Assignment 8 of Math 5302

Due Date: April 20, 2022 at 11:59pm

1. (a) Prove Property M6: If the A_k's are measurable and $A_1 \supseteq A_2 \supseteq A_3 \cdots$, and if $\lambda(A_1) < \infty$, then

$$\lambda(\bigcap_{k=1}^{\infty} A_k) = \lim_{k\to\infty} \lambda(A_k).$$

(b) Give an example to show why the assumption $\lambda(A_1) < \infty$ is needed in Property M6.

2. Let A and B be subsets of $\mathbb{R}^n$.
(a) Suppose that A and B are measurable. Prove that

$$\lambda(A) + \lambda(B) = \lambda(A \cup B) + \lambda(A \cap B).$$

(b) Prove that in general

$$\lambda^*(A) + \lambda^*(B) \geq \lambda^*(A \cup B) + \lambda^*(A \cap B).$$

3. Let $a \in \mathbb{R}$ be fixed. Prove that

$$\lambda(\{a\} \times \mathbb{R}^{n-1}) = 0.$$

4. (a) Let $\epsilon > 0$. Prove that there exists an open set $G \subset \mathbb{R}$ such that $\mathbb{Q} \subset G$ and $\lambda(G) < \epsilon$.
(b) Construct a closed subset of $[0, 1]$ whose measure is positive and whose interior is empty.
(Hint: Try the complement of G in (a).)

5. Prove that if $E \subset \mathbb{R}^n$ and $\lambda^*(E) < \infty$, then there exists a measurable set A such that $E \subseteq A$ and $\lambda^*(E) = \lambda(A)$.