LECTURE 23 — MATH 6301

Let $(X, \mathfrak{S}, \mu)$ be a measure space, $E \in \mathfrak{S}$ and $f : E \to [0,\infty]$ be a μ-measurable function. Then we put

$$\int_E f d\mu = \sup_{\substack{E = \bigcup_{n=1}^{\infty} E_n \\ E_n \cap E_m = \emptyset \\ n \neq m \\ E_n \in \mathfrak{S}}} \sum_{n=1}^{\infty} \left(\inf_{x \in E_n} f(x)\right) \cdot \mu(E_n)$$

conventions

$$\boxed{0 \cdot \infty = 0}$$

(1) $\displaystyle\int_E \alpha f \, d\mu = \alpha \int_E d\mu \qquad \alpha \geq 0$

(2) $f, g : E \to [0,\infty]$ μ-measurable functions s.t. $f(x) \leq g(x)$ $\forall x \in E$

$$\int_E f d\mu \leq \int_E g \, d\mu$$

(3) MVT

$$\inf_{x \in E} f(x) \cdot \mu(E) \leq \int_E f d\mu \leq \sup_{x \in E} f(x) \cdot \mu(E)$$

(4) $\displaystyle\int_E c \, d\mu = c\mu(E) \qquad c \geq 0$

(5) $\displaystyle\mu(E) = 0 \Rightarrow \int_E f d\mu = 0$

THEOREM: For a μ-measurable function $f : E \to [0,\infty]$, the function $\lambda : \mathfrak{S}_E \to \overline{\mathbb{R}}$ $\quad (\mathfrak{S}_E := \{A \in \mathfrak{S} : A \subset E\})$

defined by

$$\lambda(A) := \int_A f d\mu$$

is a measure on $\mathfrak{S}_E$.

(6) If $f = g$ a.e $\quad$ ($f, g : E \to [0,\infty]$ two μ-measurable functions)

then

$$\int f d\mu = \int g \, d\mu$$

E

PROOF: Put $E_0 := \{x \in E : f(x) \neq g(x)\}$. By assumption $\mu(E_0) = 0$

and by ⑤ $\int_{E_0} f d\mu = 0 = \int_{E_0} g\, d\mu$

So

$$\int_E f d\mu = \int_{E \setminus E_0} f d\mu + \int_{E_0} f d\mu. \quad \text{(by Theorem)}$$

$$= \int_{E \setminus E_0} g d\mu + 0 = \int_{E \setminus E_0} g d\mu + \int_{E_0} g d\mu = \int_E g d\mu \quad \square$$

⑦ If $\mu(E) < \infty$ then

$$\int_E f d\mu < \infty \implies f(x) < \infty \text{ a.e.}$$

PROOF: Put $E_\infty := \{x \in E : f(x) = \infty\}$. If $\mu(E_\infty) > 0$, then

$$\infty > \int_E f d\mu = \int_{E \setminus E_\infty} f d\mu + \int_{E_\infty} f d\mu \geqslant \int_{E_\infty} f d\mu = \infty \cdot \mu(E_\infty) = \infty$$

so we get a contradiction.. So $\mu(E_\infty) = 0$. $\square$

⑧ If $\int_E f d\mu = 0$ then $f(x) = 0$ a.e.

PROOF: Put $E_n := \{x \in E : f(x) \geqslant \frac{1}{n}\}$. and $E_0 := \{x \in E : f(x) = 0\}$.

then we have that
- E_n are measurably
- $E_n \subset E_{n+1}$
- $\bigcup_{n=0}^{\infty} E_n = E$

If $f(x) > 0$ then $\exists_n \; f(x) \geqslant \frac{1}{n}$ so $x \in E_n$

$x \in \bigcup_{n=1}^{\infty} E_n$. (*)

Not necessary

(Then we have

$$0 = \lambda(E) = \lambda\left(\bigcup_{n=0}^{\infty} E_n\right) \leq \sum_{n=0}^{\infty} \lambda(E_n) \quad)$$

Notice that

$$\forall_{n \geqslant 1} \quad 0 = \int_{E_n} f d\mu \geqslant \frac{1}{n} \mu(E_n) \geqslant 0$$

$f \geqslant \chi_{E_n} \cdot f$

$0 = \lambda(E) = \int f d\mu \geqslant \int \chi_E \cdot f d\mu$

$$\forall_{n\geq 1} \quad 0 = \int_{E_n} f\,d\mu \geq \frac{1}{n}\mu(E_n) \geq 0$$

$$\Downarrow$$

$$\mu(E_n) = 0$$

$$0 = \lambda(E) = \int_E f\,d\mu \geq \int_E \chi_{E_n} \cdot f\,d\mu = \int_{E_n} f\,d\mu = \lambda(E_n) \geq 0$$

So

$$0 \leq \mu\{x : f(x) > 0\} = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) \overset{(*)}{\leq} \sum_{n=1}^{\infty} \mu(E_n) = 0 \qquad \square$$

(9) If $f(x) > 0$ a.e. on E i.e $\left(E_0 = \{x \in E : f(x) = 0\},\ E_0 = \emptyset\right)$ and $\mu(E) > 0$ then

$$\int_E f\,d\mu > 0$$

PROOF $\forall_{n\in\mathbb{N}}$ $\quad \frac{1}{n}\mu(E_n) \leq \int_{E_n} f\,d\mu$ and

since $0 < \mu(E) \leq \sum_{n=0}^{\infty} \mu(E_n)$

if $\forall_{n=1,2,\ldots}$ $\quad \frac{1}{n}\mu(E_n) \leq \lambda(E_n) = \int_{E_n} f\,d\mu = 0$

then $\mu(E_n) = 0$ for $n = 1, 2, \ldots$, and we obtain that

$0 < \mu(E) \leq \mu(E_0) \leq \mu(E)$, which is a contradiction with the assumption that $f(x) > 0$ a.e. on E.

THEOREM: (LEBESGUE MONOTONE CONVERGENCE THEOREM "LMCT")

Let $f_n : E \to [0,\infty)$ be an increasing sequence of μ-measurable functions, i.e. $\left(\forall_{x\in E}\ 0 \leq f_1(x) \leq f_2(x) \leq \ldots\ldots\right)$ then the limit $f(x) := \lim_{n\to\infty} f_n(x)$, $x \in E$ exists; $f : E \to [0,\infty]$ and we have

$$\lim_{n\to\infty} \int_E f_n\,d\mu = \int_E f\,d\mu$$

PROOF: Since, f_n is an increasing sequence, $f_n(x) \leq f_{n+1}(x) \leq f(x)$ $\forall_{n\in\mathbb{N}}\ \forall_{x\in E}$

thus (by (2)) $\int f_n\,d\mu \leq \int f_{n+1}\,d\mu \leq \int f\,d\mu$

$\forall x \in E$

thus (by (2))
$$\int_E f_n\,d\mu \le \int_E f_{n+1}\,d\mu \le \int_E f\,d\mu$$

so
$$\lim_{n\to\infty} \int_E f_n\,d\mu \le \int_E f\,d\mu$$

<u>Case 1:</u> Suppose $f(x) < \infty$ for all $x \in E$. then take $0 < \Theta < 1$ and define

$$E_n := \{ x \in E : f_n(x) \geq \Theta f(x) \}$$

$\Theta f(x)$ $f_n(x)$ $f(x)$

so E_n is a measurable set such that $E_n \subset E_{n+1}$, and
$$E = \bigcup_{n=1}^{\infty} E_n$$

Then we also have
$$\int_E f_n\,d\mu \geq \int_{E_n} f_n\,d\mu \geq \int_{E_n} \Theta f\,d\mu = \Theta \int_{E_n} f\,d\mu = \Theta\,\lambda(E_n)$$

Since λ is a measure, then
$$\lambda(E) = \lim_{n\to\infty} \lambda(E_n)$$

therefore
$$\lim_{n\to\infty} \int_E f_n\,d\mu \geq \Theta \lim_{n\to\infty} \lambda(E_n) = \Theta\,\lambda(E) = \Theta \int_E f\,d\mu$$
$\forall\, \Theta \in (0,1)$

So by passing to the limit as $\Theta \nearrow 1$ we obtain
$$\lim_{n\to\infty} \int_E f_n\,d\mu \geq \int_E f\,d\mu$$

<u>Case 2</u>: $f(x) = \infty$ for all $x \in E$. For a given $m \in \mathbb{N}$ put
$$F_n := \{ x \in E : f_n(x) \geq m \}$$

$f_n(x) \longrightarrow \infty$ as $m \to \infty$

so F_n is a measurable set, $F_n \subset F_{n+1}$ so

and
$$E = \bigcup_{n=1}^{\infty} F_n$$

$$\int_E f_n\,d\mu \geq \int_{F_n} f_n\,d\mu \geq m \cdot \mu(F_n)$$

$\checkmark$ so
$$\lim \int f_n\,d\mu \geq m \cdot \lim \mu(F_n) = m \cdot \mu(E)$$

$\forall m$ so

$$\lim_{n\to\infty} \int_E f_n \, d\mu \geq m \cdot \lim_{n\to\infty} \mu(F_n) = m \cdot \mu(E)$$

Then

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \infty \quad \text{and} \quad \lim_{n\to\infty} \int_E f_n \, d\mu \geq \int_E f \, d\mu = \infty$$

follows.

In the general case, put

$$E = E_\infty \cup E' \qquad E' := E \setminus E_\infty$$

$$E_\infty := \{x \in E : f(x) = \infty\}$$

Then we have

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \lim_{n\to\infty} \int_{E_\infty} f_n \, d\mu + \lim_{n\to\infty} \int_{E'} f_n \, d\mu \geq \int_{E_\infty} f \, d\mu + \int_{E'} f \, d\mu$$

$$= \int_E f \, d\mu . \qquad \square$$

THEOREM: Assume $g_n : E \longrightarrow [0, \infty]$ is a sequence of μ-measurable functions such that $\lim_{n\to\infty} g_n(x) = f(x)$ exists for all $x \in E$ and

$$(*) \quad \forall_{x \in E} \quad 0 \leq g_n(x) \leq f(x)$$

Then

$$\lim_{n\to\infty} \int_E g_n \, d\mu = \int_E f \, d\mu$$

PROOF: Define $f_n(x) = \inf_{k \geq n} g_k(x)$, $x \in E$

Then $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \inf g_n(x) = \lim_{n\to\infty} g_n(x) = f(x)$, and we have that $f_n(x) \leq f_{n+1}(x) \leq f(x)$, and f_n is μ-measurable. Thus by LMCT the statement follows.

$$\lim_{n\to\infty} \int_E g_n \, d\mu = \int_E f \, d\mu$$

$$f_n(x) \leq g_n(x)$$

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu \qquad \text{LMCT.}$$

$$\int_E \qquad \qquad \int_E$$

thus $\displaystyle \int_E g_n \, d\mu \geq \int_E f_n \, d\mu$

$$\liminf_{n\to\infty} \int_E g_n \, d\mu \geq \lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu \qquad (\text{LMCT})$$

$$\wedge\!|$$

$$\limsup_{n\to\infty} \int_E g_n \, d\mu$$

$$\wedge\!|$$

$$\int_E f \, d\mu . \qquad \Longrightarrow \qquad \lim_{n\to\infty} \int_E g_n \, d\mu = \int_E f \, d\mu \qquad \square$$