MATH 6301 Real Analysis I
Homework 1

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Contents

Instructions:

1. Print this booklet
2. Use the space provided to write your solutions in this booklet
3. Hand in your assignment to your instructor on the due date during the class time.

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Problem 1

PROBLEM:

A set A in a metric space (X, d) is called *bounded* if

$$\exists_{R>0}\exists_{x_0\in X}\ A \subset B_R(x_0)$$

Use mathematical induction to show that if a set $A \subset X$ is unbounded (i.e. it is not bounded), then there exists a sequence $x_n \subset A$ such the $d(x_n, x_m) \forall m \neq n$.

SOLUTION:

Definition 1. *Let $A \subset X$ be a set in metric space (X, d). The open ball of radius R centered at x_0, denoted as $B_R(x_0)$, is defined by*

$$\{x \in X\ :\ d(x, x_0) < R\}$$

Proposition 1. *Let $A \subset X$ be unbounded, then there exists a sequence $x_n \subset A$ such the $d(x_n, x_m) \geq 1 \forall m \neq n$.*

Proof. For $n = 0$ and $m = 1$, select $x_m \in A\ :\ d(x_n, x_m) \geq 1$.
Next, for each n, since A is unbounded we know that $\exists_{x_m\in A}\forall_{i=1,\ldots,n} d(x_i, x_m) \geq 1$. We know this by the definition of unbounded, bounded, and an open ball:

$$\begin{aligned}\exists_{R>0}\exists_{x_0\in X}\ A \subset B_R(x_0) &\iff \forall_{R>0}\forall_{x_0\in X}\ A \subset B_R(x_0)\\ &\implies \forall_{R>0}\forall_{x_0\in X} A \supseteq B_R(x_0)\\ &\implies \exists_{R\geq 1}\forall_{m=1,\ldots,n} d(x_n, x_m) \geq R\end{aligned}$$

By mathematical induction, we can say that a sequence $x_n \subset A$ can be constructed such the $d(x_n, x_m) \geq 1 \forall m \neq n$. $\square$

Problem 2

PROBLEM:

Let $(V, \|\cdot\|)$ be a normed vector Space. Show that an open unit ball

$$B_1(0) \coloneqq \{v \in V \ : \ \|v\| < 1\}$$

is a convex set, i.e.

$$\forall_{u,v \in B_1(0)} \forall_{t \in [0,1]} tu + (1-t)v \in B_1(0)$$

SOLUTION:

Proposition 2. *$B_1(0)$ is a convex set.*

Proof. Select arbitrary $u, v \in B_1(0)$ and $t \in [0,1]$. By definition of $x \in B_1(0) \iff \|x\| < 1$. We have $\|v\| < 1$ and $\|u\| < 1$.

$$\|tu + (1-t)v\|$$

By the triangle inequality,

$$\leq \|tu\| + \|(1-t)v\|$$

Since t is a scaler,

$$\leq (t)\|u\| + (1-t)\|v\|$$

Since $\|v\| < 1$ and $\|u\| < 1$,

$$< (t)(1) + (1-t)(1) = t - t + 1 = 1$$

Therefore,

$$\|tu + (1-t)v\| < 1$$

which means

$$\forall_{u,v \in B_1(0)} \forall_{t \in [0,1]} \ tu + (1-t)v \in B_1(0)$$

$\square$

Problem 3

PROBLEM:

Suppose that (X, d) is a metric space and $A, B \subset X$ are such that $A \subset B$. Show that $\int(A) \subset \int(B)$ and $\overline{A} \subset \overline{B}$.

SOLUTION:

Definition 2. *Let A, B be sets in metric space (X, d). We define the* closure of A *is given by*

$$\overline{A} := \{x \in X \vee x = \lim_{k\to\infty} x_k \ : \ [x_k] \in A\}$$

<u>Difference of A and B</u>

$$A \backslash B = A \cap B^c := \{x \in \mathbb{R}^n \ : \ x \in A \wedge x \neq B\}$$

TO DO: correct definitions and then prove it simply by definition (really simple solution)

Problem 4

Let (X_1, d_1) and (X_2, d_2) be two metric spaces. Show that each of the following functions d is a metric on $X = X_1 \times X_2$ (Hint: Knowing that d_1 and d_2 satisfies the conditions of a metric, show that d also satisfies these conditions)

PRELIMINARIES:

Definition 3. *A* **Metric Space**, *(X, d) consists of a space X and* <u>metric</u>, *$d : X \times X \to \mathbb{R}$, that satisfies the following:*

1. *Distance to itself is zero*
$$\forall_{x \in X} d(x, x) = 0$$
2. <u>Positivity:</u>
$$\forall_{x,y \in X} d(x, y) \geq 0$$
3. <u>Symmetry:</u>
$$\forall_{x,y \in X} d(x, y) = d(y, x)$$
4. *Satisfies* <u>triangle inequality</u>
$$\forall_{x,y,z \in X} d(x, z) \leq d(x, y) + d(y, z)$$

a)

PROBLEM:

Show that the following satisfies the conditions of a metric:

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

SOLUTION:

Proposition 3. *Let (X_1, d_1) and (X_2, d_1) be metric spaces. We propose that (X, d) with $X = X_1 \times X_2$ and $d : X_1 \times X_2 \to \mathbb{R}$ is defined as*

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

Proof. d satisfies the metric conditions:

1. Distance to itself is zero
$$d((x_1, x_2), (x_1, x_2)) = \max\{d_1(x_1, x_1), d_2(x_2, x_2)\} = \max\{0, 0\} = 0$$
2. <u>*Positivity:*</u>
$$d_1(x_1, y_1) \geq 0 \wedge d_2(x_2, y_2) \geq 0 \implies \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} \geq 0$$
3. <u>*Symmetry:*</u>
$$d_1(x_1, y_1) = d_1(y_1, x_1) \wedge d_2(x_2, y_2) = d_2(y_2, x_2)$$
$$\implies \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} = \max\{d_1(y_1, x_1), d_2(y_2, x_2)\}$$
4. Satisfies <u>*triangle inequality*</u>
$$d_1(x_1, z_1) \leq d_1(x_1, y_1) + d_1(y_1, z_1) \wedge d_2(x_2, z_2) \leq d_2(x_2, y_2) + d_2(y_2, z_2)$$
$$\implies \max\{d_1(x_1, z_1), d_2(x_2, z_2)\} \leq \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} + \max\{d_1(y_1, z_1), d_2(y_2, z_2)\}$$

$\square$

b)

PROBLEM:

Show that the following satisfies the conditions of a metric:

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2}$$

SOLUTION:

Proposition 4. *Let (X_1, d_1) and (X_2, d_1) be metric spaces. We propose that (X, d) with $X = X_1 \times X_2$ and $d : X_1 \times X_2 \to \mathbb{R}$ is defined as*

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2}$$

Proof. d satisfies the metric conditions:

1. Distance to itself is zero

$$d((x_1, x_2), (x_1, x_2)) = \sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2)^2)} = \sqrt{(0)^2 + (0^2)} = 0$$

2. <u>*Positivity:*</u>

$$\begin{aligned} d_1(x_1, y_1) \geq 0 \wedge d_2(x_2, y_2) \geq 0 &\implies (d_1(x_1, y_1))^2 \geq 0 \wedge (d_2(x_2, y_2))^2 \geq 0 \\ &\implies (d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2 \geq 0 \\ &\implies \sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2)^2)} \geq 0 \end{aligned}$$

3. <u>*Symmetry:*</u>

$$\begin{aligned} d_1(x_1, y_1) = d_1(y_1, x_1) &\wedge d_2(x_2, y_2) = d_2(y_2, x_2) \\ \implies (d_1(x_1, y_1))^2 = (d_1(y_1, x_1))^2 &\wedge (d_2(x_2, y_2))^2 = (d_2(y_2, x_2))^2 \\ \implies (d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2 &= (d_2(y_2, x_2))^2 + (d_1(y_1, x_1))^2 \\ \implies \sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2)^2)} &= \sqrt{(d_1(y_1, x_1))^2 + (d_2(y_2, x_2)^2)} \end{aligned}$$

4. Satisfies <u>*triangle inequality*</u>

$$\left(\sqrt{(d_1(x_1, z_1))^2 + (d_2(x_2, z_2)^2)^2}\right)^2 = (d_1(x_1, z_1))^2 + (d_2(x_2, z_2)^2)^2 + 2d_1(x_1, z_1)d_2(x_2, z_2)$$

$$\begin{aligned} &\leq \left(\sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2} + \sqrt{(d_1(y_1, z_1))^2 + (d_2(y_2, z_2))^2}\right)^2 \\ &= (d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2 + (d_1(y_1, z_1))^2 + (d_2(y_2, z_2))^2 \\ &+2\sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2}\sqrt{(d_1(y_1, z_1))^2 + (d_2(y_2, z_2))^2} \end{aligned}$$

Therefore,

$$\begin{aligned} d_1(x_1, z_1) \leq d_1(x_1, y_1) + d_1(y_1, z_1) \wedge d_2(x_2, z_2) \leq d_2(x_2, y_2) + d_2(y_2, z_2) \\ \implies \sqrt{(d_1(x_1, z_1))^2 + (d_2(x_2, z_2)^2)} \leq \sqrt{(d_1(x_1, y_1))^2 + (d_2(x_2, y_2)^2)} + \sqrt{(d_1(y_1, z_1))^2 + (d_2(y_2, z_2)^2)} \end{aligned}$$

$\square$

c)

PROBLEM:

Show that the following satisfies the conditions of a metric:

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$$

SOLUTION:

Proposition 5. *Let (X_1, d_1) and (X_2, d_1) be metric spaces. We propose that (X, d) with $X = X_1 \times X_2$ and $d : X_1 \times X_2 \to \mathbb{R}$ is defined as*

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$$

Proof. d satisfies the metric conditions:

1. Distance to itself is zero

$$d((x_1, x_2), (x_1, x_2)) = d_1(x_1, y_1) + d_2(x_2, y_2) = 0 + 0 = 0$$

2. *Positivity:*

$$d_1(x_1, y_1) \geq 0 \wedge d_2(x_2, y_2) \geq 0 \implies d_1(x_1, y_1) + d_2(x_2, y_2) \geq 0$$

3. *Symmetry:*

$$d_1(x_1, y_1) = d_1(y_1, x_1) \wedge d_2(x_2, y_2) = d_2(y_2, x_2)$$
$$\implies d_1(x_1, y_1) + d_2(x_2, y_2) = d_1(y_1, z_1) + d_2(y_2, x_2)$$

4. Satisfies *triangle inequality*

$$d_1(x_1, z_1) \leq d_1(x_1, y_1) + d_1(y_1, z_1) \wedge d_2(x_2, z_2) \leq d_2(x_2, y_2) + d_2(y_2, z_2)$$
$$\implies d_1(x_1, z_1) + d_2(x_2, z_2) \leq d_1(x_1, y_1) + d_2(x_2, y_2) + d_1(y_1, z_1) + d_2(y_2, z_2)$$

$\square$

Problem 5

Let (X, d) be a metric space and $A \subset X$ a closed non-empty set. For a given point $x \in X$, we define the *distance* from x to A by the formula

$$d(x, A) \coloneqq \inf\{d(x, a) \ : \ a \in A\}$$

a)

PROBLEM:

Show that $x \in A \iff d(x, A) = 0$.

SOLUTION:

Proposition 6. $x \in A \iff d(x, A) = 0$

Proof. First we look at $\implies$,

$$x \in A \implies \exists d(x, a) \ : \ a \in A = 0 \implies \inf\{0, \dots\} = 0 \implies d(x, A) = \inf\{d(x, a) \ : \ a \in A\} = 0$$

Next we look at $\impliedby$,

$$d(x, A) = \inf\{d(x, a) \ : \ a \in A\} = 0 \implies \exists_{\{x_n\} \subset A} \ : \ \lim_{n \to \infty} d(x, a) = 0 \implies x \in A$$

$\square$

b)

PROBLEM:

Show that the function $\Xi_A : X \to \mathbb{R}$ defined by $\Xi_A(x) = d(x, A)$ is continuous on X.

PRELIMINARIES:

Definition 4. *A function $f : X \to Y$ is* continous *if*

SOLUTION:

TO DO: include the continuos definition and prove...