Assignment 7 of Math 5302

Due Date: April 13, 2022 at 11:59pm

1. Let $A \subset \mathbb{R}^n$ be a compact set. Show that A is bounded.

2. Let G be a nonempty subset of $\mathbb{R}^n$. If G is open and P is a special polygon with $P \subset G$, prove there exists a special polygon P' such that $P \subset P' \subset G$ and $\lambda(P) < \lambda(P')$. (Hint: consider $G \sim P$.)

3. Use the definition of Lebesgue measure, $\lambda(G)$, of an open set $G \subset \mathbb{R}^n$ to prove the following statements:
(a) If G is a bounded open set, then $\lambda(G) < \infty$.
(b) Let

$$G = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1 \text{ and } 0 < y < x^2\}.$$

Then $\lambda(G) = \frac{1}{3}$. (Hint: relate $\lambda(G)$ to lower and upper Darboux sums of the function $f(x) = x^2$ on $[0, 1]$. However, you cannot use the methods of calculus to the extent that $\lambda(G) = \int_0^1 x^2 dx = \frac{1}{3}$. You must use the actual definition of $\lambda(G)$.)

4. Prove that every nonempty open subset of $\mathbb{R}$ can be expressed as a countable disjoint union of open intervals:

$$G = \bigcup_k (a_k, b_k),$$

where the range on k can be finite or infinite. Furthermore, show that this expression is unique except for the numbering of the component intervals. (Hint: for any $x \in G$, show that there exists a largest open interval A_x such that $x \in A_x$ and $A_x \subseteq G$. Also note that the set of rational numbers is countable and dense in $\mathbb{R}$.)

5. In the notation of Problem 4, prove that $\lambda(G) = \sum_k (b_k - a_k)$.

6. Let C be the Cantor set. Show that $\frac{1}{4} \in C$ and that $\frac{1}{4}$ is not an end point of any of the intervals in the G_k's.