LECTURE 26 – MATH 6301

FUNDAMENTAL PROPERTIES OF INTEGRAL

THEOREM (FATOU's LEMMA) Suppose that $(X, \mathfrak{S}, \mu)$ is a measure space, $E \in \mathfrak{S}$ and $f_n : E \to [0, \infty]$, $n = 1, 2, 3, \dots$, are μ-measurable functions. Then

$$\int_E \liminf_{n\to\infty} f_n(x)\, d\mu(x) \le \liminf_{n\to\infty} \int_E f_n\, d\mu$$

PROOF: Notice that the functions

$$g_n(x) := \inf_{k \geq n} f_k(x), \qquad n \in \mathbb{N},\ x \in E$$

is μ-measurable and

$$f_n(x) \geq \inf_{k\geq n} f_k(x) = g_n(x) \qquad \forall x \in E$$

Moreover, for all $n \in \mathbb{N}$

(1) $$0 \le g_n(x) \le g_{n+1}(x) \qquad \forall x \in E$$

and

(2) $$\lim_{n\to\infty} g_n(x) = \liminf_{n\to\infty} f_n(x) \qquad \forall x \in E$$

Therefore, one has

$$\int_E \liminf_{n\to\infty} f_n(x)\, d\mu(x) \overset{(2)}{=} \int_E \lim_{n\to\infty} g_n(x)\, d\mu(x) \overset{LMCT}{=} \lim_{n\to\infty} \int_E g_n(x)\, d\mu(x)$$

$$\overset{(1)}{\le} \liminf_{n\to\infty} \int_E f_n\, d\mu. \qquad \square$$

THEOREM (LEBESGUE DOMINATED "CONVERGENCE" THEOREM: 1)

If $f_n : E \to \mathbb{R}$, $n = 1, 2, \dots$, are summable functions such that for some summable function $g : E \to \overline{\mathbb{R}}$ we have $(g(x) \geq 0)$

$$\forall_{n\in\mathbb{N}} \ \forall_{x \in E} \quad |f_n(x)| \le g(x) \qquad (*)$$

Then, one has

$$(\star) \quad \underbrace{\int_E \liminf_{n\to\infty} f_n(x)\, d\mu(x) \le \liminf_{n\to\infty} \int_E f_n\, d\mu}_{(1)!} \le \limsup_{n\to\infty} \int_E f_n\, d\mu \le \int_E \limsup_{n\to\infty} f_n(x)\, d\mu(x)$$

THEOREM: LEBESGUE DOMINATED CONVERGENCE THEOREM, ② (μ is complete)

If $f_n : E \to \overline{\mathbb{R}}$, $n = 1, 2, \dots$, are summable functions, $f : E \to \overline{\mathbb{R}}$, are such that

(a) $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. $x \in E$;

(b) there is a summable function $g : E \to \overline{\mathbb{R}}$ satisfying

$$\forall_{n\in\mathbb{N}} \ \forall_{x\in E} \quad |f_n(x)| \le g(x).$$

Then, f is summable and

$$\lim_{n\to\infty} \int_E f_n(x)\, d\mu(x) = \int_E f\, d\mu.$$

PROOFS: Notice that, since $|f_n(x)| \le g(x)$ then we have:

(i) $g(x) + f_n(x) \ge 0 \quad \forall x \in E$ and $g + f_n$ is μ-measurable

(ii) $g(x) - f_n(x) \ge 0 \quad \forall x \in E$ and $g - f_n$ is μ-measurable

Thus we can apply the Fatou Lemma to those functions

(i)

$$\int_E \liminf_{n\to\infty} (g(x) + f_n(x))\, d\mu(x) \le \liminf_{n\to\infty} \int_E (g + f_n)\, d\mu$$

$$\int_E \liminf_{n\to\infty} (g(x) + f_n(x))\, d\mu(x) = \int_E \left(g + \liminf_{n\to\infty} f_n(x)\right) d\mu(x) = \int_E g\, d\mu + \int_E \liminf_{n\to\infty} f_n(x)\, d\mu(x)$$

$$\liminf_{n\to\infty} \int_E (g + f_n)\, d\mu \ge \liminf_{n\to\infty} \left[\int_E g\, d\mu + \int_E f_n\, d\mu\right] = \int_E g\, d\mu + \liminf_{n\to\infty} \int_E f_n\, d\mu$$

which implies

$$\int_E \liminf_{n\to\infty} f_n(x)\,d\mu(x) \le \liminf_{n\to\infty} \int_E f_n d\mu \le \quad (3).$$

Again by Fatou's lemma applied to $g-f_n$ we obtain

(ii)
$$\int_E \liminf_{n\to\infty} (g(x)-f_n(x))\,d\mu(x) \le \liminf_{n\to\infty} \int_E (g-f_n)\,d\mu$$

$$\int_E \left(g(x) + \liminf_{n\to\infty} (-f_n(x))\right) d\mu(x) \qquad\qquad \int_E g\,d\mu - \limsup_{n\to\infty} \int_E f_n\,d\mu.$$

$$\int_E g\,d\mu - \int_E \limsup_{n\to\infty} f_n(x)\,d\mu(x)$$

which implies

$$\le \limsup_{n\to\infty} \int_E f_n\,d\mu \le \int_E \limsup_{n\to\infty} f_n(x)\,d\mu(x) \qquad (4)$$

Then, by (3) & (4) we have the required inequalities,

For the proof of LDCT (2) simply notice that, since $f_n(x) \to f(x)$ thus

$$\liminf_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) = f(x) \quad \text{a.e.}$$

and by (★) we have

$$\lim_{n\to\infty} \int_E f_n\,d\mu = \int_E f\,d\mu$$

THEOREM: LEBESGUE DOMINATED CONVERGENCE THEOREM (3)

Suppose $f_n, f : E \to \overline{\mathbb{R}}$, $n=1,2,\dots$, are μ-measurable functions such that

(a) there is a summable function $g : E \to [0,\infty]$ and $\forall_{n\in\mathbb{N}} \forall_{x\in E} \; |f_n(x)| \le g(x)$

(b) $f_n \xrightarrow{\mu} f$

Then f is summable and

$$\lim_{n\to\infty} \int_E f_n\,d\mu = \int_E f\,d\mu$$

PROOF: Define

$$a_n := \int_E f_n\,d\mu$$

| $g+f_n \ge 0$ summable
| $g-f_n \ge 0$ summable.

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$$a_n := \int_E f_n \, d\mu$$

$g + f_n \geq 0$ summable
$g - f_n \geq 0$ summable.

By Riesz-Theorem, since $f_n \xrightarrow{\mu} f$, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$f_{n_k}(x) \longrightarrow f(x) \quad \text{a.e } x \in E.$$

which implies, by LDCT (2), that $a_{n_k} \to \int_E f \, d\mu$, as $k \to \infty$.

Therefore,

$$\lim_{n\to\infty} a_n = \int_E f \, d\mu. \quad ! \quad \square$$

$$a_{n_l} \to b \neq \int_E f \, d\mu$$

Some remarks:

COROLLARY: If $f_n : E \longrightarrow \mathbb{R}^n$ are μ-measurable functions for $n = 1, 2, \ldots$, such that

$$\sum_{n=1}^{\infty} \int_E |f_n| \, d\mu < \infty$$

Then

$$\sum_{n=1}^{\infty} \int_E f_n \, d\mu = \int_E \sum_{n=1}^{\infty} f_n \, d\mu.$$

PROOF:

$$F_n(x) := \sum_{k=1}^{n} f_k(x) \qquad |F_n(x)| \leq \sum_{k=1}^{n} |f_k(x)| \leq \sum_{k=1}^{\infty} |f_k(x)| = g(x)$$

$$\sum_{k=1}^{n} |f_k(x)| \xrightarrow[n\to\infty]{} g(x) \quad x \in E$$

and.

$$\int_E \sum_{k=1}^{\infty} |f_k| \, d\mu = \sum_{k=1}^{\infty} \int |f_k| \, d\mu$$

On the other hand.

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x) = \sum_{k=1}^{\infty} f_k(x) \quad \text{a.e. } x \in E$$

and the conclusion follows from LDCT (2). $\square$

THEOREM Let $f_n, f : E \to \overline{\mathbb{R}}$ be summable functions, $n = 1, 2, \ldots$, where $\mu(E) < \infty$, and assume that

(a)

$$\lim_{\mu(A)\to 0} \sup_{n\in\mathbb{N}} \int_A |f_n| \, d\mu = 0$$

(a) $$\lim_{\mu(A)\to 0} \sup_{n\in\mathbb{N}} \int_A |f_n|\,d\mu = 0$$

(b) $$f_n \xrightarrow{\mu} f \text{ on } E.$$ (resp. $f_n(x) \longrightarrow f(x)$ a.e. $\Downarrow$ $f_n \xrightarrow{\mu} f$.)

Then

$$\lim_{n\to\infty} \int_E f_n\,d\mu = \int_E f\,d\mu$$

PROOF: By Absolute Continuity of Integral we have

$$\lim_{\mu(A)\to 0} \int_A |f|\,d\mu = 0 \quad \text{we have}$$

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \mu(A)<\delta,\ A\subset E \Rightarrow \int_A |f|\,d\mu < \frac{\varepsilon}{3}$$

and notice that by condition (a) we also have

$$\forall_n\ \forall_{\varepsilon>0}\ \exists_{\delta>0}\ \mu(A)<\delta,\ A\subset E \Rightarrow \int_A |f_n|\,d\mu < \frac{\varepsilon}{3}$$

Put

$$E_n = \{x : |f_n(x) - f(x)| \geq \underbrace{\frac{\varepsilon}{3\mu(E)}}_{\varepsilon'} \}$$

then

$$\lim_{n\to\infty} \mu(E_n) = 0 \quad \text{so}$$

$$\exists_N\ \forall_{n\geq N}\ \mu(E_n) < \delta,$$

which implies

$$(5) \quad \int_{E_n} |f_n - f|\,d\mu \leq \int_{E_n} |f_n|\,d\mu + \int_{E_n} |f|\,d\mu < \frac{2\varepsilon}{3}$$

and notice that for $n\geq N$

$$(6) \quad \int_{E\setminus E_n} |f_n - f|\,d\mu \leq \frac{\varepsilon}{3\mu(E)} \int_{E\setminus E_n} d\mu = \frac{\varepsilon}{3\mu(E)}\,\mu(E\setminus E_n) \leq \frac{\varepsilon}{3\mu(E)}\,\mu(E) = \frac{\varepsilon}{3}$$

Consequently, by (5) and (6) we have

$$\forall_{\varepsilon>0}\ \exists_N\ \forall_{n\geq N} \quad \left| \int_E f_n\,d\mu - \int_E f\,d\mu \right| \leq \int_E |f_n - f|\,d\mu \leq$$

$$\leq \int_{E_n} |f_n - f|\,d\mu + \int_{E\setminus E_n} |f_n - f|\,d\mu$$

$$< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \qquad \square$$

$$< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = 1 \qquad \square$$

<u>EXAMPLE</u> Condition $\mu(E) < \infty$ cannot be removed.

Take $E = (0, \infty)$ $\mu = m_1$ $f_n = \frac{1}{n} \chi_{(0,n)}$

Then $\forall_n \int_A |f_n| dx \le m(A) \longrightarrow 0$.

However $f_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$ but

$$\int_E f_n \, dm = \int_0^n \tfrac{1}{n} dx = 1 \not\longrightarrow 0 \qquad \square$$