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LECTURE #2 - MATH 6301

1c. Functions

DEFINITION: Let X and Y be two sets. We say that $f \subset X \times Y$ is a function from X to Y iff

(i) $\forall_{x \in X} \; \exists_{y \in Y} \; (x,y) \in f$

(ii) $\forall_{x \in X} \; (x,y) \in f \wedge (x,y') \in f \Longrightarrow y = y'$

intuitively: Notice that the relation f is exactly the graph $Gr(f)$

Then we put $y =: f(x) \overset{\text{def}}{\Longleftrightarrow} (x,y) \in f \Longleftrightarrow (x, f(x)) \in f$

$Gr(f) := \{(x, f(x)) : x \in X\}$

Then we write $f : X \to Y$ to indicate that f is a function, from X to Y, X is the domain, Y is co-domain of f.

DEFINITION: Let $f : X \to Y$ be a function. Then we say that f is

(i) injective (one-to-one) if
$$\forall_{x_1, x_2 \in X} \; f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$$

(ii) surjective (onto)
$$\forall_{y \in Y} \; \exists_{x \in X} \; y = f(x)$$

(iii) bijective if f is injective and surjective

DEFINITION Let $f : X \to Y$ be a function, $A \subset X$, $B \subset Y$. Then we define

(a) $f(A) := \{ y \in Y : \exists_{x \in A} \; f(x) = y \}$ image of A

image of A under f

(b) $f^{-1}(B) := \{ x \in X : \exists_{y \in B} \; f(x) = y \}$ inverse image of B

pre-image

PROPOSITION: Let $f : X \to Y$ be a given function, $A \subset X$, $B \subset Y$. Then we have

Then we have

(a) $\boxed{x \in f^{-1}(B) \iff f(x) \in B}$

(b) $f(f^{-1}(B)) \subset B$

(c) $f^{-1}(f(A)) \supset A$

In addition if $B, C \subset Y$ then

(d) $f^{-1}(B \cup C) = f^{-1}(B) \cup f^{-1}(C)$

(e) $f^{-1}(B \cap C) = f^{-1}(B) \cap f^{-1}(C)$

(f) $f^{-1}(B^c) = (f^{-1}(B))^c$

PROOF: (e) $x \in f^{-1}(B \cap C) \iff f(x) \in B \cap C \iff f(x) \in B \wedge f(x) \in C$

$\iff x \in f^{-1}(B) \wedge x \in f^{-1}(C) \iff x \in f^{-1}(B) \cap f^{-1}(C)$

1d GENERALIZED UNIONS:

For a given space X, we denote by $\mathcal{P}(X)$ the so-called power set or in other words, the set composed of all subsets of X.

$$\mathcal{P}(X) := \{ A : A \subset X \}$$

Suppose $\Lambda \subset \mathcal{P}(X)$ is a given (non-empty) collection of sets in $\mathcal{P}(X)$. Notice that for every set $A \in \Lambda$ corresponds the set $A \in \mathcal{P}(X)$, so we have a function

$$a : \Lambda \longrightarrow \mathcal{P}(X)$$
$$a(A) \longrightarrow A$$

Suppose, instead of writing letters $A, B, ..$ to denote elements of Λ, we call these elements by $\lambda \in \Lambda$ and put

$$a(\lambda) =: A_\lambda \qquad A_\lambda = A$$

This process indicates that any family of subsets in $\mathcal{P}(X)$ can be indexed by a certain function, i.e

$$\underset{\text{indices}}{a : \Lambda} \longrightarrow \mathcal{P}(X), \qquad \{A_\lambda : \lambda \in \Lambda\}, \quad \{A_\lambda\}_{\lambda \in \Lambda}$$
$$a(\lambda) =: A_\lambda$$

and we will say that $\{A_\lambda\}_{\lambda \in \Lambda}$ is a indexed by Λ family of subsets in X.

DEFINITION:

Given indexed family $\{A_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{P}(X)$. Then we put

$$\bigcup_{\lambda\in\Lambda} A_\lambda := \{x\in X : \exists_{\lambda\in\Lambda}\ x\in A_\lambda\}$$ Generalized union of $\{A_\lambda\}$

$$\bigcap_{\lambda\in\Lambda} A_\lambda := \{x\in X : \forall_{\lambda\in\Lambda}\ x\in A_\lambda\}$$ Generalized intersection of $\{A_\lambda\}$

PROPERTIES Given $\{A_\lambda\}_{\lambda\in\Lambda}$ and $\{B_\lambda\}_{\lambda\in\Lambda}$. Then we have

(a) $\forall_{\lambda_0\in\Lambda} \quad A_{\lambda_0} \subset \bigcup_{\lambda\in\Lambda} A_\lambda$ and $\bigcap_{\lambda\in\Lambda} A_\lambda \subset A_{\lambda_0}$

(b) $\forall_{\lambda\in\Lambda} (A_\lambda \subset C) \Rightarrow \bigcup_{\lambda\in\Lambda} A_\lambda \subset C$

(c) $\exists_{\lambda\in\Lambda} \quad A_\lambda \subset C \Rightarrow \bigcap_{\lambda\in\Lambda} A_\lambda \subset C$

(d) $\forall_{\lambda\in\Lambda} \quad C \subset A_\lambda \Rightarrow C \subset \bigcap_{\lambda\in\Lambda} A_\lambda$

(e) $\forall_{\lambda\in\Lambda} \quad A_\lambda \subset B_\lambda \Rightarrow \bigcup_{\lambda\in\Lambda} A_\lambda \subset \bigcup_{\lambda\in\Lambda} B_\lambda$

(f) $\forall_{\lambda\in\Lambda} \quad A_\lambda \subset B_\lambda \Rightarrow \bigcap_{\lambda\in\Lambda} A_\lambda \subset \bigcap_{\lambda\in\Lambda} B_\lambda.$

(g) $C \cup \bigcup_{\lambda\in\Lambda} A_\lambda = \bigcup_{\lambda\in\Lambda} (C\cup A_\lambda)$

(h) $C \cap \bigcup_{\lambda\in\Lambda} A_\lambda = \bigcup_{\lambda\in\Lambda} (C\cap A_\lambda)$

(i) $C \cup \bigcap_{\lambda\in\Lambda} A_\lambda = \bigcap_{\lambda\in\Lambda} (C\cup A_\lambda)$

(j) $C \cap \bigcap_{\lambda\in\Lambda} = \bigcap_{\lambda\in\Lambda} (C\cap A_\lambda)$

(k) $\bigcup_{\lambda\in\Lambda} (A_\lambda \cap B_\lambda) \subset \bigcup_{\lambda\in\Lambda} A_\lambda \cap \bigcup_{\lambda\in\Lambda} B_\lambda$

(l) $\bigcap_{\lambda\in\Lambda} A_\lambda \cup \bigcap_{\lambda\in\Lambda} B_\lambda \subset \bigcap_{\lambda\in\Lambda} (A_\lambda \cup B_\lambda)$

PROOF

$$x \in \bigcup_{\lambda\in\Lambda} (A_\lambda \cap B_\lambda) \iff \exists_{\lambda\in\Lambda} \; x \in A_\lambda \cap B_\lambda$$

$$\iff \exists_{\lambda\in\Lambda} \; \underbrace{x \in A_\lambda}_{p(x)} \wedge \underbrace{x \in B_\lambda}_{q(x)}$$

$$\implies \exists_{\lambda\in\Lambda} \; x \in A_\lambda \wedge \exists_{\lambda\in\Lambda} \; x \in B_\lambda$$

$$\implies x \in \bigcup_{\lambda\in\Lambda} A_\lambda \wedge x \in \bigcup_{\lambda\in\Lambda} B_\lambda$$

$$\implies x \in \bigcup_{\lambda\in\Lambda} A_\lambda \cap \bigcup_{\lambda\in\Lambda} B_\lambda$$

$$\bigcup_{\lambda\in\Lambda} (A_\lambda \cap B_\lambda) \subset \bigcup_{\lambda\in\Lambda} A_\lambda \cap \bigcup_{\lambda\in\Lambda} B_\lambda$$

(m) $$C \setminus \bigcup_{\lambda\in\Lambda} A_\lambda = \bigcap_{\lambda\in\Lambda} (C \setminus A_\lambda)$$

PROOF

$$x \in C \setminus \bigcup_{\lambda\in\Lambda} A_\lambda \iff x \in C \wedge x \notin \bigcup_{\lambda\in\Lambda} A_\lambda$$

$$\iff x \in C \wedge \sim \left(x \in \bigcup_{\lambda\in\Lambda} A_\lambda \right)$$

$$\iff x \in C \wedge \sim \left(\exists_{\lambda\in\Lambda} \; x \in A_\lambda \right) \iff x \in C \wedge \left(\forall_{\lambda\in\Lambda} \; x \notin A_\lambda \right)$$

$$\iff x \in C \wedge \forall_{\lambda\in\Lambda} \; x \notin A_\lambda \iff \forall_{\lambda\in\Lambda} \; (x \in C \wedge x \notin A_\lambda)$$

$$\iff \forall_{\lambda\in\Lambda} \; x \in C \setminus A_\lambda \iff x \in \bigcap_{\lambda\in\Lambda} (C \setminus A_\lambda)$$

(n) $$C \setminus \bigcap_{\lambda\in\Lambda} A_\lambda = \bigcup_{\lambda\in\Lambda} (C \setminus A_\lambda)$$

1(e) CARDINALITY AND COUNTABLE SETS

DEFINITION: Two sets X and Y are said to have the same cardinality if there exists a bijective function $f: X \to Y$. In such a case (because having the same cardinality is an equivalence relation) we can write $|X| = |Y|$ ($|X|$ cardinality of X)

In addition, we will also write $|X| \leq |Y|$ if there exists an injective function $f: X \to Y$. Then clearly $|X| = |f(X)|$

THEOREM (CANTOR-BERNSTEIN). For two sets X and Y if $|X| \leq |Y|$ and $|Y| \leq |X|$ then $|X| = |Y|$ $\square$

AXIOM OF CHOICE For any family of non-empty sets $\{A_\lambda\}_{\lambda \in \Lambda}$ there exists a function (choice function)

$$f: \Lambda \to \bigcup_{\lambda \in \Lambda} A_\lambda$$

such that $f(\lambda) \in A_\lambda$.

$$B := \{f(\lambda) : \lambda \in \Lambda\}. \qquad \forall_{\lambda \in \Lambda} \quad B \cap A_\lambda = \{f(\lambda)\}.$$