LECTURE 11 — MATH 6301

BOREL SETS:

Let (X,d) be a metric space with topology $\mathcal{T} \subset \mathcal{P}(X)$. Then the σ-algebra $\mathcal{S}(\mathcal{T})$ is called the σ-algebra of Borel sets and will be denoted by $\mathcal{B}(X)$. For an Euclidean space $\mathbb{R}^n$ we put $\mathcal{B}_n$ to denote Borel sets on $\mathbb{R}^n$.

REMARK: Notice that $\mathcal{B}_n$ contains the class of all intervals:

open intervals: $I = (a_1,b_1) \times (a_2,b_2) \times \dots \times (a_n,b_n) \subset \mathbb{R}^n$; $I \in \mathcal{J}_o$ $\quad |I| = \prod_{i=1}^{n} (b_i - a_i)$

closed intervals: $I = [a_1,b_1] \times [a_2,b_2] \times \dots \times [a_n,b_n]$, $I \in \mathcal{J}_c$

and every open set $U \subset \mathbb{R}^n$ can be represented as a countable union of open (or closed) intervals, which means (in particular) that

$$\mathcal{B}_n = \mathcal{S}(\mathcal{J}_o) = \mathcal{S}(\mathcal{J}_1)$$

Moreover, for $\overline{\mathbb{R}} = [-\infty, \infty]$ the σ-algebra $\mathcal{B}(\overline{\mathbb{R}})$ is generated by the class of the intervals:

$$\mathcal{K} := \{ (a,\infty] : |a| < \infty \} \subset \mathcal{T} \qquad \mathcal{T} \text{ topology in } \overline{\mathbb{R}}$$

i.e. we have:

THEOREM: $\mathcal{B}(\overline{\mathbb{R}}) = \mathcal{S}(\mathcal{K})$.

PROOF: $\mathcal{K} \subset \mathcal{T} \Rightarrow \mathcal{S}(\mathcal{K}) \subset \mathcal{S}(\mathcal{T}) = \mathcal{B}(\overline{\mathbb{R}})$

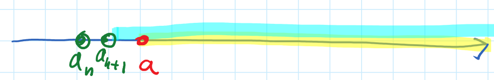
In order to show that $\mathcal{S}(\mathcal{K}) \supset \mathcal{B}(\overline{\mathbb{R}})$, it is enough to show that $\mathcal{T} \subset \mathcal{S}(\mathcal{K})$. The open sets in $\mathcal{T}$ are the countable unions of intervals

$$I = (a,b) \quad \text{or} \quad I = [-\infty, a) \checkmark \quad \text{or} \quad I = (a,\infty] \checkmark$$

which provide a base for topology $\mathcal{T}$. So we need to show that each of these intervals belong to $\mathcal{S}(\mathcal{K})$.

1) Since $(a,\infty] \in \mathcal{K}$ thus if for $a \in \mathbb{R}$ and $a_n \nearrow a$, $n \to \infty$, then $\quad a_n = a - \frac{1}{n}$ $\quad a_n < a$

$$[a,\infty] = \bigcap_{n=1}^{\infty} (a_n, \infty]$$



2) Thus $\overline{\mathbb{R}} \setminus [a,\infty] = [-\infty, a) \in \mathcal{S}(\mathcal{K})$

3) So $a < b$ then $(a,b) = [-\infty, b) \cap (a,\infty] \in \mathcal{S}(\mathcal{K})$ $\square$

PROPOSITION. Let (X,d) be a metric space and $E \in \mathcal{B}(X)$. Then the σ-algebra of Borel sets in E is the class

$$\mathcal{B}(E) = \{A \in \mathcal{B}(X) : A \subset E\} =: \mathcal{B}_E$$

PROOF: $\mathcal{T}_E := \{U \cap E : U \subset \mathcal{T}\}$. Therefore, $\mathcal{B}_E$ contains $\mathcal{T}_E$ and therefore

$$\mathcal{B}(E) = \mathcal{S}(\mathcal{T}_E) \subset \mathcal{B}_E.$$

Conversely: Assume that $A \in \mathcal{B}(X)$ and $A \subset E$, and define $\varphi : E \to X$ $\varphi(x) = x$. is a continuous map.

We need the following lemma:

! LEMMA: Let (X, d) and (Y, ρ) be two metric spaces and $f : X \to Y$ a continuous map. Then

$$\forall_{F \in \mathcal{B}(Y)} \quad f^{-1}(F) \in \mathcal{B}(X)$$

PROOF: Since

$$\mathcal{B}' := \{F \subset Y : f^{-1}(F) \in \mathcal{B}(X)\}$$

is a σ-algebra containing all open sets in Y, thus $\mathcal{B}(Y) \subset \mathcal{B}'$

which implies that if $F \in \mathcal{B}(Y)$ then $f^{-1}(F) \in \mathcal{B}(X)$ $\square$

Then by continuity of φ, we have that $A = \varphi^{-1}(A) \in \mathcal{B}(E)$. $\square$

REMARK: $\mathcal{B}_n \times \mathcal{B}_m = \mathcal{B}_{n+m}$.

MEASURABLE FUNCTIONS

Assume that $\mathcal{S} \subset \mathcal{P}(X)$ is a σ-algebra and $E \in \mathcal{S}$. A function $f : E \to \overline{\mathbb{R}}$ is called measurable relatively to $\mathcal{S}$ or called an $\mathcal{S}$-measurable function, iff

$$\forall_{a \in \mathbb{R}} \quad f^{-1}(a, \infty] := \{x \in E : f(x) > a\} \in \mathcal{S} \qquad (*)$$

Notice that, if f is $\mathcal{S}$-measurable function, then:

1) $$f^{-1}[a, \infty] = \{x \in E : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \tfrac{1}{n}\}$$
$$= \bigcap_{n=1}^{\infty} f^{-1}(a - \tfrac{1}{n}, \infty] \in \mathcal{S}$$

2) $f^{-1}[-\infty, a) = E \setminus f^{-1}[a, \infty] \in \mathcal{S}$

$\bigcup_{n=1} [a-\frac{1}{n},\infty] \in \mathcal{S}$

2) $f^{-1}[-\infty,a) = E \setminus \underbrace{f^{-1}[a,\infty]}_{\in\,\mathcal{S}} \in \mathcal{S}$

3) $f^{-1}[-\infty,a] = \bigcap_{n=1}^{\infty} f^{-1}[-\infty, a+\frac{1}{n}) \in \mathcal{S}$

COROLLARY: If $f: E \to \overline{\mathbb{R}}$, $E \in \mathcal{S}$, is $\mathcal{S}$-measurable then $\forall a \in \mathbb{R}$

1) $f^{-1}\{a\} = \{x \in E : f(x) = a\} \in \mathcal{S}$, for $f^{-1}\{a\} = f^{-1}[-\infty,a] \cap f^{-1}[a,\infty]$

2) $f^{-1}\{\infty\} = \{x \in E : f(x) = \infty\} \in \mathcal{S}$, for $f^{-1}\{\infty\} = \bigcap_{n=1}^{\infty} f^{-1}[n,\infty]$

3) $f^{-1}\{-\infty\} \in \mathcal{S}$, for $f^{-1}\{-\infty\} = \bigcap_{n=1}^{\infty} f^{-1}[-\infty,-n]$

4) $f^{-1}(a,b) \in \mathcal{S}$, for $f^{-1}(a,b) = f^{-1}[-\infty,b) \cap f^{-1}(a,\infty]$ $a<b$

5) $f^{-1}(-\infty,a) \in \mathcal{S}$, $f^{-1}(a,\infty) \in \mathcal{S}$.

THEOREM: If the function $f: E \to \overline{\mathbb{R}}$ is $\mathcal{S}$-measurable $(E \in \mathcal{S} \subset \mathcal{P}(X))$ then

$$\forall_{A \in \mathcal{B}(\overline{\mathbb{R}})} \quad f^{-1}(A) \in \mathcal{S}.$$

PROOF. Since $\mathcal{M} = \{A \in \overline{\mathbb{R}} : f^{-1}(A) \in \mathcal{S}\}$ is a σ-algebra, thus it contains $(a,\infty]$ for all $a \in \mathbb{R}$, so $\mathcal{B}(\overline{\mathbb{R}}) \subset \mathcal{M}$. $\square$

COROLLARY. For $E \in \mathcal{S} \subset \mathcal{P}(X)$, if $f: E \to \overline{\mathbb{R}}$ is $\mathcal{S}$-measurable, then

1) f is also $\mathcal{S}_E$-measurable (recall $\mathcal{S}_E := \{A : A \subset E \text{ and } A \in \mathcal{S}\}$)

2) if $A \in \mathcal{S}$, $A \subset E$, then $f_{/A}: A \to \overline{\mathbb{R}}$ is $\mathcal{S}$-measurable.

Indeed: $f_A^{-1}(a,\infty] = \{x \in A : f_A(x) > a\} = A \cap \{x \in E : f(x) > a\}$

$$= \underbrace{A}_{\in\,\mathcal{S}} \cap \underbrace{f^{-1}(a,\infty]}_{\in\,\mathcal{S}} \in \mathcal{S}.$$

COROLLARY: If $E = \bigcup_{n=1}^{\infty} A_n$, where $A_n \in \mathcal{S}$ and $f: E \to \overline{\mathbb{R}}$ is such that for every n, $f_{A_n}: A_n \to \overline{\mathbb{R}}$ is $\mathcal{S}$ measurable, then f is $\mathcal{S}$-measurable.

Indeed

$$\{x \in E : f(x) > a\} = \bigcup_{n=1}^{\infty} \underbrace{\{x : f_{A_n}(x) > a\}}_{\in\,\mathcal{S}} \in \mathcal{S} \quad \square$$

Indeed

$$\{x \in E : f(x) > a\} = \bigcup_{n=1} \underbrace{\{x : f_{A_n}(x) > a\}}_{\in\, \mathcal{S}} \in \mathcal{S} \qquad \square$$

REMARK: In particular if $E = \bigcup_{n=1}^{N} A_n$, $A_n \in \mathcal{S}$, and $f : E \to \overline{\mathbb{R}}$ is such that f_{A_n} is $\mathcal{S}$-measurable, then f is $\mathcal{S}$-measurable. $\square$

DEFINITION: A function $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ such that it is $\mathcal{B}(\overline{\mathbb{R}})$-measurable is called Baire function.

Example: $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ continuous is Baire function.

PROPOSITION: If $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a Baire function and $f : E \to \overline{\mathbb{R}}$ ($E \in \mathcal{S} \subset \mathcal{P}(X)$) is $\mathcal{S}$-measurable then the composite function

$$\varphi \circ f : E \to \overline{\mathbb{R}}$$

is $\mathcal{S}$-measurable.

PROOF Since $\forall\, a \in \mathbb{R}$ $\varphi^{-1}(a, \infty] \in \mathcal{B}(\overline{\mathbb{R}})$ (by definition) thus

$$(\varphi \circ f)^{-1}(a, \infty) = f^{-1}\big(\underbrace{\varphi^{-1}(a, \infty)}_{\in\, \mathcal{B}(\overline{\mathbb{R}})}\big) \in \mathcal{S}. \qquad \square$$

REMARK: Assume that $f : E \to \overline{\mathbb{R}}$, $E \in \mathcal{S} \subset \mathcal{P}(X)$, is $\mathcal{S}$-measurable. Then

a) $f^2 : E \to \overline{\mathbb{R}}$

b) $|f| : E \to \overline{\mathbb{R}}$

c) $\frac{1}{f} : E \to \overline{\mathbb{R}}$

d) $a \cdot f : E \to \overline{\mathbb{R}}$ $\qquad a \in \mathbb{R}$

are also $\mathcal{S}$-measurable.

CONVENTION

$$\begin{cases} \pm\infty \cdot a = \pm\infty & a > 0 \\ \pm\infty \cdot a = \mp\infty & \\ \pm\infty + a = \pm\infty & a \in \mathbb{R} \\ \dfrac{a}{\pm\infty} = 0 & a \neq 0 \\ \infty + \infty = \infty & \\ -\infty - \infty = -\infty & \\ \dfrac{0}{\infty} = 0 & \end{cases}$$

UNDEFINED EXPRESSIONS

$$\boxed{\begin{matrix} \infty - \infty \\ 0 \cdot \infty \end{matrix}} \qquad \frac{\pm\infty}{\pm\infty}$$