MATH 5302 Elementary Analysis II - Homework 8

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Contents

Preliminaries

Definition 1. <u>n-dimensional Euclidean norm-space:</u> *Let $\mathbb{R}^n$ be defined as*

$$\mathbb{R}^n := \{x = (x_1, x_2, \ldots, x_n) \ : \ x_j \in \mathbb{R}\}$$

Let $A, B \subseteq \mathbb{R}^n$.

1. <u>Compliment of A</u>
$$A^c := \{x \in \mathbb{R}^n \ : \ x \neq A\}$$
2. <u>Union of A and B</u>
$$A \cup B := \{x \in \mathbb{R}^n \ : \ x \in A \vee x \in B\}$$
3. <u>Intersection of A and B</u>
$$A \cap B := \{x \in \mathbb{R}^n \ : \ x \in A \wedge x \in B\}$$
4. <u>Difference of A and B</u>
$$A \backslash B = A \cap B^c := \{x \in \mathbb{R}^n \ : \ x \in A \wedge x \neq B\}$$
5. <u>Closure of A</u>
$$\overline{A} := \{x \in \mathbb{R}^n \ : \ x \in A \vee x = \lim_{k \to \infty} x_k \ : \ [x_k] \in A\}$$
6. <u>Euclidean norm on $\mathbb{R}^n$</u>
$$\|x\| := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$
with triangular inequality:
$$\|x + y\| \leq \|x\| + \|y\|$$
7. <u>Metric on $\mathbb{R}^n$</u>
$$d(x, y) = \|x - y\|$$
with properties
   (a) $d(x, y) \geq 0$
   (b) $d(x, y) = 0 \iff x = y$
   (c) $d(x, y) = d(y, x)$
   (d) $d(x, y) \leq d(x, z) + d(z, y)$
8. *A is considered* <u>bounded</u> *if every point in A is bounded:*
$$\exists_{M>0} \ : \ \forall_{x \in A} \|x\| \leq M$$

Definition 2. **Open and Closed Sets:** *Let $A \subseteq \mathbb{R}^n$ and $x \in A$.*

1. *Denote the* <u>open ball</u> *centered at x of radius r as:*
$$B(x, r) := \{y \in \mathbb{R}^n \ : \ d(x, y) < r\}$$
2. *x is considered an* <u>interior point of A</u> *if :*
$$\exists_{r>0} \ : \ B(x, r) \subseteq A$$
3. *A is considered* <u>open</u> *if every point $x \in A$ is an interior point of A:*
$$\forall_{x \in A} \exists_{r>0} \ : \ B(x, r) \subseteq A$$
The following properties exist for open sets:

(a) $\emptyset$ *is open*
(b) $\mathbb{R}^n$ *is open*
(c) *Union of any collection of open sets is open*
(d) *Intersection of any finite collection of open sets is open*
(e) *Any open ball is an open set*

4. *The* Interior of A *is the set of all interior points of* A

$$A^\circ := \{x \ : \ x \textit{is an interior point of } A\}$$

Properties of A°

(a) A *open* $\iff A^\circ = A$
(b) A° *open*
(c) $(A^\circ)^\circ = A^\circ$
(d) $(A \cap B)^\circ = A^\circ \cap B^\circ$
(e) $A^\circ \cup B^\circ$ *not generally equal to* $(A \cup B)^\circ$
(f) A° *is the union of all open subsets of* A
(g) A° *is the largest open subset of* A

5. *The* Boundary of A *is defined as the closure minus the Interior of* A*:*

$$\partial A := \overline{A} \backslash A^\circ$$

6. A *is considered* closed *if* A^c *is open. Properties of closed sets*

(a) $\mathbb{R}^n$ *is closed*
(b) $\emptyset$ *is closed*
(c) *The intersection of any collection of closed sets is closed*
(d) *The union of any finite collection of closed sets is closed*

Definition 3. Compact set: *Let* $A \subseteq \mathbb{R}^n$.

1. A *is called* compact *if every open cover of* A *has a finite subcover.*

2. *Properties of compact sets*

(a) $\emptyset$ *is compact*
(b) *Any finite set is compact*
(c) A *and* B *compact* $\implies A \cup B$ *compact*
(d) *Any finite union of compact sets is compact*
(e) $B(x, r)$ *is not compact*
(f) $\mathbb{R}^n$ *is not compact*
(g) *If* A *is compact then* A *is closed and bounded.*

Definition 4. Lebesgue Measure: *Let* $A \subseteq \mathbb{R}^n$. Note: *For various* n*, a Lebesgue Measure is essentially:*

1. *For* $n = 1$, $\lambda(A)$ *is a length*

2. *For* $n = 2$, $\lambda(A)$ *is an area*

3. *For* $n = 3$, $\lambda(A)$ *is a volume*

Each of the following stages define the Lebesgue measure in increasing complexity.

0. **Empty Set:**

$$\lambda(\emptyset) = 0$$

1. **Special Rectangles:**

   Let

$$I = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

   then

$$\lambda(I) = (b_1 - a_1) \cdots (b_n - a_n)$$

2. **Special Polygons:** *Special polygons are a finite union of special rectangles. They are closed and bounded subsets and therefore compact.*

   Let P be a special polygon, decomposed into the following union of nonoverlaping special rectangles:

$$P = \bigcup_{k=1}^{N} I_k$$

   The Lebesgue Measure for the special polygon is defined as the sum of the Lebesgue Measures of the special rectangles:

$$\lambda(P) = \sum_{k=1}^{N} \lambda(I_k)$$

   The following are a few properties of the Lebesgue Measure for Special Polygons:

   (a) $P_1 \subseteq P_2 \implies \lambda(P_1) \leq \lambda(P_2)$
   (b) $P_1 \cap P_2 = \emptyset \implies \lambda(P_1 \cup P_2) = \lambda(P_1) + \lambda(P_2)$

3. **Open Sets:** *Let $G \subseteq \mathbb{R}^n$ be open and $G \neq \emptyset$.*

   The Lebesgue Measure of an open set is defined as

$$\lambda(G) = \sup\{\lambda(P) \ : \ P \subset G, \ P \text{ is a special polygon}\}$$

   The following are some properties of the Lebesgue Measure for open sets:

   (a) $0 \leq \lambda(G) \leq \infty$
   (b) $\lambda(G) = 0 \iff G = \emptyset$
   (c) $\lambda(\mathbb{R}^n) = \infty$
   (d) $G_1 \subseteq G_2 \implies \lambda(G_1) \leq \lambda(G_2)$
   (e) $\lambda(\bigcup_{k=1}^{\infty} G_k) \leq \sum_{k=1}^{\infty} \lambda(G_k)$
   (f) $\bigcap_{k=1}^{\infty} G_k = \emptyset \implies \lambda(\bigcup_{k=1}^{\infty} G_k) = \sum_{k=1}^{\infty} \lambda(G_k)$
   (g) *If P is a special polygon, then* $\lambda(P) = \lambda(P^\circ)$

4. **Compact Sets:**

   Let $K \subseteq \mathbb{R}^n$ be a compact set.

   The Lebesgue Measure of a compact set is defined as

$$\lambda(K) = \inf\{\lambda(G) \ : \ K \subset G, \ G \text{ is an open set}\}$$

   The following are some properties of the Lebesgue Measure for compact sets:

   (a) $0 \leq \lambda(K) \leq \infty$
   (b) $K_1 \subseteq K_2 \implies \lambda(K_1) \leq \lambda(K_2)$

(c) $\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2)$

(d) $K_1 \cap K_2 = \emptyset \implies \lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$

5. **Finite Outer Measure:**

Definition 5. *Let $A \subseteq \mathbb{R}^n$. The* <u>outer measure of A</u> *is defined as*

$$\lambda^*(A) = \inf \{\lambda(G) \ : \ A \subseteq G \wedge G \ \textit{open}\}$$

The <u>inner measure of A</u> *is defined as*

$$\lambda_*(A) = \sup \{\lambda(K) \ : \ A \supseteq K \wedge K \ \textit{compact}\}$$

The following are properties of the outer and inner measures:

(a) $\lambda_*(A) \leq \lambda^*(A)$

(b) $A \subseteq B \implies \lambda^*(A) \leq \lambda^*(B) \wedge \lambda_*(A) \leq \lambda_*(B)$

(c) $\lambda^*(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \lambda^*(A_k)$

(d) *If all A_k's are disjoint, then*

$$\lambda_* \left(\bigcup_{k=1}^{\infty} A_k \right) \geq \sum_{k=1}^{\infty} \lambda^*(A_k)$$

(e) *If A is open or compact, then $\lambda^*(A) = \lambda_*(A) = \lambda(A)$*

Definition 6. *Let $A \subseteq \mathbb{R}^n$ be a set with finite outer measure: $\lambda^*(A) < \infty$ We say that $A \in \mathcal{L}_0$ if $\lambda^*(A) = \lambda_*(A)$ We define $\mathcal{L}_0$ as*

$$\mathcal{L}_0 = \{A \subseteq \mathbb{R}^n \ : \ \lambda^*(A) = \lambda_*(A) < \infty\}$$

If $A \in \mathcal{L}_0$ then the <u>measure of A</u> *is*

$$\lambda(A) = \lambda^*(A) = \lambda_*(A)$$

Definition 7. Properties of the Lebesgue Measure:
Let $A \subseteq \mathbb{R}^n$.

Problem 1

a)

Problem:

Prove Property M6: If the A_k's are measurable and $A_1 \supseteq A_2 \supseteq A_3 \cdots$, and if $\lambda(A_1) < \infty$, then

$$\lambda\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \lambda(A_k)$$

Solution:

Theorem 1. *Let*

b)

Problem:

Give an example to show why the assumption $\lambda(A_1) < \infty$ is needed in Property M6.

Solution: