LECTURE 22 — MATH 6301

Assume that $(X, \mathfrak{S}, \mu)$ is a measure space and $E \in \mathfrak{S}$.

THEOREM (Riesz)
Let $f_n : E \to \overline{\mathbb{R}}$ be a sequence of μ-measurable functions, finite a.e, and $f : E \to \overline{\mathbb{R}}$ a measurable function such that $f_n \xrightarrow{\mu} f$ (f_n converges in measure μ to f). Then there exists a subsequence $\{f_{n_k}\} \subset \{f_n\}$ such that $f_{n_k}(x) \to f(x)$ a.e.

PROOF: We have, by assumption, that
$$\underbrace{\forall_{\varepsilon>0} \forall_{\eta>0} \exists_{N} \forall_{n \geq N} \ \mu\{x \in E : |f_n(x) - f(x)| \geq \varepsilon\} < \eta}_{(*)}$$
$$\iff \lim_{n\to\infty} \mu\{x \in E : |f_n(x) - f(x)| \geq \varepsilon\} = 0$$

Therefore
$$\forall_{n} \exists_{k_n} \quad \mu\{x \in E : |f_{k_n}(x) - f(x)| \geq \tfrac{1}{n}\} \leq \frac{1}{2^n}$$
$\varepsilon = \frac{1}{n}$ $k_n = N$
$\eta = \frac{1}{2^n}$

and we can assume, without loss of generality, that the sequence of integers $\{k_n\}_{n=1}^{\infty}$ is strictly increasing. Then we put
$$F_m := E \setminus \{x \in E : |f(x)| = \infty\} \setminus \bigcup_{n=m}^{\infty} \{x \in E : |f_{k_n}(x) - f(x)| \geq \tfrac{1}{n}\}$$

Since $\mu\{x \in E : |f(x)| = \infty\} = 0$ we have that $E \setminus F_m \subset \bigcup_{n=m}^{\infty} \{x : |f_{k_n}(x) - f(x)| \geq \frac{1}{n}\}$

so
$$\mu(E \setminus F_m) \leq \sum_{n=m}^{\infty} \frac{1}{2^n} = \frac{1}{2^{m-1}}$$

Then notice that $f_{k_n} \rightrightarrows f$ on F_m for every m. Indeed, for every $x \in F_m$ one has for all $n \geq m$ that
$$|f_{k_n}(x) - f(x)| < \frac{1}{n}$$
which implies that $f_{k_n}(x) \to f(x)$ uniformly on F_m. $\square$

INTEGRATION (LEBESGUE INTEGRAL)

Suppose $(X, \mathfrak{S}, \mu)$ is a measure space, $E \in \mathfrak{S}$, and $f : E \to \overline{\mathbb{R}}$ is a μ-measurable function s.t. $f(x) \geq 0$ for all $x \in E$. Then we define the integral of f over E relative to μ by following formula the integral of f over E relative to μ by following formula

$$\int_E f\,d\mu = \int_E f(x)\,d\mu(x) \overset{\text{def}}{:=} \sup_{\substack{E=\bigcup_{n=1}^{\infty} E_n \\ E_n\in\mathcal{S},\, E_n\cap E_k=\phi}} \sum_{n=1}^{\infty}\left(\inf_{x\in E_n} f(x)\cdot\mu(E_n)\right) \qquad (*)$$

(here by convention we assume $\inf_{\phi} f(x) = 0$)

DEFINITION: A function $s: E\to\mathbb{R}$ is called a σ-simple if there exist disjoint sets E_n, $n=1,2,\dots$, such that

1) $E=\bigcup_{n=1}^{\infty} E_n$

2) $s(x) = \sum_{n=1}^{\infty} a_n \chi_{E_n}(x)$.

Clearly, if the sets E_n are μ-measurable, then s is also measurable.

Example: (1) Let $f: E\to[0,\infty]$ be a measurable finite a.e function, and $E=\bigcup_{n=1}^{\infty} E_n$, $E_n\in\mathcal{S}$, $E_n\cap E_k=\phi$, $\mu(E_n)>0$.

Put
$$a_n := \inf_{x\in E_n} f(x).$$

Notice that $0\le a_n<\infty$. Then

$$s(x) = \sum_{n=1}^{\infty} a_n\chi_{E_n}(x)$$

(2) If $f:E\to[0,\infty]$ is a measurable function, then there exists a sequence of simple measurable functions $s_n: E\to\mathbb{R}$, such that

(i) $\forall_n \ \forall_{x\in E}$ $\quad 0\le s_n(x)\le s_{n+1}(x)\le f(x)$

(ii) $\forall_{x\in E}$ $\quad \lim_{n\to\infty} s_n(x) = f(x)$.

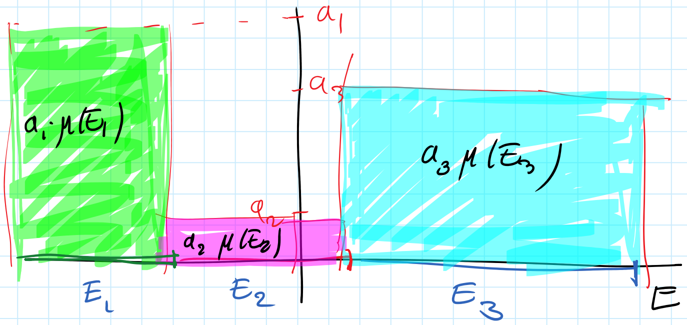
Then, since every simple function s_n is also σ-simple, we have that every non-negative measurable function $f: E\to\overline{\mathbb{R}}$ is a pointwise limit of an increasing ($\le$) a sequence of σ-simple measurable functions s_n

NOTICE: The idea of an integral for a simple measurable function

$$s(x) = \sum_{n=1}^{N} a_n\chi_{E_n}(x)$$

$$\int s\,d\mu = \sum^{N} a_n\,\mu(E_n)$$

$$\int_E s \, d\mu = \sum_{n=1}^{N} a_n \, \mu(E_n)$$



For a σ-sigma simple measurable function $s : E \to \mathbb{R}$, $s(x) \geq 0 \ \forall_{x \in E}$ we have the same formula.

$$\int_E s \, d\mu = \sum_{n=1}^{\infty} a_n \mu(E_n) \qquad s(x) = \sum_{n=1}^{\infty} a_n \chi_{E_n}(x)$$

$$E_n \cap E_m = \emptyset \quad n \neq m$$

In the case $X = \mathbb{R}^n$ and the measure μ is the Lebesgue measure m_n instead of writing

$$\int_E f \, dm_n = \int_E f(x) \, dm_n(x) \quad \text{we will simply write} \quad \int_E f(x) \, dx$$

DIRECTLY FROM THE DEFINITION $(*)$ we have the properties

(1) $\forall_{\alpha \geq 0}$ $\displaystyle \int_E \alpha f \, d\mu = \alpha \int_E f \, d\mu$

Indeed,

$$\int_E \alpha f \, d\mu = \sup_{E = \cup E_n} \sum_{n=1}^{\infty} \inf_{x \in E_n} \alpha f(x) \cdot \mu(E_n) = \alpha \sup_{E = \cup E_n} \sum_{n=1}^{\infty} f(x) \cdot \mu(E_n)$$

$$= \alpha \int_E f \, d\mu$$

(2) For two measurable functions $f, g : E \to [0, \infty]$ such that $\forall_{x \in E}$ $f(x) \leq g(x)$ $\quad (f(x) \leq g(x) \ \text{a.e})$

then

$$\int_E f \, d\mu \leq \int_E g \, d\mu \quad \text{since} \ \inf_{x \in E_n} f(x) \leq \inf_{x \in E_n} g(x)$$

③ Mean Value Theorem If $f: E \to [0,\infty]$ is measurable, then

$$\inf_{x\in E} f(x)\,\mu(E) \le \int_E f\,d\mu \le \sup_{x\in E} f(x)\cdot\mu(E)$$

By ② take

$$g_1(x) = \inf_{x\in E} f(x) = a \qquad g_2(x) = \sup_{x\in E} f(x) = b$$

then

$$g_1(x) \le f(x) \le g_2(x) \quad \forall x\in E \quad \text{so ③ follows from ②}$$

④ $$\int_E c\,d\mu = c\,\mu(E), \quad c \geq 0$$

⑤ If $\mu(E)=0$ then $$\int_E f\,d\mu = 0.$$

THEOREM: Under the above assumptions if $f: E \to [0,\infty]$ is a measurable function then the function $\lambda: \mathcal{S}_E \to [0,\infty]$ (here $\mathcal{S}_E := \{A\in\mathcal{S} : A\subset E\}$) defined by

$$\forall_{A\in\mathcal{S}_E} \qquad \lambda(A) := \int_A f\,d\mu$$

is a measure on $\mathcal{S}_E$. In particular if

$$E = \bigcup_{n=1}^{\infty} E_n, \quad E_n\cap E_m = \emptyset, \quad E_n\in\mathcal{S}$$

then

$$\int_E f\,d\mu = \sum_{n=1}^{\infty} \int_{E_n} f\,d\mu \qquad (1)$$

$\lambda(A) \geq 0$ ✓
$\lambda(\emptyset) = 0$ ✓
$\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \lambda(A_n)$
$A_n\cap A_m = \emptyset$
$A_n \in \mathcal{S}_E$

PROOF: We need to prove (1). We start with RHS: $\int_{E_n} f\,d\mu$

$\forall_n$ Take a decomposition $E_n = \bigcup_{k=1}^{\infty} E_{nk}$, $E_{nk}\in\mathcal{S}$, $E_{nk}\cap E_{nm} = \emptyset$ $k\neq m$

Then

$$E = \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} E_{nk} \qquad E = \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} E_{nk}.$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\inf_{x\in E_{nk}} f(x)\right)\cdot\mu(E_{nk}) \le \sup_{E=\bigcup E_n'} \sum_{n=1}^{\infty} \inf_{x\in E_n'} f(x)\cdot\mu(E_n')$$

$$= \int_E f\,d\mu$$

$\|$

$$= \int_E f\,d\mu$$

$$\|$$

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\inf_{x\in E_{nk}} f(x)\right) \cdot \mu(E_{nk})$$

$$\forall_m \qquad \sum_{n=1}^{m} \sum_{k=1}^{\infty} \inf_{x\in E_{nk}} f(x) \cdot \mu(E_{nk}) \quad (\geq)$$

for every every partition $E_n = \bigcup E$

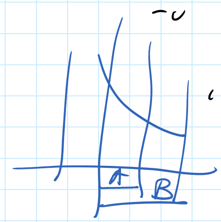
$$\sum_{n=1}^{m} \sum_{k=1}^{\infty} \inf_{x\in E_{nk}} f(x) \cdot \mu(E_{nk}) \leq \int_E f\,d\mu$$

$$\sum_{n=1}^{m} \sup_{E_n = \bigcup E_{nk}} \sum_{k=1}^{\infty} \inf f(x) \cdot \mu(E_{nk}) \leq \int_E f\,d\mu$$

$$\sum_{n=1}^{m} \int_{E_n} f\,d\mu \leq \int_E f\,d\mu$$

thus by passing to the limit $m \to \infty$

$$\sum_{n=1}^{\infty} \int_{E_n} f\,d\mu \leq \int_E f\,d\mu .$$



The reverse inequality follows from

$$\sum_{n=1}^{\infty} \inf_{x\in E_n} f(x)\, \mu(E_n) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \inf_{x\in E_{nk}} f(x) \cdot \mu(E_{nk})$$

$$\leq \sum_{n=1}^{\infty} \int_{E_n} f\,d\mu .$$

$\square$

$A \subset B$

$$\inf_A f(x) \geq \inf f(x)$$

$$\inf_{E_1 \cup E_2} f(x) \cdot \mu(E_1 \cup E_2)$$

$$= \inf_{E_1\cup E_2} f(x) \cdot \mu(E_1) + \inf_{E_1 \cup E_2} f(x)\, \mu(E_2)$$

$$\leq \inf_{E_1} f \cdot \mu(E_1) + \inf_{E_2} f \cdot \mu(E_2)$$