MATH 6301 Real Analysis I
Homework 1

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Contents

Problem 1 3

Problem 2 4

Problem 3 5

Problem 4 6
- a) 6
- b) 7
- c) 8

Problem 5 9
- a) 9
- b) 9

Instructions:

1. Print this booklet
2. Use the space provided to write your solutions in this booklet
3. Hand in your assignment to your instructor on the due date during the class time.

Question	**Weight**	**Your Score**	**Comments**
1.	10		
2.	10		
3.	10		
4.	10		
5.	10		
Total:	50		

Problem 1

PROBLEM:

A set A in a metric space (X, d) is called *bounded* if

$$\exists_{R>0}\exists_{x_0\in X}\ A \subset B_R(x_0)$$

Use mathematical induction to show that if a set $A \subset X$ is unbounded (i.e. it is not bounded), then there exists a sequence $x_n \subset A$ such the $d(x_n, x_m) \forall m \neq n$.

SOLUTION:

Definition 1. *Let $A \subset X$ be a set in metric space (X, d). The open ball of radius R centered at x_0, denoted as $B_R(x_0)$, is defined by*

$$\{x \in X\ :\ d(x, x_0) < R\}$$

Proposition 1. *Let $A \subset X$ be unbounded, then there exists a sequence $x_n \subset A$ such the $d(x_n, x_m) \geq 1 \forall m \neq n$.*

Proof. For $n = 0$ and $m = 1$, select $x_m \in A\ :\ d(x_n, x_m) \geq 1$.
Next, for each n, since A is unbounded we know that $\exists_{x_m\in A}\forall_{i=1,\ldots,n} d(x_i, x_m) \geq 1$. We know this by the definition of unbounded, bounded, and an open ball:

$$\begin{aligned}
\exists_{R>0}\exists_{x_0\in X}\ A \subset B_R(x_0) &\iff \forall_{R>0}\forall_{x_0\in X}\ A \subset B_R(x_0) \\
&\implies \forall_{R>0}\forall_{x_0\in X} A \supseteq B_R(x_0) \\
&\implies \exists_{R\geq 1}\forall_{m=1,\ldots,n} d(x_n, x_m) \geq R
\end{aligned}$$

By mathematical induction, we can say that a sequence $x_n \subset A$ can be constructed such the $d(x_n, x_m) \geq 1 \forall m \neq n$. $\square$

Problem 2

PROBLEM:

Let $(V, \|\cdot\|)$ be a normed vector Space. Show that an open unit ball

$$B_1(0) \coloneqq \{v \in V \ : \ \|v\| < 1\}$$

is a convex set, i.e.

$$\forall_{u,v \in B_1(0)} \forall_{t \in [0,1]} tu + (1-t)v \in B_1(0)$$

SOLUTION:

Proposition 2. *$B_1(0)$ is a convex set.*

Proof. Select arbitrary $u, v \in B_1(0)$ and $t \in [0,1]$. By definition of $x \in B_1(0) \iff \|x\| < 1$. We have $\|v\| < 1$ and $\|u\| < 1$.

$$\|tu + (1-t)v\|$$

By the triangle inequality,

$$\leq \|tu\| + \|(1-t)v\|$$

Since t is a scaler,

$$\leq (t)\|u\| + (1-t)\|v\|$$

Since $\|v\| < 1$ and $\|u\| < 1$,

$$< (t)(1) + (1-t)(1) = t - t + 1 = 1$$

Therefore,

$$\|tu + (1-t)v\| < 1$$

which means

$$\forall_{u,v \in B_1(0)} \forall_{t \in [0,1]} \ tu + (1-t)v \in B_1(0)$$

□

Problem 3

PROBLEM:

Suppose that (X, d) is a metric space and $A, B \subset X$ are such that $A \subset B$. Show that $\int(A) \subset \int(B)$ and $\overline{A} \subset \overline{B}$.

PRELIMINARIES:

Definition 2. *Let A, B be sets in metric space (X, d).* <u>Difference of A and B</u>

$$A \backslash B = A \cap B^c := \{x \in \mathbb{R}^n \ : \ x \in A \wedge x \neq B\}$$

We define the <u>interior of A</u> *as*

$$int(A) := \{x \in A \ : \ \exists_{\epsilon>0} B_\epsilon(x) \subset A\}$$

We define the <u>closure of A</u> *as*

$$\overline{A} := \{x \in X \ : \ B_\epsilon(x) \cap A \neq \emptyset\}$$

We define the <u>boundary of A</u> *as*

$$\partial A := \overline{A} \cap \overline{(A^c)} = \{x \in X \ : \ \forall_{\epsilon>0} B_\epsilon(x) \mathcal{A} \neq \emptyset \neq \mathcal{B}_\epsilon \cap \mathcal{A}^\rfloor\}$$

SOLUTION:

Proposition 3. *Let (X, d) be a metric space and $A, B \subset X$. If $A \subset B$ then $int(A) \subset int(B)$.*

Proof.

$$A \subset B \implies \forall_{x \in \text{int}(A)} x \in B \wedge \forall_{\epsilon>0} B_\epsilon(x) \subset A \implies B_\epsilon(x) \subset B$$

Therefore $\forall_{x \in \text{int}(A)} x \in \text{int}(B)$ which proves $\text{int}(A) \subset \text{int}(B)$ by definition. □

Proposition 4. *Let (X, d) be a metric space and $A, B \subset X$. If $A \subset B$ then $\overline{A} \subset \overline{B}$.*

Proof.

$$A \subset B \implies \forall_{x \in \bar(A)} x \in B \wedge \forall_{\epsilon>0} \cap A \neq \emptyset \implies B_\epsilon(x) \cap B \neq \emptyset$$

Therefore $\forall_{x \in \bar(A)} x \in \bar(b)$ which proves $\overline{A} \subset \overline{B}$ by definition. □

Problem 4

Let (X_1, d_1) and (X_2, d_2) be two metric spaces. Show that each of the following functions d is a metric on $X = X_1 \times X_2$ (Hint: Knowing that d_1 and d_2 satisfies the conditions of a metric, show that d also satisfies these conditions)

PRELIMINARIES:

Definition 3. *A* **Metric Space**, *(X, d) consists of a space X and* <u>metric</u>, *$d : X \times X \to \mathbb{R}$, that satisfies the following:*

1. *Distance to itself is zero*
$$\forall_{x \in X} d(x, x) = 0$$
2. <u>Positivity:</u>
$$\forall_{x,y \in X} d(x, y) \geq 0$$
3. <u>Symmetry:</u>
$$\forall_{x,y \in X} d(x, y) = d(y, x)$$
4. *Satisfies* <u>triangle inequality</u>
$$\forall_{x,y,z \in X} d(x, z) \leq d(x, y) + d(y, z)$$

a)

PROBLEM:

Show that the following satisfies the conditions of a metric:

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

SOLUTION:

Proposition 5. *Let (X_1, d_1) and (X_2, d_1) be metric spaces. We propose that (X, d) with $X = X_1 \times X_2$ and $d : X_1 \times X_2 \to \mathbb{R}$ is defined as*

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

Proof. d satisfies the metric conditions:

1. Distance to itself is zero
$$d((x_1, x_2), (x_1, x_2)) = \max\{d_1(x_1, x_1), d_2(x_2, x_2)\} = \max\{0, 0\} = 0$$
2. <u>*Positivity:*</u>
$$d_1(x_1, y_1) \geq 0 \land d_2(x_2, y_2) \geq 0 \implies \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} \geq 0$$
3. <u>*Symmetry:*</u>
$$d_1(x_1, y_1) = d_1(y_1, x_1) \land d_2(x_2, y_2) = d_2(y_2, x_2)$$
$$\implies \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} = \max\{d_1(y_1, x_1), d_2(y_2, x_2)\}$$
4. Satisfies <u>*triangle inequality*</u>
$$d_1(x_1, z_1) \leq d_1(x_1, y_1) + d_1(y_1, z_1) \land d_2(x_2, z_2) \leq d_2(x_2, y_2) + d_2(y_2, z_2)$$
$$\implies \max\{d_1(x_1, z_1), d_2(x_2, z_2)\} \leq \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} + \max\{d_1(y_1, z_1), d_2(y_2, z_2)\}$$

$\square$

b)

PROBLEM:

Show that the following satisfies the conditions of a metric:

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2}$$

SOLUTION:

Proposition 6. *Let (X_1, d_1) and (X_2, d_1) be metric spaces. We propose that (X, d) with $X = X_1 \times X_2$ and $d : X_1 \times X_2 \to \mathbb{R}$ is defined as*

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2}$$

Proof. d satisfies the metric conditions:

1. Distance to itself is zero

$$d((x_1, x_2), (x_1, x_2)) = \sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2)^2)} = \sqrt{(0)^2 + (0^2)} = 0$$

2. <u>*Positivity:*</u>

$$\begin{aligned} d_1(x_1, y_1) \geq 0 \wedge d_2(x_2, y_2) \geq 0 &\implies (d_1(x_1, y_1))^2 \geq 0 \wedge (d_2(x_2, y_2))^2 \geq 0 \\ &\implies (d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2 \geq 0 \\ &\implies \sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2)^2)} \geq 0 \end{aligned}$$

3. <u>*Symmetry:*</u>

$$\begin{aligned} & d_1(x_1, y_1) = d_1(y_1, x_1) \wedge d_2(x_2, y_2) = d_2(y_2, x_2) \\ \implies & (d_1(x_1, y_1))^2 = (d_1(y_1, x_1))^2 \wedge (d_2(x_2, y_2))^2 = (d_2(y_2, x_2))^2 \\ \implies & (d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2 = (d_2(y_2, x_2))^2 + (d_1(y_1, x_1))^2 \\ \implies & \sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2)^2)} = \sqrt{(d_1(y_1, x_1))^2 + (d_2(y_2, x_2)^2)} \end{aligned}$$

4. Satisfies <u>*triangle inequality*</u>

$$\left(\sqrt{(d_1(x_1, z_1))^2 + (d_2(x_2, z_2)^2)^2}\right)^2 = (d_1(x_1, z_1))^2 + (d_2(x_2, z_2)^2)^2 + 2d_1(x_1, z_1)d_2(x_2, z_2)$$

$$\begin{aligned} &\leq \left(\sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2} + \sqrt{(d_1(y_1, z_1))^2 + (d_2(y_2, z_2))^2}\right)^2 \\ &= (d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2 + (d_1(y_1, z_1))^2 + (d_2(y_2, z_2))^2 \\ &+2\sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2}\sqrt{(d_1(y_1, z_1))^2 + (d_2(y_2, z_2))^2} \end{aligned}$$

Therefore,

$$\begin{aligned} & d_1(x_1, z_1) \leq d_1(x_1, y_1) + d_1(y_1, z_1) \wedge d_2(x_2, z_2) \leq d_2(x_2, y_2) + d_2(y_2, z_2) \\ \implies & \sqrt{(d_1(x_1, z_1))^2 + (d_2(x_2, z_2)^2)} \leq \sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2)^2)} + \sqrt{(d_1(y_1, z_1))^2 + (d_2(y_2, z_2)^2)} \end{aligned}$$

□

c)

PROBLEM:

Show that the following satisfies the conditions of a metric:

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$$

SOLUTION:

Proposition 7. *Let (X_1, d_1) and (X_2, d_1) be metric spaces. We propose that (X, d) with $X = X_1 \times X_2$ and $d : X_1 \times X_2 \to \mathbb{R}$ is defined as*

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$$

Proof. d satisfies the metric conditions:

1. Distance to itself is zero

$$d((x_1, x_2), (x_1, x_2)) = d_1(x_1, y_1) + d_2(x_2, y_2) = 0 + 0 = 0$$

2. *Positivity:*

$$d_1(x_1, y_1) \geq 0 \wedge d_2(x_2, y_2) \geq 0 \implies d_1(x_1, y_1) + d_2(x_2, y_2) \geq 0$$

3. *Symmetry:*

$$d_1(x_1, y_1) = d_1(y_1, x_1) \wedge d_2(x_2, y_2) = d_2(y_2, x_2)$$
$$\implies d_1(x_1, y_1) + d_2(x_2, y_2) = d_1(y_1, z_1) + d_2(y_2, x_2)$$

4. Satisfies *triangle inequality*

$$d_1(x_1, z_1) \leq d_1(x_1, y_1) + d_1(y_1, z_1) \wedge d_2(x_2, z_2) \leq d_2(x_2, y_2) + d_2(y_2, z_2)$$
$$\implies d_1(x_1, z_1) + d_2(x_2, z_2) \leq d_1(x_1, y_1) + d_2(x_2, y_2) + d_1(y_1, z_1) + d_2(y_2, z_2)$$

$\square$

Problem 5

Let (X, d) be a metric space and $A \subset X$ a closed non-empty set. For a given point $x \in X$, we define the *distance* from x to A by the formula

$$d(x, A) \coloneqq \inf\{d(x, a) \ : \ a \in A\}$$

a)

PROBLEM:

Show that $x \in A \iff d(x, A) = 0$.

SOLUTION:

Proposition 8. $x \in A \iff d(x, A) = 0$

Proof. First we look at $\implies$,

$$x \in A \implies \exists d(x, a) \ : \ a \in A = 0 \implies \inf\{0, \dots\} = 0 \implies d(x, A) = \inf\{d(x, a) \ : \ a \in A\} = 0$$

Next we look at $\impliedby$,

$$d(x, A) = \inf\{d(x, a) \ : \ a \in A\} = 0 \implies \exists_{\{x_n\} \subset A} \ : \ \lim_{n \to \infty} d(x, a) = 0 \implies x \in A$$

□

b)

PROBLEM:

Show that the function $\Xi_A : X \to \mathbb{R}$ defined by $\Xi_A(x) = d(x, A)$ is continuous on X.

PRELIMINARIES:

Definition 4. *Let (X, d_x) and (Y, d_y) be metric spaces. A function $f : X \to Y$ is* <u>continous at point $a \in X$</u> *iff*

$$\forall_{\epsilon > 0} \exists_{\delta > 0} \forall_{x \in X} d_x(a, x) < \delta \implies d_y(f(x), f(a)) < \epsilon$$

f is <u>continous</u> *iff*

$$\forall_{a \in X} \forall_{\epsilon > 0} \exists_{\delta > 0} \forall_{x \in X} d_x(a, x) < \delta \implies d_y(f(x), f(a)) < \epsilon$$

Additionally, we have that f is continuous iff the inverse inverse of an open set is open, i.e

$$f \ continuous \iff \forall_{\nu \in \tau_Y} f^{-1}(\nu) \in \tau_X$$

f is <u>univformly continous</u> *iff*

$$\forall_{\epsilon > 0} \exists_{\delta > 0} \forall_{a \in X} \forall_{x \in X} d_x(a, x) < \delta \implies d_y(f(x), f(a)) < \epsilon$$

SOLUTION:

Proposition 9. *The function $\Xi_A : X \to \mathbb{R}$ defined by $\Xi_A(x) = d(x, A)$ is continuous on X.*

Proof. Ξ_A is continuous iff it is continuous at all $a \in X$.
First, let $a \in A$. Since $x \in A \iff d(x, A) = A$, we have $\forall_{x \in A} \xi_A(x) = 0$ which implies

$$\forall_{a \in A} \forall_{\epsilon > 0} \exists_{\delta > 0} \forall_{x \in A} d_x(a, x) < \delta \implies d_y(\Xi_A(x) = 0, \Xi_A(a) = 0) = 0 < \epsilon$$

Next, let $a \in X\backslash A$. The set $X\backslash A$ is open. Since $\forall_{a \in X\backslash A}$ we have that $d_A(x,a) > 0$, we define τ_Y as the open set $y > 0$. Since both $\tau_X = X\backslash A$ and $y > 0$ are open, we have that

$$\forall_{\nu \in \tau_Y} \Xi^{-1}(\nu) \in \tau_X$$

Finally, at the boundary of A, we have that for $x \notin A$,

$$\lim_{a \to A} \Xi_A(a) = 0$$

which implies Continuity at the boundary.
Therefore, Ξ_A is continuous at all $a \in X$ which means Ξ_A is a continuous function. $\square$