MATH 5302 Elementary Analysis II - Homework 2

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Problem 1

Complete the proof of Theorem 2.3 in the lecture notes by showing that a decreasing function on $[a, b]$ is integrable.

Definition 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function.*

a. f is strictly increasing *over interval I if*

$$\forall_{x_1, x_2 \in I} x_1 < x_2 \implies f(x_1) < f(x_2)$$

b. f is strictly decreasing *over interval I if*

$$\forall_{x_1, x_2 \in I} x_1 < x_2 \implies f(x_1) > f(x_2)$$

c. f is increasing *over interval I if*

$$\forall_{x_1, x_2 \in I} x_1 < x_2 \implies f(x_1) \leq f(x_2)$$

d. f is decreasing *over interval I if*

$$\forall_{x_1, x_2 \in I} x_1 < x_2 \implies f(x_1) \geq f(x_2)$$

Definition 2. *$f : \mathbb{R} \to \mathbb{R}$ is* monotone *over interval I if f is either increasing or decreasing over interval I.*

Theorem 1. *Theorem 1.4 states:*
A bounded function $f : [a, b] \to \mathbb{R}$ is integrable *iff*

$$\forall_{\epsilon > 0} \exists_{P = \{a = x_0 < x_1 < \cdots < x_n = b\}} \ : \ U(f, P) - L(f, P) < \epsilon$$

Theorem 2. *Every monotone function f on $[a, b]$ is integrable.*

Proof.

Lemma 1. *Every increasing function f on $[a, b]$ is integrable.*

Proof. Let f be increasing on $[a, b]$. Let $\epsilon > 0$. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition on

$[a, b]$ with mesh less than $\dfrac{\epsilon}{f(b) - f(a)}$.

$$
\begin{aligned}
U(f, P) - L(f, P) &= \sum_{i=1}^{n} M(f, [x_{x-1}, x_i])(x_i - x_{i-1}) - \sum_{i=1}^{n} m(f, [x_{x-1}, x_i])(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})](x_i - x_{i-1}) \\
&< \sum_{i=1}^{n} -([f(x_{i-1}) - f(x_i)]) \left(\frac{\epsilon}{f(b) - f(a)} \right) \\
&= -(f(b) - f(a)) \left(\frac{\epsilon}{f(b) - f(a)} \right) \\
&= (f(a) - f(b)) \left(\frac{\epsilon}{f(b) - f(a)} \right) = \epsilon
\end{aligned}
$$

$\square$

Lemma 2. *Every decreasing function f on $[a, b]$ is integrable.*

Proof. Let f be decreasing on $[a, b]$. Let $\epsilon > 0$. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition on $[a, b]$ with mesh less than $\dfrac{\epsilon}{f(a) - f(b)}$.

$$
\begin{aligned}
U(f, P) - L(f, P) &= \sum_{i=1}^{n} M(f, [x_{x-1}, x_i])(x_i - x_{i-1}) - \sum_{i=1}^{n} m(f, [x_{x-1}, x_i])(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})](x_i - x_{i-1}) \\
&< \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \left(\frac{\epsilon}{f(b) - f(a)} \right) \\
&= (f(a) - f(b)) \left(\frac{\epsilon}{f(a) - f(b)} \right) = \epsilon
\end{aligned}
$$

$\square$

$\square$

Problem 2

Let f be a bounded function on $[a,b]$, so that there exists $B > 0$ such that $|f(x)| \leq B$ for all $x \in [a,b]$.

a)

Show
$$U(f^2, P) - L(f^2, P) \leq 2B[U(f,P) - L(f,P)]$$
for all partitions P of $[a,b]$. Hint: $f^2(x) - f^2(y) = (f(x) + f(y))(f(x) - f(y))$

Theorem 3.
$$U(f^2, P) - L(f^2, P) \leq 2B[U(f,P) - L(f,P)]$$
for all partitions $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ *of* $[a,b]$.

Proof.

$$\begin{aligned}
U(f^2, P) - L(f^2, P) &= \sum_{i=1}^{n} M(f^2, [x_{x-1}, x_i])(x_i - x_{i-1}) - \sum_{i=1}^{n} m(f^2, [x_{x-1}, x_i])(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} f^2(x_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} f^2(x_{i-1})(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} [f^2(x_i) - f^2(x_{i-1})](x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} [(f(x_i) + f(x_{i-1}))(f(x_i) - f(x_{i-1}))](x_i - x_{i-1})
\end{aligned}$$

Since $|f(x)| \leq B \forall_{x \in [a,b]}$,

$$\begin{aligned}
&\leq \sum_{i=1}^{n} [(B + B))(f(x_i) - f(x_{i-1}))](x_i - x_{i-1}) \\
&= 2B \sum_{i=1}^{n} [(f(x_i) - f(x_{i-1}))](x_i - x_{i-1}) \\
&= 2B \left[\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) \right] \\
&= 2B \left[\sum_{i=1}^{n} M(f, [x_{x-1}, x_i])(x_i - x_{i-1}) - \sum_{i=1}^{n} m(f, [x_{x-1}, x_i])(x_i - x_{i-1}) \right] \\
&= 2B[U(f,P) - L(f,P)]
\end{aligned}$$

□

b)

Show that if f is integrable on $[a, b]$, then f^2 is also integrable on $[a, b]$.

Theorem 4. *f integrable on $[a, b] \implies f^2$ integrable on $[a, b]$.*

Proof. From Theorem 3, we have that

$$U(f^2, P) - L(f^2, P) \leq 2B[U(f, P) - L(f, P)]$$

for all partitions $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$.
f integrable on $[a, b]$ implies

$$\exists_{\{P_k\}} \ : \ \overset{k\to\infty}{\lim} [U(f, P) - L(f, P)] = 0$$

Therefore,

$$\exists_{\{P_k\}} \ : \ \overset{k\to\infty}{\lim} [U(f^2, P) - L(f^2, P)] \leq 2B[U(f, P) - L(f, P)] = 0$$

Which means that the lower and upper Darboux integrals are equal, $U(f^2) = L(f^2)$ and by definition this means f^2 is Darboux integrable. $\square$

Problem 3

Let f be a bounded function on $[a, b]$. Suppose f^2 is integrable on $[a, b]$. Must f also be integrable on $[a, b]$?
Answer: No.
A modification of the rational number indicator function can be shown as a counter example:
Let

$$f(x) = \begin{cases} 1 & \text{if} x \in \mathbb{Q} \\ -1 & \text{if} x \notin \mathbb{Q} \end{cases}$$

which is clearly not integrable due to the infinite number of discontinuities.
However, f^2 would be defined by

$$f^2(x) = 1$$

which is clearly integrable.

Problem 4

Suppose that f and g are integrable on $[a, b]$. Show that $\max(f, g)$ is also integrable on $[a, b]$. Hint: Derive and apply the formula:

$$\max(f, g) = \frac{1}{2}(f + g + |f - g|)$$

Theorem 5. *For all functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ that are integrable on $[a, b]$, $\max(f, g)$ is also integrable on $[a, b]$.*

Proof. The max function is equal to

$$\begin{aligned}
\max(f, g)(x) &= \max(f(x), g(x)) \\
&= \begin{cases} f(x) & f(x) \geq g(x) \\ g(x) & g(x) < f(x) \end{cases} \\
&= \begin{cases} g(x) + [f(x) - g(x)] & f(x) \geq g(x) \\ f(x) + [g(x) - f(x)] & g(x) < f(x) \end{cases} \\
&= \begin{cases} g(x) + |f(x) - g(x)| & f(x) \geq g(x) \\ f(x) + |g(x) - f(x)| & g(x) < f(x) \end{cases} \\
&= \frac{1}{2} \begin{cases} f(x) + g(x) + |f(x) - g(x)| & f(x) \geq g(x) \\ f(x) + g(x) + |g(x) - f(x)| & g(x) < f(x) \end{cases} \\
&= \frac{f(x) + g(x) + |f(x) - g(x)|}{2}
\end{aligned}$$

Since f and g are integrable on $[a, b]$, the following is true:

a. $U(f) = L(f)$

b. $U(g) = L(g)$

□

Problem 5

Suppose f and g are continuous functions on $[a,b]$ such that $\int_a^b f = \int_a^b g$. Prove there exists x in (a,b) such that $f(x) = g(x)$.