# LECTURE 7 – MATH 6301

**THEOREM**: (ARZELA – ASCOLI)
Let $(X,d)$ be a compact metric space and $\Phi \subset C(X;\mathbb{R})$ a bounded set. Then $\Phi$ is compact iff

(a) $\overline{\Phi} = \Phi$ (i.e. $\Phi$ is closed)

(b) $\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{x,y\in X} \; d(x,y)<\delta \Rightarrow \forall_{\varphi\in\Phi} |\varphi(x)-\varphi(y)|<\varepsilon$

the functions from $\Phi$ are uniformly equicontinuous

**PROOF**: "$\Rightarrow$" since $\Phi$ is compact, then obviously, $\Phi$ is closed, so (a) follows.
In order to show (b) take an arbitrary $\varepsilon>0$ and, since $\Phi$ is totally bounded

(1) $\exists_{\{\varphi_1,\varphi_2,\dots,\varphi_n\}\subset\Phi}$ $\quad \Phi \subset B_{\frac{\varepsilon}{3}}(\varphi_1)\cup B_{\frac{\varepsilon}{3}}(\varphi_2)\cup\dots\cup B_{\frac{\varepsilon}{3}}(\varphi_n)$
$\quad = \{\varphi_1,\varphi_2,\dots,\varphi_n\} + B_{\frac{\varepsilon}{3}}(0)$

$\frac{\varepsilon}{3}$-net

$B_\eta(x) = x + B_\eta(0)$
$\{v : \|v-x\|<\eta\}$
$x + \{w : |w|<\eta\}$
$w = v-x$
$v = w+x$

On the other hand since $(X,d)$ is compact, the functions $\varphi_1,\varphi_2,\dots,\varphi_n$ are uniformly continuous, i.e. $\forall_{i=1,2,\dots,n}$

(2) $\exists_{\delta_i>0} \forall_{x,y\in X} \; d(x,y)<\delta_i \Rightarrow |\varphi_i(x)-\varphi_i(y)|<\frac{\varepsilon}{3}$

Put $\exists\; \delta=\min\{\delta_1,\delta_2,\dots,\delta_n\}$ and consider an arbitrary function $\forall\; \varphi\in\Phi$. Then by (1), there exists $j\in\{1,2,\dots,n\}$ such that $\varphi\in B_{\frac{\varepsilon}{3}}(\varphi_j) \Leftrightarrow \|\varphi-\varphi_j\|_\infty<\frac{\varepsilon}{3}$

and consequently, $\quad (3) \Leftrightarrow \forall_{x\in X} |\varphi(x)-\varphi_j(x)|<\frac{\varepsilon}{3}$

if $d(x,y)<\delta\le\delta_j \Rightarrow |\varphi(x)-\varphi(y)| \le \overset{(3)}{|\varphi_j(x)-\varphi(x)|} + \overset{(2)}{|\varphi_j(x)-\varphi_j(y)|}$
$\quad + \overset{(3)}{|\varphi_j(y)-\varphi(y)|} < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon.$

"$\Leftarrow$" Assume that $\Phi$ is closed (a).
Since $C(X,\mathbb{R})$ is a Banach space, thus it is complete, it is sufficient to show that (a) and (b) imply that $\Phi$ is totally bounded, i.e.

? [ we need to show that $\forall_{\varepsilon>0}\exists_{\{\varphi_1,\varphi_2,\dots,\varphi_n\}\subset\Phi} \; \Phi\subset\{\varphi_1,\varphi_2,\dots,\varphi_n\}+B_\varepsilon(0)$ ] ?

$(X,d)$ metric space, $A\subset X$
The following conditions are equivalent
(i) $A$ is compact
(ii) every sequence in $A$ has convergent subsequence $\quad (*)$
(iii) $A$ is complete and $A$ is totally bounded.
We use (iii) to prove that $\Phi$ is compact.

Take an arbitrary $\forall\; \varepsilon>0$, and by condition (b)

(4) $\exists_{\delta>0}\forall_{x,y\in X} \; d(x,y)<\delta \Rightarrow \forall_{\varphi\in\Phi} |\varphi(x)-\varphi(y)|\le\frac{\varepsilon}{3}$

Since $(X,d)$ is compact, $X$ is totally bounded, i.e. there exists a $\delta$-net in $X$, i.e.

$\exists_{\{x_1,x_2,\dots,x_m\}\subset X} \quad X = B_\delta(x_1)\cup\dots\cup B_\delta(x_m)$

which implies that

(5) $\forall_{x\in X}\exists_{i\in\{1,2,\dots,m\}} \; d(x,x_i)<\delta$

Since $\varphi\in\Phi$ is continuous and $X$ is compact, $\forall_{i\in\{1,2,\dots,m\}}$ $\varphi(X)\subset\mathbb{R}$ is compact (continuous image of a compact set is compact)
so there is $M_i>0$, $\varphi(x_i)\in[-M_i,M_i]$

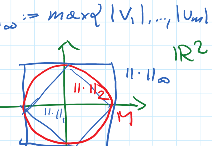
Put
$Y := [-M_1,M_1]\times[-M_2,M_2]\times\dots\times[-M_m,M_m]\subset\mathbb{R}^m$

$M=\max\{M_1,\dots,M_m\}$
$\mathbb{R}^m$, $\|\cdot\|_\infty$
$\|v\|_\infty := \max\{|v_1|,\dots,|v_m|\}$

so by Heine-Borel $Y$ is compact. Define the map
$p:\Phi\to Y$, $\quad p(\varphi) = (\underset{\in[-M_1,M_1]}{\overset{t_1}{\varphi(x_1)}}, \varphi(x_2),\dots,\underset{\in[-M_m,M_m]}{\overset{t_m}{\varphi(x_m)}})\in Y$
$\varphi \longmapsto (t_1,\dots,t_m)\in\mathbb{R}^m$

Clearly, $p$ is continuous.

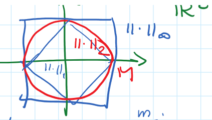
$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{\psi,\varphi\in\Phi} \; \|\varphi-\psi\|_\infty<\underset{=\varepsilon}{\delta} \Rightarrow \forall_x |\varphi(x)-\psi(x)|<\varepsilon$



$\varphi \longrightarrow (t_1,\dots,t_m) \in \mathbb{R}^m$ $\qquad [-M_1, M_1] \qquad [-M_m, M_m]$

Clearly, $p$ is continuous.

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{\psi,\varphi\in\Phi}\ \|\varphi-\psi\|_\infty < \underset{\varepsilon}{\delta} \Rightarrow \forall_x |\varphi(x_i) - \psi(x_i)| < \varepsilon$$




Then $p[\Phi] \subset Y$ ($Y$ is compact), so $p(\Phi)$ is totally bounded, $\quad \|v\|_1 = \sum_{i=1}^m |v_i|$

i.e. there exist an $\frac{\varepsilon}{3}$-net in $p(\Phi)$, $\{p(\varphi_1), p(\varphi_2), \dots, p(\varphi_n)\} \subset p(\Phi)$

satisfying

$$(6)\quad \forall_{\varphi\in\Phi}\ \exists_{k\in\{1,2,\dots,n\}}\ \forall_{i\in\{1,2,\dots,m\}}\ |\varphi_k(x_i) - \varphi(x_i)| < \frac{\varepsilon}{3}$$

$$p(\Phi) \in B_{\frac{\varepsilon}{3}}(p(\varphi_1)) \cup \dots \cup B_{\frac{\varepsilon}{3}}(p(\varphi_n))$$

$$\forall_{\varphi\in\Phi}\ \exists_{k\in\{1,\dots,n\}}\ p(\varphi) \in B_{\frac{\varepsilon}{3}}(p(\varphi_k))$$

$$\forall_{i=1,\dots,m}\ |\varphi_k(x_i) - \varphi(x_i)| < \frac{\varepsilon}{3}$$

Consequently, we have (by (6) and (5))

$$\forall_{\varphi\in\Phi}\ \exists_{k\in\{1,2,\dots,n\}}\ \forall_{x\in X}\ \exists_{i\in\{1,2,\dots,m\}}$$

$d(x, x_i) < \delta$ and

$$|\varphi(x) - \varphi_k(x)| \le |\varphi(x) - \varphi(x_i)| + |\varphi(x_i) - \varphi_k(x_i)| + |\varphi_k(x_i) - \varphi_k(x)| \overset{(4)}{<} \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

which implies that

$$\Phi \subset \{\varphi_1, \varphi_2, \dots, \varphi_n\} + B_\varepsilon(0) \quad \text{and, } \Phi \text{ is totally bounded.}$$

**REMARK:** Given a Banach space $\mathbb{V}$ and $(X, d)$ a compact metric space

Consider the space

$$\mathcal{E} := C(X; \mathbb{V}) \overset{\text{def}}{:=} \{\varphi : X \to \mathbb{V} : \varphi \text{ is continuous}\}$$

and since $\varphi(X)$ is compact, it is bounded and

$$\|\varphi\|_\infty := \sup_{x\in X} \|\varphi(x)\| \qquad (\mathbb{V}, \|\cdot\|)$$

Then, by exactly the same argument (which we applied for $C(X;\mathbb{R})$) the space $\mathcal{E}$ is a Banach space, (take $\{\varphi_n\} \subset \mathcal{E}$ a Cauchy sequence, then $\forall_{x\in X}$ $\{\varphi_n(x)\}$ is Cauchy in $\mathbb{V}$, and since $\mathbb{V}$ is complete, the limit

$$\varphi(x) := \lim_{n\to\infty} \varphi_n(x) \text{ exists}$$

and by using Uniform Convergence theorem, $\varphi \in C(X; \mathbb{V})$ (i.e. is continuous)

Then we have the following generalization of of Arzela-Ascoli Theorem

**THEOREM** (Arzela-Ascoli). Let $(X, d)$ be a compact metric space, $\mathbb{V}$ a Banach space and $\Phi \subset C(X; \mathbb{V})$ a bounded set.

Then $\Phi$ is compact in $C(X; \mathbb{V})$ iff

(a) $\Phi = \overline{\Phi}$

(b) $\forall_{x\in X}$ $\{\varphi(x) : \varphi \in \Phi\}$ is compact in $\mathbb{V}$

(c) $\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x,y\in X}\ d(x,y) < \delta \Rightarrow \forall_{\varphi\in\Phi}\ \|\varphi(x) - \varphi(y)\| < \varepsilon$

PROOF: The proof follows exactly the same constructions as in the first theorem with $|\cdot|$ replaced with $\|\cdot\|$ (in $\mathbb{V}$), and the space $Y$ replaced by

$$Y = \mathcal{M}_1 \times \mathcal{M}_2 \times \dots \times \mathcal{M}_m \ ; \quad \mathcal{M}_k := \{\varphi(x_k) : \varphi \in \Phi\}$$

which is compact by (b)

Then, $Y$ is compact.

$$Y = \mathcal{M}_1 \times \mathcal{M}_2 \times \dots \times \mathcal{M}_m \ ; \quad \mathcal{M}_k := \{ \varphi(x_k) : \varphi \in \Phi \}$$

which is compact by (b)

Then, $Y$ is compact.

$y = (y_1, y_2, \dots, y_m) \in \mathbb{V}^m$

$\{y^n\} \subset Y \ , \ \forall_k \ \{y^n_k\} \subset \mathcal{M}_k$

and the remainder of the proof is the same $\square$.