LECTURE 15 — MATH 6301

OUTER MEASURE

Consider a space X. A function $\mu^*: \mathcal{P}(X) \to \overline{\mathbb{R}}$ is called an outer measure on X iff

(μ^*1) $\quad \mu^*(A) \geq 0$

(μ^*2) $\quad \mu^*(\emptyset) = 0$

(μ^*3) $\quad A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$

(μ^*4) $\quad \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$

CARATHÉODORY CONDITION

A set $E \subset X$ is called μ^*-measurable iff it satisfies the following conditions (called Carathéodory Condition

$$(c) \quad \forall_{A \in \mathcal{P}(X)} \quad \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$$

THEOREM. Let X be a space and $\mu^*: \mathcal{P}(X) \to \overline{\mathbb{R}}$ be an outer measure on X. Then the class of sets

$$\mathfrak{S}_C := \{E \subset X : E \text{ is } \mu^*\text{-measurable}\}$$

is a σ-algebra and μ^* restricted to $\mathfrak{S}_C$ is a measure on X.

PROOF: Notice that the properties $(\mu 1)$ and $(\mu 2)$ are treadly satisfied by μ^* so we only need to show that $(\mu 3)$ is true.

We need to show that, if $\{E_n\} \subset \mathfrak{S}_C$ then $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{S}_C$ and $(\mu 3)$ is satisfied. First we will show that if $E, F \in \mathfrak{S}_C \Rightarrow E \cup F \in \mathfrak{S}_C$.

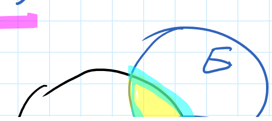
σ-algebra proof

Let $E, F \in \mathfrak{S}_C$, then $\forall_{A \in \mathcal{P}(X)}$

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$$
$$= \mu^*(A \cap E) + \mu^*((A \setminus E) \cap F) + \mu^*((A \setminus E) \setminus F)$$

Notice that

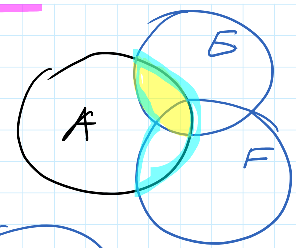
① $\quad A \cap E = A \cap (E \cup F) \cap E$

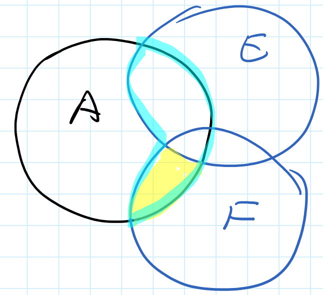


Notice that

① $A \cap E = A \cap (E \cup F) \cap E$

② $(A \setminus E) \cap F = (A \cap (E \cup F)) \setminus E)$





Thus,

$$\mu^*(A) = \mu^*(\underbrace{A \cap (E \cup F)}_{A'} \cap E) + \mu^*(\underbrace{A \cap}_{A'}((E \cup F) \setminus E)) + \mu^*((A \setminus E) \setminus F) =$$

(by assumption E is μ^*-measurable, thus in particular (because it satisfying condition (C)) one can take as $A' := A \cap (E \cup F)$, and we have $\mu^*(A') = \mu^*(A' \cap E) + \mu^*(A' \setminus E)$, therefore we have)

$$\mu^*(A) = \mu^*(A \cap (E \cup F)) + \mu^*((A \setminus E) \setminus F)$$
$$= \mu^*(A \cap (E \cup F)) + \mu^*(A \setminus (E \cup F))$$

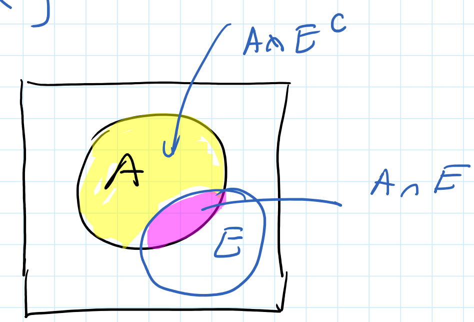
Since, $A \in \mathcal{P}(X)$ is arbitrary, we have that

$$\mu^*(A) = \mu^*(A \cap (E \cup F)) + \mu^*(A \setminus (E \cup F))$$

which means that $E \cup F$ satisfies the condition (C), i.e. $E \cup F \in \mathcal{S}_C$.

Next, notice that if $E \in \mathcal{S}_C$ then $\forall A \in \mathcal{P}(X)$

$$\mu^*(A \cap E^c) + \mu^*(A \setminus E^c)$$
$$= \mu^*(A \setminus E) + \mu^*(A \cap E) \overset{(C)}{=} \mu(A)$$



which implies $E^c \in \mathcal{S}_C$

Since we have that, if $E, F \in \mathcal{S}_C$ then

(a) $E \cup F \in \mathcal{S}_C$

(b) $E^c, F^c \in \mathcal{S}_C$

(c) $E \cap F = (E^c \cup F^c)^c \in \mathcal{S}_C$

(d) $E\setminus F = E\cap F^c \in \mathfrak{S}_C$

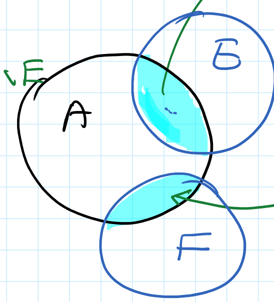
Now, assume that $E, F \in \mathfrak{S}_C$ and $E\cap F = \emptyset$. Then

$\forall$ $A\in\mathcal{P}(X)$

$$\mu^*(A\cap(E\cup F)) = \mu^*((A\cap(E\cup F))\cap E) + \mu^*((A\cap(E\cup F))\setminus E)$$

$A\cap(E\cup F) = (A\cap(E\cup F)\cap E) \cup (A\cap(E\cup F))\setminus E$

$A\cap(E\cup F)\cap E$



$(A\cap(E\cup F))\setminus F$

$$\mu^*(A\cap(E\cup F)) = \mu^*(A\cap E) + \mu^*(A\cap F)$$

Now, by induction if $\{E_k\}_{k=1}^n \subset \mathfrak{S}_C$ is a finite family of disjoint sets $\underset{k\neq l}{E_k\cap E_l = \emptyset}$, then

$$\mu^*\left(A\cap \bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mu^*(A\cap E_k) \qquad (*)$$

Consequently, if $\{E_k\}_{k=1}^\infty \subset \mathfrak{S}_C$ is such that $\underset{k\neq l}{E_k\cap E_l = \emptyset}$

then we have

$\forall$ $n\in\mathbb{N}$

$$\mu^*\left(A\cap\bigcup_{k=1}^\infty E_k\right) \overset{(\mu^*3)}{\geq} \mu^*\left(A\cap\bigcup_{k=1}^n E_k\right) \overset{(*)}{=} \sum_{k=1}^n \mu^*(A\cap E_k)$$

thus the series

$$\sum_{k=1}^\infty \mu^*(A\cap E_k)$$

converges, and

$$(1) \qquad \mu^*\left(A\cap\bigcup_{k=1}^\infty E_k\right) \geq \sum_{k=1}^\infty \mu^*(A\cap E_k)$$

But by (μ^*4)

$$(2) \qquad \mu^*\left(A\cap\bigcup_{k=1}^\infty E_k\right) = \mu^*\left(\bigcup_{k=1}^\infty (A\cap E_k)\right) \leq \sum_{k=1}^\infty \mu^*(A\cap E_k)$$

So we obtain that

$$(3) \qquad \underset{A\in\mathcal{P}(X)}{\forall} \quad \mu^*\left(A\cap\bigcup_{k=1}^\infty E_k\right) = \sum_{k=1}^\infty \mu^*(A\cap E_k).$$

Therefore,

$$\mu^*(A) = \mu^*\left(A\cap\bigcup_{k=1}^n E_k\right) + \mu^*\left(A\setminus\bigcup_{k=1}^n E_k\right) \qquad \left(\text{because } \bigcup_{k=1}^n E_k \in \mathfrak{S}_C\right.$$

Therefore,

$$\forall_{n\geq 1}\quad \mu^*(A) = \mu^*\Big(A \cap \bigcup_{k=1}^{n} E_k\Big) + \mu^*\Big(A \setminus \bigcup_{k=1}^{n} E_k\Big) \qquad \Big(\text{because } \bigcup_{k=1}^{n} E_k \in \mathcal{S}_C$$

$$\overset{(\mu^*3)}{\geq} \mu^*\Big(A \cap \bigcup_{k=1}^{n} E_k\Big) + \mu^*\Big(A \setminus \bigcup_{k=1}^{\infty} E_k\Big)$$

$$= \sum_{k=1}^{n} \mu^*(A \cap E_k) + \mu^*\Big(A \setminus \bigcup_{k=1}^{\infty} E_k\Big)$$

thus, by passing to the limit $n \to \infty$, we have

$$\forall_A\quad \mu^*(A) \geq \sum_{k=1}^{\infty} \mu^*(A \cap E_k) + \mu^*\Big(A \setminus \bigcup_{k=1}^{\infty} E_k\Big)$$

$$\overset{(3)}{=} \mu^*\Big(A \cap \bigcup_{k=1}^{\infty} E_k\Big) + \mu^*\Big(A \setminus \bigcup_{k=1}^{\infty} E_k\Big)$$ which implies

that $\bigcup_{k=1}^{\infty} E_k$ satisfies the condition (C)

REMARK: Notice that for any family of sets $\{F_n\} \subset \mathcal{S}_C$, we have

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n \qquad F_1 = E_1$$

$$E_n = F_n \setminus \bigcup_{k=1}^{n-1} F_k, \quad E_n \cap E_m = \emptyset$$

$$E_n \in \mathcal{S}_C$$

This concludes the proof that $\mathcal{S}_C$ is indeed a σ-algebra.

Notice, that μ^* restricted to $\mathcal{S}_C$ is indeed a measure, because by taking $A = X$ in (3) we obtain

$$\mu^*\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \sum_{k=1}^{\infty} \mu^*(E_k). \quad \square$$

COROLLARY: For a space X and an outer measure $\mu^* : \mathcal{P}(X) \to \overline{\mathbb{R}}$ the measure $\mu := \mu^* : \mathcal{S}_C \to \overline{\mathbb{R}}$ is complete, i.e. if $\mu^*(E) = 0$ then $E \in \mathcal{S}_C$.

PROOF Indeed let $\mu^*(E) = 0$

$$\forall_{A\in\mathcal{P}(X)}\quad \mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \setminus E) \leq 0 + \mu^*(A \setminus E) \quad (*)$$

so

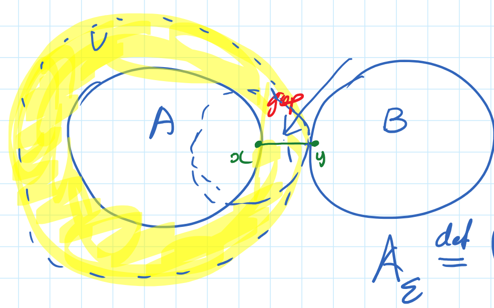
$$\mu^*(A \setminus E) \overset{(\mu^*3)}{\leq} \mu^*(A) \overset{(*)}{\leq} \mu^*(A \setminus E) \implies \mu^*(A \setminus E) = \mu^*(A).$$

so (C) is satisfied, and $E \in \mathcal{S}_C$ $\square$

<u>DEFINITION</u>: Let (X,d) be a metric space and $\mu^*: \mathcal{P}(X) \to \overline{\mathbb{R}}$ be an outer measure on X. We say that μ^* is metric outer measure iff

$(\mu^* 5)$ $\forall_{A,B \subset X}$ if $\delta(A,B) \overset{\text{def}}{:=} \inf_{\substack{x\in A\\ y \in B}} d(x,y) > 0$ then

$$\mu^*(A\cup B) = \mu^*(A) + \mu^*(B)$$



$\varepsilon = \delta(A,B) > 0$

$A_\varepsilon \cap B = \emptyset$

$A_\varepsilon \overset{\text{def}}{=} \bigcup_{x\in A} B_\varepsilon(a) = B_\varepsilon(A)$

$A_{\frac{\varepsilon}{2}} \cap B_{\frac{\varepsilon}{2}} = \emptyset$