(\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \dots\right]$$

to compute the numbers $\ln 2$ and $\ln 3$ with error at most 10^{-5}.

19. Compute the following integrals with error at most 0.001

(a) $\int_0^1 e^{-x^2}\,dx$;
(b) $\int_0^{\frac{1}{3}} \frac{dx}{\sqrt[3]{1-x^2}}$;
(c) $\int_0^1 \frac{dx}{\sqrt{1+x^4}}$;
(d) $\int_0^2 \frac{\sin x}{x}\,dx$;
(e) $\int_{10}^{100} \frac{\ln(1+x)}{x}\,dx$;
(f) $\int_0^1 \cos x^2\,dx$;
- (g) $\int_0^{\frac{1}{2}} \frac{\arctan x}{x}\,dx$;
- (h) $\int_0^1 \frac{\sinh x}{x}\,dx$;

(i) $\int_0^{\frac{1}{2}} \frac{\arcsin x}{x}\,dx$;
- (j) $\int_2^{\infty} \frac{dx}{1+x^3}$;
- (k) $\int_0^1 x^x\,dx$.

20. Find the Taylor series of $f(x) = \ln x$ at $x_o = 1$.

21. Find the Taylor series of $f(x) = \sqrt{x^3}$ at $x_o = 3$.

22. Find the Taylor series of $f(x) = \frac{1}{x}$ at $x_o = 3$.

23. Find the Taylor series of $f(x) = \frac{1}{x^2+4x+7}$ at $x_o = -2$.

12

Sequences and Series of Functions

12.1 Sequences of Functions: Pointwise and Uniform Convergence

Let us consider a general situation of a sequence of functions $f_n : X \to V$, $n \in \mathbb{N}$ defined on a metric space (X, d) with values in a normed space $(V, \|\cdot\|)$. Such a sequence will be denoted by $\{f_n\}_{n=1}^{\infty}$ (or $\{f_n\}$). In what follows we will mostly be interested in sequences of real-valued functions that are defined on an interval $U \subset \mathbb{R}$.or simply $\{f_n\}$. However, since many of the results discussed in this section are true in more general situations, we will formulate some of these results in a general setting.

Definition 12.1. We say that a sequence of functions $f_n : X \to V$, $n \in \mathbb{N}$ converges *pointwisely* to a function $f : X \to V$ if and only if

$$\forall_{x\in X} \lim_{n\to\infty} f_n(x) = f(x) \quad \overset{\text{def}}{\Longleftrightarrow} \quad \forall_{x\in X}\, \forall_{\varepsilon>0}\, \exists_{N_x>0}\, \forall_{n\geq N_x} \quad \|f_n(x) - f(x)\| < \varepsilon.$$

In such a case the function f will be called the *pointwise-limit* of $\{f_n\}$. Moreover, if for some $x_o \in X$ the sequence $\{f_n(x_o)\}$ does not converge, then we will say that it diverges and in such a case the the sequence $\{f_n\}$ will be called *divergent*. To indicate the pointwise convergence of $\{f_n\}$ to the function f, we will write: $f_n(x) \to f(x)$ as $n \to \infty$, or $\lim_{n\to\infty} f_n(x) = f(x)$.

In a similar way, we can introduce the notion of the uniform convergence of $\{f_n\}$. This notion was already introduced in section 4.3 (see Definition 4.17) in relation to the space of bounded functions $B(X, V)$ and the norm $\|\cdot\|_\infty$. Notice that in the following definition we do not require from the functions f_n to be bounded.

Definition 12.2. We say that a sequence of functions $f_n : X \to V$, $n \in \mathbb{N}$ converges *uniformly* to a function $f : X \to V$ if

$$\forall_{\varepsilon>0}\, \exists_{N_x>0}\, \forall_{n\geq N_x}\, \forall_{x\in X} \quad \|f_n(x) - f(x)\| < \varepsilon.$$

In such a case the function f will be called the *uniform limit* of $\{f_n\}$. To indicate the indicate the uniform convergence of $\{f_n\}$ to the function f, we will often write: $f_n \rightrightarrows f$ as $n \to \infty$, or $\lim_{n\to\infty} f_n = f$.

The uniform convergence is illustrated on Figure 12.1. Graphically, $f_n \rightrightarrows f$ uniformly on $[a, b]$ means that, for every $\varepsilon > 0$, there exists N such that the graphs of the functions f_n with subscript $n \geq N$ lie in the region bounded by $y = f(x) - \varepsilon$ and $y = f(x) + \varepsilon$, $x \in [a, b]$.

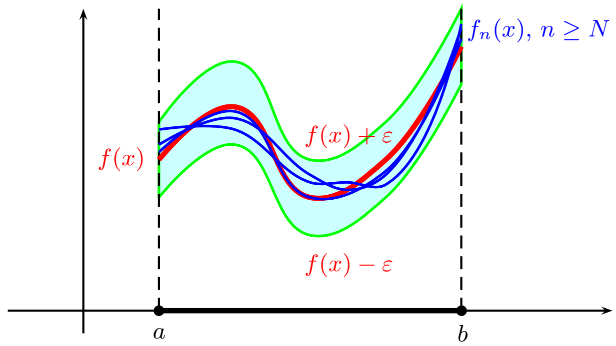

Fig. 12.1. Uniform convergent sequence $\{f_n\}$ to a function f,

It is clear that a uniformly convergent sequence of functions is pointwisely convergent, however, one can easily construct sequences that converge pointwisely but not uniformely.

Example 12.3. (a) Let $f_n(x) = x^n, n = 1, 2, \cdots,$ be a sequence of functions defined on $[0, 1]$. This sequence is pointwisely convergent to $f : [0, 1] \to br$ given by

$$f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1. \end{cases} \tag{12.1}$$

Notice that the sequence $\{f_n\}$ doesn't converge uniformly to f. Indeed, for every $N > 0$ there exist $n > N$ and $x \in (0, 1)$ such that $x^n \geq \frac{1}{2}$. Indeed, take any $n > N$ and choose $\frac{1}{2^{\frac{1}{n}}} < x < 1$.

(b) Let $f_n : [0, 1] \to br$ be defined by

$$f_n(x) = \begin{cases} 1, & \text{if } x = p/q \in \mathbb{Q}, \text{ where } \gcd(p,\, q) = 1,\, p \leq q \leq n\ , \\ 0, & \text{otherwise.} \end{cases} \tag{12.2}$$

Then $f_n(x) \to f(x)$ for all x, i.e. f_n converges pointwise on $[0,\ 1]$ to the Dirichlet function $f : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1, & \text{if } \ x \in \mathbb{Q} \cap [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the sequence $\{f_n\}$ doesn't converge uniformly to f. Indeed, for every $N > 0$ there exist $n > N$ and $x = \frac{1}{n+1} \in (0, 1)$ such that

$$|f(x) - f_n(x)| = 1 - 0 > \frac{1}{2}.$$

Notice that each of the functions f_n is bounded and has only finitely many discontinuity points.

(c) Let $f_n : [0, 1] \to \mathbb{R}$ be defined by (see Figure 12.3 below)

$$f_n(x) = \begin{cases} n^2 x, & \text{if } x \in [0,\ \frac{1}{n}), \\ 2n - n^2 x, & \text{if } x \in [\frac{1}{n},\ \frac{2}{n}), \\ 0, & \text{if } x \in [\frac{2}{n},\ 1]. \end{cases}$$

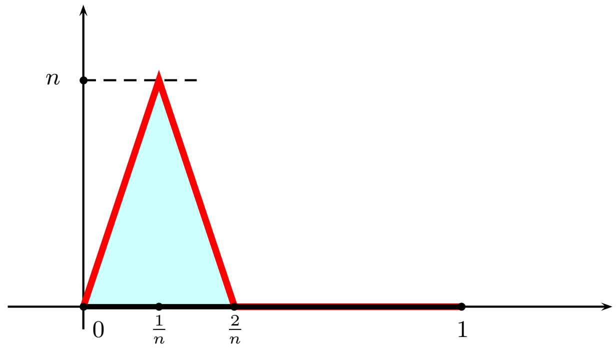

Fig. 12.2. The graph of the function f_n from Example 12.3(c)

Then we have clearly that $f_n(x)$, for $n \to \infty$, converges to the zero function $f(x) = 0$ for all $x \in [0,1]$. Indeed, (by Archimedes) for for every $x \in (0,1]$, there exists $N \in \mathbb{N}$ such $\frac{2}{N} < x$. Then for every $n \ge N$ we have $f_n(x) = 0$. From the graph of the function f_n shown on Figure 12.3, it is clear that f_n is continuous, thus it is integrable and

$$\forall_{n\in\mathbb{N}} \quad \int_0^1 f_n(x)dx = 1.$$

Remark 12.4. (a) Notice that by Example 12.3(a), the limit of pointwise convergent sequence of continuous functions doesn't need to be continuous.
(b) by Example 12.3(b), the limit of a pointwise convergent sequence of Riemann-integrable functions, doesn't need to be Riemann-integrable. Indeed, the Dirichlet function f is not integrable (see Example 8.18).
(c) It follows from Example 12.3(a) that for a pointwisely convergent sequence of Riemann integrable (continuous) functions $\{f_n : [a,b] \to \mathbb{R}\}$ to a Rieman-integrable function $f : [a,b] \to \mathbb{R}$, it may happen that

$$\lim_{n\to\infty} \int_a^b f_n(x)dx \ne \int_a^b f(x)dx, \quad \text{where } \forall_{x\in[a,b]} \lim_{n\to\infty} f_n(x) = f(x).$$

The above examples imply that ponitwise convergence of sequence of functions have very limited role for the calculus of limits of sequences. We must develop stronger convergence for sequences of functions. We will concentrate our attention on the uniform convergence of such sequences.

Since in a Banach space a sequence is convergent if and only if it is Cauchy, this allows us to formulate the following equivalent condition for uniform convergence.

Theorem 12.5. *Let (X,d) be a metric space and $(V, \|\cdot\|)$ a Banach space. A sequence of functions $\{f_n\}$, $f_n : X \to V$, converges uniformly to certain function f on X if and only if*

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$$\forall_{\varepsilon>0} \, \exists_{N\in\mathbb{N}} \, \forall_{n,m\ge N} \, \forall_{x\in X} \quad \|f_n(x) - f_m(x)\| < \varepsilon. \tag{12.3}$$

Proof: Suppose $\{f_n\}$ converges uniformly to f on X. Then, by the definition, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $k > N$ implies

$$\forall_{x\in X}\ \|f_k(x)-f(x)\|<\frac{\varepsilon}{2}. \tag{12.4}$$

Then for $m\geq N$, $n\geq N$, we have for all $x\in X$

$$\begin{aligned}\|f_m(x)-f_n(x)\| &= \|f_m(x)-f(x)+f(x)-f_n(x)\|\\ &\leq \|f_m(x)-f(x)\|+\|f(x)-f_n(x)\| \quad \text{(by (12.4))}\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

Now we assume that (12.3) is valid, which implies that for every $x\in X$, the sequence $\{f_n(x)\}$ is Cauchy in V. Since V is a Banach space, the sequence $\{f_n(x)\}$ converges to $f(x)$, which in other words means that $\{f_n\}$ converges pointwisely on X to the function f. By letting passing in (12.3) to the limit $m\to\infty$, we obtain

$$\forall_{\varepsilon>0}\ \exists_{N\in\mathbb{N}}\ \forall_{n\geq N}\ \forall_{x\in X}\ \ \|f_n(x)-f(x)|\leq\varepsilon$$

Therefore, $\{f_n\}$ converges uniformly to f on X. □

12.2 Properties of Uniformly Convergent Sequences of Functions

In this section, we discuss the continuity, differentiability and integrability of the limit function of uniform convergent sequences of functions.

Theorem 12.6. (Uniform Convergence Theorem) *Let (X,d) be a metric space and $(V,\|\cdot\|)$ a normed space. If the sequence of continuous functions $\{f_n\}$, $f_n:X\to V$, converges uniformly to $f:X\to V$ on X, then f is continuous in X.*

Proof: For every $x_0\in X$, we have

$$\begin{aligned}\|f(x)-f(x_0)\| &= \|f(x)-f_n(x)+f_n(x)-f_n(x_0)+f_n(x_0)-f(x_0)\|\\ &\leq \|f(x)-f_n(x)\|+\|f_n(x)-f_n(x_0)\|+\|f_n(x_0)-f(x_0)\|.\end{aligned} \tag{12.5}$$

Since $\{f_n\}$ converges uniformly to f on X, for every $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that for every $n\geq N$, we have

$$\forall_{x\in X}\ \ \|f(x)-f_n(x)\|<\frac{\varepsilon}{3}, \tag{12.6}$$

and in particular,

$$\|f(x_0)-f_n(x_0)\|<\frac{\varepsilon}{3}. \tag{12.7}$$

By the continuity of f_n, there exists $\delta>0$ such that $d(x,x_0)<\delta$ implies

$$\|f_n(x)-f_n(x_0)\|<\frac{\varepsilon}{3}. \tag{12.8}$$

Then by (12.5), (12.6) and (12.7) we have for every $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that for every $n\geq N$, and $x\in X$ with $d(x,x_0)<\delta$, we have

$$\|f(x)-f(x_0)\| \le \|f(x)-f_n(x)\| + \|f_n(x)-f_n(x_0)\| + \|f_n(x_0)-f(x_0)\| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon. \quad (12.9)$$

That is, f is continuous at x_0. □

Theorem 12.5 shows that, for continuous functions f_n, $n \in \mathbb{N}$, if $f_n \rightrightarrows f$, then we can change the limits with respect to n and x. i.e., for every $x_0 \in X$, we have

$$\lim_{x\to x_0}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to x_0} f_n(x) = f(x_0).$$

Remark 12.7. Note that uniform convergence is sufficient but not necessary for continuity of the limit function of sequence of functions. For example, the sequence $\{f_n\}$ with $f_n(x) = x^n$, is not uniformly convergent on $(-1,\,1)$ but its pointwise limit $f(x) = 0$ is continuous on $(-1,\,1)$.

Let us consider a sequence $f_n : [a,b] \to \mathbb{R}$ of Riemann-integrable functions on $[a,b]$ which is uniformly convergent to $f : [a,b] \to \mathbb{R}$. One can ask a natural question: *is the function f Riemann-integrable?* The following proposition gives the answer to this question.

Proposition 12.8. *Assume that $f_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, is a sequence of Riemann-integrable functions on $[a,b]$ such that f_n converges uniformly convergent to $f : [a,b] \to \mathbb{R}$. Then f is also Riemann-integrable.*

Proof: We will use the property that $f : [a,b] \to \mathbb{R}$ is Riemann-integrable, if and only if

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_P\ \ \|P\| < \delta \ \Rightarrow\ S(f,P) - s(f,P) < \varepsilon.$$

Let $P = \{x_k\}_{k=1}^m$ be a partition. Notice that if for all $x \in [a,b]$, $|f_n(x) - f(x)| < \frac{\varepsilon}{4(b-a)}$, then the inequalities $f(x) < f_n(x) + \frac{\varepsilon}{4(b-a)}$ and $f(x) < f_n(x) - \frac{\varepsilon}{4(b-a)}$ imply

$$M_k := \sup\{f(x) : x \in [x_{k-1},x_k]\} \le \sup\{f_n(x) : x \in [x_{k-1},x_k]\} + \frac{\varepsilon}{4(b-a)} =: M_k^n + \frac{\varepsilon}{4(b-a)}$$
$$m_k := \inf\{f(x) : x \in [x_{k-1},x_k]\} \ge \inf\{f_n(x) : x \in [x_{k-1},x_k]\} - \frac{\varepsilon}{4(b-a)} =: M_k^n + \frac{\varepsilon}{4(b-a)},$$

which implies that

$$S(f,P) := \sum_{k=1}^m M_k \triangle x_k \le \sum_{k=1}^m (M_k^n + \frac{\varepsilon}{4(b-a)})\triangle x_k = S(f_n,P) + \frac{\varepsilon}{4}$$
$$s(f,P) := \sum_{k=1}^m m_k \triangle x_k \ge \sum_{k=1}^m (m_k^n - \frac{\varepsilon}{4(b-a)})\triangle x_k = s(f_n,P) + \frac{\varepsilon}{4}$$

which implies that

$$S(f,P) - s(f,P) \le S(f_n,P) - s(f_n,P) + \frac{\varepsilon}{2}.$$

Then, for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $x \in [a,b]$, we have $|f_n(x) - f(x)| < \frac{\varepsilon}{4(b-a)}$, and since (by assumption) f_n is Riemann-integrable, thus

$$\exists_{\delta>0}\ \forall_P\ \ \|P\| < \delta \ \Rightarrow\ S(f_n,P) - s(f_n,P) < \frac{\varepsilon}{2},$$

and consequently

$$S(f,P) - s(f,P) \le S(f_n,P) - s(f_n,P) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Theorem 12.9. *If a sequence of Riemann-integrable functions $\{f_n\}$, $f_n : [a,b] \to \mathbb{R}$, is uniformly convergent on $[a,\, b]$ to a function $f : [a,b] \to \mathbb{R}$, then we have*

$$\int_a^b \lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx. \tag{12.10}$$

Proof: Let f be the limit function of $\{f_n\}$. Then by Proposition 12.8, f is integrable on $[a,\, b]$. Therefore, we have only to show that

$$\int_a^b f(x)dx = \lim_{n\to+\infty} \int_a^b f_n(x)dx.$$

Since $\{f_n\}$ is uniformly convergent on $[a,b]$, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$|f_n(x) - f(x)| < \varepsilon, \text{ for every } x \in [a,\, b].$$

Then we have

$$\left|\int_a^b f(x) - f_n(x)dx\right| \le \int_a^b |f(x) - f_n(x)|dx \le \int_a^b \varepsilon\; dx = (b-a)\varepsilon,$$

which proves (12.10). □

The next example illustrate that differentiation cannot exchange order with limit for a uniformly convergent sequence of differentiable functions.

Example 12.10. Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = \frac{x^n}{n}$ and $f(x) = 0$. Notice that f_n converges uniformly to $f(x) = 0$ for $x \in [0,1]$. Let $g : \mathbb{R} \to [0,1]$ be defined by

$$g(x) = \begin{cases} 0, & \text{if } x \in [0,1), \\ 1, & \text{if } x = 1. \end{cases} \tag{12.11}$$

Then we have

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} x^n = g(x), \text{ for every } x \in [0,1],$$
$$\left[\lim_{n\to\infty} f_n(x)\right]' = 0.$$

It follows that $\lim_{n\to\infty} f_n'(x) \ne [\lim_{n\to\infty} f_n(x)]'$. Namely, we cannot exchange the limit with differentiation.

Theorem 12.11. *Let $\{f_n\}$ be a sequence of continuously differentiable functions on $[a,\, b]$ with $x_0 \in [a,\, b]$. Assume that $\{f_n'\}$ is a uniformly convergent sequence on $[a,\, b]$ to a function $g : [a,b] \to \mathbb{R}$ and that the limit $\lim\limits_{n\to\infty} f_n(x_o) = c$ exists for some $c \in \mathbb{R}$. Then, the sequence $\{f_n\}$ converges pointwisely to a differentiable function $f : [a,b] \to \mathbb{R}$ such that $f'(x) = g(x)$ for all $x \in [a,b[$. In other words, we have*

$$\frac{d}{dx} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{d}{dx} f_n(x).$$

Proof: By the assumption $\{f_n'\}$ is uniformly convergent to g, i.e $\lim_{n\to\infty} \frac{d}{dx} f_n(x) = g(x)$. Then by Theorem 12.6 g is continuous on $[a, b]$. For every $n \in \mathbb{N}$ and $x \in [a, b]$, we have (by Newton-Liebnitz formula, or the so-called Fundamental Theorem of Camculus),

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(t)dt.$$

Then by Theorem 12.9, we have

$$\begin{aligned} \lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} f_n(x_0) + \lim_{n\to\infty} \int_{x_0}^{x} f_n'(t)dt \\ &= c + \int_a^x g(t)dt. \end{aligned}$$

Let $f(x) := c + \int_a^x g(t)dt$. Then $\lim_{n\to\infty} f_n(x) = f(x)$, i.e. the sequence $\{f_n\}$ converges pointwisely to f, and again by Newton-Leibnitz formula or the Fundamental Theorem of Calculus, we obtain $f' = g$. That is

$$\frac{d}{dx} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}x} f_n(x).$$

□

12.3 Series of Functions

Consider again a metric space (X, d) and a Banach space $(V, \|\cdot\|)$.

Definition 12.12. Let $\{u_n\}$, $u_n : X \to V$, be a sequence of functions defined on X. We call the expression

$$\sum_{n=1}^{\infty} u_n(x) := u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots, \quad x \in X \tag{12.12}$$

a *series of functions*. Moreover, we call

$$s_n(x) = \sum_{k=1}^{n} u_k(x), \quad n = 1, 2, 3, \ldots, \; x \in X$$

the n-th partial sum of the series (12.12). Then, the series (12.12) is said to be *convergent at* $x \in X$ if the sequence of the partial sums $\{s_n(x)\}$ converges. If the series (12.12) doesn't converge at x, then we say that the series (12.12) *diverges* at x.

If the series of functions (12.12) is convergent at every point $x \in X$, or equivalently, $\lim_{n\to\infty} s_n(x)$ exists for every $x \in X$, then we say that the series of functions (12.12) is convergent on X. Also, we will call the set E all the points $x \in X$ at which the series (12.12) converges, the *domain of convergence* of (12.12).

Definition 12.13. If the series $\sum_{n=1}^{\infty} u_n(x)$ converges on X, then the function

$$s(x) = \sum_{n=1}^{\infty} u_n(x) := \lim_{n\to\infty} s_n, \quad x \in X,$$

is called the *sum* of the series (12.12).

Since the convergence of the series (12.12) is determined by the convergence of the sequence of partial sums $\{s_n\}$, we have the following definition:

Definition 12.14. We say that the series (12.12) converges uniformly on X if the series of partial the sums $\{s_n\}$ of (12.12) converges on X uniformly.

Example 12.15. Consider the geometrical series

$$1 + x + x^2 + \cdots + x^n + \cdots .$$

Here $u_n = x^{n-1}$. Then the partial sum is $s_n(x) = \frac{1-x^n}{1-x}$ and if $|x| < 1$, we have

$$s(x) = \lim_{n\to\infty} s_n(x) = \frac{1}{1-x}.$$

If $|x| \geq 1$, $\lim_{n\to\infty} s_n(x) = \dfrac{1}{1-x}$ does not exists. That is, the geometrical series is divergent for $|x| \geq 1$.

If the partial sums $\{s_n(x)\}$ of the series (12.12) converges to $s(x)$ on X, then we put

$$r_n(x) = s(x) - s_n(x), \quad x \in X,$$

to denote the remainder of the functional series $\sum_{k=1}^{\infty} u_k(x)$.

By Definition 12.14 and Theorem 12.5, we have the following immediate result:

Theorem 12.16. *The function series $\sum_{k=1}^{\infty} u_k(x)$ is uniformly convergent on X if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n > N$ and $p \in \mathbb{N}$ we have*

$$\left\| \sum_{k=n+1}^{n+p} u_k(x) \right\| = \|s_{n+p}(x) - s_n(x)\| < \varepsilon, \qquad \textit{for every } x \in X.$$

By taking $p = 1$ in Theorem 12.16, we get

Corollary 12.17. *If the function series $\sum_{k=1}^{\infty} u_k(x)$ is uniformly convergent on X, then the sequence of functions $\{u_n(x)\}$ converges uniformly to 0.*

Theorem 12.18. *The function series $\sum_{k=1}^{\infty} u_k(x)$ is uniformly convergent to $s(x)$ on X if and only*

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$$\lim_{n\to\infty} \sup_{x\in X} \|r_n(x)\| = \lim_{n\to\infty} \sup_{x\in X} \|s(x) - s_n(x)\| = 0.$$

Proof: Suppose that the function series $\sum_{k=1}^{\infty} u_k(x)$ is uniformly convergent to $s(x)$ on X. Then by Theorem 12.16, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n > N$ and $p \in \mathbb{N}$ we have

$$\|s_{n+p}(x) - s_n(x)\| < \varepsilon, \quad \text{for every } \ x \in X. \tag{12.13}$$

By taking the limit in (12.13) as $p \to +\infty$, we get

$$\|s(x) - s_n(x)| \leq \varepsilon, \ \text{for every } \ x \in X,$$

which leads to

$$\lim_{n\to\infty} \sup_{x\in X} \|r_n(x)\| = \lim_{n\to\infty} \sup_{x\in X} \|s(x) - s_n(x)\| = 0.$$

Next, we assume that

$$\lim_{n\to\infty} \sup_{x\in X} \|r_n(x)\| = \lim_{n\to\infty} \sup_{x\in X} \|s(x) - s_n(x)\| = 0.$$

Then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n > N$, $p \in \mathbb{N}$, we have

$$\sup_{x\in X} \|s(x) - s_n(x)\| < \varepsilon,$$
$$\sup_{x\in E} \|s(x) - s_{n+p}(x)\| < \varepsilon,$$

which lead to

$$\|s(x) - s_n(x)\| \leq \sup_{x\in X} \|s(x) - s_n(x)\| < \varepsilon, \tag{12.14}$$
$$\|s(x) - s_{n+p}(x)| \leq \sup_{x\in X} \|s(x) - s_{n+p}(x)| < \varepsilon, \tag{12.15}$$

for every $x \in X$. Then by (12.14) and (12.15), we have

$$\begin{aligned}\|s_{n+p}(x) - s_n(x)| &= \|s_{n+p}(x) - s(x) + s(x) - s_n(x)\| \\ &\leq \|s_{n+p}(x) - s(x)\| + \|s(x) - s_n(x)\| \\ &\leq \varepsilon + \varepsilon \\ &= 2\varepsilon.\end{aligned}$$

□

Now we consider the properties of uniform convergent series of functions.

Corollary 12.19. *Assume that the functions $u_k : X \to V$ are continuous. If the series $\sum_{k=1}^{\infty} u_k(x)$ is uniformly convergent on X, then the sum $s(x) = \displaystyle\sum_{k=1}^{\infty} u_k(x)$ is continuous.*

Proof:

This is a direct consequence of Theorem 12.6 applied to partial sums $s_n(x)$, which are continuous functions. □ From now on we will consider the functions u_n to be real-valued functions define on a domain in $\mathbb{R}$.

Theorem 12.20. *Assume $u_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{R}$. If the series of functions $\sum_{k=1}^{+\infty} u_k(x)$ is uniformly convergent on $[a,\, b]$ and each $u_i(x)$, $i = 1,\, 2,\, \cdots$, is Riemann-integrable on $[a,\, b]$, then we have*

$$\sum_{k=1}^{\infty} \int_a^b u_k(x)dx = \int_a^b \sum_{k=1}^{\infty} u_k(x)dx.$$

Proof: Let $\{s_n(x)\}$ be the sequence of partial sums of $\sum_{k=1}^{\infty} u_k(x)$. Then, since a sum of Riemann-integrable functions is Riemann-integreable, it follows that $s_n(x)$ is Riemann-integrable. On the other hand, we have that $\{s_n\}$ converges uniformly on $[a,b]$ to $s(x)$, therefore by Theorem 12.9, we have

$$\sum_{k=1}^{\infty} \int_a^b u_k(x)dx = \lim_{n\to\infty} s_n(x)dx = \int_a^b \lim_{n\to\infty} s_n(x)dx = \int_a^b \left(\sum_{k=1}^{\infty} u_k(x) \right))dx.$$

□

Theorem 12.9 shows that we can take integral of uniformly convergent series term by term. Similarly we can show that following

Theorem 12.21. *Let $\sum_{k=1}^{\infty} u_k(x)$ be a uniformly convergent the series of functions on $[a,b]$, such that each $u_k(x)$, $k = 1,\, 2,\, \cdots$, is continuously differentiable on $[a,\, b]$. If $\sum_{k=1}^{\infty} u_k'(x)$ is uniformly convergent on $[a,b]$, then we have*

$$\sum_{k=1}^{\infty} \frac{d}{dx} u_i(x) = \frac{d}{dx} \sum_{k=1}^{\infty} u_k(x).$$

Proof: Let $\{s_n(x)\}$ be the sequence of partial sums of $\sum_{k=1}^{\infty} u_k(x)$ for $x \in [a,\, b]$. Then it is clear that $s_n(x)$ are continuously differentiable and since $s_n'(x)$ converges uniformly to $\sum_{k=1}^{\infty} u_k'(x)$ we have by Theorem 12.11,

$$\sum_{k=1}^{\infty} \frac{d}{dx} u_k(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{d}{dx} u_k(x) = \lim_{n\to\infty} \frac{d}{dx} s_n(x) = \frac{d}{dx} \sum_{k=1}^{\infty} u_k(x).$$

□

12.4 Criteria for Uniform Convergence of Function Series

To determine the uniform convergence of function series, we have the following

Theorem 12.22. (Weierstrass Criterium) *Let (X,d) be a metric space, $(V, \|\cdot\|)$ a Banach space and $\{u_k\}$, $u_k : X \to V$, $k \in \mathbb{N}$, a sequence of functions satisfying the condition*

$$\forall_{x\in X}\ \forall_{k\in\mathbb{N}}\ \ \|u_k(x)\| \le M_k, \qquad \textit{where}\ \ M_k > 0.$$

t If the series $\sum_{k=1}^{\infty} M_k$ converges then $\sum_{k=1}^{\infty} u_k(x)$ is uniformly convergent on X.

Proof: By the Cauchy criterion for convergence of series, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $i > N$ and $p \in \mathbb{N}$, we have

$$|M_{i+1} + M_{i+2} + \cdots + M_{i+p}| = M_{i+1},\, M_{i+2} + \cdots + M_{i+p} < \varepsilon.$$

Therefore, by assumption, It follows that for all $x \in X$

$$\begin{aligned}\|u_{i+1}(x) + u_{i+2}(x) + \cdots + u_{i+p}(x)\&| &\leq \|u_{i+1}(x)\| + \|u_{i+2}(x)\| + \cdots + \|u_{i+p}(x)\| \\ &\leq M_{i+1},\, M_{i+2} + \cdots + M_{i+p} < \varepsilon,\end{aligned}$$

which implies by Theoem 12.16 that $\displaystyle\sum_{k=1}^{\infty} u_k(x)$ is uniformly convergent on X. □

Example 12.23. The series of functions

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$

is uniformly convergent in $\mathbb{R}$. Indeed, since we have for every $x \in \mathbb{R}$

$$\left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2},$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, by Theorem 12.22, the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges uniformely.

Now we consider series of functions which can be written in the following form:

$$\sum_{n=1}^{\infty} u_n(x)v_n(x) = u_1(x)v_1(x) + u_2(x)v_2(x) + \cdots + u_n(x)v_n(x) + \cdots, \tag{12.16}$$

where U_k and v_k are real-valued functions defined on an interval $I \subset \mathbb{R}$.

Before the discussion of $\sum_{n=1}^{\infty} u_n(x)v_n(x)$, let us introduce the following Abel's transformation for series.

Lemma 12.24. *Let $\{a_k\}_{k=1}^n$, $\{b_k\}_{k=1}^n$ be two sets of real numbers. Let $\sigma_k = b_1 + b_2 + \cdots + b_k$, $k = 1, 2, \cdots n$. Then we have*

$$\sum_{k=1}^{n} a_k b_k = (a_1 - a_2)\sigma_1 + (a_2 - a_3)\sigma_2 + \cdots + (a_{n-1} - a_n)\sigma_{n-1} + a_n\sigma_n.$$

Proof: We have

$$\begin{aligned}a_1b_1 &= a_1\sigma_1 \\ a_2b_2 &= a_2(\sigma_2 - \sigma_1) \\ a_3b_3 &= a_3(\sigma_3 - \sigma_2) \\ \vdots &= \vdots \\ a_{n-1}b_{n-1} &= a_{n-1}(\sigma_{n-1} - \sigma_{n-2}) \\ a_nb_n &= a_n(\sigma_n - \sigma_{n-1})\end{aligned}$$

Adding each sides of the equalities we have

$$\sum_{i=1}^{n} a_i b_i = (a_1 - a_2)\sigma_1 + (a_2 - a_3)\sigma_2 + \cdots + (a_{n-1} - a_n)\sigma_{n-1} + a_n\sigma_n.$$

□

Lemma 12.25. *Let $\{a_k\}_{k=1}^n$, $\{b_k\}_{k=1}^n$ be two sets of real numbers. Let $\sigma_k = b_1 + b_2 + \cdots + b_k$, $k = 1, 2, \cdots n$. Suppose that*

(i) $\{a_k\}_{k=1}^n$ is monotone with $A = \max_i\{|a_i| : i = 1, 2, \cdots, n\}$;

(ii) $\{\sigma_k\}_{k=1}^n$, $\sigma_k = b_1 + b_2 + \cdots + b_k$, $k = 1, 2, \cdots n$, is uniformly bounded, i.e., there exists $B > 0$ such that

$$|\sigma_k| \le B, \quad \text{for every } k = 1, 2, \ldots, n.$$

Then we have

$$\left|\sum_{k=1}^{n} a_k b_k\right| \le 3AB.$$

Proof: By i) we know that $\{a_k - a_{k+1}\}_{i=1}^{n-1}$ is a sequence of real numbers which are either all non-negative or all non-positive. Then by Lemma 12.24, we have

$$\begin{aligned}
\sum_{k=1}^{n} a_k b_k &= |(a_1 - a_2)\sigma_1 + (a_2 - a_3)\sigma_2 + \cdots + (a_{n-1} - a_n)\sigma_{n-1} + a_n\sigma_n| \\
&\le |(a_1 - a_2)\sigma_1 + (a_2 - a_3)\sigma_2 + \cdots + (a_{n-1} - a_n)\sigma_{n-1}| + |a_n\sigma_n| \\
&\le B\,|(a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n-1} - a_n)| + BA \\
&= B|a_1 - a_n| + BA \\
&\le 3AB.
\end{aligned}$$

□

Theorem 12.26. (ABEL'S CRITERION) *Suppose that the series $\sum_{n=1}^{+\infty} u_n(x)v_n(x)$ satisfies*

(i) $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent on I;

(ii) for every $x \in I$, $\{v_n(x)\}$ is a monotone sequence;

(iii) $\{v_n(x)\}$ is uniformly bounded, i.e., there exists $M > 0$ such that

$$|v_n(x)| \le M$$

for every $x \in I$, $n \in \mathbb{N}$.

Then $\sum_{n=1}^{\infty} u_n(x)v_n(x)$ is uniformly convergent in I.

Proof: By (i), for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n > N$, $p \in \mathbb{N}$, $x \in I$, we have

$$|u_{n+1}(x) + u_{n+2}(x) + \cdots + u_{n+p}(x)| < \varepsilon.$$

Let $\sigma_k(x) = u_{n+1}(x) + u_{n+2}(x) + \cdots u_{n+k}(x)$, $i = k, 2, \cdots p$, and let $A = \max_i\{|v_{n+k}(x)| : k = 1, 2, \cdots, p\}$. According to Lemma 12.25, we have
(i) $\{v_{n+k}(x)\}_{k=1}^{p}$ is monotone with $A = \max_i\{|v_{n+k}(x)| : k = 1, 2, \cdots, p\}$;
(ii) $\{\sigma_k(x)\}_{k=1}^{p}$ is uniformly bounded with $|\sigma_k(x)| \leq \varepsilon$, $k = 1, 2, \cdots, p$. Then we have

$$|u_{n+1}(x)v_{n+1}(x) + u_{n+2}(x)v_{n+2}(x) + \cdots + u_{n+p}(x)v_{n+p}(x)| < 3A\varepsilon \leq 3M\varepsilon.$$

By the Cauchy Criterion (Theorem 12.16), $\sum_{n=1}^{+\infty} u_n(x)v_n(x)$ is uniformly convergent in I. □

Theorem 12.27. (Dirichlet's Criterion) *Let the series* $\displaystyle\sum_{k=1}^{\infty} u_k(x)v_k(x)$ *be such that*

(i) the sequence $s_n(x) = \sum_{k=1}^{n} u_k(x)$, $n \in \mathbb{N}$, *is uniformly bounded on* I*;*

(ii)for every $x \in I$, $\{v_k(x)\}$ *is a monotone sequence;*

(iii)$\{v_k(x)\}$ *is uniformly convergent to* 0*.*

Then the series $\displaystyle\sum_{k=1}^{\infty} u_k(x)v_k(x)$ *is uniformly convergent in* I*.*

Proof: By (i), there exists $B > 0$ such that $|s_n(x)| \leq B$ for every n, $p \in \mathbb{N}$, $x \in I$. Then we have

$$|u_{n+1}(x) + u_{n+2}(x) + \cdots + u_{n+p}(x)| = |s_{n+p}(x) - s_n(x)| \leq 2B.$$

Note by (iii), for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n > N$, $x \in I$, we have

$$|v_n(x)| < \varepsilon.$$

By Theorem 12.25, we have

$$|u_{n+1}(x)v_{n+1}(x) + u_{n+2}(x)v_{n+2}(x) + \cdots + u_{n+p}(x)v_{n+p}(x)| < 3(2B) \cdot \varepsilon = 6B\varepsilon.$$

That is, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n > N$, $x \in I$, we have

$$|u_{n+1}(x)v_{n+1}(x) + u_{n+2}(x)v_{n+2}(x) + \cdots + u_{n+p}(x)v_{n+p}(x)| < 6B\varepsilon.$$

By the Cauchy criterion (Theorem 12.16), $\displaystyle\sum_{k=1}^{\infty} u_k(x)v_k(x)$ is uniformly convergent in I. □

Example 12.28. Consider the series of functions

$$\sum_{n=1}^{+\infty} \frac{(-1)^n (x+n)^n}{n^{n+1}}, \; x \in [0, 1].$$

Let $u_n(x) = \frac{(-1)^n}{n}$, $v_n(x) = (1 + \frac{x}{n})^n$. Then by Abel's criterion (Theorem 12.26), $\sum_{n=1}^{+\infty} \frac{(-1)^n (x+n)^n}{n^{n+1}}$ is uniformly convergent.

Example 12.29. Suppose the sequence $\{a_n\}_{n=0}^{\infty}$ is monotonically convergent to zero, and $\alpha \in (0, \pi)$. Then the series of functions

$$\sum_{n=0}^{+\infty} a_n \cos nx$$

is uniformly convergent on $[\alpha, 2\pi - \alpha]$. Indeed, let $u_n(x) = \cos nx$, $v_n(x) = a_n$. Then we have

$$\begin{aligned} \left| \sum_{i=0}^{n} \cos ix \right| &= \left| \frac{\sin\left(n + \frac{1}{2}\right)x}{2\sin\frac{x}{2}} - \frac{1}{2} \right| \\ &\leq \frac{1}{2\left|\sin\frac{x}{2}\right|} + \frac{1}{2} \\ &\leq \frac{1}{2\sin\frac{\alpha}{2}} + \frac{1}{2}, \end{aligned}$$

which means the sequence of partial sum of $\sum_{n=1}^{+\infty} \cos nx$ is uniformly bounded. By Dirichlet's criterion (Theorem 12.27), $\sum_{n=0}^{+\infty} a_n \cos nx$ is uniformly convergent on $[\alpha, 2\pi - \alpha]$. In particular, we obtain that this series converges in the interval $(0, 2\pi)$.

Definition 12.30. Let $D \subset \mathbb{R}$ be an open set and assume that the series of functions $\sum_{n=1}^{\infty} u_n(x)$ converges for every $x \in D$. Then we say that $\sum_{n=1}^{\infty} u_n(x)$ converges *almost uniformly* on D if and only if for every compact set $K \subset D$ it converges uniformly on K.

Remark 12.31. Let $D \subset \mathbb{R}$ be an open set and $\sum_{n=1}^{\infty} u_n(x)$ be a convergent on D series of functions.

(a) If $\sum_{n=1}^{\infty} u_n(x)$ converges almost uniformly on D then for every interval $[a, b] \subset D$ we have

$$\int_a^b \sum_{n=1}^{\infty} u_n(x)dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)dx.$$

(b) If all the functions $u_n(x)$ are differentiable and the series $\sum_{n=1}^{\infty} u_n'(x)$ converges almost uniformly on D, then we have

$$\frac{d}{dx} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} u_n'(x).$$

Proposition 12.32. *Suppose that the power series* $\sum_{n=0}^{\infty} a_n(x - x_o)^n$, $x_o \in \mathbb{R}$ *converges for some* y_o *such that* $y_o \neq x_o$. *Then, for every the series* $\sum_{n=0}^{\infty} a_n(x - x_o)^n$ *converges almost uniformly in the set* $D = (x_o - R, x_o + R)$, *where* $R := |x_o - y_o|$. *Consequently, the domain of convergence for this series*

$$\left\{x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n(x-x_o)^n \quad \textit{converges}\right\}$$

is an interval centered at x_o*. Moreover, the power series* $\displaystyle\sum_{n=1}^{\infty} a_n n(x-x_o)^{n-1}$ *also converges almost uniformly in* D*.*

Proof: Since the series $\displaystyle\sum_{n=0}^{\infty} a_n(y_o-x_o)^n$ converges, it follows that $\displaystyle\lim_{n\to\infty} |a_n(y_o-x_o)^n| = 0$, which implies that the sequence $\{|a_n(y_o-x_o)^n|\}$ is bounded, i.e.

$$\exists_{M>0} \ \forall_{n\geq 0} \quad |a_n(y_o-x_o)^n| \leq M.$$

Notice that for all $n = 0, 1, 2, \dots$ and $x \in [x_o - r, x_o + r] \subset D$ where $0 < r < |y_o - x_o|$ is a fixed number, we have

$$\begin{aligned}|a_n(x-x_o)^n| &= |a_n(y_o-x_o)^n| \cdot \frac{|x-x_o|^n}{|y_o-x_o|^n}\\ &\leq M \cdot \left(\frac{r}{|y_o-x_o|}\right)^n.\end{aligned}$$

Since, $q := \frac{r}{|y_o-x_o|}$ satisfies $0 < q < 1$, the series $\displaystyle\sum_{n=0}^{n} Mq^n = M\frac{1}{1-q}$ converges, thus by the Weierstrass Criterium (Theorem 12.22), the series of functions $\displaystyle\sum_{n=0}^{\infty} a_n(y_o-x_o)^n$ converges uniformly in $[x_o - r, x_o + r]$. In order to show that the series of functions $\displaystyle\sum_{n=1}^{\infty} a_n n(x-x_o)^{n-1}$ also converges uniformly in $[x_o - r, x_o + r]$, we notice that for all $n = 1, 2, \dots$ and $x \in [x_o - r, x_o + r] \subset D$ where $0 < r < |y_o - x_o|$ is a fixed number, we have

$$\begin{aligned}|a_n n(x-x_o)^{n-1}| &= |a_n(y_o-x_o)^{n-1}| \cdot n \cdot \frac{|x-x_o|^{n-1}}{|y_o-x_o|^{n-1}}\\ &\leq M \cdot n \cdot \left(\frac{r}{|y_o-x_o|}\right)^{n-1} =: Mnq^{n-1}.\end{aligned}$$

Choose $p \in \mathbb{R}$ such that $q < p < 1$, and consider $b_n := p^{n-1}$. Then, we have

$$\lim_{n\to\infty} \frac{nq^{n-1}}{p^{n-1}} = \lim_{n\to\infty} \frac{n}{(p/q)^{n-1}} = \lim_{x\to\infty} \frac{x}{e^{(x-1)\ln(p/q)}} = \lim_{x\to\infty} \frac{1}{\ln(p/q)e^{(x-1)\ln(p/q)}} = 0.$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} Mnq^{n-1}$ converges by the Limit Comparison Test. Consequently, by the Weierstrass Criterium (Theorem 12.22), $\displaystyle\sum_{n=1}^{\infty} a_n n(x-x_o)^{n-1}$ converges uniformly on $[x_o - r, x_o + r]$. $\square$

Example 12.33. An important example of a series of functions is the so-called *Fourier series* of a periodic function. To be more precise, consider the following (convergent) series of functions

$$f(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(nx) + b_n \sin(nx) \Big) \tag{12.17}$$

It is clear that $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic. The series (12.17) is called the *Fourier series* of $f(x)$. Notice that

$$\int_{-\pi}^{\pi} \cos(nx)\cos(kx)dx = \begin{cases} 0 & \text{if } n \neq k \\ \pi & \text{if } n = k, \end{cases} \qquad \int_{-\pi}^{\pi} \sin(nx)\sin(kx)dx = \begin{cases} 0 & \text{if } n \neq k \\ \pi & \text{if } n = k, \end{cases},$$

and

$$\int_{-\pi}^{\pi} \cos(nx)\sin(kx)dx = 0.$$

Assume that the series (12.17) converges uniformly. Then we have

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\cos(kx)dx &= \int_{-\pi}^{\pi} \cos(kx) \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(nx) + b_n \sin(nx) \Big) \right) dx \\
&= \frac{a_0}{2} \int_{-\pi}^{\pi} \cos(kx)dx + \sum_{n=1}^{\infty} \Big(a_n \int_{-\pi}^{\pi} \cos(kx)\cos(nx)dx + b_n \int_{-\pi}^{\pi} \cos(kx)\sin(nx)dx \Big) \\
&= \pi a_k, \quad k = 0, 1, 2, \dots \\
\int_{-\pi}^{\pi} f(x)\sin(kx)dx &= \int_{-\pi}^{\pi} \sin(kx) \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(nx) + b_n \sin(nx) \Big) \right) dx \\
&= \frac{a_0}{2} \int_{-\pi}^{\pi} \sin(kx)dx + \sum_{n=1}^{\infty} \Big(a_n \int_{-\pi}^{\pi} \sin(kx)\cos(nx)dx + b_n \int_{-\pi}^{\pi} \sin(kx)\sin(nx)dx \Big) \\
&= \pi b_k, \quad k = 1, 2, 3, \dots
\end{aligned}$$

Therefore, we obtain that

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx)f(x)dx, \;\; n = 0, 1, 2, \dots, \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(kx)f(x)dx, \;\; k = 1, 2, 3, \dots$$

12.5 Problems

1. Determine whether the following sequence of functions or series of functions are uniformly convergent.
   a) $f_n(x) = \frac{x}{1+n^2x^2}$, $n = 1, 2, \cdots, x \in \mathbb{R}$;
   b)
   $$f_n(x) = \begin{cases} -(n+1)x + 1 & 0 \le x \le \frac{1}{n+1} \\ 0 & \frac{1}{1+n} \le x \le 1; \end{cases} \tag{12.18}$$
   c) $\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n+x^2}$, $x \in \mathbb{R}$;
   d) $\sum_{n=1}^{+\infty} \frac{x^2}{(1+x^2)^{n-1}}$, $x \in \mathbb{R}$.

2. Let $\{f_n\}$ be a sequence converges pointwise to f on E. $\{a_n\}$ is a sequence converges to zero. Show that if for every $n \in \mathbb{N}$, $|f_n(x) - f(x)| \leq a_n$, $x \in E$, then $\{f_n\}$ converges uniformly to f on E.
3. Suppose that $\sum u_n(x)$ converges uniformly to $S(x)$ on E and $g(x)$ is bounded on E. Show that $\sum u_n(x)g(x)$ converges uniformly to $S(x)g(x)$ on E.

Part V

APPENDICES

13

Appendix 1: Solutions to All Problems

13.1 Chapter 1: Basic Set Theory, Logic and Introduction to Proofs

1. Use the Truth table to show that the following statements are tautologies:

(a) $[(p \Rightarrow q) \Rightarrow p] \Rightarrow p$ (Pierce's Law);
(b) $(\sim p \Rightarrow p) \Rightarrow p$ (Clavius Law);
(c) $[p \Rightarrow (q \Rightarrow r)] \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$ (law of Conditional Syllogism);
(d) $(p \wedge q \Rightarrow r) \Leftrightarrow [p \Rightarrow (q \Rightarrow r)]$.

SOLUTIONS: 1.(a):

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \Rightarrow p)$	$[(p \Rightarrow q) \Rightarrow p] \Rightarrow p)$
0	0	1	0	1
0	1	1	0	1
1	0	0	1	1
1	1	1	1	1

The statement $[(p \Rightarrow q) \Rightarrow p] \Rightarrow p)$ is **true**

1.(b):

p	$\sim p$	$\sim p \Rightarrow p$	$(\sim p \Rightarrow p) \Rightarrow p)$
0	1	0	1
1	0	1	1

The statement $(\sim p \Rightarrow p) \Rightarrow p$ is **true**

1.(c): We denote by $(*)$ the statement $[p \Rightarrow (q \Rightarrow r)] \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$ (for the purpose of fitting it into the truth table). Then we have:

p	q	r	$q \Rightarrow r$	$p \Rightarrow q$	$p \Rightarrow r$	$p \Rightarrow (q \Rightarrow r)$	$(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$	$(*)$
0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	1	0	0	1	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	1	0	0	1	1	1
1	0	1	1	0	1	1	1	1
1	1	0	0	1	0	0	0	1
1	1	1	1	1	1	1	1	1

The statement $[p \Rightarrow (q \Rightarrow r)] \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$ is **true**

1.(d): We denote by $(*)$ the statement $(p \wedge q \Rightarrow r) \Leftrightarrow [p \Rightarrow (q \Rightarrow r)]$ (for the purpose of fitting it into the truth table). Then we have:

p	q	r	$q \wedge r$	$p \wedge q \Rightarrow r$	$q \Rightarrow r$	$p \Rightarrow (q \Rightarrow r)$	$(*)$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	0	1	0	1	1
0	1	1	0	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

The statement $(p \wedge q \Rightarrow r) \Leftrightarrow [p \Rightarrow (q \Rightarrow r)]$ is **true**

2. Check if the following statements are tautologies:

(a) $(p \Rightarrow q) \Rightarrow [p \Rightarrow (q \vee r)]$;
(b) $p \vee [(\sim p \wedge q) \vee (\sim p \wedge \sim q)]$;
(c) $[(p \Rightarrow q) \wedge (q \Rightarrow p)] \Rightarrow (p \vee q)$.

SOLUTIONS: **2.(a):** We have:

p	q	r	$p \Rightarrow q$	$q \vee r$	$p \Rightarrow (q \vee r)$	$(p \Rightarrow q) \Rightarrow [p \Rightarrow (q \vee r)]$
0	0	0	1	0	1	1
0	0	1	1	1	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	0	0	0	1
1	0	1	0	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

The statement $(p \Rightarrow q) \Rightarrow [p \Rightarrow (q \vee r)]$ is **true**

Another way to check if the statement $(p \Rightarrow q) \Rightarrow [p \Rightarrow (q \vee r)]$ is always **true** is to notice that it would be false only when $p \Rightarrow q$ is *true* but $p \Rightarrow (q \vee r)$ is *false*. Is it poossible? Notice that $p \Rightarrow (q \vee r)$ is false only when p is *true* but both q and r are *false*. But in such a case the implication $p \Rightarrow q$ is *false* as well. So, it is impossible .

2.(b):

p	q	$\sim p$	$\sim q$	$\sim p \wedge q$	$\sim p \wedge \sim q$	$p \vee [(\sim p \wedge q) \vee (\sim p \wedge \sim q)]$
0	0	1	1	0	1	1
0	1	1	0	1	0	1
1	0	0	1	0	0	1
1	1	0	0	0	0	1

The statement $p \vee [(\sim p \wedge q) \vee (\sim p \wedge \sim q)]$ is **true**

2.(c):

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \vee q$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$	$[(p \Rightarrow q) \wedge (q \Rightarrow p)] \Rightarrow (p \vee q)$
0	0	1	1	0	1	0
0	1	1	0	1	0	1
1	0	0	1	1	0	1
1	1	1	1	1	1	1

The statement $[(p \Rightarrow q) \wedge (q \Rightarrow p)] \Rightarrow (p \vee q)$ is **false**

3. Check whether the following statements are true or false:

(a) *"If a is a multiple of 2 and is also a multiple of 7, then if a is not a multiple of 7 implies that a is a multiple of 3;"*

(b) *"If it is not true that the line l is parallel to the line m or the line p is not parallel to the line m, then the line l is not parallel to the line m or the line p is parallel to the line m;"*

(c) *"If James doesn't know analysis, then if James knows analysis implies that James was born in the 2nd century B.C.."*

SOLUTIONS: 3(a): We use the following symbolic notation:

$$p := \{a \text{ is a multiple of } 2\}$$
$$q := \{a \text{ is a multiple of } 7\}$$
$$r := \{a \text{ is a multiple of } 3$$

Then the statement in question 3.(a) can be written symbollically as:

$$(p \wedge q) \Rightarrow (\sim q \Rightarrow r).$$

Since we have the following truth table for $(p \wedge q) \Rightarrow (\sim q \Rightarrow r)$:

p	q	r	$p \wedge q$	$\sim q \Rightarrow r$	$p \wedge q \Rightarrow (\sim q \Rightarrow r)$
0	0	0	0	0	1
0	0	1	0	1	1
0	1	0	0	1	1
0	1	1	0	1	1
1	0	0	0	0	1
1	0	1	0	1	1
1	1	0	1	1	1
1	1	1	1	1	1

So, it is **TRUE**

Another way to show that $(p \wedge q) \Rightarrow (\sim q \Rightarrow r)$ is true, is to notice that $(p \wedge q) \Rightarrow (\sim q \Rightarrow r)$ could be false only when $p \wedge q$ is *true* and $\sim q \Rightarrow r$ is *false*. But in the case $\sim q \Rightarrow r$ is false both Q and

q must be *true* which implies that the implication $\sim q \Rightarrow r$ is also *true*. That means, it is impossible that $(p \wedge q) \Rightarrow (\sim q \Rightarrow r)$ could be false. Therefore, it is **TRUE**

3(b): We use the following symbolic notation:

$$P := \{l \text{ is parallel to } m\}$$
$$Q := \{p \text{ is parallel to } m\}$$

Then the statement in question 3.b) can be written symbolically as:

$$\sim (P \vee \sim Q) \Rightarrow (\sim P \vee Q).$$

We have the following truth table for $\sim (P \vee \sim Q) \Rightarrow (\sim P \vee Q)$:

P	Q	$P \vee \sim Q$	$\sim (P \vee \sim Q)$	$\sim P \vee Q$	$\sim (P \vee \sim Q) \Rightarrow (\sim P \vee Q)$
0	0	1	0	1	1
0	1	0	1	1	1
1	0	1	0	0	1
1	1	1	0	1	1

So, it is a **TRUE** statement

3(c): We use the following symbolic notation:

$$p := \{\text{James knows analysis}\}$$
$$q := \{\text{James was born in the 2nd century B.C.}\}$$

Then the statement in question 3.(c) can be written symbolically as:

$$\sim p \Rightarrow (p \Rightarrow q)$$

We have the following truth table for $\sim p \Rightarrow (p \Rightarrow q)$:

p	q	$p \Rightarrow q$	$\sim p \Rightarrow (p \Rightarrow q)$
0	0	1	1
0	1	1	1
1	0	0	1
1	1	1	1

So, it is **TRUE**

4. Check if the following quantified statements are true or false:

(a) $\exists_x \Big(p(x) \Rightarrow q(x)\Big) \Rightarrow \Big[\exists_x\, p(x) \Rightarrow \exists_x\, q(x)\Big]$;

(b) $\exists_x\, p(x) \wedge \exists_x\, q(x) \;\Rightarrow\; \exists_x \Big(p(x) \wedge q(x)\Big)$.

Solution: 4.(a): This statement is **Not True**. For, we define the following statements depending on a varianle x:

$$p(x) := \{x < 0\}$$
$$q(x) := \{x^2 < 0\}$$

Then, the statement in question 4.(a) can be written symbolically as:

$$\exists_x \Big(x < 0 \Rightarrow x^2 < 0\Big) \Rightarrow \Big[\exists_x\ x < 0 \Rightarrow \exists_x\ x^2 < 0\Big]$$

Notice that if we take $x = 1$ then the implication $x < 0 \Rightarrow x^2 < 0$ is *True*, so the statement $\exists_x \Big(x < 0 \Rightarrow x^2 < 0\Big)$ is also true, but, on the other hand we have that $\exists_x\ x < 0$ is *True*, while $\exists_x\ x^2 < 0$ is *False*, thus the implication $\exists_x\ x < 0 \Rightarrow \exists_x\ x^2 < 0$ is *False*. Consequently, we obtain that the implication $\exists_x \Big(x < 0 \Rightarrow x^2 < 0\Big) \Rightarrow \Big[\exists_x\ x < 0 \Rightarrow \exists_x\ x^2 < 0\Big]$ is **False** in this case.

4.(b): This statement is **Not True** again. In this is case we simply take:

$$p(x) := \{x < 0\}$$
$$q(x) := \{x > 0\}$$

Then, the statement in question 4.(b) can be written symbolically as:

$$\exists_x\ x < 0 \wedge \exists_x\ x > 0 \ \Rightarrow\ \exists_x\Big(x > 0 \wedge x < 0\Big)$$

It is clear that the statement $\exists_x\ x < 0 \wedge \exists_x\ x > 0$ is *True* while the statement $\exists_x\Big(x > 0 \wedge x < 0\Big)$ is *False*. Consequently, the implication $\exists_x\ x < 0 \wedge \exists_x\ x > 0 \ \Rightarrow\ \exists_x\Big(x > 0 \wedge x < 0\Big)$ is also **False**.

5. Prove the following identities for the sets:

(a) $\displaystyle\bigcup_{t\in T} (A_t \cup B_t) = \bigcup_{t\in T} A_t \cup \bigcup_{t\in T} B_t$;

(b) $\displaystyle\bigcap_{t\in T} (A_t \cap B_t) = \bigcap_{t\in T} A_t \cap \bigcap_{t\in T} B_t$;

(c) $\displaystyle\bigcup_{t\in T} (A_t \cap B_t) \subset \bigcup_{t\in T} A_t \cap \bigcup_{t\in T} B_t$;

Solution: 5.(a): We have the following equivalences:

$$\begin{aligned} x \in \bigcup_{t\in T}(A_t \cup B_t) &\Leftrightarrow \exists_{t\in T}\ x \in A_t \cup B_t \Leftrightarrow \exists_{t\in T}\ x \in A_t \vee\ x \in B_t \\ &\Leftrightarrow \exists_{t\in T}\ x \in A_t \vee\ \exists_{t\in T} x \in B_t \Leftrightarrow x \in \bigcup_{t\in T} A_t \vee x \in \bigcup_{t\in T} B_t \\ &\Leftrightarrow x \in \bigcup_{t\in T} A_t \cup \bigcup_{t\in T} B_t \end{aligned}$$

Therefore,

$$\bigcup_{t\in T}(A_t \cup B_t) = \bigcup_{t\in T} A_t \cup \bigcup_{t\in T} B_t$$

5.(b): We have the following equivalences:

$$\begin{aligned} x \in \bigcap_{t\in T}(A_t \cap B_t) &\Leftrightarrow \forall_{t\in T}\ x \in A_t \cap B_t \Leftrightarrow \forall_{t\in T}\ x \in A_t \wedge\ x \in B_t \\ &\Leftrightarrow \forall_{t\in T}\ x \in A_t \wedge\ \forall_{t\in T} x \in B_t \Leftrightarrow x \in \bigcap_{t\in T} A_t \wedge x \in \bigcap_{t\in T} B_t \\ &\Leftrightarrow x \in \bigcap_{t\in T} A_t \cap \bigcap_{t\in T} B_t \end{aligned}$$

Therefore,

$$\bigcap_{t\in T}(A_t \cap B_t) = \bigcap_{t\in T} A_t \cap \bigcap_{t\in T} B_t$$

5.(c): We have the following logical relations:

$$\begin{aligned} x \in \bigcup_{t\in T}(A_t \cap B_t) \Leftrightarrow \exists_{t\in T}\; x \in A_t \cap B_t \Leftrightarrow \exists_{t\in T}\; x \in A_t \wedge\; x \in B_t \\ \implies \exists_{t\in T}\; x \in A_t \wedge\; \exists_{t\in T}\; x \in B_t \Leftrightarrow x \in \bigcup_{t\in T} A_t \wedge x \in \bigcup_{t\in T} B_t \\ \Leftrightarrow x \in \bigcup_{t\in T} A_t \cap \bigcup_{t\in T} B_t \end{aligned}$$

Therefore,

$$x \in \bigcup_{t\in T}(A_t \cap B_t) \implies x \in \bigcap_{t\in T} A_t \cap \bigcap_{t\in T} B_t,$$

which means $\bigcup_{t\in T}(A_t \cap B_t) \subset \bigcup_{t\in T} A_t \cap \bigcup_{t\in T} B_t$.

6. Let A, B, C be sets. Check if the following equalities are true:

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
(b) $(A \setminus B) \cup C = [(A \cup C) \setminus B] \cup (B \cap C)$;
(c) $A \setminus [B \setminus C] = (A \setminus B) \cup (A \cap C)$

Solutions: 6.(a): Notice that the equality 6.(a) is equivalent to the following statement:

$$x \in A \cap (B \cup C) \iff x \in (A \cap B) \cup (A \cap C)$$

which can be written as

$$x \in A \wedge (x \in B \vee x \in C) \iff (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$$

If we use the following symbolic notation:

$$\begin{aligned} p &:= \{x \in A\} \\ q &:= \{x \in B\} \\ r &:= \{x \in C\} \end{aligned}$$

then the equality 6.(a) can be represented by the statement:

$$(*) \qquad [p \wedge (q \vee r)] \iff [(p \wedge q) \vee (p \wedge r)].$$

Now, it is easy to notice, by using a Truth Table (see below), that (*) is in fact a tautology, thus (*) is always true, what implies that indeed we have the required equality.
Truth Table for (*)$\equiv [p \wedge (q \vee r)] \iff [(p \wedge q) \vee (p \wedge r)]$

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$	(*)
0	0	0	0	0	0	0	0	1
0	0	1	1	0	0	0	0	1
0	1	0	1	0	0	0	0	1
0	1	1	1	0	0	0	0	1
1	0	0	0	0	0	0	0	1
1	0	1	1	1	0	1	1	1
1	1	0	1	1	1	0	1	1
1	1	1	1	1	1	1	1	1

6.(b): Notice that the equality 6.(b) is equivalent to the following statement:

$$x \in (A \setminus B) \cup \iff x \in [(A \cup B) \setminus B] \cup (B \cap C)$$

We use the following symbolic notation:

$$p := \{x \in A\}$$
$$q := \{x \in B\}$$
$$r := \{x \in C\}$$

then the equality 6.(b) can be written as the statement:

$$(**) \qquad [(p \wedge \sim q) \vee r] \iff [(p \vee r) \wedge \sim q] \vee (q \wedge r).$$

Again, we can use the Truth Table in order to verify if the statement (**) is always true:
Truth Table for (**)$\equiv [(p \wedge \sim q) \vee r] \iff [(p \vee r) \wedge \sim q] \vee (q \wedge r)$

p	q	r	$(p \wedge \sim q) \vee r$	$(p \vee r) \wedge \sim q$	$[(p \vee r) \wedge \sim q] \vee (q \wedge r)$	(**)
0	0	0	0	0	0	1
0	0	1	1	1	1	1
0	1	0	0	0	0	1
0	1	1	1	0	1	1
1	0	0	1	1	1	1
1	0	1	1	1	1	1
1	1	0	0	0	0	1
1	1	1	1	0	1	1

6.(c): Notice that the equality 6.(c) is equivalent to the following statement:

$$x \in A \setminus (B \setminus C) \iff x \in (A \setminus B) \cup (A \cap C)$$

We use the following symbolic notation:

$$p := \{x \in A\}$$
$$q := \{x \in B\}$$
$$r := \{x \in C\}$$

then the equality 6.(b) can be written as the statement:

$$p\wedge \sim (q\wedge \sim r) \iff (p\wedge \sim q) \vee (p \wedge r),$$

which is equivalent to

$$(***) \qquad p \wedge (\sim q \vee r) \iff (p\wedge \sim q) \vee (p \wedge r).$$

We can use the Truth Table in order to verify if the statement (*** is always true:

Truth Table for (**)

p	q	r	$p \wedge (\sim q \vee r)$	$p\wedge \sim q$	$p \wedge r$	$(p\wedge \sim q) \vee (p \wedge r)$	(***)
0	0	0	0	0	0	0	1
0	0	1	0	0	0	0	1
0	1	0	0	0	0	0	1
0	1	1	0	0	0	0	1
1	0	0	1	1	0	1	1
1	0	1	1	1	1	1	1
1	1	0	0	0	0	0	1
1	1	1	1	0	1	1	1

7. Let $f : X \to Y$ be a function from X into Y. Show that if A and B are subsets of X, then

(a) $f(A \cap B) \subset f(A) \cap f(B)$:
(b) $(A' \subset B') \Rightarrow (f^{-1}(A') \subset f^{-1}(B'))$.
(c) $f^{-1}(A' \cap B') = f^{-1}(A') \cap f^{-1}(B')$.

SOLUTION:
7(a):

$$\begin{aligned} y \in f(A \cap B) &\Leftrightarrow \exists_x \ x \in A \cap B \ \wedge \ f(x) = y \\ &\Leftrightarrow \exists_x \ x \in A \ \wedge \ x \in B \ \wedge \ f(x) = y \\ &\Leftrightarrow \exists_x \ (x \in A \ \wedge \ f(x) = y) \ \wedge \ (x \in B \ \wedge \ f(x) = y) \\ &\Rightarrow \exists_x \ (x \in A \ \wedge \ f(x) = y) \ \wedge \ \exists_x \ (x \in B \ \wedge \ f(x) = y) \\ &\Leftrightarrow y \in f(A) \ \wedge \ y \in f(B) \\ &\Leftrightarrow y \in f(A) \cap f(B). \end{aligned}$$

7(b): We have

$$\forall_y \ y \in A' \ \Rightarrow \ y \in B'. \tag{13.1}$$

thus

$$x \in f^{-1}(A') \Leftrightarrow f(x) \in A' \overset{\text{by (13.1)}}{\Longrightarrow} f(x) \in B' \Leftrightarrow x \in f^{-1}(B').$$

7(c): We have

$$\begin{aligned} x \in f^{-1}(A' \cap B') &\Leftrightarrow f(x) \in A' \cap B' \\ &\Leftrightarrow f(x) \in A' \ \wedge \ f(x) \in B' \\ &\Leftrightarrow x \in f^{-1}(A') \ \wedge \ x \in f^{-1}(B') \\ &\Leftrightarrow x \in f^{-1}(A') \cap f^{-1}(B'). \end{aligned}$$

8. Show that the function $f : X \to Y$ is

(a) injective if and only if
$$\forall_{A\subset X}\ \ f^{-1}(f(A)) \subset A.$$

(b) surjective if and only if
$$\forall_{B'\subset Y}\ \ f(f^{-1}(B')) = B'.$$

(c) (b) Show that f is bijective if and only if
$$(f^{-1}(f(A)) = A)\ \wedge\ (f(f^{-1}(B')) = B')$$

for all $A \subset X$ and $B' \subset Y$.

SOLUTION: 8(a): Notice that by Proposition 1.1, $f(f^{-1}(B')) \subset B'$, therefore we only need to show that
$$f \quad \text{is surjective} \quad \Leftrightarrow \quad \forall_{B'\subset Y}\ \ f(f^{-1}(B')) \supset B'.$$

First we will **prove the implication** $\Longrightarrow$:
Notice that
$$f \quad \text{is surjective} \quad \Leftrightarrow \quad \forall_{y\in Y}\ \exists_{x\in X}\ \ f(x) = y. \tag{13.2}$$

Then we have
$$\begin{aligned} y \in B' &\overset{\text{by (13.2)}}{\Rightarrow} \exists_{x\in X}\ \ y \in B'\ \wedge\ f(x) = y \\ &\Leftrightarrow\ \exists_{x\in X}\ \ f(x) \in B'\ \wedge\ f(x) = y \\ &\Leftrightarrow\ \exists_{x\in X}\ \ x \in f^{-1}(B')\ \wedge\ f(x) = y \\ &\Leftrightarrow\ y \in f(f^{-1}(B'). \end{aligned}$$

We **prove the implication** $\Longleftarrow$ by **contradiction**: Suppose that f is not surjective, then we need to show that $f(f^{-1}(B')) \not\supset B'$.
$$\begin{aligned} \sim\ (f \text{ surjective})\ &\Leftrightarrow \sim (\forall_{y\in Y}\ \exists_{x\in X}\ \ f(x) = y \\ &\Leftrightarrow\ \exists_{y\in Y}\ \forall_{x\in X}\ \ f(x) \neq y. \end{aligned} \tag{13.3}$$

Put $B' := \{y\}$ (where $y \in Y$ satisfies (13.3)). Then
$$f^{-1}(B') = \emptyset \ \ \Rightarrow \ \ f(f^{-1}(B')) = f(\emptyset) = \emptyset,$$

and consequently
$$\emptyset \neq B' \not\supset f(f^{-1}(B')) = \emptyset.$$

8(b): Notice that (by definition) f is bijective if and only if it is surjective and injective. Thus we need to show that (and it was proved in class as an example) that
$$f \quad \text{injective} \quad \Longleftrightarrow \quad \forall_{A\subset X}\ \ f^{-1}(f(A)) \subset A.$$

(Notice that $f^{-1}(f(A)) \supset A$ is always satisfied – see Proposition 1.1.
We begin withe the **proof of the implication** $\Longrightarrow$:
Since we have

$$f \text{ is injective} \iff f(x) = f(x') \Rightarrow x = x', \tag{13.4}$$

we have

$$\begin{aligned} x \in f^{-1}(f(A)) &\Leftrightarrow f(x) \in f(A) \\ &\Leftrightarrow \exists_{x'}\ x' \in A \wedge f(x) = f(x') \\ &\overset{\text{by (13.4)}}{\Longrightarrow} \exists_{x'}\ x' \in A \wedge x = x' \\ &\Leftrightarrow x \in A. \end{aligned}$$

To prove the implication $\Longleftarrow$, we will argue by contradiction.
Suppose that f is not injective

$$\begin{aligned} \sim (f \text{ is injective}) &\Leftrightarrow \sim (\forall_{x,\, x'}\ f(x) = f(x') \Rightarrow x = x') \\ &\Leftrightarrow (\exists_{x,\, x'}\ f(x) = f(x') \wedge x \neq x') \end{aligned} \tag{13.5}$$

Take $A = \{x\}$, where x as well x' are specified in (13.5).
Then we have that

$$x' \in f^{-1}(f(A)) \Leftrightarrow f(x') \in f(A) \Leftrightarrow \exists_{x \in A}\ f(x') = f(x).$$

So $f^{-1}(f(A)) \not\subset A$.
Consequently, we have the following 8(c):

$$\begin{aligned} f \text{ is bijective} &\Leftrightarrow f \text{ is injective} \wedge f \text{ is surjective} \\ &\Leftrightarrow \forall_{A \subset X}\ f^{-1}(f(A)) \subset A \wedge \forall_{B' \subset Y}\ f(f^{-1}(B')) \supset B'. \end{aligned}$$

9. Let $f : X \to Y$ be a function. Write the logic negation to each of the following statements:

(a) f is surjective;
(b) f is injective;
(c) f is bijective.

SOLUTION:

$$\begin{aligned} \sim (f \text{ is injective}) &\Leftrightarrow \sim (\forall_{x,\, x'}\ f(x) = f(x') \Rightarrow x = x') \\ &\Leftrightarrow (\exists_{x,\, x'}\ f(x) = f(x') \wedge x \neq x') \end{aligned}$$

$$\begin{aligned} \sim (f \text{ surjective}) &\Leftrightarrow \sim (\forall_{y \in Y}\ \exists_{x \in X}\ \ f(x) = y \\ &\Leftrightarrow \exists_{y \in Y}\ \forall_{x \in X}\ \ f(x) \neq y. \end{aligned}$$

$$\begin{aligned} \sim (f \text{ bijective}) &\Leftrightarrow \sim \Big[(f \text{ is injective}) \wedge (f \text{ is surjective})\Big] \\ &\Leftrightarrow (f \text{ is not injective}) \vee (f \text{ is not surjective}) \\ &\Leftrightarrow (\exists_{x,\, x'}\ f(x) = f(x') \wedge x \neq x') \vee (\exists_{y \in Y}\ \forall_{x \in X}\ \ f(x) \neq y). \end{aligned}$$

Real Numbers 1. Use the axioms of an ordered field $\mathbb{F}$ to show that the following properties are always satisfied in $\mathbb{F}$:

(a) $\forall_{a,b,c,d\in\mathbb{F}}\ (a<b\wedge c<d \implies a+c<b+d)$;
(b) $\forall_{a,b,c,d\in\mathbb{F}}\ (a>b>0\wedge c>d>0 \implies ac>bd)$;
(c) $\forall_{a,b\in\mathbb{F}}\ \left(b>a>0 \implies \frac{1}{a}>\frac{1}{b}\right)$;
(d) $\forall_{x,a\in\mathbb{F}}\ (|x|\le a \iff -a\le x\le a)$;
(e) $\forall_{x,y\in\mathbb{F}}\ (|x+y|\le |x|+|y|)$;
(f) $\forall_{a,b\in\mathbb{F}}\ \Big(|a-b|\ge \Big||a|-|b|\Big|\Big)$;

SOLUTIONS: 1(a):

$$\begin{aligned} a<b\wedge c<d &\overset{(O3)}{\implies} a+c<b+c\wedge b+c<b+d\\ &\overset{(O2)}{\implies} a+c<b+d.\end{aligned}$$

1(b):

$$\begin{aligned} a>b>0\wedge c>d>0 &\overset{(O4)}{\implies} ac>bc\wedge bc>bd\\ &\overset{(O2)}{\implies} ac>bd.\end{aligned}$$

1(c): Notice that

$$(*)\qquad c>0 \implies \frac{1}{c}>0.$$

Indeed, if $\frac{1}{c}<0$ then $\forall_{a>0}\ \frac{1}{c}\cdot a<0$, but $\frac{1}{c}\cdot c=1>0$, thus $\frac{1}{c}>0$, which is a contradiction. Therefore, we have

$$\begin{aligned} b>a>0 &\overset{(*)}{\implies} \frac{1}{a}>0\wedge\frac{1}{b}>0\wedge b>a>0\\ &\overset{(O4)}{\implies} \frac{1}{a}\cdot b>\frac{1}{a}\cdot a=1\wedge\frac{1}{b}>0\\ &\overset{(O4)}{\implies} \frac{1}{a}=\frac{1}{a}\cdot b\cdot\frac{1}{b}>1\cdot\frac{1}{b}=\frac{1}{b}.\end{aligned}$$

1(d): Since $|x|=\begin{cases} x & \text{if } x>0\\ -x & \text{if } x\le 0\end{cases}$, thus $a\ge 0$ and we have

$$\begin{aligned} |x|\le a &\iff (x>0\Rightarrow x\le a)\vee(x\le 0\Rightarrow -x\le a)\\ &\iff \begin{cases} x>0\Rightarrow (x\le a\wedge -x\le a)\\ \qquad\qquad\vee\\ x\le 0\Rightarrow(-x\le a\wedge x\le a)\end{cases}\\ &\iff x\le a\wedge -x\le a.\end{aligned}$$

1(e): Since $|x|\le|x|$ we get by 1(d) that

$$-|x|\le x\le|x|\quad\wedge\quad -|y|\le y\le|y|$$

thus by 1(a), we get

$$-|x| - |y| \le x + y \le |x| + |y|,$$

which implies (by 1(d))

$$|x + y| \le |x| + |y|.$$

1(f): Since $\forall_{x,y\in\mathbb{F}} \quad |x+y| \le |x| + |y|$, by taking $x = a - b$ and $y = b$, we get

$$|a| = |(a-b) + b| \le |a-b| + |b| \iff |a-b| \ge |a| - |b|,$$

and next, by taking $x = b - a$ and $y = a$, we get

$$|b| = |(b-a) + a| \le |a-b| + |a| \iff |a-b| \ge |b| - |a|.$$

These two inequalities can be written as

$$\forall_{a,b\in\mathbb{F}} \; -|a-b| \le |a| - |b| \le |a-b| \overset{\text{by 1(d)}}{\iff} \forall_{a,b\in\mathbb{F}} \; |a-b| \ge \Big||a| - |b|\Big|.$$

2. Show that, just like the set $\mathbb{Q}$ of rational numbers, the set $\mathbb{Q}(\sqrt{n})$ of numbers of the form $a+b\sqrt{n}$, where a, $b \in \mathbb{Q}$ and n is a fixed natural number that is not the square of any integer, is an ordered set (field) satisfying the principle of Archimedes but not the axiom of completeness.

SOLUTION: We will show that $\mathbb{Q}(\sqrt{n})$ is an ordered field. Indeed, $\mathbb{Q}(\sqrt{n}) \subset \mathbb{R}$, and since the axioms of field (f1)–(f9) are satisfied in $\mathbb{R}$ we need to show that for x, $y \in \mathbb{Q}(\sqrt{n})$ we have

$$x - y \in \mathbb{Q}(\sqrt{n}), \; ' \; xy \in \mathbb{Q}(\sqrt{n}), \quad y^{-1} := \frac{1}{y} \in \mathbb{Q}(\sqrt{n}), \quad \text{for} \quad y \ne 0.$$

Consequently, the properties (f1)–(f9) will authomatically be satisfied in $\mathbb{Q}(\sqrt{n})$. Suppose therefore that

$$x := a + b\sqrt{n}, \quad y = a' + b'\sqrt{n}.$$

Clearly,

$$x - y = (a - a') + (b - b')\sqrt{n} \in \mathbb{Q}(\sqrt{n}),$$

and

$$xy := (aa' + bb'n) + (ab' + a'b)\sqrt{n},$$

so $x - y$, $xy \in \mathbb{Q}(\sqrt{n})$. On the other hand

$$\frac{1}{y} = \frac{1}{a' + b'\sqrt{n}} \cdot \frac{a' - b'\sqrt{n}}{a' - b'\sqrt{n}} = \left(\frac{a'}{a'^2 - b'^2 n}\right) + \left(\frac{-b'}{a'^2 - b'^2 n}\right)\sqrt{n},$$

thus $y^{-1} \in \mathbb{Q}(\sqrt{n})$. Notice that ${a'}^2 - {b'}^2 n \ne 0$. Indeed, by the assumption $\sqrt{n}$ is not equal to any integer, so it is possible to show that it can not be rational. (For the sake of completeness, this is the argument by contradiction: suppose that $\sqrt{n} = \frac{m}{k}$, where m, $k \in \mathbb{N}$, and the fraction $\frac{m}{k}$ is irreducible. Then $nk^2 = m^2$, and since k does not divide m, n must divide m^2, and by the assumption that $\frac{m}{k}$ is irreducible, $k = 1$ and $n = m^2$.)

In addition, we also have that $\mathbb{Q}(\sqrt{n})$ is ordered (because $\mathbb{R}$ is ordered and $\mathbb{Q}(\sqrt{n}) \subset \mathbb{R}$). Consequently, since the Archimedes axiom is satisfied in $\mathbb{R}$, and every number $a + b\sqrt{n} \in \mathbb{Q}(\sqrt{n})$ is real, we get

$$\forall_{a+b\sqrt{n}\in\mathbb{Q}(\sqrt{n})} \ \exists_{m\in\mathbb{N}} \ a + b\sqrt{n} < m.$$

However, the field $\mathbb{Q}(\sqrt{n})$ is not complete, i.e. it does not satisfy the LUB axiom. For this purpose, we choose a prime number p such that $p > n$. Then clearly $\sqrt{p}$ is irrational: indeed, suppose for contradiction that $\sqrt{p} = \frac{m}{k}$, where $\frac{m}{k}$ is irreducible, then $pk^2 = m^2$, so m is divisible by p, i.e. $m = pl$, and consequently $pk^2 = l^2p^2$, thus $k^2 = l^2p$, which implies that k is also divisible by p (which contradicts the irreducibility of $\frac{m}{k}$). We will show that

$$\sqrt{p} \notin \mathbb{Q}(\sqrt{n}).$$

Indeed, suppose for the contradiction that $a + b\sqrt{n} = \sqrt{p}$ for some a, $b \in \mathbb{Q}(\sqrt{n})$. Than, since $\sqrt{p}$ is irrational $b \neq 0$, and by a similar argument $a \neq 0$ as well (indeed, if $a = 0$, then $b^2n = p$, $b = \frac{m}{k}$ – an irreducible fraction, so $m^2n = k^2p$ and p divides m, thus it also divides k and the contradiction with irreducibility of $\frac{m}{k}$ is obtained). Therefore, by taking square of the both sides of the equality

$$a + b\sqrt{n} = \sqrt{p} \implies a^2 + 2ab\sqrt{n} + b^2n = p,$$

we obtain that

$$\sqrt{n} = \frac{p - a^2 - b^2n}{2ab},$$

which is a contradiction with the assumption that $\sqrt{n}$ is not rational.
Now, we can define the set

$$A := \{a + b\sqrt{n} \in \mathbb{Q}(\sqrt{n}) : a + b\sqrt{n} \in \mathbb{Q}(\sqrt{n}) < \sqrt{p}\},$$

and since $\mathbb{R}$ is complete, we have (it is easy to check) $\sup A = \sqrt{p}$ in $\mathbb{R}$, but since $\sqrt{p} \notin \mathbb{Q}(\sqrt{n})$, it follows that $\sup A$ does not exist in $\mathbb{Q}(\sqrt{n})$ (notice that there can be only one value $\sup A$ in $\mathbb{R}$). This proves that $\mathbb{Q}(\sqrt{n})$ is not complete.

3. Determine which axioms for the real numbers do not hold for $\mathbb{Q}(\sqrt{n})$ if the standard arthmetic operations are retained in $\mathbb{Q}(\sqrt{n})$ but order is defined by the rule $(a + b\sqrt{n}) < (a' + b'\sqrt{n})$ iff $b < b'$) or $(b = b') \ \vee \ (b = b' \ \wedge \ a < a')$. Will $\mathbb{Q}(\sqrt{n})$ now satisfy the principle of Archimedes?

SOLUTION : One can easily verify that the properties (o1), (o2) and (o3) of an ordered fieled are satisfied. However, the property (o4) is not satisfied. Indeed. If $b > 0$ then $a + b\sqrt{n} > 0$ for all $a \in \mathbb{Q}$. In particular, $b\sqrt{n} > 0$. But if $a < 0$, then

$$(a + b\sqrt{n})b\sqrt{n}) = b^2n + ab\sqrt{n} < 0,$$

so (o4) does not hold. In addition, we have for every natural number $m \in \mathbb{N}$, that

$$m = m + 0 \cdot \sqrt{n} < 0 + 1 \cdot \sqrt{n}, \quad \text{since } \ 0 < 1,$$

thus $\sqrt{n}$ is an upper bound for the set $\mathbb{N}$ in $\mathbb{Q}(\sqrt{n})$ and consequently the Archimedes axiom is not satisfied in $\mathbb{Q}(\sqrt{n})$.

4. Let us denote by $\mathbb{Q}^c$ the complement of the set of rational numbers $\mathbb{Q}$ in $\mathbb{R}$, i.e. $\mathbb{Q}^c$ stands for the set of *irrational numbers*. Show the following properties

(a) $\forall_{a,b\in\mathbb{R}}\ (a \in \mathbb{Q} \wedge b \in \mathbb{Q}^c \implies a+b \in \mathbb{Q}^c)$;
(b) $\forall_{a,b\in\mathbb{R}}\ (0 \neq a \in \mathbb{Q} \wedge b \in \mathbb{Q}^c \implies ab \in \mathbb{Q}^c)$.

SOLUTIONS: 4(a): Notice that in order to prove the statement $p \wedge q \Rightarrow r$, where $p = \{a \in \mathbb{Q}\}$, $q = \{b \in \mathbb{Q}^c\}$ and $r = \{a+b \in \mathbb{Q}^c\}$, we can use the following logical equivalences:

$$\begin{aligned} p \wedge q \Rightarrow r \quad &\equiv\ \sim(p \wedge q) \vee r \quad \equiv\ \sim p \vee \sim q \vee r \\ &\equiv\ \sim(p \wedge \sim r) \vee \sim q \quad \equiv\ \ p \wedge \sim r \Rightarrow \sim q, \end{aligned}$$

which means we need to show that

$$(**) \qquad \forall_{a,b\in\mathbb{R}}\ (a \in \mathbb{Q} \wedge a+b \in \mathbb{Q} \implies b \in \mathbb{Q})$$

However, we have

$$\begin{aligned} a \in \mathbb{Q} \quad &\equiv \quad \exists_{m,n\in\mathbb{Z}}\ a = \frac{m}{n}; \\ a+b \in \mathbb{Q} \quad &\equiv \quad \exists_{k,l\in\mathbb{Z}}\ a+b = \frac{k}{l}, \end{aligned}$$

thus

$$b = (a+b) - a = \frac{k}{l} - \frac{m}{l} = \frac{kn - ml}{ln} \in \mathbb{Q},$$

and (**) is true.

4(b): By a similar argument as in 2(a), it is sufficient to prove that

$$(\dagger) \qquad \forall_{a,b\in\mathbb{R}}\ (0 \neq a \in \mathbb{Q} \wedge ab \in \mathbb{Q} \implies b \in \mathbb{Q})$$

Since

$$\begin{aligned} 0 \neq a \in \mathbb{Q} \quad &\equiv \quad \exists_{m,n\in\mathbb{Z}}\ a = \frac{m}{n} \neq 0; \\ ab \in \mathbb{Q} \quad &\equiv \quad \exists_{k,l\in\mathbb{Z}}\ ab = \frac{k}{l}, \end{aligned}$$

thus

$$b = (ab) \cdot \frac{1}{a} = \frac{k}{l} \cdot \frac{n}{m} = \frac{kn}{lm} \in \mathbb{Q},$$

and (†) is true.

5. Use the axioms and the already proved properties of real numbers $\mathbb{R}$ to show that the following statements are true:

(a) $\forall_{a\in\mathbb{R}}\ \left[a > 0 \implies \exists_{n\in\mathbb{N}}\ \ a > \frac{1}{n} > 0\right]$;
(b) $\forall_{a,b\in\mathbb{R}}\ \left[a < b \implies \exists_{q\in\mathbb{Q}}\ \ a < q < b\right]$.

SOLUTIONS: Since $a > 0$ thus $\frac{1}{a} > 0$ and by Archimedes Axiom

$$\exists_{n\in\mathbb{N}} \;\; n > \frac{1}{a}.$$

Thus, by 1(c), we get

$$\exists_{n\in\mathbb{N}} \;\; 0 < \frac{1}{n} < a.$$

3(b): If $a < b$, then by (O3) $0 < b - a$ and by 3(a), we get

$$\exists_{n\in\mathbb{N}} \;\; 0 < \frac{1}{n} < b - a.$$

Consider first the case $0 < a < b$. We define

$$K = \{k \in \mathbb{N} : \frac{k}{n} < b\}.$$

Notice that K is not empty. Indeed, $\frac{1}{n} < b - a < b$, so $1 \in K$. The set K is also finite. Indeed, by Archimedes axiom, there exists $N \in \mathbb{N}$ such that $nb < N$, i.e. $\forall_{k\in K} \;\; k < N$. Of course any bounded set of natural numbers is finite. Put $m := \max K$. It is clear that (by definition of the set K) $\frac{m}{n} < b$, but we also have $\frac{m}{n} > a$. Indeed, notice that if $\frac{m}{n} \le a$ then $\frac{m}{n} + \frac{1}{n} = \frac{m+1}{n} \le a + \frac{1}{n} < b$, which would imply that $m+1 \in K$, so m couldn't be the maximal element of K. Consequently, the rational number $q = \frac{m}{n}$ satisfies

$$a < q < b.$$

In the general case, when the numbers a and b are not necessarily positive, we can find (by Archimedes Axiom) $N \in \mathbb{N}$ such that $0 < a' := a + N < b + N =: b'$. Since a' and b' satisfy the requirement in the previous case, it is possible to find a rational number $q' \in \mathbb{Q}$ such that $a' < q' < b'$. It is clear that the rational number $q := q' - N$ satisfies $a < q < b$, thus 3(b) is true in general case.

6. Show that the following numbers are irrational:

(a) $[\sqrt{2} - 1]^n$, where $n \in \mathbb{N}$;
(b) $\sqrt{x}$, where $x > 0$ is an irrational number;
(c) $\sqrt{p}$, where $p \in \mathbb{N}$ is a prime number.

SOLUTIONS: 6(a): By observing the patterns of $(\sqrt{2} - 1)^n$ for $n = 1, 2, 3, 4, 5$, we claim:

Lemma. *For every $n \in \mathbb{N}$ there exist two integers M and N such that*

$$(\sqrt{2} - 1)^n = M + N\sqrt{2}, \tag{13.6}$$

where M and N have opposite signs, i.e. $M \cdot N < 0$.

Proof: We apply Mathematical Induction. For $n = 1$ we have that $M = -1$ and $N = 1$, thus the formula (13.6) is true.

Assume that (13.6) is true for a certain natural number $n \ge 1$, i.e.

$$(\sqrt{2} - 1)^n = M + N\sqrt{2},$$

for two integers M and N of opposite signs. Let us consider the $(n+1)$-th power of $(\sqrt{2}-1)$. We have

$$\begin{aligned}(\sqrt{2}-1)^{n+1} &= (\sqrt{2}-1)^n \cdot (\sqrt{2}-1)\\ &= (M+N\sqrt{2})\cdot(\sqrt{2}-1)\\ &= (2N-M)+(M-N)\sqrt{2} = M'+N'\sqrt{2},\end{aligned}$$

where $M' = 2N-M$ and $N' = M-N$. It is clear that if $M<0$ and $N>0$, then $M'>0$ and $N'<0$, and similarly, if $M>0$ and $N<0$, then $M'<0$ and $N'>0$, thus M' and N' are two integers of opposite signs. Consequently, by mathematical induction (13.6) is true for all n. $\square$

In order to conclude (by 6(a)) that $(\sqrt{2}-1)^n = M+N\sqrt{2}$ is irrational, we notice that $\sqrt{2}$ is irrational so, $N\sqrt{2}$ is irrational and thus $M+N\sqrt{2}$ is also irrational.

6(b): Notice that (by 6(b)) if $\sqrt{x}$ is rational, then $x=\sqrt{x}\cdot\sqrt{x}$ is also rational. Consequently, if x is irrational, it follows that $\sqrt{x}$ must be irrational as well.

6(c): Assume that $\sqrt{p}=\frac{k}{n}$, where $\frac{k}{n}$ is an irreducible fraction. Then $pn^2=k^2$, so k is a multiple of p. Consequently, this implies that n is also a multiple of p, but this gives a contradiction.

7. Check if the following sets are bounded from above and/or below. For each bounded from above (resp, below) set, find its supremum (resp. infimum):

$$\begin{aligned}D &= \left\{x\in\mathbb{R} : \Big||x-1|-1\Big| < 1\right\},\\ E &= \left\{k+\tfrac{1}{n} : k\in\{0,1,2\},\ \ n\in\mathbb{N}\right\},\\ G &= \left\{1+\tfrac{1}{2^n} : n\in\mathbb{N}\right\},\\ I &= \left\{\tfrac{n^2+2n-3}{n+1} : n\in\mathbb{N}\right\}\end{aligned}$$

SOLUTIONS: 7(a): By 1(d), we obtain

$$\begin{aligned}D &= \{x\in\mathbb{R} : -1<|x-1|-1<1\},\\ &= \{x\in\mathbb{R} : 0<|x-1|<2\},\\ &= \{x\in\mathbb{R} : 0<|x-1|\ \wedge\ -2<x-1<2\},\\ &= \{x\in\mathbb{R} : x\neq 1\ \wedge\ -1<x<3\},\\ &= (-1,1)\cup(1,3)\end{aligned}$$

so, it follows immediately that

$$\sup D = 3 \quad\text{ and }\quad \inf D = -1.$$

7(b): Since $\forall_{k\in\{0,1,2\}}$ and $\forall_{n\in\mathbb{N}}$ we have $k\le 2$ and $1\ge\frac{1}{n}>0$, thus we have

$$\forall_{k\in\{0,1,2\}}\forall_{n\in\mathbb{N}}\ \ 0<k+\frac{1}{n}\le 3 \iff \forall_{x\in E}\ 0<x\le 3.$$

It is easy to see that $3=2+\frac{1}{1}$ is the maximal element in E, so $\sup E=3$. On the other hand, 0 is the lower bound of E, and by Archimeadian Axiom $\forall_{\varepsilon>0}\exists_{n\in\mathbb{N}}\ n>\frac{1}{\varepsilon}$, so we get

$$\forall_{\varepsilon>0}\exists_{n\in\mathbb{N}}\, \varepsilon > 0 + \frac{1}{n} \in E,$$

and consequently we obtain that

$$\inf E = 0.$$

7(c): Notice that $\forall_{n\in\mathbb{N}}\ n < 2^n$. Indeed, this inequality can be easily proved using Mathematical Indiction: For $n = 1$ we have $1 < 2^1 = 2$. Assume that $n < 2^n$ for some $n \in \mathbb{N}$, then we have

$$n + 1 < 2^n + 1 < 2^n + 2^n = 22^n = 2^{n+1},$$

thus this inequality is true for all $n \in \mathbb{N}$. Since, by the above inequality $2^n \geq 2$, we have that $\frac{1}{2^n} \leq \frac{1}{2}$ so

$$\forall_{n\in\mathbb{N}}\ 1 < 1 + \frac{1}{2^n} \leq 1 + \frac{1}{2} = \frac{3}{2},$$

which means that 1 is a lower bound of G, while $\frac{3}{2}$ is the maximum of G (notice that $1 + \frac{1}{2} \in G$). In order to show that 1 is the infimum of G we notice that by Archimeadian Axiom we have

$$\forall_{\varepsilon>0}\exists_{n\in\mathbb{N}}\ \ 2^n > n > \frac{1}{\varepsilon} \iff \forall_{\varepsilon>0}\exists_{n\in\mathbb{N}}\ \ 1 + \varepsilon > \frac{1}{2^n} + 1 \in G,$$

thus

$$1 = \inf\, G.$$

7(d): Notice that

$$\begin{aligned} I &= \left\{ \tfrac{n^2+2n-3}{n+1} : n \in \mathbb{N} \right\} \\ &= \left\{ \tfrac{(n^2+2n+1)-4}{n+1} : n \in \mathbb{N} \right\} \\ &= \left\{ n + 1 - \tfrac{4}{n+1} : n \in \mathbb{N} \right\} \\ &= \left\{ n - \tfrac{4}{n} : n \in \mathbb{N}\ \wedge\ n \geq 2 \right\} \end{aligned}$$

Since $\frac{4}{n} \leq 2$ for $n \geq 2$, it follows that $n - \frac{4}{n} \geq n - 2$, thus the set I is unbounded from above. On the other hand, we have that $n \geq 2$ implies $n^2 \geq 4$, thus $n - \frac{4}{n} \geq 0$ and consequently

$$\forall_{n\in\mathbb{N}}\ n \geq 2 \impliedby 0 \leq n - \frac{4}{n}$$

what implies that 0 is the lower bound of I. However, $0 = 2 - \frac{4}{2} \in I$, thus

$$0 = \inf\ I = \min I.$$

8. Let $A + B$ be the set of numbers of form $a + b$, $A \cdot B$ the set of numbers of form $a \cdot b$, and $-A$ the set of numbers $-a$, where $a \in A \subset \mathbb{R}$ and $b \in B \subset \mathbb{R}$.
Determine whether it is always true that

(a) $\sup(A + B) = \sup A + \sup B$
(b) $\sup(A \cdot B) = \sup A \cdot \sup B$

(c) $\sup(-A) = -\inf A$.

SOLUTION:

(a) It is always true. Indeed, suppose that

$$\alpha = \sup A \quad \Leftrightarrow \quad \forall_{a\in A}\ a \le \alpha \tag{13.7}$$
$$\forall_{\varepsilon>0}\ \exists_{a\in A}\ \alpha - \varepsilon < a \tag{13.8}$$

$$\beta = \sup B \quad \Leftrightarrow \quad \forall_{b\in B}\ b \le \beta \tag{13.9}$$
$$\forall_{\varepsilon>0}\ \exists_{b\in B}\ \beta - \varepsilon < b \tag{13.10}$$

$$\gamma = \sup(A+B) \quad \Leftrightarrow \quad \forall_{a\in A}\ \forall_{b\in B}\ a + b \le \gamma \tag{13.11}$$
$$\forall_{\varepsilon>0}\ \exists_{a\in A}\ \exists_{b\in B}\ \gamma - \varepsilon < a + b \tag{13.12}$$

Since by (13.7) and (13.9),

$$\forall_{a\in A}\ \forall_{b\in B}\ a \le \alpha\ \wedge\ b \le \beta \implies \forall_{a\in A}\ \forall_{b\in B}\ a + b \le \alpha + \beta$$

thus $\alpha + \beta$ is an upper bound of $A + B$, so by definition of $\sup(A + B)$ (by (13.12))

$$\alpha + \beta \ge \gamma.$$

We will show that $\alpha + \beta = \gamma$.
Assume for contradiction that $\alpha + \beta > \gamma$, i.e. $\alpha + \beta - \gamma > 0$. Put $\varepsilon = \frac{\alpha+\beta-\gamma}{2} > 0$. And by (13.8) and (13.10)

$$\begin{aligned}
&\exists_{a\in A}\ \alpha - \varepsilon < a\ \wedge\ \exists_{b\in B}\ \beta - \varepsilon < b \\
\Leftrightarrow\ &\exists a \in A \exists_{b\in B}\ (\alpha - \varepsilon) + (\beta - \varepsilon) < a + b \\
\Leftrightarrow\ &\exists a \in A \exists_{b\in B}\ \alpha + \beta - 2\varepsilon < a + b \\
\Leftrightarrow\ &\exists a \in A \exists_{b\in B}\ (\alpha + \beta) - (\alpha + \beta - \gamma) < a + b \\
\Leftrightarrow\ &\exists a \in A \exists_{b\in B}\ \gamma < a + b
\end{aligned}$$

which contradicts (13.12).

(b) Put $A = \{-2, 1\}$, $B = \{-2, 1\}$, then $A \cdot B = \{-2, 1, 4\}$. Clearly, $\sup A = \sup B = 1$, and $\sup A \cdot \sup B = 1$. Thus the equality in (b) does not hold.

9. Verify that $\mathbb{Z}$ and $\mathbb{Q}$ are inductive sets.

SOLUTION: We have

$$\mathbb{Z} = \mathbb{N} \cup (-\mathbb{N}) \cup \{0\}.$$

Since $\mathbb{N}$ is (by definition) the smallest inductive set, and $\mathbb{N} \subset \mathbb{Z}$, thus
(a) $1 \in \mathbb{Z}$.

Suppose that $m \in \mathbb{Z}$. If $m \in \mathbb{N} \cup \{0\}$, then clearly $m+1 \in \mathbb{N}$ (since $\mathbb{N}$ is inductive). Thus we need only to show that if $m \in (-\mathbb{N})$, then $m+1 \in \mathbb{Z}$.
Indeed, if $m = -n$, $n \in \mathbb{N}$, then $m+1 = -n+1 = -(n-1)$, and $n-1 \in \mathbb{N} \cup \{0\}$. So it follows
(b) $m \in \mathbb{Z} \;\Rightarrow\; m+1 \in \mathbb{Z}$.
So $\mathbb{Z}$ is an inductive set.
$\mathbb{Q}$ is also an inductive set. Indeed, $1 \in \mathbb{N} \subset \mathbb{Q}$ and if $a \in \mathbb{Q}$, i.e. $a = \frac{m}{n}$, $m,\ n \in \mathbb{Z}$, $n \neq 0$, then

$$a+1 = \frac{m}{n} + 1 = \frac{m}{n} + \frac{n}{n} = \frac{m+n}{n} \in \mathbb{Q},$$

because $m+n \in \mathbb{Z}$.
(c) Let $-A$ be the set of all numbers of the form $-a$, where $a \in A$, we have

$$\begin{aligned}\alpha = \sup(-A) \;\;&\Leftrightarrow\;\; \begin{cases} \forall_{-a\in -A} \; -a \le \alpha \\ \forall_{\varepsilon>0}\, \exists_{-a\in -A}\; \alpha - \varepsilon < -a \end{cases} \\ &\Leftrightarrow\;\; \begin{cases} \forall_{a\in A}\; a \ge -\alpha \\ \forall_{\varepsilon>0}\, \exists_{a\in A} \; -\alpha + \varepsilon > a \end{cases} \\ &\Leftrightarrow -\alpha = \inf A \;\Leftrightarrow\; \alpha = -\inf A.\end{aligned}$$

10. Give examples of inductive sets different from $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$.

SOLUTION:
$P_1 = [0,\infty)$ in $\mathbb{R}$
$P_2 = \mathbb{N} \cup \{-3,-2,-1,0\}$
$P_3 = \mathbb{Q}(\sqrt{n})$ (see Problem # 21)
$P_4 = \mathbb{N} \cup \{\sqrt{2}, \sqrt{2}+1, \sqrt{2}+2, \dots\}$

11. Show that an inductive set is not bounded above.

SOLUTION: Let P be an inductive set in $\mathbb{R}$, then by definition

(a) $1 \in P$
(b) $a \in P \;\Rightarrow\; a+1 \in P$.

Assume for contradiction that P is bounded above, i.e.

$$\exists_{\gamma\in\mathbb{R}}\, \forall_{a\in P}\; a \le \gamma.$$

Then by LUB axiom, there exists $\alpha = \sup P$, i.e.

(i) $\forall_{a\in P}\; a \le \alpha$
(ii) $\forall_{\varepsilon>0}\, \exists_{a\in P}\; \alpha - \varepsilon < a$.

By (ii) with $\varepsilon = 1$, we have
$$\exists_{a\in P}\ \alpha - 1 < a \ \Leftrightarrow\ \exists_{a\in P}\ \alpha < a+1 \tag{13.13}$$
Since $a \in P$, (by (b)) $a+1 \in P$, but it follows from (13.13) that it contradicts (i). Consequently, it is impossible that P is bounded above.

12. Show that for every natural number n the number $5^n + 2\cdot 3^{n-1} + 1$ is a multiple of 8.

SOLUTION: For $n = 1$, we have
$$5^1 + 2\cdot 3^{1-1} + 1 = 8\ ,$$
so it is a multiple of 8. Consequently, the above statement is true for $n = 1$.

Now assume that $n \geq 1$ is a certain natural number such that the integer $5^n + 2\cdot 3^{n-1} + 1$ is a multiple of 8, i.e. $5^n + 2\cdot 3^{n-1} + 1 = M\cdot 8$ for some integer M. Consider the number $n+1$. We have

$$\begin{aligned}
5^{n+1} + 2\cdot 3^{n+1-1} + 1 &= 5^{n+1} + 2\cdot 3^n + 1\\
&= 5^n\cdot 5 + 2\cdot 3^{n-1}\cdot 3 + 1\\
&= 5^n\cdot 5 + 2\cdot 3^{n-1}\cdot 3 + 1 + 2\cdot 3^{n-1}\cdot 2 + 4 - 2\cdot 3^{n-1}\cdot 2 - 4\\
&= 5(5^n + 2\cdot 3^{n-1} + 1) - 4\cdot 3^{n-1} - 4\\
&= 5(M\cdot 8) - 4(3^{n-1} + 1)
\end{aligned}$$

Notice that since 3 is an odd number , so is 3^{n-1} and therefore $3^{n-1}+1$ is an even number, i.e. $3^{n-1} = P\cdot 2$ for some natural number P. Consequently, we have $5^{n+1} + 2\cdot 3^{n+1-1} + 1 = 5(M\cdot 8) - 4(P\cdot 2) = 8(5\cdot M - P)$, thus the number $5^{n+1} + 2\cdot 3^{n+1-1} + 1$ is also a multiple of 8, and the conclusion follows from mathematical induction.

13. Show that for every $n \in \mathbb{N}$ we have
$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

SOLUTION: We apply the mathematical induction.
We denote by $P(n)$ the sentence $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$. We need to check that (i) $P(1)$ *is true*, (ii) *if* $P(n)$ *is true, then* $P(n+1)$ *(for* $n \geq 1$*) is also true.*
(i): Clearly, for $n = 1$, we have $\frac{1}{1^2} = 1 = 2 - \frac{1}{1}$, so $P(1)$ is evidently true.
(ii): Assume for the induction that $P(n)$ is true, then we need to prove that $P(n+1)$:
$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}$$
is true. Indeed, we have (by applying the induction assumption)

$$\begin{aligned}\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{n^2}+\frac{1}{(n+1)^2} &\le 2-\frac{1}{n}+\frac{1}{(n+1)^2}\\ &= 2-\frac{(n+1)^2-n}{(n+1)^2 n}\\ &= 2-\frac{n^2+n+1}{(n+1)^2 n}\\ &\le 2-\frac{n^2+n}{(n+1)^2 n}\\ &= 2-\frac{1}{n+1}\end{aligned}$$

Consequently, $P(n+1)$ is true, and the inequality $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

14. Show that for every natural number $n \ge 2$ the number $2^{2^n} - 6$ is a multiple of 10.

SOLUTION: Let us formulate the property $P(n)$ as: $\exists\, m \in \mathbb{N}\ \ 2^{2^n} - 6 = 10m$. We apply the principle of mathematical induction: $P(2)$ is true; indeed, we have for $n = 2$:

$$2^{2^2} - 6 = 2^4 - 6 = 16 - 6 = 10 \cdot 1.$$

Assume for induction that $P(n)$ is true ($n \ge 2$). Then we have

$$2^{2^{n+1}} - 6 = \left[2^{2^n}\right]^2 - 6 = [10m+6]^2 - 6 = 100m^2 + 120m + 36 - 6 = 10(10m^2 + 12m + 3),$$

i.e.

$$2^{2^{n+1}} - 6 = 10m', \quad \text{where} \quad m' = 10m^2 + 12m + 3.$$

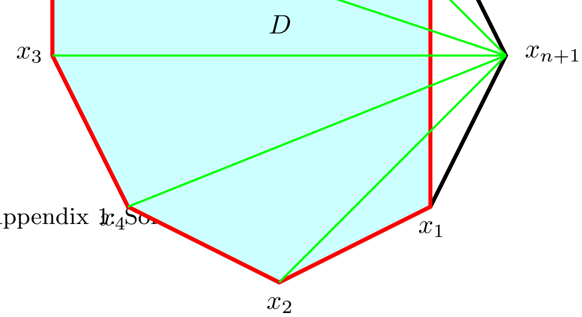
Therefore, $P(n+1)$ is also true, so the property $P(n)$ is true for all $n \in \mathbb{N}$, $n \ge 2$.

15. Find the number of diagonals in a convex polygon with n sides.

SOLUTION: By inspection, one can easily notice that in a convex n-gon, there are excatly $n-3$ diagonals meeting at each vertex. Since, each diagonal is joining two vertices, thus the total number of diagonals seems to be $\frac{n(n-3)}{2}$. This claim needs to be proved. To be more precise, we formulate:

$P(n)$: *for* $n \ge 4$*: every convex* n*-gon has exactly* $\dfrac{n(n-3)}{2}$ *diagonals.*

The claim $P(4)$ is clearly true. Assume for the induction that $P(n)$ is true for some $n \ge 4$ and consider a convex $n+1$-gon P. Notice that by removing one vertex, say x_{n+1} from P we obtain a convex n-gon D (see the picture below).



By induction assumption, the n-gon D has exactly $\frac{n(n-3)}{2}$ diagonals, which are also diagonal of $n+1$-gon P. On the other hand, there are $n-2$ diagonals from the vertex x_{n+1} and the additional diagonal is the side joining x_1 with x_n, i.e. in total, we have

$$\frac{n(n-3)}{2}+n-2+1=\frac{n(n-3)+2(n-1)}{2}=\frac{n^2-n-2}{2}=\frac{(n+1)(n-2)}{2}.$$

That means, we proved that P has $\frac{(n+1)(n-2)}{2}$, which means the property $P(n+1)$ is also true. In this way, we obtain by the principle of mathematical induction that a convex n-gon has $\frac{n(n-3)}{2}$ diagonals.

16. Use the fact that if a product $m \cdot n$ of natural numbers is divisible by a prime number p, then either m or n is divisible by p, to show that if k^m (k, $m \in \mathbb{N}$) is divisible by a oprime number p, then k is divisible by p.

SOLUTION: We will write $p|k$ to denote that p *divides* k. Our statement is $P(m)$: $p|k^m \;\Rightarrow\; p|k$. We assume that (*): $p|m \cdot n \;\Rightarrow\; (p|m \;\vee\; p|n)$. It is clear that $P(1)$ is true: $p|k \;\Rightarrow\; p|k$. Assume for induction that $P(m)$ is true for some $m \geq 1$. Then we have:

$$p|k^{m+1} \;\Leftrightarrow\; p|k^m \cdot k \overset{(*)}{\Rightarrow} p|k^m \;\vee\; p|k \overset{(P(m))}{\Rightarrow} p|k \;\vee\; p|k \;\Leftrightarrow\; p|k.$$

Consequently, $P(m+1)$ is true and the statement $P(m)$ is true by the principle of mathematical induction.

17. Show that for every natural number $n \geq 2$ we have the inequality

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}>\sqrt{n}.$$

SOLUTION: We apply mathematical induction to show that for every natural number $n \geq 2$ we have the inequality

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}>\sqrt{n}. \tag{13.14}$$

Notice that since $2>1$ it follows that $\sqrt{2}>1$ and therefore $\sqrt{2}+1>2$. By dividing the last inequality by $\sqrt{2}$ we obtain

$$1+\frac{1}{\sqrt{2}}>\frac{2}{\sqrt{2}}=\sqrt{2},$$

and therefore the inequality (13.14) is true for $n=2$.

Suppose that for certain natural number $n \geq 2$ we have

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}>\sqrt{n}.$$

Then, by the above inequality, we have

$$\begin{aligned}
1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}} &> \sqrt{n}+\frac{1}{\sqrt{n+1}}\\
&= \frac{\sqrt{n}\sqrt{n+1}+1}{\sqrt{n+1}} > \frac{\sqrt{n}\sqrt{n}+1}{\sqrt{n+1}}\\
&= \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}
\end{aligned}$$

and consequently the inequality (13.14) is also true for $n+1$. Therefore, by mathematical induction, (13.14) is true for all natural numbers.

18. Show that for every natural number $n \geq 2$ we have the inequality

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} > 2(\sqrt{n+1}-1).$$

SOLUTION: For $n=2$, we have

$$1+\frac{1}{\sqrt{2}} > 1+\frac{1}{\sqrt{3}} = 1+\frac{\sqrt{3}}{3}\ .$$

Since we have

$$\begin{aligned}
\frac{81}{25} > 3 \Rightarrow \sqrt{\frac{81}{25}} > \sqrt{3} \Rightarrow \frac{9}{5} &> \sqrt{3}\\
&\Rightarrow 9 > 5\sqrt{3}\\
&\Rightarrow 3 > \frac{5\sqrt{3}}{3}\\
&\Rightarrow 1+\frac{\sqrt{3}}{3} > \frac{6\sqrt{3}}{3}-2\ ,
\end{aligned}$$

we have

$$1+\frac{1}{\sqrt{2}} > 1+\frac{\sqrt{3}}{3} > \frac{6\sqrt{3}}{3}-2 > 2(\sqrt{3}-1) = 2(\sqrt{2+1}-1)\ .$$

Thus the statement is true for $n=2$.

Now assume that the statement is true for a certain natural number $n=k \geq 1$, i.e

$$1+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}} > 2(\sqrt{k+1}-1)\ .$$

Consider the number $n=k+1$. We have

$$\begin{aligned}
\text{L.H.S.} &= 1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \\
&> 2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}} \\
&= 2(\sqrt{k+2} - 1) + \frac{1}{\sqrt{k+1}} + 2\sqrt{k+1} - 2\sqrt{k+2} \\
&= 2(\sqrt{k+2} - 1) + \sqrt{(\frac{1}{\sqrt{k+1}} + 2\sqrt{k+1})^2} - 2\sqrt{k+2} \\
&= 2(\sqrt{k+2} - 1) + \sqrt{\frac{1}{k+1} + 4(k+1) + 4} - 2\sqrt{k+2} \\
&= 2(\sqrt{k+2} - 1) + \sqrt{\frac{1}{k+1} + 4(k+2)} - 2\sqrt{k+2} \\
&= 2(\sqrt{k+2} - 1) + \sqrt{\frac{1}{k+1} + (2\sqrt{k+2})^2} - 2\sqrt{k+2} \\
&> 2\sqrt{k+2} - 1 + \sqrt{(2\sqrt{k+2})^2} - 2\sqrt{k+2} \\
&= 2\sqrt{k+1+1} - 1 = \text{R.H.S.}.
\end{aligned}$$

Thus the statement is also true for $n = k + 1$. By mathematical induction, the statement is true for any natural number $n \geq 2$.

19. Show that for every natural number n we have the inequality

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1.$$

SOLUTION: For $n = 1$, we have

$$\frac{1}{1+1} + \frac{1}{1+2} + \frac{1}{1+3} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{26}{24} > 1\ .$$

Thus the statement is true for $n = 1$.

Now assume that the statement is true for a certain natural number $n = k \geq 1$, i.e.

$$\frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{3k+1} > 1\ .$$

Consider the number $n = k + 1$. We have

$$\begin{aligned}
&\frac{1}{(k+1)+1}+\frac{1}{(k+1)+2}+\frac{1}{(k+1)+3}+\ldots+\frac{1}{3(k+1)+1}\\
&=\frac{1}{k+2}+\frac{1}{k+3}+\frac{1}{k+4}+\ldots+\frac{1}{3k+4}>1-\frac{1}{k+1}+\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}\\
&=1-\frac{2}{3k+3}+\frac{1}{3k+2}+\frac{1}{3k+4}\\
&=1-\left(\frac{2(3k+2)(3k+4)-(3k+3)(3k+4)-(3k+3)(3k+2)}{(3k+3)(3k+2)(3k+4)}\right)\\
&=1-\left(\frac{18k^2+36k+16-9k^2-21k-12-9k^2-15k-6}{(3k+3)(3k+2)(3k+4)}\right)\\
&=1-\left(\frac{-2}{(3k+3)(3k+2)(3k+4)}\right)=1+\frac{2}{(3k+3)(3k+2)(3k+4)}>1.
\end{aligned}$$

Therefore the statement is also true for $n=k+1$. By mathematical induction, the statement is true for any natural number n.

20. Show that for $n\in\mathbb{N}$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0.$$

SOLUTION: From the binomial theorem, we have

$$(a+b)^n=\binom{n}{0}a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\ldots+\binom{n}{n-1}ab^{n-1}+\binom{n}{n}b^n\ .$$

Substituting $a=1$, $b=-1$ into the above formula, we then have

$$\begin{aligned}
&(1-1)^n=\binom{n}{0}+\binom{n}{1}(-1)+\ldots+\binom{n}{n-1}(-1)^{n-1}+\binom{n}{n}(-1)^n\\
&=\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0.
\end{aligned}$$

21. Let $n\geq 2$ be a natural number. Show that

$$n!<\left(\frac{n+1}{2}\right)^n.$$

SOLUTION: It is convenient to write the above inequality in the form

$$p(n):\quad 2^n n!<(n+1)^n.$$

For $n=2$, we have

$$8=2^2\cdot 2!<3^2=9,$$

so $p(2)$ it is true. Assume (for the induction) that $p(n)$ is true for certain $n\geq 2$. We will show that $p(n+1)$ is also true. We have the following:

$$\begin{aligned}
2^{n+1}\cdot(n+1)! &= 2^n n!\cdot 2(n+1)\\
&< (n+1)^n\cdot 2(n+1) \quad \text{by induction assumption}\\
&= (n+1)^{n+1}\cdot 2\\
&= (n+2)^{n+1}\cdot 2\left(\frac{n+1}{n+2}\right)^{n+1}\\
&= (n+2)^{n+1}\cdot \frac{2}{\left(1+\frac{1}{n+1}\right)^{n+1}}.
\end{aligned}$$

Notice that, by the binomial theorem, we have for $n \geq 2$,

$$\left(1+\frac{1}{n+1}\right)^{n+1} = 1+\binom{n+1}{1}\frac{1}{n+1}+\cdots \geq 2,$$

consequently, we obtain

$$2^{n+1} < (n+2)^{n+1}\cdot\frac{2}{\left((1+\frac{1}{n+1}\right)^{n+1}} \leq (n+2)^{n+1},$$

which proves that $p(n+1)$ is also true.

22. Show that for $n \in \mathbb{N}$ we have

$$\sqrt[n]{n} \leq 1+\sqrt{\frac{2}{n}}.$$

SOLUTION: To show that

$$\sqrt[n]{n} \leq 1+\sqrt{\frac{2}{n}},$$

we put $\sqrt[n]{n} = 1+d$. Then $d \geq 0$ and, by binomial formula we have

$$\begin{aligned}
n = (1+d)^n &= \sum_{k=0}^{n}\binom{n}{k}d^k \geq 1+\binom{n}{2}d^2\\
&= 1+\frac{n(n-1)}{2}d^2,
\end{aligned}$$

and consequently we obtain

$$\frac{2}{n} \geq d^2 = (\sqrt[n]{n}-1)^2,$$

so

$$\sqrt[n]{n} \leq 1+\sqrt{\frac{2}{n}}.$$

23. Show that

(a) every infinite set contains a countable subset;
(b) the set of even integers has the same cardinality as the set of all natural numbers.
(c) the union of an infinite set and an at most countable set has the same cardinality as the original set.

SOLUTION: (a) Suppose that A is an infinite set. We will show that it is possible to construct for every natural number $n \in \mathbb{N}$ a subset C_n of A such that

(i) $|C_n| = n$, i.e. C_n is a finite subset of A containing exactly n elements;
(ii) $C_n \subset C_{n+1}$ for all $n = 1, 2, 3, \dots$.

Indeed, since A is infinite, it is non-empty, so

$$\exists_{x_1} \;\; x_1 \in A, \qquad \text{we put} \quad C_1 := \{x_1\}.$$

We will apply the principle of mathematical induction that a sequence of subsets C_n satisfying (i) and (ii) can be constructed. Assume that for the induction, that the sets $C_k := \{x_1, x_2, \dots, x_k\}$ satisfying (i) and (ii) were already constructed for $k = 1, 2, \dots, n$. Since $C_k \subset C_n \subset A$, $k = 1, 2, \dots, n-1$, and the set C_n is finite, while A is infinite (thus they can not be equal), it follows that $A \setminus C_n \neq \emptyset$. Consequently,

$$\exists_{x_{n+1}} \;\; x_{n+1} \in A \setminus C_n, \qquad \text{and we put} \quad C_{n+1} := C_n \cup \{x_{n+1}\}.$$

Clearly, C_{n+1} contains $n+1$ elements and since $C_n \subset C_{n+1}$, it follows from the principle of mathematical induction that such a sequence of subsets C_n of A can be constructed. Now, we can put

$$C := \bigcup_{n=1}^{\infty} C_n, \quad C := \{x_1, x_2, x_3, \dots\},$$

thus the required countable set $C \subset A$ was constructed.
(b): In order to show that $|2\mathbb{Z}| = |\mathbb{N}|$, we need to construct a bijection φ from $2\mathbb{Z} := \{0, \pm 2, \pm 4, \pm 6, \dots\}$ to $\mathbb{N} = \{1, 2, 3, \dots\}$. One can easily verify that for $a \in 2\mathbb{Z}$

$$\varphi(a) := \begin{cases} a & \text{if } a > 0 \\ -a + 1 & \text{if } a \leq 0, \end{cases}$$

is the required function. Consequently, the conclusion follows.
(c): Assume that A is an infinte set and $Z := \{b_1, b_2, b_3, \dots\}$ is a countable set (infinite). We will show that there exists a bijection $\psi : A \cup Z \to A$, thus $|A \cup Z| = |A|$. Indeed, since A is infinite, by (a), there exists a countable subset $C := \{x_1, x_2, x_3, \dots\}$ of A, and consequently

$$A \cup Z = (A \setminus C) \cup C \cup Z, \qquad \text{and} \qquad A = (A \setminus C) \cup C.$$

Then we can define $\psi : (A \setminus C) \cup C \cup Z \to (A \setminus C) \cup C$ by

$$\psi(x) := \begin{cases} x & \text{if } x \in A \setminus C \\ x_{2n} & \text{if } x = x_n \in C \\ x_{2n-1} & \text{if } x = b_n \in Z. \end{cases}$$

One can easily check that $\psi : A \cup Z \to A$ is a required bijection.

13.2 Chapter 3: Elementary Theory of Metric Spaces

1. A set A in a metric space (S, d) is called *bounded* if

$$\exists_{R>0}\,\exists_{x_o\in S}\ \ A\subset B_R(x_o).$$

Use the mathematical induction to show that if a set $A\subset S$ is *unbounded* (i.e. it is not bounded), then there exists a sequence $\{x_n\}\subset A$ such that $d(x_n,x_m)\geq 1$ for all $m\neq n$.

SOLUTION: The set A is unbounded, thus we have

$$A \text{ is unbounded} \iff \forall_{R>0}\,\forall_{x_o\in S}\ A\not\subset B_R(x_o). \tag{13.15}$$

Since $A\not\subset B_R(x_o)$ is equivalent to $A\cap[B_R(x_o)]^c$, which means $\exists_{x\in A}\ x\in[B_R(x_o)]^c\ \Leftrightarrow\ \exists_{x\in A}\ d(x_o,x)\geq R$, we obtain that (13.1) is equivalent to

$$A \text{ is unbounded} \iff \forall_{R>0}\,\forall_{x_o\in S}\,\exists_{x\in A}\ d(x_o,x)\geq R \tag{13.16}$$

We choose as x_1 an arbitrary element of A. Put $P_1:=\{x_1\}$. By (13.16) (with $x_o=x_1$ and $R=1$), there exists $x_2\in A$ (i.e. $\exists_{x\in A}\ d(x_o,x)\geq R=1$, $x_2=x$) such that $d(x_1,x_2)\geq 1$. We put $P_2:=\{x_1,x_2\}$. Our claim is the following:

$P(n)$For every natural number n there exists a subset (or a finite sequence) $P_n:=\{x_1,x_2,\dots,x_n\}$ satisfying: $P_n\subset P_{n+1}$ and for all x_l, $x_k\in P_n$ with $k\neq l$, $d(x_k,x_l)\geq 1$.

It is obvious that $P(1)$ (as well as $P(2)$) are true. We assume for the induction that the set $P_n:=\{x_1,x_2,\dots,x_n\}$ satisfies the condition $P(n)$. We will construct the set P_{n+1}. Indeed, let $R:=\max\{d(x_1,x_k):k=1,2,\dots,n\}+1$. Then by (13.16) (with $x_o=x_1$ and R), there exists $x_{n+1}\in A$ (i.e. $\exists_{x\in A}\ d(x_1,x)\geq R$, $x_{n+1}:=x$) such that $d(x_1,x_{n+1})\geq R$. Consequently, for every $k=1,2,\dots,n$, we have

$$d(x_k,x_{n+1})\geq d(x_1,x_{n+1})-d(x_1,x_k)\geq d(x_1,x_k)+1-d(x_1,x_k)=1.$$

Since $d(x_k,x_l)\geq 1$ for $k\not l$, $l,k\in\{1,2,\dots,n\}$ (by the induction assumption), we obtain that the set $P_{n+1}:=P_n\cup\{x_{n+1}\}$ satisfies $P(n+1)$. In this way, we get that such sets $P_1\subset P_2\subset\dots\subset P_n\subset\dots$ can be constructed. We put

$$P:=\bigcup_{n=1}^{\infty}P_n=\{x_1,x_2,\dots,x_n,\dots\}.$$

It isclear that P contains the required sequence $\{x_1,x_2,\dots,x_n,\dots\}$.

2. Let $(V,\|\cdot\|)$ be a normed vector space. Show that an open unit ball

$$B_1(0):=\{v\in V:\|v\|<1\}$$

is a convex set, i.e.

$$\forall_{u,v\in B_1(0)}\,\forall_{t\in[0,1]}\ \ tu+(1-t)v\in B_1(0).$$

SOLUTION: We have

$$v,u\in B_1(0)\ \Leftrightarrow\ \|v\|\leq 1\ \wedge\ \|u\|\leq 1 \tag{13.17}$$

Thus,

$$\begin{aligned}\forall_{u,v\in B_1(0)}\,\forall_{t\in[0,1]}\qquad \|tu+(1-t)v\| &\le \|tu\|+\|(1-t)u\|\\ &= t\|u\|+(1-t)\|u\| \overset{\text{by (13.17)}}{<} t+(1-t)=1\end{aligned}$$

$$\Rightarrow \forall_{u,v\in B_1(0)}\forall_{t\in[0,1]}\; tu+(1-t)v\in B_1(0),$$

which implies that $B_1(0)$ is convex.

3. Consider the set $B\subset\mathbb{R}^2$ defined by

$$B:=\{(x,y)\in\mathbb{R}^2:(x-1)^2+(y-1)^2<4\}$$

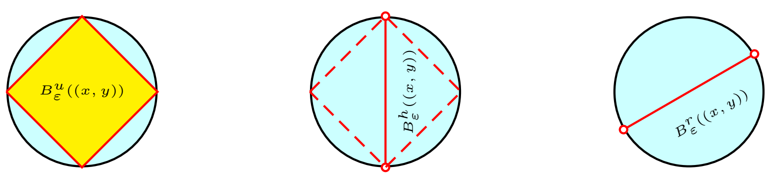
Verify if the set B is open in the following metric spaces

(a) urban metric space (S_u,d_1);
(b) highway metric space (S_h,d_h);
(c) railway metric space (S_r,d_r).

SOLUTION: Notice that the set B is an open ball $B_2^e((1,1))$ (the index e indicates that the ball is considered in the *euclidean* space) in the euclidean metric space $S_2:=\mathbb{R}^2$. Since an open ball in a metric space is an open set, it follows that B is an open set i.e.

$$\forall_{(x,y)\in B}\;\exists_{\varepsilon>0}\;B_\varepsilon^e((x,y))\subset B.$$

However, notice that an euclidean ball $B_\varepsilon^e((x,y))$ contains the balls $B_\varepsilon^h((x,y))$ (in the highway metric space), $B_\varepsilon^u((x,y))$ (in the urban metric space) and $B_\varepsilon^2((x,y))$ (in the railway metric space), doesn't matter where the point (x,y) is located (see the picture below).



$B_\varepsilon^e((x,y))$ $B_\varepsilon^e((x,y))$ $B_\varepsilon^e((x,y))$

Consequently we obtain

$$\begin{aligned}&\forall_{(x,y)\in B}\;\exists_{\varepsilon>0}\;B_\varepsilon^u((x,y))\subset B\\&\forall_{(x,y)\in B}\;\exists_{\varepsilon>0}\;B_\varepsilon^h((x,y))\subset B\\&\forall_{(x,y)\in B}\;\exists_{\varepsilon>0}\;B_\varepsilon^r((x,y))\subset B\end{aligned}$$

which implies that B is open in the urban, highway and railway metric spaces.

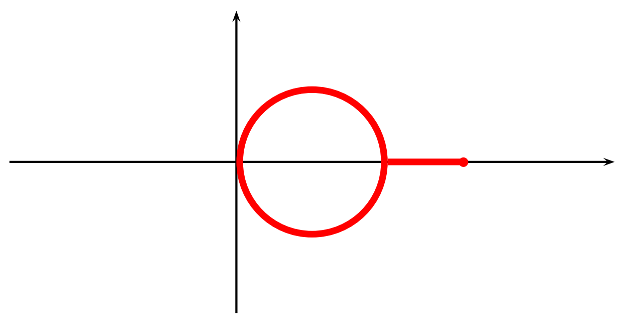
4. Identify which of the following sets are open, which are closed, and which are neither, in the metric space $\mathbb{R}^2$ equipped with the Euclidean metric:

See the notes for the precise definition of these metric spaces

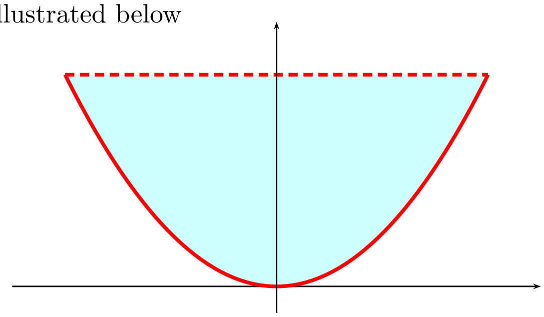
(a) $A = \{(x, y) : x^2 - 2x + y^2 = 0\} \cup \{(x, 0) : x \in [2, 3]\}$
(b) $B = \{(x, y) : y \geq x^2,\ 0 \leq y < 1\}$
(c) $C = \{(x, y) : x^2 - y^2 < 1,\ -1 < y < 1\}$

Find the interior, closure and boundary of the sets A, B, C (no proof is necessary)

SOLUTION: (a): We can illustrate the set A (using a red line)



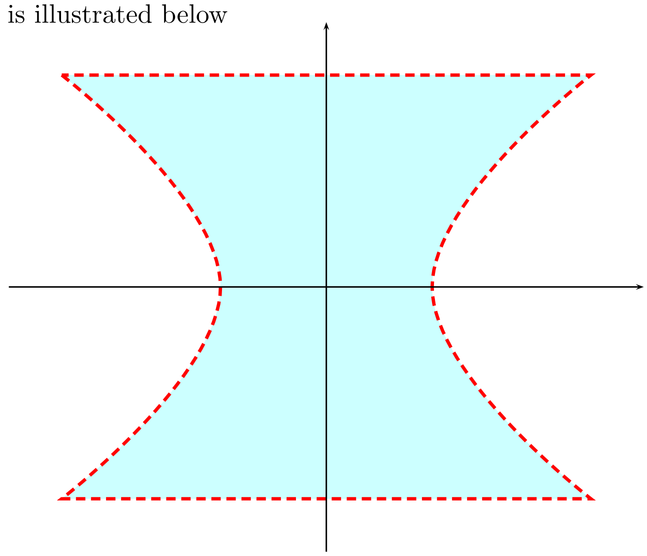
By inspection (based on the definition of a closed set) one can easily recognize that A is closed, $\text{int}(A) = \emptyset$, $\overline{A} = A$, $\partial A = A$.
(b): The set B is illustrated below



Clearly, B is not closed or open. We have

$$\begin{aligned} \text{int}(B) &= \{(x, y) : y > x^2,\ y < 1\} \\ \overline{B} &= \{(x, y) : y \geq x^2,\ 0 \leq y \leq 1\} \\ \partial B &= \{(x, y) : y = x^2,\ 0 \leq y \leq 1\} \cup \{(x, 1) : x \in [-1, 1]\}. \end{aligned}$$

(c): The set C is illustrated below



One can easily verify that boundary ∂C of the set C, which is

$$\partial C = \{(x,y) : x^2 - y^2 = 1,\ -1 \le y \le 1\} \cup \{(x,y) : (x, \pm 1),\ -\sqrt{2} \le x \le \sqrt{2}\}$$

is not included in C ($C \cap \partial C = \emptyset$), i.e. the set C is open. In particular $C = \operatorname{int}(C)$ and

$$\overline{C} = \{(x,y) : x^2 - y^2 \le 1,\ -1 \le y \le 1\}.$$

5. Show that in a metric space (S,d) a closed ball is a closed set.

SOLUTION: Let $\overline{B}_r(x_o) = \{x \in S : d(x, x_o) \le r\}$ be a closed ball in (S,d). In order to show that $\overline{B}_r(x_o)$ is closed, it is sufficient to show that the complement

$$[\overline{B}_r(x_o)]^c := \{x \in S : d(x, x_o) > r\},$$

is open in S, i.e. we need to show that

$$\begin{aligned} &\forall_{x \in \overline{B}_r(x_o)]^c}\ \exists_{\varepsilon > 0}\ B_\varepsilon(x) \subset \overline{B}_r(x_o)]^c \\ \Leftrightarrow\ &\forall_x\ [d(x, x_o) > r\ \Rightarrow\ \exists_{\varepsilon>0}\ (d(x', x) < \varepsilon\ \Rightarrow\ d(x_o, x') > r)] \end{aligned}$$

Indeed, if $d(x, x_o) > r$, then we can take $\varepsilon := d(x, x_o) - r > 0$, and we have

$$\begin{aligned} \forall_{x'}\ d(x', x) < \varepsilon\ \Rightarrow\ d(x_o, x') &\ge d(x_o, x) - d(x, x') \\ &> d(x_o, x) - \varepsilon = d(x_o, x) - d(x_o, x) + r = r \end{aligned}$$

Consequently, $[\overline{B}_r(x_o)]^c$ is open, i.e. $\overline{B}_r(x_o)$ is closed.

6. Suppose that (S,d) is a metric space and A, $B \subset S$ are such that $A \subset B$. Show that $\operatorname{int}(A) \subset \operatorname{int}(B)$ and $\overline{A} \subset \overline{B}$.

SOLUTION: We have that $A \subset B$. Thus

$$\begin{aligned} x \in \operatorname{int}(A)\ \wedge\ A \subset B\ &\Leftrightarrow\ \exists_{\varepsilon>0}\ B_\varepsilon(x) \subset A\ \wedge\ A \subset B \\ &\Rightarrow\ \exists_{\varepsilon>0}\ B_\varepsilon(x) \subset B\ \Leftrightarrow\ x \in \operatorname{int}(B) \end{aligned}$$

$$\begin{aligned} x \in \overline{A}\ \wedge\ A \subset B\ &\Leftrightarrow\ \forall_{\varepsilon>0}\ B_\varepsilon(x) \cap A \ne \emptyset\ \wedge\ A \subset B \\ &\Rightarrow\ \forall_{\varepsilon>0}\ B_\varepsilon(x) \cap B \ne \emptyset\ \Leftrightarrow\ x \in \overline{B} \end{aligned}$$

thus we obtain $\overline{A} \subset \overline{B}$.

7. Give an example of two sets A and B in a metric space such that $(\operatorname{int}(A \cup B) \ne \operatorname{int}(A) \cup \operatorname{int}(B)$. Use this example to show that the equality $\overline{C \cap D} = \overline{C} \cap \overline{D}$ does not hold in general.

SOLUTION: Consider the space $S = \mathbb{R}$ with the usual metric. We put $A = [0,1]$ and $B = [1,2]$. Then $\operatorname{int}(A) = (0,1)$, $\operatorname{int}(B) = (1,2)$, and $\operatorname{int}(A \cup B) = \operatorname{int}([0,2]) = (0,2)$. Clearly

$$(\mathrm{int}(A \cup B) = (0,2) \neq (0,1) \cup (1,2) = \mathrm{int}(A) \cup \mathrm{int}(B).$$

Put $C = A^c$ and $D = B^c$, i.e. $C = (-\infty, 0) \cup (1, \infty)$, $D = (-\infty, 1) \cup (2, \infty)$. We have $C \cap D = (2, \infty)$, thus $\overline{C \cap D} = [2, \infty]$, and $\overline{C} \cap \overline{D} = \{1\} \cup [2, \infty)$. Clearly, $\overline{C \cap D} \neq \overline{C} \cap \overline{D}$.

8. Let (S_1, d_1) and (S_2, d_2) be two metric spaces. Show that each of the following functions d is a metric on $S = S_1 \times S_2$ (Hint: Knowing that d_1 and d_2 satisfies the three conditions of a metric, show that d also satisfies these conditions)

(a) $d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$;
(b) $d((x_1, x_2), (y_1, y_2)) = \sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2}$;
(c) $d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$;

SOLUTION: Since (S_1, d_1) and (S_2, d_2) are metric spaces, then

(i$_1$): $\forall_{x_1, y_1 \in S_1}$ $d_1(x_1, y_1) \geq 0$,
(ii$_1$): $\forall_{x_1, y_1 \in S_1}$ $d_1(x_1, y_1) = 0 \Leftrightarrow x_1 = y_1$,
(iii$_1$): $\forall_{x_1, y_1 \in S_1}$ $d_1(x_1, y_1) = d_1(y_1, x_1)$,
(iv$_1$): $\forall_{x_1, y_1, z_1 \in S_1}$ $d_1(x_1, y_1) \leq d_1(x_1, z_1) + d_1(z_1, y_1)$,

(i$_2$): $\forall_{x_2, y_2 \in S_2}$ $d_2(x_2, y_2) \geq 0$,
(ii$_2$): $\forall_{x_2, y_2 \in S_2}$ $d_2(x_2, y_2) = 0 \Leftrightarrow x_2 = y_2$,
(iii$_2$): $\forall_{x_2, y_2 \in S_2}$ $d_2(x_2, y_2) = d_2(y_2, x_2)$,
(ii$_2$): $\forall_{x_2, y_2, z_2 \in S_2}$ $d_2(x_2, y_2) \leq d_2(x_2, z_2) + d_2(z_2, y_2)$.

(a): Since $\forall_{(x_1,x_2),(y_1,y_2) \in S_1 \times S_2}$

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} \geq d_1(x_1, y_1) \tag{13.18}$$
$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} \geq d_2(x_2, y_2), \tag{13.19}$$

it follows from (i$_1$) (or (i$_2$)) that

(i): $\forall_{(x_1,x_2),(y_1,y_2) \in S_1 \times S_2}$ $d((x_1, x_2), (y_1, y_2)) \geq 0$.

On the other hand

(ii): $\forall_{(x_1,x_2),(y_1,y_2) \in S_1 \times S_2}$ we have that $d((x_1, x_2), (y_1, y_2)) = 0$ implies (by (1) and (2), and then by (ii$_1$) and (ii$_2$))

$$0 = d((x_1, x_2), (y_1, y_2)) \geq d_1(x_1, y_1) \geq 0 \ \Rightarrow \ d_1(x_1, y_1) = 0 \ \Rightarrow \ x_1 = y_1$$
$$0 = d((x_1, x_2), (y_1, y_2)) \geq d_2(x_1, y_2) \geq 0 \ \Rightarrow \ d_2(x_2, y_2) = 0 \ \Rightarrow \ x_2 = y_2$$

thus $(x_1, x_2) = (y_1, y_2)$.

(iii): $\forall_{(x_1,x_2),(y_1,y_2) \in S_1 \times S_2}$ we have (by (iii$_1$) and (iii$_2$)) that

$$\begin{aligned} d((x_1, x_2), (y_1, y_2)) &= \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} \\ &= \max\{d_1(y_1, x_1), d_2(y_2, x_2)\} = d((y_1, y_2), (x_1, x_2)). \end{aligned}$$

Finally,

(iv): $\forall_{(x_1,x_2),(y_1,y_2),(z_1,z_2) \in S_1 \times S_2}$ we have (by (iv$_1$) and (iv$_2$))

$$\begin{aligned} d((x_1, x_2), (y_1, y_2)) &= \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} \\ &\leq \max\{d_1(x_1, z_1) + d_1(z_1, y_1), d_2(x_2, z_2) + d_2(z_2, y_2)\} \\ &\leq \max\{d_1(x_1, z_1), d_2(x_2, z_2)\} + \max\{d_1(z_1, y_1), d_2(z_2, y_2)\} \\ &= d((x_1, x_2), (z_1, z_2)) + d((z_1, z_2), (y_1, y_2)). \end{aligned}$$

(b): Since $\forall_{(x_1,x_2),(y_1,y_2) \in S_1 \times S_2}$

$$d((x_1,x_2),(y_1,y_2)) = \sqrt{(d_1(x_1,y_1))^2 + (d_2(x_2,y_2))^2} \geq d_1(x_1,y_1) \tag{13.20}$$
$$d((x_1,x_2),(y_1,y_2)) = \sqrt{(d_1(x_1,y_1))^2 + (d_2(x_2,y_2))^2} \geq d_2(x_2,y_2), \tag{13.21}$$

it follows from (i$_1$) (or (i$_2$)) that

(i): $\forall_{(x_1,x_2),(y_1,y_2)\in S_1\times S_2}\ \ d((x_1,x_2),(y_1,y_2)) \geq 0$.

Next,

(ii): $\forall_{(x_1,x_2),(y_1,y_2)\in S_1\times S_2}$ we have that $d((x_1,x_2),(y_1,y_2)) = 0$ implies (by (3) and (4), and then by (ii$_1$) and (ii$_2$))

$$0 = d((x_1,x_2),(y_1,y_2)) \geq d_1(x_1,y_1) \geq 0 \ \Rightarrow\ d_1(x_1,y_1) = 0 \ \Rightarrow\ x_1 = y_1$$
$$0 = d((x_1,x_2),(y_1,y_2)) \geq d_2(x_1,y_2) \geq 0 \ \Rightarrow\ d_2(x_2,y_2) = 0 \ \Rightarrow\ x_2 = y_2$$

thus $(x_1,x_2) = (y_1,y_2)$.

(iii): $\forall_{(x_1,x_2),(y_1,y_2)\in S_1\times S_2}$ we have (by (iii$_1$) and (iii$_2$)) that

$$\begin{aligned} d((x_1,x_2),(y_1,y_2)) &= \sqrt{(d_1(x_1,y_1))^2 + (d_2(x_2,y_2))^2} \\ &= \sqrt{(d_1(y_1,x_1))^2 + (d_2(y_2,x_2))^2} = d((y_1,y_2),(x_1,x_2)). \end{aligned}$$

Finally, notice that by Cauchy-Schwarz inequality, we have $\forall_{(x_1,x_2),(y_1,y_2),(z_1,z_2)\in S_1\times S_2}$

$$d_1(x_1,z_1)d_1(z_1,y_1) + d_2(x_2,z_2)d_2(z_2,y_2) \leq \sqrt{(d_1(x_1,z_1))^2 + (d_2(x_2,z_2))^2}\cdot\sqrt{(d_1(z_1,y_1))^2 + (d_2(z_2,y_2))^2}$$

so, by (iv$_1$) and (iv$_2$)

(iv): $\forall_{(x_1,x_2),(y_1,y_2),(z_1,z_2)\in S_1\times S_2}$ we have (by (iv$_1$) and (iv$_2$))

$$\begin{aligned} [d((x_1,x_2),(y_1,y_2))]^2 &= (d_1(x_1,y_1))^2 + (d_2(x_2,y_2))^2\} \\ &\leq (d_1(x_1,z_1) + d_1(z_1,y_1))^2 + (d_2(x_2,z_2) + d_2(z_2,y_2))^2 \\ &= (d_1(x_1,z_1))^2 + 2[d_1(x_1,z_1)d_1(z_1,y_1)] + (d_1(z_1,y_1))^2 \\ &+ (d_2(x_2,z_2))^2 + 2[d_2(x_2,z_2)d_2(z_2,y_2)] + (d_2(z_2,y_2))^2 \\ &= (d_1(x_1,z_1))^2 + (d_1(z_1,y_1))^2 + (d_2(x_2,z_2))^2 + (d_2(z_2,y_2))^2 \\ &+ 2[d_1(x_1,z_1)d_1(z_1,y_1) + d_2(x_2,z_2)d_2(z_2,y_2)] \\ &\leq (d_1(x_1,z_1))^2 + (d_1(z_1,y_1))^2 + (d_2(x_2,z_2))^2 + (d_2(z_2,y_2))^2 \\ &+ 2\sqrt{(d_1(x_1,z_1))^2 + (d_2(x_2,z_2))^2}\cdot\sqrt{(d_1(z_1,y_1))^2 + (d_2(z_2,y_2))^2} \\ &= \left[\sqrt{(d_1(x_1,z_1))^2 + (d_2(x_2,z_2))^2} + \sqrt{(d_1(z_1,y_1))^2 + (d_2(z_2,y_2))^2}\right]^2 \\ &= [d((x_1,x_2),(z_1,z_2)) + d((z_1,z_2),(y_1,y_2))]^2 \end{aligned}$$

Therefore, we get

$$d((x_1,x_2),(y_1,y_2)) \leq d((x_1,x_2),(z_1,z_2)) + d((z_1,z_2),(y_1,y_2))$$

(c): Since $\forall_{(x_1,x_2),(y_1,y_2)\in S_1\times S_2}$

$$d((x_1,x_2),(y_1,y_2)) = d_1(x_1,y_1) + d_2(x_2,y_2) \geq d_1(x_1,y_1) \tag{13.22}$$
$$d((x_1,x_2),(y_1,y_2)) = d_1(x_1,y_1) + d_2(x_2,y_2) \geq d_2(x_2,y_2), \tag{13.23}$$

it follows from (i_1) (or (i_2)) that

(i): $\forall_{(x_1,x_2),(y_1,y_2)\in S_1\times S_2}\ d((x_1,x_2),(y_1,y_2)) \geq 0$.

On the other hand

(ii): $\forall_{(x_1,x_2),(y_1,y_2)\in S_1\times S_2}$ we have that $d((x_1,x_2),(y_1,y_2)) = 0$ implies (by (5) and (6), and then by (ii_1) and (ii_2))

$$0 = d((x_1,x_2),(y_1,y_2)) \geq d_1(x_1,y_1) \geq 0 \ \Rightarrow\ d_1(x_1,y_1) = 0 \ \Rightarrow\ x_1 = y_1$$
$$0 = d((x_1,x_2),(y_1,y_2)) \geq d_2(x_1,y_2) \geq 0 \ \Rightarrow\ d_2(x_2,y_2) = 0 \ \Rightarrow\ x_2 = y_2$$

thus $(x_1,x_2) = (y_1,y_2)$.

(iii): $\forall_{(x_1,x_2),(y_1,y_2)\in S_1\times S_2}$ we have (by (iii_1) and (iii_2)) that

$$\begin{aligned} d((x_1,x_2),(y_1,y_2)) &= d_1(x_1,y_1) + d_2(x_2,y_2) \\ & \qquad d_1(y_1,x_1) + d_2(y_2,x_2) = d((y_1,y_2),(x_1,x_2)). \end{aligned}$$

Finally,

(iv): $\forall_{(x_1,x_2),(y_1,y_2),(z_1,z_2)\in S_1\times S_2}$ we have (by (iv_1) and (iv_2))

$$\begin{aligned} d((x_1,x_2),(y_1,y_2)) &= d_1(x_1,y_1) + d_2(x_2,y_2) \\ &\leq \Big(d_1(x_1,z_1) + d_1(z_1,y_1)\Big) + \Big(d_2(x_2,z_2) + d_2(z_2,y_2)\Big) \\ &= \Big(d_1(x_1,z_1) + d_2(x_2,z_2)\Big) + \Big(d_1(z_1,y_1) + d_2(z_2,y_2)\Big) \\ &= d((x_1,x_2),(z_1,z_2)) + d((z_1,z_2),(y_1,y_2)). \end{aligned}$$

9. Let $S = \mathbb{R}^2$ and $d_h : S \times S \to \mathbb{R}$ be the *highway metric* given by

$$d_h((x_1,y_1),(x_2,y_2)) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2, \\ |y_1| + |x_1 - x_2| + |y_2| & \text{if } x_1 \neq x_2. \end{cases}$$

Check which of the following sets are open or closed in (S, d_h):

(a) $A = \{(x,y) : x^2 + y^2 < 1\}$;
(b) $B = \{(x,y) : y = 0 \ \wedge\ -1 \leq x \leq 1\}$;
(c) $C = \{(x,y) : x = 0 \ \wedge\ 0 < y < 1\}$;
(d) $D = \{(x,y) : -1 \leq x \leq 1 \ \wedge\ 1 < y < 2\}$.

HINT: Show that an open, with respect to the Euclidean metric d_2 on $\mathbb{R}^2$ (do not mix up this metric with the question 1, where this symbol was standing for something completely different), set U is also open with respect to the metric d_h. Use the definition of an open set for this question.

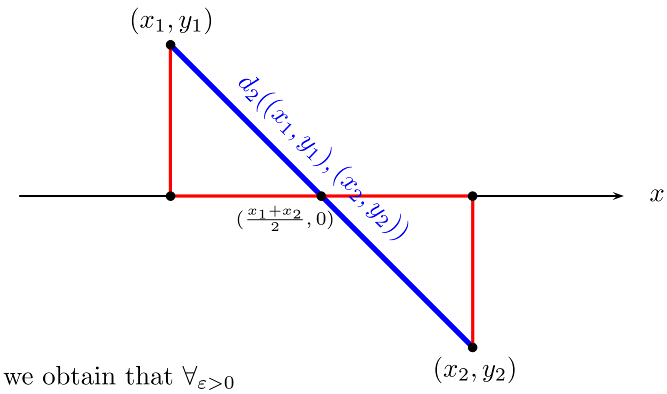
SOLUTION: Notice that $\forall_{(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2}$, $x_1 \neq x_2$, we have the following inequality (which bfollows from the triangle inequality (iv)applied twice):

$$\begin{aligned} d_2((x_1,y_1),(x_2,y_2)) &\leq d_2((x_1,y_1),(\frac{x_1+x_2}{2},0)) + d_2((\frac{x_1+x_2}{2},0),(x_2,y_2)) \\ &\leq d_2((x_1,y_1),(x_1,0)) + d_2((x_1,0),(\frac{x_1+x_2}{2},0)) \\ &+ d_2((\frac{x_1+x_2}{2},0),(x_2,0)) + d_2((x_2,0),(x_2,y_2)) \\ &= |y_1| + |x_2 - x_1| + |y_2| = d_h((x_1,y_1),(x_2,y_2)). \end{aligned}$$

That means, $\forall_{(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2}$ we have

$$d_h((x_1,y_1),(x_2,y_2)) \geq d_2((x_1,y_1),(x_2,y_2)). \tag{7}$$

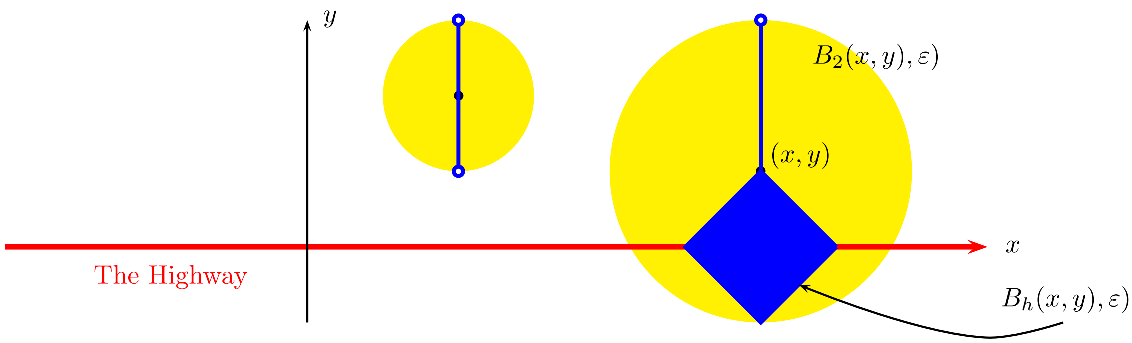
See the picture below:



Therefore, from (7) we obtain that $\forall_{\varepsilon>0}$

$$B_h((x,y),\varepsilon) \subset B_2((x,y),\varepsilon). \tag{8}$$

See the picture below:



Proposition: *If $U \subset \mathbb{R}^2$ is an open set with respect to the metric d_2 (Euclidean metric) then it is also open with respect to the metric d_h (highway metric).*

Proof: Assume that $U \subset \mathbb{R}^2$ is an open set with respect to d_2, then $\forall_{(x,y)\in U}\ \exists_{\varepsilon>0}\ \ B_2((x,y),\varepsilon) \subset U$. Since $B_h((x,y),\varepsilon) \subset B_2((x,y),\varepsilon)$, thus we get

$$\forall_{(x,y)\in U}\ \exists_{\varepsilon>0}\ \ B_h((x,y),\varepsilon) \subset U,$$

thus U is also open with respect to d_h. $\square$

Corollary: *If $A \subset \mathbb{R}^2$ is a closed set with respect to the metric d_2 (Euclidean metric) then it is also closed with respect to the metric d_h (highway metric).*

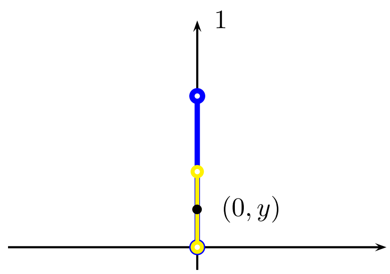
Proof: Since A is closed if and only if $\mathbb{R}^2\setminus A$ is open, then the conclusion follows from the Proposition. $\square$

Now, we can proceed to solving the Problem #2:

(a): Since the set $A = B_2((0,0),1)$, and every open ball in a metric space is an open set, it follows that A is open with respect to d_2, thus, by Proposition, it is also open with respect to d_h.

(b): Notice that the set $U := \mathbb{R}^2 \setminus B$ is open with respect to the Euclidean metric d_2. Indeed, $(x,y) \in U \quad \Leftrightarrow \quad y \neq 0 \vee |x| > 0$, thus if we choose for $(x,y) \in U$ the number $\varepsilon = \max\{|y|, |x|-1\}$, then $B_2((x,y),\varepsilon) \subset U$. Indeed, if $(x',y') \in B_2((x,y),\varepsilon)$ and $y' = 0$, then $|x'| = |x'-x+x| \geq |x|-|x'-x| > |x|-|x|+1 = 1$ so $(x',y') \in U$. This implies that B is closed with respect to d_2, and consequently, it is also closed with respect to d_h.

(c): The set C is open with respect to d_h. Indeed, $C = \{(0,y) : 0 < y < 1\}$, so for a point $(0,y) \in C$, we can choose $\varepsilon = \min\{y, 1-y\}$. Then $(x',y') \in B_h((0,y),\varepsilon) \;\Leftrightarrow\; x' = 0$ and $|y'-y| < \varepsilon$ (see the picture below):



Consequently, the set C is open with respect to d_h.

(d): It is easy to notice that for every point $(x,y) \in D$ we have that $-1 \leq x \leq 1$ and $1 < y < 2$. Assume that $(x,y) \in D$ and put $\varepsilon = \min\{2-y, y-1\}$. Then for every $(x',y') \in B_h((x,y),\varepsilon)$ we have that $x' = x$ and $|y'| = |y'-y+y| \geq y - |y'-y| > y - \varepsilon \geq y - y - 1 = 1$ and $|y'| \leq |y'-y| + y < \varepsilon + y \leq 2 - y + y = 2$, so $(x',y') \in D$, which implies that

$$\forall_{(x,y)\in D}\ \exists_{\varepsilon=\min\{2-y,y-1\}}\ \ B_h((x,y),\varepsilon) \subset D.$$

Consequently, D is open with respect to d_h.

10. Let $S = \mathbb{R}^2$ and $d_2 : S \times S \to \mathbb{R}$ be the usual Euclidean metric on $\mathbb{R}^2$. Verify which of the sets A, B, C, and D listed in Problem 2, are open or closed in (S, d_2). In addition, find the boundaries of the sets A, B, C and D.

SOLUTION: Notice, that we have already proved in Problem #2 that A is open with respect to d_2 and B is closed with respect to d_2. On the other hand, the sets C and D are not open nor closed with respect to d_2. Indeed, notice that $(0,0)$ and $(0,1)$ are limit points of C which does not belong to C (so C is not closed), while the every point of C is a limit point of the complement $C^c = \mathbb{R}^2 \setminus C$ (so the complement of C is not closed). On the other hand, the points $(x,1)$, $x \in [-1,1]$ are limit points of D, which do not belong to D, while $(1,y)$, $y \in (1,2)$, are limit points of the complent of D, which belong to D, thus again, D is not closed or open. It is not hard to see that the boundaries of the sets A, B, C and D are the following sets:

$\partial A = \{(x,y) : x^2 + y^2 = 1\}$ – the unit circle centered at $(0,0)$;

$\partial B = B = \{(x,0) : -1 \leq x \leq 1\}$ – the same as the set B;

$\partial C = \{(0,y) : 0 \leq y \leq 1\}$ – the segment from $(0,0)$ to $(0,1)$ including the end points.

$\partial D = \{(x,y) : (x = \pm 1 \ \wedge \ 1 \le y \le 2) \ \vee \ (y = 1 \ \wedge \ |x| \le 1) \ \vee \ (y = 2 \ \wedge \ |x| \le 1)\}$ – the rectangle with vertices $(-1,1)$, $(-1,2)$, $(1,2)$, and $(1,1)$.

11. Use the definition of a limit and the properties of real numbers (including the Binomial Theorem) to show that

$$\forall_{a>0} \quad \lim_{n\to\infty} a^{\frac{1}{n}} = 1.$$

SOLUTION: We consider first the case $a > 1$, so $a^{\frac{1}{n}} > 1$, and we can put $\alpha := a^{\frac{1}{n}} - 1 > 0$. Then, by applying the Binomial Theorem, we obtain:

$$\begin{pmatrix} a = (1+\alpha)^n = 1 + n\alpha + \\ n2\alpha^2 + \cdots + \alpha^n \ge 1 + n\alpha. \end{pmatrix}$$

which leads to the following inequality

$$(*) \qquad \frac{a-1}{n} \ge a^{\frac{1}{n}} - 1 > 0.$$

Now, $\forall_{\varepsilon>0}$, by Archimedian axiom, $\exists_{N\in\mathbb{N}}$ such that $N > \frac{a-1}{\varepsilon}$, and thus, $\forall_{n\ge N}$ we also have $n > \frac{a-1}{\varepsilon} \Leftrightarrow \varepsilon > \frac{a-1}{n}$. On the other hand, by (*), we have that $\frac{a-1}{n} \ge a^{\frac{1}{n}} - 1 > 0$, so we obtain

$$\forall_{\varepsilon>0} \exists_{N\in\mathbb{N}} \forall_{n\ge N} \ \varepsilon > a^{\frac{1}{n}} - 1 > 0,$$

i.e.

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1.$$

Now, assume that $0 < a < 1$ (notice that for $a = 1$, the statement is trivial). Then clearly, $b := \frac{1}{a} > 1$ and we have

$$1 - a^{\frac{1}{n}} = 1 - \frac{1}{b^{\frac{1}{n}}} = \frac{b^{\frac{1}{n}} - 1}{b^{\frac{1}{n}}} \le b^{\frac{1}{n}} - 1.$$

Since, by the previous case, we know that $\lim_{n\to\infty} b^{\frac{1}{n}} = 1$, thus

$$\forall_{\varepsilon>0} \exists_{N\in\mathbb{N}} \forall_{n\ge N} \ \varepsilon > b^{\frac{1}{n}} - 1 > 1 - a^{\frac{1}{n}} > 0,$$

i.e.

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1.$$

12. Let $\{\vec{x}_n\} \subset \mathbb{R}^m$ be a bounded sequence and $\{a_n\} \subset \mathbb{R}$ be a sequence convergent to zero. Show that the sequence $\{\vec{y}_n\}$ defined by $\vec{y}_n = a_n \vec{x}_n \in \mathbb{R}^m$ converges to $\vec{0} \in \mathbb{R}^m$.

SOLUTION: Since $\{\vec{x}_n\}$ is bounded, $\exists_{M>0}$ such that $\forall_{n\in\mathbb{N}}$ $\|\vec{x}_n\| \le M$. On the other hand, since $\{a_n\}$ converges to zero, we have

$$(**) \qquad \forall_{\varepsilon>0} \exists_{N\in\mathbb{N}} \forall_{n\ge N} \ \ |a_n| < \frac{\varepsilon}{M}.$$

Therefore, by (**), we obtain

$$\forall_{\varepsilon>0}\exists_{N\in\mathbb{N}}\forall_{n\geq N}\ \ \|a_n\vec{x}_n\| < \frac{\varepsilon}{M}\cdot M = \varepsilon,$$

so $\lim_{n\to\infty} a_n\vec{x}_n = \vec{0}$.

13. Show that a convergent sequence in a metric space is a Cauchy sequence.

SOLUTION: Let (S,d) be a metric space and assume that $\{x_n\}\subset S$ is a sequence convergent to the limit $a\in S$. Then, we have (by applying twice the convergence condition)

$$\forall_{\varepsilon>0}\exists_{N\in\mathbb{N}}\forall_{n\geq N}\ \ d(a,x_n) < \frac{\varepsilon}{2}$$
$$\forall_{\varepsilon>0}\exists_{N\in\mathbb{N}}\forall_{m\geq N}\ \ d(a,x_m) < \frac{\varepsilon}{2}$$

what clearly implies

$$\forall_{\varepsilon>0}\exists_{N\in\mathbb{N}}\forall_{n,m\geq N}\ \ d(x_n,x_m)\leq d(x_n,a)+d(a,x_n) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

The last condition means exactly that $\{x_n\}$ is a Cauchy sequence.

14. Let (X,d) be a metric space and $A\subset X$ a closed non-empty set. For a given point $x\in X$ we define the *distance from x to A* by the formula

$$d(x,A) := \inf\{d(x,a) : a\in A\}. \tag{13.24}$$

(a) Show that $x\in A \iff d(x,A)=0$;
(b) Show that the function $\chi_A : X\to\mathbb{R}$ defined by $\chi_A(x)=d(x,A)$ is continuous on X.

SOLUTION: (a): Suppose $d(x,A)=0$, then we have, by the definition of infimum

$$d(x,A)=0 \ \Leftrightarrow\ \begin{cases}\forall_{a\in A}\ 0\leq d(x,a)\\ \forall_{\varepsilon>0}\exists_{a'\in A}\ d(x,a')<\varepsilon\end{cases} \quad \Leftrightarrow\ \forall_{\varepsilon>0}\ B_\varepsilon(x)\cap A\neq\emptyset$$
$$\Leftrightarrow\ x\in\overline{A}\ \Rightarrow\ x\in A \text{ (since } A \text{ is closed.)}$$

(b): We will show the following lemma:

Lemma 13.1. *We have the following inequality*

$$\forall_{x,y\in X}\ |d(x,A)-d(y,A)|\leq d(x,y)$$

i.e. the function χ_A is Lipschitzian with constant $L=1$.

Proof: Let x,y be two arbitrary points in X. Then we have

$$d(x,A)=\inf\{d(x,a) : a\in A\}\ \Leftrightarrow (i)\ \forall_{a\in A}\ d(x,A)\leq d(x,a)$$
$$(ii)\ \forall_{\varepsilon>0}\ \exists_{a'\in A}\ d(x,A)+\varepsilon > d(x,a')$$

$$d(y,A) = \inf\{d(y,a) : a \in A\} \;\Leftrightarrow (iii)\; \forall_{a\in A}\; d(y,A) \le d(y,a)$$
$$(iv)\; \forall_{\varepsilon>0}\; \exists_{a''\in A}\; d(y,A) + \varepsilon > d(y,a'')$$

Then we have, by (i) and (iv) and the triangle inequality

$$\begin{aligned}\forall_{\varepsilon>0}\; d(x,A) - d(y,A) &< d(x,a'') - d(y,a'') + \varepsilon\\ &\le d(x,y) + d(y,a'') - d(y,a'') + \varepsilon\\ &= d(x,y) + \varepsilon.\end{aligned}$$

By interchanging x and y, the above inequality is true for all x, y, we obtain

$$\forall_{\varepsilon>0}\; d(y,A) - d(x,A) < d(x,y) + \varepsilon,$$

i.e.

$$\forall_{\varepsilon>0}\; -(d(x,y) + \varepsilon) < d(x,A) - d(y,A) < d(x,y) + \varepsilon,$$

so

$$\forall_{\varepsilon>0}\; |d(x,A) - d(y,A)| < d(x,y) + \varepsilon. \tag{13.25}$$

Since $\varepsilon > 0$ in (13.25) is arbitrary, by passing to the limit $\varepsilon \to 0^+$, we obtain

$$\forall_{\varepsilon>0}\; |d(x,A) - d(y,A)| \le d(x,y).$$

□

In order to conclude that χ_A is (uniformly) continuous, it is sufficient to refer to Question 6.

15. Prove or give a counterexample to the following statement: *Let $f : X \to Y$ be a continuous map between two metric spaces. Then for every open set $U \subset X$ the set $f(U)$ is open in Y.*

SOLUTION: Consider the following counter example:
Let $f : \mathbb{R} \to \mathbb{R}$ be a constant function (which is clearly continuous), $f(x) = 0$. Since $\mathbb{R} = U$ is open and $f(U) = \{0\}$ is not open, it is not true that the image of an open set under a continuous map is open.

16. Let (X, d) be a metric space and $f, g : X \to \mathbb{R}$ two continuous maps. Show that

$$\psi(x) := \max\{f(x), g(x)\} \quad \text{and} \quad \phi(x) := \min\{f(x), g(x)\},$$

are continuous on X.

SOLUTION: We have

$$f(x) \le \psi(x) \quad \text{and} \quad g(x) \le \psi(x).$$

By assumption f and g are continuous at x, thus

$$\begin{aligned}
\forall_{\varepsilon>0}\,\exists_{\delta>0}\,\forall_{x'\in X}\; d(x,x')<\delta \;&\Rightarrow\; |f(x)-f(x')|<\varepsilon \,\wedge\, |g(x)-g(x')|<\varepsilon\\
&\Leftrightarrow\; \begin{cases} f(x')-\varepsilon<f(x)<f(x')+\varepsilon\\ g(x')-\varepsilon<g(x)<g(x')+\varepsilon\end{cases}\\
&\Rightarrow\; \max(f(x'),g(x'))-\varepsilon<\max(f(x),g(x))<\max(f(x'),g(x'))+\varepsilon\\
&\Leftrightarrow\; |\psi(x)-\psi(x')|<\varepsilon,
\end{aligned}$$

so

$$\forall_{\varepsilon>0}\,\exists_{\delta>0}\,\forall_{x'\in X}\; d(x,x')<\delta \;\Rightarrow\; |\psi(x)-\psi(x')|<\varepsilon.$$

Since $\phi(x)=\min\{f(x),g(x)\}=-\max\{-f(x),-g(x)\}$, the continuity of $\phi(x)$ follows from the previous case.

17. Apply the definition of continuity to show that the function $f:[0,\infty)\to\mathbb{R}$, $f(x)=\sqrt{x}$ is continuous.

SOLUTION: We will show that the function $f:[0,\infty)\to\mathbb{R}$, $f(x)=\sqrt{x}$ is continuous. We will consider two cases:

Case (i) $x=0$:

$$\forall_{\varepsilon>0}\,\exists_{\delta=\varepsilon^2}\,\forall_{x'>0}\; x'<\delta \;\Rightarrow\; \sqrt{x'}<\sqrt{\delta}=\sqrt{\varepsilon^2}=\varepsilon.$$

Case (ii) $x>0$:

$$\forall_{\varepsilon>0}\,\exists_{\delta=\varepsilon\sqrt{x}}\,\forall_{x'\in[0,\infty)}\; |x-x'|<\delta \;\Rightarrow\; |\sqrt{x}-\sqrt{x'}|=\frac{|x-x'|}{\sqrt{x'}+\sqrt{x}}<\frac{|x-x'|}{\sqrt{x}}<\frac{\delta}{\sqrt{x}}=\varepsilon.$$

Therefore, f is continuous.

18. Give an example of a subspace X in $\mathbb{R}$ and a Cauchy sequence $\{x_n\}$ in X such that $\{x_n\}$ is not convergent. Verify the statement by proving it or giving a counterexample: *If $f:X\to\mathbb{R}$ is continuous, then for every Cauchy sequence $\{x_n\}$ in X the sequence $\{f(x_n)\}$ is Cauchy in $\mathbb{R}$.*

SOLUTION: Let $X=(0,\infty)$, $Y=\mathbb{R}$ and $f:X\to Y$ be $f(x)=\frac{1}{x}$, $x>0$.

Consider the sequence $\{x_n\}$, where $x_n=\frac{1}{n}\in X$, $n\in\mathbb{N}$. Since x_n converges to 0 in $\mathbb{R}$, it is therefore a Cauchy sequence. On the other hand, $0\notin X$, thus $\{x_n\}$ is a Cauchy sequence in X, but it is not convergent. Notice that $f(x_n)=\frac{1}{x_n}=n\to\infty$, thus the sequence $\{f(x_n)\}$ is not Cauchy.

19. Let X and Y be two metric spaces. We say that a function $f:X\to Y$ is *Lipschitzian* with a constant $L\geq 0$ if

$$\forall_{x,y\in X}\;\; d(f(x),f(y))\leq L\,d(x,y).$$

Show that every Lipschitzian function is uniformly continuous.

SOLUTION: Notice that if $L=0$, then f is constant function, thus it is clearly continuous. Assume therefore that $L>0$, we then have

$$\forall_{\varepsilon>0}\ \exists_{\delta=\frac{\varepsilon}{L}>0}\ \forall_{x,y\in X}\ d(x,y)<\delta\ \Rightarrow\ d(f(x),f(y))\le Ld(x,y)<L\delta=\varepsilon.$$

20. Check which of the following functions are uniformly continuous on the indicated sets:

(a) $f:(0,\infty)\to\mathbb{R},\ f(x)=\frac{1}{x}$;
(b) $f:\mathbb{R}\to\mathbb{R},\ f(x)=\frac{x}{|x|+1}$;
(c) $f:\mathbb{R}\to\mathbb{R},\ f(x)=x^2$;
(d) $f:[-1,1]\to\mathbb{R},\ f(x)=x^2$.

SOLUTION: (a) f is not uniformly continuos, i.e.

$$\exists_{\varepsilon>0}\ \forall_{\delta>0}\ \exists_{x,x'\in X}\ d(x,x')<\delta\ \Rightarrow\ |f(x)-f(x')|\ge\varepsilon.$$

Take $x_n=\frac{1}{n+1}$, $x'_n=\frac{1}{n}$, then

$$|x_n-x'_n|=\frac{1}{n(n+1)}<\frac{1}{n}$$

and $|f(x_n)-f(x'_n)|=n+1-n=1=:\varepsilon$.

Consequently,

$$\exists_{\varepsilon=1}\ \forall_{\delta>0} \text{ we choose } n \text{ such that } \frac{1}{n}<\delta, \text{ so} \exists_{x_n,x'_n\in X}\ |x,x'|<\frac{1}{n}<\delta\ \wedge\ |f(x_n)-f(x'_n)|\ge 1=\varepsilon.$$

(b) We need the following lemma:

Lemma 13.2. *If f is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that f is uniformly continuous both on $[0,\infty)$ and $(-\infty,0]$, then f is uniformly continuous on $\mathbb{R}$.*

Proof: Indeed,

$$\forall_{\varepsilon>0}\ \exists_{\delta_1>0}\ \forall_{x,x'\ge 0}\ |x-x'|<\delta_1\ \Rightarrow\ |f(x)-f(x')|<\frac{\varepsilon}{2}, \tag{13.26}$$

$$\forall_{\varepsilon>0}\ \exists_{\delta_2>0}\ \forall_{x,x'\le 0}\ |x-x'|<\delta_2\ \Rightarrow\ |f(x)-f(x')|<\frac{\varepsilon}{2}. \tag{13.27}$$

Then we have

$$\forall_{\varepsilon>0}\ \exists_{\delta=\min\{\delta_1,\delta_2\}>0}\ \forall_{x,x'}\ |x-x'|<\delta\ \Rightarrow$$

$$\begin{cases} x\cdot x'\ge 0, \text{ by (13.26)-(13.27) } |f(x)-f(x')|<\frac{\varepsilon}{2}<\varepsilon \\ x\cdot x'<0\ |f(x)-f(x')|\le|f(x)-f(0)|+|f(x')-f(0)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \text{ since } |x|,|x'|\le|x-x'|<\delta. \end{cases}$$

□

We will show that $f(x)=\frac{x}{|x|+1}$ is uniformly continuous on $[0,\infty)$ and $(-\infty,0]$.

Indeed, we have

$$\forall_{\varepsilon>0}\ \exists_{\delta=\varepsilon}\ \forall_{x,x'\ge 0 \text{ or } x,x'\le 0}\ |x-x'|<\delta\ \Rightarrow$$

$$|f(x)-f(x')|\le\left|\frac{x}{|x|+1}-\frac{x'}{|x'|+1}\right|=\frac{|x-x'|}{(|x|+1)(|x'|+1)}<|x-x'|<\delta=\varepsilon.$$

Thus by the above Lemma 2, we have the statement.

Remark: If $X = X_1 \cup X_2$ and $f : X \to Y$ is a continuous function such that f is uniformly continuous both on X_1 and X_2, it does NOT necessarily imply (in general) that f is uniformly continuous on X.

Counter example, $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \frac{x}{|x|}$.

Indeed,

$$\exists_{\varepsilon=2} \, \forall_{\delta>0} \, \exists_{x=-\frac{\delta}{4}, x'=-\frac{\delta}{4}} \, |x - x'| < \delta \, \wedge \, |f(x) - f(x')| \geq \varepsilon.$$

(c) $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. Indeed,

$$\exists_{\varepsilon=2} \, \forall_{\delta>0} \, \exists_{x=n+\frac{1}{n}, x'=n, n<\delta} \, |x - x'| < \frac{1}{n} < \delta \, \wedge \, |f(x) - f(x')| = |(n + \frac{1}{n})^2 - n^2| = 2 + \frac{1}{n^2} \geq 2 = \varepsilon.$$

(d) Since $[-1, 1]$ is compact and $f(x) = x^2$ is continuous, then by Cantor-Heine Theorem, f is uniformly continuous.

21. Show that a function $f : X \to Y$ between two metric spaces is continuous if and only if for every closed set C in Y, the set $f^{-1}(C)$ is closed in X.

SOLUTION: First we notice that

$$f^{-1}(A^c) = [f^{-1}(A)]^c.$$

Indeed

$$\begin{aligned} x \in f^{-1}(A^c) \; &\Leftrightarrow \; f(x) \in A^c \; \Leftrightarrow \; f(x) \notin A \\ &\Leftrightarrow \; x \notin f^{-1}(A) \; \Leftrightarrow \; x \in [f^{-1}(A)]^c. \end{aligned}$$

Since f is continuous if and only if $f^{-1}(U)$ is open for any open U in Y, we have for every closed set C, C^c is open, so $f^{-1}(C^c)$ is open, thus

$$[f^{-1}(C^c)]^c = [[f^{-1}(C)]^c]^c = f^{-1}(C)$$

is closed.

22. Which of the following sets in $\mathbb{R}^2$ (with the Euclidean metric) are compact (explain why!)

(a) $A := \{(x, y) : x^2 - y^2 \leq 1\}$
(b) $B := \left\{\left(\frac{1}{n}, 1\right) : n \in \mathbb{N}\right\} \cup \{(0, 1)\}$
(c) $C := \{(x, y) : 0 < x^2 + y^2 \leq 1\}$;
(d) $D := \{(x, y) : x^2 + y^4 \leq 1\}$.

SOLUTION: By Heine-Borel, $M \subset \mathbb{R}^2$ is compact if and only if it is closed and bounded.

(a) Since $(0, n) \in A$, for all $n \in \mathbb{N}$, A is not bounded. So A is not compact.
(b) The set B is bounded and its only limit point is $(0, 1)$, which belongs to B, i.e. B is compact.
(c) C is not closed, since $x_n = (\frac{1}{n}, 0) \in C$, but $\lim_n x_n = (0, 0) \notin C$. So C is not compact.

(d) D is closed and bounded ($D \subset \overline{B_2(0,0)}$), thus it is compact.

23. Show that the union of two compact sets is compact.

SOLUTION: Assume $C = C_1 \cup C_2$, where C_1 and C_2 are two compact sets. We can use the fact that a set is compact if and only if any sequence in this set contains a convergent subsequence to a limit point which belongs to this set.

Let $\{x_n\} \subset C$ be an arbitrary sequence. Since $C = C_1 \cup C_2$, one of the sets C_1 or C_2 (say C_1) must contain an infinite number of elements of $\{x_n\}$, i.e. there exists a subsequence $\{x_{n_k}\} \subset C_1$. Then by compactness of C_1, it contains a convergent subsequence $\{x_{n_{k_l}}\}$, to a limit $a = \lim_l x_{n_{k_l}} \in C_1$. Thus the sequence $\{x_n\}$ contains a convergent subsequence $\{x_{n_{k_l}}\}$ to a limit $a \in C_1 \subset C_1 \cup C_2 = C$. Therefore, C is compact.

24. Let (S, d) be a metric space. Show that

(a) the intersection of any number of compact sets in S is compact;
(b) the union of any finite number of compact sets is compact;
(c) the union of infinitely many compact sets may not be compact (even if it is bounded).

SOLUTION: Recall one of the properties of compact sets proved in class:

Proposition: *Let (S, d) be a metric space, B a closed subset of S, and A a compact subset of S such that $B \subset A$. Then B is also compact.*

(a): Since every compact set is closed and the intersection of closed sets is closed, it follows that the intersection $\bigcap_{i\in I} C_i$ of compact sets C_i, $i \in I$, is a closed set, which is contained in every compact set C_i, thus the conclusion that $\bigcap_{i\in I} C_i$ is compact follows from Proposition.

(b): Let $A = C_1 \cup \cdots \cup C_N$ be a the union of the compact sets $C_1, \ldots, C_N$. Then $\forall_{\{x_n\}\subset A}$ at least one of the sets C_j, $j \in \{1, 2, \ldots, N\}$ contains infinitely many element of the sequence $\{x_n\}$, which means, there exists a subsequence $\{x_{k_n}\} \subset \{x_n\}$ such that $\{x_{k_n}\} \subset C_j$ for some j. But C_j is assumed to be compact, thus there exists a subsequence $\{x_{k_{l_n}}\}$ of $\{x_{k_n}\}$ convergent to a point x_o in C_j. Clearly, $\{x_{k_{l_n}}\}$ is a subsequence of $\{x_n\}$ and since $x_o \in C_j \subset A$, it follows that A is compact.

(c): Consider the space $S = \mathbb{R}$ equipped with the usual metric $d(x, y) = |x - y|$. Since a set in an Euclidean space (for example in $\mathbb{R}$) is compact if and only if it is bounded and closed, it follows immediately that the sets $C_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ are compact for all $n \in \mathbb{N}$, while the set $B = (0, 1)$ is not compact but still bounded. Notice that

$$\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n}\right] = (0, 1) = B.$$

25. Let A be a compact set in a metric space (S, d). Show that

(a) for any closed set $B \subset S$ the intersection $A \cap B$ is compact;
(b) ∂A is compact;
(c) any finite set is compact.

SOLUTION:

(a): Again, the set $A \cap B$ is a closed set (remember every compact set is closed) and since $A \cap B \subset A$, thus by Proposition it follows that $A \cap B$ is compact.

(b): Since the boundary ∂A satisfies the property

$$\partial A = \overline{A} \cap \overline{A^c},$$

where A^c denotes the complement of A, thus again it is an intersection of two closed sets such that $\partial A \subset \overline{A} = A$ (remember A is compact, thus it is closed), so, by Proposition, ∂A is compact.

(c): Notice that a space consisting of a single point is compact. Indeed, in such a space every sequence is constant, so it is also convergent. Since a finite set is a finite union of sets consisting of ecaxtly one point, which are compact, the statement follows from Problem 7(b).

26. Consider the metric space (S, d_h), where $S = \mathbb{R}^2$ and $d_h : S \times S \to \mathbb{R}$ is defined in Problem 2. Verify if in this metric space closed balls are always compact. Present examples and justify them.

SOLUTION: Notice that the closed ball $B := \overline{B}_h((0,3), 1$ consists exactly of a segment in $\searrow^2$ joining the points $(0,2)$ and $(0,4)$. Since the distance in B coincides with the usual Euclidean distance (notice that $d_h((0,x),(0,y)) = d_2((0,x),(0,y)))$, and B is compact with respect to the Euclidean metric (it is closed and bounded), thus it is also compact with respect to d_h.

On the other hand, the closed ball $A = \overline{B}((0,0), 2)$, is not compact. Choose the sequence $\{\frac{1}{n}, 1)\} \subset A$ contains no convergent subsequence; Indeed, for $m \neq n$ we have

$$d_h((\frac{1}{n}, 1), (\frac{1}{m}, 1)) = |\frac{1}{n} - \frac{1}{m}| + 1 + 1 > 1.$$

Therefore the answer to this questions is that some of the closed balls in the space (S, d_h) are compact (if their interior does not intersect the x-axis) and some are not compact (if their interior intersects the x-axis).

27. Let $f : [a, b) \to \mathbb{R}$ be a bounded continuous function such that f is one-to-one. Show that the limit $\lim_{x \to b} f(x)$ exists.

SOLUTION: You can find the following Theorem in the notes:

Theorem 13.3. *Consider the interval* $[a, b)$ *and let* $f : [a, b) \to \mathbb{R}$ *be a continuous function. If* $f(x)$ *is a one-to-one function, then it is strictly monotone.*

Proof: We will show that $f(x)$ is monotonic on every closed subinterval $[a, b'] \subset [a, b)$. Let $[a, b']$ be such interval. Suppose that $f(a) < f(b')$ (the proof for the other case $f(b') > f(a)$ is the same).

We will show that $f(x)$ has to be an increasing function. Indeed, suppose that contrary to our claim, the function $f(x)$ is not increasing. Then there exist c, $d \in [a, b']$ such that $c < d$ and $f(c) > f(d)$. If $f(a) > f(c)$ then $f(d) < f(b')$. Put $\alpha_1 = \min(f(c), f(b'))$. By the Intermediate Value Theorem, for every β such that $f(d) < \beta < \alpha_1$ there exist $x_1 \in (c, d)$ and $x_2 \in (d, b')$ such that $f(x_1) = f(x_2) = \beta$. But this is a contradiction with the assumption that $f(x)$ is one-to-one. Suppose therefore, that $f(a) < f(c)$. Let $\alpha_2 = \max(f(a), f(d))$. Again, by the Intermediate Value Theorem, for every β such that $\alpha_2 < \beta < f(c)$ there exist $x_1 \in (a, c)$ and $x_2 \in (c, d)$ such that $f(x_1) = f(x_2) = \beta$, and again we obtain the contradiction. Consequently, the function $f(x)$ must be increasing. ☐ Notice that, by Theorem 13.2, the function $f(x)$ is either increasing or decreasing. Assume for example that it is increasing. Then by the assumption, it is bounded, so the set $A = f([a,))$ is also bounded. By LUB, there exists

$$\alpha := \sup\{f(x) : x \in [a, b)\} \iff \begin{cases} (i) \ \forall_{x\in[a,b)} \ f(x) \le \alpha, \\ (ii) \ \forall_{\varepsilon>0} \ \exists_{x'\in[a,b)} \ \alpha - \varepsilon < f(x'). \end{cases}$$

Therefore, we obtain (by (ii) and the fact that $f(x)$ is increasing)

$$\begin{aligned} \forall_{\varepsilon>0} \ \exists_{\delta=b-x'} \ \forall_{x\in[a,b)} \ 0 < |x - b| < \delta \ &\Rightarrow \ 0 < b - x < b - x' = \delta \\ &\Rightarrow \ b > x > x' \ \Rightarrow \ f(x) > f(x') \ \Rightarrow \ \alpha - \varepsilon < f(x') < f(x) \\ &\Rightarrow \ \alpha - f(x) < 0 \ \overset{\text{by (i)}}{\Rightarrow} \ |\alpha - f(x)| < \varepsilon. \end{aligned}$$

In other words

$$\forall_{\varepsilon>0} \ \exists_{\delta=b-x'} \ \forall_{x\in[a,b)} \ 0 < b - x < \delta \ \Rightarrow \ |\alpha - f(x)| < \varepsilon \iff \lim_{x\to b} f(x) = \alpha.$$

28. Let $\{a_n\} \subset \mathbb{R}$ be an increasing bounded sequence. Show that

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$

SOLUTION: Since the sequence $\{a_n\}$ is increasing, we have $a_n < a_m$ for $n < m$, and $\alpha = \sup\{a_n : n \in \mathbb{N}\}$ implies that (i) $\forall_{n\in\mathbb{N}} \ a_n \le \alpha$, and (ii) $\forall_{\varepsilon>0}\exists_{m\in\mathbb{N}} \ \alpha - \varepsilon < a_m$. Since $0 \le \alpha - a_m < \alpha - a_n$ for $n \ge m$, consequently we get

$$\forall_{\varepsilon>0}\exists_{m\in\mathbb{N}}\forall_{n\ge m} \ 0 \le \alpha - \varepsilon = |\alpha - a_n| < \varepsilon.$$

Therefore,

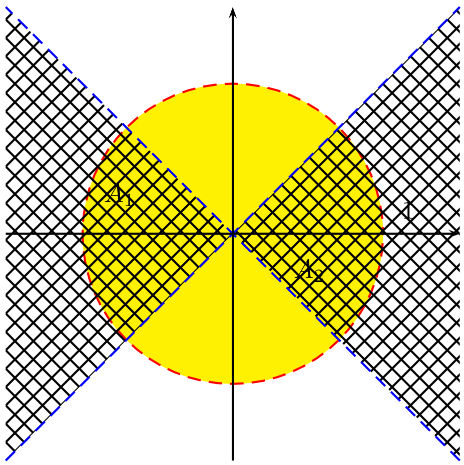
$$\lim_{n\to\infty} a_n = \alpha = \sup\{a_n : n \in \mathbb{N}\}.$$

29. Sketch the following sets in $\mathbb{R}^2$ and decide whether they are connected or not. Justify your answer.

(a) $A := \{(x, y) : x^2 + y^2 < 1 \ \wedge \ |y| < |x|\}$;
(b) $B := \{(x, y) : xy > 1\}$;

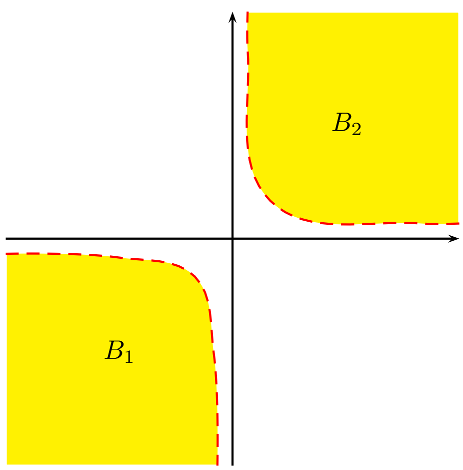
(c) $C := \{(x, y) : \dfrac{x^2}{4} + \dfrac{y^2}{9} < 1 \quad \wedge \quad x^2 + y^2 > 1\}$.

SOLUTIONS: (a): $A := \{(x, y) : x^2 + y^2 < 1 \quad \wedge \quad |y| < |x|\}$;



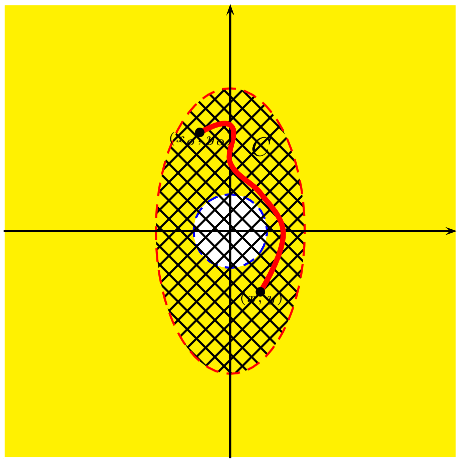
Let $A_1 := \{(x, y) : x^2+y^2 < 1 \ \wedge \ |y| < x\}$ and $A_2 := \{(x, y) : x^2+y^2 < 1 \ \wedge \ |y| < -x\}$. Then $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. Notice that in fact $\overline{A_1} \cap A_2 = \emptyset$ and $\overline{A_2} \cap A_1 = \emptyset$. Indeed, $\overline{A_1} \cap \overline{A_2} = \{(0, 0)\}$ but $(0, 0) \notin A_1$ and $(0, 0) \notin A_2$. Since A_1 and A_2 are not empty, that means the set A is **not connected**, because it can be represented as a union of two disjoint non-empty subsets A_1 and A_2 such that A_1 does not contain any limit point of A_2, neither A_2 contains any limit point of A_1.

(b): $B := \{(x, y) : xy > 1\}$;



Let $B_1 := \{(x, y) : xy < 1 \quad \wedge \quad x, y < 0\}$ and $B_2 := \{(x, y) : xy < 1 \quad \wedge \quad x, y > 0\}$. Then $B = B_1 \cup B_2$. Notice that B_1 and B_2 are nonempty and satisfy $\overline{B_1} \cap \overline{B_2} = \emptyset$.Since B_1 does not contain any limit point of B_2, neither B_2 contains any limit point of B_1, the set B is **not connected**.

(c): $C := \{(x,y) : \dfrac{x^2}{4} + \dfrac{y^2}{9} < 1 \;\; \wedge \;\; x^2 + y^2 > 1\}$.



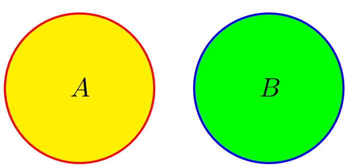
It is clear that any two points (x,y) and (x_o, y_o) in C can be connected by a path, therefore C is path connected, so it is also connected.

30. Give examples of sets A and B in $\mathbb{R}^2$ which satisfy the following properties:

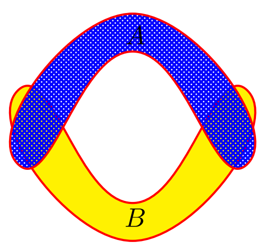
(a) The sets A and B are connected but $A \cup B$ is not connected.
(b) The sets A and B are connected but $A \cap B$ is not connected.
(c) The sets A and B are not connected but $A \cup B$ is connected.
(d) The sets A and B are not connected but $A \cap B$ is connected.
(e) The sets A and B are connected but $A \setminus B$ is not connected.

SOLUTIONS:

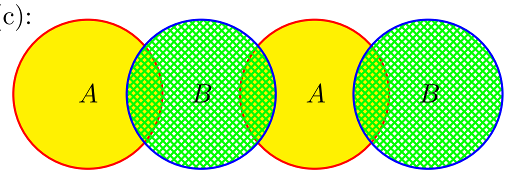
(a):



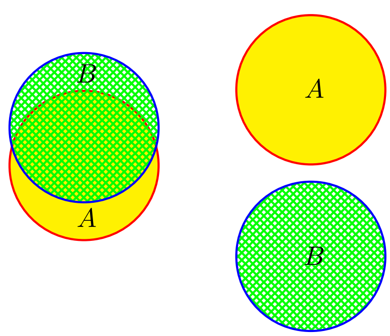
(b):



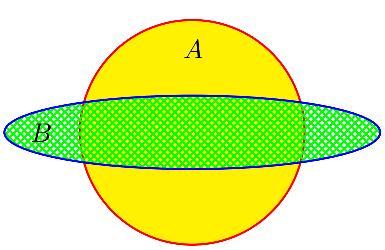
(c):



(d)



(e):

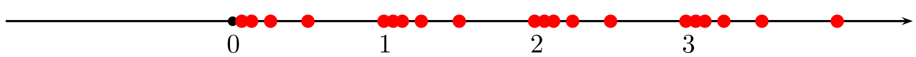


31. Give an example of a set in $\mathbb{R}^1$ with exactly four limit points.

SOLUTION: Inspired by the last assignment (the set E), we define

$$A := \left\{ k + \frac{1}{n} : k \in \{0, 1, 2, 3\},\ n \in \mathbb{N} \right\}.$$

Since for every $k \in \{0, 1, 2, 3\}$ the sequence $x_n := k + \frac{1}{n}$ converges to k as $n \to \infty$ (clearly $\{x_n\} \subset\in A$), thus k is a limit point of A. Notice that there are no other limit points of A (see the picture below).

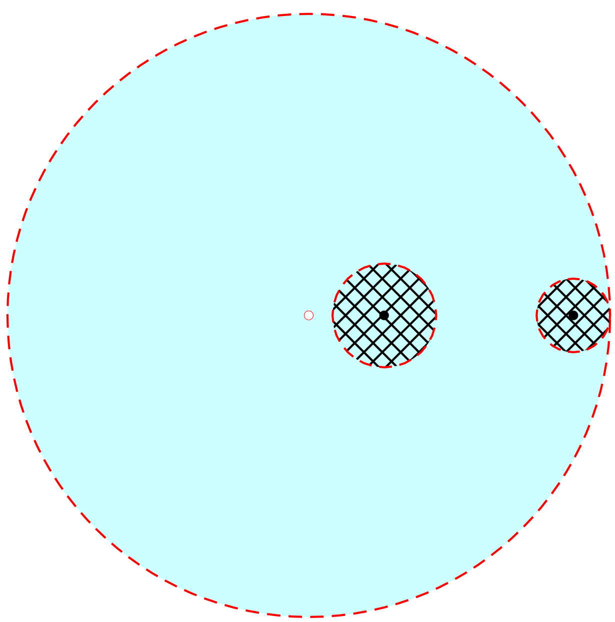


32. Given $\mathbb{R}^2$ with the Euclidean metric. Show that the set

$$S = \left\{ (x, y) : 0 < x^2 + y^2 < 1 \right\}$$

is open. Describe the sets $S^{(0)}$, S', ∂S, S^c.

SOLUTION: Let us illustrate the set S.



We will show that

$$\begin{aligned}
S^{(0)} &= \{(x, y) : 0 < x^2 + y^2 < 1\} = S,\\
S' &= \{(x, y) : x^2 + y^2 \le 1\} = \overline{B(0, 1)},\\
\partial S &= \{(0, 0)\} \cup \{(x, y) : x^2 + y^2 = 1\},\\
S^c &= \{(x, y) : x^2 + y^2 \ge 1\} \cup \{(0, 0)\}.
\end{aligned}$$

Indeed, for every $(x,y) \in S$, we put $\varepsilon = \min\{1-\sqrt{x^2+y^2}, \sqrt{x^2+y^2}\}$. Then it is clear that

$$B((x,y),\varepsilon) \subset S.$$

Indeed, put $v=(x,y)$ and $v'=(x',y')$. We have for $v' \in\in B((x,y),\varepsilon)$ that $\|v-v'\| < \varepsilon \le 1-\|v\|$ and $\|v-v'\| < \varepsilon < \|v\|$. Therefore we have

$$\|v\| - \|v'\| \le \|v-v'\| < \|v\| \quad \iff \quad \|v'\| > 0,$$

and

$$\|v'\| = \|v'-v+v\| \le \|v\| + \|v-v'\| < \|v\| + \varepsilon \le \|v\| + 1 - \|v\| = 1.$$

Consequently, $S^{(0)} = S$. Notice that every point $v \in \overline{B(0,1)}$ is a limit point of S. Indeed, if $1 \ge \|v\| > 0$, then the sequence $v_n = \left(1-\frac{1}{n}\right) v$ satisfies $\|v_n\| = \left(1-\frac{1}{n}\right)\|v\| \le 1-\frac{1}{n} < 1$, i.e. $v_n \in S$, i.e. v is a limit point of S, and for $v=\overline{0}$, the sequence $\left(\frac{1}{n},\frac{1}{n}\right) \in S$, $n=2,3,\dots$, converges to $(0,0)$. Therefore,

$$\partial S = \overline{S} \setminus S^{(0)} = (S \cup S') \setminus S = S' \setminus S,$$

which implies that

$$\partial S = \overline{B(0,1)} \setminus S = \{0,0\} \cup \{(x,y) : x^2+y^2 = 1\}.$$

Finally, the set S^c (by definition of S) is given by

$$S^c = \{(x,y) : x^2+y^2 \ge 1\} \cup \{(0,0)\}.$$

33. In $\mathbb{R}^2$ with the Euclidean metric, find an infinite collection of open sets $\{A_n\}$ such that $\bigcap\limits_n A_n$ is the closed ball $\overline{B(0,1)}$.

SOLUTION: For every $n \in \mathbb{N}$ we define

$$A_n := B\left(0, 1+\frac{1}{n}\right).$$

Since A_n is an open ball in $\mathbb{R}^2$ of radius $1+\frac{1}{n}$ centered at zero, it is an open set. In addition, we have for all n

$$\overline{B(0,1)} \subset A_n,$$

thus

$$\overline{B(0,1)} \subset \bigcap_{n=1}^{\infty} A_n.$$

We will show that $\bigcap_{n=1}^{\infty} A_n \subset \overline{B(0,1)}$. Assume, for contradiction, that $\bigcap_{n=1}^{\infty} A_n \not\subset \overline{B(0,1)}$, i.e. there is an element $x \in \bigcap_{n=1}^{\infty} A_n$ such that $x \notin \overline{B(0,1)}$, i.e. $\|x\| > 1$. Since $\|x\|-1 > 0$, by Archimedes axiom, there exists $n_o \in \mathbb{N}$ such that $0 < \frac{1}{n_o} < \|x\| - 1$. Consequently, $\|x\| > 1 + \frac{1}{n_o}$, i.e. $x \notin A_{n_o}$, thus $x \notin \bigcap_{n=1}^{\infty} A_n$. But, this is a contradiction with the assumption that $x \in \bigcap_{n=1}^{\infty} A_n$. Therefore,

$$\bigcap_{n=1}^{\infty} A_n \subset \overline{B(0,1)},$$

and we obtain

$$\bigcap_{n=1}^{\infty} A_n = \overline{B(0,1)}.$$

34. In $\mathbb{R}^1$ with the Euclidean metric, give an example of a sequence $\{p_n\}$ with two subsequences converging to different limits. Give an example of a sequence which has infinitely many subsequences converging to different limits.

SOLUTION: Take $x_n = (-1)^n$, then the subsequence $x_{2n} = 1$ converges (because it is constant) to 1, and $x_{2n+1} = -1$ converges (it is also constant) to -1.

In order to construct an example with infinitely many subsequences converging to different limits, we arrange all the positive fractions into a "infinite" matrix as follows:

$$\begin{array}{ccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \cdots \\
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \frac{5}{6} & \cdots \\
\frac{6}{1} & \frac{6}{2} & \frac{6}{3} & \frac{6}{4} & \frac{6}{5} & \frac{6}{6} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array} \tag{13.28}$$

It is possible to arrange all the fractions in the above infinite matrix into a sequence. For this purpose, we consider the fraction $\frac{1}{1}$ to be the first element in this sequence, then we go to the next diagonal (marked in green), and put the fractions $\frac{2}{1}$ and $\frac{1}{2}$ as the subsequent elements in this sequence. Then we go to the next diagonal, and put $\frac{3}{1}$, $\frac{2}{2}$, and $\frac{1}{3}$. In this way, by moving from one diagonal to another, we obtain the sequence

$$\left\{\frac{1}{1}, \underbrace{\frac{2}{1}, \frac{1}{2}}, \underbrace{\frac{3}{1}, \frac{2}{2}, \frac{1}{3}}, \underbrace{\frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}}, \underbrace{\frac{5}{1}, \frac{4}{2}, \frac{3}{3}, \frac{2}{4}, \frac{1}{5}}, \underbrace{\frac{6}{1}, \frac{5}{2}, \frac{4}{3}, \frac{3}{4}, \frac{2}{5}, \frac{1}{6}}, \dots\right\}$$

It is not hard to notice that each fraction $\frac{m}{n}$ from the array (13.28) will appear in this sequence. In fact one can check that it will be the k-th element x_k in this sequence, where

$$k = \sum_{i=1}^{m+n-1} i + n = \frac{(m+n)(m+n-1)}{2} + n. \tag{13.29}$$

Now, for a given fraction $\frac{m}{n}$, appearing as the element x_k, with k given by (13.29), the fractions $\frac{ml}{nl}$ have the same value as $\frac{m}{n}$, but they appear as the elements x_{k_l}, with k_l given by

$$k_l = \frac{(lm+ln)(lm+ln-1)}{2} + ln.$$

Since $k_l \to \infty$ as $l \to \infty$, we obtain a subsequence $\{x_{k_l}\}$ of $\{x_k\}$ such that $x_{k_l} = \frac{ml}{nl} = \frac{m}{n}$, i.e. it is a constant subsequence. Consequently, $x_{k_l} \to \frac{m}{n}$ as $l \to \infty$. Since such a subsequence can be constructed for any fraction $\frac{m}{n}$, we conclude that the sequence $\{x_k\}$ satisfies the requirements specified in this question.

13.3 Chapter 5: Real Functions

1. Show that the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \frac{1}{1+x^2}$ is uniformly continuous.
HINT: Show that

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{x,y\in\mathbb{R}} \;\; |x-y| < \delta \implies \left| \frac{1}{1+x^2} - \frac{1}{1+y^2} \right| < \varepsilon.$$

SOLUTION: Notice that

$$\forall_{t\in\mathbb{R}} \quad \frac{|t|}{1+t^2} \le 1. \tag{1}$$

Indeed, if $|t| \le 1$ then clearly the inequality (1) is true. Assume that $|t| > 1$. Then we have $t^2+1 > t^2 > |t| > 0$, so $\frac{|t|}{1+t^2} < 1$.

Notice that $\forall_{\varepsilon>0}$ we can take $\delta = \frac{1}{2}\varepsilon > 0$ (which means that by indicating the exact value of δ we show that it definitely exists), so $\forall_{x,y\in\mathbb{R}}$ $|x-y| < \delta \implies$ (so, we get)

$$\begin{aligned}\left|\frac{1}{1+x^2} - \frac{1}{1+y^2}\right| &= \frac{|x^2-y^2|}{(1+x^2)(1+y^2)} = |x-y|\frac{|x+y|}{(1+x^2)(1+y^2)}\\
&\le |x-y|\frac{|x|+|y|}{(1+x^2)(1+y^2)}\\
&= |x-y|\left[\frac{|x|}{(1+x^2)(1+y^2)} + \frac{|y|}{(1+x^2)(1+y^2)}\right]\\
&\le |x-y|\left[\frac{|x|}{1+x^2} + \frac{|y|}{1+y^2}\right] \le |x-y|(1+1) \quad \text{by (1)}\\
&< 2\cdot\delta = 2\cdot\frac{1}{2}\varepsilon = \varepsilon.\end{aligned}$$

□

2. Let $b > a$ be two real numbers and $f : (a,b) \to \mathbb{R}$ be a uniformly continuous function. Show that the set of values of $f(x)$ is bounded, i.e. $\exists_{N>0}\forall_{x\in(a,b)}$ $|f(x)| < N$.

SOLUTION: Since $f(x)$ is uniformely continuous, we have

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{x,y\in(a,b)} \quad |x-y|<\delta \implies |f(x)-f(y)|<\varepsilon. \tag{2}$$

In particular, by taking $\varepsilon = 1$, we get by (2)

$$\exists_{\delta>0}\forall_{x,y\in(a,b)} \quad |x-y|<\delta \implies |f(x)-f(y)|<1. \tag{3}$$

By Archimedes, there exists $n \in \mathbb{N}$ such that $n > \frac{b-a}{\delta}$, which means $\frac{b-1}{n} < \delta$. Let's divide the interval (a,b) into n equal subintervals: (a_{k-1}, a_k), where $a_k = a+k\frac{b-a}{n}$, for $k = 0, 1, \dots, n$. Notice that $a_0 = a$ and $a_n = b$. Since for every $x \in (a, a_2]$, $|x - a_1| \le \frac{b-a}{n} < \delta$, it follows by (3) that

$$|f(x)| - |f(a_1)| \le |f(x) - f(a_1)| < 1 \implies |f(x)| < |f(a_1)| + 1, \tag{4}$$

and in particular for $x = a_2$ we get

$$|f(a_2)| < |f(a_1)| + 1. \tag{5}$$

Next, $\forall_{x\in(a_2,a_3]}$, we have $|x-a_2|| \le \frac{b-a}{n} < \delta$, thus by (3) we get

$$|f(x)| - |f(a_2)| \le |f(x) - f(a_2)| < 1 \implies |f(x)| < |f(a_2)| + 1,$$

so, by (5) we get

$$|f(x)| < |f(a_1)| + 2.$$

In particular, for $x = a_3$, we get

$$|f(a_3)| < |f(a_1)| + 2.$$

Next, we procede by mathematical induction. Assume that for all $k = 2, 3, .., m$, where $m < n$, we have

$$\forall_{x\in(a_{k-1}-a_k]}\ \ |f(x)| < |f(a_1)| + (k-1), \tag{6}$$

in particular

$$|f(a_k)| < |f(a_1)| + (k-1), \tag{7}$$

thus $\forall_{x\in(a_k,a_{k+1}]}$ (if $a_k = b$, we should replace this condition by $\forall_{x\in(a_k,a_{k+1})}$ – remember the function $f(x)$ is defined only on the open interval (a, b)) we have

$$|f(x)| - |f(a_k)| \le |f(x) - f(a_k)| < 1 \implies |f(x)| < |f(a_k)| + 1, \tag{8}$$

so (8) and (7) imply

$$|f(x)| < |f(a_k) + 1 < |f(a_1)| + (k-1) + 1 = |f(a_1)| + k.$$

Since $|f(a_1)| + 1 < |f(a_1)| + 2 < \dots < |f(a_1)| + n$, we get that

$$\forall_{x\in(a,b)}\ \ |f(x)| < |f(a_1)| + n =: N,$$

Thus $f(x)$ is bounded. □

3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$. Show that the maximal value of $f(x)$ is 1, while there is no minimal value of $f(x)$. **HINT:** Show that $f(x) > 0$ and $\inf f(x) = 0$.

SOLUTION: It is clear that $\forall_{x\in\mathbb{R}}\ \ 1 + x^2 \ge 1 > 0$, thus (by the properties of an ordered field) $\forall_{x\in\mathbb{R}}\ \ f(x) = \frac{1}{1+x^2} > 0$. In particular, that means that 0 is a lower bound for the values of the function $f(x)$. We will show that $\inf f(x) = 0$. For thus purpose, we need to prove that

$$\forall_{\varepsilon>0}\exists_{x\in\mathbb{R}}\ \ \varepsilon > \frac{1}{1+x^2}.$$

Indeed, by Archimedes, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$. On the other hand, by taking $x := n$ (so it exists) we get $1 + x^2 = 1 + n^2 > n^2 \ge n > \frac{1}{\varepsilon}$ thus

$$\varepsilon > \frac{1}{1+x^2}.$$

□

4. Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{3}\sin(2x + 1)$ is a contraction with constant $k = \frac{2}{3}$. Show that the equation $x = \frac{1}{3}\sin(2x + 1)$ has a unique solution.

SOLUTION: Notice that we have the following inequality (proven in class):

$$\forall_{s,t\in\mathbb{R}} \quad |\sin s - \sin t| \le |s-t|. \tag{9}$$

Consequently, by applying the inequality (9) we get $\forall_{x,y\in\mathbb{R}}$

$$\begin{aligned} |f(x)-f(y)| &= \left|\frac{1}{3}\sin(2x+1) - \frac{1}{3}\sin(2y+1)\right| \\ &= \frac{1}{3}|\sin(2x+1)-\sin(2y+1)| \\ &\le \frac{1}{3}|(2x+1)-(2y+1)| \\ &= \frac{2}{3}|x-y|, \end{aligned}$$

so, $f(x)$ is a contraction with constant $k=\frac{2}{3}$. By the Banch contraction principle (the so called Banach Fixed Point Theorem), there exist a unique point $x_o \in \mathbb{R}$ such that

$$x_o = \frac{1}{3}\sin(2x_o+1),$$

i.e. the equation $x = \frac{1}{3}\sin(2x+1)$ has a unique solution. □

5. Compute the following limits

(a) $\lim\limits_{x\to 1} \dfrac{\sqrt[m]{x}-1}{\sqrt[n]{x}-1}$, where m and n are natural numbers;
(b) $\lim\limits_{x\to -\infty} \sqrt{(x+a)(x+b)}+x$;
(c) $\lim\limits_{x\to\infty} (\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x})$;
(d) $\lim\limits_{x\to\infty} (\sqrt[3]{x^3+x^2+1}-\sqrt[3]{x^3-x^2+1})$;

SOLUTION:
(a): Let $y = x^{mn}$, then $x\to 1 \Rightarrow y\to 1$, and

$$\lim_{x\to 1}\frac{\sqrt[m]{x}-1}{\sqrt[n]{x}-1} = \lim_{y\to 1}\frac{y^m-1}{y^n-1} = \lim_{x\to 1}\frac{x^m-1}{x^n-1}.$$

Then we have

$$\begin{aligned} \lim_{x\to 1}\frac{x^m-1}{x^n-1} &= \lim_{x\to 1}\frac{(x-1)(1+x+x^2+\cdots x^{m-1})}{(x-1)(1+x+x^2+\cdots+x^{n-1})} \\ &= \lim_{x\to 1}\frac{1+x+x^2+\cdots+x^{m-1}}{1+x+x^2+\cdots+x^{n-1}} \\ &= \frac{\lim_{x\to 1}(1+x+x^2+\cdots+x^{m-1})}{\lim_{x\to 1}(1+x+x^2+\cdots+x^{n-1})} \\ &= \frac{m}{n}. \end{aligned}$$

(b):

$$\begin{aligned}\lim_{x\to-\infty}(\sqrt{(x+a)(x+b)}+x) &= \lim_{x\to-\infty}\frac{(\sqrt{(x+a)(x+b)}+x)(\sqrt{(x+a)(x+b)}-x)}{\sqrt{(x+a)(x+b)}-x}\\ &= \lim_{x\to-\infty}\frac{(x+a)(x+b)-x^2}{\sqrt{x^2+(a+b)x+ab}-x}\\ &= \lim_{x\to-\infty}\frac{(a+b)x+ab}{\sqrt{x^2+(a+b)x+ab}-x}\\ &= \lim_{x\to-\infty}\frac{(a+b)+\frac{ab}{x}}{-\sqrt{1+\frac{a+b}{x}+\frac{ab}{x^2}}-1}=\frac{a+b}{-2}.\end{aligned}$$

(c):

$$\begin{aligned}\lim_{x\to\infty}(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}) &= \lim_{x\to\infty}\frac{x+\sqrt{x+\sqrt{x}}-x}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}\\ &= \lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}\\ &= \lim_{x\to\infty}\frac{\sqrt{1+\sqrt{\frac{1}{x}}}}{\sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}+1}=\frac{1}{2}.\end{aligned}$$

(d):

$$\begin{aligned}&\lim_{x\to\infty}(\sqrt[3]{x^3+x^2+1}-\sqrt[3]{x^3-x^2+1})\\ &= \lim_{x\to\infty}\frac{(x^3+x^2+1)-(x^3-x^2+1)}{(x^3+x^2+1)^{\frac{2}{3}}+(x^3+x^2+1)^{\frac{1}{3}}(x^3-x^2+1)^{\frac{1}{3}}+(x^3-x^2+1)^{\frac{2}{3}}}\\ &= \lim_{x\to\infty}\frac{2x^2}{(x^3+x^2+1)^{\frac{2}{3}}+(x^3+x^2+1)^{\frac{1}{3}}(x^3-x^2+1)^{\frac{1}{3}}+(x^3-x^2+1)^{\frac{2}{3}}}\\ &= \lim_{x\to\infty}\frac{2}{(1+\frac{1}{x}+\frac{1}{x^3})^{\frac{2}{3}}+(1+\frac{1}{x}+\frac{1}{x^3})^{\frac{1}{3}}(1-\frac{1}{x}+\frac{1}{x^3})^{\frac{1}{3}}+(1-\frac{1}{x}+\frac{1}{x^3})^{\frac{2}{3}}}\\ &= \frac{2}{1+1+1}=\frac{2}{3}.\end{aligned}$$

6. Let $b > a$ and $f, g : [a, b] \to \mathbb{R}$ be two continuous functions. Show that the functions

$$\varphi(x) = \max\{f(x), g(x)\}$$

and

$$\psi(x) = \min\{f(x), g(x)\},$$

are also continuous on $[a, b]$.

SOLUTION: Define the function

$$\alpha(t) = \frac{1}{2}\left[t + \sqrt{t^2}\right].$$

Then it is easy to notice that

$$\varphi(x) = \max\{f(x), g(x)\} = \alpha(f(x) - g(x)) + g(x)$$
$$\psi(x) = \min\{f(x), g(x)\} = f(x) - \alpha(f(x) - g(x))$$

Since $\alpha(t)$ is an elementary function, thus it is continuous, therefore, by the fact that sum, difference and composition of continuous functions is continuous, it follows that $\varphi(x)$ and $\psi(x)$ are continuous.

7. Let $f : [a, \infty) \to \mathbb{R}$ be a continuous function which is neither bounded from above nor from below. Show that for every real number α there exist infinitely many numbers t_n, $n = 1, 2, \ldots$, in the interval $[a, \infty)$ such that $f(t_n) = \alpha$, i.e. the function $f(x)$ assumes every real value infinitely many times. Give an example of such a function.

SOLUTION: Let α be any real number. Since $f(x)$ is not bounded from below, there exists $c_0 \in (a, \infty)$ such that $f(c_0) < \alpha - 1$; similarly, since $f(x)$ is not bounded from above, there exists $d_0 \in (a, \infty)$ such that $f(d_0) > \alpha + 1$. Thus, by intermediate value theorem, there exists $t_1 \in [\min\{c_0, d_0\}, \max\{c_0, d_0\}] \subset [a, \infty)$, such that $f(t_1) = \alpha$. Now, consider the interval $[t_1, \infty) \subset [a, \infty)$. Since in $[a, t_1]$, $f(x)$ is always bounded, it follows that $f(x)$ is neither bounded from above nor from below. Thus, for the same reason, there exists $t_2 \in [t_1, \infty)$, such that $f(t_2) = \alpha$. Repeating this procedure yields an infinite sequence $\{t_i\} \subset [a, \infty)$ such that $\forall n$, $f(t_n) = \alpha$. Therefore, the statement is proven.

8. Show that every function $f : (-a, a) \to \mathbb{R}$, where $a > 0$ can be represented as a sum of an even and an odd functions.

SOLUTION: We define

$$\varphi(x) = \frac{1}{2}[f(x) + f(-x)],$$

and

$$\psi(x) = \frac{1}{2}[f(x) - f(-x)].$$

It is clear that $\varphi(x)$ is even and $\psi(x)$ is odd and

$$f(x) = \varphi(x) + \psi(x).$$

9. Let $f(x) = ax^2 + bx + c$. Show that

$$f(x+3) - 3f(x+2) + 3f(x+1) - f(x) = 0.$$

SOLUTION: $f(x) = ax^2 + bx + c$, thus

$$\begin{aligned}
f(x+3)-&3f(x+2)+3f(x+1)-f(x)\\
&=a(x+3)^2+b(x+3)+c-3(x(x+2)^2+b(x+2)+c)\\
&\qquad\qquad +3(a(x+1)^2+b(x+1)+c)-(ax^2+bx+c)\\
&=ax^2+(6a+b)x+9a+3b+c\\
&\qquad\qquad -(3ax^2+(12a+3b)x+(12a+6b+3c))\\
&\qquad\qquad +3ax^2+(6a+3b)x+3(a+b+c)-(ax^2+bx+c)\\
&=(a-3a+3a-a)x^2+(6a+b-12a-3b+6a+3b-b)x\\
&\qquad\qquad +(9a+3b+c-12a-6b-3c+3a+3b+3c-c)\\
&=0x^2+0x+0=0
\end{aligned}$$

10. Let $f(x)=\frac{1}{2}\Big(a^x+a^{-x}\Big)$, where $a>0$. Show that

$$f(x+y)+f(x-y)=2f(x)f(y).$$

SOLUTION:

$$\begin{aligned}
&f(x+y)+f(x-y)-f(x)f(y)\\
&=\frac{1}{2}(a^{x+y}+a^{-(x+y)})+\frac{1}{2}(a^{x-y}+a^{-(x-y)})\\
&-2\cdot\frac{1}{2}(a^x+a^{-x})\frac{1}{2}(a^y+a^{-y})\\
&=\frac{1}{2}\Big[a^{x+y}+a^{-(x+y)}+a^{x-y}+a^{y-x}\\
&-(a^{x+y}+a^{x-y}+a^{y-x}+a^{-(x+y)})\Big]\\
&=0.
\end{aligned}$$

11. Find $f(x)$ if $f(x+1)=x^2-3x+2$.

SOLUTION: By substituting $t=x+1$, i.e. $x=t-1$, we obtain

$$f(t)=(t-1)^2-3(t-1)+2=t^2-5t+6.$$

Consequently,

$$f(x)=x^2-5x+6.$$

12. Find $f(x)$ if $f\left(x+\frac{1}{x}\right)=x^2+\frac{1}{x^2}$ for $|x|\geq 2$.

SOLUTION: Since

$$f(x+\frac{1}{x})=x^2+\frac{1}{x^2}=(x+\frac{1}{x})^2-2,$$

it follows that

$$f(x) = x^2 - 2.$$

13. Find $f(x)$ if $f\left(\frac{1}{x}\right) = x + \sqrt{1+x^2}$ for $x > 0$.

SOLUTION: Since

$$f(\frac{1}{x}) = x + \sqrt{1+x^2} = \frac{1}{\frac{1}{x}} + \sqrt{1 + \frac{1}{(\frac{1}{x})^2}},$$

it follows that

$$f(x) = \frac{1}{x} + \sqrt{1 + \frac{1}{x^2}} = \frac{1}{x}(1 + \sqrt{1+x^2}) = \frac{1 + \sqrt{1+x^2}}{x}.$$

14. Find $f(x)$ if $f\left(\frac{x}{1+x}\right) = x^2$.

SOLUTION: We make a substitution, $t = \frac{x}{1+x}$, i.e. $x = \frac{t}{1-t}$, and obtain

$$f(t) = \left(\frac{t}{1-t}\right)^2.$$

Consequently,

$$f(x) = \left(\frac{x}{1-x}\right)^2.$$

13.4 Chapter 6: Differentiable Functions of Real Variable

1. Discuss the differentiability at $x = 0$ of the function

$$f(x) = \begin{cases} x \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

SOLUTION: Notice that the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \sin\frac{1}{x}, \tag{13.30}$$

does not exist. Indeed, in the case a limit $\lim_{x\to a} F(x) = L$ exists, for every sequence $\{x_n\}$, such that $x_n \to a$ as $n \to \infty$, we have that $\lim_{n\to\infty} F(x_n) = L$. Consider the sequence $x_n := \frac{1}{2\pi n + \alpha}$, where $\alpha \in \mathbb{R}$ is an arbitrary number. It is clear that $x_n \to 0$ as $n \to \infty$, but

$$\sin\frac{1}{x_n} = \sin(2\pi n + \alpha) = \sin\alpha.$$

Since α was chosen as an arbitrary number, it is clear that the limit (13.30) does not exist. Therefore, the derivative of f at $x = 0$ does not exist. However, it is easy to notice that the function f is continuous at $x = 0$. Indeed, by applying the squeeze theorem, one easily shows that

$$\lim_{x\to 0} f(x) = 0 = f(0).$$

2. Discuss the differentiability at $x = 0$ of the function

$$f(x) = \begin{cases} x^n \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

where n is an integer larger than 1. For what values of k, does the k-th derivative exist at $x = 0$?

SOLUTION: We begin with the first derivative at $x = 0$. Since

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} x^{n-1} \sin\frac{1}{x}, \tag{13.31}$$

by the same argument, which was used in Problem 1, we obtain that $f'(0)$ exists if $n > 1$. On the other hand, the function f is differentiable at every point $x \neq 0$ and we have

$$f'(x) = nx^{n-1} \sin\frac{1}{x} - x^{n-2} \cos\frac{1}{x}.$$

Notice that for the differentiabilty of the function

$$g(x) = \begin{cases} x^m \cos\frac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

at $x = 0$, exactly the same condition, i.e. $m > 1$, has to be satisfied. Therefore, since f' is just a sum of two functions of the exactly the same type as f and g, one obtains that the function f is twice differentiable at the point $x = 0$, if and only if $n - 2 > 1$. Then, the derivative f'' is the function

$$f''(x) = \begin{cases} -x^{n-4} \sin\frac{1}{x} - 2(n-1)x^{n-3} \cos\frac{1}{x} + n(n-1)x^{n-2} \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

We obtain again, that f'' is a linear combinations of the functions of the same type as f and g. Therefore, by applying the above argument, we conclude that the derivative of f'' exists at $x = 0$ if and only if $n - 4 > 1$. By observing the pattern shown by the derivatives f' and f'', we can conjecture that for $x \neq 0$ we have for $k \in \mathbb{N}$

$$f^{(k)}(x) = \begin{cases} (-1)^l x^{n-2k} \sin\frac{1}{x} + \sum_{j=0}^{2k-1} x^{n-j}(a_j \sin\frac{1}{x} + b_j \cos\frac{1}{x}) & k = 2l, \\ (-1)^l x^{n-2k} \cos\frac{1}{x} + \sum_{j=0}^{2k-1} x^{n-j}(a_j \sin\frac{1}{x} + b_j \cos\frac{1}{x}) & k = 2l+1, \end{cases} \tag{13.32}$$

for some real numbers a_j and b_j. The formula (13.32) can be proved by induction. Indeed, for $k = 1$ it is true (see above). Assume therefore that (13.32) is true, and compute $f^{(k+1)}(x)$. We have for $k = 2l$

$$\begin{aligned} f^{(k+1)}(x) = {} & (-1)^l x^{n-2(k+1)} \cos\frac{1}{x} + (-1)^l (n-2k)x^{n-2k-1} \sin\frac{1}{x} \\ & + \sum_{j=0}^{2k-1} (n-j)x^{n-j-1}(a_j \sin\frac{1}{x} + b_j \cos\frac{1}{x}) \\ & + \sum_{j=0}^{2k-1} x^{n-j-2}(a_j \cos\frac{1}{x} - b_j \sin\frac{1}{x}). \end{aligned}$$

It is clear that the derivative $f^{(k+1)}(x)$ is of the required form. For $k = 2l + 1$ the computations of $f^{k+1}(x)$ are similar. In this way, we immediately obtain that $f^{(k)}(0)$ exists if and only if $n-2(k-1) > 1$, i.e. $\frac{n+1}{2} > k$. In summary, the function f is $\left[\frac{n}{2}\right]$-differentiable, i.e. it has $\left[\frac{n}{2}\right]$ derivatives.

3. Evaluate the following limit:

$$\lim_{x\to 0} \frac{\tan x - x}{x^3}.$$

SOLUTION: We can apply the l'Hôpital's Rule (twice) to compute this limit (notice that it is a limit of the type $\frac{0}{0}$):

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} \overset{\text{HR}}{=} \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \overset{\text{HR}}{=} \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} = \frac{1}{3}.$$

4. Suppose that f is defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

(a) Show that $f^{(n)}(0)$ exists for every positive integer n and has the value 0;

SOLUTION: First, let us observe that for any natural number n we have the limit

$$\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = 0.$$

Indeed, by applying the l'Hôpital Rule, we obtain

$$\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = \lim_{x\to 0} \frac{x^{-n}}{e^{\frac{1}{x^2}}} \overset{\text{HR}}{=} \lim_{x\to 0} \frac{-nx^{-n-1}}{e^{\frac{1}{x^2}} \frac{-2}{x^3}} = \frac{n}{2} \lim_{x\to 0} \frac{x^{-n+2}}{e^{\frac{1}{x^2}}}.$$

It is clear that we can repeat the application of the l'Hôpital Rule for $k := \left[\frac{n+1}{2}\right]$ times, which will lead to

$$\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = \frac{n(n-2)\dots(n-2(k-1))}{2^k} \lim_{x\to 0} \frac{x^{-n+2k}}{e^{\frac{1}{x^2}}} = 0,$$

because $n - 2k \ge 0$. Our second observation is that, for $x > 0$ we have

$$\begin{aligned} f'(x) &= \frac{2}{x^3}\, e^{-\frac{1}{x^2}} \\ f''(x) &= \left(-\frac{6}{x^4} + \frac{4}{x^6}\right) e^{-\frac{1}{x^2}} \\ f'''(x) &= \left(\frac{24}{x^5} - \frac{36}{x^7} + \frac{8}{x^9}\right) e^{-\frac{1}{x^2}}. \end{aligned}$$

By observing the pattern, we conjecture that the derivative $f^{(k)}(x)$, for $x > 0$, is always of the type

$$f^{(k)}(x) = \left(\sum_j \frac{a_j}{x^j} \right) e^{-\frac{1}{x^2}}, \tag{13.33}$$

where $j > 0$, the sum is finite and a_j are real numbers. Indeed, we can check the above formula by applying the mathematical induction. For $k = 1$ it is clearly true. Assume that (13.33) is true, then we have

$$\begin{aligned} f^{(k+1)}(x) &= \left(\sum_j \frac{-ja_j}{x^{j+1}} \right) e^{-\frac{1}{x^2}} + \left(\sum_j \frac{a_j}{x^j} \right) \frac{2}{x^3} e^{-\frac{1}{x^2}} \\ &= \left(\sum_j \frac{-ja_j}{x^{j+1}} + \frac{a_j}{x^{j+3}} \right) e^{-\frac{1}{x^2}} \end{aligned}$$

Now, we are in the position to check the differentiability of $f^{(k)}$ at $x = 0$. For $k = 1$, we obtain that

$$f'(0) = \lim_{x \to 0^+} \frac{e^{\frac{1}{x^2}}}{x} = 0.$$

Assume for the induction, that the derivative $f^{(k)}(0)$ exists and is equal zero. Then we have (by the formula (13.33)) that

$$\begin{aligned} f^{(k+1)}(0) &= \lim_{x \to 0^+} \frac{\left(\sum_j \frac{a_j}{x^j} \right) e^{-\frac{1}{x^2}}}{x} \\ &= \lim_{x \to 0^+} \left(\sum_j \frac{a_j}{x^{j+1}} e^{-\frac{1}{x^2}} \right) \\ &= \left(\sum_j a_j \lim_{x \to 0^+} \frac{e^{-\frac{1}{x^2}}}{x^{j+1}} \right) \\ &= \sum_j a_j 0 = 0. \end{aligned}$$

Consequently, by the principle of mathematical induction, the derivative $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$.

5. Let $1 > a > 0$ and $f(x) = (a^x + 1)^{\frac{1}{x}}$, show that the function f is decreasing for $x > 0$.

SOLUTION: We need to evaluate the derivative of $f(x)$. Notice that

$$f(x) = e^{\frac{1}{x} \ln(a^x+1)}, \quad a^x = e^{x \ln a},$$

thus, by applying the chain and product rules, we get

$$f'(x) = (a^x + 1)^{\frac{1}{x}} \left(-\frac{1}{x^2} \ln(a^x + 1) + \frac{a^x \ln a}{x(a^x + 1)} \right).$$

Since $0 < a < 1$, $\ln a < 0$, thus

$$\left(-\frac{1}{x^2}\ln(a^x+1)+\frac{a^x\ln a}{x(a^x+1)}\right)<0$$

for $x>0$, which implies $f'(x)<0$, thus f is decreasing on $(0,\infty)$.

6. Use result in Problem 19 to prove the following inequality

$$(x^a+y^a)^{\frac{1}{a}}>(x^b+y^b)^{\frac{1}{b}}, \tag{13.34}$$

for all $x>0$, $y>0$ and $0<a<b$.

SOLUTION: Notice that when $x=y=:t$, then we have

$$(x^a+y^a)^{\frac{1}{a}}=(2t^a)^{\frac{1}{a}}=2^{\frac{1}{a}}t, \quad \text{and} \quad (x^b+y^b)^{\frac{1}{b}}=(2t^b)^{\frac{1}{b}}=2^{\frac{1}{b}}t.$$

Since $\frac{1}{a}>\frac{1}{b}$ and function $x\mapsto 2^x$ is increasing, we get

$$2^{\frac{1}{a}}>2^{\frac{1}{b}},$$

so the inequatlity (13.34) is clearly true in this case. Suppose therefore that (for example) $x>y$. Then we can write the expression

$$(x^a+y^a)^{\frac{1}{a}}=x\left(1+\left(\frac{y}{x}\right)^a\right)^{\frac{1}{a}}.$$

Since, $0<\frac{y}{x}<1$, by Problem 5, the function

$$a\mapsto\left(1+\left(\frac{y}{x}\right)^a\right)^{\frac{1}{a}}, \quad a>0$$

is decreasing, we obtain for $0<a<b$ that

$$(x^a+y^a)^{\frac{1}{a}}=x\left(1+\left(\frac{y}{x}\right)^a\right)^{\frac{1}{a}}>x\left(1+\left(\frac{y}{x}\right)^b\right)^{\frac{1}{b}}=(x^b+y^b)^{\frac{1}{b}}.$$

7. Prove the following inequality for all $x>0$

$$1+2\ln x\le x^2.$$

SOLUTION: Consider the function

$$\varphi(x):=x^2-2\ln x-1, \quad x>0.$$

We have

$$\varphi'(x)=2x-\frac{2}{x}=\frac{2(x^2-1)}{x}=\frac{2(x-1)(x+1)}{x}.$$

By inspection, we get that

$$\varphi'(x)<0, \quad \text{for} \quad x\in(0,1),$$

and

$$\varphi'(x) > 0, \quad \text{for} \quad x \in (1, \infty),$$

which means that the function φ is decreasing on the interval $(0, 1)$ and increasing on $(1, \infty)$. Since $\varphi(1) = 1^2 - 2\ln 1 - 1 = 0$, in particular we obtain that

$$\begin{aligned} \varphi(x) > \varphi(1) \;&\Rightarrow\; x^2 - 2\ln x - 1 > 0 \quad \text{for} \quad x \in (0, 1), \\ \varphi(1) < \varphi(x) \;&\Rightarrow\; x^2 - 2\ln x - 1 > 0 \quad \text{for} \quad x \in (1, \infty). \end{aligned}$$

Consequently, we obtain

$$x^2 - 2\ln x - 1 \geq 0 \;\Leftrightarrow\; 1 + 2\ln x \leq x^2$$

for all $x > 0$.

8. Compute the limit

$$\lim_{x \to 1} \frac{\ln(\cosh x)}{\ln(\cos x)}.$$

SOLUTION: Notice that function

$$f(x) := \frac{\ln(\cosh x)}{\ln(\cos x)}$$

is well defined at $x = 1$, and therefore (since it is an elementary function) it is continuous at $x = 1$. Consequently we get

$$\lim_{x \to 1} \frac{\ln(\cosh x)}{\ln(\cos x)} = \frac{\ln(\cosh 1)}{\ln(\cos 1)}.$$

Notice, in addition, that application of the l'Hôpital's Rule for this limit is not correct. It would lead to a wrong result (the necessary condition — the limit of the type $\frac{0}{0}$ — is not satisfied!).

9. Prove the following theorem:

Theorem 13.4. *Let $f : (a, b) \to \mathbb{R}$ be a differentiable function which is unbounded. Then the derivative $f' : (a, b) \to \mathbb{R}$ is also unbounded.*

SOLUTION: In order to prove this theorem, we will prove the contrapositive statement: *If $f' : (a, b) \to \mathbb{R}$ is bounded then $f : (a, b) \to \mathbb{R}$ is also bounded.* Let $x_o \in (a, b)$ be a fixed point. By assumption

$$\exists_{M>0}\, \forall_{t \in (a,b)} \quad |f'(t)| \leq M. \tag{13.35}$$

Then, by Lagrange Theorem (the Mean Value Theorem), we have that for every $x \in (a, b)$, $x \neq x_o$, there exists $c(x)$ such that

$$\frac{f(x) - f(x_o)}{x - x_o} = f'(c(x)).$$

In particular, we obtain that

$$\left| \frac{f(x) - f(x_o)}{x - x_o} \right| = |f'(c(x))| \quad \Rightarrow \quad |f(x) - f(x_o)| < |x - x_o||f'(c(x))|.$$

By triangle inequality and (13.35), we obtain

$$|f(x)| < |f(x_o)| + |x - x_o||f'(c(x))| \le |f(x_o)| + (b-a)M =: M_1,$$

which implies that

$$\forall_{x\in(a,b)} \quad |f(x)| \le M_1.$$

The last statement shows that the function f is bounded on (a,b).

10. Give an example of a bounded differentiable function $f : (a,b) \to \mathbb{R}$ such that the derivative $f' : (a,b) \to \mathbb{R}$ is unbounded. What it has to do with the fact stated in the previous question?

SOLUTION: Let $f : (0,1) \to \mathbb{R}$ be defined by the formula

$$f(x) = \sin\frac{1}{x}, \quad x \in (0,1).$$

It is clear that the function f is bounded on $(0,1)$. Indeed, we have

$$|f(x)| \le 1 \quad \forall_{x\in(0,1)}.$$

However, the derivative

$$f'(x) = -\frac{1}{x^2}\cos\frac{1}{x}, \quad x \in (0,1),$$

is not bounded. Indeed, choose $x_n = \frac{1}{\pi(2n+1)} \in (0,1)$, $n = 1, 2, \dots$, then

$$f'(x_n) = -\frac{1}{x_n^2}\cos\frac{1}{x_n} = -(\pi(2n+1))^2\cos(\pi(2n+1)) = (\pi(2n+1))^2 \to \infty$$

as $n \to \infty$. This example shows that the inverse theorem, to Theorem 13.4 is **not true**.

11. Prove the following theorem:

Theorem 13.5. *Let $f : (a,b) \to \mathbb{R}$ be a differentiable function such that $f' : (a,b) \to \mathbb{R}$ is bounded. Then the function $f : (a,b) \to \mathbb{R}$ is uniformly continuous.*

SOLUTION: By assumption, the derivative $f' : (a,b) \to \mathbb{R}$ is bounded, thus

$$\exists_{M>0} \ \forall_{x\in(a,b)} \quad |f'(x)| \le M. \tag{13.36}$$

Notice that, by Lagrange Theorem, for all x, $y \in (a,b)$, $x \ne y$, there exists a point $c(x,y)$ in between the points x and y such that

$$\frac{f(x) - f(y)}{x - y} = f'(c(x,y)), \tag{13.37}$$

thus, it follows from (13.36) and (13.37) that

$$|f(x) - f(y)| = |x-y||f'(c(x,y))| \le |x-y|M. \tag{13.38}$$

Assume therefore, that $\varepsilon > 0$ is an arbitrary number (i.e. we make a statement with $\forall_{\varepsilon>0}$) and choose $\delta = \frac{\varepsilon}{M}$ (i.e. we show that there exists a $\delta > 0$, which is explicitly indicated). Then we have for all x, $y \in (a,b)$ (i.e. we make a statement for $\forall_{x,y\in(a,b)}$)

$$|x-y|<\delta \quad \Rightarrow \quad |f(x)-f(y)| \le |x-y|M < \delta M = \frac{\varepsilon}{M}M = \varepsilon.$$

In this way, we have proved that

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{x,y\in(a,b)} \;\; |x-y|<\delta \quad \Rightarrow \quad |f(x)-f(y)|<\varepsilon,$$

which means that the function f is uniformly continuous on (a,b).

12. Prove the following theorem:

Theorem 13.6. *Let $a<b$ (a and b are finite) and $f:(a,b)\to\mathbb{R}$ be a uniformly continuous function. Show that f is bounded.*

SOLUTION: By assumption the function $f:(a,b)\to\mathbb{R}$ is uniformly continuous, i.e.

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{x,y\in(a,b)} \;\; |x-y|<\delta \quad \Rightarrow \quad |f(x)-f(y)|<\varepsilon.$$

In particular, for $\varepsilon=1$ there exists $\delta>0$ such that for all $x,y\in(a,b)$ we have

$$|x-y|<\delta \quad \Longrightarrow \quad |f(x)-f(y)|<1. \tag{13.39}$$

Let us put $n := \left[\frac{2(b-a)}{\delta}\right]$ and

$$a_0 := a, \;\; a_1 := a + k\frac{b-a}{n}, \;\; \dots, \;\; a_k := a + k\frac{b-a}{n}, \;\; \dots, a_n = b.$$

Then we have

$$(a,b) := (a_0,a_1)\cup[a_1,a_2)\cup[a_2,a_3)\cup\dots\cup[a_{n-2},a_{n-1})\cup[a_{n-1},a_n). \tag{13.40}$$

Notice that if $x\in[a_{k-1},a_k)$ and $y\in[a_k,a_{k+1})$ then

$$|x-y|<|a_{k+1}-a_{k-1}| \le \frac{2(b-a)}{n} \le \frac{2(n-1)\delta}{2(b-a)} = \delta,$$

thus, by (13.39)

$$|f(x)-f(y)|<1.$$

By (13.40), for every $x\in(a,b)$, there exists $1\le k\le n$ such that $x\in[a_{k-1},a_k)$. Thus, by triangle inequality and (13.39) we have

$$\begin{aligned}
|f(x)| &= |f(x)-f(a_{k-1})+f(a_{k-1})-f(a_{k-2})+f(a_{k-2})-\dots-f(a_1)+f(a_1)| \\
&\le |f(x)-f(a_{k-1})|+|f(a_{k-1})-f(a_{k-2})|+\dots+|f(a_2)-f(a_1)|+|f(a_1)| \\
&\le k+|f(a_1)| \le n+|f(a_1)| =: M.
\end{aligned}$$

In this way, we have shown that

$$\exists_{M>0}\,\forall_{x\in(a,b)} \;\; |f(x)|\le M,$$

which means the function f is bounded on (a,b).

13. Compute the derivative of the following functions:

(a) $f(x) = e^x + e^{e^x} + e^{e^{e^x}}$;
(b) $f(x) = \left(\frac{a}{b}\right)^x \left(\frac{b}{x}\right)^a \left(\frac{x}{a}\right)^b$, where $a > 0$ and $b > 0$;
(c) $f(x) = \ln \frac{b + a\cos x + \sqrt{b^2 - a^2}\sin x}{a + b\cos x}$, where $0 \le |a| < |b|$;

SOLUTION: (a):

$$f(x) = e^x + e^{e^x} + e^{e^{e^x}},$$

let $f_1(x) = e^x$, $f_2(x) = e^{e^x}$, $f_3(x) = e^{e^{e^x}}$, then

$$\begin{aligned} f_1'(x) &= (e^x)' = e^x \\ f_2'(x) &= (e^{e^x})' = e^{e^x}(e^x)' = e^{e^x}e^x \\ f_3'(x) &= (e^{e^{e^x}})' = e^{e^{e^x}}(e^{e^x})' = e^{e^{e^x}}e^{e^x}e^x. \end{aligned}$$

Thus,

$$\begin{aligned} f'(x) &= f_1'(x) + f_2'(x) + f_3'(x) \\ &= e^x(1 + e^{e^x} + e^{e^x}e^{e^{e^x}}) \\ &= e^x(1 + e^{e^x}(1 + e^{e^{e^x}})). \end{aligned}$$

(b):

$$\begin{aligned} f'(x) &= \left(\frac{a}{b}\right)^x \ln\left(\frac{a}{b}\right) \cdot \left(\frac{x}{a}\right)^b \left(\frac{b}{x}\right)^a + \frac{\left(\frac{a}{b}\right)^x b \left(\frac{x}{a}\right)^b \left(\frac{b}{x}\right)^a}{x} \\ &\quad - \frac{\left(\frac{a}{b}\right)^x \left(\frac{x}{a}\right)^b \left(\frac{b}{x}\right)^a a}{x} \\ &= \frac{\left(\frac{a}{b}\right)^x \left(\frac{x}{a}\right)^b \left(\frac{b}{x}\right)^a \left(x \ln\frac{a}{b} + b - a\right)}{x}. \end{aligned}$$

(c):

$$\begin{aligned} f'(x) &= \frac{a + b\cos x}{b + a\cos x + \sqrt{b^2 - a^2}\sin x} \cdot \frac{1}{(a + b\cos x)^2} \\ &\quad \cdot \Big[(-a\sin x + \sqrt{b^2 - a^2}\cos x)(a + b\cos x) \\ &\quad -(b + a\cos x + \sqrt{b^2 - a^2}\sin x)(-b\sin x)\Big] \\ &= \frac{-a^2\sin x + b\sqrt{b^2 - a^2} + a\sqrt{b^2 - a^2}\cos x + b^2\sin x}{(b + a\cos x + \sqrt{b^2 - a^2}\sin x)(a + b\cos x)} \\ &= \frac{(b + a\cos x + \sqrt{b^2 - a^2}\sin x)\sqrt{b^2 - a^2}}{(b + a\cos x + \sqrt{b^2 - a^2}\sin x)(a + b\cos x)} \\ &= \frac{\sqrt{b^2 - a^2}}{a + b\cos x}. \end{aligned}$$

14. For what values of the integers n and m, where $m > 0$, the function

$$f(x) = \begin{cases} |x|^n \sin \frac{1}{|x|^m} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at $x = 0$.

SOLUTION: Since $\forall x \neq 0$,

$$f(-x) = |-x|^n \sin \frac{1}{|-x|^m} = |x|^n \sin \frac{1}{|x|^m} = f(x),$$

it follows that $f(x)$ is differentiable at $x_0 = 0$ iff

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

exists, i.e.

$$\lim_{x \to 0^+} x^{n-1} \sin \frac{1}{x^m}$$

exists. However, since

$$\lim_{x \to 0^+} x^{n-1} \sin \frac{1}{x^m} = \lim_{x \to 0^+} y^{\frac{n-1}{m}} \sin \frac{1}{y}$$

exists only if $\frac{n-1}{m} > 0$, i.e. $n > 1$. Hence, for $n > 1$, $f(x)$ is differentiable.

15. Find $y^{(n)}$ for the following functions

(a) $y = \sin ax \cos bx$;
(b) $y = x^2 \sin ax$;
(c) $y = \cosh ax \cos bx$;

SOLUTION: (a): Since $\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$, we have that

$$y = \sin ax \, \cos bx = \tfrac{1}{2}[\sin(a + b)x + \sin(a - b)x],$$

thus

$$y^{(n)} = \frac{1}{2} \Big[(a+b)^n \sin \Big((a+b)x + \tfrac{n\pi}{2} \Big) + (a-b)^n \sin \Big((a-b)x + \tfrac{n\pi}{2} \Big) \Big].$$

You could also use the Leibnitz formula:

$$\begin{aligned} y^{(n)} &= \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k} \sin ax \frac{d^{n-k}}{dx^{n-k}} \cos bx \\ &= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \sin(ax + \tfrac{k\pi}{2}) \cos(bx + \tfrac{(n-k)\pi}{2}). \end{aligned}$$

(b): We apply the Leibnitz formula to the function

$$y = x^2 \sin ax.$$

We have

$$y^{(n)} = \sum_{k=0}^{n} \binom{n}{k} a^k \sin\left(ax + \frac{k\pi}{2}\right) \cdot \frac{d^{n-k}}{dx^{n-k}} x^2.$$

Thus, for $n = 1$

$$y' = 2a \sin ax + x^2 a \sin\left(ax + \tfrac{\pi}{2}\right),$$

and for $n \geq 2$

$$\begin{aligned} y^{(n)} &= a^n \sin\left(ax + \tfrac{n\pi}{2}\right) \cdot x^2 + n a^{n-1} \sin\left(ax + \tfrac{(n-1)\pi}{2}\right) \cdot 2x \\ &+ \frac{n(n-1)}{2} a^{n-2} \sin\left(ax + \tfrac{(n-2)\pi}{2}\right) \cdot 2 \\ &= a^n x^2 \sin\left(ax + \tfrac{n\pi}{2}\right) + 2na^{n-1} x \sin\left(ax + \tfrac{(n-1)\pi}{2}\right) \\ &+ n(n-1) a^{n-2} \sin\left(ax + \tfrac{(n-2)\pi}{2}\right). \end{aligned}$$

Thus

$$y^{(n)} = a^{n-2} \sin\left(ax + \tfrac{n\pi}{2}\right) \left((ax)^2 - n(n-1)\right) + 2nxa^{n-1} \cos\left(ax + \tfrac{n\pi}{2}\right).$$

(c): Let $f(x) = \cosh ax$ and $g(x) = \cos bx$, then

$$f^{(n)}(x) = \begin{cases} a^n \sinh ax & n = 1, 3, 5, ... \\ a^n \cosh ax & n = 0, 2, 4, 6, ... \end{cases}$$

and $g^{(n)}(x) = b^n \cos(bx + \frac{n\pi}{2})$. Then, for even number n,

$$\begin{aligned} y^{(n)}(x) &= \sum_{k=0}^{\frac{n}{2}} \binom{n}{2k} a^{n-2k} b^{2k} \cosh ax \cos(bx + k\pi) \\ &+ \sum_{k=0}^{\frac{n}{2}-1} \binom{n}{2k+1} a^{n-2k-1} b^{2k+1} \sinh ax \cos(bx + \frac{2k+1}{2}\pi), \end{aligned}$$

and for odd number n

$$\begin{aligned} y^{(n)}(x) &= \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} a^{n-2k} b^{2k} \sinh ax \cos(bx - k\pi) \\ &+ \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k+1} a^{n-2k-1} b^{2k+1} \cosh ax \cos(bx + \frac{2k+1}{2}\pi), \end{aligned}$$

where $[\alpha]$ denotes the greatest integer m such that $m \leq \alpha$.

16. Explain why the functions $f(x) = x^2$ and $g(x) = x^3$ do not satisfy the assumptions of the Cauchy's Theorem (the conclusion of this theorem is also not true) on the interval $[-1, 1]$.

SOLUTION: Since $g'(x) = 3x^2$, it follows that $g'(0) = 0$. Thus the assumption of $g'(x) \neq 0$ for all $x \in [-1, 1]$ is not satisfied. Consequently, Cauchy theorem does not apply in this case.

17. Show that, if a differentiable function $f : (a, b) \to \mathbb{R}$ is unbounded, then $f' : (a, b) \to \mathbb{R}$ is also unbounded. Show by giving an example that the converse statement is not true.

SOLUTION: Assume that $f(x)$ is unbounded but $f'(x)$ is bounded on (a, b), say $\forall x \in (a, b)$, $|f'(x)| < M$ for some $M > 0$. Then by Lagrange Theorem, given $x_0 \in (a, b)$ and $\forall x \in (a, b)$, there exists a point $c \in (a, b)$ such that

$$|\frac{f(x) - f(x_0)}{x - x_0}| = |f'(c)| \le M,$$

i.e.

$$|f(x) - f(x_0)| \le M|x - x_0| \le M(b - a).$$

Therefore,

$$|f(x)| \le |f(x_0)| + M(b - a),$$

i.e. $f(x)$ is bounded. This leads to a contradiction. Thus $f'(x)$ is unbounded too. The converse statement is not true. For example, $f(x) = \sqrt{x}$. $f'(x) = \frac{1}{2\sqrt{x}}$ is unbounded on $(0, 1)$, but $f(x)$ is bounded on $(0, 1)$.

Another Proof: If $f'(x)$ is bounded on (a, b), then by Problem 6, $f(x)$ is uniformly continuous on (a, b), thus by Problem 2, Assignment 4, we get that $f(x)$ is bounded.

18. Show that if $f : (a, b) \to \mathbb{R}$ is a differentiable function such that $f'(x)$ is bounded, then $f(x)$ is uniformly continuous on (a, b).

SOLUTION: We need to show that

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x, y \in (a, b),\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Indeed, since $f'(x)$ is bounded on (a, b), there exists $M > 0$, such that $\forall x \in (a, b)$, $|f'(x)| \le M$. Thus, $\forall \varepsilon > 0$, let $\delta = \frac{\varepsilon}{M}$, then $\forall x, y \in (a, b)$, by Lagrange Theorem,

$$\exists x_0 \in (a, b), \quad \text{such that} \quad |\frac{f(x) - f(y)}{x - y}| = |f'(x_0)| \le M$$

therefore, $|f(x) - f(y)| \le M|x - y| < M\frac{\varepsilon}{M} = \varepsilon$. Consequently, $f(x)$ is uniformly continuous on (a, b).

19. Prove the following inequalities

(a) $x - \frac{x^2}{2} < \ln(1 + x) < x$ for all $x > 0$;
(b) $x - \frac{x^3}{6} < \sin x < x$ for all $x > 0$;
(c) $(x^a + y^a)^{\frac{1}{a}} > (x^b + y^b)^{\frac{1}{b}}$ for $x > 0$, $y > 0$ and $0 < a < b$;
(d) $x^a - 1 > a(x - 1)$ for $a \ge 2$, $x > 1$;

SOLUTION: (a): First, we consider the function $f(x) = \ln(1 + x) - x\frac{x^2}{2}$. For $x > 0$, we have

$$f'(x) = \frac{1}{1 + x} - 1 + x = \frac{1 - (1 - x)(1 + x)}{1 + x} = \frac{x^2}{1 + x} > 0.$$

Thus $f(x)$ is increasing on $(0,+\infty)$. Also notice that $f(0)=0$, it then follows that $f(x)>0$ on $(0,+\infty)$. Consequently, $\ln(1+x)>x-\frac{x^2}{2}$.

Next, we consider the function $g(x)=x-\ln(1+x)$. For $x>0$,

$$g'(x)=1-\frac{1}{1+x}=\frac{1+x-1}{1+x}=\frac{x}{1+x}>0.$$

Thus $g(x)$ is increasing on $(0,+\infty)$. Also notice that $g(0)=0$, it then follows $g(x)>0$ on $(0,+\infty)$. Consequently, $x>\ln(1+x)$ on $(0,+\infty)$.

(b): We consider the function $\varphi(x)=x-\sin x$. We have $\varphi'(x)=1-\cos x$, therefore $\varphi'(x)=0$ only at the points $x=2k\pi$, $k=0,\pm1,\pm2,\dots$. Therefore, for $x\in(2(k-1)\pi,2k\pi)$, $\varphi'(x)>0$, so the function $\varphi(x)$ is increasing on all intervals $[2(k-1)\pi,2k\pi]$, and consequently it is an increasing function. Since $\varphi(x)$ is increasing, it follows for $x>0$ that $\varphi(x)>\varphi(0)=0$, and the inequality

$$\sin x<x$$

follows for $x>0$. Notice that the function $\phi(x)=\cos x-1+\frac{x^2}{2}$ satisfies $\phi(0)=0$, and since $\phi'(x)=-\sin x+1=\psi(x)>0$, except at the points $x=\frac{\pi}{2}+2k\pi$, $k=0,\pm1,\pm2,\dots$, therefore the function $\phi(x)$ is increasing. Consequently, for $x>0$ we have $\phi(x)>\phi(0)=0$, so $\phi(x)=\cos x-1+\frac{x^2}{2}>0$. In order to prove the second inequality, we consider the function

$$\eta(x)=\sin x-x+\frac{x^3}{6}.$$

Notice that $\eta(0)=0$ and $\eta'(x)=\cos x-1+\frac{x^2}{2}=\phi(x)$. Since for $x>0$ we have that $\phi(x)>0$, thus $\eta'(x)>0$, and the function $\eta(x)$ is increasing on the interval $[0,\infty]$. That means, in particular that for $x>0$ we have $\eta(x)>\eta(0)=0$, i.e.

$$\sin x-x+\frac{x^3}{6}>0$$

and the above inequality follows.

(c): Define $f(z,x)=(1+z^x)^{\frac{1}{x}}$ and

$$g(z,x)=\frac{d}{dx}(\ln f(z,x))=-\frac{1}{x^2}\ln(1+z^x)+\frac{1}{x}\frac{z^x\ln z}{1+z^x}$$

for $z>0$ and $x>0$. As

$$(1+zx)^{1+z^x}>(1+z^x)^{z^x}>(z^x)^{z^x},$$

we have

$$\begin{aligned}
&(1+z^x)\ln(1+z^x)>z^x x\ln z\\
&\Rightarrow\frac{z^x\ln z}{1+z^x}-\frac{1}{x}\ln(1+z^x)<0\\
&\Rightarrow g(z,x)=\frac{1}{x}\left(\frac{z^x\ln z}{1+z^x}-\frac{1}{x}\ln(1+z^x)\right)<0\\
&\Rightarrow\frac{d}{dx}f(z,x)=f(z,x)g(z,x)<0,
\end{aligned}$$

i.e. given $z > 0$, $f(z,x)$ is a decreasing function of x for $x > 0$. Therefore, if $0 < a < b$, we have $f(z,a) > f(z,b)$. Now take $z = \frac{y}{x}$, then it follows that

$$\left(1 + \left(\frac{y}{x}\right)^a\right)^{\frac{1}{a}} > \left(1 + \left(\frac{y}{x}\right)^b\right)^{\frac{1}{b}} \Rightarrow (x^a + y^a)^{\frac{1}{a}} > \left(x^b + y^b\right)^{\frac{1}{b}}.$$

(d): Let $f(x) = x^a - 1 - a(x-1)$. $f'(x) = ax^{a-1} - a = a(x^{a-1} - 1)$. Thus, for $a \geq 2$ and $x > 1$, $f'(x) > 0$, i.e. $f(x)$ is increasing for $a \geq 2$ on $(1, \infty)$. In particular, $f(x) > f(1) = 0$. Consequently, $x^a - 1 - a(x-1) > 0$, i.e. $x^a - 1 > a(x-1)$ for $a \geq 2$ and $x > 1$.

20. Let $f(x)$ be a twice differentiable function in an interval $[a, \infty)$ such that (i) $f(a) = A > 0$; (ii) $f'(a) < 0$; (iii) $f''(x) \leq 0$ for $x > a$. Show that the equation $f(x) = 0$ has exactly one root in the interval (a, ∞).

SOLUTION: We assume that $f(x)$ is twice differentiable on the interval $[a, \infty)$ and satisfies
(i): $f(a) = A > 0$;
(ii): $f'(a) < 0$;
(iii): $f''(x) \leq 0$ for $x \geq a$.
We will show that there exists a solution to the equation $f(x) = 0$.

By assumption (iii) the function $f(x)$ is concave and that implies that the tangent line $y = g(x)$to the graph of $f(x)$ at $(a, f(a))$ is above the the graph of $f(x)$. In other words we have the following inequality:

$$g(x) = f'(a)(x-a) + f(a) \geq f(x), \quad \text{for} \quad x \geq a.$$

Indeed, by Taylor Theorem (we use here the Lagrange form of the remainder),

$$(1) \qquad \forall_{x>a}\ \exists_{c_x \in (a,x)}\ f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

Since $f''(x) \leq 0$ for all $x \geq a$, we get (1).

Since for $b = -\frac{f(a)}{f'(a)} + a > a$, we have that $g(b) = 0$, thus $f(b) \leq g(b) = 0$. On the other hand, $f(a) > 0$, therefore, by the Intermediate Value Theorem there exists a point $x_o \in [a,b]$ such that $f(x_o) = 0$.

In order to show that there is only one solution to the equation $f(x) = 0$ for $x > a$, assume that there are x_1 and x_2 such that $a < x_1 < x_2$ and $f(x_1) = f(x_2) = 0$. Then, by Rolle's theorem, there exists $c \in (x_1, x_2)$ such that $f'(c) = 0$. Since $f(x)$ is twice differentiable function in an interval $[a, \infty)$, we can apply Lagrange Theorem to $f'(x)$ on the interval $[a, c]$, i.e. we get

$$\exists_{d \in (a,c)} \quad \frac{f'(c) - f'(a)}{c - a} = f''(d) < 0 \iff -f'(a) < 0,$$

which is a contradiction with the assumption (ii). Consequently, it is impossible for the equation $f(x) = 0$ to have two solutions.

21. Use l'Hôpital's Rule to compute the following limits

(a) $\lim_{x \to 0} \frac{\cosh x - \cos x}{x^2}$;

(b) $\lim_{x\to 0} \frac{\arcsin 2x - 2\arcsin x}{x^3}$;
(c) $\lim_{x\to\infty} \frac{x^{\ln x}}{(\ln x)^x}$.

SOLUTIONS: (a):

$$\begin{aligned}\lim_{x\to 0} \frac{\cosh x - \cos x}{x^2} &\overset{HR}{=} \lim_{x\to 0} \frac{\sinh x + \sin x}{2x} \\ &\overset{HR}{=} \lim_{x\to 0} \frac{\cosh x + \cos x}{2} = \frac{1}{2}[\cosh 0 + \cos 0] = 1.\end{aligned}$$

(b):

$$\begin{aligned}&\lim_{x\to 0} \frac{\arcsin(2x) - 2\arcsin x}{x^3} \\ &= \lim_{x\to 0} \frac{\frac{2}{\sqrt{1-4x^2}} - \frac{2}{\sqrt{1-x^2}}}{3x^2} \\ &= \lim_{x\to 0} \frac{1}{\sqrt{1-4x^2}\sqrt{1-x^2}} \cdot \lim_{x\to 0} \frac{2\sqrt{1-x^2} - 2\sqrt{1-4x^2}}{3x^2} \\ &= \lim_{x\to 0} \frac{2[\frac{-x}{\sqrt{1-x^2}} - \frac{-4x}{\sqrt{1-4x^2}}]}{6x} \\ &= \lim_{x\to 0} \frac{1}{3}[\frac{4}{\sqrt{1-4x^2}} - \frac{1}{\sqrt{1-x^2}}] = \frac{3}{3} = 1.\end{aligned}$$

(c): Assume $\lim_{x\to\infty} \frac{x^{\ln x}}{(\ln x)^x}$ exists and $\lim_{x\to\infty} \frac{x^{\ln x}}{(\ln x)^x} = A$. Then since $0 < x^{\ln x} < (\ln x)^x$ for $x > 1$, it follows $0 \le A \le 1$. Thus,

$$\begin{aligned}\lim_{x\to\infty} \frac{x^{\ln x}}{(\ln x)^x} &= \lim_{x\to\infty} \frac{x^{\ln x}}{(\ln x)^x} \frac{2\ln x \frac{1}{x}}{\ln(\ln x) + \frac{1}{\ln x}} \\ &= 2A \cdot \lim_{x\to\infty} \frac{(\ln x)^2}{x \ln x \ln(\ln x) + 1} \\ &= 2A \lim_{x\to\infty} \frac{2\ln x \frac{1}{x}}{\ln x \ln(\ln x) + x\frac{1}{x}\ln(\ln x) + x\ln x \frac{1}{\ln x}\frac{1}{x}} \\ &= 2A \lim_{x\to\infty} 2\frac{\ln x}{x} \cdot \lim_{x\to\infty} \frac{1}{1 + \ln(\ln x) + \ln x \ln(\ln x)} \\ &= 2A \cdot 0 \cdot 0 = 0.\end{aligned}$$

In order to show that the limit $A = \lim_{x\to\infty} \frac{x^{\ln x}}{(\ln x)^x}$ exists. Notice that $f(x) = \frac{x^{\ln x}}{(\ln x)^x} = e^{\ln^2 x - x \ln\ln x}$, therefore it is enough to show (by continuity of the natural exponential function e^x) that the function the limit of the function $\varphi(x) = \ln^2 x - x\ln\ln x$ is decreasing. Indeed, if $\varphi(x)$ is decreasing, then $f(x) = e^{\varphi(x)}$ is also decreasing and bounded from below, so $A = \lim_{x\to\infty} f(x) = \inf_x f(x)$. Notice that

$$\begin{aligned}\varphi'(x) &= 2\frac{\ln x}{x} - \ln\ln x - \frac{1}{\ln x}, \\ \varphi''(x) &= 2\frac{1}{x^2} - 2\frac{\ln x}{x^2} - \frac{1}{x\ln x} + \frac{1}{x\ln^2 x} \\ &= 2\frac{1 - \ln x}{x^2} + \frac{1 - \ln x}{x\ln^2 x}.\end{aligned}$$

Since $\varphi'(e) = \frac{2}{e} - 1 < 0$ and for all $x > e$, we have that

$$\varphi''(x) = 2\frac{1-\ln x}{x^2} + \frac{1-\ln x}{x\ln^2 x} < 0,$$

it follows that $\varphi'(x)$ is decreasing on the interval $[e, \infty)$, so $\varphi'(x) < \varphi'(e) < 0$ for $x > e$, and consequently $\varphi(x)$ is also decreasing. Therefore, the limit $A = \lim_{x\to\infty} \frac{x^{\ln x}}{(\ln x)^x}$ exists.

Let us point out that this question can be solved without using l'Hôpital Rule. Since for $x > e$ we have that $\varphi'(x) < \varphi'(e) =: -\alpha < 0$, it follows that

$$\forall_{x>e}\ \exists_{c\in(e,x)}\ \ \varphi(x) = \varphi(e) + \varphi'(c)(x-e) < \varphi(e) - \alpha(x-e),$$

so

$$\lim_{x\to\infty} \varphi(x) \le \lim_{x\to\infty} [\varphi(e) - \alpha(x-e)] = -\infty$$

i.e.

$$\lim_{x\to\infty} \varphi(x) = -\infty \quad\Longrightarrow\quad \lim_{x\to\infty} e^{\varphi(x)} = e^{-\infty} = 0.$$

22. Show that

$$\int_0^a x^3 f(x^2)dx = \tfrac{1}{2}\int_0^{a^2} x f(x)\,dx,$$

where $a > 0$ and $f(x)$ is a continuous function on the interval $[0, a^2]$.

SOLUTION: Let $x^2 = t$. Then

$$\begin{aligned}\int_0^a x^3 f(x^2)dx &= \frac{1}{2}\int_0^a x^2 f(x^2) 2x dx = \frac{1}{2}\int_0^{a^2} t f(t)dt\\ &= \frac{1}{2}\int_0^{a^2} f(x)dx.\end{aligned}$$

13.5 Chapter 8: Riemann Integral

1. Give an example of a function $f : [a, b] \to \mathbb{R}$ which is bounded but not integrable. Include a formal argument showing that your function is indeed not integrable.

SOLUTION: We can define the function $f : [a, b] \to \mathbb{R}$ by the formula

$$f(x) = \begin{cases} 1 & \text{if } \ x \in \mathbb{Q}, \\ -1 & \text{if } \ x \in \mathbb{Q}^c, \end{cases} \tag{13.41}$$

Consider any partition $P = \{x_k\}_{k=1}^n$ of the interval $[a, b]$. Since in every interval $[x_{k-1}, x_k]$ there are rational and irrational numbers, we have

$$\begin{aligned} M_k(f) &= \sup\{f(x) : x \in [x_{k-1}, x_k]\} = 1,\\ m_k(f) &= \inf\{f(x) : x \in [x_{k-1}, x_k]\} = -1.\end{aligned}$$

Consequently,

$$s_+(f,P) - s_-(f,P) = \sum_{k=1}^{n}(M_k(f) - m_k(f))\triangle x_k = 2\sum_{k=1}^{n}\triangle x_k = 2(b-a).$$

thus

$$\lim_{\|P\|\to 0}\Big(s_+(f,P) - s_-(f,P)\Big) = 2(b-a) \neq 0,$$

so $f : [a,b] \to \mathbb{R}$ is not integrable.

2. If f is increasing on the interval $I = \{x : a \le x \le b\}$, show that f is integrable. *Hint:* Use the formula

$$S(f,P) - s(f,P) = \sum_{k=1}^{n}\big(f(x_k) - f(x_{k-1})\big)\triangle x_k.$$

Replace each $\triangle x_k$ by the longest subinterval getting an inequality, and then observe that the terms "telescope." (Notice that we are using here slightly different notation than the textbook.) Support your claims by refering to appropriate theorems (which should be stated).

SOLUTION: The function $f : [a,b] \to \mathbb{R}$ is integrable if and only if (by Theorem which was proved in class) it is bounded and

$$\lim_{\|P\|\to 0}\Big(S(f,P) - s(f,P)\Big) = 0.$$

If f is increasing on the interval $[a,b]$ then clearly we have

$$f(a) \le f(x) \le f(b), \quad \text{for all} \quad x \in [a,b],$$

which means f is bounded. On the other hand, for every partition $P = \{x_k\}_{k=1}^{n}$, with $\|P\| = \max\{\triangle x_k = x_k - x_{k-1} : k = 1,2,\dots,n\}$ we have

$$\begin{aligned} M_k(f) &= \sup\{f(x) : x \in [x_{k-1}, x_k]\} = f(x_k),\\ m_k(f) &= \inf\{f(x) : x \in [x_{k-1}, x_k]\} = f(x_{k-1}). \end{aligned}$$

thus

$$\begin{aligned} S(f,P) - s(f,P) &= \sum_{k=1}^{n}(M_k(f) - m_k(f))\triangle x_k\\ &\le \sum_{k=1}^{n}(M_k(f) - m_k(f))\|P\|\\ &= \|P\|\Big(f(x_1) - f(x_0) + f(x_2) - f(x_1) + \dots\\ &\quad + f(x_{k-1}) - f(x_{k-2}) + f(x_k) - f_(x_{k-1})\Big)\\ &= \|P\|(b-a). \end{aligned}$$

Therefore, we have

$$0 \le \lim_{\|P\|\to 0}\Big(S(f,P) - s(f,P)\Big) \le \lim_{\|P\|\to 0}\|P\|(b-a) = 0,$$

thus, by squeeze theorem,

$$\lim_{\|P\|\to 0}\Big(S(f,P)-s(f,P)\Big)=0,$$

and consequently f is integrable.

3. Give an example of a function f defined on $I=\{x: 0\le x\le 1\}$ such that $|f|$ is integrable but f is not. (Include a formal argument showing that your function f is indeed not integrable.)

SOLUTION: We use the same function $f:[a,b]\to\mathbb{R}$, which was defined in Problem 1, with $a=0$ and $b=1$. The function f is not integrable, but we have $|f(x)|=1$ for all $x\in[0,1]$, which implies that $|f|$ is integrable on the interval $[0,1]$.

22. Assume that $f:[a,b]\to\mathbb{R}$ is an integrable function. Define the following functions f_+, f_- : $[a,b]\to\mathbb{R}$, by

$$f_+(x)=\begin{cases} f(x) & \text{if } \ f(x)\ge 0,\\ 0 & \text{if } \ f(x)<0\end{cases}\qquad f_-(x)=\begin{cases} -f(x) & \text{if } \ f(x)\le 0,\\ 0 & \text{if } \ f(x)>0\end{cases}$$

Show that the functions f_+ and f_- are integrable on $[a,b]$. *Hint:* Prove the following inequality

$$S(f_+,P)-s(f_+,P)\le S(f,P)-s(f,P).$$

SOLUTION: Consider any partition $P:=\{x_k\}_{k=1}^n$ of the interval $[a,b]$. Then we clearly have

$$\begin{aligned}
&M_k(f_+)=M_k(f), && \text{if } \ M_k(f)>0,\\
&M_k(f_+)=0=m_k(f_+), && \text{if } \ M_k(f)\le 0,\\
&m_k(f_+)=m_k(f), && \text{if } \ m_k(f)>0,\\
&m_k(f_+)=0, && \text{if } \ m_k(f)\le 0.
\end{aligned}$$

All the above cases imply that

$$M_k(f_+)-m_k(f_+)\le M_k(f)-m_k(f),\quad \text{for } k=1,2,\dots,n,$$

and consequently, we have

$$\begin{aligned}
0\le S(f_+,P)-s(f_+,P)&=\sum_{k=1}^n\Big(M_k(f_+)-m_k(f_+)\Big)\triangle x_k\\
&\le\sum_{k=1}^n\Big(M_k(f)-m_k(f)\Big)\triangle x_k
\end{aligned}$$

and since f is integrable, it follows

$$\lim_{\|P\|\to 0}\Big(S(f,P)-s(f,P)\Big)=0,$$

thus, by squeeze theorem,

$$\lim_{\|P\|\to 0}\Big(S(f_+,P)-s(f_+,P)\Big)=0,$$

which implies that f_+ is integrable on $[a,b]$. In order to show that f_- is also integrable, we notice that $f_-=(-f)_+$, and the conclusion follows from the linearity property of the definite integrals and the previous statement.

4. Use the previous problem to prove the following statement:

Theorem 13.7. *Let $f:[a,b]\to\mathbb{R}$ be an integrable function, then $|f|:[a,b]\to\mathbb{R}$ is also integrable and we have*

$$\left|\int_a^b f(x)dx\right|\le\int_a^b |f(x)|dx.$$

SOLUTION: Since $f:[a,b]\to\mathbb{R}$ is integrable, by Problem 3, the functions f_+ and f_- are also integrable. Since $|f|=f_++f_-$, the function $|f|$ is integrable. Now we notice that we have the following inequality

$$-|f(x)|\le f(x)\le |f(x)|,\quad \text{for all}\quad x\in[a,b],$$

thus, by the appropriate property of the definite integral, we have

$$-\int_a^b |f(x)|dx\le\int_a^b f(x)dx\le\int_a^b |f(x)|dx,$$

which implies

$$\left|\int_a^b f(x)dx\right|\le\int_a^b |f(x)|dx.$$

5. Suppose that $f,\ g:[a,b]\to\mathbb{R}$ are integrable.

(a) Show that $f^2,\ g^2,\ fg:[a,b]\to\mathbb{R}$ are integrable;
Hint: Consider first the case where f and g are non-negative, and show that

$$M_k(fg)\le M_k(f)M_k(g),\quad m_k(fg)\ge m_k(f)m_k(g),$$

then observe

$$\begin{aligned}M_k(fg)-m_k(fg)&\le M_k(f)M_k(g)-m_k(f)m_k(g)\\&=M_k(f)\Big(M_k(g)-m_k(g)\Big)+m_k(g)\Big(M_k(f)-m_k(f)\Big)\\&\le M(f)\Big(M_k(g)-m_k(g)\Big)+M(g)\Big(M_k(f)-m_k(f)\Big),\end{aligned}$$

where

$$M(f)=\sup\{f(x):x\in[a,b]\},\quad M(g)=\sup\{g(x):x\in[a,b]\},$$

and use the appropriate results to show that

$$\lim_{\|P\|\to 0}\Big(S(fg,P)-s(fg,P)\Big)=0.$$

Next, consider the general case, where f and g are not necessarily non-negative, and use the representation (see Problem 4 and appropriate theorems)

$$fg=(f_+-f_-)(g_+-g_-)=f_+g_+-f_+g_--f_-g_++f_-g_-,$$

to conclude that fg is integrable.

(b) Define the "dot-product" for the functions f and g by

$$f \bullet g := \int_a^b f(x)g(x)dx.$$

Show that the product $f \bullet g$ satisfies the following conditions

(i) $f \bullet f \geq 0$;

(ii) $f \bullet g = g \bullet f$;

(iii) $f \bullet (g_1 + g_2) = f \bullet g_1 + f \bullet g_2$;

SOLUTION: (a): We consider the case where f and g are non-negative. Let $P = \{x_k\}_{k=1}^n$ be a partition of $[a, b]$, with

$$M_k(f) = \sup\{f(x) : x \in [x_{k-1}, x_k]\}, \qquad M_k(g) = \sup\{g(x) : x \in [x_{k-1}, x_k]\}$$
$$m_k(f) = \inf\{f(x) : x \in [x_{k-1}, x_k]\}, \qquad m_k(g) = \inf\{g(x) : x \in [x_{k-1}, x_k]\}$$

Since both functions f and g are non-negative, we have that the inequalities

$$f(x) \leq M_k(f), \quad g(x) \leq M_k(g),$$
$$f(x) \geq m_k(f), \quad g(x) \geq m_k(g),$$

for $x \in [x_{k-1}, x_k]$, imply that

$$m_k(f)m_k(g) \leq f(x)g(x) \leq M_k(f)M_k(g), \quad x \in [x_{k-1}, x_k],$$

i.e. $m_k(f)m_k(g)$ is a lower bound of $f(x)g(x)$, and $M_k(f)M_k(g)$ is an upper bound of $f(x)g(x)$, on $[x_{k-1}, x_k]$, which implies that

$$M_k(fg) \leq M_k(f)M_k(g), \quad m_k(fg) \geq m_k(f)m_k(g).$$

Therefore, we obtain

$$\begin{aligned}
S(fg, P) - s(fg, P) &= \sum_{k=1}^n \Big(M_k(fg) - m_k(fg)\Big)\triangle x_k \\
&\leq \sum_{k=1}^n \Big(M_k(f)M_k(g) - m_k(f)m_k(g)\Big)\triangle x_k \\
&= \sum_{k=1}^n M_k(f)\big(M_k(g) - m_k(g)\big)\triangle x_k \\
&\quad + \sum_{k=1}^n m_k(g)\big(M_k(f) - m_k(f)\big)\triangle x_k \\
&\leq M(f)\sum_{k=1}^n \Big(M_k(g) - m_k(g)\Big)\triangle x_k \\
&\quad + M(g)\sum_{k=1}^n \Big(M_k(f) - m_k(f)\Big)\triangle x_k \\
&= M(f)\Big(S(g, P) - s(g, P)\Big) \\
&\quad + M(g)\Big(S(f, P) - s(f, P)\Big).
\end{aligned}$$

Since the functions f and g are assumed to be integrable, it follows that

$$\lim_{\|P\|\to 0}\Big(S(f,P)-s(f,P)\Big)=\lim_{\|P\|\to 0}\Big(S(g,P)-s(g,P)\Big)=0$$

Thus, by squeeze theorem,

$$\begin{aligned}0&\le \lim_{\|P\|to0}\Big(S(fg,P)-s(fg,P)\Big)\\&\le M(f)\lim_{\|P\|\to 0}\Big(S(g,P)-s(g,P)\Big)+M(g)\lim_{\|P\|\to 0}\Big(S(g,P)-s(g,P)\Big)=0,\end{aligned}$$

which implies that

$$\lim_{\|P\|to0}\Big(S(fg,P)-s(fg,P)\Big)=0$$

thus fg is integrable. Consider now the case f and g are not necessarily positive, but integrable. Then, by Problem 3, the functions f_+, f_-, g_+ and g_- are also integrable (and non-negative). In particular, it follows from the previous case that all the products $f_\pm g_\pm$ and $f_\mp g_\pm$ are also integrable. On the other hand, since

$$fg=(f_+-f_-)(g_+-g_-)=f_+g_+-f_+g_--f_-g_++f_-g_-,$$

it follows, by the linearity property if the definite integrals, that the product fg is also integrable. Of course the integrability of f^2 (and of g^2) follows from the above statement, where we take $g=f$.
(b): Since $f^2(x)\ge 0$, it follows that

$$f\bullet f=\int_a^b f^2(x)dx\ge 0.$$

It is also clear that we have

$$f\bullet g=\int_a^b f(x)g(x)dx=\int_a^b g(x)f(x)dx=g\bullet f$$

and

$$\begin{aligned}f\bullet(\alpha_1g_1+\alpha_2g_2)&=\int_a^b f(x)\big(\alpha_1g_1(x)+\alpha_2g_2(x)\big)dx\\&=\alpha_1\int_a^b f(x)g_1(x)dx+\alpha_2\int_a^b f(x)g_2(x)dx\\&=\alpha_1(f\bullet g_1)+\alpha_2(f\bullet g_2).\end{aligned}$$

6. Use result in Problem 5 to prove the following Cauchy-Schwarz inequality:

$$\left[\int_a^b f(x)g(x)dx\right]^2\le\left[\int_a^b f^2(x)dx\right]\left[\int_a^b g^2(x)dx\right],$$

where $f,\ g:[a,b]\to\mathbb{R}$ are integrable functions.
Hint: Use the dot product $f\bullet g$, which was defined in Problem 6, in the same way it was used in the proof of the Cauchy-Schwarz inequality for sequences.

SOLUTION: Let f and g be two integrable on $[a,b]$ functions. We define the following scalar function

$$\varphi(t) = (f - tg) \bullet (f - tg) = \int_a^b (f(x) - tg(x))^2 dx \geq 0.$$

Since, by the property of the product $f \bullet g$ (which were proved in Problem 6) we have

$$\varphi(t) = (f \bullet f) - 2t(f \bullet g) + t^2(g \bullet g),$$

which implies that $\varphi(t)$ is a quadratic function, such that it is not negative. In particular, it means that its discriminant is non-positive:

$$\triangle := 4(f \bullet g)^2 - (f \bullet f)(g \bullet g) \leq 0,$$

i.e.

$$(f \bullet g)^2 \leq (f \bullet f)(g \bullet g),$$

which means

$$\left[\int_a^b f(x)g(x)dx\right]^2 \leq \left[\int_a^b f^2(x)dx\right]\left[\int_a^b g^2(x)dx\right].$$

7. Show that if $f : [a,b] \to \mathbb{R}$ is a positive function, which is integrable on $[a,b]$, then for every $n \in \mathbb{N}$ the function $f^n : [a,b] \to \mathbb{R}$ is also integrable. Is this result true for functions which are not necessarily positive? *Hint:* Use the following inequality

$$M_k^n - m_k^n = (M_k - m_k)(M_k^{n-1} + M_k^{n-2}m_k + \cdots + M_k m_k^{n-2} + m_k^{n-2}) \leq nM^{n-1}(M_k - m_k),$$

where

$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}, \quad m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\},$$
$$M = \sup\{f(x) : x \in [a,b]\}.$$

SOLUTION: Since in Problem 5, it was proved that if f and g are integrable, then it follows that fg is also integrable, we can apply the mathematical induction to prove that f^n is always integrable for all $n \in \mathbb{N}$. Indeed, this statement is true (by assumption) for $n = 1$. Assume that the function $g := f^n$ is integrable. Then, by the result in Problem 5 (a), the function $f^{n+1} = fg = ff^n$ is also integrable. Therefore, by the principle of mathematical induction, the function f^n is integrable for all $n \in \mathbb{N}$.
The statement can be also proved (independently of the result of Problem 5), by applying the hint. Namely, it is clear (for f non-negative) that

$$\begin{aligned}
S(f^n, P) - s(f^n, P) &= \sum_{k=1}^m \Big(M_k(f^n) - m_k(f^n)\Big)\triangle x_k \\
&\leq \sum_{k=1}^m \Big(M_k(f)^n - m_k(f)^n\Big)\triangle x_k \\
&\leq nM^{n-1}\sum_{k=1}^m \Big(M_k(f) - m_k(f)\Big)\triangle x_k \\
&= nM^{n-1}\Big(S(f,P) - s(f,P)\Big).
\end{aligned}$$

Since f is integrable, it follows

$$\lim_{\|P\|\to 0}\Big(S(f,P)-s(f,P)\Big)=0,$$

and consequently, by the squeeze theorem

$$\lim_{\|P\|\to 0}\Big(S(f^n,P)-s(f^n,P)\Big)=0,$$

which implies that f^n is also integrable.

8. Prove the following theorem:

Theorem 13.8. *Let $f:[a,b]\to\mathbb{R}$ be a continuous non-negative function such that*

$$\int_a^b f(x)dx=0,$$

then $f(x)=0$ for all $x\in[a,b]$.

SOLUTION: Assume for contradiction that $f(x_o)>0$ for some $x_o\in[a,b]$. We put $\varepsilon=\frac{f(x_o)}{2}$. Then by continuity of f, we have that

$$\exists_{\delta>0}\ \forall_{x\in[a,b]}\quad |x-x_o|<\delta \implies |f(x)-f(x_o)|<\varepsilon.$$

We can always assume that $\delta<\max\{b-x_o,x_o-a\}$ (notice that the point x_o may be the end-point of the interval $[a,b]$, and such a case should also be considered). Then we have for $x\in[a,b]$ such that $|x-x_o|<\delta$, that

$$f(x_o)-f(x)\le|f(x)-f(x_o)|<\varepsilon=\frac{f(x_o)}{2}\quad\implies\quad f(x)>\frac{f(x_o)}{2}.$$

Let us define the function $g:[a,b]\to\mathbb{R}$ by

$$g(x)=\begin{cases}\frac{f(x_o)}{2}, & \text{if}\ \ |x-x_o|<\delta,\\ 0, & \text{if}\ \ |x-x_o|\ge\delta,\end{cases}$$

where $x\in[a,b]$. Then clearly we have $f(x)\ge g(x)$ for all $x\in[a,b]$, therefore,

$$\int_a^b f(x)dx\ge\int_a^b g(x)dx\ge\delta\frac{f(x_o)}{2}>0,$$

which is a contradiction with the assumption that $\int_a^b f(x)dx=0$. Consequently, we have that $f(x)=0$ for all $x\in[a,b]$.

9. For a continuous function $f:[a,b]\to\mathbb{R}$ we define the *norm* of f by

$$\|f\|=\sqrt{f\bullet f}.$$

Show that this norm $\|\cdot\|$ satisfies the following properties:

(i) $\|f\| \geq 0$ and $\|f\| = 0 \Leftrightarrow f(x) = 0$ for all $x \in [a, b]$ (we assume here that f is continuous);
(ii) $\|\alpha f\| = |\alpha| \, \|f\|$, for $\alpha \in \mathbb{R}$;
(iii) $\|f + g\| \leq \|f\| + \|g\|$.

SOLUTION: (i): It is clear that

$$f(x) \geq 0 \quad \text{thus} \quad \int_a^b f^2(x)dx \geq 0.$$

Since f is continuous, the function $f^2 : [a, b] \to \mathbb{R}$ is a non-negative and continuous. By Bonus Problem 1, if $\int_a^b f^2(x)dx = 0$ then $f^2(x) = 0$ for all $x \in [a, b]$, which implies that $f(x) = 0$ for all $x \in [a, b]$. Consequently, we have

$$\|f\| = 0 \quad \Longrightarrow \quad f(x) = 0 \quad \text{for all} \quad x \in [a, b].$$

(ii): Notice that

$$\|\alpha f\| = \left(\int_a^b \alpha^2 f^2(x)dx\right)^{\frac{1}{2}} = |\alpha| \left(\int_a^b f^2(x)dx\right)^{\frac{1}{2}} = \alpha\|f\|.$$

(iii): By the Cauchy-Schwarz inequality, which was proved in Problem 7, we have

$$\begin{aligned}
\|f + g\|^2 &= (f + g) \bullet (f + g) \leq (f \bullet f) + 2(f \bullet g) + (g \bullet g) \\
&= \|f\|^2 + 2(f \bullet g) + \|g\|^2 \leq \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 \\
&= (\|f\| + \|g\|)^2,
\end{aligned}$$

thus

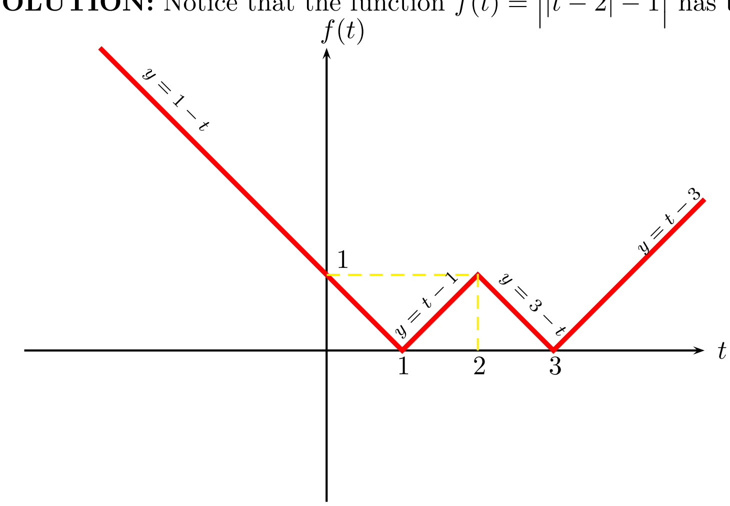
$$\|f + g\| \leq \|f\| + \|g\|.$$

10. We define the function $F : [0, \infty) \to \mathbb{R}$ by the formula

$$F(x) = \int_0^x \Big||t - 2| - 1\Big|dt.$$

Find the explicit formula for $F(x)$.

SOLUTION: Notice that the function $f(t) = \Big||t - 2| - 1\Big|$ has the graph illustrated below:



Therefore, by direct computation we obtain:

$$F(x) = \begin{cases} x - \frac{1}{2}x^2 & \text{if } \; x \le 1, \\ \frac{1}{2}x^2 - x + 1 & \text{if } \; 1 \le x \le 2, \\ 3x - \frac{1}{2}x^2 - 3 & \text{if } \; 2 \le x \le 3 \\ \frac{1}{2}x^2 - 3x + 6 & \text{if } x \ge 3. \end{cases}$$

11. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that we have the following equality

$$\int_0^{\pi} x\, f(\sin x)dx = \pi \int_0^{\frac{\pi}{2}} f(\sin x)dx.$$

SOLUTION: By making a substitution $t = x - \frac{\pi}{2}$ we obtain

$$\int_0^{\pi} xf(\sin x)dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(t + \frac{\pi}{2}\right) f(\cos t)dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} tf(\cos t)dt + \frac{\pi}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\cos t)dt.$$

Notice that the function $t \mapsto tf(\cos t)$ is odd, thus

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} tf(\cos t)dt = 0,$$

and the function $t \mapsto f(\cos t)$ is even, thus

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\cos t)dt = 2 \int_0^{\frac{\pi}{2}} f(\cos t)dt.$$

Consequently, we get

$$\int_0^{\pi} x\, f(\sin x)dx = \pi \int_0^{\frac{\pi}{2}} f(\sin x)dx.$$

12. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that

(a) $\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx$;
(b) $\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x)dx$.

SOLUTION: (a): We make the substitution $t = \frac{\pi}{2} - x$ to get

$$\begin{aligned} \int_0^{\frac{\pi}{2}} f(\sin x)dx &= \int_{\frac{\pi}{2}}^{0} f(\sin(\tfrac{\pi}{2} - t)(-dt) \\ &= \int_0^{\frac{\pi}{2}} f(\cos t)dt = \int_0^{\frac{\pi}{2}} f(\cos x)dx. \end{aligned}$$

(b): We apply the substitution $x = \pi - t$, $\sin x = \sin t$, hence

$$\begin{aligned} I = \int_0^{\pi} xf(\sin x)dx &= \int_{\pi}^{0} (\pi - t)f(\sin t)(-dt) \\ &= -\int_0^{\pi} tf(\sin t)dt + \pi \int_0^{\pi} f(\sin t)dt, \end{aligned}$$

thus

$$I = -I + \pi \int_0^\pi f(\sin x)dx,$$

so

$$I = \int_0^\pi x f(\sin x)dx = \frac{\pi}{2} \int_0^\pi f(\sin x)dx.$$

13. Let $f(x)$ be a continuous function in the interval $[0, 1]$. Show that

$$\int_0^\pi x f(\sin x)dx = \pi \int_0^{\frac{\pi}{2}} f(\sin x)dx.$$

SOLUTION: By Prob. 11(b) and 11(a) we have

$$\begin{aligned} I &= \int_0^\pi x f(\sin x)dx = \frac{\pi}{2} \int_0^\pi f(\sin x)dx \\ &= \frac{\pi}{2} \left[\int_0^{\frac{\pi}{2}} f(\sin x)dx + \int_{\frac{\pi}{2}}^\pi f(\sin x)dx \right] \qquad (\text{ substitution } x = \frac{\pi}{2} + t\ ) \\ &= \frac{\pi}{2} \left[\int_0^{\frac{\pi}{2}} f(\sin x)dx + \int_0^{\frac{\pi}{2}} f(\cos t)dt \right] \\ &= \frac{\pi}{2} \left[\int_0^{\frac{\pi}{2}} f(\sin x)dx + \int_0^{\frac{\pi}{2}} f(\sin t)dt \right] \\ &= \pi \int_0^{\frac{\pi}{2}} f(\sin x)dx. \end{aligned}$$

14. Show that

$$\int_0^1 x^m (1-x)^n dx = \int_0^1 x^n (1-x)^m dx.$$

SOLUTION: Let $t = 1 - x$. Then we have

$$\begin{aligned} \int_0^1 x^m (1-x)^n dx &= \int_1^0 (1-t)^m t^n d(1-t) \\ &= -\int_1^0 (1-t)^m t^n dt = \int_0^1 (1-t)^m t^n dt \\ &= \int_0^1 (1-x)^m x^n dx. \end{aligned}$$

15. Let $f(x)$ be a continuously differentiable function on the interval $[a, b]$ such that $f(a) = 0$.Prove that

$$\sup_{x \in [a,b]} |f(x)| \leq \sqrt{(b-a) \int_a^b [f'(x)]^2 \, dx}.$$

SOLUTION: Since $f(a) = 0$ and $f'(t)$ is continuous, we have $f(x) = \int_a^x f'(t)dt$. Thus, by Fundamental Theorem of Calculus and Cauchy-Schwartz inequality, we have for all $x \in [a, b]$

$$\begin{aligned} |f(x)| &= \left| \int_a^x f'(t)dt \right| \le \int_a^x |f'(t)|dt \\ &\le \int_a^b 1 \cdot |f'(t)|dt \le \sqrt{\int_a^b dt}\sqrt{\int_a^b |f'(t)|^2 dt} \\ &= \sqrt{(b-a)\int_a^b |f'(x)|^2 dx}. \end{aligned}$$

Consequently, $\sqrt{(b-a)\int_a^b |f'(x)|^2 dx}$ is an upper bound for the function $|f(x)|$, hence

$$\sup_{x \in [a,b]} |f(x)| \le \sqrt{(b-a)\int_a^b |f'(x)|^2 dx}$$

16. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that there exists the limit $f(\infty) = \lim_{x\to\infty} f(x)$. Show that

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = \Big(f(0) - f(\infty)\Big) \ln \tfrac{b}{a},$$

where $a > 0$ and $b > 0$.

SOLUTION: We have

$$\begin{aligned} I = \int_0^\infty \frac{f(ax) - f(bx)}{x} dx &= \lim_{T\to\infty,\ \varepsilon\to 0^+} \int_\varepsilon^T \frac{f(ax) - f(bx)}{x} dx \\ &= \lim_{T\to\infty,\ \varepsilon\to 0^+} \left[\int_\varepsilon^T \frac{f(ax)}{x} dx - \int_\varepsilon^T \frac{f(bx)}{x} dx \right]. \end{aligned}$$

By applying the substitution $t = ax$ to the first integral and the substitution $t = bx$ to the second integral, we obtain

$$\begin{aligned} I &= \lim_{T\to\infty,\ \varepsilon\to 0^+} \left[\int_{\varepsilon a}^{Ta} \frac{f(t)}{t} dt - \int_{\varepsilon b}^{Tb} \frac{f(t)}{t} dt \right] \\ &= \lim_{T\to\infty,\ \varepsilon\to 0^+} \left[\int_{\varepsilon a}^{\varepsilon b} \frac{f(t)}{t} dt - \int_{Ta}^{Tb} \frac{f(t)}{t} dt \right] \\ &= \lim_{\varepsilon\to 0} \int_{\varepsilon a}^{\varepsilon b} \frac{f(t)}{t} dt - \lim_{T\to\infty} \int_{Ta}^{Tb} \frac{f(t)}{t} dt. \end{aligned}$$

By the Generalized Mean Value Theorem, there exist $c_\varepsilon \in (\varepsilon a, \varepsilon b)$ and $c_T \in (Ta, Tb)$ such that

$$\int_{\varepsilon a}^{\varepsilon b} \frac{f(t)}{t} dt = f(c_\varepsilon) \int_{\varepsilon a}^{\varepsilon b} \frac{dt}{t} = f(c_\varepsilon) \ln \frac{b}{a},$$

and

$$\int_{aT}^{bT} \frac{f(t)}{t} dt = f(c_T) \int_{aT}^{bT} \frac{dt}{t} = f(c_T) \ln \frac{b}{a}.$$

Since $\lim_{\varepsilon \to 0} c_\varepsilon = 0$ and $\lim_{T\to\infty} c_T = \infty$, we get

$$I = \lim_{\varepsilon \to 0} f(c_\varepsilon) \ln \frac{b}{a} - \lim_{T\to\infty} f(c_T) \ln \frac{b}{a} = [f(0) - f(\infty)] \ln \frac{b}{a}.$$

17. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that for every $A > 0$ the integral $\int_A^\infty \frac{f(x)}{x} dx$ converges. Show that

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = f(0) \ln \tfrac{b}{a},$$

where $a > 0$ and $b > 0$.

SOLUTION: Assume that $\forall A > 0$, $\int_A^\infty \frac{f(x)}{x} dx$ converges. Then $\forall A_0 > 0$, $A_0 = \int_{A_0}^\infty \frac{f(t)}{t} dt < \infty$ and for function $g(x) : \quad [A_0, \infty) \longrightarrow \mathbb{R}$ defined by $g(x) = \int_x^\infty \frac{f(t)}{t} dt$, it follows that

$$\begin{aligned}
\lim_{x\to\infty} g(x) &= \lim_{x\to\infty} (\int_{A_0}^\infty \frac{f(t)}{t} dt - \int_{A_0}^x \frac{f(t)}{t} dt) \\
&= I_0 - \lim_{x\to\infty} \int_{A_0}^x \frac{f(t)}{t} dt = I_0 - \int_{A_0}^\infty \frac{f(t)}{t} dt \\
&= I_0 - I_0 = 0.
\end{aligned}$$

Now, from ???, $I = \int_0^\infty \frac{f(ax)-f(bx)}{x} dx = I_1 + I_2$ with $I_1 = \lim_{\varepsilon\to 0} \int \frac{f(t)}{t} dt = f(0) \ln \frac{b}{a}$ and

$$\begin{aligned}
I_2 &= \lim_{T\to 0} \int \frac{f(t)}{t} dt = \lim_{T\to 0} (g(aT) - g(bT)) \\
&= \lim_{T\to 0} g(aT) - \lim_{T\to 0} g(bT) = 0 - 0 = 0.
\end{aligned}$$

We conclude that $I = I_1 + I_2 = I_1 = f(0) \ln \frac{b}{a}$.

18. Compute the following improper integrals:

(a) $\int_1^\infty \frac{dx}{x^4}$;
(b) $\int_0^\infty x e^{-x^2} dx$;
(c) $\int_0^\infty e^{-x} \sin x \, dx$.

SOLUTION: (a): $\displaystyle\int_1^\infty \frac{dx}{x^4} = \lim_{T\to\infty} \frac{-1}{3x^3}\bigg|_1^T = \frac{1}{3} - \lim_{T\to\infty} \frac{1}{3T^3} = \frac{1}{3}.$

(b): We apply the substitution $t = x^2$ to get:

$$\begin{aligned}
\int_0^\infty x e^{-x^2} dx &= \frac{1}{2} \int_0^\infty e^t dt = \frac{1}{2} \lim_{T\to\infty} \frac{-1}{e^t}\bigg|_0^T \\
&= \frac{1}{2} - \frac{1}{2} \lim_{T\to\infty} \frac{1}{e^T} = \frac{1}{2}.
\end{aligned}$$

(c): We have

$$\begin{aligned}\int_0^\infty e^{-x}\sin x dx &= \lim_{T\to\infty} \frac{-\sin x-\cos x}{2}e^{-x}\Big|_0^T\\ &= \frac{1}{2} - \lim_{T\to\infty}\frac{\sin T+\cos T}{2}e^{-T} = \frac{1}{2}.\end{aligned}$$

19. Check which of the following improper integrals converge

(a) $\int_0^\infty \sqrt{x}e^{-x}dx$,
(b) $\int_{e^2}^\infty \frac{dx}{x\ln\ln x}$.

SOLUTION: (a): Since $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x}e^{-x}}{\frac{1}{x^2+1}} = 0$ and the integral $\displaystyle\int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}$ converges, thus, by Limit Comparison Test, the integral $\displaystyle\int_0^\infty \sqrt{x}e^{-x}dx$ converges.

(b): Since $\frac{1}{x\ \ln\ln x} \geq \frac{1}{x\ \ln x\ \ln\ln x}$ for $x\geq e^2$, and

$$\int_{e^2}^\infty \frac{dx}{x\ln x\ln\ln x} = \int_{\ln 2}^\infty \frac{dt}{t} = \infty, \qquad \text{where } t=\ln\ln x,$$

Therefore, by Comparison Test, the integral $\displaystyle\int_{e^2}^\infty \frac{dx}{x\ln\ln x}$ diverges.

20. Evaluate the following improper integrals

(a) $\int_1^3 \frac{xdx}{\sqrt{x-1}}$,
(b) $\int_0^1 x\ln x\,dx$,
(c) $\int_{-1}^0 \frac{e^{\frac{1}{x}}}{x^3}dx$.

SOLUTION: (a): We have

$$\begin{aligned}\int_1^3 \frac{xdx}{\sqrt{x-1}} &= \int_0^2 \frac{1+t}{\sqrt{t}}dt \qquad \text{substitution } t=x-1\\ &= \lim_{\varepsilon\to 0^+}\left[2t^{\frac{1}{2}}+\frac{2}{3}t^{\frac{3}{2}}\right]_\varepsilon^2 = 2\sqrt{2}+\frac{4}{3}\sqrt{2} - \lim_{\varepsilon\to 0^+}\left(2\varepsilon^{\frac{1}{2}}+\frac{2}{3}\varepsilon^{\frac{3}{2}}\right)\\ &= \frac{10}{3}\sqrt{2}.\end{aligned}$$

(b): By integration by parts, we obtain

$$\int_0^1 x\ln x\,dx = \frac{1}{2}x^2\ln x\Big|_0^1 - \frac{1}{2}\int_0^1 xdx = -\frac{1}{2}\lim_{\varepsilon\to 0^+}\varepsilon^2\ln\varepsilon - \frac{1}{4}x^2\Big|_0^1 = \frac{1}{4}.$$

(c): By applying the substitution $t=-\frac{1}{x}$, we obtain

$$\begin{aligned}\int_{-1}^0 \frac{e^{\frac{1}{x}}}{x^3}dx &= -\int_1^\infty te^{-t}dt = -\left(-te^{-t}\big|_1^\infty + \int_1^\infty e^t dt\right)\\ &= \lim_{T\to\infty}\frac{T}{e^T} - e^{-1} + \lim_{T\to\infty} e^{-T} - e^{-1} = -2e^{-1}.\end{aligned}$$

21. Check which of the following improper integrals converge:

(a) $\int_0^1 \frac{\sqrt{x}}{\sqrt{1-x^4}} dx$;
(b) $\int_0^1 \frac{dx}{e^{\sqrt{x}}-1}$.

SOLUTION: (a): We have

$$0 \le \frac{\sqrt{x}}{\sqrt{1-x^4}} \le \frac{1}{\sqrt{1-x^2}}, \qquad \text{for } x \in [0,1],$$

and $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \arcsin x\Big|_0^1 = \frac{\pi}{2}$, so the integral $\int_0^1 \frac{\sqrt{x}}{\sqrt{1-x^4}} dx$ converges by Comparison Test.

(b): Since for $x \in [0,1]$ we have $\frac{1}{e^{\sqrt{x}}-1} \ge \frac{1}{e^x-1}$, and

$$\begin{aligned}\int_0^1 \frac{1}{e^x-1} dx &= \int_1^e \frac{dt}{t(t-1)} \qquad \text{(substitution } t = e^x\text{)}\\ &= \int_1^e \left(\frac{1}{t-1} - \frac{1}{t}\right) dt = \ln(t-1)\Big|_1^e - 1\\ &= \ln(e-1) - \lim_{\varepsilon \to 0^+} \ln \varepsilon - 1 = \infty,\end{aligned}$$

i.e. the integral $\int_0^1 \frac{1}{e^x-1} dx$ diverges, thus, by Comparison Test, we obtain that the integral $\int_0^1 \frac{dx}{e^{\sqrt{x}}-1}$ diverges.

22. Consider the so-called Dirichlet function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational.} \end{cases}$$

Show that for any interval $[a,b]$ $(a < b)$ the function f is not Riemann integrable on $[a,b]$.

SOLUTION: We will show that

$$\lim_{\|P\| \to 0} \Big(S(f,P) - s(f,P)\Big) > 0,$$

and consequently by Theorem 6.1, $f(x)$ is not Riemann integrable. Indeed, consider an arbitrary partition $P = \{x_k\}_{k=0}^n$

$$P : a = x_0 < x_1 < \dots < x_{k-1} < x_k < \dots < x_{n-1} < x_n = a,$$

and notice that

$$\begin{aligned} M_k &= \max\{f(x) : x \in [x_{k-1}, x_k]\} = 1,\\ m_k &= \min\{f(x) : x \in [x_{k-1}, x_k]\} = 0. \end{aligned}$$

Then we have

$$S(f,P) - s(f,P) = \sum_{k=1}^n (M_k - m_k)(x_k - x_{k-1}) = \sum_{k=1}^n (x_k - x_{k-1}) = b - a > 0,$$

thus

$$\lim_{\|P\|\to 0}\Big(S(f,P)-s(f,P)\Big)=b-a>0.$$

23. Consider the so-called Riemann function $f:\mathbb{R}\to\mathbb{R}$, defined by

$$f(x)=\begin{cases}0 & \text{if}\quad x \text{ is irrational or } x=0\\ \frac{1}{n} & \text{if}\quad x=\frac{m}{n} \text{ where the fraction } \frac{m}{n} \text{ is irreducible and } n\ge 1.\end{cases}$$

Show that the function f is Riemann integrable on $[0,1]$. (Actually f is integrable on every $[a,b]$).
Hint: Show that for given $\varepsilon>0$ there exists only finitely many $x\in(0,1]$ such that $x=\frac{m}{n}$ and $\frac{1}{n}\ge\frac{\varepsilon}{2(b-a)}$. Use this fact to find $\delta>0$ such that for every partition P satisfying $\|P\|<\delta$ we have $S(f,P)-s(f,P)<\varepsilon$.

SOLUTION: We will show that

$$\lim_{\|P\|\to 0}\Big(S(f,P)-s(f,P)\Big)=0. \tag{13.42}$$

Notice that for any $\eta>0$ there exists only finitely many rational numbers $x=\frac{m}{n}$ in $(0,1)$ such that $\frac{m}{n}$ is an irreducible fraction and $\frac{1}{n}<\eta$. Indeed, there is only finitely many natural numbers n such that $\frac{1}{\eta}\ge n$. Clearly, for a fixed natural number n there exists only finitely many natural numbers k such that $\frac{k}{n}\in(0,1)$ (i.e. $k=1,2,\dots,n-1$ – clearly, not every such fraction $\frac{k}{n}$ is irreducible), thus this statement is true.

We need to show (14.1), i.e.

$$\forall_{\varepsilon>0}\,\exists_{\delta>0}\,\forall_{P=\{x_k\}_{k=1}^n}\quad \|P\|<\delta \quad\Longrightarrow\quad \sum_{k=1}^n(M_k-m_k)\triangle x_k<\varepsilon$$

Assume that $\varepsilon>0$ is an arbitrary number. Put

$$X:=\left\{\frac{m}{n}:\frac{m}{n}\in(0,1),\quad \frac{m}{n}\text{ is irreducible and }\quad \frac{\varepsilon}{2(b-a)}\ge\frac{1}{n}\right\}$$

The set X is finite. Put $N=|X|$ denotes the number of elements in X. Assume that $\delta:=\frac{\varepsilon}{4N}$ and let $P:=\{x_k\}_{k=1}^n$ be an arbitrary partition of $[a,b]$ such that $\|P\|\le\delta$. Put

$$K_1:=\{k\in\{1,2,\dots,n\}:X\cap[x_{k-1},x_k]=\emptyset\},$$

and

$$K_2:=\{k\in\{1,2,\dots,n\}:X\cap[x_{k-1},x_k]\ne\emptyset\}$$

Notice that $m_k=0$ for all $k=1,\dots,n$, and $M_k\le\frac{\varepsilon}{2(b-a)}$ if $k\in K_1$. If $k\in K_2$, then $M_k\le 1$. Also notice that a point from X can belong to at most two different subintervals $[x_{k'-1},x_{k'}]$ and $[x_{k''-1},x_{k''}]$. Thus, we have

$$\begin{aligned}\sum_{k=1}^{n}(M_k-m_k)\triangle x_k &= \sum_{k\in K_1}(M_k-m_k)\triangle x_k+\sum_{k\in K_2}(M_k-m_k)\triangle x_k\\ &\le \sum_{k\in K_1}\left(\frac{\varepsilon}{2(b-a)}-0\right)\triangle x_k+\sum_{k\in K_2}(1-0)\triangle_k\\ &\le \sum_{k=1}^{n}\frac{\varepsilon}{2(b-a)}\triangle_k+(2N)\|P\|\\ &< \frac{\varepsilon}{2(b-a)}(b-a)+(2N)\frac{\varepsilon}{4N}=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

24. Show that the function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}\frac{1}{x}-\left\lfloor\frac{1}{x}\right\rfloor & \text{if } 0<x<1\\ 0 & \text{if } x=0,\end{cases}$$

is Riemann integrable on $[0,1]$.

SOLUTION: In order to show that f is Riemann integrable on $[0,1[$ it is sufficient to show that

$$\lim_{\|P\|\to 0}\Big(S(f,P)-s(f,P)\Big)=0$$

i.e.

$$\forall_{\varepsilon.0}\,\exists_{\delta>0}\,\forall_P\quad \|P\|<\delta \quad\Rightarrow\quad S(f,P)-s(f,P)<\varepsilon. \tag{13.43}$$

By Theorem 6.3, a bounded function with finitely many discontinuity points is Riemann integrable. It is clear that $0\le a-\lfloor a\rfloor<1$, for all $a\in\mathbb{R}$, thus the function f is bounded. It is easy to notice that f has discontinuity points at $x_n=\frac{1}{n}$, $n=1,2,\dots$. Notice that if $c\in(0,1)$ then the function $f:[c,1]\to\mathbb{R}$ has only finitely many discontinuity points in $[c,1]$, thus it is Riemann integrable on $[c,1]$. Therefore, it satisfies a similar to (13.43) condition, namely

$$\forall_{\varepsilon.0}\,\exists_{\delta>0}\,\forall_{P_1}\quad \|P_1\|<\delta \quad\Rightarrow\quad S(f,P_1)-s(f,P_1)<\frac{\varepsilon}{2}. \tag{13.44}$$

where P_1 is a partition of the interval $[c,1]$. Assume therefore, that $\varepsilon>0$ and let $c=\frac{\varepsilon}{2}$. Consider an arbitrary partition P of $[0,1]$ such that $\|P\|<\delta$, where $\delta>0$ is given by the condition (13.44). We can assume without loss of generality that c belongs to the partition P. Then we can write $P=P_1\cup P_2$, where P_1 is a partition of $[0,c]$ and P_2 is a partition of $[c,1]$. Moreover, we have

$$S(f,P)=S(f,P_1)+S(f,P_2),\quad s(f,P)=s(f,P_1)+s(f,P_2)$$

therefore we have

$$\begin{aligned}S(f,P)-s(f,P)&=S(f,P_1)-s(f,P_1)+S(f,P_2)-s(f,P_2)\\ &<\frac{\varepsilon}{2}+S(f,P_2)-s(f,P_2)\\ &<\frac{\varepsilon}{2}+(1-0)c=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\end{aligned}$$

where we used the fact that

$$\sup\{f(x):x\in[0,c]\}-\inf\{f(x):x\in[0,c]\}<1-0=1.$$

Consequently the function f is Riemann integrable.

13.6 Chapter 9: Riemann-Stieltjes Integral

1. Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = x^4 - 4x^3 + 4x^2 + 1, \quad x \in \mathbb{R}.$$

Compute $\operatorname*{Var}_{[0,4]}(g)$.

SOLUTION: Notice that

$$g'(x) = 4x(x-1)(x-2)$$

therefore the function $g(x)$ is

- decreasing on the interval $(-\infty, 0)$ $(g'(x) < 0)$;
- increasing on the interval $(0, 1)$ $(g'(x) > 0)$;
- decreasing on the interval $(1, 2)$ $(g'(x) < 0)$;
- increasing on the interval $(2, \infty)$ $(g'(x) > 0)$.

Therefore, we have

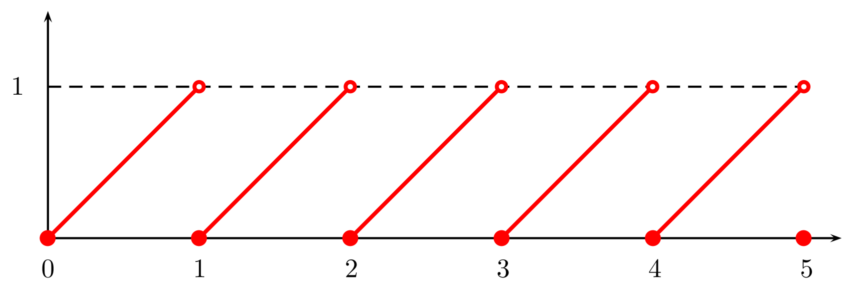
$$\begin{aligned}
\operatorname*{Var}_{[0,4]}(g) &= \operatorname*{Var}_{[0,1]}(g) + \operatorname*{Var}_{[1,2]}(g) + \operatorname*{Var}_{[2,4]}(g) \\
&= (g(1) - g(0)) + (g(1) - g(2)) + (g(4) - g(2)) \\
&= (2-1) + (2-1) + (65-1) = 66
\end{aligned}$$

2. Given the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(x) = x - \lfloor x \rfloor, \quad x \in \mathbb{R}.$$

For given $a < b$, compute $\operatorname*{Var}_{[a,b]}(g)$.

SOLUTION: The function g can be illustrated by the following graph



There are exactly $m := \lfloor b - a \rfloor$ integers in the interval $(a, b]$, which are the discontinuity points of the function g. Denote these points by $x_1 := n$, $x_2 = n + 1$, $\dots$, $x_m = n + m - 1$. Then we have

$$\begin{aligned}
\operatorname*{Var}_{[a,b]}(g) &= \operatorname*{Var}_{[a,x_1]}(g) + \operatorname*{Var}_{[x_1,x_2]}(g) + \dots + \operatorname*{Var}_{[x_{m-1},x_m]}(g) + \operatorname*{Var}_{[x_m,b]}(g) \\
&= (x_1 - a + 1) + (x_2 - x_1 + 1) + \dots + (x_m - x_{m-1} + 1) + (b - x_m + 1) \\
&= b - a + m = b - a + \lfloor b \rfloor - \lfloor a \rfloor.
\end{aligned}$$

3. Let $g : [a,b] \to \mathbb{R}$ be a function of bounded variation. Show that

$$|g(x)| \le |g(a)| + \operatorname*{Var}_{[a,b]}(g).$$

SOLUTION: Let fix $x \in (a, b]$ and consider the partition $P_o : a = x_0 < x = x_1 \le x_2 = b$. Then we have

$$\begin{aligned}
|g(x)| - |g(a)| &\le |g(x) - g(a)| \\
&\le |g(x) - g(a)| + |g(b) - g(x)| = v(g, P_o) \\
&\le \sup\{v(g, P) : PP \text{ is a partition of } [a,b]|\} = \operatorname*{Var}_{[a,b]}(g).
\end{aligned}$$

Therefore,

$$|g(x)| \le |g(a)| + \operatorname*{Var}_{[a,b]}(g).$$

4. Let $g : [a,b] \to \mathbb{R}$ be a function of bounded variation. Show that

$$\operatorname*{Var}_{[a,b]}(|g|) \le \operatorname*{Var}_{[a,b]}(g).$$

Give an example of a function g such that $\operatorname*{Var}_{[a,b]}(|g|) < \operatorname*{Var}_{[a,b]}(g)$.

SOLUTION: Consider an arbitrary partition P of $[a,b]$:

$$P : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Then we have

$$\begin{aligned} v(|g|,P) &= \sum_{k=1}^{n} \Big| |g(x_k)| - |g(x_{k-1}| \Big| \\ &\le \sum_{k=1}^{n} |g(x_k) - g(x_{k-1}| \\ &= v(g,P), \end{aligned}$$

therefore,

$$\begin{aligned} \operatorname*{Var}_{[a,b]}(|g|) &= \sup\{v(|g|,P) : \text{ is a partition of } [a,b]|\} \\ &\le \sup\{v(g,P) : \text{ is a partition of } [a,b]|\} = \operatorname*{Var}_{[a,b]}(g). \end{aligned}$$

As an example consider the function $g : [-1,1] \to \mathbb{R}$,

$$g(x) = \begin{cases} -1 & \text{if } x \in [-1,0) \\ 1 & \text{if } x \in [0,1]. \end{cases}$$

Then, since g is non-decreasing, $\operatorname*{Var}_{[-1,1]}(g) = g(1) - g(-1) = 2$, and since $|g(x)| = 1$ for all $x \in [-1,1]$, thus $\operatorname*{Var}_{[a,b]}(|g|) = 0$.

5. Let $f, g : [a,b] \to \mathbb{R}$ be two functions of bounded variations. Show that

$$\operatorname*{Var}_{[a,b]}(f+g) \le \operatorname*{Var}_{[a,b]}(f) + \operatorname*{Var}_{[a,b]}(g).$$

Give an example of two functions f and g such that $\operatorname*{Var}_{[a,b]}(f+g) < \operatorname*{Var}_{[a,b]}(f) + \operatorname*{Var}_{[a,b]}(g)$.

SOLUTION: Consider an arbitrary partition P of $[a,b]$:

$$P : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Then we have

$$\begin{aligned} v(f+g,P) &= \sum_{k=1}^{n} |(f(x_k) + g(x_k)) - (f(x_{k-1}) + g(x_{k-1}))| \\ &= \sum_{k=1}^{n} |(f(x_k) - f(x_{k-1})) + (g(x_k)) - g(x_{k-1}))| \\ &\le \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| + |g(x_k)) - g(x_{k-1})| \\ &\le \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| + \sum_{k=1}^{n} |g(x_k)) - g(x_{k-1})| \\ &= v(f,P) + v(g,P). \end{aligned}$$

Therefore,

$$\begin{aligned}\operatorname*{Var}_{[a,b]}(f+g) &= \sup\{v(f+g,P): \text{ is a partition of } [a,b]|\} \\ &\le \sup\{v(f,P)+v(g,P): \text{ is a partition of } [a,b]|\} \\ &\le \sup\{v(f,P): \text{ is a partition of } [a,b]|\} \\ &+ \sup\{v(g,P): \text{ is a partition of } [a,b]|\} \\ &= \operatorname*{Var}_{[a,b]}(f)+\operatorname*{Var}_{[a,b]}(g).\end{aligned}$$

As an example, take $f, g : [-1,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in [-1,0) \\ 1 & \text{if } x \in [0,1] \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if } x \in [-1,0) \\ 0 & \text{if } x \in [0,1]. \end{cases}$$

Then clearly, $f(x)+g(x) = 1$ for all $x \in [-1,1]$ thus $\operatorname*{Var}_{[a,b]}(f+g) = 0$, and since f and g are monotonic, we have

$$\operatorname*{Var}_{[a,b]}(f)+\operatorname*{Var}_{[a,b]}(g) = 1+1 = 2.$$

6. Evaluate the following Riemann-Stieltjes integrals:

(a): $\displaystyle\int_{-1}^{1} x dg(x)$, where $g(x) = e^{|x|}$ for $x \in [-1,1]$.

SOLUTION: The functions f and g are of bounded variation thus the above Riemann-Stieltjes exists, therefore by the properties we have

$$\begin{aligned}\int_{-1}^{1} x dg(x) &= \int_{-1}^{0} x dg(x) + \int_{0}^{1} x dg(x) \\ &= \int_{-1}^{0} x d(e^{-x}) + \int_{0}^{1} x d(e^{x}) \\ &= \int_{-1}^{0} x(-e^{-x})dx + \int_{0}^{1} x(e^{x})dx \quad \text{(Riemann integrals)} \\ &= (1+x)e^{-x}\Big|_{-1}^{0} + (1-x)e^{x}\Big|_{0}^{1} = 2.\end{aligned}$$

(b): $\displaystyle\int_{-1}^{1} f(x)dg(x)$, where $f,\ g : [-1,1] \to \mathbb{R}$ are defined by

$$f(x(= \begin{cases} x^2 & \text{if } \ -1 \le x \le 0 \\ x^3 & \text{if } \quad 0 < x \le 1, \end{cases} \quad g(x) = \begin{cases} 1 & \text{if } \quad x = -1 \\ 2x^2 & \text{if } \ -1 < x < 1 \\ -1 & \text{if } \quad x = 1. \end{cases}$$

SOLUTION: The functions f and g are of bounded variation thus the above Riemann-Stieltjes exists. The function g has discontinuities at $x = -1, 1$, and

$$g(-1^{+}) - g(-1) = 2-1 = 1, \quad g(1) - g(1^{-}) = -1-2 = -3.$$

Therefore, by the properties of the Riemann-Stieltjes integral we have

$$\begin{aligned}\int_{-1}^{1} f(x)dg(x) &= \int_{-1}^{1} f(x)g'(x)dx \\ &\quad + f(-1)(g(-1^+) - g(-1)) + f(1)(g(1) - g(1^-)) \\ &= \int_{-1}^{1} f(x)4xdx + f(-1) - 3f(1) \\ &= \int_{-1}^{0} f(x)4xdx + \int_{0}^{1} f(x)4xdx - 2 \\ &= 4\left(\int_{-1}^{0} x^3dx + \int_{0}^{1} x^4dx\right) - 2 \\ &= 4\left(-\frac{1}{4} + \frac{1}{5}\right) - 2 = -\frac{11}{5}.\end{aligned}$$

7. Let $g(x) = \begin{cases} 0 & \text{for } x \in [0,1] \\ 1 & \text{for } x \in (1,2]. \end{cases}$.

Show that $\int_0^2 gdg$ does not exist.

SOLUTION: Notice that the function g is non-decreasing, thus we need to show that

$$\lim_{\|P\|\to 0} \Big(S(g,P,g) - s(g,P,g)\Big) > 0.$$

Since for every partition $P = \{x_k\}_{k=1}^n$ and the choice of the points $\xi_k \in [x_{k-1}, x_k]$, we have

$$S(g,P,g) \le \sigma(g,P,g,\{\xi_k\}) \le s(g,P,g),$$

where

$$\sigma(g,P,g,\{\xi_k\}) = \sum_{k=1}^{n} g(\xi_k)(g(x_k - g(x_{k_1})),$$

therefore it is sufficient to show that there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exists a partition P satisfying $\|P\| < \delta$ and there are two sets of points $\{\xi_k\}$ and $\{\xi'_k\}$, such that

$$|\sigma(g,P,g,\{\xi_k\}) - \sigma(g,P,g,\{\xi'_k\})| \ge \varepsilon$$

We claim that we can take $\varepsilon = 1$. Let $\delta > 0$ be an arbitrary number and consider a partition $P = \{x_k\}_{k=1}^n$ such that there is $j \in \{1,2,\dots,n-1\}$ such that $x_{j-1} = 1$. Then for all $k \ne j$, $k = 1,2,\dots,n$, we can choose arbitrary points $\xi_k = \xi'_k \in [x_{k-1}, x_k]$ and we choose for $k = j$, $\xi_j = 1 = x_{j-1}$ and $\xi'_j = x_j > 1$. Then we have

$$\begin{aligned}|\sigma(g,P,g,\{\xi_k\}) - \sigma(g,P,g,\{\xi'_k\})| &= \left|\sum_{k=1}^{n} \Big(g(\xi_k) - g(\xi'_k)\Big)(g(x_k - g(x_{k_1}))\right| \\ &= \left|\Big(g(x_j) - g(x_{j-1})\Big)(g(x_j - g(x_{j_1}))\right| \\ &= 1.\end{aligned}$$

8. Let $\lfloor x \rfloor$ denote the largest integer less than or equal to the number x. Find the value of

$$\int_0^a (x^2+1)d(\lfloor x \rfloor).$$

where $a > 2$ is an arbitrary non-integer number.

SOLUTION: The function $g(x) = \lfloor x \rfloor$, $x \in [0,a]$ has discontinuity point $c_k := k$ for $k = 1, 2, \dots, \lfloor a \rfloor$, therefore by applying Theorem 7.9 (and noticing that $g'(x) = 0$ for $x \neq c_k$, $g(c_k^+) - g(c_k^-) = 1$), we have

$$\begin{aligned}\int_0^a (x^2+1)d(\lfloor x \rfloor) &= \int_0^a (x^2+1)g'(x)dx + \sum_{k=1}^{\lfloor a \rfloor} f(c_k)(g(c_k^+) - g(c_k^-))\\ &= 0 + \sum_{k=1}^{\lfloor a \rfloor}(1+k^2)\\ &= \lfloor a \rfloor + \frac{1}{6}\lfloor a \rfloor(\lfloor a \rfloor + 1)(2\lfloor a \rfloor + 1)\end{aligned}$$

9. (a): Let $g : [0,a] \to \mathbb{R}$ be the function $g(x) = x^\alpha$, where $\alpha \in (0,1)$. Show that g is of bounded variation but it is **not Lipschitzian**. Compute $\operatorname*{Var}_{[0,a]}(g)$

SOLUTION: It is sufficient to notice that $g'(x) = \alpha x^{\alpha-1} > 0$, $x > 0$, thus the function g is increasing and therefore it is of bounded variation on $[0,a]$. In order to show that g is not Lipschitzian, it is sufficient to show that

$$\forall_{L>0}\ \exists_{x\in(0,a]}\ \ g(x) - g(0) = |g(x) - g(0)| > L|x - 0| \ \ \Leftrightarrow \forall_{L>0}\ \exists_{x\in(0,a]}\ \ x^{\alpha-1} > L. \tag{13.45}$$

Since

$$\lim_{x\to 0^+} x^{\alpha-1} = \lim_{x\to 0^+} \frac{1}{x^{1-\alpha}} = \frac{1}{0^+} = \infty,$$

it is clear that for every $L > 0$ one can find $x > 0$, $x \in (0,a]$, such that (13.45) is satisfied.

(b): Define $g : [0,\pi] \to \mathbb{R}$ by $g(x) = \sin x$. Find the explicit formula for the function $m : [0,\pi] \to \mathbb{R}$ defined by

$$m(x) := \operatorname*{Var}_{[0,x]}(g), \quad x \in [0,\pi].$$

SOLUTION: Since the function $g(x)$ is increasing on the interval $[0, \frac{\pi}{2}]$ and decreasing on the interval $[\frac{\pi}{2}, \pi]$ we have

$$\begin{aligned}m(x) &= \begin{cases}\operatorname*{Var}_{[0,x]}(g) = & \text{for } \ x \in [0, \frac{\pi}{2}]\\ \operatorname*{Var}_{[0,x]}(g) = \operatorname*{Var}_{[0,\frac{\pi}{2}]}(g) + \operatorname*{Var}_{[\frac{\pi}{2},x]}(g) & \text{for } \ x \in [\frac{\pi}{2}, \pi]\end{cases}\\ &= \begin{cases}\sin(x) & \text{for } \ x \in [0, \frac{\pi}{2}]\\ 2 - \sin(x) & \text{for } \ x \in [\frac{\pi}{2}, \pi].\end{cases}\end{aligned}$$

10. Compute $\operatorname*{Var}_{[a,b]}(f)$ for the following functions $f : [a,b] \to \mathbb{R}$

(a) $f(t) = \begin{cases} 1 & \text{for } \ t = a; \\ 2 & \text{for } a < t < b; \\ 3 & \text{fort } t = b, \end{cases}$

(b) $f(t) = e^{t^2}$, where $a < 0 < b$.

11. Let $a < c < b$ be given numbers, $f : [a, b] \to \mathbb{R}$ a continuous function, and $g : [a, b] \to \mathbb{R}$ be defined by

$$g(t) = \begin{cases} \alpha & \text{for } \ t \leq c \\ \beta & \text{for } t > c, \end{cases}, \qquad \text{where } \ \alpha \neq \beta.$$

Compute the Riemann-Stieltjes integral

$$\int_a^b f(t)dg(t).$$

12. Compute:

(a) $V_0^4(f)$ for $f(t) = \begin{cases} t^2 & \text{for } \ 0 \leq t \leq 1; \\ 4 & \text{for } 1 < t \leq 2; \\ 4 - t & \text{for } 2 < t \leq 4. \end{cases}$

(b) $V_0^{\sqrt{\pi}}(f)$ for $f(t) = \sin t^2$.

13. Consider the function $f : [0, 2] \to \mathbb{R}$, $f(t) = e^{t^2}$ and the function $g : [0, 2] \to \mathbb{R}$ be defined by

$$g(t) = \begin{cases} t^2 & \text{for } \ 0 \leq t \leq 1 \\ 2 & \text{for } 1 < t \leq 2, \end{cases}.$$

Compute the Riemann-Stieltjes integral

$$\int_0^2 f(t)dg(t).$$

14. Compute $\underset{[0,\pi]}{\mathrm{Var}}(f)$ for $f(x) = \sin^2(nx)$, where $n \in \mathbb{N}$.

15. Given

$$f(x) = \begin{cases} \alpha & \text{if } 0 \leq x < 1 \\ \beta & \text{if } 1 \leq x < 2 \\ \gamma & \text{if } 2 \leq x \leq 3. \end{cases}$$

Find $\underset{[0,3]}{\mathrm{Var}}(f)$.

13.7 Chapter ??: Lebesgue Integral

1. Assume that $I_0 := [a, b]$ is a fixed interval. Use the "sub-additivity property" of the outer measure to prove that for A, $B \subset I_0$ we have

$$|\mu^*(A) - \mu^*(B)| \leq \mu^*(A \triangle B),$$

where

$$A \triangle B := (A \setminus B) \cup (B \setminus A)$$

denotes the symmetric difference of sets A and B.
SOLUTION: Notice that we have the following inclusions

$$A \subset A \cup B = (A \triangle B) \cup (A \cap B) \subset (A \triangle B) \cup B$$
$$B \subset A \cup B = (A \triangle B) \cup (A \cap B) \subset (A \triangle B) \cup A.$$

Therefore, by sub-additivity property of the outer mneasure

$$\mu^*(A) \leq \mu^*((A \triangle B) \cup B) \leq \mu^*(A \triangle B) + \mu^*(B)$$
$$\mu^*(B) \leq \mu^*((A \triangle B) \cup B) \leq \mu^*(A \triangle B) + \mu^*(A)$$

which implies

$$|\mu^*(A) - \mu^*(B)| \leq \mu^*(A \triangle B).$$

2. Assume that $I_0 := [a, b]$ is a fixed interval. Show that if the set $A \subset I_0$ satisfies the property $\mu^*(A) = 0$ then A is measurable and $\mu(A) = 0$.
SOLUTION: Since $0 \leq \mu_*(A) \leq \mu^*(A) = 0$, thus $\mu^*(A) = \mu^*(A) = 0$, which implies that A is measurable and $\mu(A) = 0$.

3. Assume that $I_0 := [a, b]$ is a fixed interval. Show that for every $A \subset I_0$ such that $\mu^*(A) = \mu(I_0)$, we have

$$\overline{A} = I_0.$$

SOLUTION: Assume for contradiction that $\overline{A} \neq I_0$, i.e. there exists $x_o \in I_0 \setminus \overline{A}$, which implies that

$$\exists_{\varepsilon > 0} \;\; (x_o - \varepsilon, x_o + \varepsilon) \cap A = \emptyset.$$

Denote the interval $(x_o - \varepsilon, x_o + \varepsilon) \cap I_0$ by $\tilde{I}$. Since $x_o \in I_0$, it follows that $\mu(\tilde{I}) \geq \varepsilon$. Therefore, since $A \subset \tilde{I}^c$, it follows that

$$\mu^*(A) \leq \mu(\tilde{I}^c) = \mu(I_0) - \mu(\tilde{I}) \leq \mu(I_0) - \varepsilon < \mu(I_0).$$

which implies the contradiction with the assumption that

$$\mu^*(A) = \mu(I_0).$$

4. Assume that $I_0 := [a, b]$ is a fixed interval. Let $K \subset I_0$ be a compact set. Show that for every $\varepsilon > 0$ there exists an open set U such that

$$K \subset U \quad \text{and} \quad \mu(U) - \mu(K) < \varepsilon.$$

SOLUTION: Since K is compact, thus it is measurable (see Corollary **??**). Therefore (see Theorem **??** (i)),

$$\mu(K) = \mu^*(K) = \inf\{\mu(U) : U \text{ is open and } K \subset U\}.$$

Then, by definition of the infimum

$$\forall_{\varepsilon>0}\ \exists_{U\subset I_0} \quad U \text{ is open in } I_0,\ K \subset U \text{ and } \mu(K) + \varepsilon > \mu(U).$$

5. Use the additivity property of the Lebesgue measure μ to show that for every open set $U \subset I_0$ (here $I_0 = [a, b]$ is a fixed interval) there exists a cover $\{I_k\}_{k=1}^{\infty}$ of U by open intervals $I_k = (\alpha_k, \beta_k) \cap I_0$ such that

$$\mu(U) = \sum_{k=1}^{\infty} \mu(I_k).$$

SOLUTION: By Theorem 10.23, the set U can be represented as a union of pairwise disjoint open in I_0 intervals I_k, $k = 1, 2, \dots$, i.e. $U = \bigcup_{k=1}^{\infty} I_k$. Then, by Corollary **??**, or by the Additivity Property in Theorem **??** (iv) (since the sets I_k are measurable and pairwise disjoint), we have

$$\mu(U) = \sum_{k=1}^{\infty} \mu(I_k).$$

6. (a) Let $I_0 = [a, b]$ be a fixed interval. Consider an open set U in I_0 such that for some $\lambda \in \mathbb{R}$ the set $\lambda U := \{\lambda x : x \in U\}$ is contained in I_0. Use the result in Problem 1 to show that

$$\mu(\lambda U) = |\lambda|\mu(U).$$

SOLUTION: Notice that for every interval $I = (\alpha, \beta)$ and $\lambda \in \mathbb{R}$ we have (by inspection)

$$\mu(\lambda I) = |\lambda|(\beta - \alpha) = |\lambda|\mu(I),$$

therefore, if we consider a representation of the open set U as the pairwise disjoint union of open in I_0 intervals I_k, i.e. $U = \bigcup_{k=1}^{\infty} I_k$, then we also have $\lambda U = \bigcup_{k=1}^{\infty} \lambda I_k$, where the intervals λI_k are also disjoint, therefore we obtain (by applying the result from Problem 5)

$$\begin{aligned}
\mu(\lambda U) &= \mu\left(\bigcup_{k=1}^{\infty} \lambda I_k\right) = \sum_{k=1}^{\infty} \mu(\lambda I_k) \\
&= \sum_{k=1}^{\infty} |\lambda|\mu(I_k) = |\lambda| \sum_{k=1}^{\infty} \mu(I_k) = |\lambda|\mu(U).
\end{aligned}$$

(b) Assume that $K \subset I_0$ is a compact set such that $\lambda K \subset I_0$ for some $\lambda \in \mathbb{R}$. Show that

$$\mu(\lambda K) = |\lambda|\mu(K).$$

SOLUTION: Put $U := I_0 \setminus K$. It is clear that, by (a), $\mu(\lambda U) = |\lambda|\mu(U)$, $\mu(\lambda I_0) = |\lambda|\mu(I_0)$. Since U and K are measurable, λU and λK are also measurable and since we have the following disjoint unions $I_0 = U \cup K$ and $\lambda I_0 = \lambda U \cup \lambda K$, therefore we have by (a),

$$\mu(I_0) = \mu(K) + \mu(U) \quad \implies \quad |\lambda|\mu(I_0) = |\lambda|\mu(U) + |\lambda|\mu(K) \quad \text{and}$$
$$|\lambda|\mu(I_0) = \mu(\lambda I_0) = \mu(\lambda U \cup \lambda K) = \mu(\lambda U) + \mu(\lambda K) = |\lambda|\mu(U) + \mu(\lambda K),$$

therefore, $\lambda|\mu(K) = \mu(\lambda K)$.

7. Consider a set $A \subset I_0$ (here $I_0 = [a, b]$ is a fixed interval) such that for some $\lambda \in \mathbb{R}$, we have

$$\lambda A := \{\lambda x : x \in A\} \subset I_0.$$

Show that

(a) $\mu^*(\lambda A) = |\lambda|\mu^*(A)$.
(b $\mu_*(\lambda A) = |\lambda|\mu_*(A)$.
(c) if A is measurable, then λA is measurable and $\mu(A) = |\lambda|\mu(A)$.

SOLUTION: (a): Assume $\lambda \neq 0$ (the case for $\lambda = 0$ is trivial). By Theorem **??**, we have that $\mu^*(A) = \inf\{\mu(U) : U \text{ is open and } A \subset U\}$. Therefore, we have

$$\begin{aligned}
\mu^*(\lambda A) &= \inf\{\mu(U') : U' \text{ is open and } \lambda A \subset U'\} \\
&= \inf\{\mu(\lambda U) : U \text{ is open and } \lambda A \subset \lambda U\} \quad \text{where } U = \frac{1}{\lambda}U' \\
&= \inf\{|\lambda|\mu(U) : U \text{ is open and } A \subset U\} \quad \text{by Problem 6} \\
&= |\lambda| \inf\{\mu(U) : U \text{ is open and } A \subset U\} \\
&= |\lambda|\mu^*(A).
\end{aligned}$$

On the other hand, again by Theorem **??**, $\mu_*(A) = \sup\{\mu(K) : K \text{ is compact and } K \subset A\}$, thus

$$\begin{aligned}
\mu_*(\lambda A) &= \sup\{\mu(K') : K' \text{ is compact and } K' \subset \lambda A\} \\
&= \sup\{\mu(\lambda K) : K \text{ is compact and } \lambda K \subset \lambda A\} \quad \text{where } K = \frac{1}{\lambda}K' \\
&= \sup\{|\lambda|\mu(K) : K \text{ is compact and } K \subset A\} \quad \text{by Problem 2} \\
&= |\lambda| \sup\{\mu(K) : K \text{ is compact and } K \subset A\} \\
&= |\lambda|\mu^*(A).
\end{aligned}$$

8. Let $f : [a, b] \to \mathbb{R}$ be a measurable function. Show that

(a) the function $f_+ : [a, b] \to \mathbb{R}$, defined by $f_+(x) := \max\{0, f(x)\}$, $x \in [a, b]$, is measurable

(b) the function $f_- : [a,b] \to \mathbb{R}$, defined by $f_-(x) := f_+(x) - f(x)$, $x \in [a,b]$, is measurable
(c) the function $|f| : [a,b] \to \mathbb{R}$, defined by $|f|(x) := |f(x)|$, $x \in [a,b]$, is measurable

SOLUTION: By Proposition 10.25, we have for every $\alpha \in \mathbb{R}$

$$f_+^{-1}(-\infty, \alpha) = \begin{cases} \emptyset, & \text{if } \alpha \le 0 \\ f^{-1}(-\infty, \alpha), & \text{if } \alpha > 0, \end{cases}$$

and since f is measurable, it follows that f_+ is measurable. On the other hand, since f_- is a difference of two measurable functions, thus it is measurable (by Proposition 10.26). On the other hand, since $|f| = f_+ + f_-$, therefore $|f|$ is measurable because it is a sum of two measurable functions.

9. Let $f_n : [a,b] \to \mathbb{R}$, $n = 1, 2, \dots,$, be a sequence of measurable functions. Suppose there is a function $f : [a,b] \to \mathbb{R}$ such that

$$f(x) := \inf\{f_n(x) : n \in \mathbb{N}\}.$$

Show that $f : [a,b] \to \mathbb{R}$ is measurable.
SOLUTION: Since

$$\begin{aligned} -f(x) &= -\inf\{f_n(x) : n \in \mathbb{N}\} \\ &= \sup\{-f_n(x) : n \in \mathbb{N}\}, \end{aligned}$$

thus

$$f(x) = -\sup\{-f_n(x) : n \in \mathbb{N}\}.$$

Therefore, since $-f_n$ are measurable functions (by Proposition 10.26(a) and (f)) the function f is measurable.

10. Assume that $\{A_k\}_{k=1}^{\infty}$ is a family of measurable sets in $I_0 = [a,b]$. Show that

$$\lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right).$$

SOLUTION:
Put $B_1 := A$, and for $n \ge 2$,

$$B_n := A_n \setminus \bigcup_{k=1}^{n-1} A_k$$

Then, $\{B_k\}_{k=1}^{\infty}$ are measurable and piecewise disjoint. Moreover,

$$\bigcup_{k=1}^{n} B_k = \bigcup_{k=1}^{n} A_k \quad \text{and} \quad \bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k$$

Therefore, by the additivity property of the measure μ, we have

$$\begin{aligned}\mu\left(\bigcup_{k=1}^{\infty} A_k\right) &= \mu\left(\bigcup_{k=1}^{\infty} B_k\right)\\ &= \sum_{k=1}^{\infty} \mu(B_k)\\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(B_k)\\ &= \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{\infty} B_k\right)\\ &= \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{\infty} A_k\right)\end{aligned}$$

11. Let

$$E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots \subset I_0 =: [a,b],$$

be a sequence of measurable sets and let $f : [a,b] \to \mathbb{R}$ be a summable function. Show that

$$\lim_{n\to\infty} \int_{E_n} f(x)d\mu(x) = \int_E f(x)d\mu(x) \quad \text{where} \quad E = \bigcup_{k=1}^{\infty} E_k.$$

SOLUTION:

Consider first the case when $f \geq 0$. We define the functions $f_n\colon E \to \mathbb{R}$ by

$$f_n(x) := \chi_{E_n}(x) f(x), \qquad x \in E$$

Clearly, f_n are measurable as the product of two measurable functions and it follows that

$$\int_E f_n(x)d\mu(x) = \int_E \chi_{E_n}(x) f(x) d\mu(x) = \int_{E_n} f(x)d\mu(x)$$

Notice that:

1. $0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq f_n(x) \leq \cdots \leq f(x)$, $x \in E$. Indeed, since $E_n \subset E_m$ for $n \leq m$, it follows that $\chi_{E_n}(x) \leq \chi_{E_m}(x)$ for $x \in E$, thus

$$f_n(x) = \chi_{E_n}(x) f(x) \leq \chi_{E_m} f(x) = f_m(x) \leq f(x), \quad x \in E$$

2. For every $x \in E$, $\lim_{n\to\infty} f_n(x) = f(x)$. Indeed, if

$$x \in E = \bigcup_{k=1}^{\infty} E_k$$

thus, there is an n such that $x \in E_n$ and for every $m \geq n$, we have $x \in E_n \subset E_m$; that is, $\chi_{E_m}(x) = 1$.

Thus,

$$\exists_n \forall_{m \geq n} \quad f_m(x) = \chi_{E_m}(x) f(x) = f(x)$$

for all $x \in E$ which implies 2. Consequently, by the Monotone Convergence Theorem, we have

$$\lim_{n\to\infty} \int_{E_n} f(x) d\mu(x) = \lim_{n\to\infty} \int_E f_n(x) d\mu(x) = \int_E f(x) d\mu(x)$$

12. Consider a function $s : [a, b] \to \mathbb{R}$ with finite number of values, i.e.

$$s([a, b]) = \{c_1, c_2, \dots, c_N\}.$$

Show that s is measurable if and only if for all $k = 1, 2, \dots, N$, the set $A_k := s^{-1}(\{c_k\})$ is measurable.
SOLUTION:

If s is measurable, then it follows that $s^{-1}(\alpha, \beta)$ is measurable for all $\alpha < \beta$. Choose $\alpha < \beta$ such that $(\alpha, \beta) \cap \{c_1, \dots, c_N\} = \{c_k\}$. Then

$$s^{-1}(\alpha, \beta) = s^{-1}(\{c_k\})$$

is measurable. Conversely, suppose that $s^{-1}(\{c_k\})$ is measurable for all k. Then,

$$s^{-1}(\alpha, \beta) = \bigcup_{c_k \in (\alpha, \beta)} s^{-1}(\{c_k\})$$

and therefore, $s^{-1}(\alpha, \beta)$ is also measurable as the finite union of measurable sets.

13. For the following functions $f : [-1, 1] \to \mathbb{R}$ compute the Lebesgue integrals

$$\int_{-1}^{1} f(x) d\mu(x).$$

(a): $f(x) = \begin{cases} 1 & \text{if } x \in [-1, 1] \cap \mathbb{Q} \\ x^2 & \text{if } x \in [-1, 1] \setminus \mathbb{Q} \end{cases}$
SOLUTION: Define $g(x) = x^2$, $x \in [-1, 1]$. It is clear that $f(x) = g(x)$ a.e., thus

$$\int_{-1}^{1} f(x) d\mu(x) = \int_{-1}^{1} g(x) d\mu(x) = \int_{-1}^{1} x^2 \, dx = \frac{2}{3}$$

(b): $f(x) = \begin{cases} 1 & \text{if } x \in [-1, 0] \\ \frac{1}{n+2} & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n} \right] \end{cases}$, $n \in \mathbb{N}$.
SOLUTION:

$$\int_{-1}^{1} f(x) d\mu(x) = \int_{-1}^{0} f(x) d\mu(x) + \int_{0}^{1} f(x) d\mu(x) = \int_{-1}^{0} 1 d\mu(x) + \int_{0}^{1} f(x) d\mu(x)$$
$$= 1 + \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{1}{n+2} d\mu(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n+2} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= 1 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right) = 1 + \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right)$$

$$= 1 + \frac{1}{2}\lim_{n\to\infty}\left(1 - \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}\right)$$

$$= 1 + \frac{1}{4} = \frac{5}{4}.$$

Remark: the series $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right)$ is telescopic.

14. Let $A \subset [a,b]$ be a measurable set and assume that K is a compact set and U is an open set in $[a,b]$ satisfying the properties

(i) $K \subset A \subset U$;
(ii) $\mu(U) - \mu(K) < \varepsilon$ for some $\varepsilon > 0$.

Define two functions d_K, $d_{U^c} : [a,b] \to \mathbb{R}$ by

$$d_K(x) := \inf\{|x-k| : k \in K\} \quad \text{and} \quad d_{U^c}(x) := \inf\{|x-v| : v \in U^c\}$$

(a): Show that

$$\forall_{x,x'\in[a,b]} \ |d_K(x) - d_K(x')| \le |x - x'| \quad \text{and} \quad |d_{U^c}(x) - d_{U^c}(x')| \le |x - x'|$$

SOLUTION: Notice that

$$\forall_{k\in K} \quad |x-k| \le |x-x'| + |x'-k|$$

Thus, we have that

$$d_K(x) = \inf\{|x-k| \colon k \in K\} \le |x-x'| + \inf\{|x'-k| \colon k \in K\} = |x-x'| + d_K(x')$$

which implies that $d_K(x) - d_K(x') \le |x - x'|$. By interchanging x and x', we get $d_K(x') - d_K(x) \le |x - x'|$, so it follows that

$$|d_K(x') - d_K(x)| \le |x - x'|$$

(b): Show that

$$\forall_{x\in[a,b]} \ d_K(x) + d_{U^c}(x) > 0.$$

SOLUTION: Notice that for any closed set $S \subset [a,b]$, we can define $d_S(x) = \inf\{|x-s| \colon s \in S\}$, where $x \in [a,b]$. Then, we have

$$d_S(x) = 0 \Rightarrow \forall_{\varepsilon>0} \ \exists_{s\in S} \ |x-s| < \varepsilon \iff \forall_{\varepsilon>0} \ B_\varepsilon(x) \cap S \ne \emptyset \iff x \in \overline{S}$$

Since K and U^c are closed, we have that $d_K(x) = 0 \iff x \in K$ and $d_{U^c}(x) = 0 \iff x \in U^c$. But $K \cap U^c = \emptyset$, thus $d_K(x) + d_{U^c}(x) > 0$ for all $x \in [a,b]$.

(c): Show that the function $f : [a,b] \to \mathbb{R}$ defined by $f(x) = \frac{d_{U^c}(x)}{d_K(x)+d_{U^c}(x)}$, $x \in [a,b]$ is continuous and satisfies the inequality

$$\int_a^b |f(x) - \chi_A(x)| d\mu(x) < \varepsilon.$$

where $\chi_A : [a, b] \to \mathbb{R}$ is the characteristic function of the set A.

SOLUTION: Notice that

1. $f(x) = \begin{cases} 1 & x \in K \\ 0 & x \in U^c \end{cases}$
2. $|f(x)| \leq 1$ for all $x \in [a, b]$
3. $|f(x) - \chi_A(x)| = \begin{cases} 0 & x \in K \\ 0 & x \in U^c \end{cases}$
4. $|f(x) - \chi_A(x)| \leq 1.$

Therefore, we have that if $|f(x) - \chi_A(x)| > 0$, then $x \in U \setminus K$, and thus

$$\begin{aligned} \int_a^b |f(x) - \chi_A(x)| d\mu(x) &= \int_{U \setminus K} |f(x) - \chi_A(x)| d\mu(x) \\ &\leq \int_{U \setminus K} d\mu(x) \\ &= \mu(U \setminus K) \\ &= \mu(U) - \mu(K) < \varepsilon. \end{aligned}$$

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Appendix 2: Integration Techniques

14.1 Indefinite Integral

The basic formulae for the computation of indefinite integrals are shown in Table 14.1.

Derivatives	**Integrals**
$\frac{d}{dx}C = 0$	$\int 0dx = C$
$\frac{d}{dx}x = 1$	$\int 1dx = x + C$
$\frac{d}{dx}((x^{\mu}) = \mu x^{\mu-1}$	$\int x^{\mu}dx = \frac{1}{\mu+1}x^{\mu+1} + C,\ \mu \neq -1$
$\frac{d}{dx}\ln\lvert x\rvert = \frac{1}{x}$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = \arctan x + C$
$\frac{d}{dx}a^x = a^x \ln a,\ a > 0$	$\int a^x dx = \frac{a^x}{\ln a} + C$
$\frac{d}{dx}e^x = e^x$	$\int e^x dx = e^x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x dx = -\cos x + C$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dx}\cot x = \frac{-1}{\sin^2 x}$	$\int \frac{dx}{\sin^2 x} = -\cot x + C$
$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}$	$\int \frac{dx}{\cos^2 x} = \tan x + C$
$\frac{d}{dx}\cosh x = \sinh x$	$\int \sinh x dx = \cosh x + C$
$\frac{d}{dx}\sinh x = \cosh x$	$\int \cosh x dx = \sinh x + C$
$\frac{d}{dx}\coth x = \frac{-1}{\sinh^2 x}$	$\int \frac{dx}{\sinh^2 x} = -\coth x + C$
$\frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x}$	$\int \frac{dx}{\cosh^2 x} = \tanh x + C$

Table 14.1. Differentiation and Integration

Problems 14.1 Use basic algebraic or trigonometric transformations and the table of integrals to compute the following indefinite integrals:

(a) $\displaystyle\int \frac{dx}{\sin^2 x \cos^2 x}$;

(b) $\displaystyle\int (x^2+5)^3 dx$;

(c) $\displaystyle\int \tan^2 x dx$.

Solution 14.1. Problem 14.1(a): We have

$$\int \frac{dx}{\sin^2 x \cos^2 x} = -2\int \frac{-2dx}{\sin^2 2x} = -2\cot 2x + C.$$

Problem 14.1(b): We have

$$\int (x^2+5)^3 dx = \int (x^6 + 15x^4 + 75x^2 + 125)dx = \frac{1}{7}x^7 + 3x^5 + 25x^3 + 125x + C.$$

Problem 14.1(c): We have

$$\int \tan^2 x dx = \int \left(\frac{1}{\cos^2 x} - 1\right) dx = \tan x - x + C.$$

14.2 Integration Rules

All the basic rules of differentiation can be translated into corresponding rules of integration.

Proposition 14.2. *We have the following properties*

(a) (Linearity Rule)

$$\int (a_1 f_1(x) + a_2 f_2(x))\, dx = a_1 \int f_1(x)dx + a_2 \int f_2(x)dx,$$

(b) (Substitution Rule)

$$\int f(\omega(x))\omega'(x)dx = \int f(t)dt,$$

where $t = \omega(x)$*,*

(c) (Integration by Parts Rule)

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx,$$

where all the above equalities are assumed to be satisfied up to a constant, i.e. the difference between the right hand and left hand sides of each of these equalities is a constant.

Proof: It is enough to show that for the equalities (a), (b) and (c), the right hand side and left hand side represent two antiderivatives of the same function. Indeed, for (a), it follows from the Linearity Property for derivatives that

$$\frac{d}{dx}\left[\int (a_1 f_1(x) + a_2 f_2(x))\right] dx = a_1 f_1(x) + a_2 f_2(x)$$

and

$$\begin{aligned}\frac{d}{dx}\left[a_1 \int f_1(x)dx + a_2 \int f_2(x)dx\right] &= a_1 \frac{d}{dx}\int f_1(x)dx + a_2 \frac{d}{dx}\int f_2(x)dx\\ &= a_1 f_1(x) + a_2 f_2(x),\end{aligned}$$

so the inequality (a) is true (up to a constant).

For (b), we have

$$\frac{d}{dx}\left[\int f(\omega(x))\omega'(x)dx\right] = f(\omega(x))\omega'(x),$$

and by the Chain Rule,

$$\begin{aligned}\frac{d}{dx}\left[\int f(t)dt\right] &= \frac{dt}{dx}\cdot\frac{d}{dt}\left[\int f(t)dt\right]\\ &= \frac{dt}{dx}f(t)\\ &= f(\omega(x))\omega'(x),\end{aligned}$$

where $t = \omega(x)$, (b) is satisfied.

Finally, for (c), by the Product Property, we have

$$\frac{d}{dx}\left[\int f(x)g'(x)dx\right] = f(x)g'(x),$$

and

$$\begin{aligned}\frac{d}{dx}\left[f(x)g(x) - \int f'(x)g(x)dx\right] &= \frac{d}{dx}\Big(f(x)g(x)\Big) - \frac{d}{dx}\left[\int f'(x)g(x)dx\right]\\ &= f'(x)g(x) + f(x)g'(x) - f'(x)g(x) = f(x)g'(x).\end{aligned}$$

and consequently (c) holds. □

Example 14.3. As an illustration, we consider the following integrals:

(a) $I = \int \frac{ax+b}{cx+d}dx$, where a, b, c and d are arbitrary constants. By dividing the polynomial in the numerator by the polynomial in the denominator (with a remainder) we obtain

$$I = \int\left(\frac{a}{c} + \frac{bc-ad}{c}\cdot\frac{1}{cx+d}\right)dx,$$

and by the Linearity Rule

$$I = \int\frac{a}{c}dx + \frac{bc-ad}{c}\int\frac{dx}{cx+d}.$$

Next, we put $t = cx+d$, and apply the Substitution Rule. Since $dx = \frac{dt}{c}$ we have

$$\begin{aligned}I &= \frac{a}{c}x + \frac{bc-ad}{c}\int\frac{1}{t}\frac{dt}{c}\\ &= \frac{a}{c}x + \frac{bc-ad}{c^2}\ln|t| + C\\ &= \frac{a}{c}x + \frac{bc-ad}{c}\ln|cx+d| + C.\end{aligned}$$

(b) $I = \int\frac{xdx}{1+x^4}$. We put $t = x^2$, i.e. $2xdx = dt$, therefore by the Substitution Rule

$$I = \frac{1}{2}\int\frac{dt}{1+t^2} = \frac{1}{2}\arctan t + C = \frac{1}{2}\arctan x^2 + C.$$

(c) Notice that the integral $I=\int\frac{f'(x)dx}{f(x)}$ can be written as $I=\int\frac{df(x)}{f(x)}$. Therefore, by applying the substitution $t=f(x)$ we obtain

$$I=\int\frac{df(x)}{f(x)}=\int\frac{dt}{t}=\ln|t|+C=\ln|f(x)|+C.$$

In this way we obtain

$$\begin{aligned}
\int\cot x dx &= \int\frac{d\sin x}{\sin x}=\ln|\sin x|+C,\\
\int\frac{e^{2x}}{e^{2x}+1}dx &= \frac{1}{2}\int\frac{d(e^{2x}+1)}{e^{2x}+1}=\frac{1}{2}\ln(e^{2x}+1)+C,\\
\int\frac{dx}{\sin x\cos x} &= \int\frac{\frac{dx}{\cos^2 x}}{\tan x}=\int\frac{d(\tan x)}{\tan x}=\ln|\tan x|+C,\\
\int\frac{dx}{\sin x} &= \int\frac{d(\frac{1}{2}x)}{\sin\frac{1}{2}x\cos\frac{1}{2}x}=\ln|\tan\frac{1}{2}x|+C.
\end{aligned}$$

(d) We apply the Integration by Parts Rule to compute the integral $I=\int x\cos x\,dx$. We have

$$\begin{aligned}
I=\int x\cos x\,dx &= \int x\frac{d}{dx}\sin x\,dx\\
&= x\sin x-\int\sin x dx\\
&= x\sin x+\cos x+C.
\end{aligned}$$

(e) We consider the following integral: $I=\int x^3\ln x\,dx$. By the Integration by Parts Rule we obtain

$$\begin{aligned}
I=\int x^3\ln x\,dx &= \int\frac{d}{dx}\Big(\frac{1}{4}x^4\Big)\ln x\,dx\\
&= \frac{1}{4}x^4\ln x-\frac{1}{4}\int x^3dx\\
&= \frac{1}{4}x^4\ln x-\frac{1}{16}x^4+C.
\end{aligned}$$

(f) In order to compute $I=\int\ln x\,dx$ we use again the Integration by Parts Rule:

$$\begin{aligned}
I=\int\frac{d}{dx}(x)\ln x\,dx &= x\ln x-\int x\frac{d}{dx}\ln x\,dx\\
&= x\ln x-\int dx=x(\ln x-1)+C.
\end{aligned}$$

(g) Finally, we consider $I=\int\arctan x\,dx$. We have

$$\begin{aligned}
I &= \int\frac{d}{dx}(x)\arctan x\,dx=x\arctan x-\int x\frac{d}{dx}(\arctan x)dx\\
&= x\arctan x-\int\frac{x}{x^2+1}dx\\
&= x\arctan x-\frac{1}{2}\int\frac{\frac{d}{dx}(x^2+1)}{x^2+1}dx\\
&= x\arctan x-\frac{1}{2}\ln(x^2+1)+C.
\end{aligned}$$

A repeating application of the Integration by Parts Rule leads to the so called GENERALIZED INTEGRATION BY PARTS RULE. Suppose $f(x)$ and $g(x)$ are two $n+1$-differentiable functions. Then, by The Integration by Parts Rule

$$\int f(x)g^{(n+1)}(x)dx = f(x)g^{(n)}(x) - \int f'(x)g^{(n)}(x)dx.$$

In the same way

$$\begin{aligned}
\int f'(x)g^{(n)}(x)dx &= f'(x)g^{(n-1)}(x) - \int f''(x)g^{(n-1)}(x)dx\\
\int f''(x)g^{(n-1)}(x)dx &= f''(x)g^{(n-2)}(x) - \int f^{(3)}(x)g^{(n-2)}(x)dx\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\int f^{(n)}(x)g'(x)dx &= f(x)^{(n)}g(x) - \int f^{(n+1)}(x)g(x)dx.
\end{aligned}$$

By taking the sum of these expressions (multiplied appropriately by ± 1) we obtain

$$\begin{aligned}
\int f(x)g^{(n+1)}(x)dx &= f(x)g^{(n)}(x) - f'(x)g^{(n-1)}(x) + f''(x)g^{(n+2)}(x) - \dots\\
&\quad + (-1)^n f^{(n)}(x)g(x) + (-1)^{n+1}\int f^{(n+1)}g(x)dx. \qquad (14.1)
\end{aligned}$$

The Integration by Parts Rule are usually useful to integrate functions that can be represented as a product of two functions with one of the factors being a polynomial. For example, it can be applied to evaluate the integrals

$$\int x^l \ln^m x dx,\ \int x^k \sin bx\,dx,\ \int x^k \cos bx\,dx,\ \int x^k e^{ax} dx.$$

Example 14.4. We will evaluate the following integrals

(1) $I = \int x^3 \ln x\,dx$. We apply the Integration by Parts Rule:

$$\begin{aligned}
I &= \int \frac{d}{dx}\Big(\frac{1}{4}x^4\Big)\ln x\,dx = \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3 dx\\
&= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C.
\end{aligned}$$

(2) $I = \int \ln^2 x\,dx$. We again apply the Integration by Parts Rule

$$\begin{aligned}
I &= \int \frac{d}{dx}(x)\ln^2 x\,dx = x\ln^2 x - \int x\frac{d}{dx}(\ln^2 x)dx = x\ln^2 x - \int x\cdot 2\ln x\cdot\frac{1}{x}dx\\
&= x\ln^2 x - 2\int \ln x dx = x\ln^2 x - 2x(\ln x - 1) + C.
\end{aligned}$$

(3) $I = \int \arctan x\,dx$. We have

$$I = x\arctan x - \int \frac{x}{x^2+1}dx = x\arctan x - \frac{1}{2}\ln(x^2+1) + C.$$

(4) $I = \int x^2 \sin x\, dx$. By applying the Integration by parts Rule we obtain:

$$I = \int x^2 \frac{d}{dx}(-\cos x)dx = -x^2 \cos x - \int 2x(-\cos x)dx = -x^2 \cos x + 2\int x \cos x\, dx.$$

Since the integral $\int x \cos x\, dx$ was evaluated in Example 14.3(d), we get

$$I = -x^2 \cos x + 2(x \sin x + \cos x) + C.$$

By applying the Generalized Integration by Parts Rule we can compute the integrals of type

$$\int P(x)e^{ax}dx, \ \int P(x)\sin bx\, dx, \ \int P(x)\cos bx\, dx,$$

where $P(x)$ is a polynomial of degree n. Indeed, if we assume that $f(x) = P(x)$ and $g^{(n+1)} = e^{ax}$, we have

$$g^{(n)}(x) = \frac{e^{ax}}{a}, \ g^{(n-1)}(x) = \frac{e^{ax}}{a^2}, \ \dots, g(x) = \frac{e^{ax}}{a^{n+1}},$$

and therefore we obtain

$$\int P(x)e^{ax}dx = e^{ax}\left[\frac{P(x)}{a} - \frac{P'(x)}{a^2} + \frac{P''(x)}{a^3} + \dots + (-1)^n \frac{P^{(n)}(x)}{a^{n+1}}\right] + C.$$

Similarly, assume that $g^{(n+1)}(x) = \sin bx$, then

$$g^{(n)}(x) = \frac{\sin(bx - \frac{\pi}{2})}{b}, \ g^{(n-1)}(x) = \frac{\sin(bx - 2\frac{\pi}{2})}{b^2}, \ \dots, g(x) = \frac{\sin(bx - n\frac{\pi}{2})}{b^{n+1}},$$

so

$$\begin{aligned}\int P(x)\sin bx\, dx &= \sum_{k=0}^{n}(-1)^k \sin(bx - k\frac{\pi}{2})\frac{P^{(k)}(x)}{b^{k+1}} + C\\ &= \sin bx\left[\frac{P'(x)}{b^2} - \frac{P^{(3)}(x)}{b^4} + \dots\right] + \cos bx\left[\frac{P(x)}{b} - \frac{P''(x)}{b^3} + \dots\right] + C.\end{aligned}$$

In a similar way we can derive the formula

$$\begin{aligned}\int P(x)\cos bx\, dx &= \sum_{k=0}^{n}(-1)^k \cos(bx - k\frac{\pi}{2})\frac{P^{(k)}(x)}{b^{k+1}} + C\\ &= \cos bx\left[\frac{P'(x)}{b^2} - \frac{P^{(3)}(x)}{b^4} + \dots\right] + \sin bx\left[\frac{P(x)}{b} - \frac{P''(x)}{b^3} + \dots\right] + C.\end{aligned}$$

In the following examples we will illustrate how to apply the Integration by Parts Rule to obtain a *Reduction Formula* that can be used to evaluate certain integrals.

Example 14.5. Let $I_n = \int x^\mu \ln^n x\, dx$, where μ is an arbitrary real number satisfying $\mu \neq -1$ and n is a natural number. By applying the Integration by parts Rule, we obtain

$$\begin{aligned}I_n &= \int \frac{d}{dx}\left(\frac{x^{\mu+1}}{\mu+1}\right)\ln^n x\, dx = \frac{1}{\mu+1}x^{\mu+1}\ln^n x - \frac{n}{\mu+1}\int x^\mu \ln^{n-1} x\, dx\\ &= \frac{1}{\mu+1}x^{\mu+1}\ln^n x - \frac{n}{\mu+1}I_{n-1}.\end{aligned}$$

Since $I_0 = \int x^\mu dx = \frac{1}{\mu+1}x^{\mu+1} + C$ we can also compute

$$I_1 = \frac{1}{\mu+1}x^{\mu+1}\ln x - \frac{1}{\mu+1}I_0 = \frac{1}{\mu+1}x^{\mu+1}\ln x - \frac{1}{(\mu+1)^2}x^{\mu+1} + C,$$

and

$$\begin{aligned}
I_2 &= \frac{1}{\mu+1}x^{\mu+1}\ln^2 x - \frac{2}{\mu+1}I_1 \\
&= \frac{1}{\mu+1}x^{\mu+1}\ln^2 x - \frac{2}{\mu+1}\left(\frac{1}{\mu+1}x^{\mu+1}\ln x - \frac{1}{(\mu+1)^2}x^{\mu+1}\right) + C \\
&= \frac{1}{\mu+1}x^{\mu+1}\ln^2 x - \frac{2}{(\mu+1)^2}x^{\mu+1}\ln x + \frac{2}{(\mu+1)^3}x^{\mu+1} + C.
\end{aligned}$$

Since $I_3 = \frac{1}{\mu+1}x^{\mu+1}\ln^3 x - \frac{3}{\mu+1}I_2$, we obtain

$$\begin{aligned}
I_3 &= \frac{1}{\mu+1}x^{\mu+1}\ln^3 x - \frac{3}{\mu+1}\left(\frac{1}{\mu+1}x^{\mu+1}\ln^2 x - \frac{2}{(\mu+1)^2}x^{\mu+1}\ln x + \frac{2}{(\mu+1)^3}x^{\mu+1}\right) + C \\
&= \frac{1}{\mu+1}x^{\mu+1}\ln^3 x - \frac{3}{(\mu+1)^2}x^{\mu+1}\ln^2 x + \frac{6}{(\mu+1)^3}x^{\mu+1}\ln x - \frac{6}{(\mu+1)^4}x^{\mu+1}.
\end{aligned}$$

Example 14.6. The following two interesting integrals

$$I = \int e^{ax}\cos bx\,dx, \quad \text{and} \quad \int e^{ax}\sin bx\,dx.$$

can also be evaluated by applying the Integration by Parts Rule twice. Indeed, we have

$$\begin{aligned}
I &= \int e^{ax}\cos bx = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\cdot J \\
J &= \int e^{ax}\sin bx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\cdot I.
\end{aligned}$$

Therefore, we obtain the following system of equations

$$\begin{cases} I = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\cdot J \\ J = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\cdot I \end{cases}$$

which can be easily solved with respect to I and J, so we obtain

$$\begin{aligned}
\int e^{ax}\cos bx\,dx &= \frac{b\sin bx + a\cos bx}{a^2+b^2}e^{ax} + C, \\
\int e^{ax}\sin bx\,dx &= \frac{a\sin bx - b\cos bx}{a^2+b^2}e^{ax} + C'.
\end{aligned}$$

We conclude this section with the following example

Example 14.7. We will derive a reduction formula for the integral $J_n = \int \frac{dx}{(x^2+a^2)^n}$, where n is a natural number. Assume that $f(x) = \frac{1}{(x^2+a^2)^n}$ and $g'(x) = 1$. Then $f'(x) = -\frac{2nx}{(x^2+a^2)^{n+1})}$. so we obtain

$$J_n = \int \frac{dx}{(x^2+a^2)^n} = \frac{x}{(x^2+a^2)^n} + 2n \int \frac{x^2}{(x^2+a^2)^{n+1}} dx. \tag{14.2}$$

The second integral can be transformed as follows

$$\begin{aligned}\int \frac{x^2}{(x^2+a^2)^{n+1}} dx &= \int \frac{(x^2+a^2)-a^2}{(x^2+a^2)^{n+1}} dx \\ &= \int \frac{dx}{(x^2+a^2)^n} - a^2 \int \frac{dx}{(x^2+a^2)^{n+1}} \\ &= J_n - a^2 J_{n+1}.\end{aligned}$$

By substituting the last expression to (14.2) we obtain

$$J_n = \frac{x}{(x^2+a^2)^n} + 2nJ_n - 2na^2 J_{n+1},$$

hence

$$J_{n+1} = \frac{1}{2na^2} \cdot \frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2n} \cdot \frac{1}{a^2} J_n. \tag{14.3}$$

Since, by using the substitution $t = \frac{x}{a}$ we have

$$J_1 = \int \frac{dx}{x^2+a^2} = \frac{1}{a} \int \frac{\frac{d}{dx}\left(\frac{x}{a}\right)dx}{\left(\frac{x}{a}\right)^2+1} = \frac{1}{a} \int \frac{dt}{t^2+1} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C,$$

we can easily compute

$$J_2 = \int \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^2} \cdot \frac{x}{x^2+a^2} + \frac{1}{2a^3}\arctan\left(\frac{x}{a}\right) + C$$

and

$$J_3 = \int \frac{dx}{(x^2+a^2)^3} = \frac{1}{4a^2} \cdot \frac{x}{(x^2+a^2)^2} + \frac{3}{8a^4} \cdot \frac{x}{x^2+a^2} + \frac{3}{8a^5}\arctan\left(\frac{x}{a}\right) + C.$$

Problems 14.2 (a) Compute the integral $I = \int \frac{x^2+3}{\sqrt{(2x-5)^3}} dx$;

(b) Compute the integral $I = \int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$;
(c) Compute the integral $\int \sqrt[3]{x} (\ln x)^2 dx$;
(d) Compute the integral $\int 3^x \cos x dx$;
(e) Derive the reduction formula for the integration of

$$I_n = \int \frac{dx}{\sin^n x}$$

and use it for calculating the integral $I_3 = \int \frac{dx}{\sin^3 x}$.

Solution 14.8. Problem 14.2(a): We apply the substitution $t = 2x-5$, i.e. $x = \frac{t+5}{2}$ and $dx = \frac{dt}{2}$, to get

$$\begin{aligned}I &= \int \frac{x^2+3}{\sqrt{(2x-5)^3}} dx = \frac{1}{8} \int \frac{t^2+10t+37}{t^{\frac{3}{2}}} dt \\ &= \frac{1}{8} \int (t^{\frac{1}{2}} + 10t^{-\frac{1}{2}} + 37t^{-\frac{3}{2}}) dt = \frac{1}{4}\left[\frac{t^2+30t-111}{3t^{\frac{1}{2}}}\right] + C \\ &= \frac{x^2+10x-59}{3\sqrt{2x-5}} + C.\end{aligned}$$

Problem 14.2(b): We apply the substitution $t = \tan(x)$, i.e. $dt = \frac{dx}{\cos^2 x}$, as follows

$$\begin{aligned}
I &= \int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \int \frac{1}{a^2 \tan^2 x + b^2} \cdot \frac{dx}{\cos^2 x}\\
&= \int \frac{dt}{a^2 t^2 + b^2} = \frac{1}{a^2} \int \frac{dt}{t^2 + \left(\frac{b}{a}\right)^2}\\
&= \frac{1}{ab} \arctan\left(\frac{a}{b}\tan x\right) + C.
\end{aligned}$$

Problem 14.2(c): We apply the integration by parts method:

$$\begin{aligned}
\int \sqrt[3]{x}(\ln x)^2 dx &= \frac{3}{4} x^{\frac{4}{3}} (\ln x)^2 - \frac{3}{2} \int x^{\frac{1}{3}} \ln x dx\\
&= \frac{3}{4} x^{\frac{4}{3}} (\ln x)^2 - \frac{3}{2} \left[\frac{3}{4} x^{\frac{4}{3}} \ln x - \frac{3}{4} \int x^{\frac{1}{3}} dx\right]\\
&= x^{\frac{4}{3}} \left(\frac{3}{4}(\ln x)^2 - \frac{9}{8} \ln x + \frac{27}{32}\right) + C\\
&= \frac{3}{32} x^{\frac{4}{3}} \left(8(\ln x)^2 - 12 \ln x + 9\right) + C.
\end{aligned}$$

Problem 14.2(d): We can use the formula from Example 14.6:

$$\int 3^x \cos x dx = \int e^{x \ln 3} \cos x\, dx = \frac{1}{4} e^{x \ln 3} (\sin x + \ln x\, \cos x) + C.$$

Problem 14.2(e): Assume $n \geq 2$ and apply the Integration by Parts Rule:

$$\begin{aligned}
I_n &= \int \frac{dx}{\sin^n x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^n x} dx\\
&= I_{n-2} + \int \cos \frac{\cos x}{\sin^n x} dx\\
&= I_{n-2} - \frac{1}{n-1} \frac{\cos x}{\sin^{n-1} x} - \frac{1}{n-1} I_{n-2}\\
&= \frac{n-2}{n-1} I_{n-2} - \frac{\cos x}{(n-1)\sin^{n-1} x}.
\end{aligned}$$

Notice that by applying the substitution $t = \cos x$ we get

$$\begin{aligned}
I_1 &= \int \frac{dx}{\sin x} = -\int \frac{-\sin x\, dx}{1 - \cos^2 x}\\
&= -\int \frac{dt}{1 - t^2} = \frac{1}{2} \int \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt\\
&= \frac{1}{2} \ln \left|\frac{1-t}{1+t}\right| + C = \frac{1}{2} \ln \left|\frac{(1 - \cos x)^2}{\sin^2 x}\right| + C\\
&= \ln \left|\frac{1 - \cos x}{\sin x}\right| + C = \ln|\csc x - \cot x| + C.
\end{aligned}$$

Consequently, by the reduction formula

$$I_3 = \frac{1}{2} I_1 - \frac{\cos x}{2 \sin^2 x} = \frac{1}{2} \ln|\csc x - \cot x| - \frac{\cos x}{2 \sin^2 x} + C.$$

14.3 Integration of Rational Functions

Among the elementary functions, the rational functions played a special role in the development of integration techniques. One of the important features of rational functions is the fact that their indefinite integrals can be expressed by elementary functions, and that means that theoretically, as long as we are able to effectuate related algebraic computations (like computations of roots of the denominator), we will also be able to find an exact formula for the integral. Moreover, many other types of integrals can be reduced, by special substitutions, to integrals of rational functions. However, this is not the case for all elementary functions. There are many known functions with integrals that can not be expressed by elementary functions, i.e. it is impossible to express them by an analytic formula using basic elementary functions, arithmetic operations, their compositions and inverses. The following are just few well known integrals that can not be expressed by elementary functions:

$$\int e^{-x^2}dx, \;, \int \sin x^2 dx, \; \int \cos^2 x dx$$
$$\int \frac{\sin x}{x}dx, \; \int \frac{\cos x}{x}dx, \; \int \frac{dx}{\ln x},$$

but there are many more.

A rational function $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are two polynomials, is called a *proper rational function* if $\deg P(x) < \deg Q(x)$. By using the *division algorithm* for polynomials, every rational function $f(x) = \frac{P_1(x)}{Q(x)}$ can be represented as a sum of a polynomial and a proper rational function, i.e.

$$f(x) = \frac{P_1(x)}{Q(x)} = M(x) + \frac{P(x)}{Q(x)},$$

where

$$P_1(x) = M(x)Q(x) + P(x),$$

with $P(x)$ being the remainder of the division of $P_1(x)$ by $Q(x)$, i.e. $\deg P(x) < \deg Q(x)$. We say that a proper rational function $\frac{P(x)}{Q(x)}$ is *irreducible* if $P(x)$ and $Q(x)$ have no common linear or quadratic factors.

Lemma 14.9. *Suppose that $\frac{P(x)}{Q(x)}$ is a proper rational function such that $Q(x) = (x-\alpha)^k Q_o(x)$, where α is not a root of $Q_o(x)$ and $k \in \mathbb{N}$. Then there exist a constant A and a polynomial $P_o(x)$ such that*

$$\frac{P(x)}{Q(x)} = \frac{A}{(x-\alpha)^k} + \frac{P_o(x)}{(x-\alpha)^{k-1}Q_o(x)},$$

where $\deg P_o(x) < \deg Q(x) - 1$.

Proof: Suppose that $A = \frac{P(\alpha)}{Q_o(\alpha)}$, thus the polynomial $P(x) - AQ_o(x)$ has the root α. Then we have that $P(x) - AQ_o(x)$ is divisible by $(x-\alpha)$ so there is $P_o(x)$ such that $\deg P_o(x) = \max\{\deg P(x), \deg Q(x)\} - 1 < \deg Q(x)-$ and

$$P(x) - AQ_o(x) = (x-\alpha)P_o(x). \tag{14.4}$$

If we divide (14.4) by $Q(x)$ we obtain

$$\frac{P(x)}{Q(x)} = \frac{A}{(x-\alpha)^k} + \frac{P_o(x)}{(x-\alpha)^{k-1}Q_o(x)}$$

and the conclusion follows. □

Lemma 14.10. *Let x^2+px+q be an irreducible quadratic polynomial, i.e. $p^2-4q<0$. Suppose that $\frac{P(x)}{Q(x)}$ is a proper rational function such that $Q(x)=(x^2+px+q)^m Q_o(x)$, where $Q_o(x)$ is not divisible by (x^2+px+q) and $m\in\mathbb{N}$. Then there exist constants M, N and a polynomial $P_o(x)$, with $\deg P_o(x) < \deg Q(x) - 2$, such that $\frac{P(x)}{Q(x)}$ can be represented as the following sum of proper rational functions*

$$\frac{P(x)}{Q(x)} = \frac{Mx+N}{(x^2+px+q)^m} + \frac{P_o(x)}{(x^2+px+q)^{m-1}Q_o(x)}.$$

Proof: By applying the Division Algorithm, we can represent the polynomials $P(x)$ and $Q_o(x)$ as

$$\begin{aligned} P(x) &= (x^2+px+q)R(x) + (ax+b),\\ Q_o(x) &= (x^2+px+q)S(x) + (cx+d) \end{aligned} \tag{14.5}$$

We need to find the numbers M and N such that the polynomial $P(x)-(Mx+N)Q_o(x)$ is divisible by (x^2+px+q). By using the expressions (14.5), we have to choose the numbers M and N in such a way that the polynomial

$$T(x) = ax+b-(Mx+N)(cx+d)$$

is divisible by (x^2+px+q). Since $T(x)=ax+b-(Mx+N)(cx+d) = -Mcx^2+(a-dM-cN)x+(b-dN)$, we see that the remainder obtained in dividing $T(x)$ by (x^2+px+q) should be equal to zero. Since this remainder is equal to

$$[(pc-d)M-cN+a]x + [qcM-dN+b]$$

we obtain the following system of linear equations (with M and N being unknown):

$$\begin{cases} (pc-d)M-cN &= -a\\ qcM-dN &= -b. \end{cases} \tag{14.6}$$

The determinant of this system is equal

$$\triangle = \det\begin{bmatrix} pc-d & -c\\ qc & -d\end{bmatrix} = d^2 - pcd + qc^2.$$

We need to show that the determinant $\triangle$ is different from zero, so the existence of solutions to the equations (14.6) will follow. If $c\neq 0$ then $\triangle = c^2\left[\left(-\frac{d}{c}\right)^2 + p\left(-\frac{d}{c}\right)+q\right]\neq 0$, since $x^2+px+q\neq 0$ (by the assumption that it is irreducible quadratic polynomial). On the other hand, if $c=0$, $\triangle = d^2$, and because $Q_o(x)$ is not divisible by (x^2+px+q), c and d can not both be 0, so $d\neq 0$. Therefore, in either case $\triangle\neq 0$, and there exist the unique numbers M and N, such that

$$P(x)-(Mx+N)Q_o(x) = (x^2+px+q)P_o(x) \tag{14.7}$$

so, dividing (14.7) by $Q(x)$ yields

$$\frac{P(x)}{Q(x)} = \frac{Mx+N}{(x^2+px+q)^m} + \frac{P_o(x)}{(x^2+px+q)^{m-1}Q_o(x)},$$

and the conclusion follows. □

The following types of proper rational functions are called *partial fractions:*

$$\text{I. } \frac{A}{x-\alpha}, \quad \text{II. } \frac{A}{(x-\alpha)^k} \; (k=2,3,\dots),$$
$$\text{III. } \frac{Mx+N}{x^2+px+q}, \quad \text{IV. } \frac{Mx+N}{(x^2+px+q)^m} \; (m=2,3,\dots),$$

where A, M, N, α, p and q are real numbers.

The main result of this section is the following PARTIAL FRACTION DECOMPOSITION THEOREM which will enable us to develop a general method for integration of rational functions:

Theorem 14.11. (PARTIAL FRACTION DECOMPOSITION THEOREM) *Every proper rational function* $\frac{P(x)}{Q(x)}$ *can be represented as a finite sum of partial fractions. In particular, if* $\frac{P(x)}{Q(x)}$ *is an irreducible proper fraction and*

$$Q(x) = (x-\alpha_1)^{r_1}\dots(x-\alpha_l)^{r_l}(x^2+p_1x+q_1)^{m_1}\dots(x^2+p_kx+q_k)^{m_k},$$

then

$$\begin{aligned}
\frac{P(x)}{Q(x)} &= \frac{A_{1,1}}{(x-\alpha_1)^1} + \frac{A_{1,2}}{(x-\alpha_1)^2} + \dots + \frac{A_{1,r_1}}{(x-\alpha_1)^{r_1}} \\
&+ \frac{A_{2,1}}{(x-\alpha_2)^1} + \frac{A_{2,2}}{(x-\alpha_2)^2} + \dots + \frac{A_{2,r_2}}{(x-\alpha_2 r_2} + \dots \\
&+ \frac{A_{l,1}}{(x-\alpha_l)^1} + \frac{A_{l,2}}{(x-\alpha_l)^2} + \dots + \frac{A_{l,r_l}}{(x-\alpha_l)^{r_l}} \\
&+ \frac{M_{1,1}x+N_{1,1}}{(x^2+p_1x+q_1)^1} + \frac{M_{1,2}x+N_{1,2}}{(x^2+p_1x+q_1)^2} + \dots + \frac{M_{1,m_1}x+N_{1,m_1}}{(x^2+p_1x+q_1)^{m_1}} \\
&+ \frac{M_{2,1}x+N_{2,1}}{(x^2+p_2x+q_2)^1} + \frac{M_{2,2}x+N_{2,2}}{(x^2+p_2x+q_2)^2} + \dots + \frac{M_{2,m_2}x+N_{2,m_2}}{(x^2+p_2x+q_2)^{m_2}} + \dots \\
&+ \frac{M_{k,1}x+N_{k,1}}{(x^2+p_kx+q_k)^1} + \frac{M_{k,2}x+N_{k,2}}{(x^2+p_kx+q_k)^2} + \dots + \frac{M_{k,m_k}x+N_{k,m_k}}{(x^2+p_kx+q_k)^{m_k}}
\end{aligned}$$

where $A_{i,j}$, $M_{i,j}$, $N_{i,j}$, α_j, p_j *and* q_j *are real constants.*

Proof: The proof is based on repeated applications of Lemmas 14.9 and 14.10.

Assume that $(x-\alpha)^r$ is one of the factors of $Q(x)$. By Lemma 14.9, we have that there exists a constant A_r such that we have the following decomposition

$$\frac{P(x)}{Q(x)} = \frac{A_r}{(x-\alpha)^r} + \frac{P_1(x)}{Q_1(x)}$$

where $Q_1(x)$ now has $(x-\alpha)^{r-1}$ as its linear factor. By applying Lemma 14.9 again to the proper fraction $\frac{P_1(x)}{Q_1(x)}$ we obtain that

$$\frac{P_1(x)}{Q_1(x)} = \frac{A_{r-1}}{(x-\alpha)^{r-1}} + \frac{P_2(x)}{Q_2(x)},$$

and by applying the induction principle we will finally obtain the decomposition

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-\alpha)^1} + \frac{A_2}{(x-\alpha)^2} + \cdots + \frac{A_r}{(x-\alpha)^r} + \frac{P_o(x)}{Q_o(x)},$$

where $Q_o(x)$ is not divisible by $(x-\alpha)$. Next we repeat the above construction with respect to $\frac{P_o(x)}{Q_o(x)}$ and other linear factors. The same principle is applied with respect to irreducible quadratic factors of $Q(x)$. Assume that $(x^2 - px + q)^m$ is an irreducible quadratic factor of $Q(x)$ then, by Lemma 14.10 we have

$$\frac{P(x)}{Q(x)} = \frac{M_m + N_m}{(x^2+px+q)^m} + \frac{\tilde{P}_1(x)}{\tilde{Q}_1(x)}$$

where $\tilde{Q}_1(x)$ has now $(x^2+px+q)^{m-1}$ as its quadratic factor. Next we reapply Lemma 5.3.3 to the proper rational function $\frac{\tilde{P}_1(x)}{\tilde{Q}_1(x)}$, to extract the fraction $\frac{M_{m-1}x+N_{m-1}}{(x^2+px+q)^{m-1}}$, i.e.

$$\frac{\tilde{P}_1(x)}{\tilde{Q}_1(x)} = \frac{M_{m-1}x + N_{m-1}}{(x^2+px+q)^{m-1}} + \frac{\tilde{P}_2(x)}{\tilde{Q}_2(x)},$$

where $\tilde{Q}_2(x)$ has now $(x^2+px+q)^{m-2}$ as its quadratic factor. By repeating this process $m-2$ more times, we will obtain that $\frac{P(x)}{Q(x)}$ can be represented as

$$\frac{P(x)}{Q(x)} = \frac{M_1x + N_1}{(x^2+px+q)} + \cdots + \frac{M_m + N_m}{(x^2+px+q)^m} + \frac{\tilde{P}_o(x)}{\tilde{Q}_o(x)}$$

where $\tilde{Q}_o(x)$ has no more (x^2+px+q) as its quadratic factor. Therefore, by applying the above construction to each of the factors of $Q(x)$ we will obtain the required partial fraction decomposition of $\frac{P(x)}{Q(x)}$. □

By decomposing the function $f(x) = \frac{P(x)}{Q(x)}$ into a sum of partial fractions of types (I) - (IV), we reduce the computation of the integral

$$\int \frac{P(x)}{Q(x)}dx$$

to the computation of integrals of types

$$\int \frac{A}{(x-\alpha)^k}dx, \qquad \int \frac{Mx+N}{(x^2+px+q)^m}dx.$$

The integrals of partial fractions corresponding to linear factors can be easily evaluated. Indeed, we have

$$\int \frac{A}{x-\alpha}dx = A\int \frac{dx}{x-\alpha} = A\ln|x-\alpha| + C,$$

and for $k > 1$

$$\int \frac{A}{(x-\alpha)^k}dx = A\int \frac{dx}{(x-\alpha)^k} = -\frac{A}{(k-1)} \cdot \frac{1}{(x-\alpha)^{k-1}} + C.$$

Integration of partial fractions associated with quadratic factors requires more work. Consider an irreducible quadratic polynomial $x^2 + px + q$, which we can represent in the form of a full square, i.e.

$$x^2+px+q = x^2+2\frac{p}{2}x+\left(\frac{p}{2}\right)^2+\left[q-\left(\frac{p}{2}\right)^2\right] = \left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right).$$

Since by the assumption $\left(q-\frac{p^2}{4}\right)>0$, we can define

$$a=+\sqrt{\left(q-\frac{p^2}{4}\right)}.$$

Next, we use the substitution $t=x+\frac{p}{2}$ to evaluate the following integral

$$\begin{aligned}
\int \frac{Mx+N}{x^2+px+q}dx &= \int \frac{Mx+N}{\left(x+\frac{p}{2}\right)^2+a^2}dx \\
&= \int \frac{Mt+(N-\frac{1}{2}Mp)}{t^2+a^2}dt \\
&= \frac{M}{2}\int \frac{2tdt}{t^2+a^2}+\left(N-\frac{Mp}{2}\right)\int \frac{dt}{t^2+a^2} \\
&= \frac{M}{2}\ln(t^2+a^2)+\frac{1}{a}\left(N-\frac{Mp}{2}\right)\arctan\frac{t}{a}+C.
\end{aligned}$$

and consequently we obtain

$$\int \frac{Mx+N}{x^2+px+q}dx = \frac{M}{2}\ln(x^2+px+q)+\frac{2N-Mp}{\sqrt{4q-p^2}}\arctan\left(\frac{2x+p}{\sqrt{4q-p^2}}\right)+C.$$

In order to integrate a partial fraction of type (IV), for $m>1$, we use the same substitution $t=x+\frac{p}{2}$

$$\begin{aligned}
\int \frac{Mx+N}{(x^2+px+q)^m}dx &= \int \frac{Mt+\left(N-\frac{1}{2}Mp\right)}{(t^2+a^2)^m}dt \\
&= \frac{M}{2}\int \frac{2tdt}{(t^2+a^2)^m}+\left(N-\frac{Mp}{2}\right)\int \frac{dt}{(t^2+a^2)^m}.
\end{aligned}$$

In order to evaluate the integral $\int \frac{2tdt}{(t^2+a^2)^m}$ we apply the substitution $t^2+a^2=u$, thus $2tdt=du$, so

$$\int \frac{2tdt}{(t^2+a^2)^m} = \int \frac{du}{u^m} = -\frac{1}{m-1}\cdot\frac{1}{u^{m-1}}+C = -\frac{1}{m-1}\cdot\frac{1}{(t^2+a^2)^{m-1}}+C.$$

In order to compute the integral $\int \frac{dt}{(t^2+a^2)^m}$ we use the reduction formula (14.3). Finally, we return to the original variable x by making the substitution $t=\frac{2x+p}{2}$.

We can see that the only difficulty with the computations of the integrals associated with the partial fractions is related to the integration of expressions of type (IV). In fact we have proved above that in the case $m>1$, we have

$$\int \frac{Mx+N}{(x^2+px+q)^m}dx = \frac{M_1x+N_1}{(x^2+px+q)^{m-1}}+\lambda_1\int \frac{dx}{(x^2+px+q)^{m-1}}.$$

Since the same formula can be applied to $\lambda_1\int \frac{dx}{(x^2+px+q)^{m-1}}$, we obtain that

$$\int \frac{\lambda_1 dx}{(x^2+px+q)^{m-1}} = \frac{M_2x+N_2}{(x^2+px+q)^{m-2}} + \lambda_2 \int \frac{dx}{(x^2+px+q)^{m-2}},$$

and by repeating this process again and again, eventually we would obtain that

$$\int \frac{Mx+N}{(x^2+px+q)^m} dx = \frac{R(x)}{(x^2+px+q)^{m-1}} + \lambda \int \frac{dx}{x^2+px+q},$$

where $R(x)$ is a certain polynomial of degree smaller than $2(m-1)$, i.e. $\frac{R(x)}{(x^2+px+q)^{m-1}}$ is a proper rational function.

This observation enables us to develop another purely algebraic method for the computation of rational integrals. Indeed, since the same statement is true with respect to any quadratic factor $(x^2+px+q)^m$, $m>0$, of $Q(x)$ and for the factors $(x-\alpha)^k$ we have

$$\int \frac{A}{(x-\alpha)^k} dx = \frac{\tilde{R}(x)}{(x-\alpha)^{k-1}} + \tilde{\lambda} \int \frac{dx}{x-\alpha},$$

we obtain the following

Proposition 14.12. *Let* $\frac{P(x)}{Q(x)}$ *be an irreducible proper fraction such that*

$$Q(x) = (x-\alpha)^k \cdot \dots \cdot (x^2+px+q)^m \cdot \dots .$$

Then

$$\int \frac{P(x)}{Q(x)} dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)} dx, \tag{14.8}$$

where

$$Q_1(x) = (x-\alpha)^{k-1} \cdot \dots \cdot (x^2+px+q)^{m-1} \cdot \dots,$$
$$Q_2(x) = (x-\alpha) \cdot \dots \cdot (x^2+px+q) \cdot \dots$$

and $\frac{P_1(x)}{Q_1(x)}$ *and* $\frac{P_2(x)}{Q_2(x)}$ *are proper fractions.*

The formula (14.8), called the *Ostrogradsky Formula*, can be used to find the polynomials $P_1(x)$ and $P_2(x)$, so we can avoid elaborate calculations.

By differentiating (14.8) we obtain

$$\frac{P}{Q} = \left[\frac{P_1}{Q_1}\right]' + \frac{P_2}{Q_2},$$

thus

$$\frac{P_1'Q_1 - P_1Q_1'}{Q_1^2} + \frac{P_2}{Q_2} = \frac{P}{Q}.$$

Thus

$$\frac{P_1'Q_1 - P_1\frac{Q_1'Q_2}{Q_1}}{Q_1Q_2} + \frac{P_2Q_1}{Q_1Q_2} = \frac{P}{Q},$$

and since $Q_1Q_2 = Q$, we get

$$\frac{P_1'Q_2 - P_1H + P_2Q_1}{Q} = \frac{P}{Q},$$

where $H = \frac{Q_1'Q_2}{Q_1}$, thus

$$P_1'Q_2 - P_1H + P_2Q_1 = P. \tag{14.9}$$

In order to find P_1 and P_2 it is sufficient to find their coefficients which can be determined from the equation (14.9).

Example 14.13. We will illustrate the method of Ostrogradsky by evaluating the following the integral

$$\int \frac{4x^4 + 4x^3 + 16x^2 + 12x + 8}{(x+1)^2(x^2+1)^2} dx.$$

We have that

$$Q_1(x) = Q_2(x) = (x+1)(x^2+1) = x^3 + x^2 + x + 1.$$

Thus $H(x) = \frac{(3x^2+2x+1)Q_2(x)}{Q_1(x)} = 3x^2 + 2x + 1$. Both $P_1(x)$ and $P_2(x)$ are of degree 2, i.e.

$$P_1(x) = ax^2 + bx + c, \quad P_2(x) = ex^2 + fx + g.$$

Thus the equation (14.9) can be written as

$$\begin{aligned} &4x^4 + 4x^3 + 16x^2 + 12x + 8 \\ =&(2ax + b)(x^3 + x^2 + x + 1) - (ax^2 + bx + c)(3x^2 + 2x + 1) + (dx^2 + ex + f)(x^3 + x^2 + x + 1) \end{aligned}$$

so

$$\begin{aligned} dx^5 + (-a+e)x^4 + (-2b + e + f)x^3 + (a - b - 3c + e + f)x^2 + (2a - 2c + e + f)x + (b - c + f) = \\ 4x^4 + 4x^3 + 16x^2 + 12x + 8. \end{aligned}$$

As a result, we obtain the following system of linear equations

$$\begin{cases} d & = 0 \\ -a + e & = 4 \\ -2b + e + f & = 4 \\ a - b - 3c + e + f & = 16 \\ 2a - 2c + e + f & = 12 \\ b - c + f & = 8, \end{cases}$$

which we can solve by using the substitution method and obtain the following solutions

$$a = -1, \ b = 1, \ c = -4, \ d = 0, \ e = 3, \ f = 3.$$

hence

$$\begin{aligned} \int \frac{4x^4 + 4x^3 + 16x^2 + 12x + 8}{(x+1)^2(x^2+1)^2} dx &= -\frac{x^2 - x + 4}{x^3 + x^2 + x + 1} + 3\int \frac{dx}{x^2+1} \\ &= -\frac{x^2 - x + 4}{x^3 + x^2 + x + 1} + 3\arctan x + C. \end{aligned}$$

We conclude this section with few examples.

Example 14.14. We will integrate the following rational functions:

(a) $I=\int \frac{dx}{x^2(1+x^2)^2}$. In order to get the partial fraction decomposition of the rational function $\frac{1}{x^2(1+x^2)^2}$ we make the following transformations:

$$\begin{aligned}\frac{1}{x^2(1+x^2)^2}&=\frac{(1+x^2)-x^2}{x^2(1+x^2)^2}=\frac{1}{x^2(1+x^2)}-\frac{1}{(1+x^2)^2}\\&=\frac{(1+x^2)-x^2}{x^2(1+x^2)}-\frac{1}{(1+x^2)^2}=\frac{1}{x^2}-\frac{1}{1+x^2}-\frac{1}{(1+x^2)^2}.\end{aligned}$$

Consequently,

$$\begin{aligned}I&=\int\left[\frac{1}{x^2}-\frac{1}{1+x^2}-\frac{1}{(1+x^2)^2}\right]dx\\&=\int\frac{dx}{x^2}-\int\frac{dx}{1+x^2}-\int\frac{dx}{(1+x^2)^2}\\&=-\frac{1}{x}-\frac{1}{2}\cdot\frac{x}{1+x^2}-\frac{3}{2}\arctan x+C.\end{aligned}$$

(b) $I=\int\frac{4x^2+4x-11}{(2x-1)(2x+3)(2x-5)}dx$. First, we need to find the partial fraction decomposition of $\frac{4x^2+4x-11}{(2x-1)(2x+3)(2x-5)}$. We have

$$\begin{aligned}\frac{4x^2+4x-11}{(2x-1)(2x+3)(2x-5)}&=\frac{\frac{1}{2}x^2+\frac{1}{2}x-\frac{11}{8}}{(x-\frac{1}{2})(x+\frac{3}{2})(x-\frac{5}{2})}\\&=\frac{A}{x-\frac{1}{2}}+\frac{B}{x+\frac{3}{2}}+\frac{C}{x-\frac{5}{2}}.\end{aligned}$$

Thus

$$\frac{1}{2}x^2+\frac{1}{2}-\frac{11}{8}=A(x+\frac{3}{2})(x-\frac{5}{2})+B(x-\frac{1}{2})(x-\frac{5}{2})+C(x-\frac{1}{2})(x+\frac{3}{2}). \tag{14.10}$$

Instead of comparing the coefficients of the polynomials on the left hand side and the right hand side of we substitute successively $x=\frac{1}{2}$, $-\frac{3}{2}$, $\frac{5}{2}$ into this equation. We obtain $A=\frac{1}{4}$, $B=-\frac{1}{8}$ and $C=\frac{3}{8}$. Therefore,

$$\begin{aligned}I&=\int\left[\frac{\frac{1}{4}}{\frac{1}{4}x-12}+\frac{-\frac{1}{8}}{x+\frac{3}{2}}+\frac{\frac{3}{8}}{x-\frac{5}{2}}\right]dx\\&=\frac{1}{4}\ln\left|x-\frac{1}{2}\right|-\frac{1}{8}\ln\left|x+\frac{3}{2}\right|+\frac{3}{8}\ln\left|x-\frac{5}{2}\right|+C\\&=\frac{1}{8}\ln\left|\frac{(2x-1)^2(2x-5)^3}{2x+3}\right|+C.\end{aligned}$$

(c) $I=\int\frac{dx}{x^4+1}$. Since

$$\begin{aligned}x^4+1&=(x^4+2x^2+1)-2x^2=(x^2+1)^2-(x\sqrt{2})^2\\&=(x^2+x\sqrt{2}+1)(x^2-x\sqrt{2}+1),\end{aligned}$$

we look for the following partial fraction decomposition

$$\frac{1}{x^4+1} = \frac{Ax+B}{x^2+x\sqrt{2}+1} + \frac{Cx+D}{x^2-x\sqrt{2}+1},$$

and consequently we get

$$1 = (Ax+B)(x^2-x\sqrt{2}+1) + (Cx+D)(x^2+x\sqrt{2}+1),$$

thus

$$1 = (A+C)x^3 + (-\sqrt{2}A+B+\sqrt{2}C+D)x^2 + (A-\sqrt{2}B+C+\sqrt{2}D)x + (B+D),$$

which implies the following system of equations

$$\begin{cases} A+C & = 0, \\ -\sqrt{2}A+B+\sqrt{2}C+D & = 0, \\ A-\sqrt{2}B+C+\sqrt{2}D & = 0, \\ B+D & = 1. \end{cases}$$

The solutions of the above system are:

$$A = -C = \frac{1}{2\sqrt{2}}, \quad B = D = \frac{1}{2},$$

so

$$\begin{aligned} I = \int \frac{dx}{x^4+1} &= \frac{1}{2\sqrt{2}} \int \frac{x+\sqrt{2}}{x^2+x\sqrt{2}+1} dx - \frac{1}{2\sqrt{2}} \int \frac{x-\sqrt{2}}{x^2-x\sqrt{2}+1} dx \\ &= \frac{1}{4\sqrt{2}} \ln \frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1} + \frac{1}{2\sqrt{2}} \arctan(x\sqrt{2}+1) \\ &+ \frac{1}{2\sqrt{2}} \arctan(x\sqrt{2}-1) + C. \end{aligned}$$

(d) $I = \int \frac{x^6-x^5+x^4+2x^3+3x^2+3x+3}{(x+1)^2(x^2+x+1)^3} dx$. In order to avoid elaborated partial fraction computations and subsequent computations of the integrals of expressions of type (IV), we will apply the Method of Ostrogradsky. In our example we have:

$$Q_1(x) = (x+1)(x^2+x+1)^2, \quad Q_2(x) = (x+1)(x^2+x+1),$$

and

$$\begin{aligned} H(x) = \frac{Q_1'(x)Q_2(x)}{Q_1(x)} &= \frac{(x^2+x+1)(5x^2+7x+1)(x+1)(x^2+x+1)}{(x+1)(x^2+x+1)^2} \\ &= 5x^2+7x+3. \end{aligned}$$

Put

$$P_1(x) = ax^4+bx^3+cx^2+dx+e, \; P_2(x) = fx^2+gx+h.$$

Then, by (14.9), we have

$$(4ax^3+3bx^2+2cx+d)(x+1)(x^2+x+1)-(ax^4+bx^3+cx^2+dx+e)(5x^2+7x+1)+$$
$$(fx^2+gx+h)(x+1)(x^2+x+1)^2=x^6-x^5+x^4+2x^3+3x^2+3x+3.$$

which can be reduced to the following system of linear equations

$$\begin{cases} f & =0, \\ -a+3f+g & =1, \\ -a-2b-+5f+3g+h & =-1, \\ 5a-b-3c+5f+5g+3h & =1, \\ 3b+4a-3c-4d+3f+5g+5h & =2, \\ c+f+3b-5d-5e+3g+5h & =3, \\ -d+g+2c-7e+3h & =3, \\ h+d-3e & =3. \end{cases}$$

Since $f=0$ the above system can be reduced to the following system of linear equations

$$\begin{cases} f & =0, \\ -a+g & =1, \\ -a-2b+3g+h & =-1, \\ 5a-b-3c+5g+3h & =1, \\ 4a+3b-3c-4d+5g+5h & =2, \\ 3b+c-5d-5e+3g+5h & =3, \\ 2c-d-7e+g+3h & =3, \\ d-3e+h & =3. \end{cases}$$

By solving these equations we obtain:

$$a=-1,\ b=0,\ c=-2,\ d=0,\ e=-1,\ f=g=h=0,$$

thus

$$I=\frac{-x^4-2x^2-1}{(x+1)(x^2+x+1)^2}+C.$$

Problems 14.3 (a) Find the integral $I=\int\frac{5x^3+9x^2-22x-8}{x^3-4x}dx$. Use the Partial Fraction Decomposition.

(b) Find the integral $I=\int\frac{x^3+1}{x(x-1)^3}dx$. Use the method of Ostrogradsky.

(c) Find the integral $I=\int\frac{x^3+3}{(x+1)(x^2+1)}dx$.

Solution 14.15. Problem 14.3(a): We have that

$$\begin{aligned}\frac{5x^3+9x^2-22x-8}{x^3-4x} =& 5+\frac{9x^2-2x-8}{x^3-4x}=5+\frac{9x^2-2x-8}{x(x-2)(x+2)}\\ =&5-\frac{2}{x}+\frac{5}{x-2}+\frac{6}{x+2}.\end{aligned}$$

Thus

$$\begin{aligned}\int \frac{5x^3+9x^2-22x-8}{x^3-4x}dx =& \int \left(5-\frac{2}{x}+\frac{5}{x-2}+\frac{6}{x+2}\right)dx\\ =&5x-2\ln|x|+5\ln|x-2|+6\ln|x+2|+C\\ =&5x+\ln\left|\frac{(x-2)^5(x+2)^6}{x^2}\right|+C\end{aligned}$$

Problem 14.3(b): We look for the polynomials $P_1(x)=Ax+B$ and $P_2(x)=Cx+D$ such that

$$\int \frac{x^3+1}{x(x-1)^3}dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)}dx,$$

where $Q_1(x)=(x-1)^2$ and $Q_2(x)=x(x-1)$. We put $H(x)=\frac{Q_1'(x)Q_2(x)}{Q_1(x)}=2x$ and we find the coefficients A, B, C, D from the equation

$$P_1'(x)Q_2(x)-P_1(x)H(x)+P_2(x)Q_1(x)=x^3+1,$$

which can be written as

$$Ax(x-1)-(Ax+B)2x+(Cx+D)(x-1)^2=x^3+1.$$

After simplification, we get

$$Cx^3+(-A+D-2C)x^2+(-A-2B-2D+C)x+D=x^3+1,$$

so $A=-1$, $B=0$, $C=1$, and $D=1$. Consequently,

$$\begin{aligned}\int \frac{x^3+1}{x(x-1)^3}dx =&\frac{-x}{(x-1)^2}+\int \frac{x+1}{x(x-1)}dx = \frac{-x}{(x-1)^2}+\int\left(\frac{-1}{x}+\frac{2}{x-1}\right)dx\\ =&\frac{-x}{(x-1)^2}+\ln\left|\frac{(x-1)^2}{x}\right|+C.\end{aligned}$$

Problem 14.3(c): We have the follwing partial fraction decomposition:

$$\frac{x^3+3}{(x+1)(x^2+1)} = 1+\frac{-x^2-x+2}{(x+1)(x^2+1)} = 1+\frac{1}{x+1}+\frac{1-2x}{x^2+1}$$

thus

$$\begin{aligned}\int \frac{x^3+3}{(x+1)(x^2+1)}dx =& \int\left(1+\frac{1}{x+1}-\frac{2x}{x^2+1}+\frac{1}{x^2+1}\right)dx\\ =&x+\ln|x+1|-\ln|x^2+1|+\arctan(x)+C\\ =&x+\ln\left|\frac{x+1}{x^2+1}\right|+\arctan(x)+C.\end{aligned}$$

14.4 Integration of Certain Irrational Expressions

We begin with integrals of type

$$I = \int R\left(x, \sqrt[m]{\frac{\alpha x + \beta}{\gamma x + \delta}}\right) dx,$$

where m is a natural number and R is a rational function of two variables, i.e.

$$R(x, y) = \frac{\sum a_{ji} x^i y^j}{\sum b_{ls} x^l y^s},$$

where the sums standing in the numerator and the denominator are finite.

We apply the substitution

$$t = \omega(x) = \sqrt[m]{\frac{\alpha x + \beta}{\gamma x + \delta}}, \quad \text{i.e.} \quad t^m = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

Therefore,

$$x = \varphi(t) = \frac{\delta t^m - \beta}{\alpha - \gamma t^m},$$

and we obtain that

$$I = \int R\left(x, \sqrt[m]{\frac{\alpha x + \beta}{\gamma x + \delta}}\right) dx = \int R(\varphi(t), t)\varphi'(t) dt.$$

Since $\varphi(t)$, $\varphi'(t)$ and $R(x, t)$ are rational functions, the integrant $R(\varphi(t), t)\varphi'(t)$ is also a rational function, therefore the computation of the integral I is reduced to the computation of a rational integral $\int R(\varphi(t), t)\varphi'(t) dt$. It is clear that the established method of integration of rational functions can be effectively applied as long as we are able to represent $R(\varphi(t), t)\varphi'(t)$ in a form $\frac{P(t)}{Q(t)}$, where $Q(t)$ can be factorized, i.e. we are able to compute its roots and irreducible quadratic factors.

Example 14.16. We will compute the following integrals

(a) $I = \int \frac{\sqrt{x+1}+2}{(x+1)^2 - \sqrt{x+1}} dx$. In this example the rational function $\frac{\alpha x + \beta}{\gamma x + \delta}$ is reduced to $x + 1$ and $m = 2$, therefore we apply the substitution $t = \sqrt{x+1}$, i.e. $t^2 = x + 1$, thus $dx = 2t dt$ and we obtain

$$\begin{aligned} I = \int \frac{\sqrt{x+1}+2}{(x+1)^2 - \sqrt{x+1}} dx &= 2 \int \frac{t+2}{t^3 - 1} dt \\ &= \int \left(\frac{2}{t-1} - \frac{2t+2}{t^2+t+1} \right) dt \\ &= \ln \frac{(t-1)^2}{t^2+t+1} - \frac{2}{\sqrt{3}} \arctan \frac{2t+1}{\sqrt{3}} + C \\ &= \ln \frac{(\sqrt{x+1}-1)^2}{x + \sqrt{x+1} + 2} - \frac{2}{\sqrt{3}} \arctan \frac{2\sqrt{x+1}+1}{\sqrt{3}} + C. \end{aligned}$$

(b) $I = \int \frac{dx}{\sqrt[3]{(x-1)(x+1)^2}} = \int \sqrt[3]{\frac{x+1}{x-1}} \cdot \frac{dx}{x+1}$. We apply the substitution $t = \sqrt[3]{\frac{x+1}{x-1}}$, i.e. $x = \frac{t^3+1}{t^3-1}$, so $dx = -\frac{6t^2 dt}{(t^3-1)^2}$. We obtain

$$\begin{aligned}
I = \int \sqrt[3]{\frac{x+1}{x-1}} \cdot \frac{dx}{x+1} &= \int \frac{-3dt}{t^3-1} \\
&= \int \left(-\frac{1}{t-1} + \frac{t+2}{t^2+t+1} \right) dt \\
&= \frac{1}{2} \ln \frac{t^2+t+1}{(t-1)^2} + \sqrt{3}\arctan \frac{2t-1}{\sqrt{3}} + C \\
&= \frac{1}{2} \ln \frac{\left(\sqrt[3]{\frac{x+1}{x-1}}\right)^2 + \sqrt[3]{\frac{x+1}{x-1}} + 1}{(\sqrt[3]{\frac{x+1}{x-1}} - 1)^2} + \sqrt{3}\arctan \frac{2\sqrt[3]{\frac{x+1}{x-1}} - 1}{\sqrt{3}} + C.
\end{aligned}$$

The next type of integral, which can sometimes be reduced to an integral of a rational function, is the integral

$$I = \int x^m (a + bx^n)^p dx,$$

where a and b are constants and the exponents m, n and p are rational numbers. Notice, that if p is an integer then the integral I is of the type that was considered above. Indeed, if k is the least common multiple of the denominators of the fractions m and n, then the expression $x^m(a+bx^n)^p$ can be transformed to $R(\sqrt[k]{x})$, where $R(t)$ is a rational function.

Assume therefore that p is not an integer and apply the substitution $z = x^n$. Then we have

$$\int x^m (a + bx^n)^p dx = \frac{1}{n} \int (a + bz)^p z^{\frac{m+1}{n} - 1} dz.$$

We denote $q = \frac{m+1}{n} - 1$, i.e. we have

$$I = \int x^m (a + bx^n)^p dx = \frac{1}{n} \int (a + bz)^p z^q dz.$$

If q is an integer, then again this integral is of the type studied above. Indeed, assume that $p = \frac{u}{v}$, so $\int (a + bz)^p z^q dz = \int \left(\sqrt[v]{a + bz}\right)^u z^q dz$. Assume therefore, that q is not an integer. Then we can write the integral I in the form

$$I = \frac{1}{n} \int (a + bz)^p z^q dz = \frac{1}{n} \int \left(\frac{a + bz}{z} \right)^p z^{p+q} dz.$$

If $p + q$ is an integer, then this integral is again an integral of the previous type. Consequently, the integral I can be transformed to an integral of a rational function if there is an integer among the numbers

$$p, \ q, \ p + q,$$

i.e. there is an integer among

$$p, \ \frac{m+1}{n}, \ \frac{m+1}{n} + p.$$

The above method of reduction to an integral of rational function was already known to I. Newton, but only 150 years ago Chebyshev proved that these are the only cases in which the integral of $x^m(a + bx^n)^p$ can be computed by means of elementary functions.

Example 14.17. We will compute the following three integrals:

(a) $I = \int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx = \int x^{-\frac{1}{2}}(1+x^{\frac{1}{4}})^{\frac{1}{3}}dx$. In this case we have $m = -\frac{1}{2}$, $n = \frac{1}{4}$ and $p = \frac{1}{3}$. Since

$$\frac{m+1}{n} = \frac{-\frac{1}{2}+1}{\frac{1}{4}} = 2,$$

we make the substitution $z = x^{\frac{1}{4}}$, i.e. $dx = 4z^3dz$, and transform the integral I as follows

$$\begin{aligned} I &= \int x^{-\frac{1}{2}}(1+x^{\frac{1}{4}})^{\frac{1}{3}}dx = \int \frac{\sqrt[3]{1+z}}{z^2}4z^3dz \\ &= 4\int \sqrt[3]{1+z}z dz. \end{aligned}$$

We make another substitution $t = \sqrt[3]{1+z}$, i.e. $z = t^3 - 1$ and $dz = 3t^2dt$, thus

$$I = 12\int t(t^3-1)t^2dt = 12\int (t^6 - t^3)dt = \frac{3}{7}t^4(4t^3-7) + C,$$

thus

$$\begin{aligned} I &= \frac{3}{7}\left(\sqrt[3]{1+z}\right)^4(4z+2) + C \\ &= \frac{3}{7}\left(\sqrt[3]{1+x^{\frac{1}{4}}}\right)^4(4x^{\frac{1}{4}}+2) + C \end{aligned}$$

(b) $I = \int \frac{dx}{\sqrt[4]{1+x^4}} = \int x^0(1+x^4)^{-\frac{1}{4}}dx$. This time we have $m = 0$, $n = 4$, $p = -\frac{1}{4}$. Since

$$\frac{m+1}{n} + p = \frac{1}{4} - \frac{1}{4} = 0,$$

we have again a case the integral I can be reduced to an integral of a rational function. First, we apply the substitution $z = x^4$, i.e. $x = z^{\frac{1}{4}}$, so $dx = \frac{1}{4}z^{-\frac{3}{4}}dz$ and we have

$$I = \frac{1}{4}\int (1+z)^{-\frac{1}{4}}z^{-\frac{3}{4}}dz = \frac{1}{4}\int \sqrt[4]{\frac{z}{1+z}}\cdot\frac{1}{z}dz.$$

We make the next substitution $t = \sqrt[4]{\frac{z}{1+z}}$, i.e. $z = \frac{t^4}{1-t^4}$ and $dz = \frac{4t^3(1-t^4)+4t^7}{(1-t^4)^2}$, hence

$$\begin{aligned} I &= \frac{1}{4}\int t\cdot\frac{1-t^4}{t^4}\cdot\frac{4t^3(1-t^4)+4t^7}{(1-t^4)^2}dt \\ &= -\int \frac{dt}{t^4-1} = -\frac{1}{2}\left(\int \frac{dt}{t^2+1} - \frac{1}{2}\int \frac{dt}{t+1} + \frac{1}{2}\int \frac{dt}{t-1}\right) \\ &= -\frac{1}{2}\arctan t - \frac{1}{4}\ln\frac{t+1}{t-1} + C \\ &= -\frac{1}{2}\arctan\sqrt[4]{\frac{z}{1+z}} - \frac{1}{4}\ln\frac{\sqrt[4]{\frac{z}{1+z}}+1}{\sqrt[4]{\frac{z}{1+z}}-1} + C \\ &= -\frac{1}{2}\arctan\sqrt[4]{\frac{x^4}{1+x^4}} - \frac{1}{4}\ln\frac{\sqrt[4]{\frac{x^4}{1+x^4}}+1}{\sqrt[4]{\frac{x^4}{1+x^4}}-1} + C \end{aligned}$$

(c) $I = \int \frac{dx}{x\sqrt[3]{1+x^5}} = \int x^{-1}(1+x^5)^{-\frac{1}{3}}dx$. In this case we have $m=-1$, $n=5$ and $p=-\frac{1}{3}$, thus $\frac{m+1}{n}=0$, so it is the second case considered earlier for the integral $\int x^m(a+bx^n)^p dx$. We apply the substitution $z=x^5$, i.e. $x=z^{\frac{1}{5}}$ and $dx=\frac{1}{5}z^{-\frac{4}{5}}dz$, so

$$I = \int z^{-\frac{1}{5}}(1+z)^{-\frac{1}{3}}\frac{1}{5}z^{-\frac{4}{5}}dz = \frac{1}{5}\int (1+z)^{-\frac{1}{3}}z^{-1}dz.$$

The last integral is of the type studied in the beginning of this section, so we can use the substitution $t=\sqrt[3]{1+z}$, i.e. $z=t^3-1$ and $dz=3t^2dt$. Consequently,

$$\begin{aligned} I &= \frac{3}{5}\int \frac{t^2dt}{t(t^3-1} = \frac{3}{5}\frac{tdt}{(t-1)(t^2+t+1)} \\ &= \frac{1}{5}\int \left(\frac{1}{t-1} - \frac{t-1}{t^2+t+1}\right)dt = \frac{1}{10}\ln\frac{(t-1)^2}{t^2+t+1} + \frac{\sqrt{3}}{5}\text{arctan}\,\frac{2t+1}{\sqrt{3}} + C \\ &= \frac{1}{10}\ln\frac{(\sqrt[3]{1+z}-1)^2}{\sqrt[3]{1+z}^2+t+1} + \frac{\sqrt{3}}{5}\text{arctan}\,\frac{2\sqrt[3]{1+z}+1}{\sqrt{3}} + C \\ &= \frac{1}{10}\ln\frac{(\sqrt[3]{1+x^5}-1)^2}{\sqrt[3]{1+x^5}^2+t+1} + \frac{\sqrt{3}}{5}\text{arctan}\,\frac{2\sqrt[3]{1+x^5}+1}{\sqrt{3}} + C \end{aligned}$$

The next type of integral is

$$\boxed{\int R\left(x, x^{\frac{p_1}{q_1}}, \ldots, x^{\frac{p_k}{q_k}}\right)dx,}$$

where R is a rational expression of $k+1$ variables and $p_1, \ldots, p_k, \ldots, q_k$, are natural numbers. Let m be the least common multiple of the numbers $q_1, \ldots, q_k$. This type of the integral can be 'rationalized' by applying the substitution $x=t^m$.

Example 14.18. We will evaluate the following integrals:

(a) $I = \int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx$. The least common multiple of 3 and 6 is $m=6$, therefore we make the substitution $x=t^6$, so $dx=6t^5dt$ and we obtain

$$\begin{aligned} I &= 6\int \frac{(t^6+t^4+t)t^5}{t^6(1+t^2)}dt = 6\int \frac{t^5+t^3+1}{1+t^2}dt \\ &= 6\int t^3dt + 6\int \frac{dt}{t^2+1}dt \\ &= \frac{3}{2}t^4 + 6\text{arctan}\,t + C \\ &= \frac{3}{2}x^{\frac{2}{3}} + 6\text{arctan}\,\sqrt[6]{x} + C. \end{aligned}$$

(b) $I = \int \frac{\sqrt{2x-3}}{\sqrt[3]{2x-3}+1}dx$. The least common multiple of 2 and 3 is 6, thus we put $m=6$ and we apply the substitution $t^6=2x-3$, i.e. $x=\frac{1}{2}(t^6+3)$, so $dx=3t^5dt$. We have

$$\begin{aligned}I &= \int \frac{3t^8}{t^2+1}dt = 3\int (t^6 - t^4 + t^2 - 1)dt + 3\int \frac{dt}{1+t^2}\\ &= \frac{3}{7}t^7 - \frac{3}{5}t^5 + t^3 - 3t + 3\arctan t + C\\ &= 3\left[\frac{1}{7}(2x-3)^{\frac{7}{6}} - \frac{1}{5}(2x-3)^{\frac{5}{6}} + \frac{1}{3}(2x-3)^{\frac{1}{2}} - (2x-3)^{\frac{1}{6}}\right.\\ &\left. + \arctan (2x-3)^{\frac{1}{6}}\right] + C.\end{aligned}$$

Integrals of the form

$$\boxed{\int R\left(x, \sqrt{ax^2+bx+c}\right) dx}$$

are calculated with the aid of one of the three Euler substitutions:

(1) $\sqrt{ax^2+bx+c} = t \pm x\sqrt{a}$ if $a > 0$;
(2) $\sqrt{ax^2+bx+c} = tx \pm \sqrt{c}$ if $c > 0$;
(3) $\sqrt{ax^2+bx+c} = (x-\alpha)t$ if $ax^2+bx+c = a(x-\alpha)(x-\beta)$, i.e. if α is a real root of the trinomial ax^2+bx+c.

Example 14.19. We will apply the Euler substitutions to compute the following integrals:

(a) $I = \int \frac{dx}{1+\sqrt{x^2+2x+2}}$. Here $a = 1 > 0$, therefore we make the substitution

$$\sqrt{x^2+2x+2} = t - x.$$

Thus

$$x = \frac{t^2-2}{2(1+t)}; \quad dx = \frac{t^2+2t+2}{2(1+t)^2}dt.$$

Since

$$1 + \sqrt{x^2+2x+2} = 1 + t - \frac{t^2-2}{2(1+t)} = \frac{t^2+4t+4}{2(1+t)},$$

we obtain

$$I = \int \frac{2(1+t)(t^2+2t+2)}{(t^2+4t+4)2(1+t)^2}dt = \int \frac{(t^2+2t+2)dt}{(1+t)(t+2)^2}.$$

Now, we find a partial fraction decomposition for

$$\frac{t^2+2t+2}{(1+t)(t+2)^2} = \frac{A_1}{t+1} + \frac{A_2}{t+2} + \frac{A_3}{(t+2)^2}.$$

We have the following equation

$$t^2+2t+2 = A_1(t+2)^2 + A_2(t+1)(t+2) + A_3(t+1),$$

so $A_1 = 1$, $A_2 = 0$ and $A_3 = -2$. Hence

$$\begin{aligned} I &= \int \frac{(t^2+2t+2)dt}{(1+t)(t+2)^2} = \int \frac{dt}{t+1} - 2\int \frac{dt}{(t+2)^2} \\ &= \ln|t+1| + \frac{2}{t+2} + C \\ &= \ln\left|x+1+\sqrt{x^2+2x+2}\right| + \frac{2}{x+2+\sqrt{x^2+2x+2}} + C. \end{aligned}$$

(b) $I = \int \frac{dx}{x+\sqrt{x^2-x+1}}$. Here $c = 1 > 0$, so we can apply the second Euler substitution

$$\sqrt{x^2-x+1} = tx - 1.$$

We have

$$(2t-1)x = (t^2-1)x^2; \quad x = \frac{2t-1}{t^2-1};$$

so

$$dx = -2\frac{t^2-t+1}{(t^2-1)^2}; \quad x + \sqrt{x^2-x+1} = \frac{t}{t-1},$$

and we obtain

$$I = \int \frac{dx}{x+\sqrt{x^2-x+1}} = \int \frac{-2t^2+2t-2}{t(t-1)(t+1)^2} dt.$$

We need the partial fraction decomposition of $\frac{-2t^2+2t-2}{t(t-1)(t+1)^2}$, i.e.

$$\frac{-2t^2+2t-2}{t(t-1)(t+1)^2} = \frac{A_1}{t} + \frac{A_2}{t-1} + \frac{A_3}{(t+1)^2} + \frac{A_4}{t+1}.$$

We deduce from the equation

$$-2t^2+2t-2 = A_1(t-1)(t+1)^2 + A_2 t(t+1)^2 + A_3 t(t-1) + A_4 t(t-1)(t+1),$$

that $A_1 = 2$, $A_2 = -\frac{1}{2}$, $A_3 = -3$ and $A_4 = -\frac{3}{2}$. Hence

$$\begin{aligned} I &= 2\int \frac{dt}{t} - \frac{1}{2}\int \frac{dt}{t-1} - 3\int \frac{dt}{(t+1)^2} - \frac{3}{2}\int \frac{dt}{t+1} \\ &= 2\ln|t| - \frac{1}{2}\ln|t-1| + \frac{3}{t+1} - \frac{3}{2}\ln|t+1| + C \\ &= 2\ln\left|\frac{\sqrt{x^2-x+1}+1}{x}\right| - \frac{1}{2}\ln\left|\frac{\sqrt{x^2-x+1}+1}{x} - 1\right| \\ &+ \frac{3}{\frac{\sqrt{x^2-x+1}+1}{x}+1} - \frac{3}{2}\ln\left|\frac{\sqrt{x^2-x+1}+1}{x}+1\right| + C. \end{aligned}$$

(c) $I = \int \frac{x\,dx}{(\sqrt{7x-10-x^2})^3}$. In this case $a < 0$ and $c < 0$, therefore neither the first nor the second Euler substitution can be applied. Since the quadratic equation $7x - 10 - x^2 = 0$ has real roots $\alpha = 2$ and $\beta = 5$, we can use the third Euler substitution:

$$\sqrt{7x-10-x^2} = \sqrt{(x-2)(5-x)} = (x-2)t.$$

Then

$$5 - x = (x-2)t^2, \quad x = \frac{5+2t^2}{1+t^2},$$

so

$$(x-2)t = \left(\frac{5+2t^2}{1+t^2} - 2\right)t = \frac{3t}{1+t^2}.$$

Hence

$$\begin{aligned}
I &= -\frac{6}{27}\int \frac{5+2t^2}{t^2}dt = -\frac{2}{9}\int\left(\frac{5}{t^2}+2\right) \\
&= -\frac{2}{9}\left(-\frac{5}{t}+2t\right)+C \\
&= -\frac{2}{9}\left(-\frac{5(x-2)}{\sqrt{7x-10-x^2}}+2\frac{\sqrt{7x-10-x^2}}{x-2}\right)+C
\end{aligned}$$

The Euler substitutions can sometimes lead to elaborate calculations, so it is useful to have other methods available for these type of integrals, or particular cases of them.

Integrals of the form

$$\boxed{I = \int \frac{Mx+N}{\sqrt{ax^2+bx+c}}dx}$$

are reduced by the substitution $x + \frac{b}{2a} = t$ to the form

$$I = M_1\int \frac{tdt}{\sqrt{at^2+K}} + N_1\int\frac{dt}{\sqrt{at^2+K}}$$

where M_1, N_1 and K are new coefficients.

Example 14.20. Consider the integral $I = \int \frac{x+3}{\sqrt{4x^2+4x-3}}dx$. We apply the substitution $2x+1=t$, i.e. $x = \frac{t-1}{2}$ and $dx = \frac{1}{2}dt$. hence

$$\begin{aligned}
I &= \frac{1}{4}\int\frac{(t+5)dt}{\sqrt{t^2-4}} = \frac{1}{4}\sqrt{t^2-4}+\frac{5}{4}\ln\left|t+\sqrt{t^2-4}\right|+C \\
&= \frac{1}{4}\sqrt{4x^2+4x-3}+\frac{5}{4}\ln\left|2x+1+\sqrt{4x^2+4x-3}\right|+C.
\end{aligned}$$

Integrals of the form

$$\boxed{\int \frac{P_m(x)}{\sqrt{ax^2+bx+c}}dx,}$$

where $P_m(x)$ is a polynomial of degree m, are calculated by the reduction formula

$$\int\frac{P_m(x)dx}{\sqrt{ax^2+bx+c}} = P_{m-1}(x)\sqrt{ax^2+bx+c}+K\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

where $P_{m-1}(x)$ is a polynomial of degree $m-1$ and K a constant.

Example 14.21. We will illustrate this method with an example. We consider the integral: $I = \int \frac{x^3-x-1}{\sqrt{x^2+2x+2}}dx$. Here $P_m(x) = x^3 - x - 1$, so $P_{m-1}(x)$ is an unknown polynomial of degree 2, i.e.

$$P_{m-1}(x) = ax^2 + bx + c.$$

We want to transform the integral I into the form

$$I = (ax^2 + bx + c)\sqrt{x^2 + 2x + 2} + K \int \frac{dx}{\sqrt{x^2 + 2x + 2}}.$$

By differentiating this equality we obtain

$$\begin{aligned} I' &= \frac{x^3 - x - 1}{\sqrt{x^2 + 2x + 2}} \\ &= (2ax + b)\sqrt{x^2 + 2x + 2} + (ax^2 + bx + c)\frac{x + 1}{\sqrt{x^2 + 2x + 2}} + \frac{K}{\sqrt{x^2 + 2x + 2}}. \end{aligned}$$

By transforming the right hand side of this equality into the form of a fraction over $\sqrt{x^3 + 2x + 2}$ and comparing its denominator with the denominator of the expression on the left hand side of this equality, we get

$$x^3 - x - 1 = (2ax + b)(x^2 + 2x + 2) + (ax^2 + bx + c)(x + 1) + K,$$

so

$$x^3 - x - 1 = 3ax^3 + (5a + 2b)x^2 + (4a + 3b + c)x + (2b + d + K).$$

By solving the system of linear equations

$$\begin{cases} 3a & = 1, \\ 5a + 2b & = 0, \\ 4a + 3b + c & = -1, \\ 2b + d + H & = -1. \end{cases}$$

we obtain $A = \frac{1}{3}$, $b = -\frac{5}{6}$, $c = \frac{1}{6}$ and $K = \frac{1}{2}$. Thus

$$I = \left(\frac{1}{3}x^2 - \frac{5}{6}x + \frac{1}{6}\right)\sqrt{x^2 + 2x + 2} + \frac{1}{2}\int \frac{dx}{\sqrt{x^2 + 2x + 2}},$$

where

$$I_1 = \int \frac{dx}{\sqrt{x^2 + 2x + 2}} = \int \frac{dx}{\sqrt{(x+1)^2 + 1}} = \ln(x + 1 + \sqrt{x^2 + 2x + 2}) + C.$$

The integral of the form

$$\boxed{I = \int \frac{dx}{(x - d)^m \sqrt{ax^2 + bx + c}},}$$

where m is a natural number, can be reduced to the preceding type by the substitution

$$x - d = \frac{1}{t}.$$

Example 14.22. To illustrate this method we consider the integral $I = \int \frac{dx}{(x+1)^5\sqrt{x^2+2x}}$. We apply the substitution $x+1 = \frac{1}{t}$, i.e. $x = \frac{1}{t} - 1$ and $dx = -\frac{1}{t^2}dt$, so

$$I = \int \frac{-\frac{1}{t^2}dt}{t^{-5}\sqrt{\frac{1}{t^2}-1}}dt = -\int \frac{t^4 dt}{\sqrt{1-t^2}}.$$

Since $P_m(t) = t^4$, we assume $P_{m-1}(t) = at^3 + bt^2 + ct + d$ and we determine these constants and the constant K by assuming

$$\int \frac{t^4 dt}{\sqrt{1-t^2}} = P_{m-1}(t)\sqrt{1-t^2} + K\int \frac{dt}{\sqrt{1-t^2}}.$$

By differentiating this inequality we obtain

$$\begin{aligned}\frac{-t^4 dt}{\sqrt{1-t^2}} &= (3at^2 + 2bt + c)\sqrt{1-t^2} \\ &\quad + (at^3 + bt^2 + ct + d)\frac{-t}{\sqrt{1-t^2}} + K\frac{1}{\sqrt{1-t^2}}.\end{aligned}$$

Thus

$$-t^4 = (3at^2 + 2bt + c)(1-t^2) - t(at^3 + bt^2 + ct + d) + K,$$

so

$$-t^4 = (-4a)t^4 + (-3b)t^3 + (2a-c)t^2 + (2b-d)t + (c+K),$$

what leads to the following system of linear equations

$$\begin{cases} -4a & = -1, \\ -3b & = 0, \\ 3a - 2c & = 0, \\ 2b - d & = 0, \\ c + K & = 0. \end{cases}$$

We have the solutions $a = \frac{1}{4}$, $b = 0$, $c = \frac{3}{8}$, $d = 0$ and $K = -\frac{3}{8}$, so

$$\begin{aligned} I &= (\frac{1}{4}t^3 + \frac{3}{8}t)\sqrt{1-t^2} - \frac{3}{8}\int \frac{dt}{\sqrt{1-t^2}} \\ &= (\frac{1}{4}t^3 - +\frac{3}{8}t)\sqrt{1-t^2} - \frac{3}{8}\arcsin t + C \\ &= \frac{\sqrt{x^2+2x}}{4(1+x)^3} + \frac{3\sqrt{x^2+2x}}{8(1+x)} - \frac{3}{8}\arcsin\frac{1}{1+x} + C. \end{aligned}$$

Trigonometric and hyperbolic substitutions are also very useful to integrate integrals of the type $I = \int R(x, \sqrt{ax^2+bx+c}))dx$, which will be discussed in the next section.

Problems 14.4 (a) Evaluate the integral $I = \int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}$;

(b) Evaluate the integral $I = \int \frac{\sqrt{x}+\sqrt[3]{x}}{\sqrt[4]{x^5}-\sqrt[6]{x^7}}dx$;

(c) Evaluate the integral $I = \int \frac{dx}{\sqrt{1-x^2-x}}$. Use Euler substitutions;

(d) Evaluate the integral $I = \int \frac{dx}{x-\sqrt{x^2+2x+4}}$. Use Euler substitutions;
(e) Evaluate the integral $I = \int \frac{5x+4}{\sqrt{x^2+2x+5}}dx$;
(f) Evaluate the integral $I = \int x^5(1+x^2)^{\frac{2}{3}}dx$.

Solution 14.23. Problem 14.4(a): Since

$$I = \int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}} = \int \frac{\sqrt[3]{\frac{x+1}{x-1}}}{(x^2-1)}dx,$$

we can use the substitution

$$t^3 = \frac{x+1}{x-1} \;\Rightarrow\; x = \frac{t^3+1}{t^3-1} \;\text{ and }\; dx = \frac{-6t^2}{(t^2-1)^2}dt.$$

Notice that

$$x^2-1 = \frac{(t^3+1)^2}{(t^3-1)^2} - 1 = \frac{t^6+2t^3+1-t^6+2t^3-1}{(t^3-1)^2} = \frac{4t^3}{(t^3-1)^2},$$

so

$$\begin{aligned} I &= \int \frac{t(t^3-1)^2(-6t^2)}{4t^3(t^3-1)^2}dt = -\frac{3}{2}\int dt = -\frac{3}{2}t + C \\ &= -\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}} + C \end{aligned}$$

Problem 14.4(b): We use the substitution $t^{12} = x$, i.e. $12t^{11}dt = dx$,

$$\begin{aligned} \int \frac{\sqrt{x}+\sqrt[3]{x}}{\sqrt[4]{x^5}-\sqrt[6]{x^7}}dx & \\ &= \int \frac{(t^6+t^4)12t^{11}dt}{t^{15}-t^{14}} = 12\int \frac{(t^2+1)t}{(t-1)}dt \\ &= 12\int \left(t^2+t+1+\frac{2}{t-1}\right)dt = 4t^3+6t^2+12t+2\ln|t-1| + C \\ &= 4\sqrt[4]{x} + 6\sqrt[6]{x} + 12\sqrt[12]{x} + 2\ln|\sqrt[12]{x}-1| + C \end{aligned}$$

Problem 14.4(c): We apply the Euler substitution

$$\sqrt{1-x^2-x} = tx+1 \quad \text{i.e} \quad 1-x^2-x = t^2x^2+2tx+1,$$

so

$$x = -\frac{2t+1}{t^2+1} \quad \text{and} \quad dx = \frac{2(t^2+t-1}{t^2+1)^2}dt.$$

Notice that

$$\sqrt{1-x^2-x} = tx+1 = \frac{-t(2t+1)}{t^2+1} + 1 = \frac{-(t^2+t-1)}{(t^2+1)},$$

thus

$$\begin{aligned} I =& \int \frac{(t^2+1)\cdot 2(t^2+t-1)}{-(t^2+t-1)\cdot(t^2+1)^2}dt = -2\int \frac{dt}{t^2+1} \\ =& -2\text{arctan}\,(t) + C = -2\text{arctan}\left(\frac{\sqrt{1-x^2}-x-1}{x}\right) + C \end{aligned}$$

Problem 14.4(d): We apply the Euler substitution $\sqrt{x^2+2x+4} = t + x$, so $x^2+2x+4 = t^2+2tx+x^2$, thus $x = \frac{4-t^2}{2(t-1)}$, $x - \sqrt{x^2+2x+4} = -t$ and

$$dx = \frac{-2t\cdot 2(t-1) - 2(4-t^2)}{4(t-1)^2}dt = -\frac{t^2-2t+4}{2(t-1)^2}dt.$$

Therefore

$$\begin{aligned} I =& \int \frac{t^2-2t+4}{2t(t-1)^2}dt = \int\left(\frac{-\frac{3}{2}}{t-1} + \frac{-\frac{9}{2}}{(t-1)^2} + \frac{2}{t}\right)dt \\ =& -\frac{3}{2}\ln|t-1| + \frac{9}{2}\frac{1}{t-1} + 2\ln|t| + C \\ =& -\frac{3}{2}\ln|\sqrt{x^2+2x+4}-x-1| + \frac{9}{2}\frac{1}{\sqrt{x^2+2x+4}-x-1} \\ &+2\ln|\sqrt{x^2+2x+4}-x| + C \end{aligned}$$

Problem 14.4(e): We apply the Euler substitution $\sqrt{x^2+2x+5} = t + x$, so $x = \frac{t^2-5}{2(1-t)}$ and $dx = -\frac{1}{2}\frac{t^2-2t+5}{(1-t)^2}dt$.

We also have

$$\begin{aligned} 5x+4 =& \frac{5}{2}\frac{(t^2-5)}{(1-t)} + \frac{8(1-t)}{2(1-t)} = \frac{5t^2-8t-17}{2(1-t)} \\ \sqrt{x^2+2x+5} =& t + \frac{t^2-5}{2(1-t)} = \frac{2t-2t^2+t^2-5}{2(1-t)} = \frac{t^2-2t+5}{2(1-t)}. \end{aligned}$$

Therefore,

$$\begin{aligned} I =& \int \frac{(5t^2-8t-17)2(1-t)(t^2-2t+5)}{2(1-t)(t^2-2t+5)\cdot 2(1-t)^2}dt = \frac{1}{2}\int\frac{5t^2-8t-17}{(1-t)^2}dt \\ =& \frac{1}{2}\int\left(5 + \frac{2t-22}{(t-1)^2}\right)dt = \int\left(\frac{5}{2} + \frac{1}{t-1} - \frac{10}{(t-1)^2}\right)dt \\ =& \frac{5}{2}t + \ln|t-1| + \frac{10}{t-1} + C \\ =& \frac{5}{2}(\sqrt{x^2+2x+5}-x) + \ln|\sqrt{x^2+2x+5}-x-1| + \frac{10}{\sqrt{x^2+2x+5}-x-1} + C \end{aligned}$$

Problem 14.4(f): Here we have $m = 5$, $n = 2$ and $p = \frac{2}{3}$. Notice that $\frac{m+1}{n} = 3$, therefore the integral I can be integrated. We apply the substitution $z = x^2$, i.e $x = z^{1/2}$ and $dx = \frac{1}{2}z^{-1/2}dz$,so

$$I = \frac{1}{2}\int z^{5/2}(1+z)^{3/2}z^{1/2}dz = \frac{1}{2}\int z^2(1+z)\sqrt{1+z}\,dz.$$

Next, we apply the substitution $t^2 = 1+z$, i.e $z = t^2-1$ and $dz = 2tdt$, so

$$\begin{aligned}I =&\frac{1}{2}\int (t^2-1)^2 2\cdot t^4 dt = \int (t^2-1)^2 t^4 dt\\ =&\int (t^8-2t^6+t^4)dt = \frac{1}{9}t^9 - \frac{2}{7}t^7 + \frac{1}{5}t^5 + C\\ =&\frac{1}{9}(1+x^2)^{\frac{9}{2}} - \frac{2}{7}(1+x^2)^{\frac{7}{2}} + \frac{1}{5}(1+x^2)^{\frac{5}{2}} + C\end{aligned}$$

14.5 Integration of Trigonometric and Hyperbolic Expressions

We begin this section with a discussion of integrals of the type

$$I = \int \sin^m x \cos^n x\, dx, \tag{14.11}$$

where m and n are rational numbers. We can apply the substitution $t = \sin x$, i.e. $x = \arcsin t$ and $dx = (1-t^2)^{-\frac{1}{2}} dt$, to obtain the integral

$$I = \int t^m (1-t^2)^{\frac{n-1}{2}} dt,$$

which is of the type discussed in the previous section.

In the case m, n or $m+n$ being integers, we can rationalize the integral (14.11) using the following substitutions:

(1) If n is odd (m does not need to be an integer), then $\frac{n-1}{2}$ is an integer. We apply the substitution $t = \sin t$. The expression $(1-t^2)^{\frac{n-1}{2}}$ is a rational function.
(2) If m is odd (N does not need to be an integer), then $\frac{m+1}{2}$ is an integer. We apply the substitution $t = \cos x$ and obtain the integral $\int t^n (1-t^2)^{\frac{m+1}{2}} dt$, where the expression $(1-t^2)^{\frac{m+1}{2}}$ is a rational function.
(3) If $m+n$ is even (m and n do not need to be integers), then $\frac{m+1}{2} + \frac{n-1}{2}$ is an integer. In this case we apply the substitution $\tan x = t$ or $\cot x = t$. In particular, this kind of substitution is convenient for integrals of the form

$$\tan^n x\, dx, \quad \text{or} \quad \int \cot^n x\, dx,$$

where n is a positive integer.

The substitution (3) is inconvenient if both m and n are positive integers (in particular, if there are both even). In this case it is more convenient to use the method of reducing the power with the aid of trigonometric transformations:

$$\cos^2 x = \frac{1}{2}(1+\cos 2x), \quad \sin^2 x = \frac{1}{2}(1-\cos 2x),$$

or

$$\sin x \cos x = \frac{1}{2} \sin 2x.$$

Example 14.24. We will compute the following trigonometric integrals:

(a) $I = \int \frac{\sin^3 x}{\sqrt[3]{\cos^2 x}} dx$. In this case $m = 3$ is an odd number, so we use the substitution $\cos x = t$, i.e. $\sin x dx = -dt$, which gives

$$\begin{aligned} I &= -\int (1-t^2) t^{-\frac{2}{3}} dt = -3t^{\frac{1}{3}} + \frac{3}{7} t^{\frac{7}{3}} + C \\ &= 3\sqrt[3]{\cos x} \left(\frac{1}{7} \cos^2 x - 1 \right) + C. \end{aligned}$$

(b) $I = \int \sin^4 x \cos^6 \, dx$. Here both m and n are positive even numbers. We will apply trigonometric transformations

$$\begin{aligned} I &= \frac{1}{16} \int (2 \sin x \cos x)^4 \cos^2 x dx = \frac{1}{32} \int \sin^4 2x (1 + \cos 2x) dx \\ &= \frac{1}{32} \int \sin^4 2x \, dx + \frac{1}{32} \int \sin^4 2x \, \cos 2x \, dx =: I_1 + I_2. \end{aligned}$$

The integral I_2 can be evaluated using the substitution

$$\sin 2x = t, \quad \cos 2x \, dx = \frac{1}{2} dt,$$

hence

$$I_2 = \frac{1}{64} \int t^4 dt = \frac{1}{320} t^5 + C = \frac{1}{320} \sin^4 2x + C.$$

To compute the integral I_1 we use trigonometric identities (half angle formulas) to reduce the power, i.e.

$$\begin{aligned} I_1 &= \frac{1}{32} \int \sin^4 2x dx = \frac{1}{128} \int (1 - \cos 4x)^2 dx \\ &= \frac{1}{128} \left(x - \frac{1}{2} \sin 4x \right) + \frac{1}{256} \int (1 + \cos 8x) dx \\ &= \frac{3}{256} x - \frac{1}{256} \sin 4x + \frac{1}{2048} \sin 8x + C. \end{aligned}$$

Finally, we obtain

$$I = \frac{3}{256} x - \frac{1}{256} \sin 4x + \frac{1}{2048} \sin 8x + \frac{1}{320} \sin^4 2x + C.$$

(c) $I = \int \frac{\sin^2 x}{\cos^6 x} dx$. Here both m and n are even integers, but one of them is negative. Therfore, we apply the substitution

$$\tan x = t; \ \frac{1}{\cos^2 x} = 1 + t^2; \ \frac{dx}{\cos^2 x} = dt.$$

Hence,

$$I = \int t^2 (1 + t^2) dt = \frac{1}{3} t^3 + \frac{1}{5} t^5 + C = \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C.$$

(d) $I = \int \frac{dx}{\sqrt[3]{\sin^{11} x \cos x}}$. Here both exponents $m = -\frac{11}{3}$ and $n = -\frac{1}{3}$ are negative and their sum $-\frac{11}{3} - \frac{1}{3} = -4$ is an even number, therfore we put

$$\tan x = t, \ \frac{dx}{\cos^2 x} = dt.$$

Hence,

$$\begin{aligned} I &= \int \frac{dx}{\cos^4 x \sqrt[3]{\tan^{11} x}} = \int \frac{1+t^2}{\sqrt[3]{t^{11}}} dt \\ &= \int \left(t^{-\frac{11}{3}} + t^{-\frac{5}{3}} \right) dt \\ &= -\frac{3}{8} t^{-\frac{8}{3}} - \frac{3}{2} t^{-\frac{2}{3}} + C \\ &= -\frac{3(1+4\tan^2 x)}{8\tan^2 x \sqrt[3]{\tan^2 x}} + C. \end{aligned}$$

(e) We use the substitution $t = \cos x$ to integrate $\int \tan x dx$. We have

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + C.$$

Similarly, we apply the substitution $t = \sin x$ to compute $\int \cot x \, dx$. We have

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \ln|\sin x| + C.$$

Now we will consider integrals of the type

$$\int R(\sin x, \cos x) dx, \tag{14.12}$$

where R is a rational function of $\sin x$ and $\cos x$.

The so called *universal substitution*

$$\tan\left(\frac{x}{2}\right) = t, \quad -\pi < x < \pi,$$

can always transform the integral (5.5.2) into a rational integral. In this case we have

$$\sin x = \frac{2t}{1+t^2}; \quad \cos x = \frac{1-t^2}{1+t^2},$$

and

$$x = 2\arctan t; \quad dx = \frac{2dt}{1+t^2}.$$

Sometimes, instead of the substitution $\tan \frac{x}{2} = t$, it is more advantageous to make the substitution $\cot \frac{x}{2} = t$ (for $0 < x < 2\pi$).

The universal substitution may lead to very cumbersome calculations and that's why it should be avoided if possible. However, in several special cases, it is possible to apply simpler substitutions. These cases are:

(1) If the equality

$$R(-\sin x, \cos x) = -R(\sin x, \cos x)$$

holds, apply the substitution $\cos x = t$;

(2) If the equality

$$R(\sin x, -\cos x) = -R(\sin x, \cos x)$$

holds, apply the substitution $\sin x = t$;

(3) If the equality

$$R(-\sin x, -\cos x) = R(\sin x, \cos x)$$

holds, apply the substitution $\tan x = t$ or $\cot x = t$. This is for example the case for integrals of type $\int R(\tan x)dx$;

Example 14.25. We will evaluate the following trigonometric integrals:

(a) $I = \int \frac{dx}{\sin x(2+\cos x-2\sin x)}$. We use the substitution $\tan\frac{x}{2} = t$; them we have

$$I = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}\left(2 + \frac{1-t^2}{1+t^2} - \frac{4t}{1+t^2}\right)} = \int \frac{(1+t^2)dt}{t(t^2-4t+3)}.$$

We need to find the partial fractions

$$\frac{1+t^2}{t(t-3)(t-1)} = \frac{a}{t} + \frac{b}{t-3} + \frac{c}{t-1}.$$

We have

$$1 + t^2 = a(t-3)(t-1) + bt(t-1) + ct(t-3), \tag{14.13}$$

and by substituting the values $t = 0$, $t = 3$ and $t = 1$ to the equation (14.13) we obtain $a = \frac{1}{3}$, $b = \frac{5}{3}$ and $c = -1$, therefore

$$\begin{aligned} I &= \frac{1}{3}\int \frac{dt}{t} + \frac{5}{3}\int \frac{dt}{t-3} - \int \frac{dt}{t-1} \\ &= \frac{1}{3}\ln|t| + \frac{5}{3}\ln|t-3| - \ln|t-1| + C \\ &= \frac{1}{3}\ln|\tan\frac{x}{2}| + \frac{5}{3}\ln|\tan\frac{x}{2} - 3| - \ln|\tan\frac{x}{2} - 1| + C. \end{aligned}$$

(b) $I = \int \frac{dx}{\sin x(2\cos^2 x-1)}$. Notice that

$$\frac{1}{-\sin x(2\cos^2 x - 1)} = -\frac{1}{\sin x(2\cos^2 x - 1)},$$

hence, we can take advantage of the substitution $t = \cos x$, i.e. $dt = -\sin x dx$. This leads to

$$I = -\int \frac{dt}{(1-t^2)(2t^2-1)}.$$

Since

$$\frac{1}{(1-t^2)(2t^2-1)} = \frac{(2-2t^2)-(1-2t^2)}{(1-t^2)(1-2t^2)} = \frac{2}{1-2t^2} - \frac{1}{1-t^2},$$

thus

$$\begin{aligned}
I &= -2\int \frac{dt}{1-2t^2} + \int \frac{dt}{1-t^2} \\
&= -\frac{1}{\sqrt{2}} \ln\left|\frac{1+t\sqrt{2}}{1-t\sqrt{2}}\right| + \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + C \\
&= -\frac{1}{\sqrt{2}} \ln\left|\frac{1+\sqrt{2}\cos x}{1-\sqrt{2}\cos x}\right| - \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C \\
&= \frac{1}{\sqrt{2}} \ln\left|\frac{1-\sqrt{2}\cos x}{1+\sqrt{2}\cos x}\right| + \ln\left|\cot\frac{x}{2}\right| + C.
\end{aligned}$$

(c) $I = \int \frac{\sin^2 x \cos x}{\sin x + \cos x} dx$. Notice that

$$\frac{(-\sin x)^2(-\cos x)}{(-\sin x)+(-\cos x)} = \frac{\sin^2 x \cos x}{\sin x + \cos x},$$

we can apply the substitution

$$t = \tan x, \quad dt = \frac{dx}{\cos^2 x}.$$

Hence,

$$I = \int \frac{\tan^2 x \cdot \cos^4 x \, dx}{(\tan x + 1)\cos^2 x} = \int \frac{t^2 dt}{(t+1)(t^2+1)^2}.$$

From the equation

$$\frac{t^2}{(t+1)(t^2+1)^2} = \frac{a}{t+1} + \frac{bt+c}{t^2+1} + \frac{dt+e}{(t^2+1)^2},$$

we find out that $a = \frac{1}{4}$, $b = -\frac{1}{4}$, $c = \frac{1}{4}$, $d = \frac{1}{2}$ and $e = -\frac{1}{2}$. Hence,

$$\begin{aligned}
I &= \frac{1}{4}\int \frac{dt}{t+1} - \frac{1}{4}\int \frac{t-1}{t^2+1}dt + \frac{1}{2}\int \frac{t-1}{(t^2+1)^2}dt \\
&= \frac{1}{4}\ln\frac{1+t}{\sqrt{1+t^2}} - \frac{1}{4}\cdot\frac{1+t}{1+t^2} + C \\
&= \frac{1}{4}\ln|\sin x + \cos x| - \frac{1}{4}\cos x(\sin x + \cos x) + C.
\end{aligned}$$

Integrals containing hyperbolic functions of the type

$$\boxed{\int R(\sinh x, \cosh x) dx,}$$

where R is a rational function of $\sinh x$ and $\cosh x$, are integrated in the same way as trigonometric functions. It is a good idea to remember some basic formulas:

$$\cosh^2 x - \sinh^2 x = 1; \quad \sinh^2 x = \frac{1}{2}(\cosh 2x - 1);$$

and

$$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1); \quad \sinh x \cosh x = \frac{1}{2}\sinh 2x.$$

We have also that if $\tanh\frac{x}{2} = t$, then $\sinh x = \frac{2t}{1-t^2}$ and $\cosh x = \frac{1+t^2}{1-t^2}$. In addition

$$x = 2\operatorname{arctanh} t = \ln\left(\frac{1+t}{1-t}\right), \quad (-1 < t < 1)$$

and

$$dx = \frac{2dt}{1-t^2}.$$

Example 14.26. We will evaluate the integral

$$I = \int \cosh^3 x dx.$$

Since the power of $\cosh x$ is odd, we apply the substitution $\sinh x = t$, so $\cosh x dx = dt$ and we have

$$\begin{aligned} I &= \int \cosh^2 x \cosh^2 x\, dx = \int (1+t^2) dt \\ &= t + \frac{1}{3}t^3 + C \\ &= \sinh x + \frac{1}{3}\sinh^2 x + C. \end{aligned}$$

We conclude this section with a few remarks on integrals of the type

$$\boxed{I = \int R(x, \sqrt{ax^2+bx+c}) dx}$$

that were studied in the previous section.

Integration of functions rationally depending on x and $\sqrt{ax^2+bx+c}$ can be reduced to finding integrals of one of the following types:

(1) $I_1 = \int R(t, \sqrt{p^2t^2+q^2}) dt$;
(2) $I_2 = \int R(t, \sqrt{p^2t^2-q^2}) dt$;
(3) $I_3 = \int R(t, \sqrt{q^2-p^2t^2}) dt$,

where $t = x + \frac{b}{2a}$ and $ax^2+bx+c = \pm p^2t^2 \pm q^2$ by completing to a full square.

The integrals I_1, I_2 and I_3 can be reduced to integrals of expressions rational with respect to sine or cosine functions by making the following substitutions:

(1) $t = \frac{q}{p}\tan\theta$ or $t = \frac{q}{p}\sinh\theta$;
(2) $t = \frac{q}{p}\sec\theta$ or $t = \frac{q}{p}\cosh\theta$;
(3) $t = \frac{q}{p}\sin\theta$ or $t = \frac{q}{p}\tanh\theta$.

Example 14.27. We will evaluate the following integrals

(a) $I=\int \frac{dx}{\sqrt{(5+2x+x^2)^3}}$. Since $5+2x+x^2=4+(x+1)^2$, we put $t=x+1$ and obtain

$$I=\int \frac{dx}{\sqrt{(4+t^2)^3}}.$$

We apply the trigonometric substitution $t=2\tan\theta$, so $dt=\frac{2d\theta}{\cos^2\theta}$ and $\sqrt{4+t^2}=2\sqrt{4+t^2}=2\sqrt{1+\tan^2\theta}=\frac{2}{\cos\theta}$. We get

$$\begin{aligned} I &= \frac{1}{4}\int \cos\theta\, d\theta = \frac{1}{4}\sin\theta + C \\ &= \frac{1}{4}\frac{\tan\theta}{\sqrt{1+\tan^2\theta}} + C \\ &= \frac{1}{4}\frac{\frac{t}{2}}{\sqrt{1+\frac{t^2}{4}}} + C \\ &= \frac{x+1}{4\sqrt{5+2x+x^2}} + C. \end{aligned}$$

(b) $I=\int \frac{dx}{(x+1)^2\sqrt{x^2+2x+2}}$. We have $x^2+2x+2=(x+1)^2+1$. We put $t=x+1$ to get

$$I=\int \frac{dt}{t^2\sqrt{t^2+1}}.$$

We make the substitution $t=\sinh\theta$. Then $dt=\cosh\theta d\theta$, $\sqrt{t^2+1}=\sqrt{1+\sinh^2\theta}=\cosh\theta$. Hence,

$$\begin{aligned} I &= \int \frac{\cosh\theta\, d\theta}{\sinh^2\theta\,\cosh\theta} = \int \frac{d\theta}{\sinh^2\theta} \\ &= -\coth\theta + C = -\frac{\sqrt{1+\sinh^2\theta}}{\sinh\theta} + C \\ &= -\frac{\sqrt{1+t^2}}{t} + C = -\frac{\sqrt{x^2+2x+2}}{x+1} + C. \end{aligned}$$

Problems 14.5 (a) Evaluate the integral $I=\int \frac{\cos^3 x}{\sin^6 x}dx$;
(b) Evaluate the integral $I=\int \frac{\sin^2 x}{\cos^6 x}dx$;
(c) Evaluate the integral $I=\int \frac{dx}{5+\sin x+3\cos x}$;
(d) Evaluate the integral $I=\int \frac{2\tan x+3}{\sin^2 x+2\cos^2 x}dx$.

Solution 14.28. Problem 14.5(a): We subtitute $t=\sin x$, i.e $dt=\cos x dx$, to obtain

$$\begin{aligned} I &= \int \frac{\cos^3 x}{\sin^6 x}dx = \int \frac{(1-\sin^2 x)\cos x dx}{\sin^6 x} \\ &= \int \frac{1-t^2}{t^6}dt = -\frac{1}{5}t^{-5} + \frac{1}{3}t^{-3} + C \\ &= -\frac{1}{5}\csc^5 x + \frac{1}{3}\csc^3 x + C \end{aligned}$$

Problem 14.5(b): We substitute $t=\tan x$, i.e. $dt=\frac{dx}{\cos^2 x}$.

$$\begin{aligned}
I &= \int \frac{\sin^2 x}{\cos^6 x} dx = \int \frac{\sin^2 x(\sin^2 x + \cos^2 x}{\cos^4 x} \frac{dx}{\cos^2 x} \\
&= \int t^2(t^2+1)dt = \frac{1}{5}t^5 + \frac{1}{3}t^3 + C = \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C
\end{aligned}$$

Problem 14.5(c): We substitute $t = \tan\frac{x}{2}$, i.e. $x = 2\arctan t$ and $dx = \frac{2}{t^2+1}dt$. Then we have

$$\begin{aligned}
I &= \int \frac{\sin^2\frac{x}{2} + \cos^2 x2}{5\sin^2\frac{x}{2} + 5\cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} + 3\cos^2\frac{x}{2} - 3\sin^2\frac{x}{2}} dx \\
&= \int \frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}}{2\sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} + 8\cos^2\frac{x}{2}} \\
&= \int \frac{(t^2+1)2dt}{(2t^2+2t+8)(t^2+1)} = \int \frac{dt}{t^2+t+4} = \int \frac{dt}{(t+\frac{1}{2})^2 + \frac{15}{4}} \\
&= \frac{2}{\sqrt{15}}\arctan\left(\frac{2t+1}{\sqrt{15}}\right) + C = \frac{2}{\sqrt{15}}\arctan\left(\frac{2\tan\frac{x}{2}+1}{\sqrt{15}}\right) + C
\end{aligned}$$

Problem 14.5(d): We apply the substitution $t = \tan x$ and $dt = \frac{dx}{\cos^2 x}$, to get

$$\begin{aligned}
I &= \int \frac{(2\tan x + 3)dx}{(\tan^2 x + 2)\cos^2 x} = \int \frac{(2t+3)dt}{t^2+2} = \\
&= \ln(t^2+2) + \frac{3}{\sqrt{2}}\arctan\left(\frac{t}{\sqrt{2}}\right) + C \\
&= \ln(\tan^2 x + 2) + \frac{3}{\sqrt{2}}\arctan\left(\frac{\tan x}{\sqrt{2}}\right) + C
\end{aligned}$$

Table 14.2. Methods of Integration

No.	Integral	Method of Integration
1.	$\int F[\varphi(x)]\varphi'(x)dx$	Substitution $\varphi(x)=t$
2.	$\int f(x)g'(x)dx$	Integration by Parts $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$ This method is applied when $f(x)$ is a polynomial and $g(x)$ one of the functions $e^{\alpha x}$,$\cos\alpha x$, $\sin\alpha x$, $\ln x$, $\arctan x$, $\arcsin x$, etc. and also to integrals of products of an exponential function and cosine or sine.
3.	$\int f(x)g^{(n)}(x)dx$	Use the generalized formula for integration by parts: $\int f(x)g^{(n)}(x)dx = f(x)g^{(n-1)}(x) - -f'(x)g^{(n-2)}(x) + f''(x)g^{(n-3)}(x) - \dots \dots + (-1)^{n-1}f^{(n-1)}(x)g(x) + +(-1)^n \int f^{(n)}(x)g(x)dx$
4.	$\int e^{\alpha x}p_n(x)dx$ where $p_n(x)$ - polynomial of degree n	Applying the generalized formula for for integration by parts, we get $\int e^{\alpha x}p_n(x)dx = = e^{\alpha x}\left[\frac{p_n(x)}{\alpha} - \frac{p_n'(x)}{\alpha^2} + \frac{p_n''(x)}{\alpha^3} - \dots + +(-1)^n \frac{p_n^{(n)}(x)}{\alpha^{(n-1}}\right] + C$
5.	$\int \frac{Mx+N}{x^2+px+q}dx$, $p^2-4q<0$	Substitution $x+\frac{p}{2}=t$
6.	$I_n = \int \frac{dx}{(x^2+1)^n}$	Use the Reduction Formula $I_n = \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}I_{n-1}$
7.	$\int \frac{P(x)}{Q(x)}dx$, where $\frac{P(x)}{Q(x)}$ is a proper rational!partial fractions function $Q(x) = (x-\alpha_1)^l(x-\alpha_2)^m\dots$ $\dots(x^2+px+q)^k\dots$	Express the integrant as a sum of $\frac{P(x)}{Q(x)} = \frac{A_1}{(x-\alpha_1)} + \frac{A_2}{(x-\alpha_1)^2} + \dots + \frac{A_l}{(x-\alpha_1)^l} + \frac{B_1}{(x-\alpha lpha_2)} + \frac{B_2}{(x-\alpha_2)^2} + \dots + \frac{B_m}{(x-\alpha_2)^m} + \dots + \frac{M_1x+N_1}{x^2+px+q} + \frac{M_2x+N_2}{(x^2+px+q)^2} + \dots + \frac{M_kx+N_k}{(x^2+px+q)^k} + \dots$

14.6 Supplementary Problems

1. Compute the following integrals using the basic integration formulas

(a) $\int(3-x^2)^3dx$;

(b) $\int x^2(5-x)^4dx$;

(c) $\int(1-x)(1-2x)(1-3x)dx$;

(d) $\int\left(\frac{1-x}{x}\right)^2dx$;

(e) $\int\left(\frac{a}{x}+\frac{a^2}{x^2}+\frac{a^3}{x^3}\right)dx$;

(f) $\int\frac{x+1}{\sqrt{x}}dx$;

(g) $\int\frac{\sqrt{x}-2\sqrt[3]{x^2}+1}{\sqrt[4]{x}}dx$;

(h) $\int\frac{(1-x)^3}{x\sqrt[3]{x}}dx$;

(i) $\int\left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}dx$;

(j) $\int\frac{(\sqrt{2x}-\sqrt[3]{3x})^2}{x}dx$;

(k) $\int\frac{\sqrt{x^4+x^{-4}+2}}{x^3}dx$;

(l) $\int\frac{x^2dx}{1+x^2}$;

Table 14.3. Methods of Integration

No.	Integral	Method of Integration
8.	$\int \frac{P(x)}{Q(x)}dx$, $\frac{P(x)}{Q(x)}$ - proper irreducible rational function $Q(x) = (x-\alpha)^k \dots (x^2+px+q)^m \dots$	Find $P_1(x)$ and $P_2(x)$ such that $\int \frac{P(x)}{Q(x)}dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)}dx$ where $Q_1(x) = (x-\alpha)^{k-1} \dots (x^2+px+q)^{m-1} \dots$ $Q_2(x) = (x-\alpha) \dots (x^2+px+q) \dots$ Find coefficients of $P_1(x)$ and $P_2(x)$ from $P_1'Q_2 - P_1H + P_2Q_1 = P$ where $H = \frac{Q_1'Q_2}{Q_1}$
9.	$\int \frac{Mx+N}{x^2+px+q}dx$	Use the substitution $t = x + \frac{p}{2}$ and reduce to a sum of two integrals $\int \frac{Mx+N}{x^2+px+q}dx = M_1 \int \frac{tdt}{\sqrt{t^2+a^2}} + N_1 \int \frac{dt}{t^2+a^2}$ where $a = \sqrt{q - \frac{p^2}{4}}$, $M_1 = \frac{M}{2}$, $N_1 = N - \frac{Mp}{2}$
10.	$\int R\left(x, x^{\frac{m}{n}}, \dots, x^{\frac{r}{s}}\right) dx$ where R is a rational function of its arguments	Use the substitution $x = t^k$, where k is a common denominator of the fractions $\frac{m}{n}, \dots, \frac{r}{s}$
11.	$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx$ where R is a rational function of its arguments	Use the substitution $\frac{ax+b}{cx+d} = t^n$
12.	$\int R(x, \sqrt{ax^2+bx+c})dx$ where R is a rational function of x and $\sqrt{ax^2+bx+c}$	Rationalize it with the Euler substitutions $\sqrt{ax^2+bx+c} = t \pm x\sqrt{a}$ if $a > 0$ $\sqrt{ax^2+bx+c} = tx \pm \sqrt{c}$ if $c > 0$ $\sqrt{ax^2+bx+c} = t(x-\alpha)$ if $b^2-4ac > 0$ where α is the root of $ax^2+bx+c=0$
13.	$\int R(x, \sqrt{ax^2+bx+c})dx$ where R is a rational function of x and $\sqrt{ax^2+bx+c}$	Or use the trigonometric substitutions For $a<0$ and $4ac-b^2<0$: $x + \frac{b}{2a} = \begin{cases} \frac{\sqrt{b^2-4ac}}{2a}\sin t \\ \frac{\sqrt{b^2-4ac}}{2a}\cos t. \end{cases}$ For $a>0$ and $4ac-b^2<0$: $x + \frac{b}{2a} = \begin{cases} \frac{\sqrt{b^2-4ac}}{2a}\sec t \\ \frac{\sqrt{b^2-4ac}}{2a}\csc t. \end{cases}$ For $a>0$ and $4ac-b^2>0$ $x + \frac{b}{2a} = \begin{cases} \frac{\sqrt{4ac-b^2}}{2a}\tan t \\ \frac{\sqrt{4ac-b^2}}{2a}\cot t. \end{cases}$

Table 14.4. Methods of Integration

No.	Integral	Method of Integration
14.	$\int \frac{P_m(x)dx}{\sqrt{ax^2+bx+c}}$, where $P_m(x)$ ia a polynomial of degree m.	Write the equality: $\int \frac{P_m(x)dx}{\sqrt{ax^2+bx+c}} = P_{m-1}(x)\sqrt{ax^2+bx+c}+$ $+K\int \frac{dx}{\sqrt{ax^2+bx+c}}$ where $Q_{m-1}(x)$ is a polynomial of degree $m-1$. Differentiate the above equality to get the equation: $P_m(x) = Q'_{m-1}(x)(ax^2+bx+c)+$ $+\frac{1}{2}Q_{m-1}(x)(2ax+b)+K$, Solve $m+1$ linear equations to get the coefficients of $Q_{m-1}(x)$ and K
15.	$\int x^m(a+bx^n)^p dx$ where m, n and p are rational numbers	This integral is expressed through elementary functions $\Longleftrightarrow$ **(i) if p is a positive integer,** apply binomial formula to the expression $(a+bx^n)^p$ and evaluate the integral directly. **(ii) if p is a negative integer** substitute $x=t^k$, where k is the common denominator of the fractions m and n, and get a rational integral **(iii) if $\frac{m+1}{n}$ is an integer,** Substitute $a+bx^n=t^k$, where k is the denominator of the fraction p; **(iv) if $\frac{m+1}{n}+p$ is an integer.** If Substitute $a+bx^n=x^nt^k$, where k is the denominator of p.
16.	$\int \sin^m x\cos^n x\,dx$, where m and n are rational numbers	**(i) If n is odd**- substitute $t=\sin t$: $\int t^n(1-t^2)^{\frac{m+1}{2}}dt$, with $\frac{n-1}{2}$ an integer. **(ii) If m is odd** - substitute $t=\cos x$: $\int t^n(1-t^2)^{\frac{m+1}{2}}dt$, with $\frac{m+1}{2}$ integer. **(iii) If $m+n$ is even** - substitute $\tan x=t$ or $\cot x=t$ **(iv) If m and n are positive even integers** use $cos^2x=\frac{1}{2}(1+\cos 2x$, $\sin^2 x=\frac{1}{2}(1-\cos 2x)$ and $\sin x\cos x=\frac{1}{2}\sin 2x$

(m) $\int \frac{x^2dx}{1-x^2}$;
(n) $\int \frac{x^2+3}{x^2-1}dx$;
(o) $\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}dx$;
(p) $\int \frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^4-1}}dx$;
(q) $\int (2^x+3^x)^2dx$;
(r) $\int \frac{2^{x+1}-5^{x-1}}{10^x}dx$;
(s) $\int \frac{e^{3x}+1}{e^x+1}dx$;
(t) $\int (1+\sin x+\cos x)dx$;
(u) $\int \sqrt{1-\sin 2x}dx$;
(v) $\int \cot^2 xdx$;
(w) $\int (a\sinh x+b\cosh x)dx$;
(x) $\int \tanh^2 xdx$;
(y) $\int \coth^2 xdx$;

2. Compute the following integrals

(a) $\int \frac{dx}{x+a}$;
(b) $\int \sqrt[3]{1-3x}dx$;
(c) $\int \frac{dx}{(5x-2)^{\frac{5}{2}}}$;

Table 14.5. Methods of Integration

No.	Integral	Method of Integration
17.	$\int R(\sin x, \cos x)dx$ where R is a rational function	Universal substitution $\tan\frac{x}{2}=t$. **(i)** If $R(-\sin x,\cos x)=-R(\sin x\cos x)$ - substitute $\cos x=t$ **(ii)** If $R(\sin x,-\cos x)=-R(\sin x,\cos x)$ - substitute $\sin x=t$ **(iii)** If $R(-\sin x,-\cos x)=R(\sin x,\cos x)$ - substitute $\tan x=t$ or $\cot x=t$
18.	$\int R(\sinh x, \cosh x)dx!$ where R is a rational function	Universal substitution $\tanh\frac{x}{2}=t$. **(i)** If $R(-\sinh x,\cosh x)=-R(\sinh x\cosh x)$ - substitute $\cosh x=t$ **(ii)** If $R(\sinh x,-\cosh x)=-R(\sinh x,\cosh x)$ - substitute $\sinh x=t$ **(iii)** If $R(-\sinh x,-\cosh x)=R(\sinh x,\cosh x)$ - substitute $\tanh x=t$ or $\coth x=t$
19.	$\int \sin^p x\,\cos^q x\,dx$, where $0<x<\frac{\pi}{2}$ p and q rational numbers	Reduce to the integral of binomial differential differential (type 15) by the substitution $\sin x=t$ $\int \sin^p x\,\cos^q x\,dx=\int t^p(1-t^2)^{q-1}dt$
20.	$\int \sin ax\,\sin bx\,dx$ $\int \sin ax\,\cos bx\,dx$ $\int \cos ax\,\cos bx\,dx$	Use trigonometric identities: $\sin ax\sin bx=\frac{1}{2}[\cos(a-b)x-\cos(a+b)x]$ $\cos ax\cos bx=\frac{1}{2}[\cos(a-b)x+\cos(a+b)x]$ $\sin ax\cos bx=\frac{1}{2}[\sin(a-b)x+\sin(a+b)x]$
21.	$\int R(e^{ax})dx$, where R is a rational function.	Transform into an integral of a rational function by the substitution $e^{ax}=t$

(d) $\int \frac{dx}{2+3x^2}$;
(e) $\int \frac{dx}{\sqrt{2-3x^2}}$;
(f) $\int (e^{-x}+e^{-2x})dx$;
(g) $\int \frac{dx}{\sin^2(2x+\frac{\pi}{4})}$;
(h) $\int \frac{dx}{1-cosx}$;
(i) $\int [\sinh(2x+1)+\cosh(2x-1)]dx$;
(j) $\int (2x-3)^{10}dx$;
(k) $\int \frac{dx}{\sqrt{2-5x}}$;
(l) $\int \frac{\sqrt[5]{1-2x+x^2}}{1-x}dx$;
(m) $\int \frac{dx}{2-3x^2}$;
(n) $\int \frac{dx}{\sqrt{3x^2-2}}$;
(o) $\int (\sin 5x-\sin 5a)dx$;
(p) $\int \frac{dx}{1+\cos x}$;
(q) $\int \frac{dx}{1+\sin x}$;
(r) $\int \frac{dx}{\cosh^2(x/2)}$;
(s) $\int \frac{dx}{\sinh^2(x/2)}$;

3. Compute the following integrals

(a) $\int \frac{xdx}{\sqrt{1-x^2}}$;
(b) $\int \frac{xdx}{3-2x^2}$;
(c) $\int \frac{xdx}{4+x^4}$;
(d) $\int \frac{dx}{(1+x)\sqrt{x}}$;
(e) $\int \frac{dx}{x\sqrt{x^2+1}}$;
(f) $\int \frac{dx}{(x^2+1)^{3/2}}$;
(g) $\int \frac{x^2dx}{(8x^2+27)^{2/3}}$;
(h) $\int \frac{dx}{\sqrt{x(1-x)}}$;
(i) $\int \frac{e^x dx}{2+e^x}$;
(j) $\int \frac{dx}{\sqrt{1+e^{2x}}}$;
(k) $\int \frac{dx}{x\ln x\ln(\ln x)}$;
(l) $\int \frac{\sin x}{\sqrt{\cos^3 x}}dx$;
(m) $\int x^2\sqrt[3]{1+x^3}dx$;
(n) $\int \frac{xdx}{(1+x^2)^2}$;

(o) $\int \frac{x^3\,dx}{x^8-2}$;
(p) $\int \sin\frac{1}{x}\frac{dx}{x^2}$;
(q) $\int \frac{dx}{x\sqrt{x^2-1}}$;
(r) $\int \frac{dx}{\sqrt{x(1+x)}}$;
(s) $\int xe^{-x^2}dx$;
(t) $\int \frac{dx}{e^x+e^{-x}}$;
(u) $\int \frac{\ln^2}{x}dx$;
(v) $\int \sin^5 x\cos x dx$;
(w) $\int \tan x dx$;
(x) $\int \cot x dx$;
(y) $\int \frac{\sin x\cos x dx}{\sqrt{a^2\sin^2 x+b^2\cos^2 x}}$;
(z) $\int \frac{\cos x}{\sqrt{\cos 2x}}dx$;

4. Compute the following integrals

(a) $\int \frac{dx}{\sin^2 x\sqrt[4]{\cot x}}$;
(b) $\int \frac{dx}{\sin x}$;
(c) $\int \frac{dx}{\sinh x}$;
(d) $\int \frac{\sinh x\cosh x}{\sqrt{\sinh^4 x+\cosh^4 x}}$;
(e) $\int \frac{\arctan x}{1+x^2}dx$;
(f) $\int \sqrt{\frac{\ln(x+\sqrt{1+x^2})}{1+x^2}}dx$;
(g) $\int \frac{x^2+1}{x^4+1}dx$;
(h) $\int \frac{x^2-1}{x^4+1}dx$;
(i) $\int \frac{x^{n/2}dx}{\sqrt{1+x^{n+2}}}$;
(j) $\int \frac{\cos x dx}{\sqrt{2+\cos 2x}}$;
(k) $\int \frac{\sin x+\cos x}{\sqrt[3]{\sin x-\cos x}}$;
(l) $\int \frac{\sin x}{\sqrt{\cos 2x}}dx$;
(m) $\int \frac{\sinh x}{\sqrt{\cosh 2x}}dx$;
(n) $\int \frac{dx}{\sin^2 x+2\cos^2 x}$;
(o) $\int \frac{dx}{\cos x}$;
(p) $\int \frac{dx}{\cosh x}$;
(q) $\int \frac{dx}{\cosh^2 x\sqrt[3]{\tanh^2 x}}$;
(r) $\int \frac{dx}{(\arcsin x)^2\sqrt{1-x^2}}$;
(s) $\int \frac{x^4 dx}{(x^5+1)^4}$;
(t) $\int \frac{1}{1-x^2}\ln\frac{1+x}{1-x}dx$;
(u) $\int \frac{\sin x\cos x dx}{\sin^4 x+\cos^4 x}$;
(v) $\int \frac{2^x\cdot 3^x}{9^x-4^x}dx$;
(w) $\int \frac{x dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}$;

5. Apply appropriate transformations to compute the integrals

(a) $\int x^2(2-3x^2)^2dx$;
(b) $\int x(1-x)^{10}dx$;
(c) $\int \frac{1+x}{1-x}$;
(d) $\int \frac{x^2}{1+x}dx$;
(e) $\int \frac{x^3}{3+x}dx$;
(f) $\int \frac{(1+x)^2}{1+x^2}dx$;
(g) $\int \frac{(2-x)^2}{2-x^2}dx$;
(h) $\int \frac{x^2 dx}{(1-x)^{100}}$;
(i) $\int \frac{x^5 dx}{x+1}$;
(j) $\int \frac{dx}{\sqrt{x+1}+\sqrt{x-1}}$;
(k) $\int x\sqrt{2-5x}dx$;
(l) $\int \frac{dx}{(x-1)(x+3)}$;
(m) $\int \frac{dx}{x^2+x-2}$;
(n) $\int \frac{dx}{(x^2+1)(x^2+2)}$;
(o) $\int \frac{dx}{(x^2-2)(x^2+3)}$;
(p) $\int \frac{dx}{(x+2)(x+3)}$;
(q) $\int \frac{dx}{x^4+3x^2+2}$;
(r) $\int \frac{dx}{(x+a)^2(x+b)^2}$, where $a\neq b$;
(s) $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$, where $a^2\neq b^2$.

6. Apply appropriate transformations to compute the integrals

(a) $\int \sin^2 x dx$;
(b) $\int \cos^2 x dx$;
(c) $\int \sin x\sin(x+\alpha)dx$;
(d) $\int \sin 3x\sin 5x dx$;
(e) $\int \cos\frac{x}{2}\cos\frac{x}{3}dx$;
(f) $\int \sin\left(2x-\frac{\pi}{6}\right)\cos\left(3x+\frac{\pi}{4}\right)dx$;
(g) $\int \sin^3 x dx$;
(h) $\int \cos^3 x dx$;
(i) $\int \sin^4 x dx$;
(j) $\int \cos^4 x dx$;
(k) $\int \cot^2 x dx$;
(l) $\int \tan^2 x dx$;
(m) $\int \sin^2 3x\sin^3 2x dx$;
(n) $\int \frac{dx}{\sin^2 x\cos^2 x}$;
(o) $\int \frac{dx}{\sin^2 x\cos x}$;
(p) $\int \frac{dx}{\sin x\cos^3 x}$;
(q) $\int \frac{\cos^3 x}{\sin x}dx$;
(r) $\int \frac{dx}{\cos^4 x}$;
(s) $\int \frac{dx}{1+e^x}$;

(t) $\int \frac{(1+e^x)^2}{1+e^{2x}} dx$;
(u) $\int \sinh^2 x dx$;
(v) $\int \cosh^2 x dx$;
(w) $\int \sinh x \sinh 2x dx$;
(x) $\int \cosh x \cosh 3x dx$;
(y) $\int \frac{dx}{\sinh^2 x \cosh^2 x}$.

7. Use the substitution rule to compute the following integrals

(a) $\int x^2 \sqrt[3]{1-x} dx$;
(b) $\int x^3(1-5x^2)^{10} dx$;
(c) $\int \frac{x^2}{\sqrt{2-x}}$;
(d) $\int \frac{x^5}{\sqrt{1-x^2}} dx$;
(e) $\int x^5(2-5x^3)^{2/3} dx$;
(f) $\int \cos^5 x \sqrt{\sin x} dx$;
(g) $\int \frac{\sin x \cos^3 x}{1+\cos^2 x} dx$;
(h) $\int \frac{\sin^2 x}{\cos^6 x} dx$;
(i) $\int \frac{dx}{x\sqrt{1+\ln x}}$;
(j) $\int \frac{dx}{e^{x/2}+e^x}$;
(k) $\int \frac{dx}{\sqrt{1+e^x}}$;
(l) $\int \frac{\arctan \sqrt{x}}{\sqrt{x}} \cdot \frac{dx}{1+x}$.

8. Use trigonometric substitutions to find the following integrals

(a) $\int \frac{dx}{(1-x^2)^{3/2}}$;
(b) $\int \frac{x^2 dx}{\sqrt{x^2-2}}$;
(c) $\int \sqrt{a^2-x^2} dx$;
(d) $\int \frac{dx}{(x^2+a^2)^{3/2}}$;
(e) $\int \sqrt{\frac{a+x}{a-x}} dx$;
(f) $\int x\sqrt{\frac{x}{2a-x}} dx$;
(g) $\int \frac{dx}{\sqrt{(x-a)(b-x)}}$ (Hint: $x-a=(b-a)\sin^2 t$);
(h) $\int \sqrt{(x-a)(x-b)} dx$.

9. Use hyperbolic substitutions to find the following integrals

(a) $\int \sqrt{a^2+x62} dx$;
(b) $\int \frac{x^2}{\sqrt{a^2+x^2}} dx$;
(c) $\int \sqrt{\frac{x-a}{x+a}} dx$;
(d) $\int \frac{dx}{\sqrt{(x+a)(x+b)}}$, Hint: $x+a=(b-a)\sinh^2 t$);
(e) $\int \sqrt{(x+a)(x+b)} dx$.

10. Apply the integration by parts rule to compute the following integrals

(a) $\int xe^{-x} dx$;
(b) $\int x^2 e^{-2x} dx$;
(c) $\int \ln x dx$;
(d) $\int x^n \ln x dx$, where $n \neq 1$;
(e) $\int \left(\frac{\ln x}{x}\right)^2 dx$;
(f) $\int \sqrt{x} \ln^2 x dx$;
(g) $\int x \cos x dx$;
(h) $\int x^2 \sin 2x dx$;
(i) $\int x \sinh x dx$;
(j) $\int x^3 \cosh 3x dx$;
(k) $\int \arctan x dx$;
(l) $\int \arcsin x dx$;
(m) $\int x\arctan x dx$;
(n) $\int x^2 \arccos x dx$;
(o) $\int \frac{\arcsin x}{x^2} dx$;
(p) $\int \ln\left(x+\sqrt{1+x^2}\right) dx$;
(q) $\int x \ln \frac{1+x}{1-x} dx$;
(r) $\int \arctan \sqrt{x} dx$;
(s) $\int \sin x \ln(\tan x) dx$;

11. Compute the following integrals

(a) $\int x^5 e^{x^3} dx$;
(b) $\int \arcsin^2 x \, dx$;
(c) $\int x\arctan^2 x \, dx$;
(d) $\int x^2 \ln \frac{1-x}{1+x} dx$;
(e) $\int \frac{x \ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}} dx$;
(f) $\int \frac{x^2}{(1+x^2)^2} dx$;
(g) $\int \frac{x^2}{(a^2+x^2)^2} dx$;
(h) $\int \sqrt{a^2-x^2} dx$;
(i) $\int \sqrt{x^2+a^2} dx$;
(j) $\int x^2 \sqrt{a^{2+x^2}} dx$;
(k) $\int x \sin^2 x \, dx$;
(l) $\int e^{\sqrt{x}} dx$;
(m) $\int x \sin \sqrt{x} \, dx$;
(n) $\int \frac{xe^{\arctan x}}{(1+x^2)^{3/2}} dx$;
(o) $\int \frac{e^{\arctan x}}{(1+x^2)^{\frac{3}{2}}} dx$;
(p) $\int \sin(\ln x) dx$;

(q) $\int \cos(\ln x)dx$;
(r) $\int e^{ax}\cos bx\,dx$;
(s) $\int e^{ax}\sin bx\,dx$;
(t) $\int e^{2x}\sin^2 x\,dx$;
(u) $\int (e^x-\cos x)^2dx$;
(v) $\int \frac{\operatorname{arccot} e^x}{e^x}dx$;
(x) $\int \frac{\ln(\sin x)}{\sin^2 x}dx$;
(y) $\int \frac{x\,dx}{\cos^2 x}$;
(z) $\int \frac{xe^x\,dx}{(x+1)^2}$.

12. Derive the following integration formulas

(a) $\int \frac{dx}{a^2+x^2}=\frac{1}{a}\arctan\frac{x}{a}+C$, where $a\neq 0$;
(b) $\int \frac{dx}{a^2-x^2}=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C$, where $a\neq 0$;
(c) $\int \frac{x\,dx}{a^2\pm x^2}=\pm\frac{1}{2}\ln\left|a^2\pm x^2\right|+C$;
(d) $\int \frac{dx}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}+C$, where $a>0$;
(e) $\int \frac{dx}{\sqrt{x^2\pm a^2}}=\ln\left|x+\sqrt{x^2\pm a^2}\right|+C$, where $a>0$;
(f) $\int \frac{x\,dx}{\sqrt{a^2\pm x^2}}=\pm\sqrt{a^2\pm x^2}+C$;
(g) $\int \sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C$, where $a>0$;
(h) $\int \sqrt{x^2\pm a^2}dx=\frac{x}{2}\sqrt{x^2\pm a^2}\pm\frac{a^2}{2}\ln\left|x+\sqrt{x^2\pm a^2}\right|+C$.

13. Compute the following integrals

(a) $\int \frac{dx}{a+bx^2}$, where $ab\neq 0$;
(b) $\int \frac{dx}{x^2-x+2}$;
(c) $\int \frac{dx}{3x^2-2x-1}$;
(d) $\int \frac{x\,dx}{x^4-2x^2-1}$;
(e) $\int \frac{x+1}{x^2+x+1}dx$;
(f) $\int \frac{x\,dx}{x^2-2x\cos\alpha+1}$;
(g) $\int \frac{x^3\,dx}{x^4-x^2-2}$;
(h) $\int \frac{x^5\,dx}{x^6-x^3-2}$;
(i) $\int \frac{dx}{3\sin^2 x-8\sin x\cos x+5\cos^2 x}$;
(j) $\int \frac{dx}{\sqrt{1-2x-x^2}}$;
(k) $\int \frac{dx}{\sin x+2\cos x+3}$;
(l) $\int \frac{dx}{\sqrt{a+bx^2}}$, where $b\neq 0$;
(m) $\int \sqrt{x+x^2}$;
(n) $\int \frac{dx}{\sqrt{2x^2-x-2}}$;
(o) $\int \frac{x\,dx}{\sqrt{5+x-x^2}}$;
(p) $\int \frac{x-1}{\sqrt{x^2+x+1}}dx$;
(q) $\int \frac{x\,dx}{\sqrt{1-3x^2-2x^4}}$;
(r) $\int \frac{\cos x\,dx}{\sqrt{1+\sin x+\cos^2 x}}$;
(s) $\int \frac{x^3\,dx}{\sqrt{x^4-2x^2-1}}$;
(t) $\int \frac{x+x^3}{\sqrt{1+x^2-x^4}}dx$;
(u) $\int \frac{dx}{x\sqrt{x^2+x+1}}$;
(v) $\int \frac{dx}{x^2\sqrt{x^2+x-1}}$;
(w) $\int \frac{dx}{(x+1)\sqrt{x^2+1}}$;
(x) $\int \frac{dx}{(x-1)\sqrt{x^2-2}}$;
(y) $\int \frac{dx}{(x+2)^2\sqrt{x^2+2x-5}}$;
(z) $\int \frac{1-x+x^2}{x\sqrt{1+x-x^2}}dx$.

14. Use the partial fraction decomposition to compute the following integrals of rational functions

(a) $\int \frac{(2x+3)dx}{(x-2)(x+5)}$;
(b) $\int \frac{x\,dx}{(x+1)(x+2)(x+3)}$;
(c) $\int \frac{x^{10}\,dx}{x^2+x-2}$;
(d) $\int \frac{x^3+1}{x^3-5x^2+6x}$;
(e) $\int \frac{x^4\,dx}{x^4+5x^2+4}$;
(f) $\int \frac{x\,dx}{x^3-3x+2}$;
(g) $\int \frac{(x^2+1)\,dx}{(x+1)^2(x-1)}$;
(h) $\int \left(\frac{x}{x^2-3x+2}\right)^2dx$;
(i) $\int \frac{dx}{(x+1)(x+2)^2(x+3)^3}$;
(j) $\int \frac{dx}{x^5+x^4-2x^3-2x^2+x+1}$;
(k) $\int \frac{x^2+5x+4}{x^4+5x^2+4}dx$;
(l) $\int \frac{dx}{(x+1)(x^2+1)}$;
(m) $\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)}$;
(n) $\int \frac{x\,dx}{(x-1)^2(x^2+2x+2)}$;
(o) $\int \frac{dx}{x(x-1)(1+x+x^2)}$;
(p) $\int \frac{dx}{x^4-1}$;
(q) $\int \frac{dx}{x^4+1}$;
(r) $\int \frac{dx}{x^4+x^2+1}$;
(s) $\int \frac{dx}{x^6+1}$;
(t) $\int \frac{dx}{(1+x)(1+x^2)(1+x^3)}$;
(u) $\int \frac{dx}{x^5-x^4+x^3-x^2+x-1}$;
(v) $\int \frac{x^2\,dx}{x^4+3x^3+\frac{9}{2}x^2+3x+1}$.

For what values of a, b and c is the integral

$$\int \frac{ax^2+bx+c}{x^3(x-1)^2}dx$$

a rational function? 15. Use the Ostrogradski method to compute the following integrals

(a) $\int \frac{x\,dx}{(x-1)^2(x+1)^3}$;
(b) $\int \frac{dx}{(x^3+1)^2}$;
(c) $\int \frac{dx}{(x^2+1)^3}$;
(d) $\int \frac{x^2\,dx}{(x^2+2x+2)^2}$;
(e) $\int \frac{dx}{(x^4+1)^2}$;
(f) $\int \frac{(x^2+3x-2)dx}{(x-1)(x^2+x+1)^2}$;
(g) $\int \frac{dx}{(x^4-1)^3}$.

16. Compute the following integrals

(a) $\int \frac{x^3\,dx}{(x-1)^{100}}$;
(b) $\int \frac{d\,dx}{x^8-1}$;
(c) $\int \frac{x^3\,dx}{x^8+3}$;
(d) $\int \frac{x^2+x}{x^6+1}dx$;
(e) $\int \frac{(x^4-3)dx}{x(x^8+3x^4+2)}$;
(f) $\int \frac{x^4\,dx}{(x^{10}-10)^2}$;
(g) $\int \frac{x^{11}dx}{x^8+3x^4+2}$;
(h) $\int \frac{x^9dx}{(x^{10}+x^5+5)^2}$;
(i) $\int \frac{x^{2n-1}}{x^n+1}dx$;
(j) $\int \frac{x^{3n-1}\,dx}{(x^{2n}+1)^2}$;
(k) $\int \frac{dx}{x(x^{10}+2)}$;
(l) $\int \frac{dx}{x(x^{10}+1)^2}$;
(m) $\int \frac{1-x^7}{x(1+x^7)}dx$;
(n) $\int \frac{x^4-1}{x(x^4-5)(x^5-5x+1)}dx$;
(o) $\int \frac{(x^2+1)dx}{x^4+x^2+1}$;
(p) $\int \frac{(x^2-1)dx}{x^4+x^3+x^2+x^1}$;
(q) $\int \frac{x^5-x}{x^8+1}dx$;
(r) $\int \frac{x^4+1}{x^6+1}dx$.

Find the reduction formula for the following integral

$$I_n = \int \frac{dx}{(ax^2+bx+c)^n},$$

where $a \neq 0$**, and use i to compute** $\int \frac{dx}{(x^2+x+1)^3}$**. 17.** Use the Taylor formula to compute the integral

$$\int \frac{P_n(x)}{(x-a)^{n+1}}dx,$$

where $P_n(x)$ is a polynomial of degree $n > 0$. **18.** Compute the integral $\int \frac{dx}{1+x^{2n}}$. **19.** Compute the following integrals of irrational functions

(a) $\int \frac{dx}{1+\sqrt{x}}$;
(b) $\int \frac{dx}{x(1+2\sqrt{x}+\sqrt[3]{x})}$;
(c) $\int \frac{x\sqrt[3]{2+x}}{x+\sqrt[3]{2+x}}dx$;
(d) $\int \frac{1-\sqrt{x+1}}{1+\sqrt[3]{x+1}}dx$;
(e) $\int \frac{dx}{(1+\sqrt[4]{x})\sqrt[3]{x}}$;
(f) $\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}dx$;
(g) $\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}$;
(h) $\int \frac{x\,dx}{\sqrt[4]{x^3(a-x)}}$, where $a > 0$;
(i) $\int \frac{dx}{\sqrt[n]{(x-a)^{n+1}(x-b)^{n-1}}}$, where n is a natural number.

20. Compute the following integrals

(a) $\int \frac{x^2}{\sqrt{1+x+x^2}}dx$;
(b) $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$;
(c) $\int \frac{dx}{(1-x)^2\sqrt{1-x^2}}$;
(d) $\int \frac{\sqrt{x^2+2x+2}}{x}dx$;
(e) $\int \frac{x\,dx}{(1+x)\sqrt{1-x-x^2}}$;
(f) $\int \frac{1-x+x^2}{\sqrt{1+x-x^2}}dx$;
(g) $\int \frac{x^3}{\sqrt{1+2x-x^2}}dx$;
(h) $\int \frac{x^{10}}{\sqrt{1+x^2}}dx$;
(i) $\int x^4\sqrt{a^2-x^2}dx$;
(j) $\int \frac{x^3-6x^2+11x-6}{\sqrt{x^2+4x+3}}dx$;
(k) $\int \frac{dx}{x^3\sqrt{x^2+1}}$;
(l) $\int \frac{dx}{x^4\sqrt{x^2-1}}$;
(m) $\int \frac{dx}{(x-1)^3\sqrt{x^2+3x+1}}$;
(n) $\int \frac{dx}{(x+1)^5\sqrt{x^2+2x}}$.

21. Use the Euler substitutions to compute the following integrals

(a) $\int \frac{dx}{x+\sqrt{x^2+x+1}}$;
(b) $\int x\sqrt{x^2-2x+2}\,dx$;
(c) $\int \frac{dx}{1+\sqrt{1-2x-x^2}}$;

(d) $\int \frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}\,dx$;
(e) $\int \frac{dx}{(1+\sqrt{x(1+x)})^2}$.

22. Compute the following integrals

(a) $\int \sqrt{x^3+x^4}\,dx$;
(b) $\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}\,dx$;
(c) $\int \frac{x\,dx}{\sqrt{1+\sqrt[3]{x^2}}}$;
(d) $\int \frac{x^5\,dx}{\sqrt{1-x^2}}$;
(e) $\int \frac{dx}{\sqrt[3]{1+x^3}}$;
(f) $\int \frac{dx}{\sqrt[4]{1+x^4}}$;
(g) $\int \frac{dx}{x\sqrt[6]{1+x^6}}$.

23. Compute the following trigonometric integrals

(a) $\int \cos^5 x\,dx$;
(b) $\int \sin^6 x\,dx$;
(c) $\int \cos^6 x\,dx$;
(d) $\int \sin^2 x\,\cos^4 x\,dx$;
(e) $\int \sin^4 x\,\cos^5 x\,dx$;
(f) $\int \sin^5 x\,\cos^5 x\,dx$;
(g) $\int \frac{\sin^3 x}{\cos^4 x}\,dx$;
(h) $\int \frac{\cos^4 x}{\sin^3 x}\,dx$;
(i) $\int \frac{dx}{\sin^3 x}$;
(j) $\int \frac{dx}{\cos^3 x}$;
(k) $\int \frac{dx}{\sin^4 x\,\cos^4 x}$;
(l) $\int \frac{dx}{\sin^3 x\,\cos^5 x}$;
(m) $\int \frac{dx}{\sin x\,\cos^4 x}$;
(n) $\int \tan^5 x\,dx$;
(o) $\int \cot^6 x\,dx$;
(p) $\int \frac{\sin^4 x}{\cos^6 x}\,dx$;
(q) $\int \frac{dx}{\sqrt{\sin^3 x\,\cos^5 x}}$;
(r) $\int \frac{dx}{\sqrt{\tan x}}$.

24. Dervive the reduction formulas for the following integrals

(a) $I_n = \int \frac{dx}{\sin^n x}$;
(b) $J_n = \int \frac{dx}{\cos^n x}$, where $n > 2$.

25. Compute the following integrals

(a) $\int \frac{dx}{2\sin x-\cos x+5}$;
(b) $\int \frac{dx}{(2+\cos x)\sin x}$;
(c) $\int \frac{\sin^2 x\,dx}{\sin x+2\cos x}$;
(d) $\int \frac{\sin^2 x\,dx}{1+\sin^2 x}$;
(e) $\int \frac{dx}{a^2\sin^2 x+b^2\cos^2 x}$;
(f) $\int \frac{\cos^2 x\,dx}{(a^2\sin^2 x+b^2\cos^2 x)^2}$;
(g) $\int \frac{\sin x\,\cos x}{\sin x+\cos x}\,dx$;
(h) $\int \frac{dx}{(a\sin x+b\cos x)^2}$;
(i) $\int \frac{\sin x\,dx}{\sin^3 x+\cos^3 x}$;
(j) $\int \frac{dx}{\sin^4 x+\cos^4 x}$;
(k) $\int \frac{\sin^2 x\,\cos^2 x}{\sin^8 x+\cos^8 x}\,dx$;
(l) $\int \frac{\sin^2 x-\cos^2 x}{\sin^4 x+\cos^4 x}\,dx$;
(m) $\int \frac{\sin x\,\cos x}{1+\sin^4 x}\,dx$;
(n) $\int \frac{dx}{\sin^6 x+\cos^6 x}$;
(o) $\int \frac{dx}{(\sin^2 x+2\cos^2 x)^2}$.

14.7 Solutions of All Problems

5.7.1. (a)

$$\begin{aligned}\int (3x^2)^3 dx &= \int (27 - 27x^2 + 9x^4 - x^6) dx \\ &= -\frac{1}{7}x^7 + \frac{9}{5}x^5 - 9x^3 + 27x + C\ .\end{aligned}$$

(b)

$$\begin{aligned}\int x^2(5x)^4 dx &= \int (625x^2 - 500x^3 + 150x^4 - 20x^5 + x^6) dx \\ &= \frac{1}{7}x^7 - \frac{10}{3}x^6 + 30x^5 - 125x^4 + \frac{625}{3}x^3 + C\ .\end{aligned}$$

(c)

$$\begin{aligned}\int (1-x)(1-2x)(1-3x) dx &= \int (1 - 6x + 11x^2 - 6x^3) dx \\ &= -\frac{3}{2}x^4 + \frac{11}{3}x^3 - 3x^2 + x + C\ .\end{aligned}$$

(d)

$$\begin{aligned}\int \frac{(1-x)^2}{x^2} dx &= \int (\frac{1}{x^2} - \frac{2}{x} + 1) dx \\ &= x - \frac{1}{x} - 2\ln x + C\ .\end{aligned}$$

(e)

$$\int (\frac{a}{x} + \frac{a^2}{x^2} + \frac{a^3}{x^3}) dx = a \ln x - \frac{a^2}{x} - \frac{1}{2}\frac{a^3}{x^2} + C\ .$$

(f)

$$\int \frac{x+1}{\sqrt{x}} dx = \int (\sqrt{x} + \frac{1}{\sqrt{x}}) dx = \frac{2}{3}x^{\frac{3}{2}} + 2\sqrt{x} + C\ .$$

(g)*:

$$\begin{aligned}\int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{\sqrt[4]{x}} dx &= \int \left(x^{\frac{1}{4}} - 2x^{\frac{5}{12}} + x^{-\frac{1}{4}} \right) dx \\ &= \frac{4}{5}x^{\frac{5}{4}} - \frac{24}{17}x^{\frac{17}{12}} + \frac{4}{3}x^{\frac{3}{4}} + C.\end{aligned}$$

(h)

$$\begin{aligned}\int \frac{(1-x)^3}{x\sqrt[3]{x}} dx &= \int (x^{-\frac{4}{3}} - 3x^{-\frac{1}{3}} + 3x^{\frac{2}{3}} - x^{\frac{5}{3}}) dx \\ &= -\frac{3}{8}x^{\frac{8}{3}} + \frac{9}{5}x^{\frac{5}{3}} - \frac{9}{2}x^{\frac{2}{3}} - 3x^{-\frac{1}{3}} + C\ .\end{aligned}$$

(i)

$$\int (1-\frac{1}{x^2})\sqrt{x\sqrt{x}}dx = \int (x^{\frac{4}{3}} - x^{-\frac{5}{4}})dx$$
$$= \frac{4}{7}x^{\frac{7}{4}} + 4x^{-\frac{1}{4}} + C \ .$$

(j)

$$\int \frac{(\sqrt{2x}-\sqrt[3]{3x})^2}{x}dx = \int \frac{2x+(3x)^{\frac{2}{3}} - 2\sqrt{2x}\sqrt[3]{3x}}{x}dx$$
$$= \int (2+3^{\frac{2}{3}}x^{-\frac{1}{3}} - 2\sqrt{2}\sqrt[3]{3}x^{-\frac{1}{6}})dx$$
$$= 2x - \frac{12}{5}\sqrt{2}\sqrt[3]{3}x^{\frac{5}{6}} + \frac{3}{2}3^{\frac{2}{3}}x^{\frac{2}{3}} + C \ .$$

(k)

$$\int \frac{\sqrt{x^4+x^{-4}+2}}{x^3}dx = \int \frac{\sqrt{(x^2+x^{-2})^2}}{x^3}dx$$
$$= \int (x^{-1}+x^{-5})dx = \ln x - \frac{1}{4x^4} + C \ .$$

(l)

$$\int \frac{x^2}{1+x^2}dx = \int (1-\frac{1}{1+x^2})dx = x - \arctan(x) + c \ .$$

(m)

$$\int \frac{x^2}{1-x^2}dx = \int (-1+\frac{1}{1-x^2})dx$$
$$= \int (-1+\frac{1}{2(1+x)} + \frac{1}{2(1-x)})dx$$
$$= -x + \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(x-1) + C \ .$$

(n)

$$\int \frac{x^2+3}{x^2-1}dx = \int (1+2\frac{2}{(x+1)(x-1)})dx$$
$$= \int (1+\frac{2}{x-1} - \frac{2}{x+1})dx = x + 2\ln(x-1) - 2\ln(x+1) + c \ .$$

(o)*:

$$\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}dx = \int \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1+x^2}}\right)dx$$
$$= \arcsin x + \text{arcsinh}\ x + C.$$

(p)

$$\int \frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^4-1}}dx = \int \frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}\sqrt{x^2-1}}dx$$
$$= \int (\frac{1}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2+1}})dx$$
$$= \ln|x+\sqrt{x^2-1}| - \ln|x+\sqrt{x^2+1}| + C \ .$$

(q)

$$\int (2^x + 3^x)^2 dx = \int (2^{2x} + 3^{2x} + 2 \cdot 6^x) dx$$
$$= \frac{2^{2x}}{2\ln 2} + \frac{3^{2x}}{2\ln 3} + \frac{2 \cdot 6^x}{\ln 6} + C \ .$$

(r)

$$\int \frac{2^{x+1} - 5^{x-1}}{10^x} dx = \int (2(\frac{1}{5})^x - \frac{1}{5}(\frac{1}{2})^x) dx$$
$$= -\frac{2}{\ln 5}(\frac{1}{5})^x + \frac{1}{5\ln 2}(\frac{1}{2})^x + C \ .$$

(s)

$$\int \frac{e^{3x} + 1}{e^x + 1} dx = \int \frac{(e^x + 1)(1 - e^x + e^{2x})}{e^x + 1} dx$$
$$= \int (1 - e^x + e^{2x}) dx = x - e^x + \frac{1}{2}e^{2x} + C \ .$$

(t)

$$\int (1 + \sin x + \cos x) dx = x - \cos x + \sin x + C \ .$$

(u)

$$\int \sqrt{1 - \sin(2x)} dx = \int \sqrt{1 - 2\sin x \cos x} dx$$
$$= \int \sqrt{(\cos x - \sin x)^2} dx = \int |\cos x - \sin x| dx$$
$$= \begin{cases} \sin x + \cos x & \cos x - \sin x \geq 0 \\ -(\sin x + \cos x) & \cos x - \sin x < 0 \end{cases}$$
$$= \mathrm{sgn}(\cos x - \sin x) \cdot (\sin x + \cos x) + C$$

(v)

$$\int \cot^2(x) dx = -\int (-1 - \cot^2(x) + 1) dx$$
$$= -x - \cot(x) + C \ .$$

(w)

$$\int (a \sinh x + b \cosh x) dx = a \cosh x + b \sinh x + C \ .$$

(x)

$$\int \tanh^2(x) dx = \int (1 - \frac{1}{\cosh^2 x}) dx$$
$$= x - \tanh(x) + C \ .$$

(y)

$$\int \coth^2(x)dx = \int (\frac{1}{\sinh^2 x} + 1)dx = x - \coth x + C \ .$$

5.7.2. (a)

$$\int \frac{1}{x+a}dx = \ln(x+a) + C \ .$$

(b)

$$\int \sqrt[3]{1-3x}dx = \int (1-3x)^{\frac{1}{3}}dx - \frac{1}{3}\int (1-3x)^{\frac{1}{3}}d(1-3x)$$
$$= -\frac{1}{3}\frac{3}{4}(1-3x)^{\frac{4}{3}} + C = -\frac{1}{4}(1-3x)^{\frac{4}{3}} + C \ .$$

(c)

$$\int \frac{1}{(5x-2)^{\frac{5}{2}}}dx = \frac{1}{5}\int (5x-2)^{-\frac{5}{2}}d(5x-2)$$
$$= \frac{1}{5}(-\frac{2}{3}) \cdot (5x-2)^{-\frac{3}{2}} + C = -\frac{2}{15}\frac{1}{(5x-2)^{\frac{3}{2}}} + C \ .$$

(d)*:

$$\int \frac{dx}{2+3x^2} = \frac{1}{3}\int \frac{dx}{\frac{2}{3}+x^2} = \frac{1}{\sqrt{6}}\arctan\left(\frac{\sqrt{3}x}{\sqrt{2}}\right) + C.$$

(e)

$$\int \frac{1}{\sqrt{2-3x^2}}dx = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{1-\frac{3}{2}x^2}}$$
$$= \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{3}}\int \frac{d(\sqrt{\frac{3}{2}}x)}{\sqrt{1-(\sqrt{\frac{3}{2}}x)^2}} = \frac{1}{\sqrt{3}}\arcsin(\sqrt{\frac{3}{2}}x) + C \ .$$

(f)

$$\int (e^{-x} + e^{-2x})dx = -e^{-x} - \frac{1}{2}e^{-2x} + C \ .$$

(g)*: (We apply the substitution $2x + \frac{p}{i}4 = t$)

$$\int \frac{dx}{\sin^2(2x+\frac{\pi}{4})} = \frac{1}{2}\int \frac{dt}{\sin^2 t} = \frac{1}{2}\cot t + C = \frac{1}{2}\cot\left(2x+\frac{\pi}{4}\right) + C.$$

(h)

$$\int \frac{dx}{1-\cos x} = \int \frac{dx}{1-(\cos^2\frac{x}{2}-\sin^2\frac{x}{2})} = \int \frac{dx}{2\sin^2\frac{x}{2}}$$
$$= \int \frac{d(\frac{x}{2})}{\sin^2\frac{x}{2}} = -\cot\frac{x}{2} + C \ .$$

(i)

$$\int (\sinh(2x+1) + \cosh(2x-1))dx = \frac{1}{2}\cosh(2x+1) + \frac{1}{2}\sinh(2x-1) + C \ .$$

(j)

$$\begin{aligned}\int (2x-3)^{10}dx &= \frac{1}{2}(2x-3)^{10}d(2x-3)\\ &= \frac{1}{2}\frac{1}{11}(2x-3)^{11}+C = \frac{1}{22}(2x-3)^{11}+C\ .\end{aligned}$$

(k)

$$\begin{aligned}\int \frac{dx}{\sqrt{2-5x}} &= -\frac{1}{5}\int (2-5x)^{-\frac{1}{2}}d(2-5x)\\ &= -\frac{1}{5}2(2-5x)^{\frac{1}{2}}+C = -\frac{2}{5}\sqrt{2-5x}+C\ .\end{aligned}$$

(l)

$$\begin{aligned}\int \frac{\sqrt[5]{1-2x+x^2}}{1-x}dx &= \int \frac{(1-x)^{\frac{2}{5}}}{1-x}dx\\ &= -\int (1-x)^{-\frac{3}{5}}d(1-x) = -\frac{5}{2}(1-x)^{\frac{2}{5}}+C\ .\end{aligned}$$

(m)

$$\begin{aligned}\int \frac{dx}{2-3x^2} &= \frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}}\int \frac{d(\sqrt{\frac{3}{2}}x)}{1-(\sqrt{\frac{3}{2}}x)^2}\\ &= \frac{1}{\sqrt{3}}\mathrm{arctanh}(\sqrt{\frac{3}{2}}x)+C\ .\end{aligned}$$

(n)

$$\int \frac{dx}{\sqrt{3x^2-2}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{x^2-\frac{2}{3}}} = \frac{1}{\sqrt{3}}\ln(x+\sqrt{x^2-\frac{2}{3}})+C\ .$$

(o)

$$\int (\sin(5x)-\sin(5a))dx = -\frac{1}{5}\cos(5x) - x\sin(5a)+C\ .$$

(p)

$$\begin{aligned}\int \frac{dx}{1+\cos x} &= \int \frac{dx}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}\\ &= \int \frac{d(\frac{x}{2})}{\cos^2\frac{x}{2}} = \tan\frac{x}{2}+C\ .\end{aligned}$$

(q)

$$\begin{aligned}\int \frac{dx}{1+\sin x} &= \int \frac{dx}{1+\cos(x-\frac{\pi}{2})} = \int \frac{dx}{2\cos^2(\frac{x}{2}-\frac{\pi}{4})}\\ &= \int \frac{d(\frac{x}{2}-\frac{\pi}{4})}{2\cos^2(\frac{x}{2}-\frac{\pi}{4})} = \tan(\frac{x}{2}-\frac{\pi}{4})+C\\ &= \frac{\tan\frac{x}{2}-1}{\tan\frac{x}{2}+1}+C\ .\end{aligned}$$

(r)

$$\int \frac{dx}{\cosh^2 \frac{x}{2}} = 2 \cdot \int \frac{d(\frac{x}{2})}{\cosh^2 \frac{x}{2}} = 2 \tanh \frac{x}{2} + C \ .$$

(s)

$$\int \frac{dx}{\sinh^2 \frac{x}{2}} = 2 \int \frac{d(\frac{x}{2})}{\sinh^2 \frac{x}{2}} = -2 \coth \frac{x}{2} + C \ .$$

5.7.3. (a)

$$\begin{aligned}\int \frac{xdx}{\sqrt{1-x^2}} &= \frac{1}{2} \int \frac{d(x^2)}{\sqrt{1-x^2}} \\ &= -\int \frac{d(1-x^2)}{2\sqrt{1-x^2}} = -\sqrt{1-x^2} + C \ .\end{aligned}$$

(b)

$$\int \frac{xdx}{3-2x^2} = -\frac{1}{4} \int d(3-2x^2)3-2x^2 = -\frac{1}{4} \ln|3-2x^2| + C \ .$$

(c)

$$\int \frac{xdx}{4+x^4} = \frac{1}{4} \int \frac{d(\frac{x^2}{2})}{1+(\frac{x^2}{2})^2} = \frac{1}{4} \arctan(\frac{x^2}{2}) + C \ .$$

(d)

$$\int \frac{dx}{(1+x)\sqrt{x}} = 2 \int \frac{d(\sqrt{x})}{1+x} = 2 \int \frac{d(\sqrt{x})}{1+(\sqrt{x})^2} = 2\arctan(\sqrt{x}) + C \ .$$

(e)

$$\begin{aligned}\int \frac{dx}{x\sqrt{x^2+1}} &= \int (\frac{\sqrt{x^2+1}+1}{x\sqrt{x^2+1}} - \frac{1}{x})dx = \int (\frac{x}{\sqrt{x^2+1}} \frac{1}{\sqrt{x^2+1}-1} - \frac{1}{x})dx \\ &= \int \frac{d(\sqrt{x^2+1}-1)}{\sqrt{x^2+1}-1} - \int \frac{1}{x} dx = \ln(\frac{\sqrt{x^2+1}-1}{x}) + C \ .\end{aligned}$$

(f) We use the substitution $x = \tan y$, $dx = \frac{1}{\cos^2 y}$.

$$\begin{aligned}\int \frac{dx}{(x^2+1)^{\frac{3}{2}}} &= \int \frac{\frac{1}{\cos^2 y} dy}{(\frac{1}{\cos^2 y})^{\frac{3}{2}}} = \int \cos y dy \\ &= \sin y + C = \sin(\arctan(x)) + C = \frac{x}{\sqrt{x^2+1}} + C \ .\end{aligned}$$

(g) $\int \frac{x^2}{(8x^2+27)^{\frac{2}{3}}} dx$ cannot be solved. Because it is of type $\int x^m (a+bx^n)^p dx$ with $m=2$, $n=2$ and $p = -\frac{2}{3}$, and none of p, $\frac{m+1}{n}$, $\frac{m+1}{n} + p$ are integers. For more details, please refer to page 294 of the text book.

(h)

$$\begin{aligned}\int \frac{dx}{\sqrt{x(1-x)}} &= \int \frac{dx}{\sqrt{(\frac{1}{2})^2 - (x-\frac{1}{2})^2}} \\ &= \int \frac{2dx}{\sqrt{1-(2x-1)^2}} = \int \frac{d(2x-1)}{\sqrt{1-(2x-1)^2}} \\ &= \arcsin(2x-1) + C \ .\end{aligned}$$

(i)

$$\int \frac{e^x}{2+e^x}dx = \int \frac{d(e^x+2)}{e^x+2} = \ln(e^x+2) + C \ .$$

(j) We use the substitution $e^x = y$.

$$\begin{aligned}\int \frac{1}{1+e^{2x}}dx &= \int \frac{dy}{y(1+y^2)} = \int (\frac{1}{y} - \frac{y}{1+y^2})dy \\ &= \ln y - \frac{1}{2}\int \frac{d(y^2+1)}{y^2+1} \\ &= \ln y - \frac{1}{2}\ln(1+y^2) + C \\ &= x - \frac{1}{2}\ln(e^{2x}+1) + C \ .\end{aligned}$$

(k)*: (Substitution $t = \ln(\ln x)$, thus $dt = \frac{dx}{x \ln x}$)

$$\int \frac{dx}{x\,\ln x\,\ln(\ln x)} = \int \frac{dt}{t} = \ln t + C = \ln(\ln(\ln x)) + C.$$

(v): (Substitution $t = \sin x$, thus $dt = \cos x\, dx$)

$$\int \sin^5 x\, \cos x\, dx = \int t^5 dt = \frac{1}{6}t^6 + C = \frac{1}{6}\sin^6 x + C.$$

(l)

$$\begin{aligned}\int \frac{\sin x}{\sqrt{\cos^3 x}}dx &= -\int \frac{d(\cos x)}{\cos^{\frac{3}{2}} x} = -(-2)\cos^{-\frac{1}{2}} x + C \\ &= \frac{2}{\sqrt{\cos x}} + C \ .\end{aligned}$$

(m)

$$\begin{aligned}&\int x^2\sqrt[3]{1+x^3}dx = \frac{1}{3}\int (1+x^3)^{\frac{1}{3}} d(1+x^3) \\ &= \frac{1}{3}\frac{3}{4}(1+x^3)^{\frac{4}{3}} + C = \frac{1}{4}(1+x^3)^{\frac{4}{3}} + C \ .\end{aligned}$$

(n)

$$\int \frac{xdx}{(1+x^2)^2} = \frac{1}{2}\int \frac{d(1+x^2)}{(1+x^2)^2} = -\frac{1}{2}\frac{1}{1+x^2} + C \ .$$

(o)

$$\begin{aligned}\int \frac{x^3}{x^8-2}dx &= \frac{1}{4}\int \frac{d(x^4)}{(x^4)^2-2} \\ &= \frac{-1}{4\sqrt{2}}\int \frac{d(\frac{x^4}{\sqrt{2}})}{1-(\frac{x^4}{\sqrt{2}})^2} = -\frac{\sqrt{2}}{8}\text{arctanh}(\frac{\sqrt{2}}{2}x^4) + C \ .\end{aligned}$$

(p)

$$\int \sin\frac{1}{x}\frac{dx}{x^2} = -\int \sin\frac{1}{x} d(\frac{1}{x}) = \cos\frac{1}{x} + C \ .$$

(q) We use the substitution $x = \frac{1}{\cos y}$, $dx = \frac{\sin y}{\cos^2 y} dy$.

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\cos y \cdot \sin y}{\cos^2 y} \frac{1}{\sqrt{\frac{1}{\cos^2 y} - 1}} dy = \int \frac{\sin y}{\cos y}\frac{\cos y}{\sin y} dy$$
$$= \int 1 \cdot dx = y + C = \arccos\frac{1}{x} + C \ .$$

(r)

$$\int \frac{1}{\sqrt{x(1+x)}} dx = \int \frac{d(x+\frac{1}{2})}{\sqrt{(x+\frac{1}{2})^2 - (\frac{1}{2})^2}}$$
$$= \ln|x + \frac{1}{2} + \sqrt{(x+\frac{1}{2})^2 - (\frac{1}{2})^2}| + C$$
$$= \ln|x + \frac{1}{2} + \sqrt{x^2+x}| + C \ .$$

(s)

$$\int xe^{-x^2} dx = -\frac{1}{2}\int e^{-x^2} d(-x^2) = -\frac{1}{2}e^{-x^2} + C \ .$$

(t)

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{d(e^x)}{(e^x)^2 + 1} = \arctan(e^x) + C \ .$$

(u)

$$\int \frac{\ln^2 x}{x} dx = \int \ln^2 x d(\ln x) = \frac{1}{3}\ln^3 x + C \ .$$

(v)

$$\int \sin^5 x \cos x dx = \int \sin^5 x d(\sin x) = \frac{1}{6}\sin^6 x + C \ .$$

(w)

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -int\frac{d(\cos x)}{\cos x} = -\ln(\cos x) + C \ .$$

(x)

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \int \frac{d(\sin x)}{\sin x} = \ln(\sin x) + C \ .$$

(y)

$$\int \frac{\sin x \cos x dx}{\sqrt{a^2\sin^2 x + b^2\cos^2 x}} = \frac{1}{2}\int \frac{d(\sin^2 x)}{\sqrt{a^2\sin^2 x + b^2(1-\sin^2 x)}}$$
$$= \frac{1}{2}\int \frac{d(\sin^2 x)}{\sqrt{(a^2-b^2)\sin^2 x + b^2}} = \frac{1}{\sqrt{a^2-b^2}}\int \frac{d(\sin^2 x + \frac{b^2}{a^2-b^2})}{2\sqrt{\sin^2 x + \frac{b^2}{a^2-b^2}}}$$
$$= \frac{1}{\sqrt{a^2-b^2}}\sqrt{\sin^2 x + \frac{b^2}{a^2-b^2}} + C \ .$$

(z)

$$\int \frac{\cos x}{\sqrt{\cos(2x)}}dx = \int \frac{\cos x}{\sqrt{1-2\sin^2 x}} = \frac{1}{\sqrt{2}}\int \frac{d(\sqrt{2}\sin x)}{\sqrt{1-(\sqrt{2}\sin x)^2}}$$
$$= \frac{1}{\sqrt{2}}\arcsin(\sqrt{2}\sin x) + C \ .$$

5.7.4. (a) We use the substitution $t = \cot x$, $dt = \frac{-1}{\sin^2 x}$.

$$\int \frac{1}{\sin^2 x\sqrt[4]{\cot x}}dx = -\int \frac{dt}{\sqrt[4]{t}} = -\frac{4}{3}(\cot x)^{\frac{3}{4}} + C \ .$$

(b) We use the substitution $t\cos x$, $dt = -\sin x dx$. Thus, $dx = \frac{1}{-\sin x}dt = \frac{1}{-\sqrt{1-t^2}}dt$ and $\sin x = \sqrt{1-t^2}$.

$$\int \frac{dx}{\sin x} = \int -\frac{1}{\sqrt{1-t^2}}\frac{1}{\sqrt{1-t^2}}dt = -\int \frac{dt}{1-t^2}$$
$$= -\text{arctanh}t + C = -\text{arctanh}(\cos x) + C \ .$$

(c)

$$\int \frac{dx}{\sinh x} = \int \frac{dx}{\frac{e^x - e^{-x}}{2}} = 2\int \frac{d(e^x)}{(e^x)^2 - 1}$$
$$= -2\int \frac{d(e^x)}{1-(e^x)^2} = -2\text{arctanh}(e^x) + C \ .$$

(d) We use the substitution $y = \cosh^2 x$.

$$\int \frac{\sinh x \cosh x}{\sqrt{\sinh^4 x + \cosh^4 x}} = \frac{1}{2}\int \frac{dy}{\sqrt{y^2 + (y-1)^2}} = \frac{1}{2}\int \frac{dy}{\sqrt{2y^2 - 2y + 1}}$$
$$= \frac{\sqrt{2}}{2}\int \frac{dy}{\sqrt{1+(2y-1)^2}} = \frac{\sqrt{2}}{4}\int \frac{d(2y-1)}{\sqrt{1+(2y-1)^2}}$$
$$= \frac{\sqrt{2}}{4}\text{arcsinh}(2y-1) + C$$
$$= \frac{\sqrt{2}}{4}\text{arcsinh}(2\cosh^2 x - 1) + C \ .$$

(e)

$$\int \frac{\arctan x}{1+x^2}dx = \int \arctan x d(\arctan x)$$
$$= \frac{1}{2}\arctan^2 x + C \ .$$

(f) For the question $\int \frac{\sqrt{\ln(x+\sqrt{1+x^2})}}{1+x^2}dx$, We use the substitution $x = \sinh y$.

$$\int \frac{\sqrt{\ln(x+\sqrt{1+x^2})}}{1+x^2}dx = \int \frac{\sqrt{\ln(\sinh y + \cosh y)}}{\cosh^2 y}\cosh y dx$$
$$= \int \frac{\sqrt{\ln(e^y)}}{\cosh y}dy = \int \frac{\sqrt{y}}{\cosh y}dy \ .$$

However, $\int \frac{\sqrt{y}}{\cosh y}dy$ cannot be computed as an elementary function.

For the question $\int \frac{(1+x)^2}{\sqrt{1+x^2}}dx$, we have

$$\begin{aligned}
&\int \frac{(1+x)^2}{\sqrt{1+x^2}}dx \\
&= \int \frac{(1+\sinh t)^2}{\cosh t}\cosh t dt \qquad \text{(substitution } \ x = \sinh t) \\
&= \int (1 + 2\sinh t + \sinh^2 t)dt \\
&= \int (2\sinh t + \frac{1}{2} + \frac{1}{2}\cosh(2t))dt \\
&= 2\cosh t + \frac{1}{2}t + \frac{1}{4}\sinh(2t) \\
&= \frac{1}{2}\arctan x + 2\sqrt{1+x^2} + \frac{1}{2}x\sqrt{1+x^2} \ .
\end{aligned}$$

(g)

$$\begin{aligned}
\int \frac{x^2+1}{x^4+1}dx &= \int \frac{x^2+1}{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)}dx \\
&= \frac{1}{2}\int \frac{x^2+x\sqrt{2}+1+x^2-x\sqrt{2}+1}{(x^2+x\sqrt{2}+1)(x^2-x\sqrt{2}+1)}dx \\
&= \int \frac{dx}{2(x^2+x\sqrt{2}+1)} + \int \frac{dx}{2(x^2-x\sqrt{2}+1)} \\
&= \int \frac{dx}{(\sqrt{2}x+1)^2+1} \int \frac{dx}{(\sqrt{2}-1)^2+1} \\
&= \frac{1}{\sqrt{2}}\arctan(\sqrt{2}x+1) + \frac{1}{\sqrt{2}}\arctan(\sqrt{2}x-1) + C \ .
\end{aligned}$$

(h)*: (Partial fraction decomposition and substitutions $t = x^2 - x\sqrt{2} + 1$ and $s = x^2 + x\sqrt{2} + 1$)

$$\begin{aligned}
\int \frac{x^2-1}{x^4+1}dx &= \frac{1}{2}\int \left(\frac{x\sqrt{2}-1}{x^2-x\sqrt{2}+1} - \frac{x\sqrt{2}+1}{x^2+x\sqrt{2}+1}\right)dx \\
&= \frac{\sqrt{2}}{4}\left(\ln(x^2-x\sqrt{2}+1) - \ln(x^2+x\sqrt{2}+1)\right) + C.
\end{aligned}$$

(i)

$$\begin{aligned}
\int \frac{x^{\frac{n}{2}}dx}{\sqrt{1+x^{n+2}}} &= \int \frac{1}{\frac{n}{2}+1}\frac{d(x^{\frac{n}{2}+1})}{\sqrt{1+(x^{\frac{n}{2}+1})^2}} \\
&= \frac{2}{n+2}\operatorname{arcsinh}(x^{\frac{n}{2}+1}) + C \ .
\end{aligned}$$

(j) We use the substitution $y = \sin x$.

$$\begin{aligned}
\int \frac{\cos x dx}{\sqrt{2+\cos x}} &= \int \frac{d(\sin x)}{\sqrt{3-2\sin^2 x}} = \int \frac{dy}{\sqrt{3-2y^2}} \\
&= \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{2}} \int \frac{d(\frac{\sqrt{2}}{\sqrt{3}}y)}{\sqrt{1-(\frac{\sqrt{2}}{\sqrt{3}}y)^2}} = \frac{1}{\sqrt{2}}\arcsin\left(\frac{\sqrt{2}}{\sqrt{3}}y\right) + C \\
&= \frac{\sqrt{2}}{2}\arcsin\left(\frac{\sqrt{6}}{3}\sin x\right) + C\ .
\end{aligned}$$

(k)

$$\begin{aligned}
\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}}dx &= \int (\sin x - \cos x)^{-\frac{1}{3}} d(\sin x - \cos x) \\
&= \frac{3}{2}(\sin x - \cos x)^{\frac{2}{3}} + C\ .
\end{aligned}$$

(l)

$$\begin{aligned}
\int \frac{\sin x}{\sqrt{\cos 2x}}dx &= -\int \frac{d(\cos x)}{\sqrt{2\cos^2 x - 1}} \\
&= -\frac{1}{\sqrt{2}} \int \frac{d(\sqrt{2}\cos x)}{\sqrt{(\sqrt{2}\cos x)^2 - 1}} \\
&= -\frac{\sqrt{2}}{2}\operatorname{arccosh}(\sqrt{2}\cos x) + C\ .
\end{aligned}$$

(m)

$$\begin{aligned}
\int \frac{\sinh x}{\sqrt{\cosh 2x}}dx &= \int \frac{d(\cosh x)}{\sqrt{\cosh^2 x + \sinh^2 x}} = \int \frac{d(\cosh x)}{\sqrt{2\cosh^2 x - 1}} \\
&= \frac{1}{\sqrt{2}} \int \frac{d(\sqrt{2}\cosh x)}{\sqrt{(\sqrt{2}\cosh x)^2 - 1}} = \frac{\sqrt{2}}{2}\operatorname{arccosh}(\sqrt{2}\cosh x) + C\ .
\end{aligned}$$

(n) We use the substitution $\tan\frac{x}{2} = t$.

$$\begin{aligned}
\int \frac{dx}{\sin^2 x + 2\cos^2 x} &= \int \frac{dx}{1+\cos^2 x} = \int \frac{\frac{2}{1+t^2}dt}{1+(\frac{1-t^2}{1+t^2})^2} \\
&= \int \frac{1+t^2}{1+t^4}dt\ .
\end{aligned}$$

Then, by **5.7.4.** (g), we have

$$\begin{aligned}
\int \frac{dx}{\sin^2 x + 2\cos^2 x} &= \int \frac{1+t^2}{1+t^4}dt \\
&= \frac{\sqrt{2}}{2}\arctan(\sqrt{2}t+1) + \frac{\sqrt{2}}{2}\arctan(\sqrt{2}t-1) + C \\
&= \frac{\sqrt{2}}{2}\arctan\left(\sqrt{2}\tan\frac{x}{2}+1\right) + \frac{\sqrt{2}}{2}\arctan\left(\sqrt{2}\tan\frac{x}{2}-1\right) + C\ .
\end{aligned}$$

(o)*: (Substitution $t = \sin x$)

$$\begin{aligned}\int \frac{dx}{\cos x} &= \int \frac{\cos x}{1-\sin^2 x} = \int \frac{dt}{1-t^2} \\ &= \frac{1}{2}\int \left(\frac{1}{t+1} - \frac{1}{t-1}\right) dt = \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| C \\ &= \frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x-1}\right| + C \\ &= \frac{1}{2}\ln\left|\frac{(\sin x+1)^2}{\cos^2 x}\right| + C = \ln|\sec x + \tan x| + C.\end{aligned}$$

(p)

$$\begin{aligned}\int \frac{dx}{\cosh x} &= \int \frac{\cosh x dx}{\cosh^2 x} dx = \int \frac{d(\sinh x)}{1+\sinh^2 x} \\ &= \arctan(\sinh x) + C \ .\end{aligned}$$

(q)

$$\begin{aligned}\int \frac{dx}{\cosh^2 x \sqrt[3]{\tanh^2 x}} &= \int (\tanh x)^{-\frac{2}{3}} d(\tanh x) \\ &= 3\sqrt[3]{\tanh x} + C \ .\end{aligned}$$

(r)

$$\begin{aligned}\int \frac{dx}{(\arcsin x)^2\sqrt{1-x^2}} &= \int (\arcsin x)^{-2} d(\arcsin x) \\ &= -(\arcsin x)^{-1} + C = -\frac{1}{\arcsin x} + C \ .\end{aligned}$$

(s)

$$\begin{aligned}\int \frac{x^4 dx}{(x^5+1)^4} &= \frac{1}{5}\int \frac{d(x^5+1)}{(x^5+1)^4} \\ &= \frac{1}{5}(-\frac{1}{3})(x^5+1)^{-3} + C = -\frac{1}{15}\frac{1}{(x^5+1)^3} + C \ .\end{aligned}$$

(t)

$$\begin{aligned}\int \frac{1}{1-x^2}\ln(\frac{1+x}{1-x})dx &= \frac{1}{2}\int \ln(\frac{[}{1-x}1+x)d(\ln(\frac{1+x}{1-x})) \\ &= \frac{1}{4}(\ln(\frac{1+x}{1-x}))^2 + C \ .\end{aligned}$$

(u) We use the substitution $y = \sin^2 x$.

$$\begin{aligned}&\int \frac{\sin x\cos x}{\sin^4 x + \cos^4 x} dx = \frac{1}{2}\int \frac{d(\sin^2 x)}{(\sin^2 x)^2 + (1-\sin^2 x)^2} = \frac{1}{2}\int \frac{dy}{y^2 + (1-y)^2} \\ &= \int \frac{dy}{(2y-1)^2+1} = \frac{1}{2}\int \frac{d(2y-1)}{1+(2y-1)^2} = \frac{1}{2}\arctan(2y-1) + C \\ &= \frac{1}{2}\arctan(2\sin^2 x - 1) + C = -\frac{1}{2}\arctan(\cos(2x)) + C \ .\end{aligned}$$

(v) First we evaluate $I_1 = \int \frac{(\frac{3}{2})^x}{1+(\frac{3}{2})^x} dx$ and $I_2 = \int \frac{(\frac{2}{3})^x}{1-(\frac{2}{3})^x} dx$.
i) Let $y = (\frac{3}{2})^x$. Then $dy = (\frac{3}{2})^x \ln(\frac{3}{2}) dx$ and

$$\begin{aligned} I_1 &= \int \frac{y}{1+y} \frac{1}{y \ln(\frac{3}{2})} dy = \frac{1}{\ln 3 - \ln 2} \int \frac{dy}{1+y} \\ &= \frac{\ln(1+y)}{\ln 3 - \ln 2} + C_1 = \frac{\ln(3^x + 2^x)}{\ln 3 - \ln 2} - \frac{x \ln 2}{\ln 3 - \ln 2} + C_1 \ . \end{aligned}$$

ii) Let $y = (\frac{2}{3})^x$. Then $dy = (\frac{2}{3})^x \ln(\frac{2}{3}) dx$ and

$$\begin{aligned} I_2 &= \int \frac{y}{1-y} \frac{dy}{y \ln(\frac{2}{3})} = \frac{1}{\ln 3 - \ln 2} \int \frac{dy}{y-1} \\ &= \frac{1}{\ln 3 - \ln 2} \ln(1-y) + C_2 = \frac{\ln(3^x - 2^x)}{\ln 3 - \ln 2} - \frac{x \ln 3}{\ln 3 - \ln 2} + C_2 \ . \end{aligned}$$

Now, notice that

$$\frac{2^x 3^x}{9^x - 4^x} = \frac{1}{2}\left(\frac{2^x}{3^x + 2^x} + \frac{3^x}{3^x - 2^x} - 1\right) .$$

We have

$$\begin{aligned} \int \frac{2^x 3^x}{9^x - 4^x} &= \frac{1}{2}\left(-x + \int \frac{2^x}{3^x + 2^x} dx + \int \frac{3^x}{3^x - 2^x} dx\right) \\ &= \frac{1}{2}\left(-x + \int \frac{1}{1 + (\frac{3}{2})^x} dx + \int \frac{1}{1 - (\frac{2}{3})^x} dx\right) \\ &= \frac{1}{2}\left(-x + \int (1 - \frac{(\frac{3}{2})^x}{1 + (\frac{3}{2})^x}) dx + \int (1 + \frac{(\frac{2}{3})^x}{1 + (\frac{2}{3})^x}) dx\right) \\ &= \frac{1}{2}(x + I_2 - I_1) \\ &= \frac{1}{2}(x + \frac{\ln(3^x - 2^x)}{\ln 3 - \ln 2} - \frac{x \ln 3}{\ln 3 \ln 2} - \frac{\ln(3^x + 2^x)}{\ln 3 - \ln 2} + \frac{x \ln 2}{\ln 3 - \ln 2} + C_2 - C_1) \\ &= \frac{1}{2(\ln 3 - \ln 2)} \ln(\frac{3^x - 2^x}{3^x + 2^x}) + C \ . \end{aligned}$$

(w) Let $y = x^2 + 1$.

$$\begin{aligned} \int \frac{x dx}{\sqrt{1 + x^2 + \sqrt{(1+x^2)^2}}} &= \frac{1}{2} \int \frac{d(1+x^2)}{\sqrt{(1+x^2) + \sqrt{(1+x^2)^3}}} = \int \frac{1}{\sqrt{1+\sqrt{y}}} \frac{dy}{2\sqrt{y}} \\ &= \int \frac{d(\sqrt{y})}{\sqrt{1+\sqrt{y}}} = \int (\sqrt{y}+1)^{-\frac{1}{2}} d(\sqrt{y}+1) \\ &= 2(\sqrt{y}+1)^{\frac{1}{2}} + C = 2\sqrt{1 + \sqrt{1+x^2}} + C \ . \end{aligned}$$

5.7.5. (a)

$$\begin{aligned} \int x^2 (2 - 3x^2)^2 dx &= \int (4x^2 - 12x^4 + 9x^6) dx \\ &= \frac{4}{3}x^3 - \frac{12}{5}x^5 + \frac{9}{7}x^7 + C \ . \end{aligned}$$

(b)

$$\begin{aligned}\int x(1-x)^{10}dx &= \int(1-x)^{10}dx - \int(1-x)^{11}dx\\ &= \int(x-1)^{10}dx + \int(x-1)^{11}dx\\ &= \frac{(x-1)^{11}}{11} + \frac{(1-x)^{12}}{12} + C\ .\end{aligned}$$

(c)

$$\int \frac{1+x}{1-x}dx = \int(-1-\frac{2}{x-1})dx = -x - 2\ln|x-1| + C\ .$$

(d)

$$\begin{aligned}\int \frac{x^2}{1+x}dx &= \int \frac{(x^2-1)+1}{1+x}dx\\ &= \int(x-1)+\frac{1}{x+1})dx = \frac{x^2}{2} - x + \ln|x+1| + C\ .\end{aligned}$$

(e)

$$\begin{aligned}\int \frac{x^3}{3+x}dx &= \int \frac{x^3+27-27}{3+x}dx\\ &= \int \frac{(x+3)(x^2-3x+9)-27}{(x+3)}dx = \int((x^2-3x+9)-\frac{27}{x+3})dx\\ &= \frac{x^3}{3} - \frac{3}{2}x^2 + 9x - 27\ln|x+3| + C\ .\end{aligned}$$

(f)*:

$$\int \frac{(1+x)^2}{1+x^2}dx = \int\left(1+\frac{2x}{1+x^2}\right)dx = x + \ln(1+x^2) + C.$$

(g)

$$\begin{aligned}\int \frac{(2-x)^2}{2-x^2}dx &= \int \frac{x^2-4x+4}{2-x^2}dx\\ &= \int(-1+\frac{4x}{x^2-2}+\frac{6}{2-x^2})dx\\ &= -x2\int\frac{d(x^2-2)}{x^2-2} + 3\int\frac{d(\frac{x}{\sqrt{2}})}{1-(\frac{x}{\sqrt{2}})^2}\cdot\sqrt{2}\\ &= -x + 2\ln|x^2-2| + 3\sqrt{2}\text{arctanh}(\frac{\sqrt{2}}{2}x) + C\ .\end{aligned}$$

(h)

$$\begin{aligned}\int \frac{x^2}{(1-x)^{100}}dx &= \int\frac{((x-1)+1)^2}{(x-1)^{100}}dx = \int(\frac{1}{(x-1)^{98}}+\frac{2}{(x-1)^{99}}+\frac{1}{(x-1)^{100}})dx\\ &= -(\frac{1}{97(x-1)^{97}}+\frac{1}{49(x-1)^{98}}+\frac{1}{99(x-1)^{99}}) + C\ .\end{aligned}$$

(i)

$$\begin{aligned}\int \frac{x^5}{x+1}dx &= \int \frac{(1+x)(x^4-x^3+x^2-x+1)-1}{1+x}dx \\ &= \int (x^4-x^3+x^2-x+1)dx - \int \frac{1}{1+x}dx \\ &= \frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - \ln|1+x| + C \ .\end{aligned}$$

(j)*:

$$\begin{aligned}\int \frac{dx}{\sqrt{x+1}+\sqrt{x-1}} &= \int \frac{\sqrt{x+1}-\sqrt{x-1}}{x+1-x+1}dx \\ &= \frac{1}{2}\int \left(\sqrt{x+1}-\sqrt{x-1}\right) dx \\ &= \frac{1}{3}\left(\sqrt{(x+1)^3}-\sqrt{(x-1)^3}\right) + C.\end{aligned}$$

(k)

$$\begin{aligned}\int x\sqrt{2-5x}dx &= \int (-\frac{1}{5}(2-5x)\sqrt{2-5x} + \frac{2}{5}\sqrt{2-5x})dx \\ &= \frac{1}{25}\int (2-5x)^{\frac{3}{2}} d(2-5x) - \frac{2}{25}\int (2-5x)^{\frac{1}{2}} d(2-5x) \\ &= \frac{2}{125}(2-5x)^{\frac{5}{2}} - \frac{4}{75}(2-5x)^{\frac{3}{2}} + C \ .\end{aligned}$$

(l)*:

$$\begin{aligned}\int \frac{dx}{(x-1)(x+3)} &= \frac{1}{4}\int \left(\frac{1}{x-1} - \frac{1}{x+3}\right) dx \\ &= \frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + C.\end{aligned}$$

(m)

$$\begin{aligned}\int \frac{1}{x^2+x-2}dx &= \int \frac{dx}{(x+2)(x-1)} \\ &= \frac{1}{3}\int (\frac{1}{x-1} - \frac{1}{x+2})dx = \frac{1}{3}(\ln|x-1| - \ln|x+2|) + C \ .\end{aligned}$$

(n)

$$\begin{aligned}\int \frac{dx}{(x^2+1)(x^2+2)} &= \int (\frac{1}{x^2+1} - \frac{1}{x^2+2})dx \\ &= \arctan x - \frac{1}{2}\sqrt{2}\int \frac{d(\frac{x}{\sqrt{2}})}{1+(\frac{x}{\sqrt{2}})^2} = \arctan x - \frac{\sqrt{2}}{2}\arctan(\frac{\sqrt{2}}{2}x) + C \ .\end{aligned}$$

(o)*:

$$\begin{aligned}\int \frac{dx}{(x^2+1)(x^2+2)} &= \int \left(\frac{1}{x^2+1} - \frac{1}{x^2+2}\right) dx \\ &= \arctan x - \frac{1}{\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right) + C.\end{aligned}$$

(p)

$$\int \frac{dx}{(x+2)(x+3)} = \int (\frac{1}{x+2} - \frac{1}{x+3}) dx = \ln|x+2| + \ln|x+3| + C \ .$$

(q)

$$\begin{aligned}\int \frac{dx}{x^4+3x^2+2} &= \int \frac{dx}{(x^2+1)(x^2+2)} \\ &= \arctan x - \frac{\sqrt{2}}{2} \arctan(\frac{\sqrt{2}}{2} x) + C \qquad \text{by } \mathbf{5.7.5.} \text{ (n)} \ .\end{aligned}$$

(r)

$$\begin{aligned}\int \frac{dx}{(x+a)^2(x+b)^2} &= \frac{1}{(b-a)^3} \int \frac{(b-a)^3}{(x+a)^2(x+b)^2} dx \\ &= \frac{1}{(b-a)^3} \int \frac{((x+b)-(x-a))^3}{(x+a)^2(x+b)^2} dx \\ &= \frac{1}{(b-a)^3} \int \left(\frac{x+b}{(x+a)^2} - \frac{3}{x+a} + \frac{3}{x+b} - \frac{x+a}{(x+b)^2}\right) dx \\ &= \frac{1}{(b-a)^3} \int (\frac{-2}{x+a} + \frac{2}{x+b} + \frac{b-a}{(x+a)^2} + \frac{b-a}{(x+b)^2}) dx \\ &= \frac{2}{(b-a)^3} (\ln|x+b| - \ln|x+a|) - \frac{1}{(b-a)^2} (\frac{1}{x+b} + \frac{1}{x+a}) + C \ .\end{aligned}$$

(s)

$$\begin{aligned}\int \frac{dx}{(x^2+a^2)(x^2+b^2)} &= \frac{1}{a^2-b^2} \int (\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}) dx \\ &= \frac{1}{a^2-b^2} \left(\frac{1}{b} \int \frac{d(\frac{x}{b})}{1+(\frac{x}{b})^2} - \frac{1}{a} \int \frac{d(\frac{x}{a})}{1+(\frac{x}{a})^2}\right) \\ &= \frac{1}{a^2-b^2} (\frac{\arctan \frac{x}{b}}{b} - \frac{\arctan \frac{x}{a}}{a}) + C \ .\end{aligned}$$

5.7.6. (a) Let $I = \int \sin^2 x dx$. Then

$$\begin{aligned}I &= -\int \sin x d\cos x = -\sin x \cos x + \int \cos x d\sin x \\ &= -\sin x \cos x + \int \cos^2 x dx = -\sin x \cos x + \int (1-\sin^2 x) dx \\ &= -\sin x \cos x + x - \int \sin^2 x dx \\ &= -\sin x \cos x + x - I \ .\end{aligned}$$

Therefore, $\int \sin^2 x dx = I = \frac{1}{2}(x - \sin x \cos x)$.

(b) From **5.7.6.** (a) we have

$$\begin{aligned}\int \cos^2 x dx &= \int (1-\sin^2 x)dx = x - \int \sin^2 x dx\\ &= x - \frac{1}{2}(x - \sin x \cos x)\\ &= \frac{1}{2}(x + \sin x \cos x)\ .\end{aligned}$$

(c)

$$\begin{aligned}\int \sin x \sin(x+\alpha)dx &= \frac{1}{2}\int (\cos\alpha - \cos(2x+\alpha))dx\\ &= \frac{1}{2}x\cos\alpha - \frac{1}{4}\sin(2x+\alpha) + C\ .\end{aligned}$$

(d)

$$\begin{aligned}\int \sin 3x \sin 5x dx &= \frac{1}{2}\int (\cos 2x - \cos 8x)dx\\ &= \frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + C\ .\end{aligned}$$

(e)

$$\begin{aligned}\int \cos\frac{x}{2}\cos\frac{x}{2}dx &= \frac{1}{2}\int (\cos\frac{x}{6} + \cos(\frac{5x}{6}))dx\\ &= 3\sin\frac{x}{6} + \frac{3}{5}\sin\frac{5x}{6} + C\ .\end{aligned}$$

(f)

$$\begin{aligned}\int \sin(2x - \frac{\pi}{6}\cos(3x + \frac{\pi}{4}dx &= \frac{1}{2}\int (\sin(5x + \frac{\pi}{12}) + \sin(-x - \frac{5\pi}{12}))dx\\ &= -\frac{1}{10}\cos(5x + \frac{\pi}{12}) + \frac{1}{2}\cos(x + \frac{5\pi}{12}) + C\ .\end{aligned}$$

(g)*: (Substitution $t = \cos x$)

$$\begin{aligned}\int \sin^3 x\, dx = \int (1-\cos^2 x)\sin x\, dx &= -\int (1-t^2)dt\\ &= -t + \frac{1}{3}t^3 + C = -\cos x + \frac{1}{3}\cos^3 x + C.\end{aligned}$$

(h)

$$\begin{aligned}\int \cos^3 x dx &= \int \cos^2 x d(\sin x) = \int (1-\sin^2 x)d(\sin x)\\ &= \sin x - \frac{1}{3}\sin^3 x + C\ .\end{aligned}$$

(i)

$$\begin{aligned}\int \sin^4 x dx &= \int (\frac{1-\cos 2x}{2})^2 dx = \int (\frac{1}{4} - \frac{\cos 2x}{2} + \frac{\cos^2 2x}{4}) dx \\ &= \int (\frac{1}{4} - \frac{\cos 2x}{2} + \frac{1}{4}\frac{1+\cos 4x}{2}) dx \\ &= \int (\frac{3}{8} - \frac{\cos 2x}{2} + \frac{\cos 4x}{8}) dx \\ &= \frac{3}{8}x - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C \ .\end{aligned}$$

(j)

$$\begin{aligned}\int \cos^4 x dx &= \int (\frac{1+\cos 2x}{2})^2 dx = \int (\frac{1}{4} + \frac{\cos 2x}{2} + \frac{\cos^2 2x}{4}) dx \\ &= \int (\frac{1}{4} + \frac{\cos 2x}{2} + \frac{1}{4}\frac{1+\cos 4x}{2}) dx \\ &= \int (\frac{3}{8} + \frac{\cos 2x}{2} + \frac{\cos 4x}{8}) dx \\ &= \frac{3}{8}x + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C \ .\end{aligned}$$

(k)

$$\begin{aligned}\int \cot^2 x dx &= \int \frac{\cos^2 x}{\sin^2 x} dx = \int (-1 + \frac{1}{\sin^2 x}) dx \\ &= -x - \cot x + C \ .\end{aligned}$$

(l)

$$\int \tan^2 x dx = \int (\frac{1}{\cos^2 x} - 1) dx = -x + \tan x + C \ .$$

(m)

$$\begin{aligned}&\int \sin^2 3x \sin^3 2x dx = \int \frac{1-\cos 6x}{2}\frac{1-\cos 4x}{2} \sin 2x dx \\ &= \frac{1}{4}\int (1 - \cos 6x - \cos 4x + \cos 6x \cos 4x) \sin 2x dx \\ &= \frac{1}{4}\int (\sin 2x - \cos 6x \sin 2x - \cos 4x \sin 2x + \frac{\cos 10x + \cos 2x}{2} \sin 2x) dx \\ &= \frac{1}{4}\int (\sin 2x - \frac{\sin 8x - \sin 4x}{2} - \frac{\sin 6x - \sin 2x}{2} + \frac{\sin 12x - \sin 8x}{4} + \frac{\sin 4x}{4}) dx \\ &= \int (\frac{3}{8}\sin 2x + \frac{3}{16}\sin 4x - \frac{1}{8}\sin 6x - \frac{3}{16}\sin 8x + \frac{1}{16}\sin 2x) dx \\ &= -\frac{3}{8}\cos 2x - \frac{3}{64}\cos 4x + \frac{1}{48}\cos 6x + \frac{3}{128}\cos 8x - \frac{1}{192}\cos 12x + C \ .\end{aligned}$$

(n)

$$\begin{aligned}\int \frac{dx}{\sin^2 x \cos^2 x} &= \int \frac{4dx}{(2\sin x \cos x)^2} \\ &= 2\int \frac{d(2x)}{\sin^2 x} = -2\cot 2x + C \ .\end{aligned}$$

(o)*: (Substitution $t = \sin x$)

$$\begin{aligned}
\int \frac{dx}{\sin^2 x\, \cos x} &= \int \frac{\cos x\, dx}{\sin^2 x(1-\sin^2 x)} \\
&= \int \frac{dt}{t^2(1-t)(1+t)} = \int \left(\frac{1}{t^2} - \frac{\frac{1}{2}}{t-1} + \frac{\frac{1}{2}}{t+1}\right) dt \\
&= -\frac{1}{t} + \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| + C \\
&= -\frac{1}{\sin x} + \frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x-1}\right| + C \\
&= -\csc x + \frac{1}{2}\ln\left|\frac{(\sin x+1)^2}{\cos^2 x}\right| + C \\
&= -\csc x + \ln|\sec x + \tan x| + C.
\end{aligned}$$

(p) notice that from **5.7.4.** (b)

$$\int \frac{dx}{\sin x} = -\operatorname{arctanh}(\cos x) + C\ .$$

Therefore,

$$\begin{aligned}
&\int \frac{dx}{\sin x \cos^3 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x} dx \\
&= \int \frac{\sin x}{\cos^3 x} dx + \int \frac{d(2x)}{\sin 2x} \\
&= -\int (\cos x)^{-3} d(\cos x) - \operatorname{arctanh}(\cos 2x) \\
&= \frac{1}{2\cos^2 x} - \operatorname{arctanh}(\cos 2x) + C\ .
\end{aligned}$$

(q)

$$\begin{aligned}
&\int \frac{\cos^3 x}{\sin x} dx = \int \frac{(1-\sin^2 x)\cos x}{\sin x} dx \\
&= \int \frac{\cos x}{\sin x} dx - \int \sin x \cos x dx \\
&= \int \frac{d(\sin x)}{\sin x} - \frac{1}{4}\int \sin 2x d(2x) \\
&= \ln(\sin x) + \frac{1}{4}\cos 2x + C\ .
\end{aligned}$$

(r)

$$\begin{aligned}
&\int \frac{dx}{\cos^4 x} = \int \frac{\sin^2 x + \cos^2 x}{\cos^4 x} dx = \int \frac{\sin^2 x}{\cos^4 x} dx + \int \frac{1}{\cos^2 x} dx \\
&= \int \tan^2 x \frac{dx}{\cos^2 x} + \int \frac{dx}{\cos^2 x} = \int \tan^2 x d(\tan x) + \tan x \\
&= \frac{1}{3}\tan^3 x + \tan x + C\ .
\end{aligned}$$

(s)

$$\int \frac{dx}{1+e^x} = \int (1 - \frac{e^x}{1+e^x})dx$$
$$= x - \int \frac{d(e^x)}{1+e^x} = x - \ln(1+e^x) + C \ .$$

(t)

$$\int \frac{(1+e^x)^2}{1+e^{2x}}dx = \int \frac{1+e^{2x}+2e^x}{1+e^{2x}}dx$$
$$= \int (1 + 2\frac{e^x}{1+(e^x)^2})dx = x + 2\int \frac{d(e^x)}{1+(e^x)^2}$$
$$= x + 2\mathrm{arctan}\,(e^x) + C \ .$$

(u)

$$\int \sinh^2 x dx = \int (\frac{e^x - e^{-x}}{2})^2 dx = \int \frac{e^{2x}+e^{-2x}-2}{4}dx$$
$$= \int (-\frac{1}{2} + \frac{1}{2}\cosh 2x)dx$$
$$= -\frac{1}{2}x + \frac{1}{4}\sinh 2x + C \ .$$

(v)

$$\int \cosh^2 x dx = \int (\frac{e^x + e^{-x}}{2})^2 dx$$
$$= \int \frac{e^{2x}+e^{-2x}+2}{4}dx = \int (\frac{1}{2} + \frac{1}{2}\cosh 2x)dx$$
$$= \frac{1}{2}x + \frac{1}{4}\sinh 2x + C \ .$$

(w)

$$\int \sinh x \sinh 2x dx = \int \frac{\cosh 3x - \cosh x}{2}dx$$
$$= \frac{1}{6}\sinh 3x - \frac{1}{2}\sinh x + C \ .$$

(x)

$$\int \cosh x \cosh 3x dx = \int \frac{\cosh 4x + \cosh 2x}{2}dx$$
$$= \frac{1}{8}\sinh 4x + \frac{1}{4}\sinh 2x + C \ .$$

(y)

$$\int \frac{dx}{\sinh^2 x \cosh^2 x} = \int \frac{\cosh^2 x - \sinh^2 x}{\sinh^2 x \cosh^2 x}dx$$
$$= \int \frac{1}{\sinh^2 x}dx - \int \frac{1}{\cosh^2 x}dx = -\coth x - \tanh x + C \ .$$

5.7.7. (a) (substitution $(1-x)^{\frac{1}{3}} = y$, $x = 1-y^3$, $dx = -3y^2dy$)

$$\begin{aligned}\int x^2\sqrt[3]{1-x}dx &= \int (1-y^3)^2 y(-3y^2)dy \\ &= \int (-3y^3+6y^6-3y^9)dy \\ &= -\frac{3}{10}y^{10}+\frac{6}{7}y^7-\frac{3}{4}y^4+C \\ &= -\frac{3}{10}(\sqrt[3]{1-x})^{10}+\frac{6}{7}(\sqrt[3]{1-x})^7-\frac{3}{4}(\sqrt[3]{1-x})^4+C\ .\end{aligned}$$

(b) (substitution $1-5x^2 = y$)

$$\begin{aligned}\int x^3(1-5x^2)^{10}dx &= -\frac{1}{10}\int(-5x^2)(1-5x^2)^{10}d(x^2) \\ &= \frac{1}{50}\int(-5x^2)(1-5x^2)^{10}d(1-5x^2) \\ &= \frac{1}{50}\left(\int(1-5x^2)^{11}d(1-5x^2)-\int(1-5x^2)^{10}d(1-5x^2)\right) \\ &= \frac{1}{50}\int y^{11}dy-\int y^{10}dy \\ &= \frac{y^{12}}{600}-\frac{y^{11}}{550}+C = \frac{(1-5x^2)^{12}}{600}-\frac{(1-5x^2)^{11}}{550}+C\ .\end{aligned}$$

(c) (substitution $\sqrt{2-x} = y$, $x = 2-y^2$)

$$\begin{aligned}\int \frac{x^2}{\sqrt{2-x}}dx &= \int\frac{(2-y^2)^2}{y}(-2y)dy \\ &= -2\int(4-4y^2+y^4)dy = -8y+\frac{8}{3}y^3-\frac{2}{5}y^5+C \\ &= -8\sqrt{2-x}+\frac{8}{3}(\sqrt{2-x})^3-\frac{5}{2}(\sqrt{2-x})^5+C\ .\end{aligned}$$

(d) (substitution $\sqrt{1-x^2} = y$, $x^2 = 1-y^2$)

$$\begin{aligned}\int \frac{x^5}{\sqrt{1-x^2}}dx &= \frac{1}{2}\int\frac{x^4d(x^2)}{\sqrt{1-x^2}} = \frac{1}{2}\int\frac{(1-y^2)^2}{y}(-2y)dy \\ &= -\int(1-2y^2+y^4)dy = -y+\frac{2}{3}y^3-\frac{1}{5}y^5+C \\ &= -\sqrt{1-x^2}+\frac{2}{3}(1-x^2)^{\frac{3}{2}}-\frac{1}{5}(1-x^2)^{\frac{5}{2}}+C\ .\end{aligned}$$

(e) (substitution $(2-5x^3)^{\frac{1}{3}} = y$, $x^3 = \frac{2-y^3}{5}$)

$$\begin{aligned}\int x^5(2-5x^3)^{\frac{2}{3}}dx &= \frac{1}{3}\int x^3(2-5x^3)^{\frac{2}{3}}d(x^3) = \frac{1}{3}\int\frac{2-y^3}{5}y^2(-\frac{3}{5}y^2)dy \\ &= -\frac{1}{25}\int(2y^4-y^7)dy = -\frac{2}{125}y^5+\frac{1}{200}y^8+C \\ &= -\frac{2}{125}(2-5x^3)^{\frac{5}{3}}+\frac{1}{200}(2-5x^3)^{\frac{8}{3}}+C\ .\end{aligned}$$

(f) (substitution $\sqrt{\sin x} = y$)

$$\int \cos^5 x\sqrt{\sin x}dx = \int \cos^4 \sqrt{\sin x}d(\sin x) = \int (1-\sin^2 x)^2\sqrt{\sin x}d(\sin x)$$
$$= \int (1-y^4)^2 y2ydy = \int (2y^2 - 4y^6 + 2y^{10})dy$$
$$= \frac{2}{3}y^3 - \frac{4}{7}y^7 + \frac{2}{11}y^{11} + C$$
$$= \frac{2}{3}(\sin x)^{\frac{3}{2}} - \frac{4}{7}(\sin x)^{\frac{7}{2}} + \frac{2}{11}(\sin x)^{\frac{11}{2}} + C\ .$$

(g) (substitution $\cos x = y$)

$$\int \frac{\sin x\cos^3 x}{1+\cos^2 x}dx = -\int \frac{\cos^3 xd(\cos x)}{1+\cos^3 x} = -\int \frac{y^3dy}{1+y^2}$$
$$= \int (-y + \frac{y}{1+y^2})dy = -\frac{1}{2}y^2 + \frac{1}{2}\int \frac{d(y^2)}{1+y^2}$$
$$= -\frac{1}{2}y^2 + \frac{1}{2}\ln(1+y^2) + C$$
$$= -\frac{1}{2}\cos^2 x + \frac{1}{2}\ln(1+\cos^2 x) + C\ .$$

(h) (substitution $\tan x = y$)

$$\int \frac{\sin^2 x}{\cos^6 x}dx = \int \tan^2 x\frac{1}{\cos^4 x}dx = \int \tan^2 x\frac{\sin^2 x + \cos^2 x}{\cos^2 x}\frac{1}{\cos^2 x}dx$$
$$= \int \tan^2 x(1+\tan^2 x)d(\tan x) = \int (y^2 + y^4)dy$$
$$= \frac{1}{3}y^3 + \frac{1}{5}y^5 + C = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C\ .$$

(i) (substitution $y = \ln x + 1$)

$$\int \frac{dx}{x\sqrt{1+\ln x}} = \int \frac{d(\ln x + 1)}{\sqrt{\ln x + 1}} = \int y^{-\frac{1}{2}}dy$$
$$= 2\sqrt{y} + C = 2\sqrt{1+\ln x} + C\ .$$

(j) (substitution $e^{\frac{x}{2}} = y$)

$$\int \frac{dx}{e^{\frac{x}{2}} + e^x} = \int \frac{e^{\frac{x}{2}}dx}{e^{\frac{x}{2}}(e^{\frac{x}{2}} + (e^{\frac{x}{2}})^2)} = 2\int \frac{dy}{y(y+y^2)}$$
$$= 2\int (-\frac{1}{y} + \frac{1}{y^2} + \frac{1}{1+y})dy = 2(-\ln y - \frac{1}{y} + \ln(1+y)) + C$$
$$= -x - \frac{2}{e^{\frac{x}{2}}} + 2\ln(1+e^{\frac{x}{2}}) + C\ .$$

(k) (substitution $\sqrt{1+e^x} = y$, $e^x = y^2 - 1$)

$$\int \frac{dx}{\sqrt{1+e^x}} = \int \frac{e^x dx}{e^x\sqrt{1+e^x}} = \int \frac{d(e^x)}{e^x\sqrt{1+e^x}}$$
$$= \int \frac{d(y^2-1)}{(y^2-1)y} = -2\int \frac{dy}{1-y^2}$$
$$= -2\text{arctanh}\, y + C = -2\text{arctanh}\,(\sqrt{1+e^x}) + C\ .$$

(l) (substitution $\sqrt{x} = y$)

$$\begin{aligned}
\int \frac{\arctan(\sqrt{x})}{\sqrt{x}} \frac{1}{1+x} dx &= 2 \int \frac{\arctan(\sqrt{x}) d(\sqrt{x})}{1+(\sqrt{x})^2} = 2 \int \frac{\arctan y dy}{1+y^2} \\
&= 2 \int \arctan y d(\arctan y) = \arctan^2 y + C \\
&= \arctan^2(\sqrt{x}) + C \ .
\end{aligned}$$

5.7.8. (a) (substitution $x = \cos t$, $dx = -\sin t$, $t = \arccos x$)

$$\begin{aligned}
\int \frac{dx}{(1-x^2)^{\frac{3}{2}}} &= \int \frac{-\sin t dx}{\sin^3 t} = -\int \frac{dt}{\sin^2 t} = \cot t + C \\
&= \frac{\cos(\arccos x)}{\sin(\arccos x)} + C = \frac{x}{\sqrt{1-x^2}} + C \ .
\end{aligned}$$

(b) (substitution $x = \frac{\sqrt{2}}{\sin t}$ and $y = \cos t$)

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^2-2}} dx &= \int \frac{\frac{2}{\sin^2 t}(-\sqrt{2})\frac{\cos t}{\sin^2 t}}{\sqrt{2}\frac{\cos t}{\sin t}} dt = -2 \int \frac{1}{\sin^3 t} dt \\
&= 2 \int \frac{d(\cos t)}{\sin^4 t} = 2 \int \frac{dy}{(1-y^2)^2} \\
&= 2 \int (\frac{1}{4}\frac{1}{(1-y)^2} - \frac{1}{4}\frac{-1}{1-y} + \frac{1}{4}\frac{1}{(1+y)^2} + \frac{1}{4}\frac{1}{1+y}) dy \\
&= 2(\frac{1}{4}\ln(\frac{1+y}{1-y}) + \frac{1}{2}\frac{y}{1-y^2}) + C \\
&= \frac{1}{2}\ln(\frac{1+\cos t}{1-\cos t}) + \frac{\cos t}{\sin^2 t} + C \\
&= \frac{1}{2}\ln(\frac{1+\cos(\arcsin(\frac{\sqrt{2}}{x}))}{1-\cos(\arcsin(\frac{\sqrt{2}}{x}))}) + \frac{x^2\cos(\arcsin(\frac{\sqrt{2}}{x}))}{2} + C \ .
\end{aligned}$$

(c) (substitution $x = |a| \sin t$)

$$\begin{aligned}
I = \int \sqrt{a^2-x^2} dx &= \int \sqrt{a^2 - a^2\sin^2 t} d(|a|\sin t) \\
&= a^2 \int \cos t d(\sin t) = a^2 \cos t \sin t + a^2 \int \sin^2 t dt \\
&= a^2 \cos t \sin t + a^2 t - I \ .
\end{aligned}$$

That is

$$\begin{aligned}
I &= \frac{a^2}{2}(\cos t \sin t + t) + C \\
&= a^2(\frac{\sin 2t}{4} + \frac{t}{2}) + C \\
&= a^2(\frac{\sin(2\arcsin(\frac{x}{|a|}))}{4} + \frac{\arcsin(\frac{x}{|a|})}{2}) + C \ .
\end{aligned}$$

(d) (substitution $x = |a| \tan t$)

$$\begin{aligned}
\int \frac{dx}{(\sqrt{a^2+x^2})^3} &= \int \frac{|a|(1+\tan^2 t)}{|a|^3(\sqrt{1+\tan^2 t})^3}dt \\
&= \frac{1}{a^2}\int \frac{1}{\sqrt{1+\tan^2 t}}dt = \frac{1}{a^2}\int \frac{1}{\sqrt{\frac{\sin^2 t+\cos^2 t}{\cos^2 t}}}dt \\
&= \frac{1}{a^2}\int \cos t dt = \frac{1}{a^2}\sin t + C \\
&= \frac{1}{a^2}\sin(\arctan(\frac{x}{|a|})) + C \ .
\end{aligned}$$

(e) (substitution $x = a\cos t$)

$$\begin{aligned}
\int \sqrt{\frac{a+x}{a-x}}dx &= \int -a\sin t\sqrt{\frac{a+a\cos t}{a-a\cos t}}dt \\
&= -a\int 2\sin\frac{t}{2}\cos\frac{t}{2}\sqrt{\frac{2\cos^2\frac{t}{2}}{2\sin^2\frac{t}{2}}}dt \\
&= -a\int 2\cos^2\frac{t}{2}dt = -a\int(1+\cos t)dt \\
&= -a(t+\sin t) + C \\
&= -a(\arccos\frac{x}{a} + \sin(\arccos\frac{x}{a})) + C \ .
\end{aligned}$$

(f) (substitution $x = 2a\cos^2 t$, thus $dx = -4a\sin t\cos t$)

$$\begin{aligned}
\int x\sqrt{\frac{x}{2a-x}}dx &= \int 2a\cos^2 t\sqrt{\frac{2a\cos^2 t}{2a-2a\cos^2 t}}(-4a\sin t\cos t)dt \\
&= -8a^2\int \cos^3 t\sin t\frac{\cos t}{\sin t}dt \\
&= -8a^2\int \cos^4 t dt \ .
\end{aligned}$$

Therefore, by **5.7.6.** (j).

$$\begin{aligned}
&\int x\sqrt{\frac{x}{2a-x}}dx \\
&= -8a^2(\frac{3}{8}t + \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t) + C \\
&= -a^2(3t + 2\sin 2t + \frac{1}{4}\sin 4t) + C \\
&= -a^2(3\arccos\sqrt{\frac{x}{2a}} + 2\sin(2\arccos\sqrt{\frac{x}{2a}}) + \frac{1}{4}\sin(4\arccos\sqrt{\frac{x}{2a}})) + C \ .
\end{aligned}$$

(g) (substitution $x - a = (b-a)\sin^2 t$)

$$\begin{aligned}
\int \frac{dx}{\sqrt{(x-a)(b-x)}} &= \int \frac{2(b-a)\sin t\cos t dt}{\sqrt{(b-a)\sin^2 t((b-a)-(b-a)\sin^2 t)}} \\
&= \int \frac{2(b-a)\sin t\cos t dt}{(b-a)\sin t\cos t} = \int 2dt = 2t + C \\
&= 2\arcsin\sqrt{\frac{x-a}{b-a}} + C \ .
\end{aligned}$$

(h) (substitution $x-a=(b-a)\sin^2 t$)

$$\begin{aligned}
\int \sqrt{(x-a)(b-x)}dx &= \int 2(b-a)\sin t\cos t(b-a)\sin t\cos t dt \\
&= \frac{(b-a)^2}{2}\int \sin^2 2t dt = \frac{(b-a)^2}{4}\int (1-\cos 4t)dt \\
&= \frac{(b-a)^2}{4}(4-\frac{\sin 4t}{4})+C \\
&= \frac{(b-a)^2}{4}\text{arccos}\sqrt{\frac{x-a}{b-a}} - \frac{(b-a)^2}{16}\sin(4\text{arccos}\sqrt{\frac{x-a}{b-a}})+C \ .
\end{aligned}$$

5.7.9. (a) (substitution $x=a\sinh t$)

$$\begin{aligned}
\int \sqrt{a^2+x^2}dx &= \int \sqrt{a^2+a^2\sinh^2 t}d(a\sinh t) \\
&= a^2\int \cosh^2 t dt = a^2\int \frac{1+\cosh 2t}{2}dt \\
&= a^2(\frac{t}{2}+\frac{1}{4}\sinh 2t)+C \\
&= \frac{a^2}{2}(\arctan\frac{x}{a}+\frac{1}{2}\sinh(2\arctan\frac{x}{a}))+C \ .
\end{aligned}$$

(b) (substitution $x=a\sinh t$)

$$\begin{aligned}
\int \frac{x^2}{\sqrt{a^2+x^2}}dx &= \int \frac{a^2\sinh^2 t}{\sqrt{a^2+a^2\sinh^2 t}}d(a\sinh t) \\
&= a^2\int \sinh^2 t dt = a^2\int \frac{\cosh 2t-1}{2}dt \\
&= \frac{a^2}{2}(\frac{\sinh 2t}{2}-t)+C \\
&= \frac{a^2}{2}(\frac{\sinh(2\arctan\frac{x}{a})}{2}-\arctan\frac{x}{a})+C \ .
\end{aligned}$$

(c) (substitution $x=a\cosh t$)

$$\begin{aligned}
\int \sqrt{\frac{x-a}{x+a}}dx &= \int \sqrt{\frac{a(\cosh t-1)}{a(\cosh t+1)}}d(a\cosh t) \\
&= a\int \sinh t\sqrt{\frac{2\sinh^2\frac{t}{2}}{2\cosh^2\frac{t}{2}}}dt = a\int 2\sinh\frac{t}{2}\cosh\frac{t}{2}\frac{\sinh\frac{t}{2}}{\cosh\frac{t}{2}}dt \\
&= 4a\int \sinh^2\frac{t}{2}d(\frac{t}{2}) \ .
\end{aligned}$$

Now from **5.7.9.** (b) $\int \sinh^2 t dt = \frac{\sinh 2t}{4}-\frac{t}{2}+C$, we have

$$\begin{aligned}
4a\int \sinh^2\frac{t}{2}d(\frac{t}{2}) &= 4a(\frac{\sinh t}{4}-\frac{t}{4})+C \\
&= a\sinh t - at + C \ .
\end{aligned}$$

Therefore

$$\begin{aligned}\int \sqrt{\frac{x-a}{x+a}}dx &= a\sinh t - at + C \\ &= a\sinh(\arctan\frac{x}{a}) - a\arctan\frac{x}{a} + C\ .\end{aligned}$$

(d) (substitution $x+a=(b-a)\sinh^2 t$)

$$\begin{aligned}&\int \frac{dx}{\sqrt{(x+a)(x+b)}} = \int \frac{dx}{\sqrt{(x+a)((x+a)+(b-a))}} \\ &= \int \frac{2(b-a)\sinh t\cosh t dt}{\sqrt{(b-a)\sinh^2 t((b-a)(1+\sinh^2 t))}} = \int \frac{2(b-a)\sinh t\cosh t}{(b-a)\sinh t\cosh t}dt \\ &= \int 2dt = 2t + C = 2\arctan\sqrt{\frac{x+a}{b-a}} + C\ .\end{aligned}$$

(e) (substitution $x+a=(b-a)\sinh^2 t$)

$$\begin{aligned}&\int \sqrt{(x+a)(x+b)}dx \\ &= \int \sqrt{(b-a)\sinh^2 t((b-a)(1+\sinh^2 t))}2(b-a)\sinh t\cosh t dt \\ &= 2(b-a)^2\int \sinh^2 t\cosh^2 t dt = \frac{(b-a)^2}{2}\int \sinh^2 2t dt \\ &= \frac{(b-a)^2}{2}\int \frac{\cosh 4t - 1}{2}dt \\ &= \frac{(b-a)^2}{2}(\frac{\sinh 4t}{8} - \frac{t}{2}) + C \\ &= \frac{(b-a)^2}{16}\sinh(4\arctan\sqrt{\frac{x+a}{b-a}}) - \frac{(b-a)^2}{4}\arctan\sqrt{\frac{x+a}{b-a}} + C\ .\end{aligned}$$

5.7.10. (a)

$$\begin{aligned}\int xe^{-x}dx &= -\int xd(e^{x}) \\ &= -xe^{x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C\ .\end{aligned}$$

(b)

$$\begin{aligned}&\int x^2e^{-2x}dx = -\frac{1}{2}\int x^2 d(e^{-2x}) = -\frac{1}{2}x^2e^{-2x} + \frac{1}{2}\int e^{-2x}d(x^2) \\ &= -\frac{1}{2}x^2e^{-2x} + \int xe^{-2x}dx = -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xd(e^{-2x}) \\ &= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx \\ &= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C\ .\end{aligned}$$

(c)

$$\begin{aligned}\int \ln x dx &= x\ln x - \int x d(\ln x) = x\ln x - \int x\frac{1}{x}dx \\ &= x\ln x - x + C\ .\end{aligned}$$

(d)

$$\begin{aligned}\int x^n \ln x dx &= \int \frac{\ln x d(x^{n+1})}{n+1} \\ &= \frac{x^{n+1}\ln x}{n+1} - \frac{1}{n+1}\int x^{n+1}d(\ln x) \\ &= \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\int x^n dx \\ &= \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1} + C \\ &= \frac{x^{n+1}}{n+1}(\ln x - \frac{1}{n+1}) + C\ .\end{aligned}$$

(e)

$$\begin{aligned}\int (\frac{\ln x}{x})^2 dx &= -\int \ln^2 x d(\frac{1}{x}) = -\frac{\ln^2 x}{x} + \int \frac{1}{x}d(\ln^2 x) \\ &= -\frac{\ln^2 x}{x} + \int \frac{2\ln x}{x^2}dx = -\frac{\ln^2 x}{x} - 2\int \ln x d(\frac{1}{x}) \\ &= -\frac{\ln^2 x}{x} - \frac{2\ln x}{x} + 2\int \frac{1}{x}d(\ln x) \\ &= -\frac{\ln^2 x}{x} - 2\frac{\ln x}{x} + 2\int \frac{1}{x^2}dx \\ &= -\frac{\ln^2 x}{x} - 2\frac{\ln x}{x} - \frac{2}{x} + C\ .\end{aligned}$$

(f)

$$\begin{aligned}\int \sqrt{x}\ln^2 x dx &= \frac{2}{3}\int \ln^2 x d(x^{\frac{3}{2}}) = \frac{2}{3}x^{\frac{3}{2}}\ln^2 x - \frac{2}{3}\int x^{\frac{3}{2}}d(\ln^2 x) \\ &= \frac{2}{3}x^{\frac{3}{2}}\ln^2 x - \frac{4}{3}\int \sqrt{x}\ln x dx \\ &= \frac{2}{3}x^{\frac{3}{2}}\ln^2 x - \frac{8}{9}\int \ln x d(x^{\frac{3}{2}}) \\ &= \frac{2}{3}x^{\frac{3}{2}}\ln^2 x - \frac{8}{9}x^{\frac{3}{2}}\ln x + \frac{8}{9}\int x^{\frac{3}{2}}d(\ln x) \\ &= \frac{2}{3}x^{\frac{3}{2}}\ln^2 x - \frac{8}{9}x^{\frac{3}{2}}\ln x + \frac{8}{9}\int \sqrt{x}dx \\ &= \frac{2}{3}x^{\frac{3}{2}}\ln^2 x - \frac{8}{9}x^{\frac{3}{2}}\ln x + \frac{16}{27}x^{\frac{3}{2}} + C \\ &= x^{\frac{3}{2}}(\frac{2}{3}\ln^2 x - \frac{8}{9}\ln x + \frac{16}{27}) + C\ .\end{aligned}$$

(g)

$$\int x \cos x dx = \int x d(\sin x) = x \sin x - \int \sin x dx$$
$$= x \sin x + \cos x + C \ .$$

(h)

$$\int x^2 \sin 2x dx = -\frac{1}{2} \int x^2 d(\cos 2x) = -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} \int \cos(2x) \cdot 2x dx$$
$$= -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} \int x d(\sin 2x)$$
$$= -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x - \frac{1}{2} \int \sin 2x dx$$
$$= -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + C \ .$$

(i)

$$\int x \sinh x dx = \int x d(\cosh x) = x \cosh x - \int \cosh x dx$$
$$= x \cosh x - \sinh x + C \ .$$

(j)

$$\int x^3 \cosh 3x dx dx = \frac{1}{3} \int x^3 d(\sinh 3x) = \frac{1}{3} x^3 \sinh 3x - \int \sinh(3x) \cdot x^2 dx$$
$$= \frac{1}{3} x^3 \sinh 3x - \frac{1}{3} \int x^2 d(\cosh 3x)$$
$$= \frac{1}{3} x^3 \sinh 3x - \frac{1}{3} x^2 \cosh 3x + \frac{2}{3} \int \cosh(3x) \cdot x dx$$
$$= \frac{1}{3} x^3 \sinh 3x - \frac{1}{3} x^2 \cosh 3x + \frac{2}{9} x \sinh 3x - \frac{2}{9} \sinh 3x dx$$
$$= \frac{1}{3} x^3 \sinh 3x - \frac{1}{3} x^2 \cosh 3x + \frac{2}{9} x \sinh 3x - \frac{2}{27} \cosh 3x + C \ .$$

(k)

$$\int \arctan x dx = x \arctan x - \int x d(\arctan x)$$
$$= x \arctan x - \int \frac{x}{x^2+1} dx$$
$$= x \arctan x - \frac{1}{2} \int \frac{d(x^2)}{x^2+1}$$
$$= x \arctan x - \frac{1}{2} \ln(x^2+1) + C \ .$$

(l)

$$\int \arcsin x dx = x \arcsin x - \int x d(\arcsin x) = x \arcsin x - \frac{1}{2} \int \frac{d(x^2)}{\sqrt{1-x^2}}$$
$$= x \arcsin x + \frac{1}{2} \int \frac{d(1-x^2)}{\sqrt{1-x^2}} = x \arcsin x + \sqrt{1-x^2} + C \ .$$

(m)

$$\int x\text{arctan}\,x dx = \frac{1}{2}x^2\text{arctan}\,x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx$$
$$= \frac{1}{2}x^2\text{arctan}\,x - \frac{1}{2}x + \frac{1}{2}\text{arctan}\,x + C\ .$$

(n)

$$\int x^2\text{arccos}\,x dx = \frac{1}{3}x^3\text{arccos}\,x - \frac{1}{3}\int x^3 d(\text{arccos}\,x)$$
$$= \frac{1}{3}x^3\text{arccos}\,x + \frac{1}{3}\frac{1}{2}\int \frac{x^2 d(x^2)}{\sqrt{1-x^2}}$$
$$= \frac{1}{3}x^3\text{arccos}\,x - \frac{1}{6}\int \sqrt{1-x^2}d(x^2) + \frac{1}{6}\int \frac{d(x^2)}{\sqrt{1-x^2}}$$
$$= \frac{1}{3}x^3\text{arccos}\,x - \frac{1}{6}\frac{2}{3}(1-x^2)^{\frac{3}{2}} - \frac{1}{6}2\sqrt{1-x^2} + C$$
$$= \frac{1}{3}x^3\text{arccos}\,x - \frac{x^2}{9}\sqrt{1-x^2} - \frac{2}{9}\sqrt{1-x^2} + C\ .$$

(o)

$$\int \frac{\arcsin x}{x^2}dx = -\int \arcsin x d(\frac{1}{x}) = -\frac{\arcsin x}{x} + \int \frac{d(\arcsin x)}{x}$$
$$= -\frac{\arcsin x}{x} + \int \frac{dx}{x\sqrt{1-x^2}} = -\frac{\arcsin x}{x} - \int \frac{d(\frac{1}{x})}{\sqrt{(\frac{1}{x})^2 - 1}}$$
$$= -\frac{\arcsin x}{x} - \text{arctan}\,(\frac{1}{x}) + C\ .$$

(p)

$$\int \ln(x+\sqrt{1+x^2})dx = x\ln(x+\sqrt{1+x^2}) - \int xd(\ln(x+\sqrt{1+x^2}))$$
$$= x\ln(x+\sqrt{1+x^2}) - \int \frac{x(1+\frac{x}{\sqrt{1+x^2}})}{x+\sqrt{1+x^2}}dx$$
$$= x\ln(x+\sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}}dx$$
$$= x\ln(x+\sqrt{1+x^2}) - \int \frac{d(x^2+1)}{2\sqrt{1+x^2}}$$
$$= x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C\ .$$

(q)

$$\int x\ln(\frac{1+x}{1-x})dx = \frac{1}{2}x^2\ln(\frac{1+x}{1-x}) - \frac{1}{2}\int x^2 d(\ln(\frac{1+x}{1-x}))$$
$$= \frac{1}{2}x^2\ln(\frac{1+x}{1-x}) - \frac{1}{2}\int x^2\frac{1-x}{1+x}(\frac{1}{1-x} + \frac{x+1}{(1-x)^2})dx$$
$$= \frac{1}{2}x^2\ln(\frac{1+x}{1-x}) + x - \int \frac{1}{1-x^2}dx$$
$$= \frac{1}{2}x^2\ln(\frac{1+x}{1-x}) + x - \text{arctanh}\,(x) + C\ .$$

(r)

$$\begin{aligned}\int \arctan\sqrt{x}dx &= x\arctan\sqrt{x} - \int xd(\arctan\sqrt{x})\\ &= x\arctan\sqrt{x} - \int x\frac{1}{1+x}\frac{1}{2\sqrt{x}}dx\\ &= x\arctan\sqrt{x} - \frac{1}{2}\int\frac{\sqrt{x}}{1+(\sqrt{x})^2}dx\\ &= x\arctan\sqrt{x} - \frac{1}{2}\int(\frac{-1}{\sqrt{x}(1+x)} + \frac{1}{\sqrt{x}})dx\\ &= x\arctan\sqrt{x} - \sqrt{x} + \int\frac{d(\sqrt{x})}{1+(\sqrt{x})^2}\\ &= x\arctan\sqrt{x} - \sqrt{x} + \arctan\sqrt{x} + C\ .\end{aligned}$$

(s) Notice that by **5.7.4.** (b), $\int \frac{1}{\sin x}dx = \ln\frac{1-\cos x}{\sin x}$, we have

$$\begin{aligned}\int \sin x\ln(\tan x)dx &= -\int \ln(\tan x)d(\cos x)\\ &= -\cos x\ln(\tan x) + \int \cos x d(\ln(\tan x))\\ &= -\cos x\ln(\tan x) + \int \cos x\frac{1+\tan^2 x}{\tan x}dx\\ &= -\cos x\ln(\tan x) + \int\frac{1}{\sin x}dx\\ &= -\cos x\ln(\tan x) + \ln\frac{1-\cos x}{\sin x} + C\ .\end{aligned}$$

5.7.11. (a) (substitution $x^3 = y$)

$$\begin{aligned}\int x^5e^{x^3}dx &= \frac{1}{3}\int x^3e^{x^3}d(x^3) = \frac{1}{3}\int ye^ydy\\ &= \frac{1}{3}\int yd(e^y) = \frac{1}{3}ye^y - \frac{1}{3}\int e^ydy\\ &= \frac{1}{3}ye^y - \frac{1}{3}e^y + C = \frac{1}{3}e^y(y-1) + C\\ &= \frac{1}{3}e^{x^3}(x^3-1) + C\ .\end{aligned}$$

(b)

$$\begin{aligned}
\int \arcsin^2 x dx &= x\arcsin^2 x - \int x d(\arcsin^2 x) \\
&= x\arcsin^2 x - 2\int \frac{x}{\sqrt{1-x^2}} \arcsin x dx \\
&= x\arcsin^2 x + 2\int \arcsin x d(\sqrt{1-x^2}) \\
&= x\arcsin^2 x + 2\sqrt{1-x^2}\arcsin x - 2\int \sqrt{1-x^2} d(\arcsin x) \\
&= x\arcsin^2 x + 2\sqrt{1-x^2}\arcsin x - 2\int \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} dx \\
&= x\arcsin^2 x + 2\sqrt{1-x^2}\arcsin x - 2x + C \ .
\end{aligned}$$

(c)

$$\begin{aligned}
\int x\arctan^2 x dx &= \frac{1}{2}x^2\arctan^2 x - \frac{1}{2}\int x^2 d(\arctan^2 x) \\
&= \frac{1}{2}x^2\arctan^2 x - \int \frac{x^2}{1+x^2}\arctan x dx \\
&= \frac{1}{2}x^2\arctan^2 x - x\arctan x + \int \frac{x}{1+x^2}dx + \int \arctan x d(\arctan x) \\
&= \frac{1}{2}x^2\arctan^2 x - x\arctan x + \frac{1}{2}\ln(1+x^2) + \frac{1}{2}\arctan^2 x + C \ .
\end{aligned}$$

(d)

$$\begin{aligned}
\int x^2 \ln\frac{1-x}{1+x}dx &= \frac{1}{3}x^3\ln\frac{1-x}{1+x} - \frac{1}{3}\int x^3 d(\ln\frac{1-x}{1+x}) \\
&= \frac{1}{3}x^3\ln\frac{1-x}{1+x} + \frac{1}{3}\int \frac{2x^3}{1-x^2}dx \\
&= \frac{1}{3}x^3\ln\frac{1-x}{1+x} + \frac{1}{3}\int(-2x - \frac{1}{x-1} - \frac{1}{x+1})dx \\
&= \frac{1}{3}x^3\ln\frac{1-x}{1+x} - \frac{x^2}{3} - \ln(1-x) - \ln(1+x) + C \ .
\end{aligned}$$

(e)

$$\begin{aligned}
\int \frac{x\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}dx &= \int \ln(x+\sqrt{1+x^2})d(\sqrt{1+x^2}) \\
&= \sqrt{1+x^2}\ln(x+\sqrt{1+x^2}) - \int \sqrt{1+x^2}d(\ln(x+\sqrt{1+x^2})) \\
&= \sqrt{1+x^2}\ln(x+\sqrt{1+x^2}) - \int \sqrt{1+x^2}\frac{1+\frac{1}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}dx \\
&= \sqrt{1+x^2}\ln(x+\sqrt{1+x^2}) - \int \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}dx \\
&= \sqrt{1+x^2}\ln(x+\sqrt{1+x^2}) - x + C \ .
\end{aligned}$$

(f)

$$\begin{aligned}\int \frac{x^2}{(1+x^2)^2}dx &= -\frac{1}{2}\int xd(\frac{1}{1+x^2}) = -\frac{1}{2}\frac{x}{1+x^2} + \frac{1}{2}\int \frac{1}{1+x^2}dx \\ &= \frac{1}{2}\arctan x - \frac{1}{2}\frac{x}{1+x^2} + C \ .\end{aligned}$$

(g)

$$\begin{aligned}\int \frac{x^2}{(a^2+x^2)^2}dx &= -\frac{1}{2}\int xd(\frac{1}{a^2+x^2}) = -\frac{1}{2}\frac{x}{a^2+x^2} + \frac{1}{2a}\int \frac{d(\frac{x}{a})}{1+(\frac{x}{a})^2} \\ &= \frac{1}{2a}\arctan\frac{x}{a} - \frac{1}{2}\frac{x}{a^2+x^2} + C \ .\end{aligned}$$

(h) (substitution $x = |a|\sin t$)

$$\begin{aligned}I = \int \sqrt{a^2-x^2}dx &= \int \sqrt{a^2-a^2\sin^2 t}d(|a|\sin t) \\ &= a^2\int \cos t d(\sin t) = a^2\cos t\sin t + a^2\int \sin^2 t dt \\ &= a^2\cos t\sin t + a^2 t - I \ .\end{aligned}$$

That is

$$\begin{aligned}I &= \frac{a^2}{2}(\cos t\sin t + t) + C \\ &= a^2(\frac{\sin 2t}{4} + \frac{t}{2}) + C \\ &= a^2(\frac{\sin(2\arcsin(\frac{x}{|a|}))}{4} + \frac{\arcsin(\frac{x}{|a|})}{2}) + C \ .\end{aligned}$$

(i) (substitution $x = a\sinh t$)

$$\begin{aligned}\int \sqrt{a^2+x^2}dx &= \int \sqrt{a^2+a^2\sinh^2 t}d(a\sinh t) \\ &= a^2\int \cosh^2 t dt = a^2\int \frac{1+\cosh 2t}{2}dt \\ &= a^2(\frac{t}{2} + \frac{1}{4}\sinh 2t) + C \\ &= \frac{a^2}{2}(\arctan\frac{x}{a} + \frac{1}{2}\sinh(2\arctan\frac{x}{a})) + C \ .\end{aligned}$$

(j) Let $I = \int x^2\sqrt{a^2+x^2}dx$. Then from **5.7.11.** (h) we have

$$\begin{aligned}I &= \frac{1}{3}\int xd((a^2+x^2)^{\frac{3}{2}}) = \frac{1}{3}x(a^2+x^2)^{\frac{3}{2}} - \frac{1}{3}\int (a^2+x^2)^{\frac{3}{2}}dx \\ &= \frac{1}{3}x(a^2+x^2)^{\frac{3}{2}} - \frac{a^2}{3}\int \sqrt{a^2+x^2}dx - \frac{1}{3}\int x^2\sqrt{a^2+x^2}dx \\ &= \frac{1}{3}x(a^2+x^2)^{\frac{3}{2}} - \frac{a^2}{3}(\frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\arctan\frac{x}{a}) - \frac{1}{3}I \ .\end{aligned}$$

Thus

$$I = \frac{1}{4}x(a^2+x^2)^{\frac{3}{2}} - \frac{a^2}{8}x\sqrt{a^2+x^2} - \frac{a^4}{8}\arctan\frac{x}{a} + C \ .$$

(k)

$$\begin{aligned}\int x\sin^2 x dx &= \int x\frac{1-\cos 2x}{2}dx = \frac{x^2}{4} - \frac{1}{4}\int xd(\sin 2x)\\ &= \frac{x^2}{4} - \frac{1}{4}x\sin 2x + \frac{1}{4}\int \sin 2x dx\\ &= \frac{1}{4}x^2 - \frac{1}{4}x\sin x - \frac{1}{8}\cos 2x + C\ .\end{aligned}$$

(l) (substitution $\sqrt{x} = t$)

$$\begin{aligned}\int e^{\sqrt{x}}dx &= \int e^t d(t^2) = 2\int td(e^t) = 2te^t - 2\int e^t dt\\ &= 2te^t - 2e^t + C = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C\ .\end{aligned}$$

(m) (substitution $\sqrt{x} = t$)

$$\begin{aligned}\int x\sin(\sqrt{x})dx &= \int t^2 \sin t\cdot 2tdt = 2\int t^3 \sin tdt\\ &= -2\int t^3 d(\cos t) = -2t^3\cos t + 2\int \cos t\cdot 3t^2 dt\\ &= -2t^3\cos t + 6t^2\sin t - 6\int \sin t\cdot 2tdt\\ &= -2t^3\cos t + 6t^2\sin t + 12t\cos t - 12\int \cos tdt\\ &= -2t^3\cos t + 6t^2\sin t + 12t\cos t - 12\sin t + C\\ &= -2x^{\frac{3}{2}}\cos\sqrt{x} + 6x\sin\sqrt{x} + 12\sqrt{x}\cos\sqrt{x} - 12\sin\sqrt{x} + C\ .\end{aligned}$$

(n) Let $I = \int x\frac{e^{\arctan x}}{(1+x)^{\frac{3}{2}}}dx$ Then

$$\begin{aligned}\int \frac{x}{\sqrt{1+x}}e^{\arctan x}d(\arctan x) &= \int \frac{x}{\sqrt{1+x^2}}d(e^{\arctan x})\\ &= \frac{e^{\arctan x}}{\sqrt{1+x^2}}x - \int e^{\arctan x}d(\frac{x}{\sqrt{1+x^2}})\\ &= \frac{e^{\arctan x}}{\sqrt{1+x^2}}x - \int \frac{e^{\arctan x}}{(1+x^2)^{\frac{3}{2}}}dx\\ &= \frac{e^{\arctan x}}{\sqrt{1+x^2}}x - \int \frac{1}{\sqrt{1+x^2}}d(e^{\arctan x})\\ &= \frac{e^{\arctan x}}{\sqrt{1+x^2}}x - \frac{e^{\arctan x}}{\sqrt{1+x^2}} + \int e^{\arctan x}d(\frac{1}{\sqrt{1+x^2}})\\ &= \frac{e^{\arctan x}}{\sqrt{1+x^2}}(x-1) - I\ .\end{aligned}$$

Therefore,

$$I = \frac{1}{2}\frac{e^{\arctan x}}{\sqrt{1+x^2}}(x-1)\ .$$

(o)

$$\begin{aligned}\int \frac{e^{\arctan x}}{(1+x^2)^{\frac{3}{2}}}dx &= \int \frac{1}{\sqrt{1+x^2}}d(e^{\arctan x}) \\ &= \frac{1}{\sqrt{1+x^2}}e^{\arctan x} - \int e^{\arctan x} d(\frac{1}{\sqrt{1+x^2}}) \\ &= \frac{e^{\arctan x}}{\sqrt{1+x^2}} + \int \frac{xe^{\arctan x}}{(1+x^2)^{\frac{3}{2}}}dx \ .\end{aligned}$$

Thus, by **5.7.11.** (n),

$$\begin{aligned}\int \frac{e^{\arctan x}}{(1+x^2)^{\frac{3}{2}}}dx &= \frac{e^{\arctan x}}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}}e^{\arctan x} \cdot \frac{x-1}{2} + C \\ &= \frac{e^{\arctan x}}{\sqrt{1+x^2}}\frac{x+1}{2} \ .\end{aligned}$$

(p) Let $I = \int \sin(\ln x)dx$. Then

$$\begin{aligned}I &= x\sin(\ln x) - \int xd(\sin(\ln x)) = x\sin(\ln x) - \int \frac{x\cos(\ln x)}{x}dx \\ &= x\sin(\ln x) - \int \cos(\ln x)dx \\ &= x\sin(\ln x) - x\cos(\ln x) + \int xd(\cos(\ln x)) \\ &= x\sin(\ln x) - x\cos(\ln x) - I \ .\end{aligned}$$

Therefore,

$$\int \sin(\ln x)dx = \frac{x}{2}(\sin(\ln x) - \cos(\ln x)) \ .$$

(q) Let $I = \int \cos(\ln x)dx$. Then

$$\begin{aligned}I &= x\cos(\ln x) - \int \frac{x(-\sin(\ln x))}{x}dx \\ &= x\cos(\ln x) + \int \sin(\ln x)dx \ .\end{aligned}$$

Therefore, by **5.7.11.** (p),

$$\begin{aligned}\int \cos(\ln x)dx &= x\cos(\ln x) + \frac{x}{2}(\sin(\ln x) - \cos(\ln x)) \\ &= \frac{x}{2}(\sin(\ln x) + \cos(\ln x)) \ .\end{aligned}$$

(r) Let $A = \int e^{ax}\cos bxdx$ and $B = \int e^{ax}\sin bxdx$. Then

$$\begin{aligned}A &= \frac{1}{a}\cos bx \cdot e^{ax} + \frac{b}{a}\int e^{ax}\sin bxdx \\ &= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}B \\ B &= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bxdx \\ &= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}A \ .\end{aligned}$$

Solving the above equations for A and B yields

$$\int e^{ax}\cos bx dx = A = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$
$$\int e^{ax}\sin bx dx = B = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\ .$$

(s) See **5.7.11.** (r).

(t)

$$\begin{aligned}\int\ e^{2x}\sin^2 x dx &= \int e^{2x}\frac{1-\cos 2x}{2}dx\\ &= \frac{e^{2x}}{4} - \frac{1}{2}\int e^{2x}\cos 2x dx\ .\end{aligned}$$

By **5.7.11.** (r) with $a=b=2$, we have

$$\int e^{2x}\sin^2 x dx = \frac{e^{2x}}{4} - \frac{e^{2x}}{8}(\cos 2x + \sin 2x) + C\ .$$

(u)

$$\begin{aligned}\int\ (e^x-\cos x)^2 dx &= \int(e^{2x} - 2e^x\cos x + \cos^2 x)dx\\ &= \frac{e^{2x}}{2} - 2\int e^x\cos x dx + \int\frac{1+\cos 2x}{2}dx\\ &= \frac{1}{2}e^{2x} + \frac{1}{2}x + \frac{1}{4}\sin 2x - 2\int e^x\cos x dx\ .\end{aligned}$$

Thus, by **5.7.11.** (r) with $a=b=1$, we have

$$\int(e^x-\cos x)^2 dx = \frac{1}{2}e^{2x} + \frac{1}{2}x + \frac{1}{4}\sin 2x - 2e^x(\cos x + \sin x) + C\ .$$

(v) (substitution $e^x = y$, thus $dx = \frac{1}{e^x}dy = \frac{dy}{y}$)

$$\begin{aligned}\int\ \frac{\operatorname{arccot} e^x}{e^x}dx &= \int\frac{\operatorname{arccot} y}{y^2}dy = -\int\operatorname{arccot} y d(\frac{1}{y})\\ &= -\frac{\operatorname{arccot} y}{y} + \int\frac{1}{y}d(\operatorname{arccot} y) = -\frac{\operatorname{arccot} y}{y} - \int\frac{1}{y}\frac{1}{1+y^2}dy\\ &= -\frac{\operatorname{arccot} y}{y} - \int(\frac{1}{y} - \frac{y}{1+y^2})dy\\ &= -\frac{\operatorname{arccot} y}{y} - \ln y + \frac{1}{2}\ln(1+y^2) + C\\ &= -\frac{\operatorname{arccot} e^x}{e^x} - x + \frac{1}{2}\ln(1+e^{2x}) + C\ .\end{aligned}$$

(x)

$$\begin{aligned}\int \frac{\ln(\sin x)}{\sin^2 x}dx &= -\int \ln(\sin x)d(\cot x)\\ &= -\ln(\sin x)\cot x + \int \cot x d(\ln(\sin x))\\ &= -\ln(\sin x)\cot x + \int \cot^2 x dx\\ &= -\ln(\sin x)\cot x - \int(-1-\cot^2 x)dx - \int 1dx\\ &= -\ln(\sin x)\cot x - \cot x - x + C \ .\end{aligned}$$

(y)

$$\begin{aligned}\int \frac{x}{\cos^2 x}dx &= \int xd(\tan x) = x\tan x - \int \tan x dx\\ &= x\tan x - \int \frac{\sin x}{\cos x}dx = x\tan x + \int \frac{d(\cos x)}{\cos x}\\ &= x\tan x + \ln(\cos x) + C \ .\end{aligned}$$

(z)

$$\begin{aligned}\int \frac{xe^x}{(1+x)^2}dx &= -\int xe^x d(\frac{1}{1+x})\\ &= -\frac{x}{1+x}e^x + \int \frac{1}{1+x}d(e^x) = -\frac{x}{1+x}e^x + \int \frac{e^x+xe^x}{1+x}dx\\ &= -\frac{x}{1+x}e^x + \int e^x dx = (-\frac{x}{1+x}+1)e^x + C\\ &= \frac{e^x}{1+x} + C \ .\end{aligned}$$

5.7.12. (a) As

$$\frac{d}{dx}(\frac{1}{a}\arctan\frac{x}{a}) = \frac{1}{a}\frac{1}{1+(\frac{x}{a})^2}\frac{1}{a} = \frac{1}{a^2+x^2} \ ,$$

we have

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\frac{x}{a} + C \ .$$

(b)

$$\begin{aligned}&\frac{d}{dx}\ (\frac{\ln|\frac{a+x}{a-x}|}{2a})\\ &\quad = \frac{1}{2a}|\frac{a-x}{a+x}|\mathrm{sign}(\frac{a+x}{a-x})\frac{(a-x)+(a+x)}{(a-x)^2}\\ &\quad = \frac{1}{2a}\frac{a-x}{a+x}\frac{2a}{(a-x)^2} = \frac{1}{a^2-x^2} \ .\end{aligned}$$

It follows that

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln|\frac{a+x}{a-x}| + C \ .$$

(c)

$$\frac{d}{dx}\quad (\pm\frac{1}{2}\ln|a^2\pm x^2|) = \pm\frac{a}{2}\frac{1}{|a^2\pm x^2|}\mathrm{sign}(a^2\pm x^2)(\pm 2x)$$
$$= \pm\frac{a}{2}\frac{\pm 2x}{a^2\pm x^2} = \frac{x}{a^2\pm x^2}\ .$$

It follows that
$$\int\frac{x}{a^2\pm x^2}dx \pm \frac{1}{2}\ln|a^2\pm x^2| + C\ .$$

(d) As
$$\frac{d}{dx}(\arcsin\frac{x}{a}) = \frac{1}{\sqrt{1-(\frac{x}{a})^2}}\frac{1}{a} = \frac{1}{\sqrt{a^2-x^2}}\ ,$$

it follows that
$$\int\frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C\ .$$

(e) As
$$\frac{d}{dx}\quad \ln|x+\sqrt{x^2\pm a^2}| = \frac{1}{|x+\sqrt{x^2\pm a^2}|}\mathrm{sign}(x\sqrt{x^2\pm a^2})(1+\frac{x}{\sqrt{x^2\pm a^2}})$$
$$= \frac{\frac{x+\sqrt{x^2\pm a^2}}{\sqrt{x^2\pm a^2}}}{x+\sqrt{x^2\pm a^2}} = \frac{1}{\sqrt{x^2\pm a^2}}\ ,$$

it follows that
$$\frac{dx}{\sqrt{x^2\pm a^2}} = \ln|x+\sqrt{x^2\pm a^2}| + C\ .$$

(f) As
$$\frac{d}{dx}(\pm\sqrt{a^2\pm x^2}) = \frac{\pm 2x}{\pm 2\sqrt{a^2\pm x^2}} = \frac{x}{\sqrt{a^2\pm x^2}}\ ,$$

it follows that
$$\int\frac{xdx}{\sqrt{a^2\pm x^2}} = \pm\sqrt{a^2\pm x^2} + C\ .$$

(g) As
$$\frac{d}{dx}\quad (\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a})$$
$$= \frac{\sqrt{a^2-x^2}}{2} + \frac{x}{2}\frac{-2x}{2\sqrt{a^2-x^2}} + \frac{a^2}{2}\frac{1}{\sqrt{a^2-x^2}}$$
$$= \frac{\sqrt{a^2-x^2}}{2} + \frac{1}{2}\frac{1}{\sqrt{a^2-x^2}}(a^2-x^2) = \frac{\sqrt{a^2-x^2}}{2} + \frac{\sqrt{a^2-x^2}}{2}$$
$$= \sqrt{a^2-x^2}\ ,$$

it follows that
$$\int\sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C\ .$$

(h) As

$$
\begin{aligned}
&\frac{d}{dx} \quad (\frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln|x\sqrt{x^2 \pm a^2}|) \\
&= \frac{\sqrt{x^2 \pm a^2}}{2}\frac{x}{2}\frac{2x}{2\sqrt{x^2 \pm a^2}} \pm \frac{a^2}{2}\frac{\operatorname{sign}(x+\sqrt{x^2 \pm a^2})}{|x+\sqrt{x^2 \pm a^2}|}(1+\frac{2x}{2\sqrt{x^2 \pm a^2}}) \\
&= \frac{\sqrt{x^2 \pm a^2}}{2} + \frac{x^2}{2\sqrt{x^2 \pm a^2}} \pm \frac{a^2}{2\sqrt{x^2 \pm a^2}} \\
&= \frac{\sqrt{x^2 \pm a^2}}{2} + \frac{x^2 \pm a^2}{2\sqrt{x^2 \pm a^2}} = \frac{\sqrt{x^2 \pm a^2}}{2} + \frac{\sqrt{x^2 \pm a^2}}{2} \\
&= \sqrt{x^2 \pm a^2} \ ,
\end{aligned}
$$

it follows that

$$\int \sqrt{x^2 \pm a^2}dx = \frac{x}{a}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln|x+\sqrt{x^2 \pm a^2}| + C \ .$$

5.7.13. (a) For $ab > 0$, from **5.7.12.** (a), we have

$$
\begin{aligned}
\int \frac{1}{a+bx^2}dx &= \int \frac{d(bx)}{(\sqrt{ab})^2 + (bx)^2} \\
&= \frac{1}{\sqrt{ab}}\arctan(\frac{bx}{\sqrt{ab}}) + C \ .
\end{aligned}
$$

For $ab < 0$, from **5.7.12.** (b), we have

$$
\begin{aligned}
&\int \frac{1}{a+bx^2}dx = \int \frac{bx}{ab+bx^2}dx = -\int \frac{bx}{(\sqrt{-ab})^2 - (bx)^2}dx \\
&= -\int \frac{d(bx)}{(\sqrt{-ab})^2 - (bx)^2} = -\frac{1}{2\sqrt{-ab}}\ln|\frac{\sqrt{-ab}+bx}{\sqrt{-ab}-bx}| + C \ .
\end{aligned}
$$

(b)

$$
\begin{aligned}
&\int \frac{dx}{x^2-x+2} = \int \frac{d(x-\frac{1}{2})}{(x-\frac{1}{2})^2 + (\frac{\sqrt{7}}{2})^2} = \frac{2}{\sqrt{7}\arctan(\frac{x-\frac{1}{2}}{\frac{\sqrt{7}}{2}})} + C \\
&= \frac{2\sqrt{7}}{7}\arctan(\frac{\sqrt{7}}{7}(2x-1)) + C \ .
\end{aligned}
$$

(c)

$$
\begin{aligned}
&\int \frac{dx}{3x^2-x+2} = \frac{1}{3}\int \frac{dx}{(x-\frac{1}{3})^2 + \frac{2}{9}} = \frac{1}{3}\int \frac{d(x-\frac{1}{3})}{(x-\frac{1}{3})^2 + (\frac{\sqrt{2}}{3})^2} \\
&= \frac{1}{3}\frac{3}{\sqrt{2}}\arctan(\frac{x-\frac{1}{2}}{\frac{\sqrt{2}}{3}}) + C \\
&= \frac{\sqrt{2}}{2}\arctan(\frac{\sqrt{2}}{2}(3x-1)) + C \ .
\end{aligned}
$$

(d) (substitution $x^2 = t$)

$$\int \frac{x}{x^4-2x^2-1}dx = \frac{1}{2}\int \frac{d(x^2)}{x^4-2x^2-1} = \frac{1}{2}\int \frac{dt}{t^2-2t-1}$$
$$= \frac{1}{2}\int \frac{dt}{(t-1)^2-(\sqrt{2})^2} = \frac{1}{2}\frac{1}{2\sqrt{2}}\ln|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}| + C$$
$$= \frac{\sqrt{2}}{8}\ln|\frac{\sqrt{2}+x^2-1}{\sqrt{2}-x^2+1}| + C\ .$$

(e)

$$\int \frac{x+1}{x^2+x+1}dx = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx + \frac{1}{2}\int \frac{1}{x^2+x+1}dx$$
$$= \frac{1}{2}\int \frac{d(x^2+x+1)}{x^2+x+1} + \frac{1}{2}\int \frac{1}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$
$$= \frac{1}{2}\ln(x^2+x+1) + \frac{1}{2}\frac{2}{\sqrt{3}}\arctan(\frac{(x+\frac{1}{2})2}{\sqrt{3}}) + C$$
$$= \frac{1}{2}\ln(x^2+x+1) + \frac{\sqrt{3}}{3}\arctan(\frac{\sqrt{3}}{3}(2x+1)) + C\ .$$

(f)

$$\int \frac{x}{x^2-2x\cos\alpha+1}dx = \int \frac{xdx}{(x-\cos\alpha)^2+\sin^2\alpha}$$
$$= \int \frac{(x-\cos\alpha)dx}{(x-\cos\alpha)^2+\sin^2\alpha} + \cos\alpha\int \frac{dx}{(x-\cos\alpha)^2+\sin^2\alpha}$$
$$= \frac{1}{2}\ln(x^2-2x\cos\alpha+1) + \cot\alpha\arctan(\frac{x-\cos\alpha}{\sin\alpha}) + C\ .$$

(g) (substitution $x^2 = t$)

$$\int \frac{x^3}{x^4-x^2-2}dx = \frac{1}{2}\int \frac{x^2d(x^2)}{x^4-x^2-2} = \frac{1}{2}\int \frac{tdt}{t^2-t-2}$$
$$= \frac{1}{2}\int \frac{tdt}{(t-2)(t+1)} = \frac{1}{2}\int (\frac{2}{3}\frac{1}{t-2} + \frac{1}{3}\frac{1}{t+1})dt$$
$$= \frac{1}{3}\ln|t-2| + \frac{1}{6}\ln|1+t| + C$$
$$= \frac{1}{3}\ln|x^2-2| + \frac{1}{6}\ln(x^2+1) + C\ .$$

(h) (substitution $x^3 = t$)

$$\int \frac{x^5}{x^6-x^3-2}dx = \frac{1}{3}\int \frac{x^3d(x^3)}{x^6-x^3-2} = \frac{1}{3}\int \frac{tdt}{(t-2)(t+1)}$$
$$= \frac{1}{3}\int (\frac{2}{3}\frac{1}{t-2} + \frac{1}{3}\frac{1}{t+1})dt$$
$$= \frac{2}{9}\ln|t-2| + \frac{1}{9}\ln|1+t| + C$$
$$= \frac{2}{9}\ln|x^3-2| + \ln|x^3+1| + C\ .$$

(i) (substitution $\tan\frac{x}{2} = t$)

$$\begin{aligned}
&\int \frac{dx}{3\sin^2 x - 8\sin x\cos x + 5\cos^2 x} = \int \frac{\frac{2dt}{1+t^2}}{\frac{3(2t)^2}{(1+t^2)^2} - 8\frac{2t(1-t^2)}{(1+t^2)^2} + \frac{5(1-t^2)^2}{(1+t^2)^2}} \\
&= \int \frac{2(1+t^2)dt}{2t^2 - 16t + 16t^3 + 5 + 5t^4} = \int \frac{2(1+t^2)dt}{(t^2+2t-1)(5t^2+6t-5)} \\
&= \int -\frac{t+1}{t^2+2t-1}dt + \int \frac{3+3t}{5t^2+6t-5}dt \\
&= -\frac{1}{2}\int \frac{d(t^2+2t-1)}{t^2+2t-1} + \frac{1}{2}\int \frac{d(5t^2+6t-5)}{5t^2+6t-5} \\
&= -\frac{1}{2}\ln|t^2+2t-1| + \frac{1}{2}\ln|5t^2+6t-5| + C \\
&= \frac{1}{2}\ln\left|\frac{5\tan^2\frac{x}{2} + 6\tan\frac{x}{2} - 5}{\tan^2\frac{x}{2} + 2\tan\frac{x}{2} - 1}\right| + C\ .
\end{aligned}$$

(j)

$$\begin{aligned}
\int \frac{dx}{\sqrt{1-2x-x^2}} &= \int \frac{d(x+1)}{\sqrt{(\sqrt{2})^2 - (x+1)^2}} \\
&= \arcsin\frac{x+1}{\sqrt{2}} + C = \arcsin\left(\frac{\sqrt{2}}{2}(x+1)\right) + C\ .
\end{aligned}$$

(k) (substitution $\tan\frac{x}{2} = t$)

$$\begin{aligned}
&\int \frac{dx}{\sin x + 2\cos x + 3} = \int \frac{\frac{2}{1+t^2}dt}{\frac{2t}{1+t^2} + 2\frac{1-t^2}{1+t^2} + 3} = \int \frac{2dt}{t^2+2t+5} \\
&= 2\int \frac{dt}{(t+1)^2 + 2^2} = \frac{2}{2}\arctan\left(\frac{t+1}{2}\right) + C \\
&= \arctan\left(\frac{1}{2}\tan\frac{x}{2} + \frac{1}{2}\right) + C\ .
\end{aligned}$$

(l) For $b > 0$,

$$\begin{aligned}
\int \frac{dx}{\sqrt{a+bx^2}} &= \int \frac{dx}{\sqrt{b}\sqrt{x^2 + \frac{a}{b}}} \\
&= \frac{1}{\sqrt{b}}\ln\left|x + \sqrt{x^2 + \frac{a}{b}}\right| + C\ .
\end{aligned}$$

For $b < 0$,

$$\begin{aligned}
&\int \frac{dx}{\sqrt{a+bx^2}} = \int \frac{dx}{(\sqrt{a})^2 - (\sqrt{-b}x)^2} \\
&= \frac{1}{\sqrt{-b}}\int \frac{d(\sqrt{-b}x)}{(\sqrt{a})^2 - (\sqrt{-b}x)^2} = \frac{1}{\sqrt{-b}}\arcsin\left(\frac{\sqrt{-b}}{\sqrt{a}}x\right) + C\ .
\end{aligned}$$

(m)

$$\int \frac{dx}{\sqrt{x+x^2}} = \int \frac{d(x+\frac{1}{2})}{\sqrt{(x+\frac{1}{2})^2-(\frac{1}{2})^2}}$$
$$= \ln|x+\frac{1}{2}+\sqrt{x+x^2}|+C\ .$$

(n)

$$\int \frac{dx}{\sqrt{2x^2-x-2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{(x-\frac{1}{4})^2-\frac{17}{16}}}$$
$$= \frac{\sqrt{2}}{2}\ln|x-\frac{1}{4}+\sqrt{x^2-\frac{x}{2}-1}|+C\ .$$

(o)

$$\int \frac{x}{\sqrt{5+x-x^2}}dx = \int \frac{x}{\sqrt{\frac{21}{4}-(x-\frac{1}{2})^2}}dx$$
$$= \int \frac{x-\frac{1}{2}}{\sqrt{\frac{21}{4}-(x-\frac{1}{2})^2}}dx + \frac{1}{2}\int \frac{dx}{\sqrt{\frac{21}{4}-(x-\frac{1}{2})^2}}$$
$$= -\sqrt{5+x-x^2}+\frac{1}{2}\arcsin\left(\frac{x-\frac{1}{2}}{\sqrt{\frac{21}{4}}}\right)+C$$
$$= -\sqrt{5+x-x^2}+\frac{1}{2}\arcsin\left(\frac{\sqrt{21}}{21}(2x-1)\right)+C\ .$$

(p)

$$\int \frac{x-1}{\sqrt{x^2+x+1}}dx = \int \frac{x-1}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}dx$$
$$= \int \frac{x+\frac{1}{2}}{\sqrt{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}}dx - \frac{3}{2}\int \frac{dx}{\sqrt{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}}$$
$$= \sqrt{x^2+x+1}-\frac{3}{2}\ln|x+\frac{1}{2}+\sqrt{x^2+x+1}|+C\ .$$

(q) (substitution $x^2=t$)

$$\int \frac{x}{\sqrt{1-3x^2-2x^4}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1-3t-2t^2}} = \frac{1}{2\sqrt{2}}\int \frac{dt}{\sqrt{\frac{17}{16}-(t+\frac{3}{4})^2}}$$
$$= \frac{1}{2\sqrt{2}}\arcsin\left(\frac{4}{\sqrt{17}}(t+\frac{3}{4})\right)+C$$
$$= \frac{\sqrt{2}}{4}\arcsin\left(\frac{\sqrt{17}}{17}(4t+3)\right)+C$$
$$= \frac{\sqrt{2}}{4}\arcsin\left(\frac{\sqrt{17}}{17}(4x^2+3)\right)+C\ .$$

(r) (substitution $\sin x = t$)

$$\begin{aligned}
\int \frac{\cos x}{\sqrt{1+\sin x+\cos^2 x}}dx &= \int \frac{d(\sin x)}{\sqrt{2+\sin x-\sin^2 x}} = \int \frac{dt}{\sqrt{2+t-t^2}} \\
&= \int \frac{dt}{\sqrt{(\frac{3}{2})^2-(t-\frac{1}{2})^2}} = \arcsin\left(\frac{2}{3}(t-\frac{1}{2})\right)+C \\
&= \arcsin\left(\frac{2}{3}(\sin x-\frac{1}{2})\right)+C\ .
\end{aligned}$$

(s) (substitution $x^2=t$)

$$\begin{aligned}
\int \frac{x^3}{\sqrt{x^4-2x^2-1}}dx &= \frac{1}{2}\int \frac{x^2 d(x^2)}{\sqrt{x^4-2x^2-1}} = \frac{1}{2}\int \frac{tdt}{\sqrt{t^2-2t-1}} \\
&= \frac{1}{2}\int \frac{tdt}{\sqrt{(t-1)^2-(\sqrt{2})^2}} = \frac{1}{2}\int \frac{(t-1)dt}{\sqrt{(t-1)^2-2}} + \frac{1}{2}\int \frac{dt}{\sqrt{(t-1)^2-2}} \\
&= \frac{1}{2}\sqrt{t^2-2t-1} + \frac{1}{2}\ln|t-1+\sqrt{t^2-2t-1}|+C \\
&= \frac{1}{2}\sqrt{x^4-2x^2-1} + \frac{1}{2}\ln|x^2-1+\sqrt{x^4-2x^2-1}|+C\ .
\end{aligned}$$

(t) (substitution $x^2=t$)

$$\begin{aligned}
\int \frac{x+x^3}{\sqrt{1+x^2-x^4}}dx &= \frac{1}{2}\int \frac{(1+x^2)d(x^2)}{\sqrt{1+x^2-x^4}} = \frac{1}{2}\int \frac{(1+t)dt}{\sqrt{1+t-t^2}} \\
&= \frac{1}{2}\int \frac{(t-\frac{1}{2})dt}{\sqrt{\frac{5}{4}-(t-\frac{1}{2})^2}} + \frac{3}{4}\int \frac{dt}{\sqrt{\frac{5}{4}-(t-\frac{1}{2})^2}} \\
&= -\frac{1}{2}\sqrt{1+t-t^2} + \frac{3}{4}\arcsin\left(\frac{\sqrt{5}}{5}(2t-1)\right)+C \\
&= \frac{3}{4}\arcsin\left(\frac{\sqrt{5}}{5}(2x^2-1)\right) - \frac{1}{2}\sqrt{1+x^2-x^4}+C\ .
\end{aligned}$$

(u) (substitution $\frac{1}{x}=t$)

$$\begin{aligned}
\int \frac{1}{x\sqrt{x^2+x+1}}dx &= \int \frac{dx}{x^2\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}} = -\int \frac{d(\frac{1}{x})}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}} \\
&= -\int \frac{dt}{\sqrt{1+t+t^2}} = -\int \frac{dt}{\sqrt{(t+\frac{1}{2})^2+\frac{3}{4}}} = -\ln|t+\frac{1}{2}+\sqrt{1+t+t^2}|+C \\
&= -\ln|\frac{1}{x}+\frac{1}{2}+\frac{\sqrt{x^2+x+1}}{x}|+C\ .
\end{aligned}$$

(v) (substitution $\frac{1}{x}=t$)

$$\int \frac{dx}{x^2\sqrt{x^2+x-1}} = -\int \frac{d(\frac{1}{x})}{x\sqrt{1+\frac{1}{x}-\frac{1}{x^2}}} = -\int \frac{tdt}{\sqrt{1+t-t^2}}$$

$$= -\int \frac{(t-\frac{1}{2})dt}{\sqrt{\frac{5}{4}-(t-\frac{1}{2})^2}} - \frac{1}{2}\int \frac{dt}{\sqrt{\frac{5}{4}-(t-\frac{1}{2})^2}}$$

$$= \sqrt{1+t-t^2} - \frac{1}{2}\arcsin\left(\frac{2}{\sqrt{5}}(t-\frac{1}{2})\right) + C$$

$$= \frac{\sqrt{x^2+x-1}}{x} - \frac{1}{2}\arcsin\left(\frac{\sqrt{5}}{5}(\frac{2}{x}-1)\right) + C\ .$$

(w) (substitution $\frac{1}{1+x} = t$)

$$\int \frac{dx}{(x+1)\sqrt{x^2+1}} = \int \frac{d(x+1)}{(x+1)\sqrt{(x+1)^2-2(x+1)+2}}$$

$$= -\int \frac{d(\frac{1}{x+1})}{\sqrt{1-\frac{2}{x+1}+\frac{2}{(x+1)^2}}}$$

$$= -\int \frac{dt}{\sqrt{1-2t+2t^2}} = -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{(t-\frac{1}{2})^2+(\frac{1}{2})^2}}$$

$$= -\frac{\sqrt{2}}{2}\ln\left|t-\frac{1}{2}+\sqrt{t^2-t+\frac{1}{2}}\right| + C$$

$$= -\frac{\sqrt{2}}{2}\ln\left|\frac{1}{x+1}-\frac{1}{2}+\frac{\sqrt{2}}{2}\frac{\sqrt{x^2+1}}{x+1}\right| + C\ .$$

(x) (substitution $\frac{1}{x-1} = t$)

$$\int \frac{dx}{(x-1)\sqrt{x^2-2}} = \int \frac{d(x-1)}{(x-1)\sqrt{(x-1)^2+2(x-1)-1}}$$

$$= -\int \frac{d(\frac{1}{x-1})}{\sqrt{1+\frac{2}{x-1}-\frac{1}{(x-1)^2}}} = -\int \frac{dt}{\sqrt{1+2t-t^2}}$$

$$= -\int \frac{dt}{\sqrt{(\sqrt{2})^2-(t-1)^2}} = -\arcsin\left(\frac{t-1}{\sqrt{2}}\right) + C$$

$$= -\arcsin\left(\frac{\sqrt{2}}{2}(\frac{1}{x-1}-1)\right) + C\ .$$

(y) (substitution $\frac{1}{x+2} = t$)

$$\begin{aligned}\int \frac{dx}{(x+2)^2\sqrt{x^2+2x-5}} &= \int \frac{dx}{(x+2)^2\sqrt{(x+2)^2-2(x+2)-5}}\\
&= -\int \frac{d(\frac{1}{x+2})}{(x+2)\sqrt{1-\frac{2}{x+2}-\frac{5}{(x+2)^2}}}\\
&= -\int \frac{tdt}{\sqrt{1-2t-5t^2}} = -\frac{1}{\sqrt{5}}\int \frac{tdt}{\sqrt{\frac{6}{25}-(t+\frac{1}{5})^2}}\\
&= \frac{1}{5}\sqrt{1-2t-5t^2}+\frac{\sqrt{5}}{25}\arcsin\left(\frac{5}{\sqrt{6}}(t+\frac{1}{5})\right)+C\\
&= \frac{\sqrt{x^2+2x-5}}{5(x+2)}+\frac{\sqrt{5}}{25}\arcsin\left(\frac{\sqrt{6}}{6}(\frac{5}{x+2}+1)\right)+C\ .\end{aligned}$$

(z) Let $I_1 = \int \frac{(x-1)dx}{\sqrt{1+x-x^2}}$ and $I_2 = \int \frac{dx}{x\sqrt{1+x-x^2}}$.
i)

$$\begin{aligned}I_1 &= \int \frac{(x-\frac{1}{2})dx}{\sqrt{\frac{5}{4}-(x-\frac{1}{2})^2}} - \frac{1}{2}\int \frac{dx}{\sqrt{\frac{5}{4}-(x-\frac{1}{2})^2}}\\
&= -\sqrt{1+x-x^2}-\frac{1}{2}\arcsin\left(\frac{2}{\sqrt{5}}(x-\frac{1}{2})\right)+C\\
&= -\sqrt{1+x-x^2}-\frac{1}{2}\arcsin\left(\frac{\sqrt{5}}{5}(2x-1)\right)+C\ .\end{aligned}$$

ii) (substitution $\frac{1}{x}=t$)

$$\begin{aligned}I_2 &= \int \frac{dx}{x^2\sqrt{\frac{1}{x^2}+\frac{1}{x}-1}} = -\int \frac{d(\frac{1}{x})}{\sqrt{(\frac{1}{x})^2+\frac{1}{x}-1}} = -\int \frac{dt}{\sqrt{t^2+t-1}}\\
&= -\int \frac{dt}{\sqrt{(t+\frac{1}{2})^2-\frac{5}{4}}} = -\ln\left|t+\frac{1}{2}+\sqrt{t^2+t-1}\right|+C\\
&= -\ln\left|\frac{1}{x}+\frac{1}{2}+\frac{\sqrt{1+x-x^2}}{x}\right|+C\ .\end{aligned}$$

Therefore,

$$\begin{aligned}\int \frac{1-x+x^2}{x\sqrt{1+x-x^2}}dx &= \int\left(\frac{x(x-1)}{x\sqrt{1+x-x^2}}+\frac{1}{x\sqrt{1+x-x^2}}\right)dx = I_1+I_2\\
&= -\sqrt{1+x-x^2}-\frac{1}{2}\arcsin\left(\frac{\sqrt{5}}{5}(2x-1)\right)-\ln\left|\frac{1}{x}+\frac{1}{2}+\frac{\sqrt{1+x-x^2}}{x}\right|+C\ .\end{aligned}$$

5.7.14. (a) As

$$\frac{2x+3}{(x-2)(x+5)} = \frac{1}{x-2}+\frac{1}{x+5}\ ,$$

therefore,

$$\int \frac{x-2}{x+5}dx = \ln(x-2)+\ln(x+5)+C\ .$$

(b) Let $\frac{x}{(x+1)(x+2)(x+3)} = \frac{a}{x+1}+\frac{b}{x+2}+\frac{c}{x+3}$. Then we have

$$x = a(x+2)(x+3) + b(x+1)(x+3) + c(x+1)(x+2) \ . \tag{14.14}$$

By substituting successively $x = -1$, $x = -2$ and $x = -3$ into (14.14), we obtain $a = -\frac{1}{2}$, $b = 2$ and $c = -\frac{3}{2}$. Therefore,

$$\int \frac{x}{(x+1)(x+2)(x+3)} dx = -\frac{1}{2} \ln|x+1| + 2\ln|x+2| - \frac{3}{2} \ln|x+3| + C \ .$$

(c) By long division,

$$\begin{aligned} \frac{x^{10}}{x^2+x-2} &= x^8 - x^7 + 3x^6 - 5x^5 + 11x^4 + 43x^2 \\ &\quad -85x + 171 - \frac{1024}{3}\frac{1}{x+2} + \frac{1}{3}\frac{1}{x-1} \ . \end{aligned}$$

Therefore

$$\begin{aligned} \int \frac{x^{10}}{x^2+x-2} dx = \frac{1}{9}x^9 - \frac{1}{8}x^8 + \frac{3}{7}x^7 - \frac{5}{6}x^6 + \frac{11}{5}x^5 + \frac{43}{3}x^4 \\ -\frac{85}{2}x^2 + 171x - \frac{1024}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + C \ . \end{aligned}$$

(d) As

$$\frac{x^3+1}{x^3-5x^2+6x} = 1 + \frac{5x^2-6x+1}{x(x-2)(x-3)} \ ,$$

we look for the following partial fraction decomposition:

$$1 + \frac{5x^2-6x+1}{x(x-2)(x-3)} = \frac{a}{x} + \frac{b}{x-2} + \frac{c}{x-3} \ ,$$

i.e.

$$5x^2 - 6x + 1 = a(x-2)(x-3) + bx(x-3) + cx(x-2) \ . \tag{14.15}$$

By substituting successively $x = 0$, $x = 2$ and $x = 3$ into (14.15), we obtain $a = \frac{1}{6}$, $b = -\frac{9}{2}$ and $c = \frac{28}{3}$. Therefore,

$$\int \frac{x^3+1}{x^3-5x^2+6x} dx = x + \frac{1}{6}\ln|x| - \frac{9}{2}\ln|x-2| + \frac{28}{3}\ln|x-3| + C \ .$$

(e) As $x^4 + 5x^2 + 4 = (x^2+1)(x^2+4)$, we look for a and b such that

$$\frac{x^4}{x^4+5x^2+4} = 1 + \frac{a}{x^2+1} + \frac{b}{x^2+4} \ ,$$

i.e.

$$-5x^2 + 4 = a(x^2+4) + b(x^2+1) \ ,$$

which implies

$$\begin{cases} a + b = -5 \\ 4a + b = 4 \end{cases}$$

Thus $a = \frac{1}{3}$ and $b = -\frac{16}{3}$. Hence,

$$\begin{aligned}\int \frac{x^4}{x^4+5x^2+4}dx &= \int(1+\frac{1}{3}\frac{1}{x^2+1}-\frac{16}{3}\frac{1}{x^2+4})dx \\ &= x = \frac{1}{3}\arctan x - \frac{8}{3}\arctan\frac{x}{2} + C\ .\end{aligned}$$

(f) As $\frac{x}{x^3-3x+2} = \frac{x}{(x+2)(x-1)^2}$, we look for a, b and c such that

$$\frac{x}{x^3-3x+2} = \frac{a}{x+2} + \frac{b}{x-1} + \frac{c}{(x-1)^2}\ ,$$

i.e.

$$x = a(x-1)^2 + b(x-1)(x+2) + c(x+2)\ ,$$

which implies

$$\begin{cases} a+b=0 \\ -2a+b+c=1 \\ a-2b+2c=0 \end{cases}$$

Thus, $a = -\frac{2}{9}$, $b = \frac{2}{9}$ and $c = \frac{1}{3}$. Hence,

$$\begin{aligned}\int \frac{x}{x^3-3x+2}dx &= \int(-\frac{2}{9}\frac{1}{x+2}+\frac{2}{9}\frac{1}{x-1}+\frac{1}{3}\frac{1}{(x-1)^2})dx \\ &= -\frac{2}{9}\ln|x+2| + \frac{2}{9}\ln|x-1| - \frac{1}{3}\frac{1}{x-1} + C\ .\end{aligned}$$

(g) Let

$$\frac{x^2+1}{(x+1)^2(x-1)} = \frac{a}{x+1} + \frac{b}{x-1} + \frac{c}{(x+1)^2}\ ,$$

we then have

$$x^2+1 = a(x+1)(x-1) + b(x+1)^2 + c(x-1)\ ,$$

which implies

$$\begin{cases} a+b=1 \\ 2b+c=0 \\ -a+b-c=1 \end{cases}$$

Thus, $a = \frac{1}{2}$, $b = \frac{1}{2}$ and $c = -1$. Hence,

$$\int \frac{x^2+1}{(x+1)^2(x-1)}dx = \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + \frac{1}{x+1} + C\ .$$

(h) As

$$(\frac{x}{x^2-3x+2})^2 = \frac{x^2}{(x-1)^2(x-2)^2}\ ,$$

we look for a, b, c and d such that

$$(\frac{x}{x^2-3x+2})^2 = \frac{a}{x-1} + \frac{b}{x-2} + \frac{c}{(x-1)^2} + \frac{d}{(x-2)^2}\ ,$$

i.e.

$$x^2 = a(x-1)(x-2)^2 + b(x-2)(x-1)^2 + c(x-2)^2 + d(x-1)^2\ ,$$

which implies

$$\begin{cases} a+b=0 \\ -5a-4b+c+d=1 \\ 8a+5b-4c-2d=0 \\ -4a-2b+4c+d=0 \end{cases}$$

Thus, $a=4$, $b=-4$, $c=1$ and $d=4$. Hence,

$$\int (\frac{x}{x^2-3x+2})^2 dx = 4\ln|x-1| - 4\ln|x-2| - \frac{1}{x-1} - 4\frac{1}{x-2} + C \ .$$

(i) We look for a, b, c, d, e and f such that

$$\begin{aligned} \frac{1}{(x+1)(x+2)^2(x+3)^3} &= \frac{a}{x+1} + \frac{b}{x+2} + \frac{c}{x+3} + \frac{d}{(x+2)^2} \\ &\quad + \frac{e}{(x+3)^2} + \frac{f}{(x+3)^3} \ , \end{aligned}$$

i.e.

$$\begin{aligned} 1 &= a(x+2)^2(x+3)^3 + b(x+1)(x+2)(x+3)^3 + c(x+1)(x+2)^2(x+3)^2 \\ &\quad + d(x+1)(x+3)^3 + e(x+1)(x+2)^2(x+3) + f(x+1)(x+2)^2 \ , \end{aligned}$$

which implies

$$\begin{cases} a+b+c=0 \\ 13a+12b+11c+d+e=0 \\ 67a+56b+47c+10d+8e+f=0 \\ 171a+126b+97c+36d+23e+5f=0 \\ 216a+135b+96c+54d+28e+8f=0 \\ 108a+54b+36c+27d+12e+4f=1 \end{cases}$$

Thus, $a=\frac{1}{8}$, $b=2$, $c=-\frac{17}{8}$, $d=-1$, $e=-\frac{5}{4}$ and $f=-\frac{1}{2}$. Hence,

$$\begin{aligned} \int \frac{dx}{(x+1)(x+2)^2(x+3)^3} &= \frac{1}{8}\ln|x+1| + 2\ln|x+2| - \frac{17}{8}\ln|x+3| \\ &\quad + \frac{1}{x+2} + \frac{5}{4}\frac{1}{x+3} + \frac{1}{4}\frac{1}{(x+3)^2} + C \ . \end{aligned}$$

(j) As

$$\begin{aligned} x^5+x^4-2x^3-2x^2+x+1 &= (x+1)(x^4-2x^2+1) \\ &= (x+1)(x^2-1)^2 = (x+1)^3(x-1)^2 \ , \end{aligned}$$

we look for the following partial fraction decomposition:

$$\begin{aligned} \frac{1}{x^5+x^4-2x^3-2x^2+x+1} &= \frac{a}{x+1} + \frac{b}{x-1} + \frac{c}{(x+1)^2} \\ &\quad \frac{d}{(x-1)^2} + \frac{e}{(x+1)^3} \ , \end{aligned}$$

i.e.

$$1 = a(x+1)^2(x-1)^2 + b(x+1)^3(x-1) + c(x+1)(x-1)^2 \\ +d(x+1)^3 + e(x-1)^2 \ ,$$

which implies

$$\begin{cases} a+b=0 \\ 2b+c+d=0 \\ -2a-c+3d+e=0 \\ -2b-c+3d-2e=0 \\ a-b+c+d+e=1 \end{cases}$$

Thus, $a = \frac{3}{16}$, $b = -\frac{3}{16}$, $c = \frac{1}{4}$, $d = \frac{1}{8}$, and $e = \frac{1}{4}$. Hence,

$$\int \frac{dx}{x^5+x^4-2x^3-2x^2+x+1} = \frac{3}{16}\ln|x+1| - \frac{3}{16}\ln|x-1| \\ -\frac{1}{4}\frac{1}{x+1} - \frac{1}{8}\frac{1}{x-1} - \frac{1}{8}\frac{1}{(x+1)^2} + C \ .$$

(k) As

$$\frac{x^2+5x+4}{x^4+5x^2+4} = \frac{x^2+5x+4}{(x^2+1)(x^2+4)} \ ,$$

we look for a, b, c and d such that

$$\frac{x^2+5x+4}{x^4+5x^2+4} = \frac{ax+b}{x^2+1} + \frac{cx+d}{x^2+4} \ ,$$

i.e.

$$x^2+5x+4 = (ax+b)(x^2+4) + (cx+d)(x^2+1) \ ,$$

which implies

$$\begin{cases} a+c=0 \\ b+d=1 \\ 4a+c=5 \\ 4b+d=4 \end{cases}$$

Thus, $a = \frac{5}{3}$, $b = 1$, $c = -\frac{5}{3}$ and $d = 0$. Hence,

$$\int \frac{x^2+5x+4}{x^4+5x^2+4}dx = \int \frac{1}{x^2+1}dx + \frac{5}{3}\int \frac{x}{1+x^2}dx - \frac{5}{3}\int \frac{x}{x^2+4}dx \\ = \arctan x + \frac{5}{6}\ln(1+x^2) - \frac{5}{6}\ln(x^2+4) + C \ .$$

(l) We look for a, b and c such that

$$\frac{1}{(1+x)(1+x^2)} = \frac{a}{1+x} + \frac{bx+c}{1+x^2} \ ,$$

i.e.

$$1 = a(1+x^2) + (bx+c)(1+x) \ ,$$

which implies

$$\begin{cases} a+b=0 \\ b+c=0 \\ a+c=1 \end{cases}$$

Thus, $a=\frac{1}{2}$, $b=-\frac{1}{2}$ and $c=\frac{1}{2}$. Hence

$$\int \frac{1}{(1+x)(1+x^2)}dx = \frac{1}{2}\ln|1+x| - \frac{1}{2}\int \frac{x-1}{1+x^2}dx$$
$$= \frac{1}{2}\ln|1+x| - \frac{1}{4}\ln|1+x^2| + \frac{1}{2}\arctan x + C \ .$$

(m) As

$$\frac{1}{(x^2-4x+4)(x^2-4x+5)} = \frac{1}{x^2-4x+4} - \frac{1}{x^2-4x+5}$$
$$= \frac{1}{(x-2)^2} - \frac{1}{(x-2)^2+1} \ ,$$

we have

$$\int \frac{1}{(x^2-4x+4)(x^2-4x+5)}dx = \int \frac{1}{(x-2)^2}dx - \int \frac{1}{(x-2)^2+1}dx$$
$$= -\frac{1}{x-2} - \arctan(x-2) + C \ .$$

(n) We look for a, b, c and d such that

$$\frac{x}{(x-1)^2(x^2+2x+2)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{cx+d}{x^2+2x+2} \ ,$$

i.e.

$$x = a(x-1)(x^2+2x+2) + b(x^2+2x+2) + (cx+d)(x-1)^2 \ ,$$

which implies

$$\begin{cases} a+c=0 \\ a+b-2c+d=0 \\ 2b+c-2d=1 \\ -2a+2b+d=0 \end{cases}$$

Thus, $a=\frac{1}{25}$, $b=\frac{1}{5}$, $c=-\frac{1}{25}$ and $d=-\frac{8}{25}$. Hence,

$$\int \frac{x}{(x-1)^2(x^2+2x+2)}dx = \frac{1}{25}\ln|x-1| - \frac{1}{5}\frac{1}{x-1} - \frac{1}{25}\int \frac{x+8}{x^2+2x+2}dx$$
$$= \frac{1}{25}\ln|x-1| - \frac{1}{5}\frac{1}{x-1} - \frac{1}{25}\int \frac{(x+1)dx}{(x+1)^2+1} - \frac{7}{25}\int \frac{1}{(x+1)^2+1}dx$$
$$= \frac{1}{25}\ln|x-1| - \frac{1}{5}\frac{1}{x-1} - \frac{1}{50}\ln(x^2+2x+2) - \frac{7}{25}\arctan(x+1) + C \ .$$

(o) We look for a, b, c and d such that

$$\frac{1}{x(x-1)(x^2+x+1)} = \frac{a}{x} + \frac{b}{x-1} + \frac{cx+d}{x^2+x+1} \ ,$$

i.e.

$$1 = a(x-1)(x^2+x+1) + bx(x^2+x+1) + (cx+d)x(x-1) \ ,$$

which implies

$$\begin{cases} a+b+c=0 \\ b-c+d=0 \\ b-d=0 \\ -a=1 \end{cases}$$

Thus, $a=-1$, $b=\frac{1}{3}$, $c=\frac{2}{3}$ and $d=\frac{1}{3}$. Hence,

$$\begin{aligned} \int \frac{1}{x(x-1)(x^2+x+1)}dx &= -\ln|x| + \frac{1}{3}\ln|x-1| + \frac{1}{3}\int \frac{2x+1}{x^2+x+1}dx \\ &= -\ln|x| + \frac{1}{3}\ln|x-1| + \frac{1}{3}\int \frac{d(x^2+x+1)}{x^2+x+1} \\ &= -\ln|x| + \frac{1}{3}\ln|x-1| + \frac{1}{3}\ln(x^2+x+1) + C \ . \end{aligned}$$

(p) As

$$\frac{1}{x^4-1} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{2}\frac{1}{x^2-1} - \frac{1}{2}\frac{1}{x^2+1} \ ,$$

we have

$$\begin{aligned} \int \frac{dx}{x^4-1} &= \frac{1}{2}\int \frac{1}{x^2-1}dx - \frac{1}{2}\int \frac{1}{1+x^2}dx \\ &= -\frac{1}{2}\frac{1}{2}\ln|\frac{1+x}{1-x}| - \frac{1}{2}\arctan x + C \\ &= -\frac{1}{4}\ln|\frac{1+x}{1-x}| - \frac{1}{2}\arctan x + C \ . \end{aligned}$$

(q) As

$$x^4+1 = (x^2+x\sqrt{2}+1)(x^2-x\sqrt{2}+1) \ ,$$

we look for a, b, c and d such that

$$\frac{1}{x^4+1} = \frac{ax+b}{x^2-x\sqrt{2}+1} + \frac{cx+d}{x^2+x\sqrt{2}+1} \ ,$$

i.e.

$$1 = (ax+b)(x^2+x\sqrt{2}+1) + (cx+d)(x^2-x\sqrt{2}+1) \ ,$$

which implies

$$\begin{cases} a+c=0 \\ b+d+\sqrt{2}(a-c)=0 \\ a+c+\sqrt{2}(b-d)=0 \\ b+d=1 \end{cases}$$

Thus, $a=-\frac{\sqrt{2}}{4}$, $b=\frac{1}{2}$, $c=\frac{\sqrt{2}}{4}$ and $d=\frac{1}{2}$. Hence,

$$\begin{aligned}
\int \frac{1}{x^4+1}dx &= \int \frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2-x\sqrt{2}+1}dx + \int \frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+x\sqrt{2}+1}dx \\
&= \frac{\sqrt{2}}{4}\int \frac{x+\frac{\sqrt{2}}{2}}{(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}}dx - \frac{\sqrt{2}}{4}\int \frac{x-\frac{\sqrt{2}}{2}}{(x-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}dx \\
&+\frac{1}{4}\int \frac{dx}{(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}} + \frac{1}{4}\int \frac{dx}{(x-\frac{\sqrt{2}}{2})^2+\frac{1}{2}} \\
&= \frac{\sqrt{2}}{8}\ln(\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}) + \frac{\sqrt{2}}{4}\arctan(\sqrt{2}x+1) \\
&+\frac{\sqrt{2}}{4}\arctan(\sqrt{2}x-1)+C \ .
\end{aligned}$$

(r) As $x^4+x^2+1=(x^2+x+1)(x^2-x+1)$, we look for a, b, c and d such that

$$\frac{1}{x^4+x^2+1} = \frac{ax+b}{x^2+x+1} + \frac{cx+d}{x^2-x+1} \ ,$$

i.e.

$$1=(ax+b)(x^2-x+1)+(cx+d)(x^2+x+1) \ ,$$

which implies

$$\begin{cases} a+c=0 \\ -a+b+c+d=0 \\ a-b+c+d=0 \\ b+d=1 \end{cases}$$

Thus, $a=\frac{1}{2}$, $b=\frac{1}{2}$, $c=-\frac{1}{2}$ and $d=\frac{1}{2}$. Hence,

$$\begin{aligned}
\int \frac{1}{x^4+x^2+1}dx &= \frac{1}{2}\int \frac{x+1}{x^2+x+1}dx - \frac{1}{2}\int \frac{x-1}{x^2-x+1}dx \\
&= \frac{1}{2}\int \frac{(x+\frac{1}{2})dx}{(x+\frac{1}{2})^2+\frac{3}{4}} - \frac{1}{2}\int \frac{(x-\frac{1}{2})dx}{(x-\frac{1}{2})^2+\frac{3}{4}} \\
&+ \ \frac{1}{4}\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} + \frac{1}{4}\int \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}} \\
&= \frac{1}{2}\ln(\frac{x^2+x+1}{x^2-x+1}) + \frac{1}{4}\frac{2}{\sqrt{3}}\arctan(\frac{2}{\sqrt{3}}(x-\frac{1}{2})) \\
&+\frac{1}{4}\frac{2}{\sqrt{3}}\arctan(\frac{2}{\sqrt{3}}(x+\frac{1}{2}))+C \\
&= \frac{1}{2}\ln(\frac{x^2+x+1}{x^2-x+1}) + \frac{\sqrt{3}}{6}\arctan(\frac{\sqrt{3}}{3}(2x-1)) \\
&+\frac{\sqrt{3}}{6}\arctan(\frac{\sqrt{3}}{3}(2x+1))+C \ .
\end{aligned}$$

(s) As $x^6+1=(x^2+1)(x^4-x^2+1)=(x^2+1)(x^2+\sqrt{3}x+1)(x^2-\sqrt{3}x+1)$, we look for a, b, c, d, e and f such that

$$\frac{1}{x^6+1} = \frac{ax+b}{x^2+1} + \frac{cx+d}{x^2+\sqrt{3}x+1} + \frac{ex+f}{x^2-\sqrt{3}x+1} \ ,$$

i.e.

$$\begin{aligned} 1 &= (ax+b)(x^4-x^2+1)+(cx+d)(x^2)(x^2-\sqrt{3}x+1) \\ &+(ex+f)(x^2+1)(x^2+\sqrt{3}x+1)\ , \end{aligned}$$

which implies

$$\begin{cases} a+c+e=0 \\ b+d+f-\sqrt{3}(c-e)=0 \\ -a+2c+2e-\sqrt{3}(d-f)=0 \\ -b+2d+2f-\sqrt{3}(c-e)=0 \\ a+c+e-\sqrt{3}(d-f)=0 \\ b+d+f=1 \end{cases}$$

Thus, $a=0$, $b=\frac{1}{3}$, $c=\frac{\sqrt{3}}{6}$, $d=\frac{1}{3}$, $e=-\frac{\sqrt{3}}{6}$ and $f=\frac{1}{3}$. Hence,

$$\begin{aligned} &\int \frac{1}{x^6+1}dx = \frac{1}{3}\arctan x + \int \frac{\frac{\sqrt{3}}{6}x+\frac{1}{3}}{x^2+\sqrt{3}x+1}dx + \int \frac{-\frac{\sqrt{3}}{6}x+\frac{1}{3}}{x^2-\sqrt{3}x+1}dx \\ &= \frac{1}{3}\arctan x + \frac{\sqrt{3}}{6}\int \frac{x+\frac{\sqrt{3}}{2}}{(x+\frac{\sqrt{3}}{2})^2+\frac{1}{4}}dx - \frac{\sqrt{3}}{6}\int \frac{x-\frac{\sqrt{3}}{2}}{(x-\frac{\sqrt{3}}{2})^2+\frac{1}{4}}dx \\ &+\frac{1}{12}\int \frac{dx}{(x+\frac{\sqrt{3}}{2})^2+\frac{1}{4}} + \frac{1}{12}\int \frac{dx}{(x\frac{\sqrt{3}}{2})^2+\frac{1}{4}} \\ &= \frac{1}{3}\arctan x + \frac{\sqrt{3}}{12}\ln(\frac{x^2+\sqrt{3}x+1}{x^2-\sqrt{3}x+1}) + \frac{1}{6}\arctan(2x+\sqrt{3}) \\ &+\frac{1}{6}\arctan(2x-\sqrt{3})+C\ . \end{aligned}$$

(t) As $(1+x)(1+x^2)(1+x^3)=(1+x)^2(1+x^2)(x^2-x+1)$, we look for a, b, c, d, e and f such that

$$\frac{1}{(1+x)(1+x^2)(1+x^3)} = \frac{a}{x+1} + \frac{b}{(1+x)^2} + \frac{cx+d}{1+x^2} + \frac{ex+f}{x^2-x+1}\ ,$$

i.e.

$$\begin{aligned} 1 &= a(1+x)(1+x^2)(x^2-x+1)+b(1+x^2)(x^2-x+1) \\ &+(cx+d)(1+x)^2(x^2-x+1)+(ex+f)(1+x)^2(1+x^2)\ , \end{aligned}$$

which implies

$$\begin{cases} a+c+e=0 \\ d+c+b+f+2e=0 \\ d-b+2f+2e+a=0 \\ c+a+2b+2f+2e=0 \\ 2f+e-b+d+c=0 \\ a+b+d+f=1 \end{cases}$$

Thus, $a=\frac{1}{3}$, $b=\frac{1}{6}$, $c=0$, $d=\frac{1}{2}$, $e=-\frac{1}{3}$ and $f=0$. Hence,

$$\begin{aligned}\int \frac{1}{(1+x)(1+x^2)(1+x^3)} &= \frac{1}{3}\ln|1+x| - \frac{1}{6}\frac{1}{x+1} + \frac{1}{2}\arctan x\\ &\quad -\frac{1}{3}\int \frac{x}{x^2-x+1}dx\\ &= \frac{1}{3}\ln|1+x| - \frac{1}{6}\frac{1}{x+1} + \frac{1}{2}\arctan x - \frac{1}{3}\int \frac{(x-\frac{1}{2})dx}{(x-\frac{1}{2})^2+\frac{3}{4}}\\ &\quad -\frac{1}{6}\int \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}}\\ &= \frac{1}{3}\ln|1+x| - \frac{1}{6}\frac{1}{x+1} + \frac{1}{2}\arctan x - \frac{1}{6}\ln(x^2-x+1)\\ &\quad -\frac{\sqrt{3}}{9}\arctan\left(\frac{\sqrt{3}}{3}(2x-1)\right) + C\ .\end{aligned}$$

(u) As

$$\begin{aligned}x^5-x^4+x^3-x^2+x-1 &= (x-1)(x^4+x^2+1)\\ &= (x-1)(x^2+x+1)(x^2-x+1)\ ,\end{aligned}$$

we look for a, b, c, d and e such that

$$\frac{1}{x^5-x^4+x^3-x^2+x-1} = \frac{a}{x-1} + \frac{bx+c}{x^2+x+1} + \frac{dx+e}{x^2-x+1}\ ,$$

i.e.

$$\begin{aligned}1 &= a(x^4+x+1) + (bx+c+)(x-1)(x^2-x+1)\\ &\quad +(dx+e)(x-1)(x^2+x+1)\ ,\end{aligned}$$

which implies

$$\begin{cases} a+b+d=0\\ -2b+c+e=0\\ a+2b-2c=0\\ -b+2c-d=0\\ a-c-e=1\end{cases}$$

Thus, $a=\frac{1}{3}$, $b=-\frac{1}{3}$, $c=-\frac{1}{6}$, $d=0$ and $e=-\frac{1}{2}$. Hence,

$$\begin{aligned}\int \frac{1}{x^5-x^4+x^3-x^2+x+1}dx &= \frac{1}{3}\ln|x-1| - \frac{1}{6}\int\frac{2x+1}{x^2+x+1}dx\\ &\quad -\frac{1}{2}\int\frac{dx}{x^2-x+1}\\ &= \frac{1}{3}\ln|x-1| - \frac{1}{6}\int\frac{d(x^2+x+1)}{x^2+x+1} - \frac{1}{2}\int\frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}}\\ &= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}}{3}(2x-1)\right) + C\ .\end{aligned}$$

(v) As

$$\frac{x^2}{x^4+3x^3+\frac{9}{2}x^2+3x+1} = \frac{2x^2}{(x^2+2x+2)(2x^2+2x+1)}\ ,$$

we look for a, b, c and d such that

$$\frac{x^2}{x^4+3x^3+\frac{9}{2}x^2+3x+1}=\frac{ax+b}{x^2+2x+2}+\frac{cx+d}{2x^2+2x+1}\ ,$$

i.e.

$$x^2=(ax+b)(2x^2+2x+1)+(cx+d)(x^2+2x+2)\ ,$$

which implies

$$\begin{cases} 2a+c=0\\ 2a+2b+2c+d=1\\ a+2b+2c+2d=0\\ b+2d=0 \end{cases}$$

Thus, $a=\frac{4}{5}$, $b=\frac{12}{5}$, $c=-\frac{8}{5}$ and $d=-\frac{6}{5}$. Hence,

$$\begin{aligned}\int \frac{x^2}{x^4+3x^3+\frac{9}{2}x^2+3x+1}dx &= -\frac{5}{2}\int\frac{4x+3}{2x^2+2x+1}dx+\frac{4}{5}\int\frac{x+3}{x^2+2x+2}dx\\ &-\frac{2}{5}\int\frac{d(2x^2+2x+1)}{2x^2+2x+1}-\frac{2}{5}\int\frac{dx}{2x^2+2x+1}\\ &+\frac{2}{5}\int\frac{d(x^2+2x+2)}{x^2+2x+2}+\frac{8}{5}\int\frac{dx}{x^2+2x+2}\\ &=-\frac{2}{5}\ln(2x^2+2x+1)+\frac{2}{5}\ln(x^2+2x+2)\\ &-\frac{2}{5}\arctan(2x+1)+\frac{8}{5}\arctan(x+1)+C\ .\end{aligned}$$

5.7.15.* We look for partial fraction decomposition of

$$\begin{aligned}&\frac{ax^2+bx+c}{x^3(x-1)^2}\\ =&\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x-1}+\frac{E}{(x-1)^2}\\ =&\frac{(A+D)x^4+(-2A+B-D+E)x^3+(A-2B+C)x^2+(B-2C)x+C}{x^3(x-1)^2}.\end{aligned}$$

As result we obtain the following system of linear equations

$$\begin{cases} A+D &=0\\ -2A+B-D+E &=0\\ A-2B+C &=a\\ B-2C &=b\\ C &=c, \end{cases}$$

which leads to the following value of the coefficient D

$$D=-2b-3c-a.$$

Consequently, the integral $I=\int\frac{ax^2+bx+c}{x^3(x-1)^2}dx$ is rational if and only if $a+2b+3c=0$.

5.7.16. (a) We have

$$\begin{aligned}
P(x) &= x \\
Q_1(x) &= (x-1)(x+1)^2 = x^3 + x^2 - x - 1, \qquad Q_1'(x) = 3x^2 + 2x - 1 \\
Q_2(x) &= (x-1)(x+1) = x^2 - 1 \\
H(x) &= \frac{(3x^2+2x-1)(x^2-1)}{(x-1)(x+1)^2} = 3x - 1 \ .
\end{aligned}$$

Now assume that $P_1(x) = ax^2 + bx + c$ and $P_2(x) = dx + e$. Then we have

$$(2ax+b)(x^2-1) - (ax^2+bx+c)(3x-1) + (dx+e)(x^3+x^2-x-1) = x \ ,$$

which implies

$$\begin{cases} d = 0 \\ -a + d + e = - \\ a - 2b - d + e = 0 \\ -2a + b - 3c - d + e = 1 \\ -b + c - e = 0 \end{cases}$$

Thus, $a = -\frac{1}{8}$, $b = -\frac{1}{8}$, $c = -\frac{1}{4}$, $d = 0$ and $e = -\frac{1}{8}$. Hence,

$$\begin{aligned}
\int \frac{x}{(x-1)^2(x+1)^3}dx &= -\frac{1}{8}\frac{x^2+x+2}{x^3+x^2-x-1} - \frac{1}{8}\int \frac{dx}{x^2-1} \\
&= -\frac{1}{8}\frac{x^2+x+2}{x^3+x^2-x-1} + \frac{1}{16}\ln\left|\frac{1+x}{x-1}\right| + C \ .
\end{aligned}$$

(b) As $(x^3+1)^2 = (x+1)^2(x^2-x+1)^2$, we have

$$\begin{aligned}
P(x) &= 1 \\
Q_1(x) &= Q_2(x) = (x+1)(x^2-x+1) = x^3 + 1 \\
Q_1'(x) &= 3x^2 \\
H(x) &= \frac{3x^2 \cdot Q_2(x)}{Q_1(x)} = 3x^2 \ .
\end{aligned}$$

Now assume that $P_1(x) = ax^2 + bx + c$ and $P_2(x) = dx^2 + ex + f$. Then we have

$$(2ax+b)(x^3+1) - (ax^2+bx+c)(3x^2) + (dx^2+ex+f)(x^3+1) = 1 \ ,$$

which implies

$$\begin{cases} d = 0 \\ -a + e = 0 \\ -2b + f = 0 \\ -3c + d = 0 \\ 2a + e = 0 \\ b + f = 1 \end{cases}$$

Thus, $a = 0$, $b = \frac{1}{3}$, $c = d = e = 0$ and $f = \frac{2}{3}$. Hence,

$$\begin{aligned}
\int \frac{1}{(x^3+1)^2}dx &= \frac{1}{3}\frac{x}{x^3+1} + \frac{2}{3}\int \frac{dx}{x^3+1} \\
&= \frac{1}{3}\frac{x}{x^3+1} + \frac{2}{3}\int \frac{dx}{(x+1)(x^2-x+1)} = \frac{1}{3}\frac{x}{x^3+1} + \frac{2}{9}\int (\frac{1}{x+1} - \frac{x-2}{x^2-x+1})dx \\
&= \frac{1}{3}\frac{x}{x^3+1} + \frac{2}{9}\ln|x+1| - \frac{2}{9}\int \frac{x-\frac{1}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}dx + \frac{1}{3}\int \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}} \\
&= \frac{1}{3}\frac{x}{x^3+1} + \frac{2}{9}\ln|x+1| - \frac{1}{9}\ln|x^2-x+1| + \frac{2\sqrt{3}}{9}\arctan(\frac{\sqrt{3}}{3}(2x-1)) + C \ .
\end{aligned}$$

(c) We have

$$\begin{aligned}
P(x) &= 1 \\
Q_1(x) &= (1+x^2)^2 = x^4 + 2x^2 + 1 Q_1'(x) = 4x^3 + 4x^2 \\
Q_2(x) &= 1 + x^2 \\
H(x) &= \frac{(4x^3+6x^2)(1+x^2)}{(1+x^2)^2} = 4x \ .
\end{aligned}$$

Now assume that $P_1(x) = ax^3 + bx^2 + cx + d$ and $P_2(x) = ex + f$. Then we have

$$(3ax^2 + 2bx + c)(1+x^2) - (ax^3 + bx^2 + cx + d)(4x) + (ex+f)(1+x^2)^2 = 1 \ ,$$

which implies

$$\begin{cases} e = 0 \\ -a + f = 0 \\ -2b + 2e = 0 \\ 3a - 3c + 2f = 0 \\ 2b - 4d + e = 0 \\ c + f = 1 \end{cases}$$

Thus, $a = \frac{3}{8}$, $b = 0$, $c = \frac{5}{8}$, $d = e = 0$ and $f = \frac{3}{8}$. Hence,

$$\begin{aligned}
\int \frac{1}{(x^2+1)^3}dx &= \frac{1}{8}\frac{3x^3+5x}{(1+x^2)^2} + \frac{3}{8}\int \frac{1}{1+x^2}dx \\
&= \frac{1}{8}\frac{3x^3+5x}{(1+x^2)^2} + \frac{3}{8}\arctan x + C \ .
\end{aligned}$$

(d) We have

$$\begin{aligned}
P(x) &= x^2 \\
Q_1(x) &= Q_2(x) = x^2 + 2x + 2 \\
Q_1'(x) &= 2x + 2 = 2(x+1) \\
H(x) &= \frac{2(x+1)(x^2+2x+2)}{x^2+2x+2} = 2(x+1) \ .
\end{aligned}$$

Now assume that $P_1(x) = ax + b$ and $P_2(x) = cx + d$. Then we have

$$a(x^2+2x+2) - (ax+b)2(x+1) + (cx+d)(x^2+2x+2) = x^2 \ ,$$

which implies

$$\begin{cases} c = 0 \\ -a + 2c + d = 1 \\ -2b + 2c + 2d = 0 \\ 2a - 2b + 2d = 0 \end{cases}$$

Thus, $a = c = 0$ and $b = d = 1$. Hence,

$$\begin{aligned} \int \frac{x^2}{(x^2+2x+2)^2} dx &= \frac{1}{x^2+2x+2} + int \frac{1}{x^2+2x+2} dx \\ &= \frac{1}{x^2+2x+2} + \int \frac{1}{(x+1)^2+1} dx \\ &= \frac{1}{x^2+2x+2} + \arctan(x+1) + C \ . \end{aligned}$$

(e) As $(x^4+1)^2 = (x^2+\sqrt{2}x+1)^2(x^2-\sqrt{2}x+1)^2$, we have

$$\begin{aligned} P(x) &= 1 \\ Q_1(x) &= x^4 + 1 \ , \quad Q_1'(x) = 4x^3 \\ Q_2(x) &= x^4 + 1 \\ H(x) &= \frac{4x^3(x^4+1)}{x^4+1} = 4x^3 \ . \end{aligned}$$

Now assume that $P_1(x) = ax^3 + bx^2 + cx + d$ and $P_2(x) = ex^3 + fx^2 + gx + h$. Then we have

$$(3ax^2 + 2bx + c)(x^4+1) - (ax^3 + bx^2 + cx + d) \cdot 4x^3 + (ex^3 + fx^2 + gx + h)(x^4+1) = 1 \ ,$$

which implies

$$\begin{cases} e = 0 \\ -a + f = 0 \\ -2b + g = 0 \\ -3c + h = 0 \\ -4d + e = 0 \\ 3a + f = 0 \\ 2b + g = 0 \\ c + h = 1 \end{cases}$$

Thus, $a = b = d = e = f = g = 0$, $c = \frac{1}{4}$ and $h = \frac{3}{4}$. Hence,

$$\begin{aligned} \int \frac{1}{(x^4+1)^2} dx &= \frac{1}{4} \frac{x}{x^4+1} + \frac{3}{4} \int \frac{1}{x^4+1} dx \\ &= \frac{1}{4} \frac{x}{x^4+1} + \frac{3\sqrt{2}}{32} \ln \left| \frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} \right| + \frac{3\sqrt{2}}{16} \arctan(\sqrt{2}x+1) \\ &= \frac{3\sqrt{2}}{16} \arctan(\sqrt{2}x-1) + C \qquad \text{by } \textbf{5.7.14.} \text{ (q)} \ . \end{aligned}$$

(f) We have

$$\begin{aligned}
P(x) &= x^2 + 3x - 2 \\
Q_1(x) &= x^2 + x + 1 \ , \qquad Q_1'(x) = 2x + 1 \\
Q_2(x) &= (x^2 + x + 1)(x - 1) = x^3 - 1 \\
H(x) &= \frac{(2x+1)(x^2+x+1)(x-1)}{x^2+x+1} = (2x+1)(x-1) \ .
\end{aligned}$$

Now assume that $P_1(x) = ax + b$ and $P_2(x) = cx^2 + dx + e$. Then we have

$$a(x^3 - 1) - (ax + b)(2x + 1)(x - 1) + (cx^2 + dx + e)(x^2 + x + 1) = x^2 + 3x - 2 \ ,$$

which implies

$$\begin{cases} c = 0 \\ -a + c + d = 0 \\ a - 2b + c + d + e = 1 \\ a + b + d + e = 3 \\ -a + b + e = -2 \end{cases}$$

Thus, $a = \frac{5}{3}$, $b = \frac{2}{3}$, $c = 0$, $d = \frac{5}{3}$ and $e = -1$. Hence,

$$\begin{aligned}
&\int \frac{x^2+3x-2}{(x-1)(x^2+x+1)^2} dx = \frac{1}{3}\frac{5x+2}{x^2+x+1} + \frac{1}{3}\int \frac{5x-3}{x^3-1} dx \\
&= \frac{1}{3}\frac{5x+2}{x^2+x+1} + \frac{1}{3}\int (\frac{2}{3}\frac{1}{x-1} - \frac{1}{3}\frac{2x+1}{x^2+x+1} + \frac{4}{x^2+x+1}) dx \\
&= \frac{1}{3}\frac{5x+2}{x^2+x+1} + \frac{2}{9}\ln|x-1| - \frac{1}{9}\ln|x^2+x+1| + \frac{4}{3}\frac{2}{\sqrt{3}}\arctan(\frac{2}{\sqrt{3}}(x+\frac{1}{2})) + C \\
&= \frac{1}{3}\frac{5x+2}{x^2+x+1} + \frac{2}{9}\ln|x-1| - \frac{1}{9}\ln|x^2+x+1| + \frac{8\sqrt{3}}{9}\arctan(\frac{\sqrt{3}}{3}(2x+1)) + C \ .
\end{aligned}$$

(g) As $(x^4 - 1)^3 = (x^2 + 1)^3(x - 1)^3(x + 1)^3$, we have

$$\begin{aligned}
Q_1(x) &= (x^2+1)^2(x-1)^2(x+1)^2 = (x^4-1)^2, \quad Q_1'(x) = 8(x^4-1)x^3 \\
Q_2(x) &= (x^2+1)(x-1)(x+1) = x^4 - 1 \\
H(x) &= \frac{8(x^4-1)x^3(x^4-1)}{(x^4-1)^2} = 8x^3 \ .
\end{aligned}$$

Now assume that $P_1(x) = ax^7 + bx^6 + cx^5 + dx^4 + ex^3 + fx^2 + gx + h$ and $P_2(x) = ix^3 + jx^2 + kx + l$. Then we have

$$\begin{aligned}
&(7ax^6 + 6bx^5 + 5cx^4 + 4dx^3 + 3ex^2 + 2fx + g)(x^4 - 1) \\
&\quad -8x^3(ax^7 + bx^6 + cx^5 + dx^4 + ex^3 + fx^2 + gx + h) \\
&\quad +(ix^3 + jx^2 + kx + l)(x^4 - 1)^2 = 1 \ ,
\end{aligned}$$

which implies

$$\begin{cases} i = 0 \\ -a + j = 0 \\ -2b + k = 0 \\ -3c + l = 0 \\ -4d - 2i = 0 \\ -5e - 7a - 2j = 0 \\ -6f - 6b - 2k = 0 \\ -7g - 5c - 2l = 0 \\ -4d + i - 8h = 0 \\ j - 3e = 0 \\ k - 2f = 0 \\ -g + l = 1 \end{cases}$$

Thus, $a = b = d = e = f = h = i = j = k = 0$, $c = \frac{7}{32}$, $g = -\frac{11}{32}$ and $l = \frac{21}{32}$. Hence,

$$\begin{aligned} \int \frac{1}{(x^4-1)^3}dx &= \frac{1}{32}\frac{7x^5 - 11x}{(x^4-1)^2} + \frac{21}{32}\int \frac{dx}{x^4-1} \\ &= \frac{1}{32}\frac{7x^5 - 11x}{(x^4-1)^2} + \frac{21}{64}\int \frac{dx}{x^2-1} - \frac{21}{64}\int \frac{dx}{x^2+1} \\ &= \frac{1}{32}\frac{7x^5 - 11x}{(x^4-1)^2} - \frac{21}{128}\ln\left|\frac{1+x}{1-x}\right| - \frac{21}{64}\arctan x + C \ . \end{aligned}$$

5.7.17. (a)

$$\begin{aligned} \int \frac{x^3dx}{(x-1)^{100}} &= \int \frac{(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1}{(x-1)^{100}}dx \\ &= -\frac{1}{96(x-1)^{96}} - \frac{3}{97(x-1)^{97}} - \frac{3}{98(x-1)^{98}} - \frac{1}{99(x-1)^{99}} + C \ . \end{aligned}$$

(b) As $\frac{d}{(x^4+1)(x^4-1)} = \frac{d}{2}(\frac{1}{x^4-1} - \frac{1}{x^4+1})$, by **5.7.14.** (p) and (q), we have

$$\begin{aligned} \int \frac{d}{x^8-1}dx &= -\frac{d}{8}\ln\left|\frac{1+x}{1-x}\right| - \frac{d}{4}\arctan x - \frac{\sqrt{2}d}{16}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right| \\ &- \frac{\sqrt{2}d}{8}\arctan(\sqrt{2}x+1) - \frac{\sqrt{2}d}{16}\arctan(\sqrt{2}x-1) + C \ . \end{aligned}$$

(c) (substitution $x^4 = t$)

$$\begin{aligned} \int \frac{x^3dx}{x^8+3} &= \frac{1}{4}\int \frac{d(x^4)}{x^8+3} = \frac{1}{4}\int \frac{dt}{t^2+3} \\ &= \frac{1}{4}\frac{\sqrt{3}}{3}\arctan(\frac{\sqrt{3}}{3}t) + C = \frac{\sqrt{3}}{12}\arctan(\frac{\sqrt{3}}{3}x^4) + C \ . \end{aligned}$$

(d)*:

$$I = \int \frac{x^2+x}{x^6+1}dx = \int \frac{x^2}{x^6+1}dx + \int \frac{x}{x^6+1}dx.$$

For the first integral we substitute $u = x^3$, $du = 3x^2dx$ and for the second $v = x^2$, $dv = 2xdx$, so we obtain

$$\begin{aligned}
I &= \tfrac{1}{3}\int \frac{du}{u^2+1} + \tfrac{1}{2}\int \frac{dv}{v^3+1} = \tfrac{1}{3}\arctan u + \tfrac{1}{2}\int \frac{dv}{(v+1)(v^2-v+1)} \\
&= \tfrac{1}{3}\arctan x^3 + \tfrac{1}{6}\int \frac{dv}{v+1} - \tfrac{1}{6}\int \frac{v-2}{v^2-v+1} \\
&= \tfrac{1}{3}\arctan x^3 + \tfrac{1}{6}\ln(x^2+1) - \tfrac{1}{12}\ln(x^4-x^2+1) \\
&+ \tfrac{\sqrt{3}}{6}\arctan\left(\frac{2x^2-1}{\sqrt{3}}\right) + C.
\end{aligned}$$

(e)

$$\begin{aligned}
\int \frac{x^4-3}{x(x^8+3x^4+2)}dx &= \int \frac{x^4-3}{x(x^4+1)(x^4+2)}dx = \int \frac{(x^4+1)-4}{x(x^4+1)(x^4+2)}dx \\
&= \int \frac{1}{x(x^4+2)}dx - 4\int \frac{1}{x(x^4+1)(x^4+2)}dx \\
&= \int \frac{1}{x(x^4+2)}dx - 4\int \frac{(x^4+1)-x^4}{x(x^4+1)(x^4+2)}dx \\
&= -3\int \frac{1}{x(x^4+2)}dx + 4\int \frac{x^3}{(x^4+1)(x^4+2)}dx \\
&= -\frac{3}{2}\int \frac{(x^4+2)-x^4}{x(x^4+2)}dx + 4\int \frac{x^3}{x^4+1}dx - 4\int \frac{x^3}{x^4+2}dx \\
&= -\frac{3}{2}\int (\frac{1}{x} - \frac{x^3}{x^4+2})dx + \int \frac{d(x^4)}{x^4+1} - \int \frac{d(x^4)}{x^4+2} \\
&= -\frac{3}{2}(\ln|x| - \frac{1}{4}\ln(x^4+2)) + \ln(x^4+1) - \ln(x^4+2) + C \\
&= -\frac{3}{2}\ln|x| + \ln(x^4+1) - \frac{5}{8}\ln(x^4+2) + C\ .
\end{aligned}$$

(f)*: We apply the substitution $u = x^5$, $du = 5x^4dx$,

$$\begin{aligned}
I &= \int \frac{x^4dx}{(x^{10}-10)^2} = \frac{1}{5}\int \frac{du}{(u^2-10)^2} \\
&= \frac{1}{5}\left[-\tfrac{\sqrt{10}}{400}\int \frac{du}{u-\sqrt{10}} + \tfrac{1}{40}\int \frac{du}{(u-\sqrt{10})^2} + \tfrac{\sqrt{10}}{400}\int \frac{du}{u+\sqrt{10}}\right. \\
&\left.+\tfrac{1}{40}\int \frac{du}{(u+\sqrt{10})^2}\right] \\
&= \frac{\sqrt{10}}{2000}\ln|x^{10}-10| - \frac{1}{100}\frac{x^5}{x^{10}-10} + C.
\end{aligned}$$

(g) (substitution $x^4 = t$)

$$\begin{aligned}\int \frac{x^{11}dx}{x^8+3x^4+2} &= \frac{1}{4}\int \frac{x^8 d(x^4)}{x^8+3x^4+2} = \frac{1}{4}\int \frac{t^2 dt}{t^2+3t+2}\\ &= \frac{1}{4}\int (1+\frac{1}{t+1}-\frac{4}{t+2})dt\\ &= \frac{1}{4}t+\frac{1}{4}\ln|t+1| - \ln|t+2| + C\\ &= \frac{1}{4}x^4+\frac{1}{4}\ln(1+x^4)-\ln(2+x^4)+C\ .\end{aligned}$$

(h) (substitution $x^5 = y$)

$$\begin{aligned}\int \frac{x^9dx}{(x^{10}+x^5+5)^2} &= \frac{1}{10}(\int \frac{d(x^{10}+x^5+5)}{(x^{10}+x^5+5)^2} - \int \frac{d(x^5)}{(x^{10}+x^5+5)^2})\\ &-\frac{1}{10}(x^{10}+x^5+5)^{-1} - \frac{1}{10}\int \frac{dy}{(y^2+y+5)^2}\\ &= -\frac{1}{10}(x^{10}+x^5+5)^{-1} - \frac{1}{10}(\frac{2y+1}{19(y^2+y+5)} + \frac{4-3}{2-1}\frac{2}{19}\int \frac{dy}{y^2+y+5})\\ &= -\frac{1}{10}(x^{10}+x^5+5)^{-1} - \frac{1}{190}\frac{2y+1}{y^2+y+5} - \frac{1}{95}\int \frac{dy}{(y+\frac{1}{2})^2+(\frac{\sqrt{19}}{2})^2}\\ &= -\frac{1}{10}(x^{10}+x^5+5)^{-1} - \frac{1}{190}\frac{2x^5+1}{x^{10}+x^5+5} - \frac{1}{95}\frac{2}{\sqrt{19}}\arctan(\frac{2}{\sqrt{19}}(y+\frac{1}{2}))+C\\ &= -\frac{x^5+10}{95(x^{10}+x^5+5)} - \frac{2}{95\sqrt{19}}\arctan(\frac{2}{\sqrt{19}}(x^5+\frac{1}{2}))+C\ ,\end{aligned}$$

where for the computing of $\int \frac{dy}{(y^2+y+5)^2}$, we used the result from **5.7.18.** with $a=1$, $b=1$, $c=5$ and $n=2$.

(i) (substitution $x^n = t$)

$$\begin{aligned}\int \frac{x^{2n-1}}{x^n+1}dx &= \frac{1}{n}\int \frac{x^n d(x^n)}{x^n+1} = \frac{1}{n}\int \frac{tdt}{t+1}\\ &= \frac{1}{n}\int (1-\frac{1}{t+1})dt = \frac{1}{n}(t-\ln|t+1|)+C\\ &= \frac{1}{n}(x^n - \ln(1+x^n))+C\ .\end{aligned}$$

(j)*: We apply the substitution $u = x^n$, $du = nx^{n-1}dx$:

$$\begin{aligned}I = \int \frac{x^{3n-1}dx}{(x^{2n}+1)^2} &= \frac{1}{n}\int \frac{x^{2n}\cdot nx^{n-1}dx}{(x^{2n}+1)^2}\\ &= \frac{1}{n}\int \frac{u^2du}{(u^2+1)^2)} = \frac{1}{n}\int \frac{du}{u^2+1} - \frac{1}{n}\int \frac{du}{(u^2+1)^2}\\ &= \frac{1}{2n}\arctan x^n - \frac{1}{2n}\frac{x^n}{1+x^{2n}}+C.\end{aligned}$$

(k) Let $I = \int \frac{dx}{x(x^{10}+2)}$. Then

$$\begin{aligned}I &= \int \frac{x^{10}+2-1-x^{10}}{x(x^{10}+2)} = \int (\frac{1}{x} - \frac{1}{x(x^{10}+2)} - \frac{x^9}{x^{10}+2})dx\\ &= \ln|x| - \frac{1}{10}\ln(x^{10}+2) - I\ .\end{aligned}$$

Therefore,

$$I = \frac{1}{2}\ln|x| - \frac{1}{20}\ln(x^{10}+2) + C\ .$$

(l)

$$\begin{aligned}\int \frac{1}{x(x^{10}+1)^2}dx &= \int \frac{(x^{10}+1)-x^{10}}{x(x^{10}+1)^2}dx = \int (\frac{1}{x(x^{10}+1)} - \frac{x^9}{(x^{10}+1)^2})dx \\ &= \int \frac{(x^{10}+1)-x^{10}}{x(x^{10}+1)}dx - \int \frac{d(x^{10})}{10(x^{10}+1)^2} \\ &= \int (\frac{1}{x} - \frac{x^9}{x^{10}+1})dx + \frac{1}{10}(x^{10}+1)^{-1} \\ &= \ln|x| - \frac{1}{10}\ln(x^{10}+1) + \frac{1}{10(x^{10}+1)} + C\ .\end{aligned}$$

(m)

$$\begin{aligned}\int \frac{1-x^7}{x(1+x^7)}dx &= \int \frac{(1+x^7)-2x^7}{x(1+x^7)}dx \\ &= \int (\frac{1}{x} - 2\frac{x^6}{1+x^7})dx = \ln|x| - \frac{2}{7}\ln|1+x^7| + C\ .\end{aligned}$$

(n)

$$\begin{aligned}\int \frac{x^4-1}{x(x^4-5)(x^5-5x+1)}dx &= \int \frac{x^4-1}{(x^5-5x)(x^5-5x+1)}dx \\ &= \int (\frac{x^4-1}{x^5-5x} - \frac{x^4-1}{x^5-5x+1})dx \\ &= \frac{1}{5}\int \frac{d(x^5-5x)}{x^5-5x} - \frac{1}{5}\int \frac{d(x^5-5x+1)}{x^5-5x+1} \\ &= \frac{1}{5}\ln|x^5-5x| - \frac{1}{5}\ln|x^5-5x+1| + C\ .\end{aligned}$$

(o)*: We have

$$\begin{aligned}\int \frac{(x^2+1)dx}{x^4+x^2+1} &= \int \frac{(x^2+1)dx}{(x^2+x+1)(x^2-x+1)} = \frac{1}{2}\int \frac{dx}{x^2+x+1} + \frac{1}{2}\int \frac{dx}{x^2-x+1} \\ &= \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C.\end{aligned}$$

(p)

$$\begin{aligned}\int \frac{x^2-1}{x^4+x^3+x^2+x}dx &= \int \frac{(x+1)(x-1)}{(x+1)x(x^2+1)}dx \\ &= \int \frac{x-1}{x(x^2-4)}dx = \int \frac{1}{1+x^2}dx - \int \frac{x}{x^2(x^2+1)}dx \\ &= \int \frac{1}{1+x^2}dx - \int \frac{1}{x}dx + \int \frac{x}{1+x^2}dx \\ &= \arctan x - \ln|x| + \frac{1}{2}\ln(1+x^2) + C\ .\end{aligned}$$

(q) (substitution $x^2 = t$)

$$\int \frac{x^5 - x}{x^8 + 1} dx = \frac{1}{2} \int \frac{(x^4 - 1)d(x^2)}{x^8 + 1} = \frac{1}{2} \int \frac{t^2 - 1}{t^4 + 1} dt$$
$$= \frac{1}{2} \int \frac{t^2 - 1}{(t^2 + \sqrt{2}t + 1)(t^2 - \sqrt{2}t + 1)} dt \ .$$

We look for a, b, c and d such that

$$\frac{t^2 - 1}{t^4 + 1} = \frac{at + b}{t^2 + \sqrt{2}t + 1} + \frac{ct + d}{t^2 - \sqrt{2}t + 1} \ ,$$

i.e.

$$t^2 - 1 = (at + b)(t^2 - \sqrt{2}t + 1) + (ct + d)(t^2 + \sqrt{2}t + 1) \ ,$$

which implies

$$\begin{cases} a + c = 0 \\ \sqrt{2}(c - a) + d + b = 1 \\ a + c + \sqrt{2}(d - b) = 0 \\ b + d = -1 \end{cases}$$

Thus, $a = -\frac{\sqrt{2}}{2}$, $b = -\frac{1}{2}$, $c = \frac{\sqrt{2}}{2}$ and $d = -\frac{1}{2}$. Hence,

$$\int \frac{x^5 - x}{x^8 + 1} dx = \frac{1}{4} \int (\frac{\sqrt{2}t - 1}{t^2 - \sqrt{2}t + 1} - \frac{\sqrt{2}t + 1}{t^2 + \sqrt{2}t + 1}) dt$$
$$= \frac{\sqrt{2}}{8} \int \frac{d(t^2 - \sqrt{2}t + 1)}{t^2 - \sqrt{2}t + 1} - \frac{\sqrt{2}}{8} \int \frac{d(t^2 + \sqrt{2}t + 1)}{t^2 + \sqrt{2}t + 1}$$
$$= \frac{\sqrt{2}}{8} \ln |\frac{t^2 - \sqrt{2}t + 1}{t^2 + \sqrt{2}t + 1}| + C$$
$$= \frac{\sqrt{2}}{8} \ln |\frac{x^4 - \sqrt{2}x^2 + 1}{x^4 + \sqrt{2}x^2 + 1}| + C \ .$$

(r)

$$\int \frac{x^4 + 1}{x^6 + 1} dx = \int \frac{x^4 + 1}{(x^2 + 1)(x^4 - x^2 + 1)} dx$$
$$= \int \frac{1}{x^2 + 1} dx + \int \frac{x^2}{(x^2 + 1)(x^4 - x^2 + 1)} dx$$
$$= \int \frac{1}{x^2 + 1} dx + \frac{1}{3} \int \frac{d(x^3)}{x^6 + 1}$$
$$= \arctan x + \frac{1}{3} \arctan (x^3) + C \ .$$

5.7.18.* Notice

$$ax^2 + bx + c = a \left[\left(x + \frac{b}{2a} \right)^2 - \frac{b^2 - 4ac}{4a^2} \right] = a \left[u^2 - \frac{\Delta}{4a^2} \right],$$

where $u = x + \frac{b}{2a}$ and $\Delta = b^2 - 4ac$. Thus, for $n \geq 2$ we have

$$
\begin{aligned}
I_n &= \int \frac{dx}{(ax^2+bx+c)^n} = \frac{1}{a^n}\int \frac{du}{(u^2-\frac{\Delta}{4a^2})^n} \\
&= \frac{-4}{\Delta a^{n-2}}\left[\int \frac{du}{(u^2-\frac{\Delta}{4a^2})^{n-1}} - \int u \frac{udu}{(u^2-\frac{\Delta}{4a^2})^n}\right] \\
&= \frac{-4}{\Delta a^{n-2}}\left[\int \frac{du}{(u^2-\frac{\Delta}{4a^2})^{n-1}} - \left(\frac{u}{2(1-n)}\cdot\frac{1}{(u^2-\frac{\Delta}{4a^2})^n-1}\right.\right. \\
&\left.\left. -\frac{1}{2(1-n)}\int \frac{du}{(u^2-\frac{\Delta}{4a^2})^{n-1}}\right)\right] \\
&= \frac{-4a}{\Delta}\left[\frac{2n-3}{2n-2}\cdot\frac{1}{a^{n-1}}\int \frac{du}{(u^2-\frac{\Delta}{4a^2})^{n-1}} + \frac{1}{2n-2}\cdot\frac{u}{a^{n-1}(u^2-\frac{\Delta}{4a^2})^{n-1}}\right].
\end{aligned}
$$

Therefore, we have

$$I_n = \frac{4a}{-\Delta}\cdot\frac{2n-3}{2n-2}\cdot I_{n-1} + \frac{4a}{-\Delta}\cdot\frac{1}{2n-2}\frac{x+\frac{b}{2a}}{(ax^2+bx+c)^{n-1}}$$

and consequently we obtain the following reduction formula

$$I_n = \frac{2ax+b}{(n-1)(-\Delta)(ax^2+bx+c)^{n-1}} + \frac{2n-3}{n-1}\cdot\frac{2a}{-\Delta}I_{n-1}.$$

We compute

$$I_3 = \int \frac{dx}{(x^2+x+1)^3}.$$

Here we have $\Delta = -3$ and $a = 1$. Thus

$$I_1 = \int \frac{dx}{x^2+x+1} = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C,$$

so

$$I_2 = \frac{2x+1}{3(x^2+x+1)} + \frac{2}{3}\cdot\frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C,$$

hence

$$I_3 = \frac{2x+1}{6(x^2+x+1)^2} + \frac{2x+1}{3(x^2+x+1)} + \frac{4}{3\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

5.7.19.* By Taylor formula we have

$$P_n(x) = \sum_{k=0}^{n} \frac{P_n^{(k)}(a)}{k!}(x-a)^k.$$

Thus

$$\begin{aligned}\int \frac{P_n(x)}{(x-a)^{n+1}}dx &= \sum_{k=0}^{n}\int \frac{P_n^{(k)}(a)}{k!}(x-a)^{k-n-1}dx\\ &= \sum_{k=0}^{n}\frac{P_n^{(k)}(a)}{k!}\int (x-a)^{k-n-1}dx\\ &= \sum_{k=0}^{n-1}\frac{P_n^{(k)}(a)}{k!}\cdot\frac{(x-a)^{k-n}}{k-n}\\ &+ \frac{P_n^{(n)}(a)}{n!}\ln|x-a|+C.\end{aligned}$$

5.7.20.* We look for the complex roots of the equation

$$x^{2n} = -1 = \cos\pi + i\sin\pi\ .$$

The roots are

$$\alpha_k = \cos\varphi_k + i\sin\varphi_k, \qquad \bar{\alpha}_k = \cos\varphi_k - i\sin\varphi_k$$

where $k = 0,1,2,\dots,n-1$, and $\varphi_k = \frac{(2k+1)\pi}{2n}$. That means, we have exactly $2n$ roots β_j, $j = 1,2,\dots,2n$, which are

$$\beta_{2k+1} = \alpha_k, \qquad \beta_{2k+2} = \bar{\alpha}_k,$$

for $k = 0,1,2,\dots,n$. Therefore we have

$$P(x) := x^{2n}+1 = (x-\beta_1)(x-\beta_2)\dots(x-\beta_{2n}) =: \prod_{j=1}^{2n}(x-\beta_j).$$

Since all the roots β_j are not repeating, there is the following (complex) partial fraction decomposition of $\frac{1}{x^{2n}+1}$:

$$\begin{aligned}\frac{1}{x^{2n}+1} &= \sum_{j=0}^{2n}\frac{C_j}{x-\beta_j} = \frac{\sum_{j=1}^{2n}C_j\prod_{m\neq j}(x-\beta_m)}{\prod_{m=1}^{2n}(x-\beta_m)}\\ &= \frac{\sum_{j=1}^{2n}C_j\frac{P(x)-P(\beta_j)}{x-\beta_j}}{P(x)}.\end{aligned}$$

Consequently,

$$1 = \sum_{j=1}^{2n}C_m\frac{P(x)-P(\beta_j)}{x-\beta_j}.$$

In the last equality, by passing to the limit $x\to\beta_m$, $m = 1,2,\dots,2n$, we obtain

$$1 = C_m\cdot\lim_{x\to\beta_m}\frac{P(x)-P(\beta_m)}{x-\beta_m} = C_m\cdot P'(\beta_m),$$

i.e.

$$C_m = \frac{1}{P'(\beta_m)} = \frac{1}{2n\beta_m^{2n-1}} = \frac{\beta_m}{2n\beta_m^{2n}} = \frac{-\beta_m}{2n}.$$

Therefore, we have

$$\begin{aligned}\frac{1}{x^{2n}+1} &= \frac{-1}{2n}\sum_{j=1}^{2n}\frac{\beta_j}{x-\beta_j} = \frac{-1}{2n}\sum_{k=0}^{n-1}\left[\frac{\alpha_k}{x-\alpha_k}+\frac{\bar{\alpha}_k}{x-\bar{\alpha}_k}\right]\\ &= -\frac{1}{2n}\sum_{k=0}^{n-1}\frac{\alpha_k(x-\bar{\alpha}_k)+\bar{\alpha}_k(x-\alpha_k)}{(x-\alpha_k)(x-\bar{\alpha}_k)}\\ &= -\frac{1}{2n}\sum_{k=0}^{n-1}\frac{x(\alpha_k+\bar{\alpha}_k)-2\alpha_k\bar{\alpha}_k}{x^2-(\alpha_k+\bar{\alpha}_k)x+\alpha_k\bar{\alpha}_k} = -\frac{1}{2n}\sum_{k=0}^{n-1}\frac{2x\cos\varphi_k-2}{x^2-2\cos\varphi_k+1}\\ &= -\frac{1}{2n}\sum_{k=0}^{n-1}\frac{(x-\cos\varphi_k)2\cos\varphi_k+2\cos^2\varphi_k-2}{(x-\cos\varphi_k)^2+1-\cos^2\varphi_k}\\ &= -\frac{1}{2n}\sum_{k=0}^{n-1}\frac{(x-\cos\varphi_k)2\cos\varphi_k-2\sin^2\varphi_k}{(x-\cos\varphi_k)^2+\sin^2\varphi_k}.\end{aligned}$$

Thus

$$\begin{aligned}\int\frac{dx}{x^{2n}+1} &= \frac{-1}{2n}\sum_{k=0}^{n-1}\cos\varphi_k\ln(x^2-2x\cos\varphi_k+1)\\ &\quad+\frac{1}{n}\sum_{k=1}^{n-1}\sin\varphi_k\arctan\left(\frac{x-\cos\varphi_k}{\sin\varphi_k}\right)+C.\end{aligned}$$

5.7.21. (a) (substitution $\sqrt{x}=t$)

$$\begin{aligned}\int\frac{dx}{1+\sqrt{x}} &= \int\frac{d(t^2)}{1+t} = \int\frac{2t}{1+t}dt = \int(2-\frac{2}{1+t})dt\\ &= 2t-2\ln|t+1|+C = 2\sqrt{x}-2\ln(1+\sqrt{x})+C\ .\end{aligned}$$

(b) (substitution $\sqrt[6]{x}=t$)

$$\begin{aligned}\int\frac{dx}{x(1+2\sqrt{x}+\sqrt[3]{x})} &= \int\frac{6t^5dt}{t^6(1+2t^3+t^2)}\\ &= 6\int\frac{dt}{t(2t^3+t^2+1)} = \int\frac{6dt}{t(t+1)(2t^2-t+1)}\ .\end{aligned}$$

We look for a, b, c and d such that

$$\frac{6}{t(2t^3+t^2+1)} = \frac{a}{t}+\frac{b}{t+1}+\frac{ct+d}{2t^2-t+1}\ ,$$

i.e.

$$6 = a(t+1)(2t^2-t+1)+bt(2t^2-t+1)+(ct+d)t(t+1)\ ,$$

which implies

$$\begin{cases}2a+2b+c=0\\ a-b+c+d=0\\ b+d=0\\ a=6\end{cases}$$

Thus, $a = 6$, $b = -\frac{3}{2}$, $c = -9$ and $d = \frac{3}{2}$. Hence,

$$\begin{aligned}
\int \frac{dx}{x(1+2\sqrt{x}+\sqrt[3]{x})} &= 6\ln|t| - \frac{3}{2}\ln|t+1| - \frac{3}{2}\int \frac{6t-1}{2t^2-t+1}dt \\
&= 6\ln|t| - \frac{3}{2}\ln|t+1| - \frac{3}{2}\int \frac{(6t-\frac{3}{2})dt}{2t^2-t+1} - \frac{3}{4}\int \frac{dt}{2t^2-t+1} \\
&= 6\ln|t| - \frac{3}{2}\ln|t+1| - \frac{9}{4}\int \frac{d(2t^2-t+1)}{2t^2-t+1} - \frac{3}{8}\int \frac{dt}{(t-\frac{1}{4})^2+\frac{7}{16}} \\
&= 6\ln|t| - \frac{3}{2}\ln|t+1| - \frac{9}{4}\ln|2t^2-t+1| - \frac{3}{8}\frac{4}{\sqrt{7}}\arctan\left(\frac{4}{\sqrt{7}}(t-\frac{1}{4})\right) + C \\
&= \ln|x| - \frac{3}{2}\ln|\sqrt[6]{x}+1| - \frac{9}{4}\ln|2\sqrt[3]{x}-\sqrt[6]{x}+1| \\
&\quad -\frac{3\sqrt{7}}{14}\arctan\left(\frac{\sqrt{7}}{7}(4\sqrt[6]{x}-1)\right) + C\ .
\end{aligned}$$

(c) (substitution $\sqrt[3]{2+x} = t$, thus $x = t^3-2$ and $dx = 3t^2dt$)

$$\begin{aligned}
\int \frac{x\sqrt[3]{2+x}}{x+\sqrt[3]{2+x}}dx &= \int \frac{(t^3-2)t3t^2}{t^3-2+t}dt = 3\int \frac{t^3(t^2-2)}{t^3+t-2}dt \\
&= \int (3t^2-3t+\frac{3t(t-2)}{(t-1)(t^2+t+2)})dt \\
&= \frac{3}{4}t^4 - \frac{3}{2}t^2 + \int (-\frac{3}{4}\frac{1}{t-1} + \frac{3}{4}\frac{5t-2}{t^2+t+2})dt \\
&= \frac{3}{4}t^4 - \frac{3}{2}t^2 - \frac{3}{4}\ln|t-1| + \frac{15}{4}\int \frac{t+\frac{1}{2}}{(t+\frac{1}{2})^2+\frac{7}{4}}dt - \frac{27}{8}\int \frac{dt}{(t+\frac{1}{2})^2+\frac{7}{4}} \\
&= \frac{3}{4}t^4 - \frac{3}{2}t^2 - \frac{3}{4}\ln|t-1| + \frac{15}{8}\ln|t^2+t+2| - \frac{27}{8}\frac{2}{\sqrt{7}}\arctan\left(\frac{2}{\sqrt{7}}(t+\frac{1}{2})\right) + C \\
&= \frac{3}{4}(2+x)^{\frac{4}{3}} - \frac{3}{2}(2+x)^{\frac{2}{3}} - \frac{3}{4}\ln|\sqrt[3]{2+x}-1| + \frac{15}{8}\ln|(2+x)^{\frac{2}{3}}+(2+x)^{\frac{1}{3}}+2| \\
&\quad -\frac{27\sqrt{7}}{28}\arctan\left(\frac{\sqrt{7}}{7}(2(2+x)^{\frac{1}{3}}+1)\right) + C\ .
\end{aligned}$$

(d) (substitution $\sqrt[6]{x+1} = t$, thus $x = t^6-1$ and $dx = 6t^5dt$)

$$\begin{aligned}
\int \frac{1-\sqrt{x+1}}{1+\sqrt[3]{x+1}}dx &= \int \frac{6t^5(1-t^3)}{1+t^2}dt \\
&= \int (-6t^6+6t^4+6t^3-6t^2-6t+6+3\frac{2t}{1+t^2}-\frac{6}{1+t^2})dt \\
&= -\frac{6}{7}t^7 + \frac{6}{5}t^5 + \frac{3}{2}t^4 - 2t^3 - 3t^2 + 6t + 3\ln(1+t^2) - 6\arctan t + C \\
&= -\frac{6}{7}(x+1)^{\frac{7}{6}} - \frac{6}{5}(x+1)^{\frac{5}{6}} + \frac{3}{2}(x+1)^{\frac{2}{3}} - 2(x+1)^{\frac{1}{2}} - 3(x+1)^{\frac{1}{3}} \\
&\quad +6(x+1)^{\frac{1}{6}} + 3\ln(1+\sqrt[3]{x+1}) - 6\arctan\sqrt[6]{x+1} + C\ .
\end{aligned}$$

(e)*: We apply the substitution $t^{12} = x$, $12t^{11}dt = dx$ to get:

$$\begin{aligned}\int \frac{dx}{(1+\sqrt[4]{x})\sqrt[3]{x}} &= \int \frac{12t^{11}dt}{(1+t^3)t^4} = 12\int \frac{t^7dt}{t^3+1}\\ &= 12\int\left(t^4 - t - \tfrac{1}{3}\frac{1}{t+1} + \tfrac{1}{3}\frac{t+1}{t^2-t+1}\right)dt\\ &= \tfrac{12}{5}t^5 - 6t^2 - 4\ln|t+1| + 4\int\frac{t+1}{t^2-t+1}dt\\ &= \frac{12}{5}x^{\frac{5}{12}} - 6x^{\frac{1}{6}} - 4\ln(x^{\frac{1}{12}}+1)\\ &\quad + 2\ln(x^{\frac{1}{6}} - x^{\frac{1}{12}} + 1) + 4\sqrt{3}\text{arctan}\left(\frac{2x^{\frac{1}{12}}-1}{\sqrt{3}}\right) + C.\end{aligned}$$

(f)

$$\begin{aligned}&\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}dx = \int\frac{(\sqrt{x+1}-\sqrt{x-1})^2}{(x+1)-(x-1)}dx = \frac{1}{2}\int(2x - 2\sqrt{x^2-1})dx\\ &= \int(x-\sqrt{x^2-1})dx = \frac{1}{2}x^2 - \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\ln(x+\sqrt{x^2-1}) + C\ .\end{aligned}$$

(g)*: Since

$$I := \int\frac{\sqrt[3]{(x+1)^2(x-1)^4}}{dx} = \int\frac{\sqrt[3]{\frac{x+1}{x-1}}}{x^2-1}dx,$$

we apply the substitution $t^3 = \frac{x+1}{x-1}$, $x = \frac{t^3+1}{t^3-1}$, $dx = \frac{-6t^2dt}{(t^3-1)^2}$ and

$$x^2 - 1 = \frac{(t^3+1)^2}{(t^3-1)^2} - 1 = \frac{4t^3}{(t^3-1)^2}.$$

So, we get

$$I = \int\frac{t(-6t^2)dt}{\frac{4t^3}{(t^3-1)^2}(t^3-1)^2} = -\frac{3}{2}\int dt = -\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}} + C.$$

(h) (substitution $\sqrt[4]{\frac{x}{a-x}} = t$, thus $x = \frac{at^4}{1+t^4}$ and $dx = \frac{4at^3}{1+t^4}dt$)

$$\begin{aligned}\int\frac{x}{\sqrt[4]{x^3(a-x)}}dx &= \int\sqrt[4]{\frac{x}{a-x}}dx = \int\frac{4at^4}{(1+t^4)^2}dt\\ &= 4a\int\frac{1}{1+t^4}dt - 4a\int\frac{1}{(1+t^4)^2}dt\ .\end{aligned}$$

Thus, by **5.7.14.** (q),

$$\int \frac{x}{\sqrt[4]{x^3(a-x)}}dx = 4a\int \frac{1}{1+t^4}dt - 4a(\frac{1}{4}\frac{t}{1+t^4} + \frac{3}{4}\int \frac{1}{1+t^4}dt)$$
$$= a\int \frac{1}{1+t^4}dt - \frac{at}{1+t^4}$$
$$= \frac{\sqrt{2}a}{8}\ln|\frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1}| + \frac{\sqrt{2}a}{4}\arctan(\sqrt{2}t+1)$$
$$+\frac{\sqrt{2}a}{4}\arctan(\sqrt{2}t-1) - \frac{at}{1+t^4} + C$$
$$= \frac{\sqrt{2}a}{8}\ln|\frac{\sqrt{\frac{x}{a-x}}+\sqrt{2}\sqrt[4]{\frac{x}{a-x}}+1}{\sqrt{\frac{x}{a-x}}-\sqrt{2}\sqrt[4]{\frac{x}{a-x}}+1}| + \frac{\sqrt{2}a}{4}\arctan(\sqrt{2}\sqrt[4]{\frac{x}{a-x}}+1)$$
$$+\frac{\sqrt{2}a}{4}\arctan(\sqrt{2}\sqrt[4]{\frac{x}{a-x}}-1) - \frac{a\sqrt[4]{\frac{x}{a-x}}}{1+\frac{x}{a-x}} + C\ .$$

(i)*: Since

$$I := \int \frac{dx}{\sqrt[n]{(x-a)^{n+1}(x-b)^{n-1}}} = \int \frac{\sqrt[n]{\frac{x-b}{x-a}}}{(x-a)(x-b)}dx,$$

we apply the substitution $t^n = \frac{x-b}{x-a}$, $x = \frac{t^n a - b}{t^n - 1}$, $dx = \frac{nt^{n-1}(b-a)}{(t^n-1)^2}dt$, and

$$x-a = \frac{a-b}{t^n-1}, \qquad x-b = \frac{t^n(a-b)}{t^n-1}.$$

Consequently,

$$I = \frac{-n}{a-b}\int dt = \frac{n}{b-a}\sqrt[n]{\frac{x-b}{x-a}} + C.$$

5.7.22. (a)

$$\int \frac{x^2}{\sqrt{1+x+x^2}}dx = \int \frac{1+x+x^2-x-1}{\sqrt{1+x+x^2}}dx = \int \sqrt{1+x+x^2}dx$$
$$-\int \frac{x+1}{\sqrt{1+x+x^2}}dx$$
$$= \int \sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}dx - \int \frac{x+\frac{1}{2}}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}dx - \frac{1}{2}\int \frac{dx}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}$$
$$= \frac{x+\frac{1}{2}}{2}\sqrt{x^2+x+1} + \frac{3}{8}\ln|x+\frac{1}{2}+\sqrt{x^2+x+1}| - \sqrt{x^2+x+1}$$
$$-\frac{1}{2}\ln|x+\frac{1}{2}+\sqrt{x^2+x+1}| + C$$
$$= \frac{x}{2}\sqrt{x^2+x+1} - \frac{3}{4}\sqrt{x^2+x+1} - \frac{1}{8}\ln|x+\frac{1}{2}+\sqrt{x^2+x+1}| + C\ .$$

(b) (substitution $\frac{1}{x+1} = t$)

$$\int \frac{dx}{(x+1)\sqrt{x^2+x+1}} = \int \frac{dx}{(x+1)\sqrt{(x+1)^2-(x+1)+1}}$$
$$= \int \frac{dx}{(x+1)^2\sqrt{1-\frac{1}{x+1}+\frac{1}{(x+1)^2}}}$$
$$= -\int \frac{dt}{\sqrt{t^2-t+1}} = -\int \frac{dt}{\sqrt{(t-\frac{1}{2})^2+\frac{3}{4}}} = -\ln|t-\frac{1}{2}+\sqrt{t^2-t+1}|+C$$
$$= -\ln|\frac{1}{x+1}-\frac{1}{2}+\frac{\sqrt{x^2+x+1}}{x+1}|+C\ .$$

(c) (substitution $\frac{1}{1-x}=t$ and $\sqrt{2t-1}=y$)

$$\int \frac{dx}{(1-x)^2\sqrt{1-x^2}} = \int \frac{dx}{(1-x)^2\sqrt{-(x-1)^2+2(1-x)}} = \int \frac{d(\frac{1}{1-x})}{(1-x)\sqrt{\frac{2}{1-x}-1}}$$
$$= \int \frac{tdt}{\sqrt{2t-1}} = \int \frac{y^2+1}{2}dy = \frac{1}{6}y^3+\frac{y}{2}+C$$
$$= \frac{1}{6}(\sqrt{2t-1})^3+\frac{1}{2}\sqrt{2t-1}+C$$
$$= \frac{1}{6}(\sqrt{\frac{2}{1-x}-1})^3+\frac{1}{2}\sqrt{\frac{2}{1-x}-1}+C$$
$$= \frac{1}{6}(\frac{1+x}{1-x})^{\frac{3}{2}}+\frac{1}{2}(\frac{1+x}{1-x})^{\frac{1}{2}}+C\ .$$

(d) (substitution $\frac{1}{x}=t$)

$$\int \frac{\sqrt{x^2+2x+2}}{x}dx = \int \frac{x^2+2x+2}{x\sqrt{x^2+2x+2}}dx = \int \frac{x^2+x+x+2}{x\sqrt{x^2+2x+2}}dx$$
$$= \int \frac{x+1}{\sqrt{x^2+2x+2}}dx + \int \frac{1}{\sqrt{x^2+2x+2}}dx + 2\int \frac{dx}{x^2\sqrt{1+\frac{2}{x}+\frac{2}{x^2}}}$$
$$= \sqrt{x^2+2x+2}+\ln|x+1+\sqrt{x^2+2x+2}| - 2\int \frac{dt}{\sqrt{2t^2+2t+1}}$$
$$= \sqrt{x^2+2x+2}+\ln|x+1+\sqrt{x^2+2x+2}| - \frac{2}{\sqrt{2}}\int \frac{dt}{\sqrt{(t+\frac{1}{2})^2+\frac{1}{4}}}$$
$$= \sqrt{x^2+2x+2}+\ln|x+1+\sqrt{x^2+2x+2}| - \sqrt{2}\ln|t+\frac{1}{2}+\sqrt{t^2+t+\frac{1}{2}}|+C$$
$$= \sqrt{x^2+2x+2}+\ln|x+1+\sqrt{x^2+2x+2}| - \sqrt{2}\ln|\frac{1}{x}+\frac{1}{2}$$
$$+\frac{1+x+\frac{1}{2}x^2}{x}|+C\ .$$

(e) (substitution $\frac{1}{1+x}=t$)

$$\begin{aligned}
\int \frac{x}{(1+x)\sqrt{1-x-x^2}}dx &= \int \frac{dx}{\sqrt{1-x-x^2}} \\
&\quad - \int \frac{dx}{(1+x)\sqrt{-(x+1)^2+(x+1)+1}} \\
&= \int \frac{dx}{\sqrt{\frac{5}{4}-(x+\frac{1}{2})^2}} - \int \frac{dx}{(x+1)^2\sqrt{-1+\frac{1}{x+1}+\frac{1}{(x+1)^2}}} \\
&= \arcsin\left(\frac{2}{\sqrt{5}}(x+\frac{1}{2})\right) + \int \frac{dt}{\sqrt{t^2+t-1}} \\
&= \arcsin\left(\frac{1}{\sqrt{5}}(2x+1)\right) + \int \frac{dt}{\sqrt{(t+\frac{1}{2})^2-\frac{3}{4}}} \\
&= \arcsin\left(\frac{\sqrt{5}}{5}(2x+1)\right) + \ln|t+\frac{1}{2}+\sqrt{t^2+t-1}| + C \\
&= \arcsin\left(\frac{\sqrt{5}}{5}(2x+1)\right) + \ln|\frac{1}{x+1}+\frac{1}{2}+\frac{\sqrt{1-x-x^2}}{x+1}| + C \ .
\end{aligned}$$

(f)

$$\begin{aligned}
\int \frac{1-x+x^2}{\sqrt{1+x-x^2}}dx &= \int \frac{x^2-x-1+2}{\sqrt{1+x-x^2}}dx \\
&= -\int \sqrt{1+x-x^2}dx + 2\int \frac{dx}{\sqrt{1+x-x^2}} \\
&= -\int \sqrt{\frac{5}{4}-(x+\frac{1}{2})^2}dx + 2\int \frac{dx}{\sqrt{\frac{5}{4}-(x+\frac{1}{2})^2}} \\
&= -\frac{x-\frac{1}{2}}{2}\sqrt{1+x-x^2} - \frac{5}{8}\arcsin\left(\frac{2}{\sqrt{5}}(x-\frac{1}{2})\right) + 2\arcsin\left(\frac{2}{\sqrt{5}}(x-\frac{1}{2})\right) + C \\
&= \frac{1-2x}{4}\sqrt{1+x-x^2} + \frac{11}{8}\arcsin\left(\frac{\sqrt{5}}{5}(2x-1)\right) + C \ .
\end{aligned}$$

(g) (substitution $x = y + 1$ and $y = \sqrt{2}\sin t$)

$$\begin{aligned}
\int \frac{x^3}{1+2x-x^2}dx &= \int \frac{x^3}{2-(x-1)^2}dx = \int \frac{(y+1)^3 dy}{\sqrt{2-y^2}} \\
&= \int \frac{(\sqrt{2}\sin t+1)^3}{\sqrt{2}\cos t}\sqrt{2}\cos t dt \\
&= \int (\sqrt{2}\sin t+1)^3 dt \\
&= 2\sqrt{2}\int \sin^3 t dt + 6\int \sin^2 t dt + 3\sqrt{2}\int \sin t dt + \int dt \\
&= t - 3\sqrt{2}\cos t + 6\int \frac{1-\cos 2t}{2}dt - 2\sqrt{2}\int (1-\cos^2 t) d(\cos t) \\
&= t - 3\sqrt{2}\cos t + 3t - \frac{3}{2}\sin 2t - 2\sqrt{2}\cos t + \frac{2\sqrt{2}}{3}\cos^3 t + C \\
&= 4\arcsin(\frac{x-1}{\sqrt{2}}) - 5\sqrt{2}\cos(\arcsin(\frac{x-1}{\sqrt{2}})) \\
&\quad -\frac{3}{2}\sin(2\arcsin(\frac{x-1}{\sqrt{2}})) + \frac{2\sqrt{2}}{3}\cos^3(\arcsin(\frac{x-1}{\sqrt{2}})) + C \ .
\end{aligned}$$

(h) (substitution $x = \sinh t$)

$$\int \frac{x^{10}}{\sqrt{1+x^2}}dx = \int \frac{\sinh^{10} t \cosh t dt}{\cosh t} = \int \sinh^{10} t dt \ .$$

Define $I_n = \int \sinh^n t dt$, we then have

$$\begin{aligned}
I_n &= \int \sinh^{n-1} t d(\cosh t) = \sinh^{n-1} t \cosh t - \int \cosh t (n-1)\sinh^{n-2} t \cosh t dt \\
&= \sinh t \cosh t - (n-1)\int (1+\sinh^2 t)\sinh^{n-2} t dt \\
&= \sinh^{n-1} t \cosh t - (n-1)I_{n-2} - (n-1)I_n \ .
\end{aligned}$$

Thus, $I_n = \frac{1}{n}\sinh^{n-1} t \cosh t - \frac{n-1}{n}I_{n-2}$. Hence,

$$\begin{aligned}
\int \frac{x^{10}}{\sqrt{1+x^2}}dx &= I_{10} = \frac{1}{10}\sinh^9 t \cosh t - \frac{9}{10}I_8 \\
&= \frac{1}{10}\sinh^9 t \cosh t - \frac{9}{10}(\frac{1}{8}\sinh^7 t \cosh t - \frac{7}{8}I_6) \\
&= \cdots \\
&= \frac{1}{10}\sinh^9 t \cosh t - \frac{9}{80}\sinh^7 t \cosh t + \frac{21}{160}\sinh^5 t \cosh t - \frac{21}{128}\sinh^3 t \cosh t \\
&\quad + \frac{63}{256}\sinh t \cosh t - \frac{63}{256}t + C \\
&= (\frac{1}{10}x^9 - \frac{9}{80}x^7 + \frac{21}{160}x^5 - \frac{21}{128}x^3 + \frac{63}{256}x)\sqrt{1+x^2} \\
&\quad - \frac{63}{256}\arctan x + C \ .
\end{aligned}$$

(i)*: We apply the following trigonometric substitutions
$x = a\sin\varphi$, $dx = a\cos\varphi d\varphi$, $\sqrt{a^2-x^2} = a\cos\varphi$.

Then we have

$$\begin{aligned}
I &= \int x^4\sqrt{a^2-x^2}dx = a^6\int \sin^4\varphi\cos^2\varphi d\varphi \\
&= a^6\int \left(\tfrac{1}{2}\sin 2\varphi\right)^2\left(\tfrac{1}{2}\big(1-\cos 2\varphi\big)\right)d\varphi \\
&= \frac{a^6}{8}\left(\int \sin^2 2\varphi d\varphi - \int \sin^2 2\varphi\cos 2\varphi d\varphi\right) \\
&= \frac{a^6}{16}\varphi - \frac{a^6}{64}\sin 4\varphi - \frac{a^6}{8}\int \sin^2 2\varphi\cos 2\varphi d\varphi \quad (\text{ subst. } \ t=\sin 2\varphi) \\
&= \frac{a^6}{16}\varphi - \frac{a^6}{64}\sin 4\varphi - \frac{a^6}{16}\int t^2 dt = \frac{a^6}{16}\varphi - \frac{a^6}{64}\sin 4\varphi - \frac{a^6}{48}t^3 \\
&= \frac{a^6}{16}\varphi - \frac{a^6}{16}\big[\sin vp\cos\varphi(\cos^2\varphi-\sin^2\varphi)\big] - \frac{a^6}{6}\sin^3\varphi\cos^3\varphi + C \\
&= \frac{a^6}{16}\text{arcsin}\left(\frac{x}{a}\right) - \frac{a62}{16}x\sqrt{a^2-x^2}(a^2-2x^2) - \frac{a^3}{6}x^3\sqrt{(a^2-x^2)^3} + C \\
&= \frac{a^6}{16}\text{arcsin}\left(\frac{x}{a}\right) - \frac{a^2}{16}(xa^2-2x^3)\sqrt{a^2-x^2} - x^3\sqrt{(a^2-x^2)^3} + C.
\end{aligned}$$

Another method to compute the integral I: We make the substitution $z=x^2$, $\frac{1}{2}z^{-\frac{1}{2}}dz = dx$, so

$$I = \int z^2(a^2-z)^{\frac{1}{2}}\cdot\tfrac{1}{2}z^{-\frac{1}{2}}dx = \frac{1}{2}\int z^2\sqrt{\frac{-z+a^2}{z}}dz.$$

and by applying the substitution

$$t^2 = \frac{-z+a^2}{z}, \quad z = \frac{a^2}{1+t^2}, \quad dz = \frac{-a^2 2t}{(1+t^2)^2}dt,$$

we obtain that

$$\begin{aligned}
I &= -a^6\int \frac{t^2}{(1+t^2)^3}dt = -a^6\Big[-\tfrac{1}{4}\frac{t}{(1+t^2)^2} + \tfrac{1}{8}\frac{t}{1+t^2} + \tfrac{1}{8}\text{arctan}\, t\Big] + C \\
&= -a^6\left[-\tfrac{1}{4}\frac{\sqrt{\frac{a^2-x^2}{x^2}}}{\left(\frac{a^2}{x^2}\right)^2} + \tfrac{1}{8}\frac{\sqrt{\frac{a^2-x^2}{x^2}}}{\frac{a^2}{x^2}} + \tfrac{1}{8}\text{arctan}\left(\sqrt{\frac{a^2-x^2}{a^2}}\right)\right] + C \\
&= \frac{1}{8}(2a^2x^3 - a^4x)\sqrt{a^2-x^2} - \frac{a^6}{8}\text{arctan}\left(\frac{\sqrt{a^2-x^2}}{x}\right) + C.
\end{aligned}$$

(j) (substitution $x+2=y$ and $y=\cosh t$)

$$\begin{aligned}
\int \frac{x^3-6x^2+11x-6}{\sqrt{x^2+4x+3}}dx &= \int \frac{(x+2)^3-12(x+2)^2+47(x+2)-48}{\sqrt{(x+2)^2-1}}d(x+2)\\
&= \int \frac{y^3-12y+47y-48}{\sqrt{y^2-1}}dy = \int \frac{(\cosh^3 t - 12\cosh^2 t + 47\cosh t - 48)\sinh t}{\sinh t}dt\\
&= \int \cosh^2 t d(\sinh t) - 12\int \frac{1+\cosh 2t}{2}dt + 47\sinh t - 48t\\
&= \int (1+\sinh^2 t)d(\sinh t) - 6t - 3\int d(\sinh 2t) + 47\sinh t - 48t\\
&= \sinh t + \frac{1}{3}\sinh^3 t - 54t - 3\sinh 2t + 47\sinh t + C\\
&= 48\sinh t + \frac{1}{3}\sinh^3 t - 3\sinh 2t - 54t + C\\
&= \sinh t + \frac{1}{3}\sinh^3 t - 54t - 3\sinh 2t + 47\sinh t + C\\
&= 48\sinh(\arctan(x+2)) + \frac{1}{3}\sinh^3(\arctan(x+2))\\
&\quad -3\sinh 2(\arctan(x+2)) - 54(\arctan(x+2)) + C\ .
\end{aligned}$$

(k) (substitution $x = \sinh t$)

$$\begin{aligned}
\int \frac{dx}{x^3\sqrt{1+x^2}} \int \frac{\cosh t}{\sinh^3 t\cosh t}dt &= \int \frac{dt}{\sinh^3 t} := I\\
&= -\int \frac{1}{\sinh t}d(\coth t) = -\frac{\cosh t}{\sinh^2 t} + \int \frac{\cosh t}{\sinh t}d(\frac{1}{\sinh t})\\
&= -\frac{\cosh t}{\sinh^2 t} - \int \frac{\cosh^2 t}{\sinh^3 t}dt = -\frac{\cosh t}{\sinh^2 t} - \int \frac{1+\sinh^2 t}{\sinh^3 t}dt\\
&= -\frac{\cosh t}{\sinh^2 t} - I - \int \frac{1}{\sinh t}dt\ .
\end{aligned}$$

Thus,

$$\begin{aligned}
\int \frac{dx}{x^3\sqrt{1+x^2}} = I &= -\frac{1}{2}\frac{\cosh t}{\sinh^2 t} + \frac{1}{2}\int \frac{d(\cosh t)}{1-\cosh^2 t}\\
&= -\frac{1}{2}\frac{\cosh t}{\sinh^2 t} + \frac{1}{4}\ln|\frac{1+\cosh t}{1-\cosh t}| + C\\
&= -\frac{1}{2}\frac{\cosh(\arctan x)}{x^2} + \frac{1}{4}\ln|\frac{1+\cosh(\arctan x)}{1-\cosh(\arctan x)}| + C\\
&= -\frac{1}{2}\frac{\sqrt{1+x^2}}{x^2} + \frac{1}{4}\ln|\frac{1+\sqrt{1+x^2}}{1-\sqrt{1+x^2}}| + C\ .
\end{aligned}$$

(l) (substitution $x = \cosh t$)

$$\begin{aligned}
\int \frac{dx}{x^4\sqrt{x^2-1}} &= \int \frac{\sinh t dt}{\cosh^4 t \sinh t} = \int \frac{1}{\cosh^4 t} dt \\
&= \int \frac{\cosh^2 t - \sinh^2 t}{\cosh^4 t} dt = \int (1-\tanh^2 t) d(\tanh t) \\
&= \tanh t - \frac{1}{3}\tanh^3 t + C \\
&= \tanh(\arctan x) - \frac{1}{3}\tanh^3(\arctan x) + C \\
&= \frac{\sqrt{x^2-1}}{x} - \frac{1}{3}\frac{(x^2-1)^{\frac{3}{2}}}{x^3} + C\ .
\end{aligned}$$

(m)*: We begin with a remark about the strategy for integrating

$$I_n := \int \frac{x^n dx}{\sqrt{x^2+a}}, \quad a \neq 0.$$

Since

$$\begin{aligned}
I_n &= \int x^{n-1} \cdot \frac{x}{\sqrt{x^2+a}} dx \\
&= x^{n-1}\sqrt{x^2+a} - (n-1)\int x^{n-2}\sqrt{x^2+a}dx \\
&= x^{n-1}\sqrt{x^2+a} - (n-1)\int \frac{x^n + ax^{n-2}}{\sqrt{x^2+a}} dx \\
&= x^{n-1}\sqrt{x^2+a} - (n-1)I_n - a(n-1)I_{n-2},
\end{aligned}$$

thus

$$I_n = \frac{1}{n}x^{n-1}\sqrt{x^2+a} - \frac{n-1}{n}aI_{n-2}.$$

In particular we have

$$\begin{aligned}
I_0 &= \int \frac{dx}{\sqrt{x^2+a}} = \ln|x+\sqrt{x^2+a}| + C \\
I_1 &= \sqrt{x^2+a} + C \\
I_2 &= \frac{1}{2}x\sqrt{x^2+a} - \frac{1}{2}a\ln|x+\sqrt{x^2+a}| + C \\
I_3 &= \frac{1}{3}(x^2-2a)\sqrt{x^2+a} + C, \quad \text{etc. ...}
\end{aligned}$$

In order to compute the integral

$$I := \int \frac{dx}{(x-1)^3\sqrt{x^2+3x+1}}$$

we apply the substitution $x-1=\frac{1}{t}$, $t=\frac{1}{x-1}$, $x=\frac{1}{t}+1$ and $dx=-\frac{1}{t^2}dt$. Then we obtain

$$\begin{aligned}
I &= \int \frac{t^3(-1)dt}{\sqrt{\left(\frac{1}{t}+1\right)^2 + 3\left(\frac{1}{t}+1\right) + 1}t^2} = -\int \frac{tdt}{\frac{\sqrt{5t^2+5t+1}}{t}} \\
&= -\frac{1}{\sqrt{5}}\int \frac{t^2}{\sqrt{t^2+t+\frac{1}{5}}} dt.
\end{aligned}$$

Next, we substitute $u = t + \frac{1}{2}$, $t = u - \frac{1}{2}$, $du = dt$, thus

$$\begin{aligned}
I &= \frac{-1}{\sqrt{5}} \int \frac{(u-\frac{1}{2})^2}{\sqrt{u^2 - \frac{1}{4} + \frac{1}{5}}} du \\
&= \frac{-1}{\sqrt{5}} \left[\frac{1}{2} u \sqrt{u^2 - \frac{1}{20}} + \frac{1}{2} \cdot \frac{1}{20} \ln \left| u + \sqrt{u^2 - \frac{1}{20}} \right| \right. \\
&\quad \left. - \sqrt{u^2 - \frac{1}{20}} + \frac{1}{4} \ln \left| u + \sqrt{u^2 - \frac{1}{20}} \right| \right] + C \\
&= -\frac{1}{2\sqrt{5}} (u-2) \sqrt{u^2 - \frac{1}{20}} - \frac{11}{40\sqrt{5}} \ln \left| u + \sqrt{u^2 - \frac{1}{20}} \right| + C \\
&= -\frac{1}{2\sqrt{5}} (t - \frac{3}{2}) \sqrt{t^2 + t + \frac{1}{5}} \\
&\quad - \frac{11}{40\sqrt{5}} \ln \left| t + \frac{1}{2} + \sqrt{t^2 + t + \frac{1}{5}} \right| + C \\
&= \tfrac{-1}{20} \frac{5-3x}{(x-1)^2} \sqrt{x^2 + 3x + 1} - \frac{11}{40\sqrt{5}} \ln \left| \frac{\sqrt{5}(x+1) + 2\sqrt{x^2+3x+1}}{x-1} \right| + C.
\end{aligned}$$

(n) (substitution $x + 1 = y$ and $y = \cosh t$)

$$\int \frac{dx}{(x+1)^5 \sqrt{x^2 + 2x}} = \int \frac{dy}{y^5 \sqrt{y^2 - 1}} = \int \frac{\sinh t dt}{\cosh^5 t \sinh t} = \int \frac{dt}{\cosh^5 t} \; .$$

Let $I_n = \int \frac{dt}{\cosh^n t}$. Then we have

i)

$$I_1 = \int \frac{dt}{\cosh t} = \int \frac{d(\sinh t)}{1 + \sinh^2 t} = \arctan\left(\sinh t\right) \; .$$

ii)

$$\begin{aligned}
I_n &= \int \frac{1}{\cosh^{n-2} t} d(\tanh t) = \frac{\sinh t}{\cosh^{n-1} t} - \int \frac{\sinh t}{\cosh t} d(\cosh^{2-n} t) \\
&= \frac{\sinh t}{\cosh^{n-1} t} + (n-2) \int \frac{\sinh t}{\cosh t} \cosh^{-(n-2)-1} t \sinh t dt \\
&= \frac{\sinh t}{\cosh^{n-1} t} + (n-2) \int \frac{\cosh^2 t - 1}{\cosh^n t} dt \\
&= \frac{\sinh t}{\cosh^{n-1} t} + (n-2) I_{n-2} - (n-2) I_n \; .
\end{aligned}$$

Thus,

$$I_n = \frac{1}{n-1} \frac{\sinh t}{\cosh^{n-1} t} + \frac{n-2}{n-1} I_{n-2} \; .$$

Therefore,

$$\int \frac{dx}{(x+1)^5\sqrt{x^2+2x}} = I_5 = \frac{1}{4}\frac{\sinh t}{\cosh^4 t} + \frac{3}{4}(\frac{1}{2}\frac{\sinh t}{\cosh^2 t} + \frac{1}{2}\arctan(\sinh t)) + C$$
$$= \frac{1}{4}\frac{\sinh t}{\cosh^4 t} + \frac{3}{8}\frac{\sinh t}{\cosh^2 t} + \frac{3}{8}\arctan(\sinh t) + C$$
$$= \frac{1}{4}\frac{\sinh(\arctan(1+x))}{\cosh^4(\arctan(1+x))} + \frac{3}{8}\frac{\sinh(\arctan(1+x))}{\cosh^2(\arctan(1+x))}$$
$$+\frac{3}{8}\arctan(\sinh(\arctan(1+x))) + C$$
$$= \frac{1}{4}\frac{\sqrt{x(x+2)}}{(x+1)^4} + \frac{3}{8}\frac{\sqrt{x(x+2)}}{(x+1)^2} + \frac{3}{8}\arctan(\sqrt{x(x+2)})$$

.

5.7.23. (a) We use the Euler substitution $\sqrt{x^2+x+1} = t - x$. Thus $x = \frac{t^2-1}{2t+1}$, $dx = \frac{2(t^2+t+1)}{(2t+1)^2}$ and

$$\int \frac{dx}{x+\sqrt{x^2+x+1}} = \int \frac{2(t^2+t+1)}{(2t+1)^2 t}dt \ .$$

Now that

$$\frac{2(t^2+t+1)}{(2t+1)^2 t} = \frac{A}{t} + \frac{B}{2t+1} + \frac{C}{(2t+1)^2} \ ,$$

i.e. $2t^2+2t+2 = A(2t+1)^2 + B(2t+1) + Ct$. So $A = 2$, $B = -3$ and $C = -3$. Hence

$$\int \frac{dx}{x+\sqrt{x^2+x+1}} = 2\int \frac{1}{t}dt - 3\int \frac{1}{2t+1}dt - 3\int \frac{1}{(2t+1)^2}dt$$
$$= 2\ln|t| - \frac{3}{2}\ln|2t+1| + \frac{3}{2}\frac{1}{2t+1} + C$$
$$= 2\ln|x+\sqrt{x^2+x+1}| - \frac{3}{2}\ln|2x+2\sqrt{x^2+x+1}+1|$$
$$+\frac{3}{2}\frac{1}{2x+2\sqrt{x^2+x+1}+2} + C \ .$$

(b) We use the Euler substitution $\sqrt{x^2-2x+2} = t + x$. Thus $x = -\frac{1}{2}\frac{t^2-2}{1+t}$, $dx = -\frac{1}{2}\frac{t^2+2t+2}{(1+t)^2}$. Hence

$$\int x\sqrt{x^2-2x+2}dx = \int -\frac{1}{2}\frac{t^2-2}{1+t}(t - \frac{1}{2}\frac{t^2-2}{1+t})(-\frac{1}{2}\frac{t^2+2t+2}{(1+t)^2})dt$$
$$= \frac{1}{8}\int \frac{(t^2-2)(t^2+2t+2)^2}{(1+t)^4}dt$$
$$= \frac{1}{8}\int (1+t)^2 - 2(1+t) + 1 - 4(1+t)^{-1} - (1+t)^{-2} + 2(1+t)^{-3} - (1+t)^{-4}dt$$
$$= \frac{1}{24}t^3 - \frac{1}{2}\ln|1+t| + \frac{1}{8}\frac{1}{1+t} + \frac{1}{8}\frac{1}{(1+t)^2} + \frac{1}{24}\frac{1}{(1+t)^3} + C$$
$$= \frac{1}{24}(\sqrt{x^2-2x+2}-x)^3 - \frac{1}{2}\ln|1+\sqrt{x^2-2x+2}-x|$$
$$+\frac{1}{8}\frac{1}{1+\sqrt{x^2-2x+2}-x} + \frac{1}{8}\frac{1}{(1+\sqrt{x^2-2x+2}-x)^2}$$
$$+\frac{1}{24}\frac{1}{(1+\sqrt{x^2-2x+2}-x)^3} + C \ .$$

(c)*: We apply the Euler substitution to the integral

$$I := \int \frac{dx}{1+\sqrt{1-2x-x^2}},$$

i.e. we substitute $\sqrt{1-2x-x^2} = tx-1$, $x = 2\frac{t-1}{t^2+1}$, $dx = 2\frac{-t^2+2t+1}{(t^2+1)^2}dt$ and $1+\sqrt{1-2x+x^2} = 2\frac{(t-1)t}{t^2+1}$. Then we have

$$\begin{aligned} I &= \int \frac{2(-t^2+2t+1)(t^2+1)dt}{(t^2+1)^2 2(t-1)t} = \int \frac{-t^2+2t+1}{(t^2+1)(t-1)t}dt \\ &= -\int \frac{dt}{t} + \int \frac{dt}{t-1} - 2\int \frac{dt}{t^2+1} = \ln\left|\frac{t-1}{t}\right| - 2\arctan t + C \\ &= \ln\left|\frac{\sqrt{1-2x-x^2}+1-x}{\sqrt{1-2x-x^2}+1}\right| - 2\arctan(\sqrt{1-2x-x^2}+1) + C. \end{aligned}$$

(d) We use the Euler substitution $\sqrt{x^2+3x+2} = t(x+1)$. Thus $x = \frac{t^2-2}{1-t^2}$, $dx = \frac{-2t}{(1-t^2)^2}$ and

$$\begin{aligned} \int \frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}dx &= \int \frac{\frac{t^2-2}{1-t^2} - t(\frac{t^2-2}{1-t^2}+1)}{\frac{t^2-2}{1-t^2} + t(\frac{t^2-2}{1-t^2}+1)} \cdot \frac{-2t}{(1-t^2)^2}dt \\ &= -2\int \frac{t(t+2)}{(1+t)(t^2-1)(t^2-t-2)}dt = -2\int \frac{t(t+2)}{(1+t)^3(t-2)(t-1)}dt\ . \end{aligned}$$

Now that

$$\frac{-2t(t+2)}{(1+t)^3(t-2)(t-1)} = \frac{1}{t-1} + \frac{b}{t-2} + \frac{c}{t+1} + \frac{d}{(t+1)^2} + \frac{e}{(t+1)^3}\ ,$$

i.e.

$$\begin{aligned} -2(t+2)t\ &= a(t-2)(t+1)^3 + b(t-1)(t+1)^3 + c(t-1)(t-2)(t+1)^2 \\ &\quad + d(t+1)(t-1)(t-2) + e(t-1)(t-2)\ . \end{aligned}$$

So $a = \frac{3}{4}$, $b = -\frac{16}{27}$, $c = -\frac{17}{108}$, $d = \frac{5}{18}$ and $e = \frac{1}{3}$. Hence

$$\begin{aligned} \int \frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}dx &= -\frac{1}{6}\frac{1}{(1+t)^2} - \frac{5}{18}\frac{1}{1+t} - \frac{107}{108}\ln|t+1| \\ &\quad -\frac{16}{27}\ln|t-2| + \frac{3}{4}\ln|t-1| + C \\ &= -\frac{1}{6}\frac{(x+1)^2}{(\sqrt{x^2+3x+2}+x+1)^2} - \frac{5}{18}\frac{x+1}{\sqrt{x^2+3x+2}+x+1} \\ &\quad -\frac{107}{108}\ln\left|\frac{\sqrt{x^2+3x+2}+x+1}{x+1}\right| - \frac{16}{27}\ln\left|\frac{\sqrt{x^2+3x+2}}{x+1} - 2\right| \\ &\quad +\frac{3}{4}\ln\left|\frac{\sqrt{x^2+3x+2}}{x+1} - 1\right| + C\ . \end{aligned}$$

(e)*: Again, in order to compute

$$I := \int \frac{dx}{[1+\sqrt{x^2+x}]^2},$$

we apply the Euler substitution $\sqrt{x^2+x} = t + x$, $t = \sqrt{x^2+x} - x$, $x = \frac{t^2}{1-2t}$, $dx = -2\frac{t(t-1)}{(1-2t)^2}dt$, and $1 + \sqrt{x^2+x} = \frac{1-t-t^2}{1-2t}$. Then, we have

$$I = \int \frac{-2t^2+2t}{(t^2+t-1)^2}dt = \int \frac{-2t^2+2t}{\left[\left(t+\frac{1}{2}\right)^2 - \frac{5}{4}\right]^2}dt.$$

Next, we substitute $u = t + \frac{1}{2}$, $du = dt$, hence

$$\begin{aligned}
I &= \int \frac{-2u^2+4u-\frac{3}{2}}{(u^2-\frac{5}{4})^2}du = \int \frac{-2u^2+4u-\frac{3}{2}}{(u-\frac{\sqrt{5}}{2})^2(u+\frac{\sqrt{5}}{2})^2}du \\
&= \int \left[-\tfrac{2}{25}\frac{\sqrt{5}}{u-\frac{\sqrt{5}}{2}} - \tfrac{2}{5}\frac{2-\sqrt{5}}{(u-\frac{\sqrt{5}}{2})^2} + \tfrac{2}{25}\frac{\sqrt{5}}{u+\frac{\sqrt{5}}{2}} - \tfrac{2}{5}\frac{2+\sqrt{5}}{(u+\frac{\sqrt{5}}{2})^2}\right] \\
&= \frac{2}{5\sqrt{5}}\ln\left|\frac{2u+\sqrt{5}}{2u-\sqrt{5}}\right| + \tfrac{2}{5}\frac{4u-5}{4u^2-5} + C \\
&= \frac{2}{5\sqrt{5}}\ln\left|\frac{2t+1+\sqrt{5}}{2t+1-\sqrt{5}}\right| + \frac{2}{20}\cdot\frac{4t-3}{t^2+t-1} + C,
\end{aligned}$$

where $t = \sqrt{x(1+x)} - x$.

5.7.24. (a) (substitution $x + \frac{1}{2} = y$ and $y = \frac{1}{2}\cosh t$)

$$\begin{aligned}
\int \sqrt{x^3+x^4}dx &= \int x\sqrt{(x+\frac{1}{2})^2 - \frac{1}{4}}dx = \int (y-\frac{1}{2})\sqrt{y^2-\frac{1}{4}}dy \\
&= \int \frac{1}{2}(\cosh t - 1)\frac{1}{2}\sinh t\frac{1}{2}\sinh t dt \\
&= \frac{1}{8}\int \sinh^2 t d(\sinh t) - \frac{1}{8}\int \sinh^2 t dt \\
&= \frac{1}{24}\sinh^3 t - \frac{1}{8}\int \frac{\cosh 2t - 1}{2}dt \\
&= \frac{1}{24}\sinh^3 t - \frac{1}{32}\sinh 2t + \frac{1}{16}t + C \\
&= \frac{1}{24}\sinh^3(\arctan(2x+1)) - \frac{1}{32}\sinh(2\arctan(2x+1)) \\
&\quad + \frac{1}{16}\arctan(2x+1) + C\ .
\end{aligned}$$

(b) (substitution $\sqrt[6]{x} = t$, thus $x = t^6$ and $dx = 6t^5dt$)

$$\begin{aligned}
&\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}dx = \int \frac{t^3 6t^5}{(1+t^2)^2}dt = 6\int \frac{t^8}{(t^2+1)^2}dt \\
&= 6[\int (t^4-2t^2+3)dt - \int \frac{4t^2+3}{(1+t^2)^2}dt] \\
&= \frac{6}{5}t^5 - \frac{1}{3}t^3 + 18t - 24\int \frac{1}{1+t^2}dt + 6\int \frac{1}{(1+t^2)^2}dt \\
&= \frac{6}{5}t^5 - 4t^3 + 18t - 24\arctan t + 6\int (\frac{1}{2}\frac{1}{1+t^2} + \frac{1}{2}\frac{1-t^2}{(1+t^2)^2})dt \\
&= \frac{6}{5}t^5 - 4t^3 + 18t - 21\arctan t + 3\frac{t}{1+t^2} + C \\
&= \frac{6}{5}x^{\frac{5}{6}} - 4\sqrt{x} + 18\sqrt[6]{x} - 21\arctan \sqrt[6]{x} + 3\frac{\sqrt[6]{x}}{1+\sqrt[3]{x}} + C\ .
\end{aligned}$$

(c) (substitution $\sqrt[3]{x} = y$ and $y = \sinh t$)

$$\begin{aligned}
&\int \frac{xdx}{\sqrt{1+\sqrt[3]{x^2}}} = \int \frac{y^3 3y^2}{\sqrt{1+y^2}}dy = 3\int \frac{y^5 dy}{\sqrt{1+y^2}} = 3\int \frac{\sinh^5 t\cosh t dt}{\cosh t} \\
&= 3\int (\cosh^2 t - 1)^2 d(\cosh t) = \int (3\cosh^4 t - 6\cosh^2 t + 3)d(\cosh t) \\
&= \frac{3}{5}\cosh^5 t - 2\cosh^3 t + 3\cosh t + C \\
&= \frac{3}{5}\cosh^5(\arctan \sqrt[3]{x}) - 2\cosh^3(\arctan \sqrt[3]{x}) \\
&\quad +3\cosh(\arctan \sqrt[3]{x}) + C\ .
\end{aligned}$$

(d) (substitution $x = \sin t$)

$$\begin{aligned}
&\int \frac{x^5 dx}{\sqrt{1-x^2}} = \int \frac{\sin^5 t\cos t dt}{\cos t} = \int \sin^5 t dt = -\int (1-\cos^2 t)^2 d(\cos t) \\
&= \int (-1 + 2\cos^2 t - \cos^4 t)d(\cos t) \\
&= -\cos t + \frac{2}{3}\cos^3 t - \frac{1}{5}\cos^5 t + C \\
&= -\cos(\arcsin x) + \frac{2}{3}\cos^3(\arcsin x) - \frac{1}{5}\cos^5(\arcsin x) + C\ .
\end{aligned}$$

(e)*: In order to compute the integral

$$I = \int \frac{dx}{\sqrt[3]{1+x^3}} = \int \left(1+x^3\right)^{-\frac{1}{3}} dx,$$

first, we apply the substitution $z = x^3$, $x = z^{\frac{1}{3}}$, $dx = \frac{1}{3}z^{-\frac{2}{3}}dz$, to obtain

$$I = \frac{1}{3}\int \frac{1}{z}\sqrt[3]{\frac{z}{1+z}}dz,$$

and, next, the substitution $t^3 = \frac{z}{1+z}$, $z = \frac{t^3}{1-t^3}$ and $dz = \frac{-3t^2}{(1-t^3)^2}dt$, hence

$$
\begin{aligned}
I &= \frac{1}{3}\int \frac{1-t^3}{t^3}\cdot t\frac{-3t^2}{(1-t^3)^2}dt = \int \frac{dt}{t^3-1}\\
&= \frac{1}{3}\int \frac{dt}{t-1} - \frac{1}{3}\int \frac{t+2}{t^2+t+1}dt\\
&= \frac{1}{3}\ln|t-1| - \frac{1}{6}\ln(t^2+t+1) - \frac{1}{\sqrt{3}}\arctan\left(\frac{2t+1}{\sqrt{3}}\right) + C\\
&= \frac{1}{3}\ln\left|\frac{x}{\sqrt[3]{x^3+1}} - 1\right| - \frac{1}{6}\ln\left|\frac{x^2\sqrt[3]{x^2+1} + x\sqrt[3]{(x^2+1)^2}}{x^3+1} + 1\right|\\
&\quad - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x}{\sqrt{3}\sqrt[3]{x^3+1}} + \frac{1}{\sqrt{3}}\right) + C.
\end{aligned}
$$

(f) First we apply the substitution $x^4 = y$, thus $dx = \frac{1}{4}y^{-\frac{3}{4}}dy$, to obtain

$$
I = \int \frac{dx}{\sqrt[4]{1+x^4}} = \frac{1}{4}\int \frac{1}{y}\sqrt[4]{\frac{y}{1+y}}dy\ .
$$

Then, we apply the substitution $t^4 = \frac{y}{1+y}$, thus $y = \frac{t^4}{1-t^4}$ and $dy = \frac{4t^3}{(1-t^4)^2}dt$.

$$
\begin{aligned}
I &= \frac{1}{4}\int \frac{1-t^4}{t^4}\cdot t\frac{4t^3}{(1-t^4)^2}dt = \int \frac{1}{1-t^4}dt\\
&= \frac{1}{2}\int \frac{1}{1-t^2}dt + \frac{1}{2}\int \frac{1}{1+t^2}dt\\
&= \frac{1}{4}\ln|\frac{1+t}{1-t}| + \frac{1}{2}\arctan t + C\\
&= \frac{1}{4}\ln|\frac{\sqrt[4]{1+x^4}+\sqrt[4]{x}}{\sqrt[4]{1+x^4}-\sqrt[4]{x}}| + \frac{1}{2}\arctan(\sqrt[4]{\frac{x^4}{1+x^4}}) + C\ .
\end{aligned}
$$

(g)*: We have

$$
I = \int \frac{dx}{x^6\sqrt[6]{1+x^6}} = \int x^{-1}(1+x^6)^{-\frac{1}{6}}dx.
$$

We apply the substitution $z = x^6$, $x = z^{\frac{1}{6}}$, $dx = \frac{1}{6}z^{-\frac{5}{6}}dz$. Then we have

$$
I = \frac{1}{6}\int z^{-1}(1+z)^{-\frac{1}{6}}dx.
$$

We substitute $t^6 = 1+z$, $6t^5dt = dz$ to get

$$
\begin{aligned}
I &= \int \frac{t^4}{t^6-1}dt = \int \frac{t^4dt}{(t-1)(t^2+t+1)(t+1)(t^2-t+1)}\\
&= \tfrac{1}{6}\int \frac{dt}{t-1} - \tfrac{1}{6}\int \frac{dt}{t+1} - \tfrac{1}{6}\int \frac{t-1}{t^2+t+1}dt + \tfrac{1}{6}\int \frac{t+1}{t^2-t+1}dt\\
&= \tfrac{1}{6}\ln\left|\frac{t-1}{t+1}\right| + \tfrac{1}{6}\left[-\tfrac{1}{2}\ln(t^2+t+1) + \sqrt{3}\arctan\left(\frac{2t+1}{\sqrt{3}}\right)\right.\\
&\quad \left. +\tfrac{1}{2}\ln|t^2-t+1| + \sqrt{3}\arctan\left(\frac{2t-1}{\sqrt{3}}\right)\right] + C\\
&= \frac{1}{6}\ln\left|\frac{t-1}{t+1}\right| + \frac{1}{12}\ln\left|\frac{t^2-t+1}{t^2+t+1}\right| + \sqrt{3}\arctan\left(\frac{\sqrt{3}t}{1-t^2}\right) + C,
\end{aligned}
$$

where $t = \sqrt[6]{1+x^6}$.

5.7.25. (a) (substitution $\sin x = t$)

$$\begin{aligned}\int \cos^5 x dx &= \int \cos^4 x d(\sin x) = \int (1-\sin^2 x)^2 d(\sin x) = \int (1-t^2)^2 dt\\ &= \int (1-2t^2+t^4)dt = t - \frac{2}{3}t^3 + \frac{1}{5}t^5 + C\\ &= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C\ .\end{aligned}$$

(b)

$$\begin{aligned}\int \sin^6 x dx &= \int (\frac{1-\cos 2x}{2})^3 dx\\ &= \frac{1}{8}\int (1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x)dx\\ &= \frac{x}{8} - \frac{3}{16}\sin 2x + \frac{3}{8}\int \frac{1+\cos 4x}{2}dx - \frac{1}{16}\int (1-\sin^2 2x)d(\sin 2x)\\ &= \frac{5}{16}x - \frac{3}{16}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{16}\sin 2x + \frac{1}{16}\frac{1}{3}\sin^3 2x + C\\ &= \frac{5}{16}x - \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C\ .\end{aligned}$$

(c)

$$\begin{aligned}\int \cos^6 x dx &= \int (\frac{1+\cos 2x}{2})^3 dx\\ &= \frac{1}{8}\int (1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x)dx\\ &= \frac{x}{8} + \frac{3}{16}\sin 2x + \frac{3}{8}\int \frac{1+\cos 4x}{2}dx + \frac{1}{16}\int (1-\sin^2 2x)d(\sin 2x)\\ &= \frac{5}{16}x + \frac{3}{16}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{16}\sin 2x - \frac{1}{16}\frac{1}{3}\sin^3 2x + C\\ &= \frac{5}{16}x + \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C\ .\end{aligned}$$

(d)

$$\begin{aligned}\int \sin^2 x\cos^4 x dx &= \int (1-\cos^2 x)\cos^4 x dx\\ &= \int \cos^4 x dx - \int \cos^6 x dx\\ &= \int (\frac{1+\cos 2x}{2})^2 dx - \int \cos^6 x dx\\ &= \frac{1}{4}\int (1 + 2\cos 2x + \frac{1+\cos 4x}{2})dx - \int \cos^6 x dx\\ &= \frac{1}{4}\int (\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x)dx - \int \cos^6 x dx\\ &= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x - \int \cos^6 x dx\ .\end{aligned}$$

Therefore, by **5.7.25.** (c),

$$\int \sin^2 x \cos^4 x dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x - \frac{5}{16}x - \frac{1}{4}\sin 2x - \frac{3}{64}\sin 4x$$
$$+\frac{1}{48}\sin^3 2x + C$$
$$= \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C \ .$$

(e)

$$\int \sin^4 x \cos^5 x dx = \int \sin^4 x(1-\sin^2 x)^2 d(\sin x)$$
$$= \int \sin^4 x(1 - 2\sin^2 x + \sin^4 x)d(\sin x)$$
$$= \int (\sin^4 x - 2\sin^6 x + \sin^8 x)d(\sin x)$$
$$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C \ .$$

(f)

$$\int \sin^5 x \cos^5 x dx = \int (\frac{1}{2}\sin 2x)^5 dx = \frac{1}{32}\int \sin^5 2x dx$$
$$= -\frac{1}{64}\int (1-\cos^2 2x)^2 d(\cos 2x)$$
$$= -\frac{1}{64}\int (1 - 2\cos^2 2x + \cos^4 2x)d(\cos 2x)$$
$$= -\frac{1}{64}\cos 2x + \frac{1}{32}\frac{1}{3}\cos^3 2x - \frac{1}{64}\frac{1}{5}\cos^5 2x + C$$
$$= -\frac{1}{64}\cos 2x + \frac{1}{96}\cos^3 2x - \frac{1}{320}\cos^5 2x + C \ .$$

(g) (substitution $\cos x = t$)

$$\int \frac{\sin^3 x}{\cos^4 x}dx = -\int \frac{\sin^2 x}{\cos^4 x}d(\cos x) = -\int \frac{1-\cos^2 x}{\cos^4 x}d(\cos x)$$
$$= -\int \frac{1-t^2}{t^4}dt = \int \frac{1}{t^2} - \frac{1}{t^4}dt = -\frac{1}{t} + \frac{1}{3}\frac{1}{t^3} + C$$
$$= -\frac{1}{\cos x} + \frac{1}{3}\frac{1}{\cos^3 x} + C \ .$$

(h) (substitution $\cos x = t$)

$$\begin{aligned}
\int \frac{\cos^4 x}{\sin^3 x}dx &= -\int \frac{\cos^4 x}{\sin^4 x}d(\cos x) = -\int \frac{t^4}{(1-t^2)^2}dt \\
&= \int (-1 + \frac{1-2t^2}{(1-t^2)^2})dt = \int ((-1) + \frac{1}{1-t^2} - (\frac{t}{1-t^2})^2)dt \\
&= -t + \frac{1}{2}\ln|\frac{1+t}{1-t}| - \int (\frac{1}{2}\frac{1}{1-t} - \frac{1}{2}\frac{1}{1+t})^2 dt \\
&= -t + \frac{1}{2}\ln|\frac{1+t}{1-t}| - \frac{1}{4}\int (\frac{1}{(1-t)^2})dt + \frac{1}{2}\int (\frac{1}{1-t^2})dt - \frac{1}{4}\int (\frac{1}{(1+t)^2})dt \\
&= -t + \frac{1}{2}\ln|\frac{1+t}{1-t}| + \frac{1}{4}\frac{1}{t-1} + \frac{1}{4}\ln|\frac{1+t}{1-t}| + \frac{1}{4}\frac{1}{t+1} + C \\
&= -t + \frac{3}{4}\ln|\frac{1+t}{1-t}| + \frac{1}{4}\frac{1}{t-1} + \frac{1}{4}\frac{1}{1+t} + C \\
&= -\cos x + \frac{3}{4}\ln|\frac{1+\cos x}{1-\cos x}| + \frac{1}{4}\frac{1}{\cos x - 1} + \frac{1}{4}\frac{1}{\cos x + 1} + C \ .
\end{aligned}$$

(i) (substitution $\cos x = t$)

$$\int \frac{1}{\sin^3 x}dx = \int \frac{\sin x dx}{\sin^4 x} = -\int \frac{d(\cos x)}{(1-\cos^2 x)^2} = -\int \frac{dt}{(1-t^2)^2} \ .$$

As $(1-t^2)^2 = (1-t)^2(1+t)^2$, we look for a, b, c and d such that

$$\frac{-1}{(1-t^2)^2} = \frac{a}{1-t} + \frac{b}{1+t} + \frac{c}{(1-t)^2} + \frac{d}{(1+t)^2} \ ,$$

i.e.

$$-1 = a(1-t)(1+t)^2 + b(1+t)(1-t)^2 + c(1+t)^2 + d(1-t)^2 \ ,$$

which implies

$$\begin{cases} -a+b=0 \\ -a-b+c+d=0 \\ a-b+2c-2d=0 \\ a+b+c+d=-1 \end{cases}$$

Thus, $a = b = c = d = -\frac{1}{4}$. Hence,

$$\begin{aligned}
\int \frac{dx}{\sin^3 x} &= -\frac{1}{4}(-\ln|t-1| + \ln|t+1| - \frac{1}{t-1} - \frac{1}{t+1}) + C \\
&= -\frac{1}{4}\ln|\frac{t+1}{t-1}| - \frac{1}{2}\frac{t}{1-t^2} + C \\
&= -\frac{1}{4}\ln|\frac{1+\cos x}{1-\cos x}| - \frac{\cos x}{2\sin^3 x} + C \ .
\end{aligned}$$

(j) (substitution $\sin x = t$)

$$\int \frac{dx}{\cos^3 x} = \int \frac{\cos x}{\cos^4 x}dx = \int \frac{d(\sin x)}{(1-\sin^2 x)^2} = \int \frac{dt}{(1-t^2)^2} \ .$$

Thus by by **5.7.18**,

$$\begin{aligned}\int \frac{dx}{\cos^3 x} &= \int \frac{dt}{(1-t^2)^2} = \frac{2t}{-4(t^2-1)} - \frac{1}{2}\int \frac{1}{t^2-1}dt \\ &= \frac{t}{2(1-t^2)} + \frac{1}{4}\ln\left|\frac{1+t}{1-t}\right| + C \\ &= \frac{\sin x}{2\cos^2 x} + \frac{1}{4}\ln\frac{1+\sin x}{1-\sin x} + C\ .\end{aligned}$$

(k) (substitution $2x = t$)

$$\begin{aligned}\int \frac{dx}{\sin^4 x\cos^4 x} &= \int \frac{dx}{(\frac{1}{2}\sin 2x)^4} = 8\int \frac{d(2x)}{\sin^4 2x} = \int \frac{dt}{\sin^4 t} \\ &= 8\int \frac{\cos^2 t + \sin^2 t}{\sin^4 t}dt = -8\int (\cot^2 t + 1)d(\cot t) \\ &= -\frac{8}{3}\cot^3 t - 8\cot t + C \\ &= -\frac{8}{3}\cot^3 2x - 8\cot 2x + C\ .\end{aligned}$$

(l)*: We have

$$I = \int \frac{dx}{\sin^3 x\cos^5 x} = \int \frac{(1+\tan^2 x)^3 dx}{\tan^3 x\cos^2 x},$$

so by making the substitution $t = \tan x$, $dt = \frac{dx}{\cos^2 x}$ we get

$$\begin{aligned}I &= \int \frac{(1+t^2)^3}{t^3}dt = \int (t^{-3} + 3t^{-1} + 3t + t^3)dt \\ &= -\frac{-1}{2t^2} + 3\ln|t| + \frac{3}{2}t^2 + \frac{1}{4}t^4 + C \\ &= -\frac{1}{2}\cot^2 x + 3\ln|\tan x| + \frac{3}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C.\end{aligned}$$

(m)

$$\begin{aligned}\int \frac{dx}{\sin x\cos^4 x}dx &= \int \frac{\sin^2 x + \cos^2 x}{\sin x\cos^4 x}dx \\ &= -\int \frac{d(\cos x)}{\cos^4 x} + \int \frac{1}{\sin x\cos^2 x}dx \\ &= \frac{1}{3\cos^3 x} + \int \frac{\sin^2 x + \cos^2 x}{\sin x\cos^2 x}dx \\ &= \frac{1}{3\cos^3 x} - \int \frac{d(\cos x)}{\cos^2 x} + \int \frac{dx}{\sin x} \\ &= \frac{1}{3\cos^3 x} + \frac{1}{\cos x} - \int \frac{d(\cos x)}{1-\cos^2 x} \\ &= \frac{1}{3\cos^3 x} + \frac{1}{\cos x} - \frac{1}{2}\ln\frac{1+\cos x}{1-\cos x} + C\ .\end{aligned}$$

(n) (substitution $\cos x = t$)

$$\begin{aligned}
\int \tan^5 x dx &= \int \frac{\sin^5 x}{\cos^5 x} dx = -\int \frac{(1-\cos^2 x)^2 d(\cos x)}{\cos^5 x} \\
&= -\int \frac{1-2\cos^2 x+\cos^4 x}{\cos^5 x} d(\cos x) = -\int (\frac{1}{t^5} - 2\frac{1}{t^3} + \frac{1}{t}) dt \\
&= \frac{1}{4t^4} - \frac{1}{t^2} - \ln|t| + C \\
&= \frac{1}{4\cos^4 x} - \frac{1}{\cos^2 x} - \ln|\cos x| + C \ .
\end{aligned}$$

(o)*: We substitute $t = \cot x$, $x = \operatorname{arccot} t$, $dx = \frac{-dt}{1+t^2}$, so

$$\begin{aligned}
I = \int \cot^6 x \, dx &= -\int \frac{t^6}{1+t^2} dt = \int \left(-t^4 + t^2 - 1 + \frac{1}{1+t^2}\right) dt \\
&= -\frac{1}{5}t^5 + \frac{1}{3}t^3 - t - \operatorname{arccot} t + C \\
&= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - x + C.
\end{aligned}$$

(p)

$$\int \frac{\sin^4 x}{\cos^6 x} dx = \int \tan^4 x d(\tan x) = \frac{1}{5}\tan^5 x + C \ .$$

(q)*: We have

$$I = \int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}} = \int \frac{(1+\tan^2 x) dx}{\sqrt{\tan^3 x}\cos^2 x}.$$

We substitute $t = \tan x$, $dt = \frac{dx}{\cos^2 x}$, to get

$$\begin{aligned}
I &= \int \frac{1+t^2}{t^{\frac{3}{2}}} dt = \int (t^{-\frac{3}{2}} + t^{\frac{1}{2}}) dt \\
&= -2t^{-\frac{1}{2}} + \frac{2}{3}t^{\frac{3}{2}} + C = -2\sqrt{\cot x} + \frac{2}{3}\sqrt{\tan^3 x} + C.
\end{aligned}$$

(r) (substitution $\tan x = t$, thus $dx = \frac{dt}{1+t^2}$)

$$\int \frac{dx}{\sqrt{\tan x}} = \int \frac{dt}{(1+t^2)\sqrt{t}} = 2\int \frac{d(\sqrt{t})}{1+(\sqrt{t})^4} \ .$$

Thus by **5.7.14.** (q),

$$\begin{aligned}
\int \frac{dx}{\sqrt{\tan x}} &= 2\int \frac{d(\sqrt{t})}{1+(\sqrt{t})^4} = \frac{\sqrt{2}}{4}\ln\left|\frac{t+\sqrt{2t}+1}{t-\sqrt{2t}+1}\right| + \frac{\sqrt{2}}{2}\arctan(\sqrt{2t}+1) \\
&\quad + \frac{\sqrt{2}}{2}\arctan(\sqrt{2t}-1) + C \\
&= \frac{\sqrt{2}}{4}\ln\left|\frac{\tan x+\sqrt{2\tan x}+1}{\tan x-\sqrt{2\tan x}+1}\right| + \frac{\sqrt{2}}{2}\arctan(\sqrt{2\tan x}+1) \\
&\quad + \frac{\sqrt{2}}{2}\arctan(\sqrt{2\tan x}-1) + C \ .
\end{aligned}$$

5.7.26. (a) We have for $n > 2$.

$$\begin{aligned}
I_n &= \int \frac{dx}{\sin^n x} = \int \frac{\sin^2 + \cos^2 x}{\sin^n x} dx \\
&= \int \frac{1}{\sin^{n-2} x} dx + \int \cos x \frac{\cos x}{\sin^n x} dx \\
&= I_{n-2} - \frac{1}{n-1} \int \cos x d(\frac{1}{\sin^{n-1} x}) \\
&= I_{n-2} - \frac{1}{n-1} \frac{\cos x}{\sin^{n-1} x} + \frac{1}{n-1} \int \frac{1}{\sin^{n-1} x} d(\cos x) \\
&= I_{n-2} - \frac{1}{n-1} \frac{\cos x}{\sin^{n-1} x} - \frac{1}{n-1} \int \frac{1}{\sin^{n-2} x} dx \\
&= \frac{n-2}{n-1} I_{n-2} - \frac{1}{n-1} \frac{\cos x}{\sin^{n-1} x} \ .
\end{aligned}$$

Therefore, we have the reduction formula:

$$I_n = -\frac{1}{n-1} \frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1} I_{n-2} \ .$$

(b)* We have for $n \geq 2$

$$\begin{aligned}
J_n &= \int \frac{dx}{\cos^n x} = \int \frac{\sin^2 x + \cos^2 x}{\cos^n x} dx \\
&= \int \sin x \frac{\sin x}{\cos^n x} dx + \int \frac{dx}{\cos^{n-2} x} \\
&= \frac{1}{n-1} \cdot \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}.
\end{aligned}$$

Therefore, we have the reduction formula

$$J_n = \frac{1}{n-1} \cdot \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} J_{n-2}.$$

5.7.27. (a) (substitution $\tan \frac{x}{2} = t$)

$$\begin{aligned}
&\int \frac{1}{2\sin x - \cos x + 5} dx = \int \frac{2dt}{(2\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} + 5)(1+t^2)} = \int \frac{dt}{3t^2 + 2t + 2} \\
&= \frac{1}{3} \int \frac{dt}{(t + \frac{1}{3})^2 + \frac{5}{9}} = \frac{1}{3} \frac{3}{\sqrt{5}} \arctan\left(\frac{3}{\sqrt{5}}(t + \frac{1}{3})\right) + C \\
&= \frac{\sqrt{5}}{5} \arctan\left(\frac{\sqrt{5}}{5}(3\tan\frac{x}{2} + 1)\right) + C \ .
\end{aligned}$$

(b) (substitution $\tan \frac{x}{2} = t$)

$$\int \frac{dx}{\sin x(2 + \cos x)} = \int \frac{\frac{2}{1+t^2} dt}{\frac{2t}{1+t^2}(2 + \frac{1-t^2}{1+t^2})} = \int \frac{1+t^2}{t(3+t^2)} dt \ .$$

We look for a, b and c such that

$$\frac{1+t^2}{t(3+t^2)} = \frac{a}{t} + \frac{bt+c}{3+t^2} \ ,$$

i.e. $t^2 + 1 = a(3+t^2) + (bt+c)t$. Thus $a = \frac{1}{3}$, $b = \frac{2}{3}$ and $c = 0$. Hence,

$$\begin{aligned}\int \frac{dx}{\sin x(2+\cos x)} &= \frac{1}{3}\ln|t| + \frac{2}{3}\int \frac{tdt}{t^2+3}\\ &= \frac{1}{3}\ln|t| + \frac{1}{3}\ln|t^2+3| + C = frac13\ln|t(t^2+3)| + C\\ &= \frac{1}{3}\ln|\tan\frac{x}{2}(\tan^2\frac{x}{2}+3)| + C\ .\end{aligned}$$

(c) (substitution $\tan\frac{x}{2} = t$)

$$\int \frac{\sin^2 x}{\sin x + 2\cos x}dx = \int \frac{(\frac{2t}{1+t^2})^2(\frac{2}{1+t^2})}{\frac{2t}{1+t^2} + 2\frac{1-t^2}{1+t^2}}dt = \int \frac{-4t^2}{(1+t^2)^2(t^2-t-1)}dt := I\ .$$

Next, we use the Ustrogradski method to compute I. We have $Q_1(t) = 1+t^2$, $Q_1'(t) = 2t$, $Q_2(t) = (1+t^2)(t^2-t-1)$, $P(t) = -4t$ and $H(t) = 2t(t^2-t-1)$. Now assume that $P_1(at+b)$ and $P_2(t) = ct^3+dt^2+et+f$. Then it follows that

$$\begin{aligned}&a(1+t^2)(t^2-t-1) - (at+b)2t(t^2-t-1)\\ &+(ct^3+dt^2+et+f)(1+t^2) = -4t^2\ ,\end{aligned}$$

which implies

$$\begin{cases} c = 0\\ -a+d = 0\\ a-2b+c+e = 0\\ 2a+2b+d+f = -4\\ -a+2b+e = 0\\ -a+f = 0\end{cases}$$

Thus, $a = -\frac{4}{5}$, $b = -\frac{2}{5}$, $c = 0$, $d = -\frac{4}{5}$, $e = 0$ and $f = -\frac{4}{5}$. Hence,

$$\begin{aligned}\int \frac{\sin^2 x}{\sin x + 2\cos x}dx &= \frac{1}{5}\frac{-4t-2}{1+t^2} - \frac{4}{5}\int \frac{t^2+1}{(1+t^2)(t^2-t-1)}dt\\ &= \frac{1}{5}\frac{-4t-2}{1+t^2} - \frac{4}{5}\int \frac{dt}{(t-\frac{1}{2})^2 - \frac{5}{4}}\\ &= -\frac{2}{5}\frac{2t+1}{1+t^2} + \frac{4}{5}\frac{1}{2}\frac{2}{\sqrt{5}}\ln|\frac{\frac{\sqrt{5}}{2}+t-\frac{1}{2}}{\frac{\sqrt{5}}{2}-t+\frac{1}{2}}| + C\\ &= -\frac{2}{5}\frac{2t+1}{1+t^2} + \frac{4\sqrt{5}}{25}\ln|\frac{\frac{\sqrt{5}}{2}+t-\frac{1}{2}}{\frac{\sqrt{5}}{2}-t+\frac{1}{2}}| + C\\ &= -\frac{2}{5}\frac{2\tan\frac{x}{2}+1}{1+\tan^2\frac{x}{2}} + \frac{4\sqrt{5}}{25}\ln|\frac{\frac{\sqrt{5}}{2}+\tan\frac{x}{2}-\frac{1}{2}}{\frac{\sqrt{5}}{2}-\tan\frac{x}{2}+\frac{1}{2}}| + C\ .\end{aligned}$$

(d) (substitution $2x = y$ and $\tan\frac{y}{2} = t$)

$$\begin{aligned}\int \frac{\sin^2 x}{1+\sin^2 x}dx &= \int (1-\frac{1}{1+\sin^2 x})dx = x - \int \frac{1}{1+\frac{1-\cos 2x}{2}}dx\\ &= x - \int \frac{d(2x)}{3-\cos 2x} = x - \int \frac{dy}{3-\cos y} = x - \int \frac{\frac{2dt}{1+t^2}}{3-\frac{1-t^2}{1+t^2}}\\ &= x - \int \frac{2dt}{2+4t^2} = x - \frac{1}{\sqrt{2}}\int \frac{d(\sqrt{2}t)}{1+(\sqrt{2}t)^2}\\ &= x - \frac{\sqrt{2}}{2}\arctan(\sqrt{2}t) + C\\ &= x - \frac{\sqrt{2}}{2}\arctan(\sqrt{2}\tan x) + C\ .\end{aligned}$$

(e)*: Since

$$I = \int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \int \frac{dx}{(a^2\tan^2 x + b^2)\cos^2 x},$$

we apply the substitution $t = \tan x$, $dt = \frac{dx}{\cos^2 x}$, to get

$$\begin{aligned}I &= \int \frac{dt}{a^2t^2+b^2} = \frac{1}{a^2}\int \frac{dt}{t^2+(\frac{b}{a})^2} = \frac{1}{ab}\arctan\left(\frac{at}{b}\right) + C\\ &= \frac{1}{ab}\arctan\left(\frac{a\tan x}{b}\right) + C.\end{aligned}$$

(f) (substitution $\tan x = t$)

$$\begin{aligned}\int \frac{\cos^2 x dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} &= \int \frac{dx}{(a^2\tan^2 x + b^2)^2\cos^2 x}\\ &= \int \frac{d(\tan x)}{(a^2\tan^2 x + b^2)^2} = \int \frac{dt}{(a^2t^2+b^2)^2}\\ &= \frac{1}{2}\frac{t}{b^2(a^2t^2+b^2)} + \frac{1}{2}\frac{\arctan(\frac{at}{b})}{ab^3} + C\ .\end{aligned}$$

(g) (substitution $x + \frac{\pi}{4} = t$)

$$\begin{aligned}\int \frac{\sin x\cos x}{\sin x + \cos x}dx &= \frac{1}{2}\int \frac{\sin 2x}{\sqrt{2}(\sin x\cos\frac{\pi}{4} + \cos x\sin\frac{\pi}{4})}dx\\ &= \frac{\sqrt{2}}{4}\int \frac{\sin 2x}{\sin(x+\frac{\pi}{4})}dx = \frac{\sqrt{2}}{4}\int \frac{\sin(2t-\frac{\pi}{2})}{\sin t}dt\\ &= -\frac{\sqrt{2}}{4}\int \frac{1-2\sin^2 t}{\sin t}dt = -\frac{\sqrt{2}}{2}\int -\sin t dt - \frac{\sqrt{2}}{4}\int \frac{1}{\sin t}dt\\ &= -\frac{\sqrt{2}}{2}\cos t + \frac{\sqrt{2}}{4}\int \frac{d(\cos t)}{1-\cos^2 t}\\ &= -\frac{\sqrt{2}}{2}\cos t + \frac{\sqrt{2}}{4}\frac{1}{2}\ln|\frac{1+\cos t}{1-\cos t}| + C\\ &= -\frac{\sqrt{2}}{2}\cos(x+\frac{\pi}{4}) + \frac{\sqrt{2}}{8}\ln|\frac{1+\cos(x+\frac{\pi}{4})}{1-\cos(x+\frac{\pi}{4})}| + C\ .\end{aligned}$$

(h) (substitution $\tan x = t$)

$$\begin{aligned}\int \frac{dx}{(a\sin x+b\cos x)^2} &= \int \frac{dx}{(a\tan x+b)^2\cos^2 x} = \int \frac{d(\tan x)}{(a\tan x+b)^2} \\ &= \int \frac{dt}{(at+b)^2} = \frac{1}{a^2}\int \frac{dt}{(t+\frac{b}{a})^2} = -\frac{1}{a^2}\frac{1}{t+\frac{b}{a}} = -\frac{1}{a(at+b)}+C \\ &= -\frac{1}{a(a\tan x+b)}+C\ .\end{aligned}$$

(i) (substitution $\tan x = t$)

$$\begin{aligned}\int \frac{\sin x dx}{\sin^3 x+\cos^3 x} &= \int \frac{dx}{\sin^3 x+\frac{\cos^3 x}{\sin x}} = \int \frac{dx}{(\tan^2 x+\frac{1}{\tan x})\cos^2 x} \\ &= \int \frac{dt}{t^2+\frac{1}{t}} = \int \frac{t}{t^3+1}dt = \int \frac{t}{(t+1)(t^2-t+1)}dt \\ &= -\frac{1}{3}\int \frac{1}{t+1}dt + \frac{1}{3}\int \frac{t+1}{t^2-t+1}dt \\ &= -\frac{1}{3}\ln|t+1| + \frac{1}{3}\int \frac{t-\frac{1}{2}}{(t-\frac{1}{2})^2+\frac{3}{4}}dt + \frac{1}{2}\int \frac{dt}{(t-\frac{1}{2})^2+\frac{3}{4}} \\ &= -\frac{1}{3}\ln|t+1| + \frac{1}{6}\ln|t^2-t+1| + \frac{1}{2}\frac{2}{\sqrt{3}}\arctan(\frac{2}{\sqrt{3}}(t-\frac{1}{2})) + C \\ &= -\frac{1}{3}\ln|\tan x+1| + \frac{1}{6}\ln|\tan^2 x-\tan x+1| \\ &+\frac{\sqrt{3}}{3}\arctan(\frac{\sqrt{3}}{3}(2\tan x-1)) + C\ .\end{aligned}$$

(j) (substitution $2x = y$ and $\tan y = t$)

$$\begin{aligned}\int \frac{dx}{\sin^4 x+\cos^4 x} &= \int \frac{dx}{(\frac{1-\cos 2x}{2})^2+(\frac{1+\cos 2x}{2})^2} = \int \frac{d(2x)}{1+\cos^2 2x} \\ &= \int \frac{dy}{1+\cos^2 y} = \int \frac{dy}{(\frac{1}{\cos^2 y}+1)\cos^2 y} = \int \frac{d(\tan y)}{\frac{\sin^2 y+\cos^2 y}{\cos^2 y}+1} \\ &= \int \frac{dt}{2+t^2} = \frac{1}{\sqrt{2}}\arctan(\frac{t}{\sqrt{2}}) + C \\ &= \frac{\sqrt{2}}{2}\arctan(\frac{\sqrt{2}}{2}\tan 2x) + C\ .\end{aligned}$$

(k) (substitution $2x = y$ and $\tan y = t$)

$$\begin{aligned}
\int \frac{\sin^2 x \cos^2 x}{\sin^8 x + \cos^8 x} dx &= \frac{1}{4} \int \frac{\sin^2 2x}{(\frac{1-\cos 2x}{2})^4 + (\frac{1+\cos 2x}{2})^4} dx \\
&= \int \frac{\sin^2 y}{\cos^4 y + 6\cos^2 y + 1} dy = \int \frac{\tan^2 y dy}{\cos^2 y (1 + \frac{6}{\cos^2 y} + \frac{1}{\cos^4 y})} \\
&= \int \frac{t^2 dt}{1 + 6\frac{\sin^2 y + \cos^2 y}{\cos^2 y} + \frac{\sin^2 y + \cos^2 y}{\cos^4 y}} = \int \frac{t^2 dt}{7 + 6t^2 \frac{\tan^2 y}{\cos^2 y} + \frac{1}{\cos^2 y}} \\
&= \int \frac{t^2 dt}{7 + 6t^2 + (t^2+1)(t^2+1)} = \int \frac{t^2 dt}{t^4 + 8t^2 + 8} \\
&= \frac{1+\sqrt{2}}{2} \int \frac{dt}{t^2 + 4 + 2\sqrt{2}} + \frac{1-\sqrt{2}}{2} \int \frac{dt}{t^2 + 4 - 2\sqrt{2}} \\
&= \frac{1+\sqrt{2}}{2} \frac{1}{\sqrt{4+2\sqrt{2}}} \arctan\left(\frac{t}{\sqrt{4+2\sqrt{2}}}\right) \\
&\quad + \frac{1-\sqrt{2}}{2} \frac{1}{\sqrt{4-2\sqrt{2}}} \arctan\left(\frac{t}{\sqrt{4-2\sqrt{2}}}\right) + C \\
&= \frac{1+\sqrt{2}}{2\sqrt{4+2\sqrt{2}}} \arctan\left(\frac{\tan 2x}{\sqrt{4+2\sqrt{2}}}\right) + \frac{1-\sqrt{2}}{2\sqrt{4-2\sqrt{2}}} \arctan\left(\frac{\tan 2x}{\sqrt{4-2\sqrt{2}}}\right) + C \ .
\end{aligned}$$

(l)*: We have

$$\begin{aligned}
I &= \int \frac{\sin^2 x - \cos^2 x}{\sin^4 x + \cos^4 x} dx = \int \frac{\sin^2 x - \cos^2 x}{(\sin^2 x - \cos^2 x)^2 + 2\sin^2 x \cos^2 x} dx \\
&= \int \frac{-\cos 2x}{\cos^2 2x + \frac{1}{2}\sin^2 2x} dx = \int \frac{-\cos 2x \, dx}{1 - \frac{1}{2}\sin^2 2x}.
\end{aligned}$$

By substituting $t = \sin 2x$, $dt = 2\cos 2x \, dx$ we obtain

$$I = \int \frac{dt}{t^2 - 2} = \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2} - t}{\sqrt{2} + t} \right| + C = \frac{1}{2\sqrt{2}} \ln \left(\frac{\sqrt{2} - \sin 2x}{\sqrt{2} + \sin 2x} \right) + C.$$

(m) (substitution $2x = y$ and $t = \cos y$)

$$\begin{aligned}
\int \frac{\sin x \cos x}{1 + \sin^4 x} dx &= \frac{1}{2} \int \frac{\sin 2x dx}{1 + (\frac{1-\cos 2x}{2})^2} = 2 \int \frac{\sin 2x dx}{5 - 2\cos 2x + \cos^2 2x} \\
&= -\int \frac{d(\cos y)}{5 - 2\cos y + \cos^2 y} = -\int \frac{dt}{(t-1)^2 + 4} = -\frac{1}{2} \arctan\left(\frac{t-1}{2}\right) + C \\
&= -\frac{1}{2} \arctan\left(\frac{1 - \cos 2x}{2}\right) + C \\
&= -\frac{1}{2} \arctan\left(\sin^2 x\right) + C \ .
\end{aligned}$$

(n) (substitution $2x = y$ and $\tan y = t$)

$$\begin{aligned}
\int \frac{1}{\sin^6 x + \cos^6 x} dx &= \int \frac{1}{(\frac{1-\cos 2x}{2})^3 + (\frac{1+\cos 2x}{2})^3} dx \\
&= 8 \int \frac{1}{2(1+3\cos^2 2x)} dx \\
&= 2 \int \frac{dy}{1+3\cos^2 y} = 2 \int \frac{dy}{\cos^2 y (\frac{\sin^2 y + \cos^2 y}{\cos^2 y} + 3)} \\
&= 2 \int \frac{dt}{4+t^2} = \frac{2}{2} \arctan \frac{t}{2} + C \\
&= \arctan (\frac{1}{2} \tan 2x) + C \ .
\end{aligned}$$

(o) (substitution $2x = y$ and $\tan \frac{y}{2} = t$)

$$\begin{aligned}
\int \frac{dx}{(\sin^2 x + 2\cos^2 x)^2} &= \int \frac{dx}{(\frac{1-\cos 2x}{2} + 2\frac{1+\cos 2x}{2})^2} = \int \frac{4dx}{(3+\cos 2x)^2} \\
&= 2 \int \frac{dy}{(3+\cos y)^2} = 2 \int \frac{\frac{2dt}{1+t^2}}{(3 + \frac{1-t^2}{1+t^2})^2} = \int \frac{1+t^2}{(2+t^2)^2} dt \ ,
\end{aligned}$$

which can be computed by using Ostrogradski method as follows. We have $Q_1(t) = Q_2(t) = t^2 + 2$, $Q_1'(t) = 2t$, $H(t) = 2t$ and $P(t) = 1 + t^2$. Now assume that $P_1(t) = at + b$ and $P_2(t) = ct + d$. Then it follows that $a(t^2 + 2) - (at + b)2t + (ct + d)(t^2 + 2) = 1 + t^2$, which implies

$$\begin{cases} c = 0 \\ -a + d = 1 \\ 2c - 2b = 0 \\ 2a + 2d = 1 \end{cases}$$

Thus, $a = -\frac{1}{4}$, $b = c = 0$ and $d = \frac{3}{4}$. Hence,

$$\begin{aligned}
\int \frac{dx}{(\sin^2 x + 2\cos^2 x)^2} &= -\frac{1}{4} \frac{t}{2+t^2} + \frac{3}{4} \int \frac{1}{2+t^2} dt \\
&= -\frac{1}{4} \frac{t}{t^2+2} + \frac{3\sqrt{2}}{8} \arctan (\frac{\sqrt{2}}{2} t) + C \\
&= -\frac{1}{4} \frac{\tan x}{\tan^2 x + 2} + \frac{3\sqrt{2}}{8} \arctan (\frac{\sqrt{2}}{2} \tan x) + C \ .
\end{aligned}$$

1. (g):

$$\begin{aligned}
\int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{\sqrt[4]{x}} dx &= \int \left(x^{\frac{1}{4}} - 2x^{\frac{5}{12}} + x^{-\frac{1}{4}} \right) dx \\
&= \frac{4}{5} x^{\frac{5}{4}} - \frac{24}{17} x^{\frac{17}{12}} + \frac{4}{3} x^{\frac{3}{4}} + C.
\end{aligned}$$

(o):

$$\begin{aligned}
\int \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1-x^4}} dx &= \int \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1+x^2}} \right) dx \\
&= \arcsin x + \arctan x + C.
\end{aligned}$$

2. (d):

$$\int \frac{dx}{2+3x^2} = \frac{1}{3}\int \frac{dx}{\frac{2}{3}+x^2} = \frac{1}{\sqrt{6}}\arctan\left(\frac{\sqrt{3}x}{\sqrt{2}}\right) + C.$$

(g): (We apply the substitution $2x + \frac{p}{i}4 = t$)

$$\int \frac{dx}{\sin^2(2x+\frac{\pi}{4})} = \frac{1}{2}\int \frac{dt}{\sin^2 t} = \frac{1}{2}\cot t + C = \frac{1}{2}\cot\left(2x+\frac{\pi}{4}\right) + C.$$

3. (k): (Substitution $t = \ln(\ln x)$, thus $dt = \frac{dx}{x\ln x}$)

$$\int \frac{dx}{x\,\ln x\,\ln(\ln x)} = \int \frac{dt}{t} = \ln t + C = \ln(\ln(\ln x)) + C.$$

(v): (Substitution $t = \sin x$, thus $dt = \cos x\,dx$)

$$\int \sin^5 x\,\cos x\,dx = \int t^5 dt = \frac{1}{6}t^6 + C = \frac{1}{6}\sin^6 x + C.$$

4. (h): (Partial fraction decomposition and substitutions $t = x^2 - x\sqrt{2} + 1$ and $s = x^2 + x\sqrt{2} + 1$)

$$\begin{aligned}\int \frac{x^2-1}{x^4+1}dx &= \frac{1}{2}\int\left(\frac{x\sqrt{2}-1}{x^2-x\sqrt{2}+1} - \frac{x\sqrt{2}+1}{x^2+x\sqrt{2}+1}\right)dx\\ &= \frac{\sqrt{2}}{4}\left(\ln(x^2-x\sqrt{2}+1) - \ln(x^2+x\sqrt{2}+1)\right) + C.\end{aligned}$$

(o): (Substitution $t = \sin x$)

$$\begin{aligned}\int \frac{dx}{\cos x} &= \int \frac{\cos x}{1-\sin^2 x} = \int \frac{dt}{1-t^2}\\ &= \frac{1}{2}\int\left(\frac{1}{t+1} - \frac{1}{t-1}\right)dt = \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| C\\ &= \frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x-1}\right| + C\\ &= \frac{1}{2}ln\left|\frac{(\sin x+1)^2}{\cos^2 x}\right| + C = \ln|\sec x + \tan x| + C.\end{aligned}$$

5. (f):

$$\int \frac{(1+x)^2}{1+x^2}dx = \int\left(1 + \frac{2x}{1+x^2}\right)dx = x + \ln(1+x^2) + C.$$

(j):

$$\begin{aligned}\int \frac{dx}{\sqrt{x+1}+\sqrt{x-1}} &= \int \frac{\sqrt{x+1}-\sqrt{x-1}}{x+1-x+1}dx\\ &= \frac{1}{2}\int\left(\sqrt{x+1}-\sqrt{x-1}\right)dx\\ &= \frac{1}{3}\left(\sqrt{(x+1)^3} - \sqrt{(x-1)^3}\right) + C.\end{aligned}$$

(l):

$$\begin{aligned}\int \frac{dx}{(x-1)(x+3)} &= \frac{1}{4}\int \left(\frac{1}{x-1} - \frac{1}{x+3}\right) dx \\ &= \frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + C.\end{aligned}$$

(o):

$$\begin{aligned}\int \frac{dx}{(x^2+1)(x^2+2)} &= \int \left(\frac{1}{x^2+1} - \frac{1}{x^2+2}\right) dx \\ &= \arctan x - \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C.\end{aligned}$$

6. (g): (Substitution $t = \cos x$)

$$\begin{aligned}\int \sin^3 x\, dx &= \int (1-\cos^2 x)\sin x\, dx = -\int (1-t^2)dt \\ &= -t + \frac{1}{3}t^3 + C = -\cos x + \frac{1}{3}\cos^3 x + C.\end{aligned}$$

(o): (Substitution $t = \sin x$)

$$\begin{aligned}\int \frac{dx}{\sin^2 x\,\cos x} &= \int \frac{\cos x\, dx}{\sin^2 x(1-\sin^2 x)} \\ &= \int \frac{dt}{t^2(1-t)(1+t)} = \int \left(\frac{1}{t^2} - \frac{\frac{1}{2}}{t-1} + \frac{\frac{1}{2}}{t+1}\right) dt \\ &= -\frac{1}{t} + \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| + C \\ &= -\frac{1}{\sin x} + \frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x-1}\right| + C \\ &= -\csc x + \frac{1}{2}\ln\left|\frac{(\sin x+1)^2}{\cos^2 x}\right| + C \\ & -\csc x + \ln|\sec x + \tan x| + C.\end{aligned}$$

15. We look for partial fraction decomposition of

$$\begin{aligned}&\frac{ax^2+bx+c}{x^3(x-1)^2} \\ =&\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2} \\ =&\frac{(A+D)x^4 + (-2A+B-D+E)x^3 + (A-2B+C)x^2 + (B-2C)x + C}{x^3(x-1)^2}.\end{aligned}$$

As result we obtain the following system of linear equations

$$\begin{cases} A+D & =0 \\ -2A+B-D+E & =0 \\ A-2B+C & =a \\ B-2C & =b \\ C & =c, \end{cases}$$

which leads to the following value of the coefficient D

$$D=-2b-3c-a.$$

Consequently, the integral $I=\int \frac{ax^2+bx+c}{x^3(x-1)^2}dx$ is rational if and only if $a+2b+3c=0$.

17. (d):

$$I=\int \frac{x^2+x}{x^6+1}dx=\int \frac{x^2}{x^6+1}dx+\int \frac{x}{x^6+1}dx.$$

For the first integral we substitute $u=x^3$, $du=3x^2dx$ and for the second $v=x^2$, $dv=2xdx$, so we obtain

$$\begin{aligned} I &= \tfrac{1}{3}\int \frac{du}{u^2+1}+\tfrac{1}{2}\int \frac{dv}{v^3+1}=\tfrac{1}{3}\arctan u+\tfrac{1}{2}\int \frac{dv}{(v+1)(v^2-v+1)} \\ &= \tfrac{1}{3}\arctan x^3+\tfrac{1}{6}\int \frac{dv}{v+1}-\tfrac{1}{6}\int \frac{v-2}{v^2-v+1} \\ &= \tfrac{1}{3}\arctan x^3+\tfrac{1}{6}\ln(x^2+1)-\tfrac{1}{12}\ln(x^4-x^2+1) \\ &+ \tfrac{\sqrt{3}}{6}\arctan\left(\frac{2x^2-1}{\sqrt{3}}\right)+C. \end{aligned}$$

(f): We apply the substitution $u=x^5$, $du=5x^4dx$,

$$\begin{aligned} I &= \int \frac{x^4dx}{(x^{10}-10)^2}=\frac{1}{5}\int \frac{du}{(u^2-10)^2} \\ &= \frac{1}{5}\left[-\tfrac{\sqrt{10}}{400}\int \frac{du}{u-\sqrt{10}}+\tfrac{1}{40}\int \frac{du}{(u-\sqrt{10})^2}+\tfrac{\sqrt{10}}{400}\int \frac{du}{u+\sqrt{10}}\right. \\ &\left.+\tfrac{1}{40}\int \frac{du}{(u+\sqrt{10})^2}\right] \\ &= \frac{\sqrt{10}}{2000}\ln|x^{10}-10|-\frac{1}{100}\frac{x^5}{x^{10}-10}+C. \end{aligned}$$

(j): We apply the substitution $u=x^n$, $du=nx^{n-1}dx$:

$$\begin{aligned} I &= \int \frac{x^{3n-1}dx}{(x^{2n}+1)^2}=\frac{1}{n}\int \frac{x^{2n}\cdot nx^{n-1}dx}{(x^{2n}+1)^2} \\ &= \frac{1}{n}\int \frac{u^2du}{(u^2+1)^2)}=\frac{1}{n}\int \frac{du}{u^2+1}-\frac{1}{n}\int \frac{du}{(u^2+1)^2} \\ &= \frac{1}{2n}\arctan x^n-\frac{1}{2n}\frac{x^n}{1+x^{2n}}+C. \end{aligned}$$

(o): We have

$$\int \frac{(x^2+1)dx}{x^4+x^2+1} = \int \frac{(x^2+1)dx}{(x^2+x+1)(x^2-x+1)} = \frac{1}{2}\int \frac{dx}{x^2+x+1} + \frac{1}{2}\int \frac{dx}{x^2-x+1}$$
$$= \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C.$$

18. Notice

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right] = a\left[u^2 - \frac{\Delta}{4a^2}\right],$$

where $u = x + \frac{b}{2a}$ and $\Delta = b^2-4ac$. Thus, for $n \geq 2$ we have

$$\begin{aligned}
I_n &= \int \frac{dx}{(ax^2+bx+c)^n} = \frac{1}{a^n}\int \frac{du}{(u^2-\frac{\Delta}{4a^2})^n} \\
&= \frac{-4}{\Delta a^{n-2}}\left[\int \frac{du}{(u^2-\frac{\Delta}{4a^2})^{n-1}} - \int u\frac{udu}{(u^2-\frac{\Delta}{4a^2})^n}\right] \\
&= \frac{-4}{\Delta a^{n-2}}\left[\int \frac{du}{(u^2-\frac{\Delta}{4a^2})^{n-1}} - \left(\frac{u}{2(1-n)}\cdot\frac{1}{(u^2-\frac{\Delta}{4a^2})^n-1}\right.\right. \\
&\quad \left.\left. - \frac{1}{2(1-n)}\int \frac{du}{(u^2-\frac{\Delta}{4a^2})^{n-1}}\right)\right] \\
&= \frac{-4a}{\Delta}\left[\frac{2n-3}{2n-2}\cdot\frac{1}{a^{n-1}}\int \frac{du}{(u^2-\frac{\Delta}{4a^2})^{n-1}} + \frac{1}{2n-2}\cdot\frac{u}{a^{n-1}(u^2-\frac{\Delta}{4a^2})^{n-1}}\right].
\end{aligned}$$

Therefore, we have

$$I_n = \frac{4a}{-\Delta}\cdot\frac{2n-3}{2n-2}\cdot I_{n-1} + \frac{4a}{-\Delta}\cdot\frac{1}{2n-2}\frac{x+\frac{b}{2a}}{(ax^2+bx+c)^{n-1}}$$

and consequently we obtain the following reduction formula

$$I_n = \frac{2ax+b}{(n-1)(-\Delta)(ax^2+bx+c)^{n-1}} + \frac{2n-3}{n-1}\cdot\frac{2a}{-\Delta}I_{n-1}.$$

We compute

$$I_3 = \int \frac{dx}{(x^2+x+1)^3}.$$

Here we have $\Delta = -3$ and $a = 1$. Thus

$$I_1 = \int \frac{dx}{x^2+x+1} = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C,$$

so

$$I_2 = \frac{2x+1}{3(x^2+x+1)} + \frac{2}{3}\cdot\frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C,$$

hence

$$I_3 = \frac{2x+1}{6(x^2+x+1)^2} + \frac{2x+1}{3(x^2+x+1)} + \frac{4}{3\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

19. By Taylor formula we have

$$P_n(x) = \sum_{k=0}^{n} \frac{P_n^{(k)}(a)}{k!}(x-a)^k.$$

Thus

$$\begin{aligned}\int \frac{P_n(x)}{(x-a)^{n+1}}dx &= \sum_{k=0}^{n} \int \frac{P_n^{(k)}(a)}{k!}(x-a)^{k-n-1}dx \\ &= \sum_{k=0}^{n} \frac{P_n^{(k)}(a)}{k!} \int (x-a)^{k-n-1}dx \\ &= \sum_{k=0}^{n-1} \frac{P_n^{(k)}(a)}{k!} \cdot \frac{(x-a)^{k-n}}{k-n} \\ &+ \frac{P_n^{(n)}(a)}{n!} \ln|x-a| + C.\end{aligned}$$

20. We look for the complex roots of the equation

$$x^{2n} = -1 = \cos\pi + i\sin\pi .tag1$$

The roots are

$$\begin{aligned}\alpha_k &= \cos\varphi_k + i\sin\varphi_k, \\ \bar{\alpha}_k &= \cos\varphi_k - i\sin\varphi_k\end{aligned}$$

where $k = 0, 1, 2, \dots, n-1$, and $\varphi_k = \frac{(2k+1)\pi}{2n}$. That means, we have exactly $2n$ roots β_j, $j = 1, 2, \dots, 2n$, which are

$$\beta_{2k+1} = \alpha_k, \qquad \beta_{2k+2} = \bar{\alpha}_k,$$

for $k = 0, 1, 2, \dots, n$. Therefore we have

$$P(x) := x^{2n} + 1 = (x-\beta_1)(x-\beta_2)\dots(x-\beta_{2n}) =: \prod_{j=1}^{2n}(x-\beta_j).$$

Since all the roots β_j are not repeating, there is the following (complex) partial fraction decomposition of $\frac{1}{x^{2n}+1}$:

$$\begin{aligned}\frac{1}{x^{2n}+1} &= \sum_{j=0}^{2n} \frac{C_j}{x-\beta_j} \\ &= \frac{\sum_{j=1}^{2n} C_j \prod_{m\neq j}(x-\beta_m)}{\prod_{m=1}^{2n}(x-\beta_m)} \\ &= \frac{\sum_{j=1}^{2n} C_j \frac{P(x)-P(\beta_j)}{x-\beta_j}}{P(x)}.\end{aligned}$$

Consequently,

$$1 = \sum_{j=1}^{2n} C_m \frac{P(x) - P(\beta_j)}{x - \beta_j}.$$

In the last equality, by passing to the limit $x \to \beta_m$, $m = 1, 2, \dots, 2n$, we obtain

$$1 = C_m \cdot \lim_{x \to \beta_m} \frac{P(x) - P(\beta_m)}{x - \beta_m} = C_m \cdot P'(\beta_m),$$

i.e.

$$C_m = \frac{1}{P'(\beta_m)} = \frac{1}{2n\beta_m^{2n-1}} = \frac{\beta_m}{2n\beta_m^{2n}} = \frac{-\beta_m}{2n}.$$

Therefore, we have

$$\begin{aligned}
\frac{1}{x^{2n}+1} &= \frac{-1}{2n} \sum_{j=1}^{2n} \frac{\beta_j}{x - \beta_j} = \frac{-1}{2n} \sum_{k=0}^{n-1} \left[\frac{\alpha_k}{x - \alpha_k} + \frac{\bar{\alpha}_k}{x - \bar{\alpha}_k} \right] \\
&= -\frac{1}{2n} \sum_{k=0}^{n-1} \frac{\alpha_k (x - \bar{\alpha}_k) + \bar{\alpha}_k (x - \alpha_k)}{(x - \alpha_k)(x - \bar{\alpha}_k)} \\
&= -\frac{1}{2n} \sum_{k=0}^{n-1} \frac{x(\alpha_k + \bar{\alpha}_k) - 2\alpha_k \bar{\alpha}_k}{x^2 - (\alpha_k + \bar{\alpha}_k)x + \alpha_k \bar{\alpha}_k} \\
&= -\frac{1}{2n} \sum_{k=0}^{n-1} \frac{2x \cos\varphi_k - 2}{x^2 - 2\cos\varphi_k + 1} \\
&= -\frac{1}{2n} \sum_{k=0}^{n-1} \frac{(x - \cos\varphi_k) 2\cos\varphi_k + 2\cos^2\varphi_k - 2}{(x - \cos\varphi_k)^2 + 1 - \cos^2\varphi_k} \\
&= -\frac{1}{2n} \sum_{k=0}^{n-1} \frac{(x - \cos\varphi_k) 2\cos\varphi_k - 2\sin^2\varphi_k}{(x - \cos\varphi_k)^2 + \sin^2\varphi_k}.
\end{aligned}$$

Thus

$$\begin{aligned}
\int \frac{dx}{x^{2n}+1} &= \frac{-1}{2n} \sum_{k=0}^{n-1} \cos\varphi_k \ln(x^2 - 2x\cos\varphi_k + 1) \\
&\quad + \frac{1}{n} \sum_{k=1}^{n-1} \sin\varphi_k \arctan\left(\frac{x - \cos\varphi_k}{\sin\varphi_k} \right) + C.
\end{aligned}$$

21. (e): We apply the substitution $t^{12} = x$, $12t^{11}dt = dx$ to get:

$$\begin{aligned}
\int \frac{dx}{(1 + \sqrt[4]{x})\sqrt[3]{x}} &= \int \frac{12t^{11}dt}{(1 + t^3)t^4} = 12 \int \frac{t^7 dt}{t^3 + 1} \\
&= 12 \int \left(t^4 - t - \tfrac{1}{3} \frac{1}{t+1} + \tfrac{1}{3} \frac{t+1}{t^2 - t + 1} \right) dt \\
&= \tfrac{12}{5} t^5 - 6t^2 - 4\ln|t+1| + 4 \int \frac{t+1}{t^2 - t + 1} dt \\
&= \frac{12}{5} x^{\frac{5}{12}} - 6x^{\frac{1}{6}} - 4\ln(x^{\frac{1}{12}} + 1) \\
&\quad + 2\ln(x^{\frac{1}{6}} - x^{\frac{1}{12}} + 1) + 4\sqrt{3}\arctan\left(\frac{2x^{\frac{1}{12}} - 1}{\sqrt{3}} \right) + C.
\end{aligned}$$

(g): Since

$$I := \int \frac{\sqrt[3]{(x+1)^2(x-1)^4}}{dx} = \int \frac{\sqrt[3]{\frac{x+1}{x-1}}}{x^2-1}dx,$$

we apply the substitution $t^3 = \frac{x+1}{x-1}$, $x = \frac{t^3+1}{t^3-1}$, $dx = \frac{-6t^2dt}{(t^3-1)^2}$ and

$$x^2 - 1 = \frac{(t^3+1)^2}{(t^3-1)^2} - 1 = \frac{4t^3}{(t^3-1)^2}.$$

So, we get

$$I = \int \frac{t(-6t^2)dt}{\frac{4t^3}{(t^3-1)^2}(t^3-1)^2} = -\frac{3}{2}\int dt = -\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}} + C.$$

(i): Since

$$I := \int \frac{dx}{\sqrt[n]{(x-a)^{n+1}(x-b)^{n-1}}} = \int \frac{\sqrt[n]{\frac{x-b}{x-a}}}{(x-a)(x-b)}dx,$$

we apply the substitution $t^n = \frac{x-b}{x-a}$, $x = \frac{t^n a-b}{t^n-1}$, $dx = \frac{nt^{n-1}(b-a)}{(t^n-1)^2}dt$, and

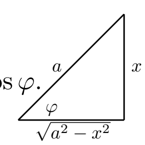
$$x - a = \frac{a-b}{t^n-1}, \qquad x - b = \frac{t^n(a-b)}{t^n-1}.$$

Consequently,

$$I = \frac{-n}{a-b}\int dt = \frac{n}{b-a}\sqrt[n]{\frac{x-b}{x-a}} + C.$$

22. (i): We apply the following trigonometric substitutions

$x = a\sin\varphi$, $dx = a\cos\varphi d\varphi$, $\sqrt{a^2-x^2} = a\cos\varphi$.



Then we have

$$\begin{aligned}
I &= \int x^4\sqrt{a^2-x^2}dx = a^6\int \sin^4\varphi\cos^2\varphi d\varphi \\
&= a^6\int\left(\tfrac{1}{2}\sin 2\varphi\right)^2\left(\tfrac{1}{2}\big(1-\cos 2\varphi\big)\right)d\varphi \\
&= \frac{a^6}{8}\left(\int \sin^2 2\varphi d\varphi - \int \sin^2 2\varphi\cos 2\varphi d\varphi\right) \\
&= \frac{a^6}{16}\varphi - \frac{a^6}{64}\sin 4\varphi - \frac{a^6}{8}\int \sin^2 2\varphi\cos 2\varphi d\varphi \quad (\text{subst. } t = \sin 2\varphi) \\
&= \frac{a^6}{16}\varphi - \frac{a^6}{64}\sin 4\varphi - \frac{a^6}{16}\int t^2 dt = \frac{a^6}{16}\varphi - \frac{a^6}{64}\sin 4\varphi - \frac{a^6}{48}t^3 \\
&= \frac{a^6}{16}\varphi - \frac{a^6}{16}\big[\sin vp\cos\varphi(\cos^2\varphi - \sin^2\varphi)\big] - \frac{a^6}{6}\sin^3\varphi\cos^3\varphi + C \\
&= \frac{a^6}{16}\arcsin\left(\frac{x}{a}\right) - \frac{a62}{16}x\sqrt{a^2-x^2}(a^2-2x^2) - \frac{a^3}{6}x^3\sqrt{(a^2-x^2)^3} + C \\
&= \frac{a^6}{16}\arcsin\left(\frac{x}{a}\right) - \frac{a^2}{16}(xa^2-2x^3)\sqrt{a^2-x^2} - x^3\sqrt{(a^2-x^2)^3} + C.
\end{aligned}$$

Another method to compute the integral I: We make the substitution $z = x^2$, $\frac{1}{2}z^{-\frac{1}{2}}dz = dx$, so

$$I = \int z^2 (a^2 - z)^{\frac{1}{2}} \cdot \tfrac{1}{2} z^{-\frac{1}{2}} dx = \frac{1}{2} \int z^2 \sqrt{\frac{-z + a^2}{z}} dz.$$

and by applying the substitution

$$t^2 = \frac{-z + a^2}{z}, \quad z = \frac{a^2}{1 + t^2}, \quad dz = \frac{-a^2 2t}{1 + t^2} dt,$$

we obtain that

$$\begin{aligned} I &= -a^6 \int \frac{t^2}{(1+t^2)^3} dt = -a^6 \Big[-\tfrac{1}{4} \frac{t}{(1+t^2)^2} + \tfrac{1}{8} \frac{t}{1+t^2} + \tfrac{1}{8} \arctan t \Big] + C \\ &= -a^6 \left[-\tfrac{1}{4} \frac{\sqrt{\frac{a^2 - x^2}{x^2}}}{\left(\frac{a^2}{x^2}\right)^2} + \tfrac{1}{8} \frac{\sqrt{\frac{a^2 - x^2}{x^2}}}{\frac{a^2}{x^2}} + \tfrac{1}{8} \arctan \left(\sqrt{\frac{a^2 - x^2}{a^2}} \right) \right] + C \\ &= \frac{1}{8} (2a^2 x^3 - a^4 x) \sqrt{a^2 - x^2} - \frac{a^6}{8} \arctan \left(\frac{\sqrt{a^2 - x^2}}{x} \right) + C. \end{aligned}$$

(m): We begin with a remark about the strategy for integrating

$$I_n := \int \frac{x^n dx}{\sqrt{x^2 + a}}, \quad a \neq 0.$$

Since

$$\begin{aligned} I_n &= \int x^{n-1} \cdot \frac{x}{\sqrt{x^2 + a}} dx \\ &= x^{n-1} \sqrt{x^2 + a} - (n-1) \int x^{n-2} \sqrt{x^2 + a} dx \\ &= x^{n-1} \sqrt{x^2 + a} - (n-1) \int \frac{x^n + a x^{n-2}}{\sqrt{x^2 + a}} dx \\ &= x^{n-1} \sqrt{x^2 + a} - (n-1) I_n - a(n-1) I_{n-2}, \end{aligned}$$

thus

$$I_n = \frac{1}{n} x^{n-1} \sqrt{x^2 + a} - \frac{n-1}{n} a I_{n-2}.$$

In particular we have:

$$\begin{aligned} I_0 &= \int \frac{dx}{\sqrt{x^2 + a}} = \ln |x + \sqrt{x^2 + a}| + C \\ I_1 &= \sqrt{x^2 + a} + C \\ I_2 &= \frac{1}{2} x \sqrt{x^2 + a} - \frac{1}{2} a \ln |x + \sqrt{x^2 + a}| + C \\ I_3 &= \frac{1}{3} (x^2 - 2a) \sqrt{x^2 + a} + C, \quad \text{etc. ...} \end{aligned}$$

In order to compute the integral

$$I := \int \frac{dx}{(x-1)^3 \sqrt{x^2 + 3x + 1}}$$

we apply the substitution $x-1=\frac{1}{t}$, $t=\frac{1}{x-1}$, $x=\frac{1}{t}+1$ and $dx=-\frac{1}{t^2}dt$. Then we obtain

$$\begin{aligned}
I &= \int \frac{t^3(-1)dt}{\sqrt{\left(\frac{1}{t}+1\right)^2+3\left(\frac{1}{t}+1\right)+1t^2}} = -\int \frac{tdt}{\frac{\sqrt{5t^2+5t+1}}{t}} \\
&= -\frac{1}{\sqrt{5}}\int \frac{t^2}{\sqrt{t^2+t+\frac{1}{5}}}dt.
\end{aligned}$$

Next, we substitute $u=t+\frac{1}{2}$, $t=u-\frac{1}{2}$, $du=dt$, thus

$$\begin{aligned}
I &= \frac{-1}{\sqrt{5}}\int \frac{(u-\frac{1}{2})^2}{\sqrt{u^2-\frac{1}{4}+\frac{1}{5}}}du \\
&= \frac{-1}{\sqrt{5}}\left[\frac{1}{2}u\sqrt{u^2-\frac{1}{20}}+\frac{1}{2}\cdot\frac{1}{20}\ln\left|u+\sqrt{u^2-\frac{1}{20}}\right|\right. \\
&\quad \left.-\sqrt{u^2-\frac{1}{20}}+\frac{1}{4}\ln\left|u+\sqrt{u^2-\frac{1}{20}}\right|\right]+C \\
&= -\frac{1}{2\sqrt{5}}(u-2)\sqrt{u^2-\frac{1}{20}}-\frac{11}{40\sqrt{5}}\ln\left|u+\sqrt{u^2-\frac{1}{20}}\right|+C \\
&= -\frac{1}{2\sqrt{5}}(t-\frac{3}{2})\sqrt{t^2+t+\frac{1}{5}} \\
&\quad -\frac{11}{40\sqrt{5}}\ln\left|t+\frac{1}{2}+\sqrt{t^2+t+\frac{1}{5}}\right|+C \\
&= \tfrac{-1}{20}\frac{5-3x}{(x-1)^2}\sqrt{x^2+3x+1}-\frac{11}{40\sqrt{5}}\ln\left|\frac{\sqrt{5}(x+1)+2\sqrt{x^2+3x+1}}{x-1}\right|+C.
\end{aligned}$$

23. (c): We apply the Euler substitution to the integral

$$I := \int \frac{dx}{1+\sqrt{1-2x-x^2}},$$

i.e. we substitute $\sqrt{1-2x-x^2}=tx-1$, $x=2\frac{t-1}{t62+1}$, $dx=$ $2\frac{-t^2+2t+1}{(t^2+1)^2}dt$ and $1+\sqrt{1-2x+x^2}=2\frac{(t-1)t}{t^2+1}$. Then we have

$$\begin{aligned}
I &= \int \frac{2(-t^2+2t+1)(t^2+1)dt}{(t^2+1)^2 2(t-1)t} = \int \frac{-t^2+2t+1}{(t^2+1)(t-1)t}dt \\
&= -\int \frac{dt}{t}+\int \frac{dt}{t-1}-2\int \frac{dt}{t^2+1} = \ln\left|\frac{t-1}{t}\right|-2\arctan t+C \\
&= \ln\left|\frac{\sqrt{1-2x-x^2}+1-x}{\sqrt{1-2x-x^2}+1}\right|-2\arctan\left(\sqrt{1-2x-x^2}+1\right)+C.
\end{aligned}$$

(e): Again, in order to compute

$$I := \int \frac{dx}{[1+\sqrt{x^2+x}]^2},$$

we apply the Euler substitution $\sqrt{x^2+x} = t + x$, $t = \sqrt{x^2+x} - x$, $x = \frac{t^2}{1-2t}$, $dx = -2\frac{t(t-1)}{(1-2t)^2}dt$, and $1 + \sqrt{x^2+x} = \frac{1-t-t^2}{1-2t}$. Then, we have

$$I = \int \frac{-2t^2+2t}{(t^2+t-1)^2}dt = \int \frac{-2t^2+2t}{\left[\left(t+\frac{1}{2}\right)^2 - \frac{5}{4}\right]^2}dt.$$

Next, we substitute $u = t + \frac{1}{2}$, $du = dt$, hence

$$\begin{aligned} I &= \int \frac{-2u^2+4u-\frac{3}{2}}{(u^2-\frac{5}{4})^2}du = \int \frac{-2u^2+4u-\frac{3}{2}}{(u-\frac{\sqrt{5}}{2})^2(u+\frac{\sqrt{5}}{2})^2}du \\ &= \int \left[-\tfrac{2}{25}\frac{\sqrt{5}}{u-\frac{\sqrt{5}}{2}} - \tfrac{2}{5}\frac{2-\sqrt{5}}{(u-\frac{\sqrt{5}}{2})^2} + \tfrac{2}{25}\frac{\sqrt{5}}{u+\frac{\sqrt{5}}{2}} - \tfrac{2}{5}\frac{2+\sqrt{5}}{(u+\frac{\sqrt{5}}{2})^2}\right] \\ &= \frac{2}{5\sqrt{5}}\ln\left|\frac{2u+\sqrt{5}}{2u-\sqrt{5}}\right| + \tfrac{2}{5}\frac{4u-5}{4u^2-5} + C \\ &= \frac{2}{5\sqrt{5}}\ln\left|\frac{2t+1+\sqrt{5}}{2t+1-\sqrt{5}}\right| + \frac{2}{20}\cdot\frac{4t-3}{t^2+t-1} + C, \end{aligned}$$

where $t = \sqrt{x(1+x)} - x$.

24. (e): In order to compute the integral

$$I = \int \frac{dx}{\sqrt[3]{1+x^3}} = \int \left(1+x^3\right)^{-\frac{1}{3}} dx,$$

first, we apply the substitution $z = x^3$, $x = z^{\frac{1}{3}}$, $dx = \frac{1}{3}z^{-\frac{2}{3}}dz$, to obtain

$$I = \frac{1}{3}\int \frac{1}{z}\sqrt[3]{\frac{z}{1+z}}dz,$$

and, next, the substitution $t^3 = \frac{z}{1+z}$, $z = \frac{t^3}{1-t^3}$ and $dz = \frac{-3t^2}{(1-t^3)^2}dt$, hence

$$\begin{aligned} I &= \frac{1}{3}\int \frac{1-t^3}{t^3}\cdot t\frac{-3t^2}{(1-t^3)^2}dt = \int \frac{dt}{t^3-1} \\ &= \frac{1}{3}\int \frac{dt}{t-1} - \frac{1}{3}\int \frac{t+2}{t^2+t+1}dt \\ &= \frac{1}{3}\ln|t-1| - \frac{1}{6}\ln(t^2+t+1) - \frac{1}{\sqrt{3}}\arctan\left(\frac{2t+1}{\sqrt{3}}\right) + C \\ &= \frac{1}{3}\ln\left|\frac{x}{\sqrt[3]{x^3+1}} - 1\right| - \frac{1}{6}\ln\left|\frac{x^2\sqrt[3]{x^2+1} + x\sqrt[3]{(x^2+1)^2}}{x^3+1} + 1\right| \\ &\quad - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x}{\sqrt{3}\sqrt[3]{x^3+1}} + \frac{1}{\sqrt{3}}\right) + C. \end{aligned}$$

(g): We have

$$I = \int \frac{dx}{x^6\sqrt[6]{1+x^6}} = \int x^{-1}(1+x^6)^{-\frac{1}{6}}dx.$$

We apply the substitution $z = x^6$, $x = z^{\frac{1}{6}}$, $dx = \frac{1}{6}z^{-\frac{5}{6}}dz$. Then we have

$$I = \frac{1}{6}\int z^{-1}(1+z)^{-\frac{1}{6}}dx.$$

We substitute $t^6 = 1+z$, $6t^5dt = dz$ to get

$$\begin{aligned}
I &= \int \frac{t^4}{t^6-1}dt = \int \frac{t^4dt}{(t-1)(t^2+t+1)(t+1)(t^2-t+1)}\\
&= \tfrac{1}{6}\int \frac{dt}{t-1} - \tfrac{1}{6}\int \frac{dt}{t+1} - \tfrac{1}{6}\int \frac{t-1}{t^2+t+1}dt + \tfrac{1}{6}\int \frac{t+1}{t^2-t+1}dt\\
&= \tfrac{1}{6}\ln\left|\frac{t-1}{t+1}\right| + \tfrac{1}{6}\left[-\tfrac{1}{2}\ln(t^2+t+1) + \sqrt{3}\text{arctan}\left(\frac{2t+1}{\sqrt{3}}\right)\right.\\
&\quad \left. + \tfrac{1}{2}\ln|t^2-t+1| + \sqrt{3}\text{arctan}\left(\frac{2t-1}{\sqrt{3}}\right)\right] + C\\
&= \frac{1}{6}\ln\left|\frac{t-1}{t+1}\right| + \frac{1}{12}\ln\left|\frac{t^2-t+1}{t^2+t+1}\right| + \sqrt{3}\text{arctan}\left(\frac{\sqrt{3}t}{1-t^2}\right) + C,
\end{aligned}$$

where $t = \sqrt[6]{1+x^6}$.

25. (l): We have

$$I = \int \frac{dx}{\sin^3 x\cos^5 x} = \int \frac{(1+\tan^2 x)^3dx}{\tan^3 x\cos^2 x},$$

so by making the substitution $t = \tan x$, $dt = \frac{dx}{\cos^2 x}$ we get

$$\begin{aligned}
I &= \int \frac{(1+t^2)^3}{t^3}dt = \int (t^{-3} + 3t^{-1} + 3t + t^3)dt\\
&= -\frac{-1}{2t^2} + 3\ln|t| + \frac{3}{2}t^2 + \frac{1}{4}t^4 + C\\
&= -\frac{1}{2}\cot^2 x + 3\ln|\tan x| + \frac{3}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C.
\end{aligned}$$

(o): We substitute $t = \cot x$, $x = arccott$, $dx = \frac{-dt}{1+t^2}$, so

$$\begin{aligned}
I &= \int \cot^6 x\,dx = -\int \frac{t^6}{1+t^2}dt = \int \left(-t^4 + t^2 - 1 + \frac{1}{1+t^2}\right)dt\\
&= -\frac{1}{5}t^5 + \frac{1}{3}t^3 - t - \text{arccot}\,t + C\\
&= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - x + C.
\end{aligned}$$

(q): We have

$$I = \int \frac{dx}{\sqrt{\sin^3 x\cos^5 x}} = \int \frac{(1+\tan^2 x)dx}{\sqrt{\tan^3 x}\cos^2 x}.$$

We substitute $t = \tan x$, $dt = \frac{dx}{\cos^2 x}$, to get

$$\begin{aligned}
I &= \int \frac{1+t^2}{t^{\frac{3}{2}}} dt = \int (t^{-\frac{3}{2}} + t^{\frac{1}{2}}) dt \\
&= -2t^{-\frac{1}{2}} + \frac{2}{3} t^{\frac{3}{2}} + C = -2\sqrt{\cot x} + \frac{2}{3}\sqrt{\tan^3 x} + C.
\end{aligned}$$

26. We have for $n \geq 2$

$$\begin{aligned}
J_n &= \int \frac{dx}{\cos^n x} = \int \frac{\sin^2 x + \cos^2 x}{\cos^n x} dx \\
&= \int \sin x \frac{\sin x}{\cos^n x} dx + \int \frac{dx}{\cos^{n-2} x} \\
&= \frac{1}{n-1} \cdot \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}.
\end{aligned}$$

Therefore, we have the reduction formula

$$J_n = \frac{1}{n-1} \cdot \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} J_{n-2}.$$

27. (e): Since

$$I = \int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \int \frac{dx}{(a^2 \tan^2 x + b^2) \cos^2 x},$$

we apply the substitution $t = \tan x$, $dt = \frac{dx}{\cos^2 x}$, to get

$$\begin{aligned}
I &= \int \frac{dt}{a^2 t^2 + b^2} = \frac{1}{a^2} \int \frac{dt}{t^2 + (\frac{b}{a})^2} = \frac{1}{ab} \arctan\left(\frac{at}{b}\right) + C \\
&= \frac{1}{ab} \arctan\left(\frac{a \tan x}{b}\right) + C.
\end{aligned}$$

(j): We have

$$\begin{aligned}
I &= \int \frac{\sin^2 x - \cos^2 x}{\sin^4 x + \cos^4 x} dx = \int \frac{\sin^2 x - \cos^2 x}{(\sin^2 x - \cos^2 x)^2 + 2\sin^2 x \cos^2 x} dx \\
&= \int \frac{-\cos 2x}{\cos^2 2x + \frac{1}{2}\sin^2 2x} dx = \int \frac{-\cos 2x \, dx}{1 - \frac{1}{2}\sin^2 2x}.
\end{aligned}$$

By substituting $t = \sin 2x$, $dt = 2\cos 2x \, dx$ we obtain

$$I = \int \frac{dt}{t^2 - 2} = \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2} - t}{\sqrt{2} + t} \right| + C = \frac{1}{2\sqrt{2}} \ln \left(\frac{\sqrt{2} - \sin 2x}{\sqrt{2} + \sin 2x} \right) + C.$$

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