LECTURE #1

REAL ANALYSIS

I. PRELIMINARIES.

1. REVIEW OF ELEMENTARY ANALYSIS:

1a: LOGIC

Logical sentence: p, q, r, s

$0 \equiv$ FALSE
$1 \equiv$ TRUE

SENTENTIAL CONNECTIVES:

Negation	$\sim p$	"NOT p"
conjunction	$p \wedge q$	"p AND q"
disjunction	$p \vee q$	"p OR q"
implication	$p \Rightarrow q$	"p IMPLIES q"
equivalence	$p \Leftrightarrow q$	"p IF AND ONLY IF q"

LAWS OF LOGIC

1. $\sim\sim p \Leftrightarrow p$ double negation
2. $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$
3. $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$ } De Morgan's
4. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
5. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ } distributive
6. $p \Rightarrow q \Leftrightarrow \sim q \Rightarrow \sim p$ contrapositive.
7. $p \Rightarrow q \Leftrightarrow \sim p \vee q$

QUANTIFIERS:

universal quantifier $\forall_x\, p(x)$: "FOR EVERY x" $p(x)$ is TRUE

existential quantifier $\exists_x\, q(x)$: "THERE EXISTS x" $q(x)$ is TRUE.

LAWS OF QUANTIFIERS

1) $\sim \exists_x\, p(x) \Leftrightarrow \forall_x \sim p(x)$

$\sim \forall_x\, p(x) \Leftrightarrow \exists_x \sim p(x)$

2) $\underset{x}{\exists}\,\underset{y}{\exists}\, p(x,y) \Leftrightarrow \underset{y}{\exists}\,\underset{x}{\exists}\, p(x,y)$, $\underset{x}{\exists}\, p(x) \Leftrightarrow \underset{t}{\exists}\, p(t)$

$\underset{x}{\forall}\,\underset{y}{\forall}\, p(x,y) \Leftrightarrow \underset{y}{\forall}\,\underset{x}{\forall}\, p(x,y)$ $\underset{x}{\forall}\, p(x) \Leftrightarrow \underset{t}{\forall}\, p(t)$

$$\forall_x \forall_y \, p(x,y) \iff \forall_y \forall_x \, p(x,y) \qquad \forall_x p(x) \iff \forall_t p(t)$$

3) $$\exists_x \, p(x) \vee q(x) \iff \exists_x p(x) \vee \exists_x q(x) \iff \exists_x p(x) \vee \exists_y q(y)$$

$$\forall_x \, p(x) \wedge q(x) \iff \forall_x p(x) \wedge \forall_x q(x)$$

4) $$\exists_x \, p(x) \wedge q(x) \implies \exists_x p(x) \wedge \exists_x q(x)$$

$$\forall_x p(x) \vee \forall_x q(x) \implies \forall_x \, p(x) \vee q(x)$$

5) $$\exists_x \, p \vee q(x) \iff p \vee \exists_x q(x) \quad \Leftarrow$$

$$\exists_x \, p \wedge q(x) \iff p \wedge \exists_x q(x)$$

$$\forall_x \, p \vee q(x) \iff p \vee \forall_x q(x)$$

$$\forall_x \, p \wedge q(x) \iff p \wedge \forall_x q(x)$$

6) $$\exists_x \, (p \Rightarrow q(x)) \iff p \Rightarrow \exists_x q(x) \qquad p \Rightarrow q \iff \sim p \vee q$$

$$\exists_x (p(x) \Rightarrow q) \iff \exists_x (\sim p(x) \vee q) \iff \left(\exists_x \sim p(x)\right) \vee q$$

$$\iff \sim \forall_x p(x) \vee q$$

$$\iff \forall_x p(x) \Rightarrow q$$

$$\forall_{p(x)} q(x) \overset{def}{\iff} \forall_x (p(x) \Rightarrow q(x))$$

$$\exists_{p(x)} q(x) \overset{def}{\iff} \exists_x (p(x) \wedge q(x))$$

REMARK All the properties (1) — (6) are valid for the quantified sentences where $\forall_x$ or $\exists_x$ is replaced by $\forall_{r(x)}$ or $\exists_{r(x)}$

1b ELEMENTARY SET THEORY

$\emptyset$ is empty set

X is a set, $x \in X$ — x is an element of X, x is in X, x belongs to X

$$X = \{x_1, x_2, \dots, x_n\}$$

x is in X

x belongs to X

$x \notin X$ x is not an element of X

$\Updownarrow$

$\sim (x \in X)$

$$X = \{x_1, x_2, \dots, x_n\}$$
$$\phi = \{\ \}$$
$$X = \{x : p(x)\}$$

It is convenient to consider a given set X as the space (of our interest) and deal with the subsets $A, B, C, \dots$ of X.

$A \subset B$ i.e. A is a subset of B

$$A \subset B \overset{\text{def}}{\Longleftrightarrow} \forall_x (x \in A \Rightarrow x \in B)$$ inclusion of sets.

$x \in A \cap B \overset{\text{def}}{\Longleftrightarrow} x \in A \wedge x \in B$

intersection

$x \in A \cup B \Longleftrightarrow x \in A \vee x \in B$

union

$x \in A \setminus B \Longleftrightarrow x \in A \wedge x \notin B$

Now assume that A, B are sets in the space X

$x \in A^c \Longleftrightarrow x \in X \wedge x \notin A$ $\qquad A^c = X \setminus A$

complement

By laws of logic we have the immediate properties of sets

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup A^c = X$$

(2) $(A \cup B)^c = A^c \cap B^c \quad (*)$

$$(A \cap B)^c = A^c \cup B^c$$

EXAMPLE:

$$(A \cup (B \cap C))^c \overset{(*)}{=} A^c \cap (B \cap C)^c$$
$$= A^c \cap (B^c \cup C^c)$$

PRODUCT SET

Let X and Y be two sets

$X \times Y := \{(x,y) : x \in X \wedge y \in Y\}$ Cartesian product.

where the pair (x,y) is defined as the set

$$(x,y) := \{\{x\}, \{x,y\}\}$$

RELATION

Definition For two sets X and Y a subset $\mathcal{R} \subset X \times Y$ is called a relation between elements of X and Y.

Then we will write

$$x \mathcal{R} y \overset{\text{def}}{\iff} (x,y) \in \mathcal{R}$$

x is in relation $\mathcal{R}$ with y

$\emptyset \times X = \emptyset$

Definition: Let X be a set and $\mathcal{R} \subset X \times X$ be a relation.
Then we say that

(i) $\mathcal{R}$ is reflexive $\underset{\text{iff}}{\iff} \forall_{x \in X}\ x \mathcal{R} x$

(ii) $\mathcal{R}$ is symmetric $\iff \forall_{x,y \in X}\ x \mathcal{R} y \Rightarrow y \mathcal{R} x$

(iii) $\mathcal{R}$ is transitive $\iff$

$$\forall_{x,y,z \in X}\ x \mathcal{R} y \wedge y \mathcal{R} z \Rightarrow x \mathcal{R} z$$

A reflexive, symmetric, transitive relation $\mathcal{R} \subset X \times X$ is called an equivalence relation. Then one can define the so-called equivalence class $[x]_{\mathcal{R}}$ of $x \in X$ by

$$[x]_{\mathcal{R}} := \{ y \in X : x \mathcal{R} y \}$$

Then notice that.

(i) $[x]_{\mathcal{R}} \cap [y]_{\mathcal{R}} \neq \emptyset \Rightarrow x \mathcal{R} y$ and $[x]_{\mathcal{R}} = [y]_{\mathcal{R}}$

(ii)

$$X = \bigcup_{x \in X} [x]_{\mathcal{R}}$$

Then, it is convenient to introduce the notion $:=$

$$X/\mathcal{R} := \{ [x]_{\mathcal{R}} : x \in \mathcal{R} \}$$

Example: Let $A \subset X$ be a given set. We define the relation $\mathcal{R} \subset X \times X$ by

$$x \mathcal{R} y \overset{\text{def}}{\iff} \begin{cases} x, y \in A & \text{or} \\ x = y \end{cases}$$

One can easily verify that $\mathcal{R}$ is an equivalence relation. and

$$[x]_{\mathcal{R}} = \begin{cases} \{x\} & \text{if } x \notin A \\ A & \text{if } x \in A \end{cases}$$

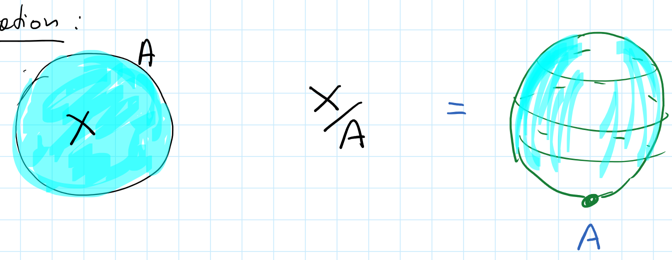
$$[x]_R = \begin{cases} \{x\} & \text{if } x \notin A \\ A & \text{if } x \in A \end{cases}$$

Then, we put

(:= def)

$$X/_A := X/_R$$ and call it the quotient set.

Illustration:



Definition Let X be a set $\mathcal{R} \subset X \times X$ be a relation. We say that $\mathcal{R}$ is anti-symmetric

$$\forall_{x,y} \quad x \mathcal{R} y \wedge y \mathcal{R} x \Rightarrow x = y$$

A relation $\mathcal{R} \subset X \times X$ which is reflexive, transitive and anti-symmetric is called an order (partial order) relation in X.

For more intuitive notation, we denote such a relation $\mathcal{R}$ by $\leq$ i.e

$$x \leq y \overset{def}{\Longleftrightarrow} x \mathcal{R} y$$

Strict order relation is denoted by $<$ and we have

$$x < y \overset{def}{\Longleftrightarrow} x \leq y \wedge x \neq y.$$

A set X together with a given order relation $\leq$ will be called an ordered space, which will be denoted by $(X, \leq)$

Zorn Lemma (Kuratowski-Zorn lemma)

Given $(X, \leq)$ an ordered space. A subset $C \subset X$ is called a chain if C is totally ordered with respect to $\leq$, i.e

$$\forall_{x,y \in C} \quad x \leq y \vee y \leq x$$

For a give subset $S \subset X$, an element $u \in X$ is called an upper bound of S iff

$\forall$

$$\forall_{x \in S} \quad x \leq u$$

An element $v \in X$ is called maximal in X iff

$$\forall_{x \in X} \quad x \geq v \Rightarrow x = v$$

THEOREM (Zorn's Lemma)

Let $(X, \leq)$ be an ordered set such that every chain in X has an upper bound. Then X contains at least one maximal element.