LECTURE 3 — MATH 6301

Cardinality: $|X| \leq |Y| \overset{\text{def}}{\iff} \underset{f:X\to Y}{\exists}$ f is injective.

THEOREM: For two sets X and Y we have

$$|X| \leq |Y| \iff \underset{g:Y\to X}{\exists} \quad g \text{ is surjective}$$

PROOF: $\Rightarrow$ If $|X| \leq |Y|$ then $\underset{\substack{f:X\to Y\\ \text{injective}}}{\exists}$. Choose $x_0 \in X$ and define

$g: Y \to X$

$$g(y) := \begin{cases} f^{-1}(y) & y \in f(X) \\ x_0 & y \notin f(X). \end{cases} \qquad f: X \underset{\text{bijection}}{\overset{f^{-1}}{\rightleftarrows}} f(X) \subset Y$$

Clearly, g is surjective

$\Leftarrow$ Let $g: Y \to X$ be a surjective map $(g(Y) = X)$. Define for $x \in X$ the set $A_x := g^{-1}(x)$

1) $A_x \neq \emptyset$ ✓

2) $x' \neq x \Rightarrow A_{x'} \cap A_x = \emptyset$

3) $\bigcup\limits_{x\in X} A_x = Y$

By axiom of choice, there exists a function $f: X \to Y$ such that

$$\forall_{x\in X} \; f(x) \in A_x.$$

By 2) we have that $x' \neq x \Rightarrow f(x') \neq f(x)$, so f is injective and the conclusion follows. $\square$

THEOREM: (CANTOR)

Let X be a set and $\mathcal{P}(X)$ its power set. Then

$$|\mathcal{P}(X)| > |X|$$

PROOF For $X = \emptyset$ the statement is obvious.

Assume $X \neq \emptyset$ and notice that there is an injective function $f: X \to \mathcal{P}(X)$ given by

$$f(x) = \{x\}$$

so we have that $|X| \leq |\mathcal{P}(X)|$

Assume for contradiction that $|\mathcal{P}(X)| \leq |X|$, then (by def) there exists an injective $g : \mathcal{P}(X) \to X$, and $A = g(\mathcal{P}(X)) \subset X$

and clearly $g : \mathcal{P}(X) \to A$ is a bijection, so we put

$$h = g^{-1} : A \to \mathcal{P}(X)$$

Then $\forall_{a \in A}$ $h(a) \subset X$ (is a subset of X) then we have two possibilities either $a \in h(a)$ or $a \notin h(a)$

Define the subset $Z \subset X$ by

$$\boxed{Z = \{a \in A : a \notin h(a)\}}, \quad Z \subset A \subset X, \; Z \in \mathcal{P}(X)$$

so by assumption $g(Z) = x_0$ for some $x_0 \in A$, or we have $h(x_0) = Z$. Notice that

if $x_0 \in h(x_0) = Z$, then $x_0 \notin h(x_0)$, so this is a contradiction

or

if $x_0 \notin h(x_0) = Z$ then $x_0 \in Z$ (by definition of Z) and we obtain again a contradiction. $\square$

REAL NUMBERS: $\mathbb{R}$

Axioms:

Axioms of a field: $\forall_{x,y,z \in \mathbb{R}}$, $0 \in \mathbb{R}$ specified element, $1 \in \mathbb{R}$ specified element

(a1) $(x+y)+z = x+(y+z)$

(a2) $x+y = y+x$

(a3) $x+0 = 0+x = x$

(a4) $\forall_x \exists_{-x}$ $x+(-x) = 0$ additive inverse

(b1) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

(b2) $x\,y = y\,x$

(b3) $x \cdot 1 = 1 \cdot x = x$

(b4) $\forall_{x \neq 0} \exists_{x^{-1}}$ $x \cdot x^{-1} = 1$ multiplicative inverse

(d) $x(y+z) = xy + xz$

$(\mathbb{R}, +, \cdot, 0, 1)$ field.

Example: $\mathbb{Q}[p] := \{a + b\sqrt{p} : a, b \in \mathbb{Q}\}$
p prime

Axioms of Order: $\mathbb{R}$ admits a total order, satisfying $\forall_{x,y,z \in \mathbb{R}}$

(o1) $x > y \Rightarrow x+z > y+z$

(o2) $x > y \wedge z > 0 \Rightarrow xz > yz$

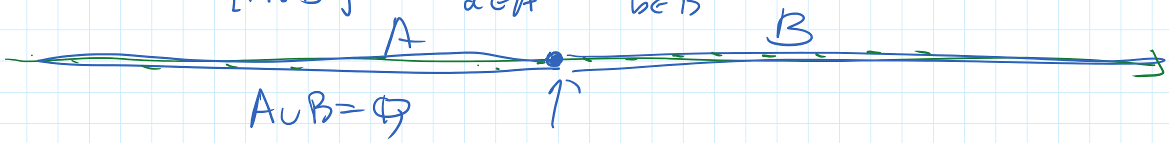
Axiom of Completeness For any bounded from above set $A \subset \mathbb{R}$ there exists $\sup A \in \mathbb{R}$

- $A \subset \mathbb{R}$ bounded from above means $\exists$ ... $\forall$ $a \leq b$

There exists $\sup A \in \mathbb{R}$

- $A \subset \mathbb{R}$ <u>bounded from above</u> means $\exists_{b \in \mathbb{R}} \forall_{a \in A}\ a \leq b$
- $A \subset \mathbb{R}$ bounded from below $\exists_{c \in \mathbb{R}} \forall_{a \in A}\ c \leq a$
- $x = \sup A \iff$ 1) $\forall_{a \in A}\ a \leq x$
  2) $\forall_{\varepsilon > 0} \exists_{a \in A}\ a > x - \varepsilon$
- $y = \inf A \iff$ 1) $\forall_{a \in A}\ a \geq y$
  2) $\forall_{\varepsilon > 0} \exists_{a \in A}\ a < y + \varepsilon$

$[A:B]$ $\forall_{a \in A}$ $\forall_{b \in B}$ $a < b$



2. METRIC SPACES

<u>DEFINITION</u>: A set X and a function $d : X \times X \to \mathbb{R}$ satisfying

(d1) $d(x,y) \geq 0$ and $d(x,y) = 0 \iff x = y$

(d2) $d(x,y) = d(y,x)$

(d3) $d(x,y) \leq d(x,z) + d(z,y)$

is called a metric space. We will write (X, d) to indicate that X is a metric space with the metric d.

<u>QUESTION</u>: Is $\emptyset$ a metric space?

$f : \underset{X}{\emptyset} \to Y$

$\emptyset \in \emptyset \times \emptyset$

$\forall_{x \in X} \exists!_{y \in Y}\ \underline{\underline{(x,y) \in f}}$

$\forall_x\ x \in \underbrace{X}_{\emptyset} \Rightarrow \exists!_{y \in Y}\ (x,y) \in f$

0

(X, d) $\quad x_0 \in X,\ \varepsilon > 0$

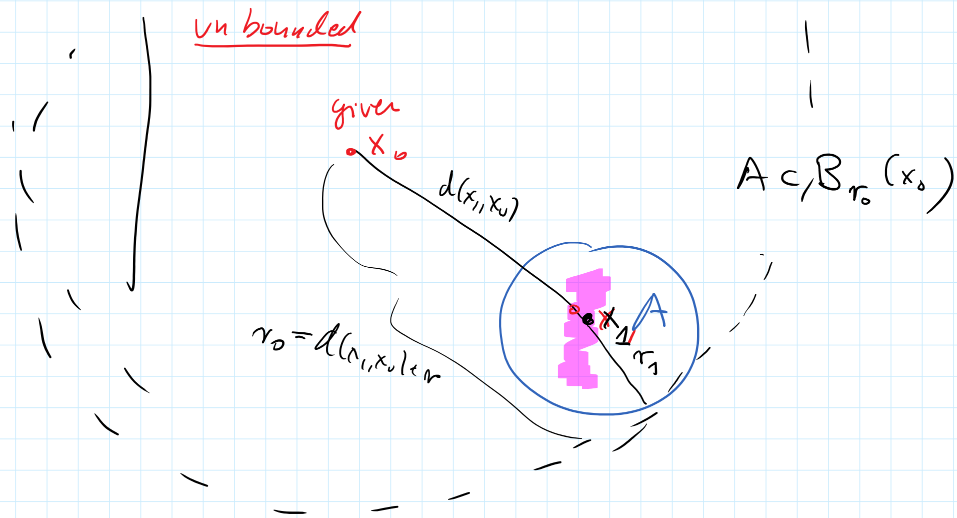
$B_\varepsilon(x_0) := \{x \in X : d(x, x_0) < \varepsilon\}$ open ball.

ε- neighborhood of x_0

DEFINITION: A set $\mathcal{U} \subset X$ is called open iff
$$\forall_{x \in \mathcal{U}} \ \exists_{\varepsilon > 0} \ B_\varepsilon(x) \subset \mathcal{U}$$

DEFINITION $A \subset X$ is bounded iff given $x_0 \in X$ $\exists_{r>0}$ $A \subset B_r(x_0)$

Example: $(\mathbb{R}, d)$; $d(x,y) = |x-y|$ is a metric space.



The set $\mathcal{T} \subset \mathcal{P}(X)$ consisting of all open sets is called a topology on X.

Properties of $\mathcal{T}$:
1) $X, \emptyset \in \mathcal{T}$
2) $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_n \in \mathcal{T} \Rightarrow \mathcal{U}_1 \cap \mathcal{U}_2 \cap \dots \cap \mathcal{U}_n \in \mathcal{T}$
3) $\{\mathcal{U}_i\}_{i \in I} \subset \mathcal{T} \Rightarrow \bigcup_{i \in I} \mathcal{U}_i \in \mathcal{T}$

DEFINITION: $S \subset X$ is closed iff $S^c := X \setminus S$ is open

DEFINITION: Let $A \subset X$, we put
$$\operatorname{diam}(A) = \sup\{d(x,y) : x, y \in A\}$$
and call it diameter of A.

PROPOSITION: $A \subset X$ is bounded $\Leftrightarrow$ $\operatorname{diam}(A) < \infty$

PROOF $\Rightarrow$ if $A \subset B_r(x_0) \Rightarrow \forall_{x,y \in A} \ d(x,y) \leq d(x, x_0) + d(y, y_0) < 2r$

$$\Rightarrow \operatorname{diam}(A) \leq 2r$$

$\Leftarrow$ $\operatorname{diam}(A) = r \Rightarrow \forall_{x,y \in A} \ d(x,y) \leq r$,

$\Rightarrow$ For given $x_0 \in A$ $\quad \forall_{y \in A} \quad d(x_0, y) \le r < r+1$

$$\Rightarrow \quad A \subset B_{r+1}(x_0) \quad \square$$

OPERATIONS ON SETS

Given $A \subset X$

$$\text{int}(A) := \{ x \in A : \exists_{\varepsilon > 0} \; B_\varepsilon(x) \subset A \}$$ interior of A

$$\overline{A} := \{ x \in X : \forall_{\varepsilon > 0} \; B_\varepsilon(x) \cap A \neq \emptyset \}$$ closure of A

$$\partial A = \overline{A} \cap \overline{(A^c)} = \{ x \in X : \forall_{\varepsilon > 0} \; B_\varepsilon(x) \cap A \neq \emptyset \neq B_\varepsilon(x) \cap A^c \}$$

boundary of A

PROPERTIES:

(a) $\text{int}(A)$ is an open set, $\quad \text{int}(A) \subset A$

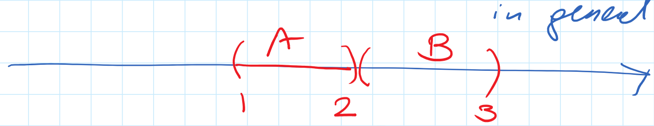
$(\overline{A})$ is a closed set $\quad A \subset \overline{A}$

1) $\text{int}(\text{int}(A)) = \text{int}(A)$ $\qquad$ 1') $\overline{\overline{A}} = \overline{A}$

2) $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$ $\qquad$ 2') $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$

3) $A \subset B \Rightarrow \text{int}(A) \subset \text{int}(B)$ $\qquad$ 3') $A \subset B \Rightarrow \overline{A} \subset \overline{B}$

$$\overline{A \cap B} \overset{\subset}{\neq} \overline{A} \cap \overline{B}$$ in general.



$\overline{A \cap B} = \emptyset \qquad \overline{A} \cap \overline{B} = \{2\}$