University of Texas at Dallas

Exam 1: Review Questions

Last Name:	**First Name and Initial:** ....
Course Name: Real Analysis 1	**Number:** MATH 6301
Instructor: Wieslaw Krawcewicz	**Due Date:** October 3, 2022
E-mail Address:	**Student's Signature:** ....

Problem 1. Let X and Y be two non-empty sets, $f : X \to Y$ a function and $\mathscr{S} \subset \mathscr{P}(X)$ a σ-algebra. Show that the family of sets given by

$$\mathscr{C} := \{F \subset Y : f^{-1}(F) \in \mathscr{S}\}$$

is a σ-algebra.

SOLUTION:

Problem 2: Let X and Y be two non-empty sets, $f : X \to Y$ a function and $\mathscr{S} \subset \mathscr{P}(X)$ a σ-algebra. Suppose that $\mathscr{K} \subset \mathscr{P}(Y)$ is a given family of sets in Y and denote by $\mathscr{S}(\mathscr{K})$ the smallest σ-algebra generated by $\mathscr{K}$. Show that, if

$$\forall_{B \in \mathscr{K}} \quad f^{-1}(B) \in \mathscr{S},$$

then

$$\forall_{F \in \mathscr{S}(\mathscr{K})} \quad f^{-1}(F) \in \mathscr{S},$$

SOLUTION:

Problem 3: Consider the following family of intervals in $\mathbb{R}$

$$\mathscr{K} := \{(-\infty, a] : a \in \mathbb{R}\}$$

Show that $\mathscr{S}(\mathscr{K}) = \mathscr{B}(\mathbb{R})$, i.e. $\mathscr{K}$ generates the σ-algebra of Borel sets in $\mathbb{R}$.

SOLUTION:

Problem 4: Suppose $\mathscr{N} \subset \mathscr{P}(X)$ i a monotone family of sets and let $\mathscr{L} \subset \mathscr{P}(X)$ be an arbitrary class of sets. Show that the class

$$\mathscr{J}(\mathscr{L}) := \{E \in \mathscr{P}(X) : \forall_{F \in \mathscr{L}} \quad E \cup F,\, E \setminus F,\, F \setminus E \in \mathscr{N}\}$$

is monotone.

SOLUTION:

Problem 5: Consider a subset A in X, $A \neq X, \emptyset$ and the family $\mathscr{K} := \Big\{\{A\}, \{A^c\}\Big\}$. Describe the σ-algebra

$$\mathscr{S}(\mathscr{K} \barwedge \mathscr{K}),$$

i.e. the smallest σ-algebra generated by $\mathscr{K} \barwedge \mathscr{K}$.

SOLUTION:

Problem 6: Let $X = \mathbb{R}$ and consider the following collection of sets $\mathcal{A} := \{\{n\} : n \in \mathbb{N}\}$. Show that

$$\mathscr{S}(\mathcal{A}) := \{\mathbb{R} \setminus S : S \subset \mathbb{N}\} \cup \{S : S \subset \mathbb{N}\}.$$

SOLUTION:

Problem 7: Let $\mathscr{S} \subset \mathscr{P}(X)$ be a σ-algebra and $f,\ g : X \to \mathbb{R}$ be $\mathscr{S}$-measurable functions. Show that $f + g : X \to \mathbb{R}$ is a $\mathscr{S}$-measurable function.

SOLUTION:

Problem 8: Let $\mathscr{S} \subset \mathscr{P}(X)$ be a σ-algebra and $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of $\mathscr{S}$-measurable functions. Show that $f : X \to \overline{\mathbb{R}}$ given by

$$f(x) := \sup_{n\in\mathbb{N}} f_n(x)$$

a $\mathscr{S}$-measurable function.

SOLUTION:

Problem 9: Let $\mathscr{S} \subset \mathscr{P}(X)$ be a σ-algebra and $f,\ g : X \to \mathbb{R}$ be $\mathscr{S}$-measurable functions. Show that $fg : X \to \mathbb{R}$ is a $\mathscr{S}$-measurable function.

SOLUTION:

Problem 10: Let $\mathscr{S} \subset \mathscr{P}(X)$ be a σ-algebra and $f,\, g : X \to \mathbb{R}$ be two $\mathscr{S}$-measurable functions. Show that the set $\{x \in X : f(x) = g(x)\}$ is measurable (i.e bel;ongs to $\mathscr{S}$.

SOLUTION:

Problem 11: Let $\mathscr{S} \subset \mathscr{P}(X)$ be a σ-algebra and $f,\, g : X \to \mathbb{R}$, $g(x) \neq 0$ for all $x \in X$, be two $\mathscr{S}$-measurable functions. Show that $\frac{f}{g} : X \to \mathbb{R}$ is a $\mathscr{S}$-measurable function.

SOLUTION:

Problem 12: Let (X, d) be a metric space and assume that $\mathscr{S} := \mathbb{B}(X)$ (i.e. $\mathscr{S}$ stands for Borel sets in X). Show that every continuous function $f : X \to \mathbb{R}$ is $\mathscr{S}$-measurable.

SOLUTION:

Problem 13: Let (X, d) be a metric space and assume that $\mathscr{S} := \mathbb{B}(X)$ (i.e. $\mathscr{S}$ stands for Borel sets in X). Show that if $f : X \to \mathbb{R}$ is continuous, except for a finite number of discontinuity points $N = \{x_1, x_2, \dots, x_n\}$, then f is $\mathscr{S}$-measurable.

SOLUTION:

Problem 14:. Let (X, d) be a metric space and A a closed set in X, show that there exists a sequence of continuous functions $f_n : X \to [0, \infty)$ such that

(a) for all $x \in X$ one has $... \geq f_{n+1}(x) \geq f_n(x) \geq \chi_A(x)$, where χ_A stands for the characteristic function of A.

(b) for all $x \in X$ one has

$$\chi_A(x) = \lim_{n \to \infty} f_n(x).$$

SOLUTION: