MATH 5302 Elementary Analysis II - Homework 2

Jonas Wagner

2022, Febuary 9^{th}

Problem 1

Show that

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$$

is well-defined for $\alpha > 0$ and $\beta > 0$.

Definition 1. *The improper integral*

$$\int_0^a f(x)\,\mathrm{d}x$$

is <u>well-defined</u> *iff*

$$\lim_{\epsilon \to 0} \int_0^a f(x)\,\mathrm{d}x$$

exists.

Definition 2. *The beta function $B(\alpha, \beta)$ is defined as*

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

for $\alpha > 0$ and $\beta > 0$.

Theorem 1. ***Limit Comparison Test:*** *Let $f, g : [a, b) \to \mathbb{R}$ be two functions such that (i) $f(x)$ and $g(x)$ are integrable on $[a, A] \subset [a, b)$, for $a < A < b$; (ii) There exists $a \leq K \leq b$ such that $\lim_{x \to b^-} \frac{f(x)}{g(x)} = K$. Then,*

a. If $0 < K < \infty$, then $\int_a^b g(x)\,\mathrm{d}x$ converges iff $\int_a^b f(x)$ converges.

b. If $K = 0$, then $\int_a^b g(x)$ converges implies $\int_a^b f(x)\,\mathrm{d}x$ converges.

c. If $K \infty 0$, then $\int_a^b g(x)$ divergent implies $\int_a^b f(x)\,\mathrm{d}x$ divergent.

Theorem 2. *The improper integral that defines the beta function*

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$$

is well-defined for $\alpha > 0$ and $\beta > 0$.

Proof. The integrand of $B(\alpha, \beta)$,

$$b(\alpha, \beta) = x^{\alpha-1}(1-x)^{\beta-1}$$

is not strictly bounded $\forall_{\alpha,\beta>0}$, but this is not necessary for convergence. $\forall \alpha, \beta \in [0, \infty)$ the $b(\alpha, \beta)$ is bounded. This makes $B(\alpha, \beta)$ a proper integral which is therefore convergent.

In the other case the integrand is not bounded, but the improper integral still converges. $\forall_{\alpha\in(0,1)}$ then $b(\alpha,\beta)$ is unbounded at $x=0$. Similarly, $\forall_{\beta\in(0,1)}$ then $b(\alpha,\beta)$ is unbounded at $x=1$.
The beta function can instead be split up into two parts:

$$B(\alpha,\beta)=\int_0^c x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x+\int_c^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$$

where $c\in(0,1)$.
For the first improper integral, $\int_0^c x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$ a discontinuity exists at $x=0$ for $\alpha\in(0,1)$. Using the Limit Comparison Test from Theorem 1 with $g(x)=x^{\alpha-1}$,

$$\begin{aligned}\lim_{x\to 0^+}\frac{f(x)}{g(x)}&=\lim_{x\to 0^+}\frac{x^{\alpha-1}(1-x)^{\beta-1}}{x^{\alpha-1}}\\&=\lim_{x\to 0^+}(1-x)^{\beta-1}\\&=1\neq 0\end{aligned}$$

Which then implies that $\int_0^c x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$ converges $\forall_{\alpha,\beta>0}$.
For the second improper integral, $\int_c^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$ a discontinuity exists at $x=1$ for $\beta\in(0,1)$. Using the Limit Comparison Test from Theorem 1 with $g(x)=(1-x)^{\beta-1}$,

$$\begin{aligned}\lim_{x\to 1^-}\frac{f(x)}{g(x)}&=\lim_{x\to 1^-}\frac{x^{\alpha-1}(1-x)^{\beta-1}}{(1-x)^{\beta-1}}\\&=\lim_{x\to 1^-}x^{\alpha-1}\\&=1\neq 0\end{aligned}$$

Which then implies that $\int_c^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$ converges $\forall_{\alpha,\beta>0}$.
Together, the convergence of $\int_0^c x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$ and $\int_c^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$ implies that $B(\alpha,\beta)$ converges $\forall_{\alpha,\beta>0}$ and therefore $B(\alpha,\beta)$ is well defined. $\square$

Problem 2

Show that f if Riemann integrable on $[a, b]$, then

$$\lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

Problem 3

Evaluate $\int_0^1 (1 - x^{\frac{2}{3}})^{\frac{3}{2}}\,\mathrm{d}x$. Hint: Express the integral in terms of the gamma function first.

Problem 4

Let

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } 0 < x \leq 1; \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f is bounded and continuous on $[0,1]$, but not of bounded variation on $[0,1]$.

Problem 5

Assume f is differentiable on $[a, b]$ with $|f'(x)| \leq M < \infty$ for $a \leq x \leq b$. Show that f is of bounded variation and $V_a^b(f) \leq M(b - a)$. (Hint: Use Mean Value Theorem)