Problem 1. Let $A \subset [0,1]$ be a non-measurable set (with respect to the one-dimensional Lebesgue measure), and let $\mu_2(\cdot)$ be the planar Lebesgue measure. Put

$$E := \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leq x, y \leq 1\}$$

Is the subset

$$B := \{(x, 0) \in E \ : \ x \in A\}$$

measurable with respect to $\mu_2(\cdot)$? Justify your answer.

Solution:
Yes. Recall the completeness of the Lebesgue measure: If $C \subset E$ is measurable with $\mu_2(C) = 0$ then for any $B \subset C$,

1. B is $\mu_2(\cdot)$ measurable
2. $\mu_2(B) = 0$

Consider the set $C := \{(x, 0) \in E \ : \ x \in [0,1]\}$ which is a rectangle (and hence measurable) with

i. $\mu_2(C) = 0$

ii. $B \subset C$

Therefore, by completeness of the Lebesgue measure, the set B is measurable with respect to $\mu_2(\cdot)$ and further $\mu_2(B) = 0$. $\square$

Problem 2. Let $I = [0, 1]$ and $A \subset I$. Show that, if $\mu(A) = 1$, then A must be dense in I.

Solution:
Let $[a, b] \subset I$ be any non-degenerate subinterval (in particular with $a < b$) and suppose $A \cap [a, b] = \emptyset$ then,

$$\mu(A \cup [a, b]) = \mu(A) + \mu([a, b]) = 1 + (b - a) > 1$$

However, $A \cup [a, b] \subset I$, and so by semi-σ-additivity of the Lebesgue measure, $\mu(A \cup [a, b]) \leq \mu(I) = 1 \ \rightarrow$ contradiction.
Therefore, for any nonempty subinterval, $[a, b] \subset I$, it must be that

$$A \cap [a, b] \neq \emptyset$$

i.e. A must be dense in I. $\square$

Problem 3. Let $A \subset \mathbb{R}^n$. Show that, if $\mu(A) = 0$, then $int(A) = \emptyset$. Is the converse true?

Solution:
Suppose $\mu(A) = 0$ and let $V \subset A$ be any open subset of A, then by completeness of the Lebesgue measure

$$\mu(V) = 0$$

Now, suppose V is nonempty and take $x \in V$. As V is open, there exists an $\epsilon > 0$ rectangular neighborhood $B_\epsilon(x)$ with

i. $x \in B_\epsilon(x)$

ii. $B_\epsilon(x) \subset V$

iii. $\mu(B_\epsilon(x)) = \epsilon$

However *ii.* and *iii.* contradict semi-σ-additivity of the Lebesgue measure. Therefore, every open subset of A is empty.

i.e. $int(A) = \emptyset$ $\square$

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The converse is NOT true.
Indeed, consider $A = [0,1] \setminus \mathbb{Q}$. As $[0,1] \cap \mathbb{Q}$ is dense in $[0,1]$

$$int(A) = \emptyset$$

However, as A is measurable, its Lebesgue measure coincides with its outer measure and

$$\mu^\star(A) = 1 = \mu(A)$$

Problem 4. Is the following statement true or false?
"If the boundary of $\Omega \subset \mathbb{R}^n$ has outer measure zero, then Ω is measurable."
Justify your answer.

Solution:
The statement is true. Suppose $\mu^\star(\partial\Omega) = 0$. Recall, that for *any* set $A \subset E$,

i. $\mu_\star(A) \leq \mu^\star(A)$

ii. $\mu_\star(A) \geq 0$

So we reason that $\mu_\star(\partial\Omega) = 0$. As the outer and inner measures of $\partial\Omega$ coincide, the set is Lebesgue measurable such that $\mu(\partial\Omega) = 0$
Now, Ω can always be expressed as the union of its interior and the complement of its interior in Ω

$$\Omega = int(\Omega) \cup (\Omega \setminus int(\Omega))$$

The collection of Lebesgue measurable sets in $\mathbb{R}^n$ forms a ring, so if we can show that both $int(\Omega)$ and $\Omega \setminus int(\Omega)$ are Lebesgue measurable, so too will Ω be Lebesuge measurable as their union.
However the measurability of these sets is clear. Indeed, $int(\Omega)$ is measurable (as are all open sets), and $\Omega \setminus int(\Omega)$ as the subset of the measure-zero set $\partial\Omega$ is measurable by completeness of the Lebesgue measure. $\square$

Problem 5. Let $K \subset \mathbb{R}$ be any compact set.
(a) Is K Lebesgue measurable? Justify your answer.
(b) Is K Jordan measurable? Justify your answer.

Solution:
(a) Yes, K is Lebesgue measurable. In any Euclidean space, compactness is equivalent to closedness + boundedness and every closed set is Lebesgue measurable. $\square$

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(b) No, K need NOT be Jordan measurable.
Recall Cantor set which can be defined constructively as follows: Take the unit interval $[0,1]$ and the open set $A_1 := (\frac{1}{3}, \frac{2}{3}) \subset [0,1]$, which is its 'middle third' sub-interval, put $C_1 := [0,1] \setminus A_1$. Next take $A_2 := (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$, which is the union of the middle thirds of each remaining connected sub-component and put $C_2 := [0,1] \setminus A_1 \cup A_2$, continue this process and define $C := [0,1] \setminus \bigcup_{n=1}^{\infty} A_n$, this is the Cantor set. Notice that the the Cantor set is indeed compact as the countable union of closed intervals bounded by $[0,1]$. Now, the Cantor set *is* Jordan measurable and is *not* our counter example but it motivates the correct counter-example to this statement.

First, however, it is important to understand *why* the Cantor set is Jordan measurable. Consider that at each step in its construction, intervals (rectangles) are removed with known lengths such that the total of the lengths removed can be easily shown to be the following convergent geometric series

$$\frac{1}{3} + \frac{2}{9} + \cdots + \frac{2^n}{3^{n+1}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{3}(\frac{2}{3})^n = \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1$$

The complement of the Cantor set in $[0,1]$ is a countable collection of rectangles and as such both $[0,1] \setminus C$ and C are Lebesgue measurable. From the above calculation $\mu([0,1] \setminus C) = 1 \to \mu(C) = 0$. Finally, by construction, the Cantor set coincides with its boundary. Indeed, take $x \in C$ then $\forall_{\epsilon > 0}$ the neighborhood $|y - x| < \epsilon$ will have non empty intersection with some $A_n \subset [0,1] \setminus C$. Therefore $\mu(C) = \mu(\partial C) = 0$, which is equivalent to the Jordan measurability of C.
Now we will construct the so-called Fat Cantor set and reason that, unlike the Cantor set, it is not Jordan measurable. Start again by taking the unit interval $[0,1]$ and its 'middle fourth' $\hat{A}_1 := (\frac{3}{8}, \frac{5}{8})$, put $\hat{C}_1 := [0,1] \setminus \hat{A}_1 = [0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$. Next take the 'middle eighths' of each remaining connected component $\hat{A}_2 := (\frac{5}{32}, \frac{7}{32}) \cup (\frac{25}{32}, \frac{27}{32})$ of length $\frac{1}{16} + \frac{1}{16} = \frac{1}{8}$ and put $\hat{C}_2 := [0,1] \setminus \hat{A}_2$. Continue in this fashion, at each step removing sub-intervals of combined lengths $\frac{1}{4}(\frac{1}{2})^n$, and define $\hat{C} := [0,1] \setminus \bigcup_{n=1}^{\infty} \hat{A}_n$.

Notice that complement has measure $\mu([0,1] \setminus \hat{C}) = \frac{\frac{1}{4}}{1-\frac{1}{2}} = \frac{1}{2}$ such that

$\mu(\hat{C}) = \frac{1}{2}$. As with the Cantor set, the Lebesgue measurability of the complement of $\hat{C}$ in $[0, 1]$ implies Lebesgue measurability of $\hat{C}$, and $\hat{C}$ coincides with its own boundary $\partial\hat{C} = \hat{C}$ such that

$$\mu^{\star}(\partial\hat{C}) = \mu(\partial\hat{C}) \neq 0$$

Therefore, compactness does not guarantee Jordan measurability.

Problem 6. Let A be the graph of the function

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 2022, & x = 0 \end{cases}$$

(a) Is A compact? Justify your answer.
(b) Is A Lebesgue measurable? Justify your answer.

Solution:
(a) No, A is NOT compact.

Indeed, recall that $A \subset \mathbb{R}^2$ is compact if and only if it is closed and bounded, and that A is closed if and only if $\overline{A} = A$. But consider the set of limit points of A (union of accumulation points and isolated points)

$$C := \{(0, y) \in \mathbb{R}^2 \ : \ -1 \leq y \leq 1\} \cup \{(0, 2022)\}$$

Observe that $C \not\subset A$, in fact we have $C \cap A = \{(0, 2022)\}$. Hence, as A is not closed it cannot be compact. $\square$

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(a) Yes, A is Lebesgue measurable.

Let $B := A \cup C$, where C is the set of limit points of A defined above, then $\overline{B} = B$, i.e. B is closed and therefore Lebesgue measurable. Now A can be expressed with respect to B as follows

$$A = (B \setminus C) \cup \{(0, 2022)\}$$

Note that the sets B, C, and $\{(0, 2022)\}$ are each Lebesgue measurable as a closed set, an interval, and a singleton respectively. As the collection of Lebesgue measurable sets in $\mathbb{R}^2$ form a ring it is in particular closed under set union and set minus. Hence, A is measurable and further as $\mu(C) = \mu(\{(0, 2022)\}) = 0$

$$\mu(A) = \mu(B) - \mu(C) + \mu(\{(0, 2022)\}) = \mu(B)$$

Problem 7. Is the set of irrational numbers belonging to the segment $[0,1]$ Jordan measurable? Justify your answer.

Solution:
No, the set described is NOT Jordan measurable.

Recall the criterion for Jordan measurability: $A := [0,1] \setminus \mathbb{Q}$ is Jordan measurable if and only if $\forall_{\epsilon>0}$ there exists elementary sets A_1, A_2 with $A_1 \subset A \subset A_2$ such that $\mu(A_2 \setminus A_1) < \epsilon$
Now, there is no non-empty interval in $[0,1]$ that contains only irrational numbers. Recall: elementary sets are defined as the finite disjoint union of rectangles. Hence, the only elementary sets which can be placed inside A must consist of a finite union of irrational singletons. We conclude for any elementary set $A_1 \in A$, it must be that $\mu(A_1) = 0$
As, for any elementary cover $A \subset A_2$, it must be the case that $A_1 \subset A_2$ such that $\mu(A_2 \setminus A_1) = \mu(A_2) - \mu(A_1)$, the question of Jordan measurablitiy has been reduced to the question of the existence of an elementary set $A_2 \supset A$ with $\mu(A_2) < \epsilon$. However, the only elementary sets covering A must cover all of $[0,1]$ such that $\mu(A_2) \geq 1$.
Therefore, as A is not Jordan measurable. $\square$

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Alternatively, we might use the following criterion for Jordan measurability: $A := [0,1] \setminus \mathbb{Q}$ is Jordan measurable if and only if $\mu^\star(\partial A) = 0$
Now, the boundary of A is exactly the unit interval $[0,1]$. Indeed, there is no ϵ-neighborhood of any point in $[0,1]$ which does not contain both rational and irrational numbers. As $\partial A = [0,1]$ is a rectangle, it is surely Lebesgue measurable and so

$$\mu^\star(\partial A) = \mu(\partial A) = \mu([0,1]) = 1 \neq 0$$

Therefore, as A is not Jordan measurable. $\square$

Problem 8. Does there exist a sequence of Lebesgue measurable functions $\{f_n : [0,1] \to \mathbb{R}\}$ which converges in measure but no subsequence converges uniformly on any subset of positive measure? Justify your answer.

Solution:
No such sequence can exist. Suppose that $\{f_n\}$ converges in measure to f, then by Riesz' Theorem there exists a subsequence $\{f_{n_k}\}$ converging almost everywhere to f.
Now by Egorov's theorem, $\forall_{\delta>0}$ there exists a set $E_\delta \subset [0,1]$ such that

1. $\mu([0,1] \setminus E_\delta) < \delta$
2. f_{n_k} converges uniformly to f on E_δ

where $\mu(E_\delta) > 1 - \delta > 0$. $\square$

Problem 9. Let a function $h : [0,1] \to \mathbb{R}$ satisfy the following conditions

i. h is differentiable in $(0,1)$

ii. h admits one-sided derivatives at 0 and 1

iii. there exists $M > 0$ such that $|h'(x)| \leq M$ for all $x \in [0,1]$

Show that:

a. h' is integrable

b. $\int\limits_{[a,b]} h' d\mu = h(b) - h(a)$

Solution:
Consider the sequence of (continuous and therefore measurable) functions $h_n := \frac{h(x+\frac{1}{n})-h(x)}{\frac{1}{n}}$. Assumptions $i.$ and $ii.$ imply the following point-wise limit

$$\forall_{x\in[0,1]} \lim_{n\to\infty} h_n(x) = h'(x)$$

Notice, by the Mean Value Theorem, $\forall_{x\in[0,1]}, \forall_n \exists_{c\in(0,1)}$ such that $h'(c) = h_n(x)$ and so assumption $iii.$ implies the uniform following bound

$$\forall_{x\in[0,1]} |h_n(x)| \leq M$$

At this point, the conditions of Lebesgue Bounded Convergence Theorem are satisfied. Indeed, the sequence h_n converges pointwise (and therefore almost everywhere) and is uniformly, absolutely dominated by a nonnegative integrable function (in this case just the constant function $\phi(x) \equiv M$). Hence it can be concluded that

1. h' is integrable
2. $\lim_{n\to\infty} \int_a^b h_n(x)dx = \int\limits_{[a,b]} h' d\mu$

Now, make the change of variable $u = x + \frac{1}{n}$, $du = dx$, and let $H(x)$ be the antiderivative of $h(x)$

$$\int_a^b h_n(x)dx = n[\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} h(u)du - \int_a^b h(x)dx] = n[\int_b^{b+\frac{1}{n}} h(x)dx - \int_a^{a+\frac{1}{n}} h(x)dx]$$

$$= \frac{\int_b^{b+\frac{1}{n}} h(x)dx}{\frac{1}{n}} - \frac{\int_a^{a+\frac{1}{n}} h(x)dx}{\frac{1}{n}} = \frac{H(b+\frac{1}{n}) - H(b)}{\frac{1}{n}} - \frac{H(a+\frac{1}{n}) - H(a)}{\frac{1}{n}}$$

In the limit, as $n \to \infty$, the difference quotients tend to $h(b) - h(a)$. $\square$

Problem 10. Let (X, Σ_X, μ) be a *finite* measure space. Take $\{f_n : X \to \mathbb{R}\}$, $\{g_n : X \to \mathbb{R}\}$, two sequences of measurable functions with f_n converging in measure to $f : X \to \mathbb{R}$ and g_n converging in measure to $g : X \to \mathbb{R}$.

Show that $\{f_n g_n\}$ converges in measure to fg. Is this true if $\mu(X) = \infty$?

Solution:
Recall that if h_n converges in measure to h, then $\forall_{\epsilon>0}$

$$\lim_{n\to\infty} \mu(\{x \in X \ : \ |h_n(x) - h(x)| \geq \epsilon\}) = 0$$

Assume, for contradiction, that $\{f_n g_n\}$ does NOT converge in measure to fg, then $\exists_{\epsilon>0}\exists_{\delta>0}$ such that $\forall_{n\in\mathbb{N}}$ sufficiently large

$$\mu(\{x \in X \ : \ |f_n(x)g_n(x) - f(x)g(x)| \geq \epsilon\}) \geq \delta$$

However, by Riesz Theorem, there exists subsequences f_{n_k}, g_{n_k} converging almost everywhere to f, g respectively, and therefore

$$\lim_{k\to\infty} \mu(\{x \in X \ : \ f_{n_k}(x)g_{n_k}(x) \neq f(x)g(x)\}) = 0$$

Which is a contradiction with our assumption. Hence, $f_n g_n$ converges in measure to fg. $\square$

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No, the result does not follow in the case that $\mu(X) = \infty$.
Take for instance $X = (0, \infty)$ and $f_n(x) = \sqrt{x^4 + \frac{x}{n}}$ then f_n converges in measure (and indeed point-wise converges) to $f(x) = x^2$.

However, $f_n^2(x) = x^4 + \frac{x}{n}$ does not converge in measure to $f^2(x) = x^4$. Indeed, take any $\epsilon > 0$ then $\forall_n$

$$\mu(\{x \in X \ : \ \frac{x}{n} > \epsilon\}) = \mu(n\epsilon, \infty) = \infty \ \square$$

Problem 11. Does there exist a non-measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $|g|$ is a measurable function and $g^{-1}(a)$ is a measurable set for each $a \in \mathbb{R}$? Justify your answer

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Solution: Let $A \subset (0, \infty)$ be a non-measurable set. Define the function $g : \mathbb{R} \to \mathbb{R}$ as follows

$$g(x) = \begin{cases} x, & x \in A \cup (-\infty, 0) \\ -x, & x \in [0, \infty] \setminus A \end{cases}$$

Then for any $a \in \mathbb{R}$ the set $g^{-1}(a)$ consists of at most two points and is therefore Lebesgue measurable.

Such a g is not measurable, $g^{-1}(0, \infty) = A$ which is not measurable by construction.

Finally $|g|$ is simply the identity function on $\mathbb{R}$, which is always measurable.

Problem 12.

(i) Show that

$$\int_{[0,1]}\int_{[0,1]} \frac{x^2-y^2}{(x^2+y^2)^2}dydx = \frac{\pi}{4} \quad \text{and} \quad \int_{[0,1]}\int_{[0,1]} \frac{x^2-y^2}{(x^2+y^2)^2}dxdy = -\frac{\pi}{4}$$

(ii) Does this contradict Fubini's theorem? Justify your answer.

Solution:
First, notice the antiderivatives

$$\int \frac{x^2-y^2}{(x^2+y^2)^2}dx = \frac{-x}{x^2+y^2} \quad \text{and} \quad \int \frac{x^2-y^2}{(x^2+y^2)^2}dy = \frac{y}{x^2+y^2}$$

Now integrate first with respect to x and then with respect to y

$$\int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dxdy = \int_0^1 \frac{-x}{x^2+y^2}\bigg|_{x=0}^{x=1} dy = -\int_0^1 \frac{1}{1+y^2}dy = -\frac{\pi}{4}$$

On the other hand, integrating in the opposite order yields the following

$$\int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dydx = \int_0^1 \frac{y}{x^2+y^2}\bigg|_{y=0}^{y=1} dx = \int_0^1 \frac{1}{1+x^2}dx = \frac{\pi}{4}$$

□

. . . .

No, this is not in contradiction with Fubini's, which requires as a condition that the integrand, $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$, is L^1 integrable.

Notice that so long as $x \geq y$, $f(x,y) = |f(x,y)|$, such that

$$\int_0^1\int_0^1 \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right|dydx \geq \int_0^1\int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}dydx = \int_0^1 \frac{y}{x^2+y^2}\bigg|_{y=0}^{y=x} dx = \int_0^1 \frac{1}{2x}$$

$$= \frac{1}{2}\int_{c\to 0}^1 \frac{1}{x} = \frac{1}{2}[ln(1) - \lim_{c\to 0} ln(c)] = \infty$$

Problem 13. Does convergence in $L^{\frac{5}{2}}([0,1])$ imply convergence in measure? Justify your answer

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Solution: Yes. Recall, we say f_n converges in measure to f if $\forall_{\epsilon>0}$

$$\mu\{x \in [0,1] \,:\, |f_n(x) - f(x)| \geq c\} \xrightarrow{n\to\infty} 0$$

We will use the Chebyshev inequality which states: if $g : [0,1] \to \mathbb{R}$ non-negative almost everywhere, then $\forall_{c>0}$

$$\mu\{x \in [0,1] \,:\, g(x) \geq c\} < \frac{1}{c} \int\limits_{[0,1]} g d\mu$$

Now, the function $|f_n(x) - f(x)|$ is non-negative, so choose $\epsilon > 0$ and by Chebyshev

$$\mu\{x \in [0,1] \,:\, |f_n(x) - f(x)| \geq c\} \leq \frac{1}{\epsilon} \int\limits_{[0,1]} |f_n(x) - f(x)| d\mu = \frac{1}{\epsilon} \|f_n - f\|_1$$

Finally, note that if f_n converges to f in L^p for then for any $q < p$ f_n must converges to f in L^q. In particular

$$\mu\{x \in [0,1] \,:\, |f_n(x) - f(x)| \geq c\} \leq \frac{1}{\epsilon} \|f_n - f\|_1 \leq \frac{1}{\epsilon} \|f_n - f\|_p \xrightarrow{n\to\infty} 0$$

Therefore, convergence in $L^p([0,1])$ for any $p \geq 1$ implies convergence in measure. $\square$

Problem 14. Assume $\{f_n\}$ is a norm bounded sequence in $L^2([0,1])$. Does the sequence $\{f_n/n\}$ converge to $f \equiv 0$ a.e.? Justify your answer

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Solution: Yes. Recall, by the corollary to Levi's monotone convergence theorem, if $\{h_k : [0,1] \to \mathbb{R}\}$ is a sequence of functions such that

1. h_k is integrable $\forall_k$
2. $h_k(x) \geq 0$ almost everywhere $\forall_k$
3. $\sum_{k=1}^{\infty}(\int_{[0,1]} h_k d\mu) \leq M < \infty$

then $\sum_{k=1}^{\infty} h_k(x)$ is integrable, converges almost everywhere and the summation and integration signs may be interchanged.
As $\{f_n\} \subset L^2([0,1])$, $\exists_{C>0}$ such that $\forall_n$

$$\int_{[0,1]} (f_n)^2 d\mu < C$$

Define the following non-negative, integrable sequence $h_n = (\frac{f_n}{n})^2$ and notice

$$\sum_{n=1}^{\infty} \int_{[0,1]} (\frac{f_n}{n})^2 d\mu \leq C \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

So, by Levi, the series $\sum_{n=1}^{\infty}(\frac{f_n}{n})^2$ must be integrable. In particular, the sequence $(\frac{f_n}{n})^2$ must converge to 0 and therefore, as the result gaurantees convergence almost everywhere, $\frac{f_n}{n} \to 0$ a.e.