Assignment 6 of Math 5302

Due Date: April 6, 2022 at 11:59pm

1. Assume f is a real-valued function defined on $[a, b]$ and f is Lipschitz continuous on $[a, b]$. Show that f is absolutely continuous on $[a, b]$.

2. If f is continuous and α is of bounded variation on $[a, b]$. Then f is Riemann-Stieltjes integrable with respect to α on $[a, b]$. Let $\beta(x) = V_a^x(\alpha)$ and $\gamma(x) = \beta(x) - \alpha(x)$, $x \in [a, b]$. Show that
(a)

$$\left| \int_a^b f(x) d\alpha(x) \right| \le \int_a^b |f(x)| d\beta(x) \le \max_{x \in [a,b]} |f| V_a^b(\alpha).$$

(b) The function α is Riemann-Stieltjes integrable with respect to f on $[a, b]$.

3. Given a positive integer n and numbers c_0, c_1, c_2, ..., c_n, let α be the step function defined on $[0, 1]$ by

$$\begin{aligned}
&\alpha(0) = 0, \\
&\alpha(x) = c_0 \text{ for } 0 < x < \frac{1}{n}, \\
&\alpha(x) = \sum_{i=0}^{k-1} c_i \text{ for } \frac{k-1}{n} \le x < \frac{k}{n}, k = 2, 3, ..., n, \\
&\alpha(1) = \sum_{i=0}^{n} c_i.
\end{aligned}$$

Show that $V_0^1(\alpha) \le \sum\limits_{i=0}^{n} |c_i|$. (Hint: Use Riemann-Stieltjes integral to estimate the variation.)

4. Let

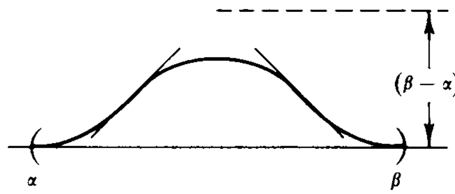
$$f(x) = \begin{cases} x^2 & \text{if } -1 \le x \le 0; \\ x^3 & \text{if } 0 < x \le 1; \end{cases} \quad \text{and } \alpha(x) = \begin{cases} 1 & \text{if } x = -1; \\ 2x^2 & \text{if } -1 < x < 1; \\ -1 & \text{if } x = 1. \end{cases}$$

Evaluate the Darboux-Stieltjes integral $\int_{-1}^1 f(x) d\alpha(x)$.

5. Let C be the Cantor set in $[0, 1]$. The Cantor set C is created by iteratively deleting the open middle third from a set of non-overlapping closed intervals. One starts by deleting the open middle third $(\frac{1}{3}, \frac{2}{3})$ from the interval $[0, 1]$, leaving two closed intervals: $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Next, the open middle third of

each of these remaining intervals is deleted, leaving four closed intervals: $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$, and $[\frac{8}{9}, 1]$. Continue this process for ever. The Cantor set contains all points in the interval $[0, 1]$ that are not deleted at any step in this infinite process. Let D be the open set deleted. Then $C = [0, 1] \sim D$.

A continuous function f is defined to be zero on C and on each component interval (α, β) of D to have its graph as shown in the figure. The exact equation is not important, but on (α, β), f' is continuous, $f'(\alpha^+) = f'(\beta^-) = 0$, $\max_{x\in(\alpha,\beta)} |f'(x)| = 1$, and $\max_{x\in(\alpha,\beta)} f(x) \leq (\beta - \alpha)^2$. Show that the Riemann



integral $\int_0^1 f'(x)dx$ doesn't exist even though $f'(x)$ exists and are bounded on $[0, 1]$.