University of Texas at Dallas

Assignment #4

Last Name:	First Name and Initial:
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Course Name: Real Analysis 1	**Number:** MATH 6301
Instructor: Wieslaw Krawcewicz	**Due Date:** October 27, 2022
E-mail Address:	**Student's Signature:** ....

Instructions:

1. Print this booklet
2. Use the space provided to write your solutions in this booklet
3. Hand in your assignment to your instructor on the due date during the class time.

Question	Weight	Your Score	Comments
1.	10		
2.	10		
3.	10		
4.	10		
5.	10		
Total:	50		

Problem 1. Assume that $U \subset \mathbb{R}^n$ is an open set and $f : U \to \mathbb{R}$ a differentiable function. Show that for every $k = 1, 2, \ldots, n$, the partial derivative

$$\frac{\partial f}{\partial x_k} : U \to \mathbb{R},$$

is $\mathscr{B}_n$-measurable (here $\mathscr{B}_n$ stands for the σ-algebra of Borel sets in $\mathbb{R}^n$).

SOLUTION:

Problem 2: Let X be a space and $\mathscr{S} \subset \mathscr{P}(X)$ a σ-algebra in X. We say that a map $f : X \to \mathbb{R}^n$ is $\mathscr{S}$-*measurable* if and only if

$$\forall_{V \in \mathscr{B}_n} \quad f^{-1}(V) \in \mathscr{S}.$$

Assume that $f : X \to \mathbb{R}^n$ is a map such that for all $v \in \mathbb{R}^n$ the function $\varphi_y(x) := f(x) \bullet v$, $x \in X$, is $\mathscr{S}$-measurable. Show that the map f is $\mathscr{S}$-measurable.

SOLUTION:

Problem 3: Let X be a bounded set in a Banach space $\mathscr{E}$. We define the following function $\mu^* : \mathscr{P}(X) \to \mathbb{R}$ by

$$\mu^*(A) := \inf \left\{ r > 0 : \exists_{x_1, x_2, \dots, x_k \in X} \;\; A \subset \bigcup_{j=1}^{k} B_r(x_j) \right\}, \quad A \subset X,$$

where $B_r(x_o) := \{x \in \mathscr{E} : \|x - x_o\| < r\}$. Verify if the function μ^* is an outer measure on X and if it is check if it is a metric outer measure.

(The function μ^* defined above is called a *measure of non-compactness*. ¿ Can you guess what would be μ^* if $\mathscr{E} = \mathbb{R}^n$?)

SOLUTION:

Problem 4: For two given spaces X and Y and assume that $\mu_1^* : \mathscr{P}(X) \to \overline{\mathbb{R}}$ and $\mu_2^* : \mathscr{P}(Y) \to \overline{\mathbb{R}}$ are two outer measures. Define the function $\nu^* : \mathscr{P}(X \times Y) \to \overline{\mathbb{R}}$ by

$$\nu^*(C) := \inf \left\{ \sum_{k=1}^{\infty} \mu_1^*(A_k)\mu_2^*(B_k) : C \subset \bigcup_{k=1}^{\infty} A_k \times B_k,\ A_k \subset X,\ B_k \subset Y \right\}$$

Check if the function ν^* is an outer measure on $X \times Y$.

SOLUTION:

Problem 5: A set $I \subset \mathbb{R}^n$ is called an *interval* in $\mathbb{R}^n$ if there exist $a_1 \le b_1$, $a_2 \le b_2$, ..., $a_n \le b_n$ such that

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \subset I \subset [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n].$$

We denote by $\mathscr{J}$ the family of all intervals in $\mathbb{R}^n$. Consider the set

$$X := [c_1, d_1] \times [c_2, d_2] \times \cdots \times [c_n, d_n], \quad c_k < d_k.$$

Is the family $\mathscr{R} \subset \mathscr{P}(X)$, given by

$$\mathscr{R} := \left\{ A \subset X : \exists_{I_1, I_2, \dots, I_N \in \mathscr{J}} \; A := \bigcup_{k=1}^{N} I_k, \; I_k \subset X \right\}.$$

an algebra of sets in X? Justify your answer.

SOLUTION: