MATH 5302 Elementary Analysis II - Homework 2

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Problem 1

Show that
$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \, \mathrm{d}x$$
is well-defined for $\alpha > 0$ and $\beta > 0$.

Definition 1. *The improper integral*
$$\int_0^a f(x) \, \mathrm{d}x$$
is <u>well-defined</u> *iff*
$$\lim_{\epsilon \to 0} \int_0^a f(x) \, \mathrm{d}x$$
exists.

Definition 2. *The* <u>gamma function</u> $\Gamma(\alpha)$ *is defined as*
$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} \, \mathrm{d}x$$
for $0 < \alpha < \infty$*.*

Definition 3. *The* <u>beta function</u> $B(\alpha, \beta)$ *is defined as*
$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \, \mathrm{d}x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$
for $\alpha > 0$ *and* $\beta > 0$*.*

Theorem 1. ***Limit Comparison Test:*** *Let* $f, g : [a, b) \to \mathbb{R}$ *be two functions such that (i)* $f(x)$ *and* $g(x)$ *are integrable on* $[a, A] \subset [a, b)$*, for* $a < A < b$*; (ii) There exists* $a \leq K \leq b$ *such that* $\lim_{x \to b^-} \frac{f(x)}{g(x)} = K$*. Then,*

a. If $0 < K < \infty$*, then* $\int_a^b g(x) \, \mathrm{d}x$ *converges iff* $\int_a^b f(x)$ *converges.*

b. If $K = 0$*, then* $\int_a^b g(x)$ *converges implies* $\int_a^b f(x) \, \mathrm{d}x$ *converges.*

c. If $K \infty 0$*, then* $\int_a^b g(x)$ *divergent implies* $\int_a^b f(x) \, \mathrm{d}x$ *divergent.*

Theorem 2. *The improper integral that defines the beta function*
$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \, \mathrm{d}x$$
is well-defined for $\alpha > 0$ *and* $\beta > 0$*.*

Proof. The integrand of $B(\alpha, \beta)$,

$$b(\alpha, \beta) = x^{\alpha-1}(1-x)^{\beta-1}$$

is not strictly bounded $\forall_{\alpha,\beta>0}$, but this is not necessary for convergence. $\forall \alpha, \beta \in [0, \infty)$ the $b(\alpha, \beta)$ is bounded. This makes $B(\alpha, \beta)$ a proper integral which is therefore convergent.
In the other case the integrand is not bounded, but the improper integral still converges. $\forall_{\alpha\in(0,1)}$ then $b(\alpha, \beta)$ is unbounded at $x = 0$. Similarly, $\forall_{\beta\in(0,1)}$ then $b(\alpha, \beta)$ is unbounded at $x = 1$.
The beta function can instead be split up into two parts:

$$B(\alpha, \beta) = \int_0^c x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x + \int_c^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$$

where $c \in (0, 1)$.
For the first improper integral, $\int_0^c x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$ a discontinuity exists at $x = 0$ for $\alpha \in (0, 1)$. Using the Limit Comparison Test from Theorem 1 with $g(x) = x^{\alpha-1}$,

$$\begin{aligned}
\lim_{x\to 0^+} \frac{f(x)}{g(x)} &= \lim_{x\to 0^+} \frac{x^{\alpha-1}(1-x)^{\beta-1}}{x^{\alpha-1}} \\
&= \lim_{x\to 0^+} (1-x)^{\beta-1} \\
&= 1 \neq 0
\end{aligned}$$

Which then implies that $\int_0^c x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$ converges $\forall_{\alpha,\beta>0}$.
For the second improper integral, $\int_c^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$ a discontinuity exists at $x = 1$ for $\beta \in (0, 1)$. Using the Limit Comparison Test from Theorem 1 with $g(x) = (1-x)^{\beta-1}$,

$$\begin{aligned}
\lim_{x\to 1^-} \frac{f(x)}{g(x)} &= \lim_{x\to 1^-} \frac{x^{\alpha-1}(1-x)^{\beta-1}}{(1-x)^{\beta-1}} \\
&= \lim_{x\to 1^-} x^{\alpha-1} \\
&= 1 \neq 0
\end{aligned}$$

Which then implies that $\int_c^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$ converges $\forall_{\alpha,\beta>0}$.
Together, the convergence of $\int_0^c x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$ and $\int_c^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$ implies that $B(\alpha, \beta)$ converges $\forall_{\alpha,\beta>0}$ and therefore $B(\alpha, \beta)$ is well defined. □

Problem 2

Show that f if Riemann integrable on $[a, b]$, then

$$\lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

Problem 3

Evaluate $\int_0^1 (1 - x^{\frac{2}{3}})^{\frac{3}{2}} \, \mathrm{d}x$. Hint: Express the integral in terms of the gamma function first.

Example 1. *Let*

$$F(x) = \int_0^1 (1 - x^{\frac{2}{3}})^{\frac{3}{2}} \, \mathrm{d}x$$

???

Problem 4

Let

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } 0 < x \le 1; \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f is bounded and continuous on $[0,1]$, but not of bounded variation on $[0,1]$.

Definition 4. *$f : (a,b) \to \mathbb{R}$ is a* <u>bounded function</u> *iff*

$$\exists_{N\in\mathbb{R}} \;:\; \forall_{x\in(a,b)} |f(x)| < N$$

Definition 5. *$f : (S_1, d_1) \to (S_2, d_2)$ is a* <u>continuous function</u> *iff*

$$\forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x,y) < \delta \implies d_2(f(x), f(y)) < \epsilon$$

For function $f : [a,b] \to \mathbb{R}$ and partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$:

a. the <u>*variaton of f over P*</u> is defined as

$$V_a^b(f,P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

b. the <u>*variaton of f from a to b*</u> is defined as

$$V_a^b(f) = \sup_P V_a^b(f,P)$$

c. f is considered of <u>*bounded variation*</u> on $[a,b]$ if $V_a^b(f)$ is finite.

d. the <u>*family of functions of bounded variaton on $[a,b]$*</u> is denoted as BV_a^b.

Definition 6.

Example 2. *Let*

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & 0 < x \le 1 \\ 0 & x = 0 \end{cases}$$

a) $f(x)$ is bounded

Proof. For $x \in \{0\}$, $f(x) = 0$. For $x \in (0,1]$,

$$f(x) = x\sin\left(\frac{1}{x}\right) \le x(1) \le 1$$

which is bounded. Therefore, $f(x)$ is bounded $\forall_{x\in[0,1]}$. □

b) $f(x)$ is continuous

Proof. For $x \in \{0\}$, $f(x) = 0$. This means that

$$\lim_{x\to 0^-} f(x) = 0$$

For $x \in (0,1]$,

$$f(x) = x\sin\left(\frac{1}{x}\right)$$

which is continuous on $(0,1]$. Additionally, this results in

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right) = 0$$

Therefore, $f(x)$ is continuous $\forall_{x\in[0,1]}$. □

c) $f(x)$ is not of bounded variation on $[0, 1]$.

Proof. In order for f to be of bounded variation on $[a, b]$, the variation

$$V_a^b(f) = \sup_P V_a^b(f, P)$$

must be finite. The existence of this bound can be demonstrated with the following counter-example: Let

$$\{a\}_k := \left\{a_k = (2k+1)\frac{\pi}{2} \forall_{k=0,\ldots,N-1}\right\}$$

For size N, define partition of $[0, 1]$

$$P_N = \left\{0 = x_0 = 0 < x_1 = \frac{1}{a_{N-1}} < x_2 = \frac{1}{a_{N-2}} < \cdots < x_{n-1} = \frac{1}{a_0} < x_n = 1\right\}$$

which can be used to construct a sequence, but that's not the point.
The variation $V_a^b(f, P_N)$ is finite only for bounded N. i.e. $\exists_{0<M_N<\infty}$ that bounds the variation $V_a^b(f, P_N)$ for a given N:

$$\begin{aligned}
V_a^b(f, P_N) &= \sum_{i=1}^{N} |f(x_i) - f(x_i - 1)| \\
&= \left|\frac{1}{a_{N-1}}\sin(a_{N-1})\right| + \sum_{i=2}^{N}\left|\frac{1}{a_{N-i}}\sin(a_{N-i}) - \frac{1}{a_{N-i-1}}\sin(a_{N-i-1})\right| \\
&= \left|\frac{1}{a_{N-1}}\right| + \sum_{i=1}^{N}\left|\frac{1}{a_{N-i}} - \frac{1}{a_{N-i-1}}\right| |(1) - (-1)|^1 \\
&= \left|\frac{1}{a_{N-1}}\right| + 2\sum_{i=1}^{N}\left|\frac{a_{N-i} - a_{N-i-1}}{a_{N-i}a_{N-i-1}}\right|
\end{aligned}$$

However, this variation over $[a, b]$ is since the sum does not converge as $N \to \infty$. $\square$

Problem 5

Assume f is differentiable on $[a, b]$ with $|f'(x)| \leq M < \infty$ for $a \leq x \leq b$. Show that f is of bounded variation and $V_a^b(f) \leq M(b - a)$. (Hint: Use Mean Value Theorem)