MATH 5302 Elementary Analysis II - Homework 5

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Problem 1

Let f be a real-valued bounded function on $[-1, 1]$. Let

$$\alpha(x) = \begin{cases} 0 & \text{if } -1 \leq x < 0; \\ 2 & \text{if } 0 \leq x \leq 1. \end{cases}$$

Assume f is Riemann-Stieljes integrable with respect to α on $[-1, 1]$. Show that

a. f is continuous at 0 from the left.

b. $\int_{-1}^{1} f(x)\,\mathrm{d}\alpha(x) = 2f(0)$.

a) f is continuous at 0 from the left

Example 1. *Let* $f : [-1, 1] \to \mathbb{R}$ *bounded. Let*

$$\alpha(x) = \begin{cases} 0 & -1 \leq x < 0 \\ 2 & 0 \leq x < 1 \end{cases}$$

If f is Riemann-Stieljes integrable w.r.t. α on $[-1, 1]$, then f is continuous at 0 from the left.

Proof. $\square$

Problem 2

Let f and α be real-valued bounded functions on $[a, b]$ and α is increasing. Let $L(f, \alpha)$ and $U(f, \alpha)$ represents the lower and upper Darboux-Stieltjes integral of f with respect to α on $[a, b]$, respectively,

a. Show that $U(f, \alpha) \leq U(|f|, \alpha)$.

b. Is it true that $L(f, \alpha) \leq L(|f|, \alpha)$?

Problem 3

Let α be a bounded real-0valued increasing function on $[a, b]$. Assume $a < c < b$ and α is continuous at c. Let

$$f(x) = \begin{cases} 1 & \text{if } x = c; \\ 0 & \text{if } x \neq c. \end{cases}$$

Show directly that f is Darboux-Stieltjes integrable on $[a, b]$ and $\int_a^b f(x)\,\mathrm{d}\alpha(x) = 0$. (Do not use Theorem 8.16.)

Problem 4

Let f and α be real-valued bounded functions on $[a, b]$ and α is increasing on $[a, b]$. Assume f is Darboux-Stieltjes integrable with respect to α on $[a, b]$. Let $[c, d] \subset [a, b]$. Show that f is Darboux-Stieltjes integrable with respect to α on $[c, d]$.

Problem 5

Let α be a real-valued bounded function on $[a, b]$ and α is increasing with $\alpha(a) < \alpha(b)$. Let

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that if α is continuous on $[a, b]$, then f is not Darboux-Stieltjes integrable with respect to α on $[a, b]$.