

MATH 5302 Elementary Analysis II - Homework 2

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Problem 1

Let

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \leq t \leq 1 \\ 4 & \text{if } t > 1 \end{cases}$$

Let $F(x) = \int_0^x f(t)dt$.

- Find $F(x)$
- Where is F continuous?
- Where is F differentiable? Calculate F' at the points of differentiability.

Definition 1. Let $f : [a, b) \rightarrow \mathbb{R}$ be integrable $\forall [a, A] \subset [a, b)$. If

$$\lim_{\epsilon \rightarrow 0^+} \int_a^{b-\epsilon} f(x) dx$$

exists, then

$$\int_a^b f(x) dx = \lim_{\epsilon \rightarrow 0^+} \int_a^{b-\epsilon} f(x) dx$$

is an improper integral from a to b .

- If $\int_a^b f(x) dx$ is finite, then the improper integral converges.
- Otherwise $\int_a^b f(x) dx$ diverges, and thus the improper integral diverges.

Definition 2. A function $f : (a, b) \rightarrow \mathbb{R}$ is continuous on $[a, b]$ if

$$\forall x \in (a, b) \forall \epsilon > 0 \exists \delta(x, \epsilon) > 0 \forall y \in \mathbb{R} |y - x| < \delta \implies |f(y) - f(x)| < \epsilon$$

Definition 3. Let $f : (a, b) \rightarrow \mathbb{R}$ be a function.

- The derivative of the function at point x_0 is defined as

$$f'(x_0) := \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

- If the derivative is defined at x_0 , then it is differentiable at x_0 .
- If the derivative is defined for all $x_0 \in (a, b)$, then the function f is said to be differentiable.

d. When f is differentiable, the derivative of $f(x)$ is defined as:

$$f'(x) := \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Example 1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$, be defined as

$$f(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 4 & t > 1 \end{cases}$$

a) Find $F(x)$

Define the integral $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is defined as

$$F(t) = \begin{cases} \frac{1}{2}t^2 & 0 \leq t \leq 1 \\ \frac{1}{2} + 4(t-1) & t > 1 \end{cases}$$

Proof. For $0 < t < 1$,

$$\begin{aligned} F(t) &= \int_0^t f(x) \, dx \\ &= \int_0^t x \, dx \end{aligned}$$

which is monotonically increasing, therefore:

$$\begin{aligned} &= \left. \frac{1}{2}x^2 \right|_0^t \\ &= \frac{1}{2}t^2 - 0 \\ &= \frac{t^2}{2} \end{aligned}$$

For $t > 1$,

$$\begin{aligned} F(t) &= \int_0^t f(x) \, dx \\ &= \int_0^1 x \, dx + \int_1^t 4 \, dx \\ &= \left. \frac{1}{2}x^2 \right|_0^1 + \left. 4x \right|_1^t \\ &= \frac{1^2}{2} - 0 + 4(t) - 4(1) \\ &= \frac{1}{2} + 4(t-1) \end{aligned}$$

For $t = 1$ and $t \geq 1$, $1 \in [0, t)$ is a discontinuity within f ; however, F remains continuous but not differentiable at the discontinuity point.

$$F(t \rightarrow 1^-) = \lim_{t \rightarrow 1^-} \frac{t^2}{2} = \frac{1}{2} = \lim_{t \rightarrow 1^+} \frac{1}{2} + 4(t-1) = F(t \rightarrow 1^+)$$

□

b) Where is F continuous?

$F(t)$ is continuous for the entire domain, $[0, \infty)$. i.e.

$$\forall_{x \in [0, \infty)} \forall_{\epsilon > 0} \exists_{\delta(x, \epsilon) > 0} : \forall_{y \in \mathbb{R}} |y - x| < \epsilon \implies |f(y) - f(x)| < \delta$$

Proof. (not required by problem...) Essentially, this is proven by demonstrating that

$$\lim_{t \rightarrow 1^-} F(t) = \lim_{t \rightarrow 1^+} F(t)$$

□

c) Where is F differentiable? Calculate F' at the points of differentiability.

$F(t)$ is differentiable in $(a, 1) \cup (1, \infty)$ which excludes 2 points from the domain: 0 and 1.

$$F' = \begin{cases} t & 0 < t < 1 \\ 4 & t > 1 \\ \textit{Undefined} & t \in \{0, 1\} \end{cases}$$

Proof. (not required by problem...) Essentially, this is proven by demonstrating that

$$\forall_{x \in (a, 1) \cup (1, \infty)} \lim_{t \rightarrow x^-} F'(t) \neq \lim_{t \rightarrow x^+} F'(t)$$

This is also true since on regions $(a, 1)$ and $(1, \infty)$, $F(t)$ is smooth continuous which by definition implies differentiability. However, this is not the case for the boundary, $x = 1$:

$$\lim_{t \rightarrow 1^-} F'(t) \neq \lim_{t \rightarrow 1^+} F'(t)$$

which by definition means that $F(x)$ not differentiable at $x = 1$.

□

Problem 2

Let f be a continuous function on \mathbb{R} and define

$$F(x) = \int_0^{\sin x} f(t) \, dt$$

Show that F is differentiable on \mathbb{R} and compute F' .

Theorem 1. If g is a continuous function on $[a, b]$ that is differentiable on (a, b) , and if g' is integrable on $[a, b]$, then

$$\int_a^b g' = g(b) - g(a)$$

Theorem 2. If u and v are continuous function on $[a, b]$ that are differentiable on (a, b) , and if u' and v' are integrable on $[a, b]$, then

$$\int_a^b u(x)v'(x) \, dx + \int_a^b u'(x)v(x) \, dx = \int_a^b u(x)v(x) \, dx = u(b)v(b) - u(a)v(a)$$

Theorem 3. Let $u : J \rightarrow I$ be differentiable with u' continuous. If f continuous on I , then $f \odot u$ is continuous on J and

$$\int_a^b f \odot u(x)u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) \, du$$

for $a, b \in J$.

Example 2. Let

$$F(x) = \int_0^{\sin x} f(t) \, dt$$

where f is some continuous function on \mathbb{R} .

a) Show that F is differentiable on \mathbb{R}

Let $u(x) = \sin x$. This definition results in $u'(x) = \cos x$. Applying Theorem 3, we have

$$u(a) = \sin a = 0 \implies a = 0$$

and

$$u(b) = \sin b = \sin x \implies b = x$$

which defines the necessary conditions for differentiability according to the theorem.

b) Compute F' .

Following,

$$\begin{aligned} F(x) &= \int_0^{\sin x} f(t) \, dt \\ &= \int_{u(a)}^{u(b)} f(t) \, dt \\ &= \int_a^b f(u(x))u'(x) \, dx \\ &= \int_0^x f(\sin(x)) \cos(x) \, dx \end{aligned}$$

Therefore, by the Fundamental Theorem of Calculus (Theorem 1),

$$\boxed{F'(x) = f(\sin(x)) \cos(x)}$$

Problem 3

Let

$$F(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & \text{if } 0 < x \leq 1; \\ 0 & \text{if } x = 0. \end{cases}$$

- Show that F has a derivative at every $x \in [0, 1]$.
- Show that F' is not Riemann Integrable on $[0, 1]$. (So F is not the integral of its derivative.)

Definition 4. Define bounded function $f : [a, b] \rightarrow \mathbb{R}$ and set $S \subseteq [a, b]$. Let $M(f, S) := \sup \{f(x) : x \in S\}$ and $m(f, S) = \inf \{f(x) : x \in S\}$. Let $U(f, P)$ and $L(f, P)$ for f w.r.t. P be defined by

$$U(f, P) = \sum_{i=1}^n M(f, [x_{i-1}, x_i]) \cdot (x_i - x_{i-1})$$

and

$$L(f, P) = \sum_{i=1}^n m(f, [x_{i-1}, x_i]) \cdot (x_i - x_{i-1})$$

The Upper Darboux Integral $U(f)$ for f over $[a, b]$ is defined as

$$U(f) = \inf \{U(f, P) : P = \{a = x_0 < x_1 < \cdots < x_n = b\}\}$$

The Lower Darboux Integral $L(f)$ for f over $[a, b]$ is defined as

$$L(f) = \sup \{L(f, P) : P = \{a = x_0 < x_1 < \cdots < x_n = b\}\}$$

Definition 5. f is Darboux Integrable on $[a, b]$ iff $L(f) = U(f)$. i.e.

$$\int_a^b f = \int_a^b f(x) dx = L(f) = U(f)$$

Example 3. Let $F : [0, 1] \rightarrow \mathbb{R}$ be defined as

$$F(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & x \in (0, 1] \\ 0 & x = 0; \end{cases}$$

a) Show that F has a derivative at every $x \in [0, 1]$.

From Definition 3, the derivative of f at point x_0 , $f'(x_0)$ is defined as:

$$f'(x_0) := \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

The derivative $F'(x_0)$ is defined $\forall_{x_0 \in [0, 1]}$.

Proof. For $x_0 \in (0, 1)$,

$$\begin{aligned}
F'(x_0) &= \lim_{x \rightarrow x_0} \frac{x^2 \sin\left(\frac{1}{x^2}\right) - (x_0)^2 \sin\left(\frac{1}{x_0^2}\right)}{x - x_0} \\
F'(x) &= \lim_{h \rightarrow 0} \frac{(x+h)^2 \sin\left(\frac{1}{(x+h)^2}\right) - (x)^2 \sin\left(\frac{1}{x^2}\right)}{h} \\
&= \lim_{h \rightarrow 0} \frac{(x^2 + 2xh + h^2) \sin\left(\frac{1}{(x+h)^2}\right) - (x)^2 \sin\left(\frac{1}{x^2}\right)}{h} \\
&= \lim_{h \rightarrow 0} \frac{(x)^2 \left(\sin\left(\frac{1}{(x+h)^2}\right) - \sin\left(\frac{1}{x^2}\right) \right) + (2xh + h^2) \sin\left(\frac{1}{(x+h)^2}\right)}{h} \\
&= \lim_{h \rightarrow 0} \frac{x^2 \left(\sin\left(\frac{1}{(x+h)^2}\right) - \sin\left(\frac{1}{x^2}\right) \right)}{h} \\
&\quad + \lim_{h \rightarrow 0} \frac{h^2 \sin\left(\frac{1}{(x+h)^2}\right)}{h} \\
&\quad + \lim_{h \rightarrow 0} \frac{2xh \sin\left(\frac{1}{(x+h)^2}\right)}{h} \\
&= x^2 \lim_{h \rightarrow 0} \frac{\sin\left(\frac{1}{x+h}\right)^2 - \sin\left(\frac{1}{x}\right)^2}{h} + 0 + \lim_{h \rightarrow 0} 2x \sin\left(\frac{1}{(x+h)^2}\right) \\
&= 2x \sin\left(\frac{1}{x^2}\right) + x^2 \frac{d}{dx} \sin\left(\frac{1}{x^2}\right) \\
&= 2x \sin\left(\frac{1}{x^2}\right) + x^2 \left(-2 \frac{1}{x^3} \cos\left(\frac{1}{x^2}\right) \right) \\
F'(x) &= 2x \sin\left(\frac{1}{x^2}\right) - 2 \frac{1}{x} \cos\left(\frac{1}{x^2}\right)
\end{aligned}$$

Which means that $F(x)$ is differentiable $\forall_{x_0 \in (0,1)}$.

This result can be expanded to the closed interval by taking the limit of $F(x)$ to the boundaries which also exist; therefore, $F(x)$ is differentiable $\forall_{x_0 \in [0,1]}$. \square

- b) **Show that F' is not Riemann Integrable on $[0, 1]$. (So F is not the integral of its derivative.)**

Problem 4

Show that for each $p > 0$, $\int_1^\infty \frac{\sin(x)}{x^p} dx$ converges. Hint: For $0 < p < 1$, you may find it helpful to use integration by parts.

Definition 6. Let $f : [a, \infty) \rightarrow \mathbb{R}$ be a function that is integrable over $[a, A] \subset [a, \infty]$. If the limit

$$\lim_{A \rightarrow \infty} \int_a^A f(x) dx$$

exists then the improper integral from $a \rightarrow \infty$ is denoted as

$$\int_a^\infty f(x) dx = \lim_{A \rightarrow \infty} \int_a^A f(x) dx$$

a. If $\int_a^\infty f(x) dx$ is finite, then it is called converging.

b. If $\int_a^\infty f(x) dx$ is not finite, then it is called diverging.

c. This definition implies for $f : (-\infty, a] \rightarrow \mathbb{R}$ and

$$\int_a^\infty f(x) dx = \lim_{A \rightarrow \infty} \int_a^A f(x) dx$$

Definition 7. Let $f : (-\infty, \infty) \rightarrow \mathbb{R}$ be a function which is integrable on $\forall [A, B] \subset (-\infty, \infty)$. If

Theorem 4. For all $p > 0$, the integral

$$\int_1^\infty \frac{\sin(x)}{x^p}$$

converges.

Proof. First, letting $p \geq 1$,

□

Problem 5

Consider $\int_1^\infty \frac{x^p}{1+x^q}$.

- a. For what values of p and q are the integral convergent?
- b. For what values of p and q are the integral absolutely convergent?

Example 4. *Define the integral*

$$\int_1^\infty \frac{x^p}{1+x^q}$$