LECTURE 6 — MATH 6301

COMPACTNESS IN METRIC SPACES

For a complete metric space (X, d) and a set $A \subset X$ the following conditions are equivalent

(a) A is compact
(b) $A = \overline{A}$ and A is totally bounded
(c) for every sequence $\{x_n\} \subset A$ there exists a subsequence $\{x_{n_k}\}$ s.t. $\lim\limits_{k\to\infty} x_{n_k} \in A$

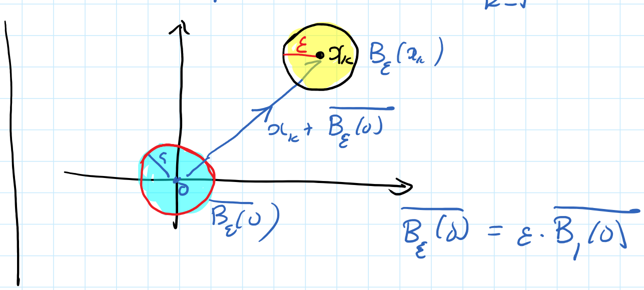
PROPOSITION (HEINE-BOREL THM)
$A \subset \mathbb{R}^n$ is compact iff $A = \overline{A}$ and A is bounded.

REMARK: If $(V, \|\cdot\|)$ is a finite-dimensional normed space then the set $A \subset V$ is compact iff $A = \overline{A}$ and A is bounded. Indeed, it is well-known that for a finite-dimensional space V, any two norms are equivalent, i.e. the topologies generated by these norms coincide. Then, one can use the identification of V with $\mathbb{R}^n$ $(n = \dim V)$, so this fact follows from Heine-Borel Thm.

PROPOSITION: Let $(V, \|\cdot\|)$ be a normed space and $B := \{x \in V : \|x\| \le 1\} = \overline{B_1(0)}$. Then B is compact iff $\dim V < \infty$.

PROOF: $\Leftarrow$ it follows from the above Remark
$\Rightarrow$ Assume B is compact, thus by (b) B is totally bounded therefore for given $0 < \varepsilon < 1$ there exists an ε-net $\{x_1, x_2, \dots, x_n\} \subset B$ satisfying

$$B \subset \bigcup_{k=1}^{n} B_\varepsilon(x_k) \subset \bigcup_{k=1}^{n} \overline{B_\varepsilon(x_k)} = \bigcup_{k=1}^{n} \left(x_k + \varepsilon B \right)$$



$$\overline{B_\varepsilon(0)} = \varepsilon \cdot \overline{B_1(0)}$$

$$(A + B = \{a + b : a \in A,\ b \in B\})$$

$$= \left(\bigcup_{k=1}^{n} \{x_k\} \right) + \varepsilon B$$

$$= \{x_1, x_2, \dots, x_n\} + \varepsilon B \subset \underbrace{\operatorname{span}\{x_1, x_2, \dots, x_n\}}_{V_0} + \varepsilon B \qquad \dim V_0 \le n$$

$$= V_0 + \varepsilon B,$$ so we got

$$\boxed{B \subset V_0 + \varepsilon B} \subset V_0 + \varepsilon\left(\boxed{V_0 + \varepsilon B}\right)$$

$$= V_0 + \varepsilon V_0 + \varepsilon^2 B = V_0 + \varepsilon^2 B \subset V_0 + \varepsilon^k B \qquad \forall\, k \in \mathbb{N}$$

Thus $$B \subset \bigcap_{k=1}^{\infty} (V_0 + \varepsilon^k B) = \bar{V}_0 + \bigcap_{k=1}^{\infty} \varepsilon^k B = \bar{V}_0 + \{0\}$$
$$= \bar{V}_0$$

$$\lim_{k\to\infty} \varepsilon^k = 0 \qquad \ln \varepsilon < 0$$
$$\varepsilon^k = e^{k \ln \varepsilon} \xrightarrow[k\to\infty]{} e^{-\infty} = 0$$

$\bar{V} = \mathrm{span}(B) \subset \mathrm{span}(\bar{V}_0) = \bar{V}_0$ so $\bar{V} = \bar{V}_0$. □

PROPOSITION: Let X and Y be two metric spaces and $f: X \to Y$ a continuous map. If X is compact then $f(X)$ is also compact.

PROOF: Take an open cover $\{V_i\}_{i\in I}$ of $f(X)$ and notice that $U_i := f^{-1}(V_i)$ is open and $\{U_i\}_{i\in I}$ is a cover of X, thus by compactness of X, there is a finite subcover $\{U_{i_k}\}_{k=1}^n$ of X, and clearly $\{V_{i_k}\}_{k=1}^n$ is a finite subcover of $\{V_i\}_{i\in I}$. □

PROPOSITION: Let X be a compact metric space and $f: X \to \mathbb{R}$ a continuous map. Then $\exists\, x_1, x_2 \in X$ such that

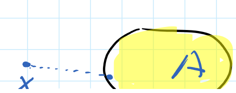
$$f(x_1) = \inf_{x\in X} f(x) = \min_{x\in X} f(x) \qquad f(x_2) = \sup_{x\in X} f(x) = \max_{x\in X} f(x)$$

PROOF: Since $f(X) \subset \mathbb{R}$ is compact thus it is closed an bounded so $\inf f(X) \in f(X)$ and $\sup f(X) \in f(X)$. □

THEOREM: Let (X,d) and (Y,ρ) be two metric spaces, where X is a compact space and $f: X \to Y$ a continuous map. Then f is uniformly continuous. (Uniform Continuity Theorem)

LEMMA #1: Let (X,d) be a metric space and $A \subset X$ a closed set. Then the function $\varphi_A : X \to \mathbb{R}$, given by

$$\varphi_A(x) = \inf_{a\in A} d(x,a) =: \mathrm{dist}(x, A)$$



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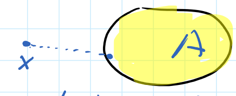


distance from A

satisfies

(a) $\varphi_A^{-1}\{0\} = A \iff (x\in A \iff \varphi(x)=0)$

(b) $\forall_{x,x'\in X} \quad |\varphi_A(x) - \varphi_A(x')| \le d(x,x')$

PROOF

$\inf_{a\in A} d(x,a) = \varphi_A(x) \iff$ 1) $\forall_{a\in A} \; \varphi_A(x) \le d(x,a')$. 2) $\forall_{\varepsilon>0} \exists_{a\in A} \; \varphi_A(x) + \varepsilon > d(x,a)$

$\inf_{a\in A} d(x',a) = \varphi_A(x') \iff$ 1') $\forall_{a\in A} \; \varphi_A(x') \le d(x',a)$

2') $\forall_{\varepsilon>0} \exists_{a'\in A} \; \varphi_A(x') + \varepsilon > d(x',a')$

$\varphi_A(x') > d(x',a') - \varepsilon$

$-\varphi_A(x') < -d(x',a') + \varepsilon \quad a'$

$$\varphi_A(x) - \varphi_A(x') < d(x,a') - d(x',a') + \varepsilon$$
$$\le d(x,x') + \varepsilon$$

$$\varphi_A(x') - \varphi_A(x) < d(x,x') + \varepsilon$$

So $\forall_{\varepsilon>0} \; |\varphi_A(x) - \varphi_A(x')| < d(x,x') + \varepsilon \implies |\varphi_A(x) - \varphi_A(x')| \le d(x,x')$.

$\square$

LEMMA 2 (Lebesgue Number Theorem) Let (X,d) be a compact metric space and $\{U_i\}_{i\in I}$ an open cover of X. Then there exists $\lambda > 0$ (Lebesgue number of the cover $\{U_i\}_{i\in I}$) such that

$$(*) \qquad \forall_{x\in X} \; \exists_{i\in I} \quad B_\lambda(x) \subset U_i$$

PROOF: By compactness of X we can assume (without loss of generality) that the considered cover is finite, i.e. $\{U_1, U_2, \dots, U_n\}$

Then define for $\forall_{k=1,2,\dots,n} \quad \varphi_k : X \to \mathbb{R}$ by

$$\varphi_k(x) := d_{A_k}(x) = dist(x, U_k^c), \; A_k := X \setminus U_k = U_k^c$$

Since $\{U_k\}_{k=1}^{n}$ is a cover, thus $\forall_{x\in X} \exists_k \; \varphi_k(x) > 0 \quad (x\in U_k)$

$\iff \varphi_k(x) \ne 0$

so the function $\varphi(x) := \max\{\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)\} > 0$

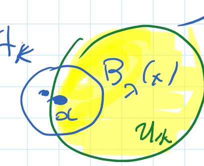
we can take

$$(**) \qquad \lambda := \min_{x\in X} \varphi(x) > 0$$

Indeed, $\varphi(x) > 0 \; \forall_{x\in X}$ so $\lambda > 0$. Suppose $(*)$ is not true (for contradiction)

i.e. $\exists_{x\in X} \forall_{k=1,\dots,n} \quad B_\lambda(x) \not\subset U_k$ i.e. $B_\lambda(x) \cap U_k^c \ne \emptyset \iff$

$\Rightarrow \forall_{k=1,\dots,n} \; \varphi_k(x) < \lambda$ and we get a contradiction with $(**)$.



$\varphi_k(x) < \lambda$

$\square$

PROOF OF THEOREM: Suppose $f: X \to \mathbb{R}$ is continuous and X is compact. Take an $\varepsilon > 0$, and consider $x \in X$. By continuity of f, we have

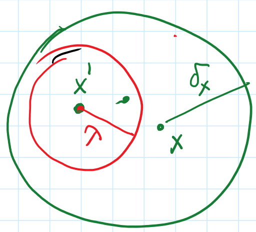
$$\underset{\delta_x > 0}{\exists} \quad \underset{\Updownarrow \atop x' \in B_{\delta_x}(x)}{d(x,x') < \delta_x} \Rightarrow |f(x) - f(x')| < \varepsilon/2$$

Consequently, we obtain an open cover $\{U_x\}_{x \in X}$, $U_x := B_{\delta_x}(x)$. By Lemma 2 there exists a Lebesgue number $\underset{=\delta}{\lambda} > 0$ for this cover. Then

$$\underset{x' \in X}{\forall} \ \underset{x \in X}{\exists} \ B_\lambda(x') \subset B_{\delta_x}(x)$$

which means that if

$$d(x', x'') < \lambda \Rightarrow d(x'', x) < \delta_x$$
$$\Downarrow$$
$$|f(x') - f(x)| < \varepsilon/2$$
$$|f(x'') - f(x)| < \varepsilon/2$$



Therefore

$$|f(x') - f(x'')| \leq |f(x') - f(x)| + |f(x'') - f(x)|$$

$$\underset{\varepsilon > 0}{\forall} \ \underset{\delta > 0}{\exists} \ \underset{x', x'' \in X}{\forall} \ d(x', x'') < \delta \Rightarrow \underset{\overset{\wedge}{\varepsilon}}{|f(x') - f(x'')|} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

ARZELA - ASCOLI THEOREM

Let (X, d) be a compact metric space. We define the following vector space

$$C(X; \mathbb{R}) = \{ u : X \to \mathbb{R} : u \text{ is continuous} \}$$

equipped with the norm

$$\|u\|_\infty := \sup_{x \in X} |u(x)| \qquad u(X) \text{ compact set in } \mathbb{R}$$

PROPOSITION: The space $(C(X; \mathbb{R}), \|\cdot\|_\infty)$ is a Banach space.

PROOF Notice that if $\{u_k\}_{k=1}^\infty \subset C(X; \mathbb{R})$ is a Cauchy sequence, ie

$$\underset{\varepsilon > 0}{\forall} \ \underset{N}{\exists} \ \underset{m,k \geq N}{\forall} \ \|u_k - u_m\|_\infty < \varepsilon \iff \underset{\varepsilon > 0}{\forall} \ \underset{N}{\exists} \ \underset{m,k \geq N}{\forall} \ \underset{x \in X}{\forall} \ |u_k(x) - u_m(x)| < \varepsilon \quad (*)$$

$$\|u_k - u_m\|_\infty = \max_{x \in X} |u_k(x) - u_m(x)|$$

which implies that $\underset{x \in X}{\forall}$ $\{u_k(x)\} \subset \mathbb{R}$ is Cauchy, so by completeness of $\mathbb{R}$, there exist the limit $u(x) := \lim_{k\to\infty} u_k(x)$.

Then by passing to the limit $k \to \infty$ in $(*)$ we obtain

$$(**) \quad \forall_{\varepsilon>0} \ \exists_{N} \ \forall_{m \geq N} \ |u(x) - u_m(x)| \leq \varepsilon \iff \sup_{x\in X} |u(x) - u_m(x)| \leq \varepsilon$$

and only detail that is left to show is that u is continuous.

Indeed, take $x_0 \in X$. Then for a fixed $\varepsilon > 0$, by $(**)$ we have

$$\exists_{N} \ \forall_{m\geq N} \ |u(x) - u_m(x)| \leq \frac{\varepsilon}{3} \qquad (1)$$

and take $m = N$, so since u_m is continuous, then

$$\exists_{\delta>0} \ \forall_{x\in X} \ d(x, x_0) < \delta \Rightarrow |u_m(x) - u_m(x_0)| < \frac{\varepsilon}{3} \qquad (2)$$

so we obtain

$$\exists_{\delta>0} \ \forall_{x\in X} \ d(x,x_0) < \delta \Rightarrow |u(x) - u(x_0)| < |u(x) - u_m(x)| +$$
$$+ |u_m(x) - u_m(x_0)| + |u_m(x_0) - u(x_0)|$$
$$\overset{(1)}{<} \frac{\varepsilon}{3} + \overset{(2)}{\frac{\varepsilon}{3}} + \overset{(1)}{\frac{\varepsilon}{3}} = \varepsilon \qquad \square$$

THEOREM: (Arzela-Ascoli Theorem) Let (X, d) be a compact metric space and $\Phi \subset C(X;\mathbb{R})$ a bounded subset. Then

$$\Phi \text{ is compact} \iff \begin{cases} (i) \ \overline{\Phi} = \Phi & \text{i.e. } \Phi \text{ is closed} \\ (ii) \ \forall_{\varepsilon>0} \ \exists_{\delta>0} \ \forall_{x,y\in X} \ d(x,y) < \delta \Rightarrow \forall_{u\in\Phi} \ |u(x) - u(y)| < \varepsilon \end{cases}$$

functions from Φ are uniformly equicontinuous