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LECTURE 4 — MATH 6301

SEQUENCES IN METRIC SPACES

Let (X,d) be a metric space. A function $a:\mathbb{N}\to X$ is called a sequence in X. For convenience we put

$$a_n := a(n),\quad n\in\mathbb{N},$$

and we list the values of a in a sequential form $\{a_1, a_2, a_3, \dots\} = \{a_n\}_{n=1}^{\infty}$

DEFINITION: A sequence $\{x_n\}_{n=1}^{\infty}\subset X$ is called convergent to $a\in X$ iff

notation: $\lim\limits_{n\to\infty} x_n = a \overset{\text{def}}{\iff} \forall_{\varepsilon>0}\ \exists_{N}\ \forall_{n\geq N}\ d(x_n, a)<\varepsilon$

limit of x_n is a

$x_n \to a$ as $n\to\infty$

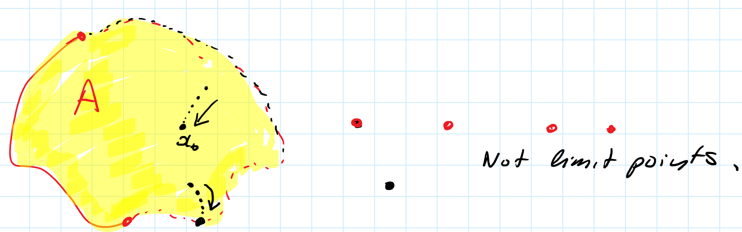
DEFINITION Let $k:\mathbb{N}\to\mathbb{N}$ be an increasing function $(n>m \Rightarrow k(n)>k(m))$ then the sequence

$$\{x_{k(n)}\}_{n=1}^{\infty} \qquad y_n := x_{k(n)}$$

is called a subsequence of $\{x_n\}$

DEFINITIONS: Let A be a set and $x_0\in X$. We say that x_0 is a limit point of A iff

$$\exists_{\{x_n\}\subset A}\quad x_n\neq x_0 \text{ and } \lim_{n\to\infty} x_n = x_0$$



$\overline{A}$

PROPOSITION: A set $A\subset X$ is closed if and only if it contains all its limit points.

Topology $\mathcal{T}\subset\mathcal{P}(X)$

Notice that metric topology can be characterized by sequences.

DEFINITION: Let (X,d) be a metric space and $\{x_n\}_{n=1}^{\infty}\subset X$ a

DEFINITION: Let (X,d) be a metric space and $\{x_n\}_{n=1}^{\infty} \subset X$ a sequence. The sequence $\{x_n\}$ is called Cauchy iff

$$\underset{\varepsilon>0}{\forall}\ \underset{N>0}{\exists}\ \underset{n,m\geq N}{\forall}\quad d(x_n,x_m)<\varepsilon .$$

REMARK Notice that every convergent sequence is Cauchy

$$\lim x_n = a \overset{\text{def}}{\iff} \underset{\varepsilon>0}{\forall}\ \underset{N}{\exists}\ \underset{n,m\geq N}{\forall}\quad \begin{matrix} d(x_n,a)<\frac{\varepsilon}{2} \\ d(x_m,a)<\frac{\varepsilon}{2} \end{matrix} \implies \underset{\varepsilon>0}{\forall}\ \underset{N}{\exists}\ \underset{n,m\geq N}{\forall}\quad d(x_n,x_m)\leq d(x_n,a)+d(x_m,a)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

DEFINITION: A metric space (X,d) is called complete iff every Cauchy sequence in X converges (to an element in X).

EXAMPLE: Take $X=\mathbb{R}$ with $d(x,y)=|x-y|$. Then (X,d) is a complete metric space. Indeed, let $\{a_n\}_{n=1}^{\infty}\subset\mathbb{R}$ be a Cauchy sequence, i.e.

$$\underset{\varepsilon>0}{\forall}\ \underset{N}{\exists}\ \underset{n,m\geq N}{\forall}\quad |a_n-a_m|<\varepsilon \quad\Updownarrow\quad a_n-a_m<\varepsilon \text{ and } a_m-a_n<\varepsilon$$

$$\Updownarrow$$

$$\underset{\varepsilon>0}{\forall}\ \underset{N}{\exists}\ \underset{n,m\geq N}{\forall}\quad \begin{cases} a_n<a_m+\varepsilon \\ a_n-\varepsilon<a_m \end{cases}$$

$$\Downarrow$$

$$\underset{\varepsilon>0}{\forall}\ \underset{N}{\exists}\ \underset{m\geq N}{\forall}\quad \begin{cases} a_n<a_N+\varepsilon \quad (1) \\ a_n-\varepsilon<a_N \end{cases} \quad (*)$$

then for $\underset{\varepsilon>0}{\forall}$ there is an $N\in\mathbb{N}$ for which $(*)$ are satisfied

Put $\mathcal{A}:=\{y\in\mathbb{R} : a_n\leq y$ for only finitely many $n\in\mathbb{N}\}$

$\Updownarrow$

$a_n>y$ for almost all n (except finitely many)

Then $a_N-\varepsilon\in\mathcal{A} \Rightarrow \mathcal{A}\neq\emptyset$. Moreover $\mathcal{A}$ is bounded from above, because

- if $y\in\mathcal{A} \Rightarrow y-r\in\mathcal{A}$ for all $r>0$
- $a_N+\varepsilon$ is an upper bound of $\mathcal{A}$.

By completeness axiom (C) of real numbers, there exists

$$a=\sup\mathcal{A} \iff \begin{cases} \underset{n}{\forall}\ a\geq a_n \\ \underset{\varepsilon>0}{\forall}\ \underset{n}{\exists}\ a-\varepsilon<a_n \end{cases}$$

and we have

$$\underset{\varepsilon>0}{\forall}\ \underset{N}{\exists}\ \underset{n\geq N}{\forall}\quad |a_n-a|\leq|a_n-a_N|+a-a_N \qquad \boxed{a_n-a_N<\varepsilon}$$

and we have

$$\forall_{\varepsilon>0} \ \exists_{N} \ \forall_{n\geq N} \quad |a_n - a| \leq |a_n - a_N| + a - a_N$$
$$\leq \varepsilon + \varepsilon = 2\varepsilon$$

$\square$

$$\boxed{a_n - a_N < \varepsilon}$$
$$a_n < \varepsilon + a_N \quad (**)$$
$$a \leq \varepsilon + a_N$$
$$a - a_N \leq \varepsilon$$

<u>REMARK</u>: The Euclidean space $(\mathbb{R}^n, d)$

$$d(x,y) := \left(\sum_{k=1}^{n}(x_k - y_k)^2\right)^{1/2} \qquad \begin{array}{l} x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \\ y = (y_1, y_2, \ldots, y_n)^T \end{array}$$

is also a complete metric space

$$\{x^m\}, \quad x^m = (x_1^m, x_2^m, \ldots, x_n^m)^T, \quad \text{is Cauchy} \iff \forall_{k=1,\ldots,n} \ \{x_k^m\} \text{ is Cauchy.}$$

<u>REMARK</u>: (X_1, d_1), (X_2, d_2) two complete metric spaces. Then one can define the metric space $(X_1 \times X_2, d)$, with

$$d((x_1, x_2), (y_1, y_2)) = \left(d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2\right)^{1/2}$$
$$(x_1, x_2) \in X_1 \times X_2$$
$$(y_1, y_2) \in X_1 \times X_2$$

<u>Then</u> $(X_1 \times X_2, d)$ is also a complete metric space.

<u>DEFINITION</u> (Normed Space) $\quad (V, \|\cdot\|)$

Let V be a real vector space and $\|\cdot\|: V \to \mathbb{R}$ be a function (called norm on V) satisfying

(n1) $\quad \|x\| \geq 0$ and $\|x\| = 0 \iff x = 0 \qquad \forall_{x \in V}$

(n2) $\quad \|rx\| = |r|\,\|x\| \qquad \forall_{x \in V} \ \forall_{r \in \mathbb{R}}$

(n3) $\quad \|x+y\| \leq \|x\| + \|y\|$

then $(V, \|\cdot\|)$ is called normed space.

<u>DEFINITION</u>: If $(V, \|\cdot\|)$ is a normed space, and

$$d(x,y) := \|x - y\| \qquad (*)$$

is the so-called associated with $\|\cdot\|$ metric on V, then $(V, \|\cdot\|)$ is called Banach space iff (V, d) is complete.

normed $\qquad\qquad$ metric

<u>PROPOSITION</u>: The space $(\mathbb{R}^n, \|\cdot\|_2)$, with $\|x\|_2 := \left(\sum_{k=1}^{n} x_k^2\right)^{1/2}$ is a Banach space.

<u>PROOF</u>: We only need to show that $\mathbb{R}^n$ is normed, i.e. $\|\cdot\|_2$ satisfies (n1), (n2), (n3). (Notice that (n1) and (n2) are obvious). To show (n3) notice that

(n1), (n2), (n3). (Notice that (n1) and (n2) are obvious). To show (n3) notice that

$$x \cdot x = \|\cdot\|_2^2 = \sum_{k=1}^{n} x_k^2 \geq 0 \qquad x \cdot y = \sum_{k=1}^{n} x_k y_k$$

and consider the function $t \in \mathbb{R}$

$$\varphi(t) = (x+ty)\cdot(x+ty) = x\cdot x + 2t(x\cdot y) + t^2 y\cdot y$$

$$= \|x\|^2 + 2t(x\cdot y) + t^2\|y\|^2 \geq 0 \qquad \varphi(t) = at^2 + bt + c, \quad \Delta = b^2 - 4ac \leq 0$$

thus $4(x\cdot y)^2 - 4\|x\|^2\|y\|^2 \leq 0$

$$\Updownarrow$$

$$(x\cdot y)^2 \leq \|x\|^2\|y\|^2$$

$$\Updownarrow$$

$$|x\cdot y| \leq \|x\|\cdot\|y\| \qquad \text{Cauchy-Schwarz inequality}$$

Thus

$$\|x+y\|^2 = (x+y)\cdot(x+y) = \|x\|^2 + 2x\cdot y + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

$$\Downarrow$$

$$\|x+y\| \leq \|x\| + \|y\| \qquad \square$$

CONCEPT OF CONTINUITY: Let (X, d_X), (Y, d_Y) be two metric spaces and consider $f : X \setminus \{a\} \to Y$ here $a \in X$ is a given limit point of X.

Then

$$\lim_{x\to a} f(x) = b \overset{\text{def}}{\Longleftrightarrow} \forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x\in X\setminus\{a\}} \quad d_X(x,a) < \delta \Rightarrow d_Y(f(x), b) < \varepsilon$$

(here $b \in Y$ a given point)

and we say that f has a limit b at a.

PROPOSITION: Let $a \in X$ be a limit point of X and $f : X\setminus\{a\} \to Y$ a function. Then

$$\lim_{x\to a} f(x) = b \Longleftrightarrow \forall_{\{a_n\}\subset X\setminus\{a\}} \quad \lim_{n\to\infty} x_n = a \Rightarrow \lim_{n\to\infty} f(x_n) = b$$

DEFINITION: Let $f : X \to Y$ be a function and $a \in X$. We say that f is continuous at a iff

(i) a is an isolated point of X.

(ii) a is a limit point of X, then $\lim_{n\to\infty} x_n = a \Rightarrow \lim_{n\to\infty} f(x_n) = f(a)$

THEOREM: Let $f: X \to Y$ be a function, $a \in X$. Then f is continuous at $a \in X$ iff

$$\forall_{\varepsilon>0} \ \exists_{\delta>0} \ \forall_{x \in X} \quad d_X(a,x) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon$$

DEFINITION: $f: X \to Y$ is called continuous if it is continuous at every $a \in X$

REMARK: $f: X \to Y$ is continuous $\Longleftrightarrow \forall_{a \in X} \ \forall_{\varepsilon>0} \ \exists_{\delta>0} \ \forall_{x \in X} \ d_X(a,x) < \delta \Rightarrow d_Y(f(a), f(x)) < \varepsilon$

$\forall_{\varepsilon>0} \ \exists_{\delta>0} \ \forall_{a \in X} \ \forall_{x \in X} \ d_X(a,x) < \delta \Rightarrow d_Y(f(a), f(x)) < \varepsilon$ the function f is called uniformly continuous

THEOREM: A function $f: X \to Y$ is continuous $\Longleftrightarrow$
$\forall_{V \in \mathcal{T}_Y} \ f^{-1}(V) \in \mathcal{T}_X$ (i.e. inverse image of an open set is open).