Thursday, October 27, 2022 5:24 PM

LECTURE 20 — MATH 6301

THEOREM (LUZIN)

Let $E \in \mathcal{L}_n$. Then a function $f: E \to \mathbb{R}$ is $\mathcal{L}_n$-measurable

$$\iff \underset{\varepsilon>0}{\forall} \ \underset{\substack{F\subset E\\ F=\overline{F}}}{\exists} \quad m_n(E\setminus F)<\varepsilon \text{ and } f|_F : F \to \mathbb{R} \text{ is continuous.}$$

PROOF: $\Leftarrow$ We need to show that $\underset{a\in\mathbb{R}}{\forall}$ the set $E_a := \{x\in E : f(x) \geq a\}$ is $\mathcal{L}_n$-measurable.

Given $\varepsilon>0$, choose $F=\overline{F}\subset E$ such that $m_n(E\setminus F)<\varepsilon$ and $f_{|F}: F\to\mathbb{R}$ is continuous. Put

$$F_a := \{x\in F : \underset{f_{|F}(x)}{\overset{\shortparallel}{f(x)}} \geq a\} = E_a \cap F \quad \text{and } F_a \in \mathcal{L}_n$$

Since $E_a\setminus F_a \subset E\setminus F$ we have

$$m_n^*(E_a\setminus F_a) \leq m_n(E\setminus F) < \varepsilon$$

which implies that

$$\underset{\varepsilon>0}{\forall} \ \underset{F=\overline{F}\subset E}{\exists} \quad m_n^*(E_a) \leq m_n^*(E_a\setminus F_a) + m_n^*(F_a)$$

$$\leq \varepsilon + m_{n*}(E_a))$$

i.e. $m_n^*(E_a) = m_{n*}(E_a)$ so E_a is $\mathcal{L}_n$-measurable

$\Rightarrow$ First, we will "make" the function $f: E\to\mathbb{R}$ bounded. Namely, we define $g: E\to\mathbb{R}$ by

$$g(x) = \arctan(f(x))$$

arctan: $\mathbb{R} \xrightarrow{} (-\frac{\pi}{2}, \frac{\pi}{2})$, $\xleftarrow{\tan}$

Then f is $\mathcal{L}_n$-measurable $\iff$ g is $\mathcal{L}_n$-measurable

Since g is $\mathcal{L}_n$-measurable and bounded by the Simple-Functions Approximation Theorem there exists a sequence of simple measurable functions $\sigma_k : E\to\mathbb{R}$ convergent uniformly to g.

REMARK

Take a simple $\mathcal{L}_n$-measurable function $\sigma: E\to\mathbb{R}$ and $E = E_1\cup\ldots\cup E_m$, $E_j\in\mathcal{L}_n$, $E_j\cap E_i=\emptyset$ and $\alpha_j\in\mathbb{R}$, $j=1,\ldots,m$

$$\sigma(x) = \sum^{m} \alpha_j \chi \ (x)$$

$E = E_1 \cup \dots \cup E_m$, $E_j \in \mathcal{L}_n$, $E_j \cap E_i = \emptyset$ and $\alpha_j \in \mathbb{R}$, $j = 1, \dots, m$

$$\sigma(x) = \sum_{j=1}^{m} \alpha_j \chi_{E_j}(x)$$

Then we have the property: For every $\varepsilon > 0$ and for every $j = 1, 2, \dots, m$ there exists a closed set $F_j \subset E_j$ such that $m_n(E_j \setminus F_j) < \frac{\varepsilon}{m}$

So we can put $F = F_1 \cup F_2 \cup \dots \cup F_m$ and we have

$$m_n(E \setminus F) < \varepsilon$$

and the function $\sigma_{|F} : F \to \mathbb{R}$ is continuous.

For a given $\varepsilon > 0$.

Take the simple function σ_k, find a closed subset $F_k \subset E$ (specified in Remark) such that $\sigma_{k|F_k} : F_k \to \mathbb{R}$ is continuous and

$$m_n(E \setminus F_k) < \frac{\varepsilon}{2^k}$$

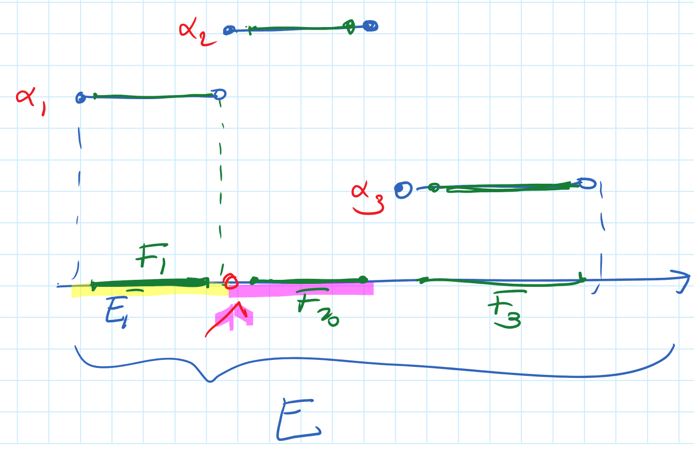
Then put

$$F = \bigcap_{k=1}^{\infty} F_k, \quad \text{so } F \text{ is closed and}$$

$$\begin{aligned} m_n(E \setminus F) &= m_n\Big(E \setminus \bigcap_{k=1}^{\infty} F_k\Big) \\ &= m_n\Big(\bigcup_{k=1}^{\infty} (E \setminus F_k)\Big) \\ &\le \sum_{k=1}^{\infty} m_n(E \setminus F_k) = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon \end{aligned}$$

Since, σ_n converges uniformly to g on E, it converges uniformly on F. Then the Uniform Convergence theorem $\lim_{k\to\infty} \sigma_k(x) = g(x)$, $x \in F$, (σ_k continuous on $F \subset F_k$) is continuous. Consequently $f(x) = \tan(g(x))$ $x \in F$ is also continuous. $\square$

$\sigma_{|F} : F \to \mathbb{R}$



E

THEOREM (Fréchet)

Let $E \in \mathcal{L}_n$ and $f : E \to \mathbb{R}$ be an $\mathcal{L}_n$-measurable function. Then there exists a sequence of continuous bounded functions $f_k : E \to \mathbb{R}$ such that

$$\lim_{k\to\infty} f_k(x) = f(x) \quad \text{a.e on } E.$$

PROOF: By Luzin Theorem, for every $k \in \mathbb{N}$, take $\varepsilon = \frac{1}{k}$, then $\exists\, F_k \subset E$, $F_k = \overline{F_k}$, $m_n(E \setminus F_k) < \frac{1}{k}$ and

$$f_{|F_k} : F_k \to \mathbb{R} \quad \text{is continuous.}$$

(X, d), $A \subset X$, $A = \overline{A}$

$$\begin{tikzcd} A \arrow[r, "g", "\text{cont}"'] \arrow[d, hook] & \mathbb{R} \\ X \arrow[ur, "F_g"', "\text{cont.}"] & \end{tikzcd}$$

Since, F_k is closed, by Tietze Extension Theorem there exists a continuous extension $\tilde{f}_k : \mathbb{R}^n \to \mathbb{R}$ of $f_{|F_k}$. Then we define

$$f_k(x) = \max\left(\min\left(\tilde{f}_k(x), k\right), -k\right)$$

and notice that

$$\forall_k \quad m_n\left(E \setminus \bigcup_{k=1}^{\infty} F_k\right) = m_n\left(\bigcap_{k=1}^{\infty} (E \setminus F_k)\right) \leq m_n(E \setminus F_k) < \frac{1}{k}$$

so

$$m_n\left(E \setminus \bigcup_{k=1}^{\infty} F_k\right) = 0$$

and

$$\forall_{x \in \bigcup_{k=1}^{\infty} F_k} \quad f_k(x) \longrightarrow f(x)$$; so the statement follows. $\square$

We assume that $(X, \mathcal{S}, \mu)$ is a measure space. Take $E \in \mathcal{S}$ and consider a sequence $f_n : E \to \mathbb{R}$ of $\mathcal{S}$-measurable (μ-measurable, measurable) Then if

$$f(x) := \lim_{n\to\infty} f_n(x) \quad \text{exists for a.e. } x \in E$$

i.e. f is a pointwise limit a.e on E of measurable functions, then if μ is a <u>complete measure</u>, then f is also μ-measurable

Assign#6

REMARK: If $f_n, g_n : E \to \mathbb{R}$ are two sequences of measurable functions such that $f_n(x) = g_n(x)$ a.e on E, and if $f(x) = \lim_{n\to\infty} f_n(x)$ for a.e. $x \in E$ then the limit $g(x) = \lim_{n\to\infty} g_n(x)$ exists for a.e. $x \in E$ and

$$f(x) = g(x) \quad \text{a.e on } E.$$

LEMMA: Let $f, f_n : E \to \overline{\mathbb{R}}$, $n = 1, 2, \ldots$, be μ-measurable functions finite a.e and $\mu(E) < \infty$. If $f_n(x) \to f(x)$ a.e on E then

$$\forall_{\varepsilon>0} \quad \lim_{n\to\infty} \mu\left(\bigcup_{k=n}^{\infty} E_k(\varepsilon)\right) = 0 \quad \text{where}$$

$$E_k(\varepsilon) = \{ x \in E : |f_k(x) - f(x)| \geq \varepsilon \}$$

PROOF: the sets $\bigcup_{k=n}^{\infty} E_k(\varepsilon)$ are decreasing, thus

$$\lim_{n\to\infty} \mu\left(\bigcup_{k=n}^{\infty} E_k(\varepsilon)\right) = \mu\left(\lim_{n\to\infty} \bigcup_{k=n}^{\infty} E_k(\varepsilon)\right)$$
$$= \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k(\varepsilon)\right)$$

Notice that in order to show that $\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k(\varepsilon)\right) = 0$ it is sufficient to notice that

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k(\varepsilon) \subset \{ x \in E : f_k(x) \not\to f(x) \} \cup \{ x \in E : |f| = \infty \}$$

Indeed, if $x_0 \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k(\varepsilon)$ then $|f_k(x_0) - f(x_0)| \geq \varepsilon$ for sufficiently large k, so $f_k(x_0) \not\to f(x_0)$. or $|f(x)| = \infty$. $\square$

DEFINITION: Let $f, f_n : E \to \overline{\mathbb{R}}$ be the same as in Lemma. We say that f_n is μ-convergent to f or (convergent in measure μ) (notation $f_n \xrightarrow{\mu} f$) iff

$$\forall_{\varepsilon>0} \quad \lim_{n\to\infty} \mu\{ x \in E : |f_n(x) - f(x)| \geq \varepsilon \} = 0$$

THEOREM (Lebesgue) Under the same assumptions as in Lemma

<u>THEOREM</u> (Lebesgue) Under the same assumptions as in Lemma and $\mu(E) < \infty$, if $f_n(x) \to f(x)$ a.e on E then $f_n \xrightarrow{\mu} f$.

(PROOF is direct consequence of Lemma).