LECTURE 8 — MATH 6301

MEASURE THEORY

1. Algebra of Sets:

Let X be a given set (we consider it as a space) and assume $\mathcal{F} \subset \mathcal{P}(X)$ is a given family of subsets in X. We say that $\mathcal{F}$ is an algebra of sets (finitely additive algebra) iff

(A1) $\emptyset \in \mathcal{F}$

(A2) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ $\qquad (A^c := X \setminus A)$

(A3) $A_1, A_2, \dots, A_n \in \mathcal{F} \Rightarrow A_1 \cup A_2 \cup \dots \cup A_n \in \mathcal{F}$ Finite

REMARKS: (a) Notice that if $\mathcal{F}_\lambda \subset \mathcal{P}(A)$ is an algebra of sets for every $\lambda \in \Lambda$, then

$$\mathcal{F}_0 := \bigcap_{\lambda \in \Lambda} \mathcal{F}$$

is also an algebra of sets. Indeed, (A1) $\forall_{\lambda \in \Lambda}\ \emptyset \in \mathcal{F}_\lambda \Rightarrow \emptyset \in \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$

(A2) $\forall_{\lambda \in \Lambda} \left(A \in \mathcal{F}_\lambda \underset{(*)}{\Rightarrow} A^c \in \mathcal{F}_\lambda \right) \Rightarrow$ if $A \in \mathcal{F}_0 \Leftrightarrow \forall_{\lambda \in \Lambda}\ A \in \mathcal{F}_\lambda \Rightarrow A^c \in \mathcal{F}_\lambda$

$\Rightarrow A^c \in \mathcal{F}_0 = \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$, and in a similar way

(A3) If $\forall_{\lambda \in \Lambda}\ A_1, A_2, \dots, A_n \in \mathcal{F}_\lambda \Rightarrow A_1 \cup A_2 \cup \dots \cup A_n \in \mathcal{F}_0$ (since $\forall_\lambda\ A_1 \cup A_2 \cup \dots \cup A_n \in \mathcal{F}_\lambda$).

(b) The smallest algebra of sets in X is the family $\{\emptyset, X\}$ and the largest algebra of sets is $\mathcal{P}(X)$.

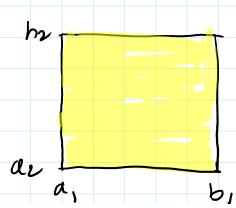
(c) Suppose $\mathcal{A} \subset \mathcal{P}(X)$ is a given family of sets and put $\mathcal{F}(\mathcal{A})$ to be the smallest algebra of sets containing $\mathcal{A}$, i.e

$$\mathcal{F}(\mathcal{A}) = \bigcap \left\{ \mathcal{F} \subset \mathcal{P}(X) : \begin{array}{l} 1)\ \mathcal{F} \text{ is algebra of sets} \\ 2)\ \mathcal{A} \subset \mathcal{F} \end{array} \right\}$$

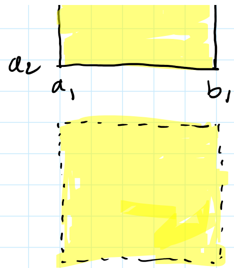
EXAMPLE: Take $\mathcal{A} = \emptyset$. What is $\mathcal{F}(\mathcal{A})$?

$$\mathcal{F}(\mathcal{A}) = \{\emptyset, X\}$$

EXAMPLE: Measurable objects in $\mathbb{R}^2$: rectangles $[a_1, b_1] \times [a_2, b_2] = R$



R has an area $= (b_1 - a_1) \cdot (b_2 - a_2)$

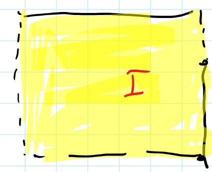


Clearly, one dimensional intervals do not contribute to area.

<u>Notation:</u> Take a rectangle R which we consider as our space X.
Then a set $I \subset R$ will be called an interval in X iff

(i) $\operatorname{int}(I) = (a,b) \times (c,d)$

(ii) $\overline{I} = [a,b] \times [c,d]$

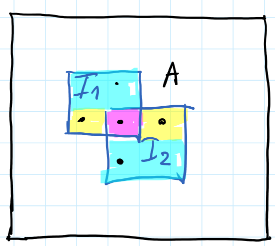


$\mu(I) = (b-a)(d-c)$

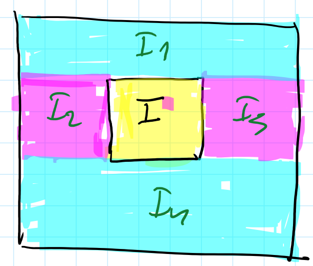
Take a family of all intervals $\mathcal{J} \subset \mathcal{P}(X)$ and consider $\mathcal{F}(\mathcal{J})$. Then notice that

$$\mathcal{F}(\mathcal{J}) = \left\{ A \subset \mathcal{P}(X) : \exists_{I_1, I_2, \ldots, I_m} \; A = I_1 \cup \ldots \cup I_m \right\}$$

(A3)



(A2)



$I^c = I_1 \cup I_2 \cup I_3 \cup I_4$

$$A = I_1 \cup I_2 \cup \ldots \cup I_m$$
$$A^c = I_1^c \cap I_2^c \cap \ldots \cap I_m^c$$

In the case of the Euclidean space $\mathbb{R}^n$, one can introduce the notion of n-interval $I \subset X$, when

$$X := [c_1, d_1] \times [c_2, d_2] \times \ldots \times [c_n, d_n] \qquad c_k < d_k \quad k = 1, 2, \ldots, n$$

if

(i) $\operatorname{int}(I) = (a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n)$,

(ii) $\overline{I} = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$.

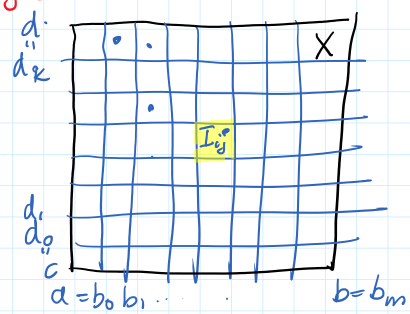
Notice, that
$$\mu(I) = \prod_{k=1}^{n} (b_k - a_k),$$

and if $A = I_1 \cup I_2 \cup \ldots \cup I_m$ $\quad I_j \cap I_i = \emptyset$

then $\mu(A) = \mu(I_1) + \mu(I_2) + \ldots + \mu(I_m)$.

The algebra $\mathcal{F}(\mathcal{I})$ is intimately connected to the notion of Riemann integral.

$f : X \to \mathbb{R}$ $\qquad x = (x_1, x_2) \in \mathbb{R}^2$



$I_{ij} = [b_{i-1}, b_i) \times [d_{j-1}, d_j)$ $\qquad \{I_{ij}\} = \mathcal{P}$

size. $\quad |\mathcal{P}| = \max_{i,j} \{ b_i - b_{i-1}, d_j - d_{j-1} \}$

we can also choose $x_*^{ij} \in I_{ij}$. $\quad \{\ldots\}$

and define the Riemann sum

$$\sigma(f, \mathcal{P}, \{x_*^{ij}\}) = \sum_{i,j} f(x_*^{ij})\, \mu(I_{ij})$$

Then $f : X \to \mathbb{R}$ is Riemann integrable

$$\int_X f(x)\,dx = \lim_{|\mathcal{P}| \to 0} \sigma(f, \mathcal{P}, \{x_*^{ij}\})$$

exists and is finite and doesn't depend on choice $\{x_*^{ij}\}$.

PROPOSITION: Darboux integrals : $\mathcal{P} = \{I_{ij}\}$

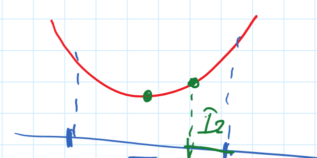
lower Darboux sum $\quad s(f, \mathcal{P}) := \sum_{i,j} m_{ij}\, \mu(I_{ij})$, $\quad m_{ij} = \inf \{ f(x) : x \in I_{ij} \}$

upper Darboux sum $\quad S(f, \mathcal{P}) := \sum_{i,j} M_{ij}\, \mu(I_{ij})$, $\quad M_{ij} = \sup \{ f(x) : x \in I_{ij} \}$

Then.

$$\underline{\int_X} f\,d\mu := \sup_{\mathcal{P}} s(f, \mathcal{P})$$

$$\overline{\int_X} f\,d\mu := \inf_{\mathcal{P}} S(f, \mathcal{P})$$



$I_2 \subset I_1$ $\qquad I_1$

$\inf_{x \in I_1} f(x) \boxed{\leq} \inf_{x \in I_2} f(x)$

$$s(f, \mathcal{P}) \leq \sigma(f, \mathcal{P}, \{x_*^{ij}\}) \leq S(f, \mathcal{P})$$

$$\xrightarrow{|\mathcal{P}| \to 0} \int_X f\,dx$$

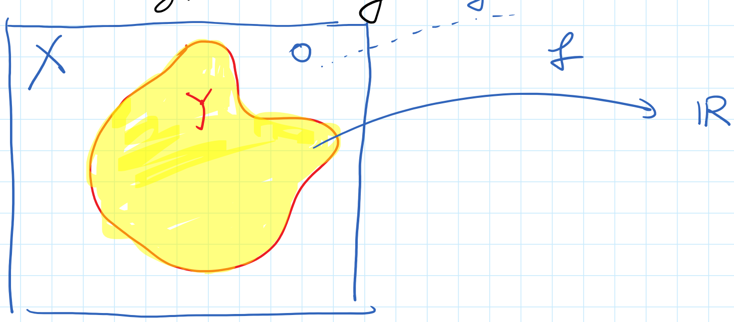
Notice that if $Y \subset \mathbb{R}^n$ is a bounded set, one (usually) define the Riemann integral

$$\int f(x)\,dx \overset{def}{=} \int \tilde{f}(x)\,dx \qquad \tilde{f}(x) = \begin{cases} f(x) & x \in Y \end{cases}$$

define the Riemann integral

$$\int_Y f(x)\,dx \overset{def}{=} \int_X \tilde{f}(x)\,dx \qquad \tilde{f}(x) = \begin{cases} f(x) & x \in Y \\ 0 & x \notin Y \end{cases}$$

where X is a rectangle containing Y.



Is the function $\tilde{f} : X \to \mathbb{R}$ Riemann integrable?

For example, for $A \subset \mathbb{R}^n$; define $\chi_A : \mathbb{R}^n \to \mathbb{R}$ by $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$ characteristic function

and consider the the integral (if it is integrable)

$$\int_X \chi_Y(x)\,dx$$

χ_Y

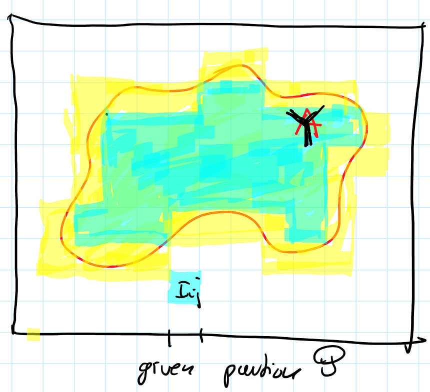
inner Jordan measure of Y

$\mu_*(Y) = \sup_{\mathcal{P}} s(\chi_Y, \mathcal{P})$

$\mu^*(Y) = \inf_{\mathcal{P}} S(\chi_Y, \mathcal{P})$

We say that Y is Jordan measurable if

$\mu_*(Y) = \mu^*(Y)$



$$s(\chi_Y, \mathcal{P}) = \sum_{\substack{i,j \\ I_{ij} \subset Y}} 1 \cdot \mu(I_{ij}) + 0$$

$$S(\chi_Y, \mathcal{P}) = \sum_{\substack{i,j \\ I_{ij} \cap Y \neq \emptyset}} 1 \cdot \mu(I_{ij}) + 0$$

In this way, we obtain a new class $\mathcal{J}$ of sets in $\mathbb{R}^n$, which are bounded and Jordan measurable. One can show that $\mathcal{J}$ is an algebra of sets.

2. σ-Algebra of Sets:

Let X be a given space and $\mathcal{S} \subset \mathcal{P}(X)$ a family of sets in X.
We say that $\mathcal{S}$ is a σ-algebra (countably additive algebra) of sets in X iff the following conditions are satisfied:

(σA1) $\quad \emptyset \in \mathcal{S}$

(σA2) $\quad A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$

(σA3) $\quad \{A_1, A_2, \dots, A_n, \dots\} \subset \mathcal{S} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{S}$

Notice that we also have the following properties of σ-algebra

(σA4) $\quad \{A_1, A_2, \dots, A_n, \dots\} \subset \mathcal{S} \Rightarrow \bigcap_{k=1}^{\infty} A_k \in \mathcal{S}$

Indeed, since $A^{cc} = A$

$$\bigcap_{k=1}^{\infty} A_k = X \setminus \Big(\bigcap_{k=1}^{\infty} A_k\Big)^c \in \mathcal{S}. \quad \square$$

$$= \underbrace{X \setminus \underbrace{\bigcup_{k=1}^{\infty} A_k^c}_{(\sigma A3)\, \in \mathcal{S}}}_{(\sigma A2)\, \in \mathcal{S}}$$