LECTURE 13 – MATH 6301

Approximation of measurable functions by simple functions

THEOREM (SIMPLE FUNCTIONS APPROXIMATION THEOREM)

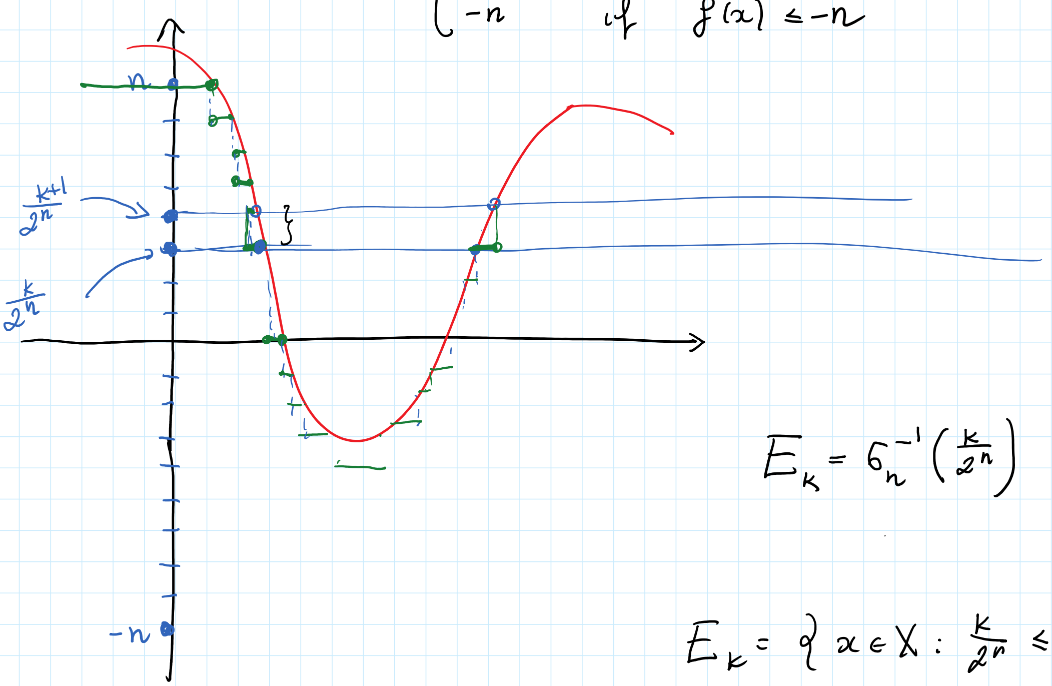
Let $\mathfrak{S} \subset \mathcal{P}(X)$ be a σ-algebra and $f : X \to \overline{\mathbb{R}}$ be an $\mathfrak{S}$-measurable function. Then there exists a sequence of simple $\mathfrak{S}$-measurable functions $\sigma_n : X \to \mathbb{R}$ such that

$$\forall_{x \in X} \quad f(x) = \lim_{n\to\infty} \sigma_n(x)$$

i.e. every $\mathfrak{S}$-measurable function is a limit of a sequence of $\mathfrak{S}$-measurable simple functions.

PROOF: We define the simple functions $\sigma_n : X \to \mathbb{R}$ by

$$(*) \qquad \sigma_n(x) = \begin{cases} n & \text{if } f(x) \geq n \\ \frac{k}{2^n} & \text{if } -n \leq \underbrace{\frac{k}{2^n}}_{\sigma_n(x)} \leq f(x) < \frac{k+1}{2^n} \leq n \\ -n & \text{if } f(x) \leq -n \end{cases} \qquad k \in \mathbb{Z}$$



$$E_k = \sigma_n^{-1}\left(\frac{k}{2^n}\right) \qquad -n2^n < k < n2^n$$

$$E_k = \left\{ x \in X : \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \right\}$$

$\mathfrak{S}$-measurable sets

In addition, if $f(x) = \pm\infty$ then $\sigma_n(x) = \pm n$ and clearly $\sigma_n(x) = \pm n \xrightarrow[n\to\infty]{} \pm\infty$.

On the other hand if $f(x) \neq \pm\infty$, then $\exists_{n \in \mathbb{N}}$ $-n \leq f(x) < n$

and therefore

On the other hand if $f(x) \neq \pm\infty$, then $\exists_{n\in\mathbb{N}}$ $-n \le f(x) < n$ and, therefore $\sigma_n(x) = \frac{k}{2^n}$ where $\frac{k}{2^n} \le f(x) < \frac{k+1}{2^n}$

which means

$$\left| f(x) - \sigma_n(x) \right| < \frac{1}{2^n} \xrightarrow[n\to\infty]{} 0$$

$\square$

REMARK: 1) Notice that if the function $f: X \to \mathbb{R}$ is a bounded $\mathcal{S}$-measurable function, then one can construct a sequence $\sigma_n : X \to \mathbb{R}$ of simple $\mathcal{S}$-measurable functions, which is uniformly convergent to f

$$\forall_{\varepsilon>0} \ \exists_{N} \ \forall_{n\ge N} \ \forall_{x\in X} \quad |f(x) - \sigma_n(x)| < \varepsilon$$

Indeed, if $-M \le f(x) \le M$ then, take σ_n defined by $(*)$ and notice that $\forall_n$ $n \ge M$ we have

$$|f(x) - \sigma_n(x)| < \frac{1}{2^n}$$

2) In the case $f: X \to \mathbb{R}$ is non-negative $\mathcal{S}$-measurable function, then the sequence $\sigma_n : X \to \mathbb{R}$ given by $(*)$ satisfies

$$0 \le \sigma_1(x) \le \sigma_2(x) \le \dots \le \sigma_n(x) \le \dots f(x) \quad \text{for all } x \in X$$

3) If $\mathcal{S} \subset \mathcal{S}'$ are two σ-algebras in X then every $\mathcal{S}$-measurable function $f: X \to \overline{\mathbb{R}}$ is also $\mathcal{S}'$-measurable.

THEOREM: Let $\mathcal{S}_1 \subset \mathcal{P}(X_1)$ and $\mathcal{S}_2 \subset \mathcal{P}(X_2)$ be two σ-algebras and $f: X_1 \times X_2 \to \overline{\mathbb{R}}$ be an $\mathcal{S}_1 \times \mathcal{S}_2$-measurable function. Then

(a) $\forall_{x_1 \in X_1}$ $f_{x_1}(x_2) := f(x_1, x_2)$ is $\mathcal{S}_2$-measurable

(b) $\forall_{x_2 \in X_2}$ $f^{x_2}(x_1) := f(x_1, x_2)$ is $\mathcal{S}_1$-measurable

PROOF: Notice that for all $a \in \mathbb{R}$ we have (for fixed $x_1 \in X_1$) we have

$$\{x_2 \in X_2 : f_{x_1}(x_2) > a\} = \{x_2 \in X_2 : f(x_1, x_2) > a\}$$
$$= \Big(\underbrace{\{(x_1', x_2) : f(x_1', x_2) > a\}}_{\mathcal{S}_1 \times \mathcal{S}_2\text{-measurable}}\Big)_{x_1} \leftarrow \mathcal{S}_2\text{-measurable}$$

$= \left(\{(x_1', x_2) : f(x_1', x_2) > a\}\right)_{x_1'} \in S_2$ measurable

$S_1 \times S_2$ - measurable

MEASURE AND OUTER MEASURE

Let $\mathcal{S} \subset \mathcal{P}(X)$ be a σ-algebra. A function $\mu : \mathcal{S} \to \overline{\mathbb{R}}$ is called a **measure** iff

(μ1) $\forall_{E \in \mathcal{S}} \quad \mu(E) \geq 0$

(μ2) $\mu(\emptyset) = 0$

(μ3) $\forall \{E_n\}_{n=1}^{\infty} \subset \mathcal{S}$ such that $E_n \cap E_m = \emptyset$ for $n \neq m$

$n=1$

Then, we will also say that $(X, \mathcal{S}, \mu)$ is the **measure space**.
To be precise, in certain situation we will also say that elements $E \in \mathcal{S}$ are **μ-measurable sets**.

REMARK: If $(X, \mathcal{S}, \mu)$ is a measure space, $E \in \mathcal{S}$, then the restriction of μ to $\mathcal{S}_E := \{F \in \mathcal{S} : F \subset E\}$ is a measure on E, which will be called **restricted to E measure**.
Moreover, if $\{E_1, E_2, \dots, E_n\} \subset \mathcal{S}$ are disjoint sets then

$$\mu(E_1 \cup E_2 \cup \dots \cup E_n) = \mu(E_1) + \mu(E_2) + \dots + \mu(E_n).$$

ADDITIONAL PROPERTIES: Assume $(X, \mathcal{S}, \mu)$ is a measure space.
Let $E, F \in \mathcal{S}$, $\{E_n\} \subset \mathcal{S}$.

(1) If $F \subset E$ then $\mu(F) \leq \mu(E)$

Indeed $E = (E \setminus F) \cup F$
Since $E \setminus F \cap F = \emptyset$, we have by (μ3)

$$\mu(E) = \mu(E \setminus F) + \mu(F) \geq \mu(F)$$

$\vee\!\!|$ (μ1)

(2) If $F \subset E$ and $\mu(E) < \infty$ then $\mu(E \setminus F) = \mu(E) - \mu(F)$

(3) If $E \subset \bigcup_{n=1}^{\infty} E_n$ then

$$\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

σ-subadditivity

$$\boxed{\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)}$$

PROOF

PROOF

Notice that

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n \ , \quad F_n = E_n \setminus \{E_1 \cup \dots \cup E_{n-1}\} \subset E_n$$

$$F_n \cap F_m = \emptyset \ , \text{ for } n \neq m$$

so we have

$$\mu(E) \leq \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) =$$

$$= \sum_{n=1}^{\infty} \mu(F_n) \leq \sum_{n=1}^{\infty} \mu(E_n) \qquad \square$$

(4) If $E_1 \subset E_2 \subset \dots\dots$ then

$$\mu\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

PROOF:

$$\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n = E_1 \cup \bigcup_{n=2}^{\infty} F_n = E_1 \cup \bigcup_{n=2}^{\infty} (E_n \setminus E_{n-1})$$

so we have

$$\mu\left(\lim_{n\to\infty} E_n\right) = \mu(E_1) + \sum_{n=2}^{\infty} \mu(E_n \setminus E_{n-1}) \qquad (*)$$

Since

$$\mu(E_n) = \mu\left(\bigcup_{k=1}^{n} E_k\right) = \mu\left(E_1 \cup \bigcup_{k=2}^{n} (E_k \setminus E_{k-1})\right)$$

$$= \mu(E_1) + \sum_{k=2}^{n} \mu(E_k \setminus E_{k-1}).$$

So, by taking the limit as $n \to \infty$ we obtain that

$$\lim_{n\to\infty} \mu(E_n) = \mu(E_1) + \sum_{k=2}^{\infty} \mu(E_k \setminus E_{k-1}) \qquad (**)$$

and by comparing $(*)$ with $(**)$, we get

$$\mu\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n) \quad \square$$

PROPOSITION: Let $(X, \mathfrak{S}, \mu)$ and suppose $E_1 \supset E_2 \supset E_3 \supset \dots \supset E_n \supset E_{n+1} \supset \dots$ is a decreasing sequence of μ-measurable sets such that

$$\exists_{m\in\mathbb{N}} \quad \mu(E_m) < \infty$$

Then

$$\mu\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

Proof

$$\mu(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$$

<u>PROOF</u>

$$\mu(\lim_{n\to\infty} E_n) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right)$$

By assumption $\mu(E_m) < \infty$ then $\forall_{n \geq m}$ $\mu(E_n) \leq \mu(E_m) < \infty$

and

$$\bigcap_{n=1}^{\infty} E_n = \bigcap_{n \geq m}^{\infty} E_n$$

$$E_m \setminus E_{m+1} \subset E_m \setminus E_{m+2} \subset \dots \subset E_m \setminus E_n \subset \dots \quad n > m+2$$

and

$$\mu(\lim_{n\to\infty} (E_m \setminus E_n)) \overset{(4)}{=} \lim_{n\to\infty} \mu(E_m \setminus E_n)$$

$$= \lim_{n\to\infty} \left(\mu(E_m) \setminus \mu(E_n)\right) = \mu(E_m) - \lim_{n\to\infty} \mu(E_n)$$

<u>Since</u>

$$\bigcup_{n=m}^{\infty} E_m \setminus E_n = E_m \setminus \bigcap_{n=m}^{\infty} E_n = E_m \setminus \lim_{n\to\infty} E_n$$

We also have

$$\mu(E_m \setminus \lim_{n\to\infty} E_n) \overset{(2)}{=} \mu(E_m) - \mu(\lim_{n\to\infty} E_n)$$

$$\shortparallel$$

$$\mu(\lim_{n\to\infty} (E_m \setminus E_n)) = \mu(E_m) - \lim \mu(E_n)$$

$$\Rightarrow \lim_{n\to\infty} \mu(E_n) = \lim \mu(\lim E_n)$$

$\square$

<u>EXAMPLE</u>: Take $X = \mathbb{N}$, $\mathcal{S} = \mathcal{P}(\mathbb{N})$, $A \subset \mathbb{N}$

$$\mu(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

$(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ is a measure space

Take $E_n := \{k \in \mathbb{N} : k \geq n\}$ $\quad \mu(E_n) = \infty \quad \overline{\mathbb{R}}$

then $\lim_{n\to\infty} \mu(E_n) = \infty$

$$E_n \supset E_{n+1} \supset \dots \quad \bigcap_{n=1}^{\infty} E_n = \emptyset$$

$$\mu(\lim E_n) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \mu(\emptyset) = 0$$

so

$$\lim_{n\to\infty} \mu(E_n) \neq \mu(\lim E_n) \quad \square$$

$n \to \infty$

For a given measure space $(X, \mathcal{S}, \mu)$ and a certain logical statement $p(x)$, $x \in X$, we say that

$p(x)$ is true almost everywhere in X $\iff \mu\{x \in X : \sim p(x)\} = 0$

true a.e

$$A = \{x \in X : p(x)\} \implies \mu(A^c) = 0$$