MATH 6301 Real Analysis I
Homework 5

Jonas Wagner
jonas.wagner@utdallas.edu

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Contents

Instructions:

1. Print this booklet
2. Use the space provided to write your solutions in this booklet
3. Hand in your assignment to your instructor on the due date during the class time.

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Problem 1

PROBLEM:

Let $(X, \mathcal{S}, \mu)$ be a measure space, $E \in \mathcal{S}$, and $f : E \to \overline{\mathbb{R}}$ a summable function. Show that

$$\mu\{x \in E \;:\; |f(x)| = \infty\} = 0$$

PRELIMINARIES:

Definition 1. *Let $f : [a, b] \to [0, \infty)$ be a measuable function. The integral of f (which can be equal to ∞) on a measurable set $E \subset [a, b]$ with respect to measure μ is defined by*

$$\int_E f(x)\mathrm{d}\mu(x) := \sup\left\{\int_E s\mathrm{d}\mu : \forall_{x\in X} 0 \leq s(x) \leq f(x)\right\}$$

Notes:

- *Each $s(x)$ can be considered a set of piecewise simple functions that approximate $f(x)$.*
- *If $\int_E f\mathrm{d}\mu < \infty$ then f is said to be summable on E.*
- *We can look at functions defined to $\mathbb{R}$ as opposed to just $[0, \infty)$ as two functions defined when positive and negative (f_+ and f_-) and then the integral is just the sum of positive minus the negative.*
- *If $E \subset [a, b]$ is a measurable set, then we can use the charectoristic/indication function of E, χ_E (1 if included in the set, otherwise 0), and do the integral as*

$$\int_E f(x)\mathrm{d}\mu(x) := \int_a^b \chi_E(x)f(x)\mathrm{d}\mu(x)$$

SOLUTION:

The solution to this problem revolves around the fact that f is defined as a summable function. Although defined unto the entire space $\overline{\mathbb{R}}$, in order for the condition to be true, $|f(x)| = \infty$ must be accompanied by $\chi_E(x) = 0$ ($x \neq E$) or be individual/distinct/(satisfying $\mu\{\cdot\} = 0$).
In a more straight forward manor, we have that f summable on E implies f_+ and f_- summable on E meaning f_+ and f_- are bounded almost anywhere on E which therefore implies $|f| = f_+ + f_-$ is also bounded almost anywhere on E. This then implies that the measure where $|f|(x) = \infty$ is 0. i.e.

$$\begin{aligned}
\int_E f\,\mathrm{d}\mu < \infty \implies & \int_E f_+\,\mathrm{d}\mu\,, \int_E f_-\,\mathrm{d}\mu < \infty \implies \\
& \implies f_+, f_- < \infty \text{ a.e. } x \in E \implies \\
& \implies |f(x)| = f_+(x) + f_-(x) < \infty \text{ a.e. } x \in E \implies \\
& \implies \mu\{x \in E \;:\; |f(x)| = \infty\} = 0
\end{aligned}$$

Problem 2

PROBLEM:

Let $(X, \mathcal{S}, \mu)$ be a measure space, $E \in \mathcal{S}$, and $f, f_n : E \to \overline{\mathbb{R}}, n = 1, 2, \ldots$ be summable functions. Show that [1]

$$\lim_{n\to\infty} \int_E |f - f_n| \mathrm{d}\mu = 0 \qquad \Longrightarrow \qquad f_n \xrightarrow{\mu} f$$

Verify if the reverse implication is also true. Justify your answer.

SOLUTION:

This problem relates much to the Lebesgue Dominated Convergence Theorems and can be thought as another version of the same concept.
By Fatou's Lemma, [2] we have that

$$\int_E \liminf_{n\to\infty} |f - f_n| \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_E |f - f_n| \mathrm{d}\mu = 0$$

For this to be true, $\liminf_{n\to\infty} |f - f_n| = 0$ almost everywhere within E. This demonstrates by definition that f_n converges to f under μ. (i.e. $f_n \xrightarrow{\mu} f$)
The reverse implication is certainly true as well. This can simply be done by constructing the sequence of $|f - f_n|$ and demonstrating that if they converge to zero almost everywhere in E then the limit of the integral will also be zero.

[1] I'm assuming there was a typo and $\lim_{n\to 0}$ should be $\lim_{n\to\infty}$. If not this can be achieved by just reversing the indices in some way (although it's weird and idk because infinity is weird)

[2] noting the with $|\cdot|$ and zero equivelence we know that lim and lim inf equivalent

Problem 3

PROBLEM:

Let (X, d) be a metric space and $A \subset X$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Find the set

$$B := \{x_0 \in X \ : \ \lim_{x \to x_0} f(x) = f(x_0)\}$$

SOLUTION:

$B \subset X$ is equivalent to $X \backslash \partial A$.
For basic metric spaces B could also be thought of as $B = \{x \in X \ : \ f'(x) = 0\}$ (everywhere that $f(x)$ is continuous).
Proofs are numerous. One potential proof could be based on the limit point definition of a set's boundary (since $f(x)$ really is just the indicator function) and then demonstrate that B is everywhere except for the boundary.

Problem 4

PROBLEM:

Let $(X, \mathcal{S}, \mu)$ be a measure space, $E \in \mathcal{S}$ and $f : E \to \overline{\mathbb{R}}$ a summable functions such that

$$\lim_{n\to\infty} \int_E |f_n - f| \mathrm{d}\mu = 0$$

and $\epsilon_k > 0$ is a given sequence such that $\lim_{k\to\infty} \epsilon_k = 0$. Show that there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$\forall_{k\in\mathbb{N}} \int_E \left|f_{n_{k+1}} - f_{n_k}\right| \mathrm{d}\mu < \epsilon_k$$

SOLUTION:

Essentially, f_n functions converge so that the distance between them and f over the set goes to zero; therefore, there will always exist another function $f_{n_{k+1}}$ 'closer' to f but close enough ($< \epsilon_k$ condition) to f_n.
More formally, from Fatou's Lemma we have $\lim_{n\to\infty} \int_E |f - f_n| \mathrm{d}\mu = 0$. We also have that this implies $f_n \xrightarrow{\mu} f$ from Problem 2.
We define the sequence $\{a_k\} := \int_E |f_n - f| \mathrm{d}\mu$ which we know converges to zero. We can then select $\{a_{n_k}\} \subset \{a_k\}$ such that $a_{n_k} + a_{n_{k+1}} < \epsilon_k$. We then use properties of the integral and triangle inequality to demonstrate this satisfies the requirements.

$$\begin{aligned}
\epsilon_k > a_{n_k} + a_{n_{k+1}} &= \int_E |f_{n_k} - f| \mathrm{d}\mu + \int_E \left|f_{n_{k+1}} - f\right| \mathrm{d}\mu \\
&= \int_E |f_{n_k} - f| + \left|-(f - f_{n_{k+1}})\right| \mathrm{d}\mu \\
&= \int_E |f_{n_k} - f| + \left|f - f_{n_{k+1}}\right| \mathrm{d}\mu \\
&\geq \int_E \left|f_{n_k} - f + f - f_{n_{k+1}}\right| \mathrm{d}\mu \\
&\geq \int_E \left|f_{n_k} - f_{n_{k+1}}\right| \mathrm{d}\mu
\end{aligned}$$

Therefore the following is satisfied for every element in the sequence

$$\int_E \left|f_{n_k} - f_{n_{k+1}}\right| \mathrm{d}\mu < \epsilon_k$$

Problem 5

PROBLEM:

Let $(X, \mathcal{S}, \mu)$ be a measure space, $E \in \mathcal{S}$ and $f : E \to \overline{\mathbb{R}}$ a summable functions such that

$$\lim_{n\to\infty} \int_E |f_n - f| \mathrm{d}\mu = 0$$

Show that there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$\lim_{k\to\infty} f_{n_k}(x) = f(x) \quad \text{a.e.} \quad x \in E$$

SOLUTION:

This follows directly from the result in Problem 2 and the Lebesgue-Reisz Theorem.
From problem 2 we have $\lim_{n\to\infty} \int_E |f - f_n| \mathrm{d}\mu = 0 \implies f_n \xrightarrow{\mu} f$. We then use the Lebesgue-Reisz Theorem to say that $f_n \xrightarrow{\mu} f \implies \lim_{k\to\infty} f_{n_k}(x) = f(x)$ almost everywhere.