University of Texas at Dallas

Assignment #2

Last Name:	**First Name and Initial:** ....
Course Name: Real Analysis 1	**Number:** MATH 6301
Instructor: Wieslaw Krawcewicz	**Due Date:** September 15, 2022
E-mail Address:	**Student's Signature:** ....

Instructions:

1. Print this booklet
2. Use the space provided to write your solutions in this booklet
3. Hand in your assignment to your instructor on the due date during the class time.

Question	Weight	Your Score	Comments
1.	10		
2.	10		
3.	10		
4.	10		
5.	10		
Total:	50		

Problem 1. Given two metric spaces (X,d) and (Y,ρ), $a \in X$ and a function $f : X \setminus \{a\} \to Y$. We denote by $A_a(f)$ the set of all accumulation values of f at a. Show that $A_a(f)$ is a closed set.

SOLUTION:

Problem 2: Assume that (X, d) is a metric space, $a \in X$ and $f : X \setminus \{a\} \to \mathbb{R}$ is a function. Put for $\delta > 0$

$$C_\delta(a) := \{x \in X : 0 < d(x, a) < \delta\}.$$

Show that

$$\limsup_{x \to a} = \inf_{\delta > 0} \sup_{x \in C_\delta(a)} f(x),$$
$$\liminf_{x \to a} = \sup_{\delta > 0} \inf_{x \in C_\delta(a)} f(x).$$

SOLUTION:

Problem 3: Given a metric space (X, d), $a \in X$ and a function $f : X \setminus \{a\} \to \mathbb{R}$ such that $A_a(f) \neq \emptyset$ is also bounded. Show that

(a) $\limsup\limits_{x\to a} f(x) < \alpha$ for some $\alpha \in \mathbb{R}$ if and only if

$$\exists_{\delta>0}\exists_{\overline{\alpha}<\alpha}\forall_{x\in X}\ \ 0 < d(x,a) < \delta \quad\Rightarrow\quad f(x) \leq \overline{\alpha};$$

(b) $\liminf\limits_{x\to a} f(x) > \alpha$ for some $\alpha \in \mathbb{R}$ if and only if

$$\exists_{\delta>0}\exists_{\overline{\alpha}>\alpha}\forall_{x\in X}\ \ 0 < d(x,a) < \delta \quad\Rightarrow\quad f(x) \geq \overline{\alpha};$$

(c) $\limsup\limits_{x\to a} f(x) \leq \alpha$ for some $\alpha \in \mathbb{R}$ if and only if

$$\forall_{\alpha'>\alpha}\exists_{\delta>0}\forall_{x\in X}\ \ 0 < d(x,a) < \delta \quad\Rightarrow\quad f(x) \leq \alpha';$$

(d) $\liminf\limits_{x\to a} f(x) \leq \alpha$ for some $\alpha \in \mathbb{R}$ if and only if

$$\forall_{\alpha'<\alpha}\exists_{\delta>0}\forall_{x\in X}\ \ 0 < d(x,a) < \delta \quad\Rightarrow\quad f(x) \geq \alpha';$$

SOLUTION:

Problem 4: Given two metric spaces (X, d) and (Y, ρ), $a \in X$ and a function $f : X \to Y$. Show that f is continuous at $a \in X$ if and only if

$$A_a(f) = \{f(a)\}.$$

SOLUTION:

Problem 5: Denote by $\overline{\mathbb{R}}$ the ordered set of extended real numbers $\{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ and define the function $\varphi : \overline{\mathbb{R}} \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} \arctan x & \text{if } x \in \mathbb{R}, \\ \pm\frac{\pi}{2} & \text{if } x = \pm\infty, \end{cases}$$

and the function $d : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \mathbb{R}$ by

$$d(x, y) := |\varphi(x) - \varphi(y)|$$

Show that

(a) the function d is a metric on $\overline{\mathbb{R}}$;

(b) the topology $\mathscr{T}$ induced by the metric d on $\overline{\mathbb{R}}$, restricted to $\mathbb{R}$, coincide with the usual topology on $\mathbb{R}$.

SOLUTION: