LECTURE 19 — MATH 6301

PROPOSITION: Let $A \in \mathcal{L}_n$. Then

$$\mathfrak{m}_n(A) = \inf \left\{ \sum_{n=1}^{\infty} |\mathring{I}_n| : \mathring{I}_n \text{ are open intervals and } A \subset \bigcup_{n=1}^{\infty} \mathring{I}_n \right\}$$

or

$$= \inf \left\{ \sum_{n=1}^{\infty} |\overline{I}_n| : I_n \text{ are closed intervals and } A \subset \bigcup_{n=1}^{\infty} \overline{I}_n \right\}$$

PROOF: Recall that for $A \in \mathcal{L}_n$ we have

$$(*) \qquad \mathfrak{m}_n(A) \overset{\text{def}}{:=} \inf \{ \lambda(\mathcal{U}) : \mathcal{U} \text{ is open and } A \subset \mathcal{U} \}$$

where $\mathcal{U}$ can be represented as a countable disjoint union of intervals J_n

$$\lambda(\mathcal{U}) = \sum_{n=1}^{\infty} |J_n| \qquad \begin{array}{|l} 1)\ J_n \cap J_m = \emptyset \quad n \neq m \\ 2)\ \mathcal{U} = \bigcup_{n=1}^{\infty} J_n \end{array}$$

By definition $(*)$ $\forall_{\varepsilon > 0}$ $\exists_{\substack{\mathcal{U} = \bigcup_{n=1}^{\infty} J_n \\ \text{open}}}$ $A \subset \bigcup_{n=1}^{\infty} J_n = \mathcal{U}$

and

$$\mathfrak{m}_n(A) + \varepsilon > \sum_{n=1}^{\infty} |J_n| = \sum_{n=1}^{\infty} |\overline{J}_n|$$

therefore, put $\overline{I}_n = \overline{J}_n$ and we have that

$$\mathfrak{m}_n(A) \leq \mathfrak{m}_n\left(\bigcup_{n=1}^{\infty} \overline{I}_n\right) \leq \sum_{n=1}^{\infty} |\overline{I}_n| < \mathfrak{m}_n(A) + \varepsilon$$

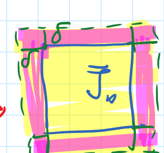
consequently we have

$$\mathfrak{m}_n(A) = \inf \left\{ \sum_{n=1}^{\infty} |\overline{I}_n| : A \subset \bigcup_{n=1}^{\infty} \overline{I}_n \right\}$$

Similarly,

$\forall_{\varepsilon > 0}$ $\exists_{\mathcal{U} = \bigcup_{n=1}^{\infty} J_n}$ $A \subset \bigcup_{n=1}^{\infty} J_n$ $\qquad \mathfrak{m}_n(A) + \frac{\varepsilon}{2} > \sum_{n=1}^{\infty} |J_n|$

$\|\cdot\|_\infty$ $\qquad \mathring{I}_n := B_\delta(\overline{J}_n)$



Chose $\delta > 0$ small enough such that

$$|\mathring{I}_n| < |J_n| + \underbrace{\frac{\varepsilon}{2^{n+1}}}$$

$J_n \subset \mathring{I}_n$ so we have

$J_n \subset \mathring{I}_n$ so we have

$\forall_{\varepsilon>0}$ $$m_n(A) \le m_n\left(\bigcup_{n=1}^{\infty} \mathring{I}_n\right) \le \sum_{n=1}^{\infty} |\mathring{I}_n| \le \sum_{n=1}^{\infty} |J_n| + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}}$$

$$\le m_n(A) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = m_n(A) + \varepsilon \quad \square$$

SUMMARY OF PROPERTIES OF LEBESGUE MEASURE

Let $\mathcal{L}_n$ be the class of Lebesgue measurable sets in $\mathbb{R}^n$ and $m_n : \mathcal{L}_n \longrightarrow [0,\infty]$ be the Lebesgue measure on $\mathbb{R}^n$ $(\mathcal{B}_n \subset \mathcal{L}_n)$

(1) m_n is complete

(2) m_n is metric

(3) $\forall_{x_0 \in \mathbb{R}^n}$ $m_n(x_0 + A) = m_n(A)$

m_n is invariant w/r to shifting of sets

(4) $\forall_{r>0}$ $m_n(rA) = r\, m_n(A)$

(5) $m_n(-A) = m_n(A)$

$x_0 \in \mathbb{R}^n$ $A \subset \mathbb{R}^n$

$x_0 + A = \{x_0 + a : a \in A\}$

$rA := \{ra : a \in A\}$

(6) $T \in GL(n;\mathbb{R})$ $m_n(T(A)) = |\det(T)| \cdot m_n(A)$

ADDITIONAL PROPERTIES: If $\mu : \mathcal{B}_n \longrightarrow [0,\infty]$ is measure such that

i) $\mu \not\equiv 0$ ii) μ satisfies (3), (4) & (5) iii) $E \subset \mathbb{R}^n$ is bounded then $\mu(E) < \infty$

Then $\exists_{\alpha>0}$ such that $\forall_{A \in \mathcal{B}_n}$ $\mu(A) = \alpha\, m_n(A)$

(7) $A \in \mathcal{L}_n$ $B \in \mathcal{L}_k$ $\mathbb{R}^n, \mathbb{R}^k$ $\mathbb{R}^n \times \mathbb{R}^k = \mathbb{R}^{n+k}$

$\mathcal{L}_{n+k}$

$$m_{n+k}(A \times B) = m_n(A) \cdot m_k(B).$$

which implies that $\mathcal{L}_n \times \mathcal{L}_k \subsetneq \mathcal{L}_{n+k}$

PROOFS: (3) $m_n(A) = \inf\left\{ \sum_{n=1}^{\infty} |\mathring{I}_n| : A \subset \bigcup_{n=1}^{\infty} \mathring{I}_n \right\}$

$|x_0 + \mathring{I}_n| = |\mathring{I}_n|$

Thus $m_n(x_0 + A) = \inf\left\{ \sum^{\infty} |x_0 + \mathring{I}_n| : x_0 + A \subset \bigcup^{\infty} (x_0 + \mathring{I}_n)\right.$

$|x_0 + I_n| = |I_n|$

Thus $\mathfrak{M}_n(x_0+A) = \inf \{ \sum_{n=1}^{\infty} |x_0 + \mathring{I}_n| : x_0 + A \subset \bigcup_{n=1}^{\infty} (x_0 + \mathring{I}_n)$

$= \inf \{ \sum_{n=1}^{\infty} |\mathring{I}_n| : A \subset \bigcup_{n=1}^{\infty} \mathring{I}_n \} = \mathfrak{M}_n(A)$

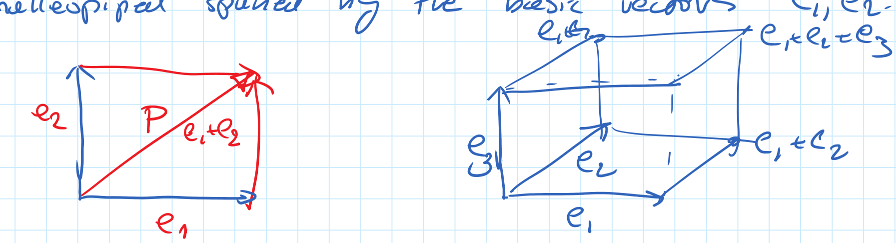
(6) Take a matrix $T \in GL(n, \mathbb{R})$

$\{e_1, e_2, \dots, e_n\}$ standard basis in $\mathbb{R}^n$

then we can consider the interval

$I = [0,1] \times [0,1] \times \dots \times [0,1]$

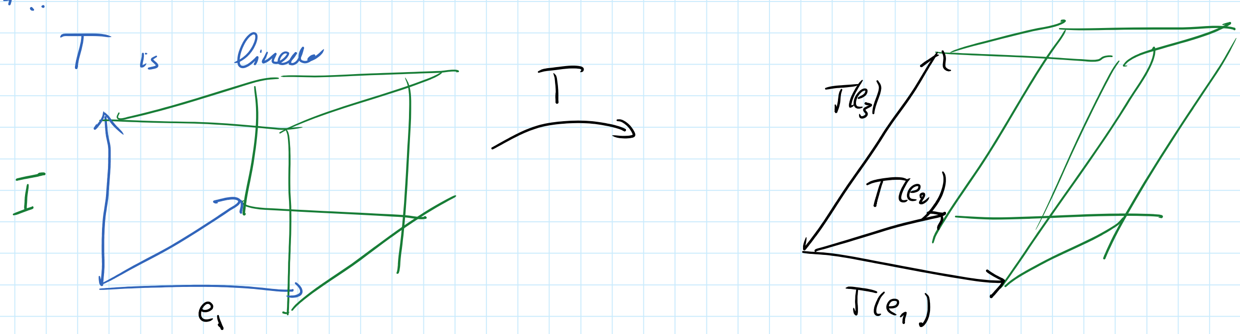
which is the parallelopiped spanned by the basic vectors $e_1, e_2 \dots e_n$



We know the volume of I : $|I| = 1$. Take the matrix T and apply it on I, $T(I)$ = image of I under T

what is it?

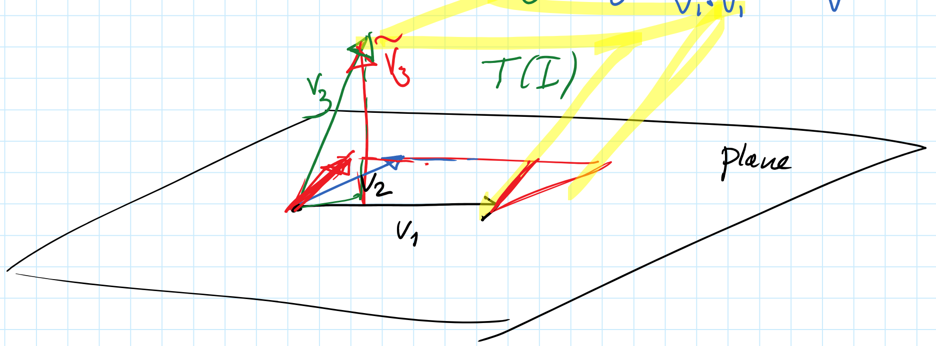
Because T is linear

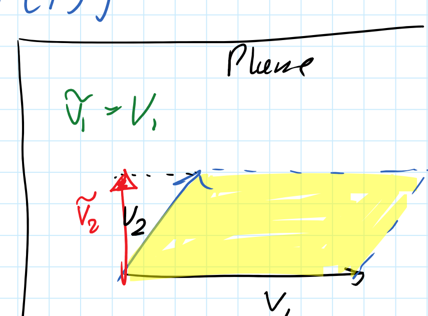


$P = T(I)$ is a parallelopiped generated by $T(e_1), T(e_2), \dots T(e_n)$

What is the volume $|T(I)|$? ANSWER: $|\det(T)|$

$\tilde{v}_3 = v_3 - \frac{v_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{v_3 \cdot v_2}{v}$

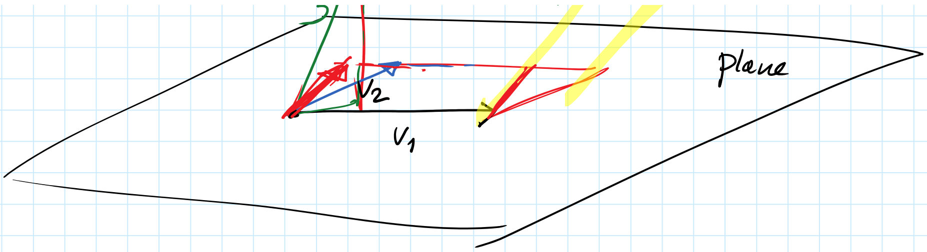


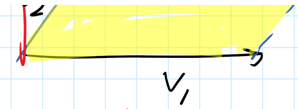


$\tilde{v}_1 = v_1$

$\tilde{v}_2 = v_2 - \frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1$

α_{11}





$$\tilde{v}_2 = v_2 - \frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

c_{11}

$$T(I) = \|\tilde{v}_1\| \cdot \|\tilde{v}_2\| \cdot \|\tilde{v}_3\|$$

$$\det \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \det \begin{bmatrix} v_1 \\ v_2 - c_{c_1} v_1 \\ v_3 - c_{31} v_1 - c_{32} v_2 \\ \vdots \\ v_n - c_{n1} v_1 - \dots - c_{n\,n-1} v_{n-1} \end{bmatrix} = \det \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \vdots \\ \tilde{v}_n \end{bmatrix}$$

$$= \det \begin{bmatrix} \|\tilde{v}_1\| \frac{\tilde{v}_1}{\|\tilde{v}_1\|} \\ \vdots \\ \|\tilde{v}_n\| \frac{\tilde{v}_n}{\|\tilde{v}_n\|} \end{bmatrix} = \|\tilde{v}_1\| \, \|\tilde{v}_2\| \cdots \|\tilde{v}_n\| \det \begin{bmatrix} \frac{\tilde{v}_1}{\|\tilde{v}_1\|} \\ \vdots \\ \frac{\tilde{v}_n}{\|\tilde{v}_n\|} \end{bmatrix} = \pm \|\tilde{v}_1\| \, \|\tilde{v}_2\| \cdots \|\tilde{v}_n\|$$

$$= \pm |T(I)|$$

Therefore,

$$m_n(T(A)) = \inf \left\{ \sum_{n=1}^{\infty} |T(I_n)| : T(A) \subset \bigcup_{n=1}^{\infty} T(I_n) \right\}$$

$$= \inf \left\{ \sum_{n=1}^{\infty} |\det T| \, |I_n| : A \subset \bigcup_{n=1}^{\infty} I_n \right\}$$

$$= |\det T| \inf \left\{ \sum_{n=1}^{\infty} |I_n| : A \subset \bigcup_{n=1}^{\infty} I_n \right\}$$

$$= |\det T| \, m_n(A).$$

VITALI'S EXAMPLE OF NON-MEASURABLE SET

Case $n=1$: We will construct a non-measurable set $M \subset [0,1]$.

Since the set $\mathbb{Q} \cap [-1,1]$ is countable, there exists a sequence of rational numbers $\{r_n\}$, $n = 1, 2, \dots$ such that

$\mathbb{Q} \cap [-1,1] = \{r_1, r_2, r_3, \dots\}$

For two real numbers $x, y \in [0,1]$ we put $x \sim y \iff x - y \in \mathbb{Q} \cap [-1,1]$

$\iff \exists_{r_n} \; x - y = r_n$

The relation "$\sim$" is an equivalence relation, so we have that

$$[0,1] = \bigcup_{x \in [0,1]} [x] \qquad [x] = \{x' : x' \sim x\}$$

By axiom of choice there exists a set $M \subset [0,1]$ such that $\forall_{x \in [0,1]}$ the intersection $M \cap [x]$ contains exactly one element,

$$\exists!_{x' \in [x]} \quad M \cap [x] = \{x'\} \qquad (*)$$

Thus we have:

i) $\forall_{r_m, r_n \in \mathbb{Q} \cap [-1,1]}$ $\quad (r_n + M) \cap (r_m + M) = \begin{cases} r_n + M & r_n = r_m \\ \emptyset & r_n \neq r_m \end{cases}$

Indeed $\quad x \in (r_n + M) \cap (r_m + M) : \exists_{x' \in M} \; \exists_{x'' \in M}$

$x = r_n + x' = r_m + x'' \iff x' - x'' = r_n - r_m \in \mathbb{Q}$

$\Downarrow$

$x' \sim x''$

$\Downarrow$

$x'' = x'$

so if $r_n \neq r_m$ we get a contradiction

ii) $\forall_n \quad r_n + M \subset [-1,1] + [0,1] = [-1,2]$

iii) $[0,1] \subset \bigcup_{n=1}^{\infty} (r_n + M)$: Indeed, take $x \in [0,1]$ then

$\forall_{x \in [0,1]} \; [x] \cap M = x'$

$x' - x = -r_n$

$x' = x - r_n$, $x' \in M$

$\exists_{r_n} \quad x - r_n \in M$

$x \in r_n + M$

<u>Assume M</u> is $\mathcal{L}_1$-measurable, then for every $n \in \mathbb{N}$

$$A_n := r_n + M$$

is also $\mathcal{L}_1$-measurable. And by (i) $\quad A_n \cap A_m = \emptyset$, $n \neq m$

and by (ii) and (iii)

$$[0,1] \subset \bigcup_{n=1}^{\infty} A_n \subset [-1,2]$$

therefore, by properties of the Lebesgue measure

$$1 = m_1([0,1]) \le m_1\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n) = \sum_{n=1}^{\infty} m(M)$$

$$\le m_1[-1,2] = 3$$

$m(M) \neq 0$

$m(M) > 0$

$= \infty$

so we get a contradiction. $\square$