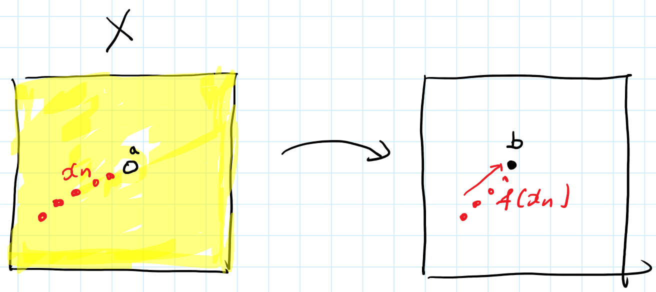
Tuesday, September 6, 2022 5:21 PM

LECTURE 5 — MATH 6301

Assume that (X,d) and (Y,ς) are two metric spaces and let $a\in X$, $b\in Y$ be two points. Then we say that a function $f: X\setminus\{a\}\longrightarrow Y$ has an accumulation value (cluster value) b at the point a, if there exists a sequence $\{x_n\}\subset X\setminus\{a\}$ such that $\lim x_n = a$ and $\lim f(x_n)=b$.



We denote by $A_a(f)$ the set of all accumulation values of f at a.

PROBLEM 1 Show that $A_a(f)$ is a closed set.

We are interested in a particular case when the set $A_a(f)$ is bounded and non-empty. Then we can put

$$\limsup_{x\to a} f(x) := \sup A_a(f)$$

$$\liminf_{x\to a} f(x) := \inf A_a(f)$$

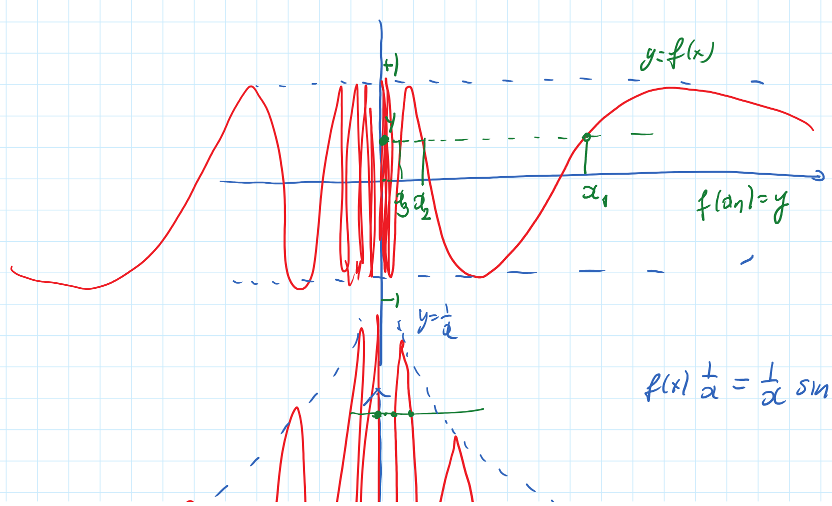
REMARK: Notice that it is possible that

(a) $A_a(f)=\emptyset$ Example: $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ $f(x)=\frac{1}{x}$, $A_0(f)=\emptyset$

(b) $A_a(f)$ can be unbounded. Example $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ $f(x)=\frac{1}{x}\sin\left(\frac{1}{x}\right)$

Notice that

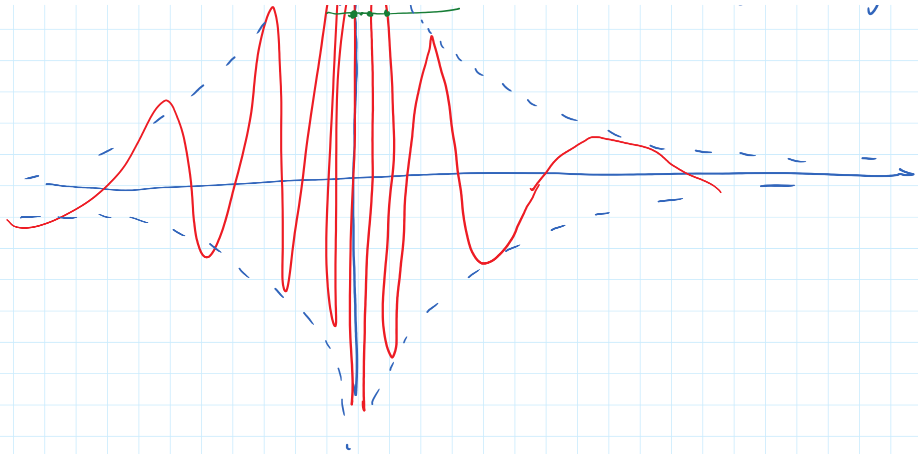
$f(x)=\sin\left(\frac{1}{x}\right)$



$A_0(f)=[-1,1]$

$f(x)\frac{1}{x}=\frac{1}{x}\sin\left(\frac{1}{x}\right)$

$A_0(f)=\mathbb{R}$



For $a \in X$ and $\delta > 0$ we put

$$C_\delta(a) := \{ x \in X : 0 < d(a,x) < \delta \}$$

and call it deleted δ-neighbourhood of a

PROBLEM 2: Suppose $f : X \setminus \{a\} \to \mathbb{R}$ is such that $A_a(f) \neq \emptyset$ is bounded. Show that

(a) $$\limsup_{x\to a} f(x) = \inf_{\delta>0} \sup_{x\in C_\delta(a)} f(x)$$

(b) $$\liminf_{x\to a} f(x) = \sup_{\delta>0} \inf_{x\in C_\delta(a)} f(x)$$

RECALL: For a given sequence $\{y_n\} \subset \mathbb{R}$ we have

$$\limsup_{n\to\infty} y_n = \inf_n \sup_{k\geq n} y_k$$
$$\liminf_{n\to\infty} \qquad = \lim_{n\to\infty} \left(\sup_{k\geq n} y_k \right)$$

PROBLEM 3: Let $f : X \setminus \{a\} \to \mathbb{R}$ be a function such that $A_a(f) \neq \emptyset$ is bounded and $\alpha \in \mathbb{R}$. Show that

(a) $\limsup_{x\to a} f(x) < \alpha \iff \exists_{\delta>0} \; \exists_{\bar\alpha<\alpha} \; \forall_{x\in X} \; 0 < d(x,a) < \delta \Rightarrow f(x) \leq \bar\alpha$

(b) $\liminf_{x\to a} f(x) > \alpha \iff \exists_{\delta>0} \; \exists_{\bar\alpha>\alpha} \; \forall_{x\in X} \; 0 < d(x,a) < \delta \Rightarrow f(x) \geq \bar\alpha$

(c) $\limsup_{x\to a} f(x) \leq \alpha \iff \forall_{\alpha'>\alpha} \; \exists_{\delta>0} \; \forall_{x\in X} \; 0 < d(x,a) < \delta \Rightarrow f(x) \leq \alpha'$

(d) $\liminf_{x\to a} f(x) \geq \alpha \iff \forall_{\alpha'<\alpha} \; \exists_{\delta>0} \; \forall_{x\in X} \; 0 < d(x,a) < \delta \Rightarrow f(x) \geq \alpha'$

Notice that if $f, g : X \to \mathbb{R}$ is a given function then by (excluding the point $a \in X$) one can consider the set of all accumulation points

$$A_a(f), \; A_a(g)$$

Then notice that if $(A_a(f) \neq \emptyset, A_a(g)$ bounded$)$ we have

$$\forall_{x \in X} \; f(x) \leq g(x) \implies \begin{cases} \limsup\limits_{x \to a} f(x) \leq \limsup\limits_{x \to a} g(x) \\ \liminf\limits_{x \to a} f(x) \leq \liminf\limits_{x \to a} g(x) \end{cases}$$

PROBLEM 4: Let (X, d) and (Y, ϱ) be two metric spaces, $a \in X$, and $f: X \to Y$ a function. Show that f is continuous at a iff

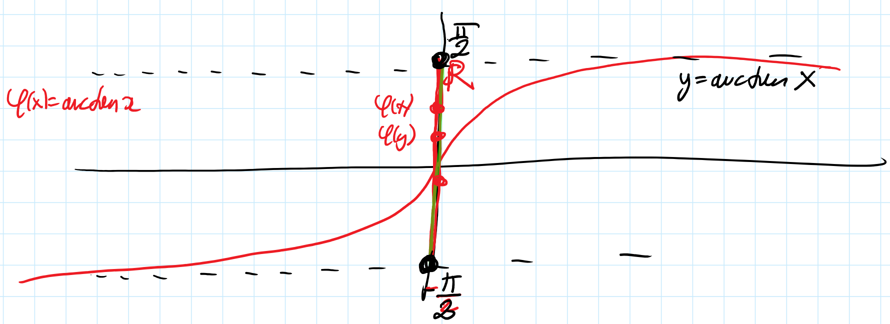
$$A_a(f) = \{f(a)\}$$

REMARK: Since it would be convenient to include the values ∞ and $-\infty$ as the possible values of function $f: X \to \mathbb{R}$ (and consequently avoid making repetitive assumptions that $A_a(f)$ is bounded), we can simply introduce the concept of the so-called extended set of real numbers: $\overline{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ ie we assume that

$$\forall_{x \in \mathbb{R}} \; -\infty < x < \infty \qquad \text{(so we extend the order on } \overline{\mathbb{R}}\text{)}$$

and we can equipe $\overline{\mathbb{R}}$ with the metric:

$$(*) \qquad d(x,y) = \begin{cases} |\arctan(x) - \arctan(y)| & \text{if } x, y \in \mathbb{R} \\ \frac{\pi}{2} - \arctan y & \text{if } x = -\infty, \; y \in \mathbb{R} \\ \frac{\pi}{2} + \arctan x & \text{if } y = \infty \\ & \text{if } x = \infty, \; y = -\infty \end{cases}$$

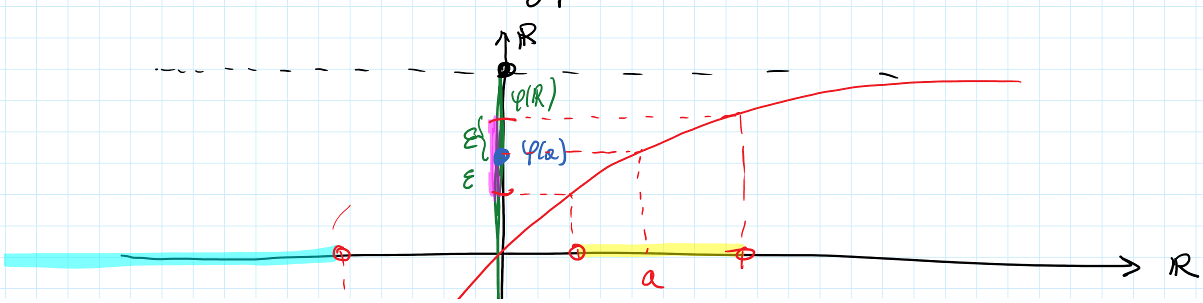


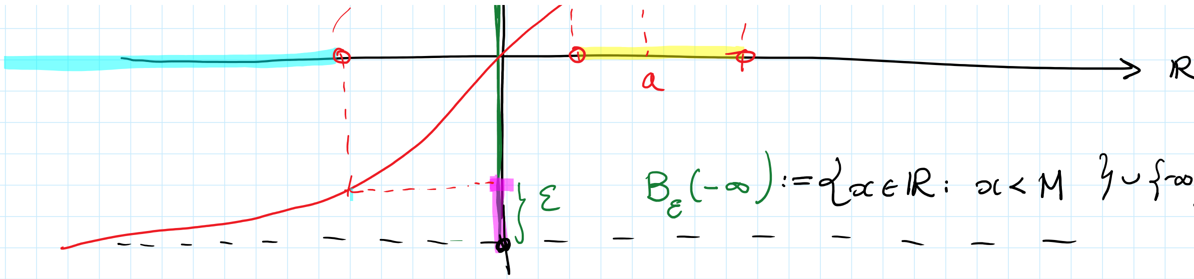
$\mathbb{R} \simeq (-\frac{\pi}{2}, \frac{\pi}{2})$

PROBLEM 5: Show that the function $d: \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \mathbb{R}$ given by $(*)$

(a) is a metric on $\overline{\mathbb{R}}$

(b) restricted to $\mathbb{R} \subset \overline{\mathbb{R}}$ generates the same topology on $\mathbb{R}$ as the standard metric $|x - y|$.





$B_\varepsilon(-\infty) := \{x \in \mathbb{R} : x < M\} \cup \{-\infty\}$ $M := \arctan\left(\frac{\pi}{2} + \varepsilon\right)$

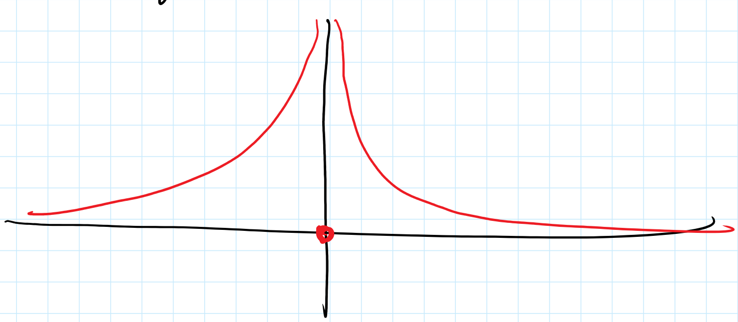
<u>REMARK</u>: Once the concept of limsup and liminf is extended to function $f : X \to \overline{\mathbb{R}} = [-\infty, \infty]$, there is no need to make additional assumptions because $A_a(f)$ is always well defined in $[-\infty, \infty]$ (we now admit $\pm\infty$ as possible accumulation values)

<u>Example</u>: (a) $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ $\limsup_{x \to 0} f(x) = \infty$, $\liminf_{x \to 0} f(x) = -\infty$

(b) $f(x) = \frac{1}{x^2}$ $x \in \mathbb{R} \setminus \{0\}$

$\limsup_{x \to 0} f(x) = \infty$ $\liminf_{x \to 0} f(x) = \infty$

(c) $f(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$ $\liminf_{x \to 0} f(x) = \infty$



<u>COMPACTNESS IN METRIC SPACES</u>

Let (X, d) be a metric space, $A \subset X$ a set. We say that $\{U_i\}_{i \in I} \subset \tilde{\mathcal{T}}$ is an open cover of A $\iff$

$$A \subset \bigcup_{i \in I} U_i$$

Any family $\{V_j\}_{j \in J} \subset \{U_i\}_{i \in I}$ such that $\{V_j\}_{j \in J}$ is an open cover of A

is called a subcover of $\{U_i\}_{i\in I}$.

DEFINITION: A set $A\subset X$ is said to be compact iff every open cover $\{U_i\}_{i\in I}$ of A contains a finite subcover, i.e $\exists\ \{U_{i_1}, U_{i_2}, \dots, U_{i_n}\}\subset\{U_i\}_{i\in I}$

$$A\subset U_{i_1}\cup U_{i_2}\cup\dots\cup U_{i_n}$$

If $A=X$ is compact, then we say that X is a compact space.

DEFINITION: We say that $A\subset X$ is totally bounded iff

$$\forall_{\varepsilon>0}\ \exists_{\underset{\substack{\varepsilon\text{-net}\\ \text{finite set}}}{\{a_1,a_2,\dots,a_n\}\subset A}}\quad A\subset \underbrace{B_\varepsilon(a_1)\cup\dots\cup B_\varepsilon(a_n)}=\bigcup_{i=1}^{n} B_\varepsilon(a_i)$$

THEOREM Let (X,d) be a metric space. The following conditions are equivalent

(a) X is a compact space

(b) Every sequence in X has a convergent subsequence, i.e

$$\forall_{\{x_n\}\subset X}\ \exists_{x_0\in X}\ \exists_{\{x_{n_k}\}\subset\{x_n\}}\quad \lim_{k\to\infty} x_{n_k}=x_0$$

(c) X is complete and totally bounded.

REMARK: (a) Notice that if $A\subset X$ is a compact set then A is closed.

(b) If $A\subset X$ is compact then it is bounded

(c) If A is compact and $B\subset A$ is closed then B is compact.

THEOREM: A set $A\subset\mathbb{R}^n$ is compact $\iff \overline{A}=A$ and A is bounded.

PROOF: $n=1$ idea x_n $x_{n_1}=x_1$

x_{n_2}