MATH 5302 Elementary Analysis II - Homework 2

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Problem 1

Let

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \leq t \leq 1 \\ 4 & \text{if } t > 1 \end{cases}$$

Let $F(x) = \int_0^x f(t)\mathrm{d}t$.

a. Find $F(x)$

b. Where is F continuous?

c. Where is F differentiable? Calculate F' at the points of differentiability.

Definition 1. *Let $f : [a, b) \to \mathbb{R}$ be integrable $\forall_[a, A] \subset [a, b)$. If*

$$\lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,\mathrm{d}x$$

exists, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,\mathrm{d}x$$

is an improper integral *from a to b.*

a. If $\int_a^b f(x)\,\mathrm{d}x$ is finite, then the improper integral converges.

b. Otherwise $\int_a^b f(x)\,\mathrm{d}x$ diverges, and thus the improper integral diverges.

Definition 2. *A function $f : (a, b) \to \mathbb{R}$ is continuous on $[a, b]$ if*

$$\forall_{x \in (a,b)} \forall_{\epsilon > 0} \exists_{\delta(x,\epsilon) > 0} \forall_{y \in \mathbb{R}} |y - x| < \epsilon \implies |f(y) - f(x)| < \delta$$

Definition 3. *Let $f : (a, b) \to \mathbb{R}$ be a function.*

a. The derivative of the function at point x_0 *is defined as*

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

b. If the derivative is defined at x_0, then it is differentiable at x_0.

c. If the derivative is defined for all $x_0 \in (a, b)$, then the function f is said to be differentialble.

d. When f is differentiable, the <u>derivative</u> *of $f(x)$ is defined as:*

$$f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Example 1. *Let $f : \mathbb{R} \to \mathbb{R}$, be defined as*

$$f(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 4 & t > 1 \end{cases}$$

a) Find $F(x)$

Define the integral $F : \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$\boxed{F(t) = \begin{cases} \frac{1}{2}t^2 & 0 \le t \le 1 \\ \frac{1}{2} + 4(t-1) & t > 1 \end{cases}}$$

Proof. For $0 < t < 1$,

$$\begin{aligned} F(t) &= \int_0^t f(x)\,\mathrm{d}x \\ &= \int_0^t x\,\mathrm{d}x \end{aligned}$$

which is monotopically increasing, therefore:

$$\begin{aligned} &= \frac{1}{2}x^2\Big|_0^t \\ &= \frac{1}{2}t^2 - 0 \\ &= \frac{t^2}{2} \end{aligned}$$

For $t > 1$,

$$\begin{aligned} F(t) &= \int_0^t f(x)\,\mathrm{d}x \\ &= \int_0^1 x\,\mathrm{d}x + \int_1^t 4\,\mathrm{d}x \\ &= \frac{1}{2}x^2\Big|_0^1 + 4x\Big|_1^t \\ &= \frac{1^2}{2} - 0 + 4(t) - 4(1) \\ &= \frac{1}{2} + 4(t-1) \end{aligned}$$

For $t = 1$ and $t \ge 1$, $1 \in [0, t)$ is a discontinuity within f; however, F remains continuous but not differentiable at the discontinuity point.

$$F(t \to 1^-) = lim_{t\to 1^1} \frac{t^2}{2} = \frac{1}{2} = \lim_{t\to 1^+} \frac{1}{2} + 4(t-1) = F(t \to 1^+)$$

$\square$

b) Where is F continuous?

$F(t)$ is continuous for the entire domain, $[0,\infty)$. i.e.

$$\forall_{x\in[0,\infty)}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0} \ : \ \forall_{y\in\mathbb{R}}|y-x|<\epsilon \implies |f(y)-f(x)|<\delta$$

Proof. (not required by problem...) Essentially, this is proven by demonstrating that

$$\lim_{t\to 1^-} F(t) = \lim_{t\to 1^+} F(t)$$

□

c) Where is F differentiable? Calculate F' at the points of differentiability.

$F(t)$ is differentiable in $(a,1)\cup(1,\infty)$ which excludes 2 points from the domain: 0 and 1.

$$F' = \begin{cases} t & 0<t<1 \\ 4 & t>1 \\ \textbf{\textit{Undefined}} & t\in\{0,1\} \end{cases}$$

Proof. (not required by problem...) Essentially, this is proven by demonstrating that

$$\forall_{x\in(a,1)\cup(1,\infty)} \lim_{t\to x^-} F'(t) \neq \lim_{t\to x^+} F'(t)$$

This is also true since on regions $(a,1)$ and $(1,\infty)$, $F(t)$ is smooth continuous which by definition implies differentiability. However, this is not the case for the boundary, $x=1$:

$$\lim_{t\to 1^-} F'(t) \neq \lim_{t\to 1^+} F'(t)$$

which by definition means that $F(x)$ not differentiable at $x=1$. □

Problem 2

Let f be a continuous function on $\mathbb{R}$ and define

$$F(x) = \int_0^{\sin x} f(t)\,\mathrm{d}t$$

Show that F is differentiable on $\mathbb{R}$ and compute F'.

Theorem 1. *If g is a continuous function on $[a, b]$ that is differentiable on (a, b), and if g' is integrable on $[a, b]$, then*

$$\int_a^b g' = g(b) - g(a)$$

Theorem 2. *If u and v are continuous function on $[a, b]$ that are differentiable on (a, b), and if u' and v' are integrable on $[a, b]$, then*

$$\int_a^b u(x)v'(x)\,\mathrm{d}x + \int_a^b u'(x)v(x)\,\mathrm{d}x = \int_a^b u(x)v(x)\,\mathrm{d}x = u(b)v(b) - u(a)v(a)$$

Theorem 3. *Let $u : J \to l$ be differentiable with u' continuous. If f continuous on l, then $f \odot u$ is continuous on J and*

$$\int_a^b f \odot u(x)u'(x)\,\mathrm{d}x = \int_{u(a)}^{ub} f(u)\,\mathrm{d}u$$

for $a, b \in J$.

Example 2. *Let*

$$F(x) = \int_0^{\sin x} f(t)\,\mathrm{d}t$$

where f is some continuous function on $\mathbb{R}$.

a) Show that F is differentiable on $\mathbb{R}$

Let $u(x) = \sin x$. This definition results in $u'(x) = \cos x$. Applying Theorem 3, we have

$$u(a) = \sin a = 0 \implies a = 0$$

and

$$u(b) = \sin b = \sin x \implies b = x$$

which defines the necessary conditions for differentiability according to the theorem.

b) Compute F'.

Following,

$$\begin{aligned}
F(x) &= \int_0^{\sin x} f(t)\,\mathrm{d}t \\
&= \int_{u(a)}^{u(b)} f(t)\,\mathrm{d}t \\
&= \int_a^b f(u(x))u'(x)\,\mathrm{d}x \\
&= \int_0^x f(\sin(x))\cos(x)\,\mathrm{d}x
\end{aligned}$$

Therefore, by the Fundamental Theorem of Calculus (Theorem 1),

$$\boxed{F'(x) = f(\sin(x))\cos(x)}$$

Problem 3

Let

$$F(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & \text{if } 0 < x \leq 1; \\ 0 & \text{if } x = 0. \end{cases}$$

a. Show that F has a derivative at every $x \in [0, 1]$.

b. Show that F' is not Riemann Integrable on $[0, 1]$. (So F is not the integral of its derivative.)

Example 3. *Let $F : [0, 1] \to \mathbb{R}$ be defined as*

$$F(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & x \in (0, 1] \\ 0 & x = 0; \end{cases}$$

a) Show that F has a derivative at every $x \in [0, 1]$.

From Definition 3, the derivative of f at point x_0, $f'(x_0)$ is defined as:

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

The derivative $F'(x_0)$ is defined $\forall_{x_0 \in [0,1]}$.

Proof. For $x_0 \in (0, 1)$,

$$\begin{aligned}
F'(x_0) &= \lim_{x \to x_0} \frac{x^2 \sin\left(\frac{1}{x^2}\right) - (x_0)^2 \sin\left(\frac{1}{x_0^2}\right)}{x - x_0} \\
F'(x) &= \lim_{h \to 0} \frac{(x+h)^2 \sin\left(\frac{1}{(x+h)^2}\right) - (x)^2 \sin\left(\frac{1}{x^2}\right)}{h} \\
&= \lim_{h \to 0} \frac{\left(x^2 + 2xh + h^2\right) \sin\left(\frac{1}{(x+h)^2}\right) - (x)^2 \sin\left(\frac{1}{x^2}\right)}{h} \\
&= \lim_{h \to 0} \frac{(x)^2 \left(\sin\left(\frac{1}{(x+h)^2}\right) - \sin\left(\frac{1}{x^2}\right)\right) + \left(2xh + h^2\right) \sin\left(\frac{1}{(x+h)^2}\right)}{h} \\
&= \lim_{h \to 0} \frac{x^2 \left(\sin\left(\frac{1}{(x+h)^2}\right) - \sin\left(\frac{1}{x^2}\right)\right)}{h} \\
&\quad + \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{(x+h)^2}\right)}{h} \\
&\quad + \lim_{h \to 0} \frac{2xh \sin\left(\frac{1}{(x+h)^2}\right)}{h} \\
&= x^2 \lim_{h \to 0} \frac{\sin\left(\frac{1}{x+h}^2\right) - \sin\left(\sin 1x^2\right)}{h} + 0 + \lim_{h \to 0} 2x \sin\left(\frac{1}{(x+h)^2}\right) \\
&= 2x \sin\left(\frac{1}{x^2}\right) + x^2 \frac{\mathrm{d}}{\mathrm{d} \sin\left(\frac{1}{x^2}\right)} \\
&= 2x \sin\left(\frac{1}{x^2}\right) + x^2 \left(-2 \frac{1}{x^3} \cos\left(\frac{1}{x^2}\right)\right) \\
F'(x) &= 2x \sin\left(\frac{1}{x^2}\right) - 2 \frac{1}{x} \cos\left(\frac{1}{x^2}\right)
\end{aligned}$$

Which means that $F(x)$ is differentiable $\forall_{x_0 \in (0,1)}$.
This result can be expanded to the closed interval by taking the limit of $F(x)$ to the boundaries which also exist; therefore, $F(x)$ is differentiable $\forall_{x_0 \in [0,1]}$. $\square$

Problem 4

Show that for each $p > 0$, $\int_1^\infty \frac{\sin(x)}{x^p}\,\mathrm{d}x$ converges. Hint: For $0 < p < 1$, you may find it helpful to use integration by parts.

Definition 4. *Let $f : [a,\infty) \to \mathbb{R}$ be a function that is integrable over $[a, A] \subset [a,\infty]$. If the limit*

$$\lim_{A\to\infty}\int_a^A f(x)\,\mathrm{d}x$$

exists then the improper integral from $a \to \infty$ *is denoted as*

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{A\to\infty}\int_a^A f(x)\,\mathrm{d}x$$

a. If $\int_a^\infty f(x)\,\mathrm{d}a$ is finite, then it is called converging.

b. If $\int_a^\infty f(x)\,\mathrm{d}a$ is not finite, then it is called diverging.

c. This definition implies for $f : (-\infty, a] \to \mathbb{R}$ and

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{A\to\infty}\int_a^A f(x)\,\mathrm{d}x$$

Definition 5. *Let $f : (-\infty,\infty) \to \mathbb{R}$ be a function which is integrable on $\forall[A, B] \subset (-\infty,\infty)$. If for some $a \in (-\infty,\infty)$ there exists $\int_{-\infty}^a f(x)\,\mathrm{d}x$ and $\int_a^\infty f(x)ddx$ converges, then*

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)ddx$$

converges.

Theorem 4. ***Comparison Test:*** *Let $f, g : [a, b) \to \mathbb{R}$ be two functions such that (i) $f(x)$ and $g(x)$ are integrable on $[a, A] \subset [a, b)$, for $a < A < b$; (ii) There exists $a < M < b$ such that $0 \le f(x) \le g(x)$ forall $x \in [M, b)$. Then,*

a. If $\int_a^b g(x)\,\mathrm{d}x$ converges then $\int_a^b f(x)\,\mathrm{d}x$ also converges;

b. If $\int_a^b f(x)\,\mathrm{d}x$ diverges then $\int_a^b f(x)\,\mathrm{d}x$ also diverges.

Theorem 5. ***Limit Comparison Test:*** *Let $f, g : [a, b) \to \mathbb{R}$ be two functions such that (i) $f(x)$ and $g(x)$ are integrable on $[a, A] \subset [a, b)$, for $a < A < b$; (ii) There exists $a \le K \le b$ such that $\lim_{x\to b^-}\frac{f(x)}{g(x)} = K$. Then,*

a. If $0 < K < \infty$, then $\int_a^b g(x)\,\mathrm{d}x$ converges iff $\int_a^b f(x)$ converges.

b. If $K = 0$, then $\int_a^b g(x)$ converges implies $\int_a^b f(x)\,\mathrm{d}x$ converges.

c. If $K\infty 0$, then $\int_a^b g(x)$ divergent implies $\int_a^b f(x)\,\mathrm{d}x$ divergent.

a) Solution:

Theorem 6. *For all $p > 0$, the integral*

$$\int_1^\infty \frac{\sin(x)}{x^p}\,\mathrm{d}x$$

converges.

Proof. By integration by parts we have

$$\begin{aligned}\int_1^\infty \frac{\sin(x)}{x^p}\,\mathrm{d}x &= \int_1^\infty \sin(x)\frac{1}{x^p}\,\mathrm{d}x\\ &= \frac{-\cos(x)}{x^p}\Big|_0^\infty - p\int_0^\infty \frac{cos(x)}{x^{p+1}}\,\mathrm{d}x\end{aligned}$$

The demonstration of convergence can be done with the by extending the upper limit up to infinity:

$$\lim_{L\to\infty}\int_1^L \frac{\sin(x)}{x^p} = \lim_{L\to\infty}\frac{-\cos(x)}{x^p}\Big|_1^L - p\int_1^L \frac{\cos(x)}{x^{p+1}}\,\mathrm{d}x$$

Since $\left|\frac{\cos(x)}{x^{p+1}}\right| \leq \left|\frac{1}{x^{p+1}}\right| \forall_{x\geq 1}$, we can use the Comparison test (Theorem 4) to conclude that $\int_1^L \frac{\cos(x)}{x^{p+1}} < \int_1^L \frac{1}{x^{p+1}}$. Further, since $\int_1^\infty \frac{1}{x^{p+1}}$ converges $\forall_{p+1>1}$, this portion converges $\forall_{p>0}$. Therefore, $\lim_{L\to\infty}\int_1^L \frac{\sin(x)}{x^p}$ converges since $\lim_{L\to\infty} \frac{-\cos(x)}{x^p}\Big|_1^L$ is finite and $\lim_{L\to\infty}\int_1^L \frac{\cos(x)}{x^{p+1}}$ converges. $\square$

Problem 5

Consider $\int_1^\infty \frac{x^p}{1+x^q}$.

a. For what values of p and q are the integral convergent?

b. For what values of p and q are the integral absolutely convergent?

Example 4. *Define the integral*

$$\int_1^\infty \frac{x^p}{1+x^q}$$