MATH 5302 Elementary Analysis II - Homework 2

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Problem 1

Let

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \leq t \leq 1 \\ 4 & \text{if } t > 1 \end{cases}$$

Let $F(x) = \int_0^x f(t) \mathrm{d}t$.

a) Find $F(x)$

Definition 1. *Let $f : [a, b) \to \mathbb{R}$ be integrable $\forall_[a, A] \subset [a, b)$. If*

$$\lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x) \, \mathrm{d}x$$

exists, then

$$\int_a^b f(x) \, \mathrm{d}x = \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x) \, \mathrm{d}x$$

is an improper integral *from a to b.*

a. If $\int_a^b f(x) \, \mathrm{d}x$ is finite, then the improper integral converges.

b. Otherwise $\int_a^b f(x) \, \mathrm{d}x$ diverges, and thus the improper integral diverges.

Solution:

Example 1. *Let $f : \mathbb{R} \to \mathbb{R}$, be defined as*

$$f(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 4 & t > 1 \end{cases}$$

Define the integral $F : \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$F(t) = \begin{cases} \frac{1}{2} t^2 & 0 \leq t \leq 1 \\ \frac{1}{2} + 4(t-1) & t > 1 \end{cases}$$

Proof. For $0 < t < 1$,

$$\begin{aligned} F(t) &= \int_0^t f(x) \, \mathrm{d}x \\ &= \int_0^t x \, \mathrm{d}x \end{aligned}$$

which is monotopically increasing, therefore:

$$\begin{aligned}
&= \frac{1}{2}x^2\bigg|_0^t \\
&= \frac{1}{2}t^2 - 0 \\
&= \frac{t^2}{2}
\end{aligned}$$

For $t > 1$,

$$\begin{aligned}
F(t) &= \int_0^t f(x)\,\mathrm{d}x \\
&= \int_0^1 x\,\mathrm{d}x + \int_1^t 4\,\mathrm{d}x \\
&= \frac{1}{2}x^2\bigg|_0^1 + 4x\bigg|_1^t \\
&= \frac{1^2}{2} - 0 + 4(t) - 4(1) \\
&= \frac{1}{2} + 4(t-1)
\end{aligned}$$

For $t = 1$ and $t \geq 1$, $1 \in [0, t)$ is a discontinuity within f; however, F remains continuous but not differentiable at the discontinuity point.

$$F(t \to 1^-) = lim_{t \to 1^1} \frac{t^2}{2} = \frac{1}{2} = \lim_{t \to 1^+} \frac{1}{2} + 4(t-1) = F(t \to 1^+)$$

$\square$