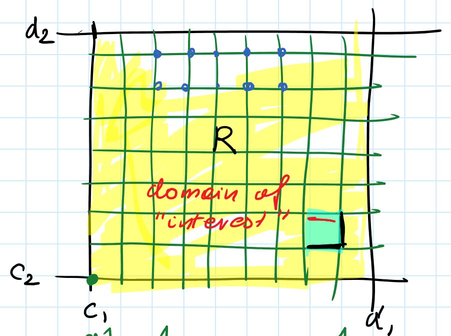
Tuesday, November 29, 2022 5:29 PM

LECTURE 27 — MATH 6301

RIEMANN INTEGRALL AND JORDAN MEASURE

Given a (rectangular) interval R in $\mathbb{R}^n$:

$$R = [c_1, d_1] \times [c_2, d_2] \times \ldots \times [c_n, d_n]$$



$n = 2$

$c_1 = a_0^1 < a_1^1 < \ldots < a_{m_1}^1 = d_1$

$c_2 = a_0^2 < a_1^2 < \ldots < a_{m_2}^2 = d_2$

Given a partition P of R, i.e. $P = \{I_i\}$ where

$$i = (i_1, i_2, \ldots, i_n) \qquad 0 < i_k \leq m_k$$

$$I_i = [a^1_{i_1-1}, a^1_{i_1}] \times [a^2_{i_2-1}, a^2_{i_2}] \times \ldots \times [a^n_{i_n-1}, a^n_{i_n}]$$

So P is a partition of R into intervals I_i satisfying:

1) $\mathring{I}_i \cap \mathring{I}_{i'} = \emptyset \quad i \neq i'$

2) $\bigcup_i I_i = R$

We put $\quad |I_i| := \max\{|a^k_{i_k-1} - a^k_{i_k}| : k = 1, \ldots, n\} \geq 0$

and call $\quad \|P\| := \max_i |I_i| \quad$ size of partition P.

REMARK: It is useful to identify the partition P with the points (Lattice points)

$(a^1_{i_1}, a^2_{i_2}, \ldots, a^n_{i_n}) =: a_i$

$1 \leq i_1 \leq m_1$
$1 \leq i_2 \leq m_2$
$\vdots$
$1 \leq i_n \leq m_n$

$P = \{a_i\}$
$i = (i_1, i_2, \ldots, i_n)$

NOTATION: We denote by $\mathcal{P}$ the collection of all partitions of R, and introduce the following relation: $P, P' \in \mathcal{P}$

$$P \leq P' \iff P \subset P'$$

P' refinement of P

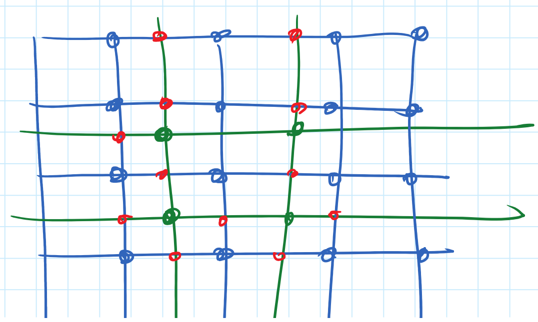
PROPOSITION: For every two partitions $P, P' \in \mathcal{P}$ there exists a partition $Q \in \mathcal{P}$ such that

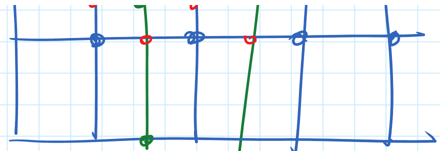
1) $P, P' \leq Q$

2) $\|P\|, \|P'\| \geqslant \|Q\|$

Take any partition Q such that

$Q \supset P \cup P'$?





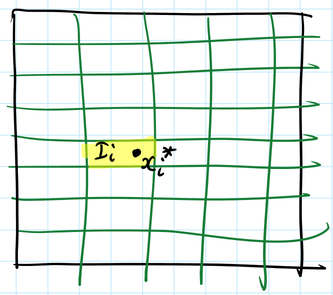
Moreover, for every $P \in \mathcal{P}$ and $\varepsilon > 0$ there exist a partition $Q \in \mathcal{P}$ such that

1) $P \leq P'$

2) $\|P'\| < \varepsilon$

<u>DEFINITION</u>: Let $f : R \to \mathbb{R}$ be a bounded function, and assume that $P \in \mathcal{P}$ is a given partition of R, and $\forall_{i=(i_1, i_2, \dots, i_n)}$ we have chosen $x_i^* \in I_i$

$x^* = \{x_i^*\}$



Then we call

$$\sigma(f, P, x^*) := \sum_i f(x_i^*)\,|I_i| \quad \text{the Riemann sum of } f \text{ for } P \text{ and } x^*$$

$$s(f, P) := \sum_i m_i(f)\,|I_i|, \quad m_i(f) := \inf_{x \in I_i} f(x) \quad \text{Lower Darboux sum of } f \text{ for } P$$

$$S(f, P) := \sum_i M_i(f)\,|I_i|, \quad M_i(f) := \sup_{x \in I_i} f(x) \quad \text{Upper Darboux sum of } f$$

Clearly, for any partition $P \in \mathcal{P}$ and associated with P choice of x^* we have:

$$s(f, P) \leq \sigma(f, P, x^*) \leq S(f, P)$$

<u>DEFINITION</u>: We say that $f : R \to \mathbb{R}$ is Riemann integrable iff

$$\underline{\int_R} f\,dx := \sup_P s(f, P) = \inf_P S(f, P) =: \overline{\int_R} f\,dx$$

lower Darboux integral. upper Darboux integral

and

$$\underline{\int_R} f\,dx = \overline{\int_R} f\,dx = \int_R f\,dx = \int_R f(x_1, x_2, \dots, x_n)\,dx_1\,dx_2 \dots dx_n \quad \text{Riemann integral.}$$

f is Riemann integrable.

Notice that the correspondance $f \longmapsto \int_R f\,dx$ satisfies the properties:

(P1) <u>Linearity</u>

(P1) Linearity

$$\int_R (\alpha f_1 + \beta f_2)\, dx = \alpha \int_R f_1 dx + \beta \int_R f_2\, dx$$

(P2) Monotonicity $\qquad f_1(x) \le f_2(x), \quad x \in R$

$$\int_R f_1 dx \le \int_R f_2 dx$$

(P3) $$\int_R c\, dx = c\,|R| \qquad |R| = \mathrm{vol}(R)$$

(P4) Mean Value $\qquad m \le f(x) \le M \quad \forall x \in R$

$$m\,|R| \le \int_R f dx \le M\,|R|$$

Then, notice that the properties (P1) — (P4) define the Riemann integral uniquely for any Riemann integrable function $f: R \to \mathbb{R}$

THEOREM: For an integrable function $f: R \to \mathbb{R}$, we have

$$\int_R f(x)dx = \int_{c_1}^{d_1} \int_{c_2}^{d_2} \cdots \int_{c_n}^{d_n} f(x_1, x_2, \dots, x_n)\, dx_n\, dx_{n-1} \dots dx_1$$

iteratted integral.

Moreover, notice that $f: R \to \mathbb{R}$ is Riemann integrable iff

(a) $$\lim_{\|P\| \to 0} \big(S(f,P) - s(f,P) \big) = 0$$

(b) $$\lim_{\|P\| \to 0} \sigma(f, P, x^*) = \int_R f dx$$

exists and is equal to and doesn't depend on choice of x^*.

RIEMANN INTEGRAL OVER ARBITRARY DOMAIN:

Let $D \subset R \subset \mathbb{R}^n$ (here we assume that D is a bounded set) and $f: D \to \mathbb{R}$ a given (bounded) function. Then we define $\tilde{f}: R \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & x \in D \\ 0 & x \notin D \end{cases}$$

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<u>DEFINITION:</u> We say that Riemann integral $\int_D f dx$ exists iff $\int_R \tilde{f} dx$ exists and we put

$$\int_D f dx := \int_R \tilde{f} dx$$

Since, it is appropriate to expect from the integral $\int_D f dx$ that it satisfies the basic properties (P1) — (P4), in particular we anticipate that

$$\int_D dx = \int_R \chi_D dx = |D| =: \nu(D)$$

exists

this number exists and represents a kind of a "measure" of D

This measure, if it exists, is called the Jordan measure of D (remember Jordan measure is not a measure in the sense of a measure space terminology)

To be more precise, notice that $\int_D dx$ exists iff

$$\sup_P s(\chi_D, P) = \inf_P S(\chi_D, P)$$

(1) $$s(\chi_D, P) = \sum_i m_i(\chi_D)\, |I_i| \qquad m_i(\chi_D) = \inf_{x \in I_i} \chi_D(x) = \begin{cases} 1 & I_i \subset D \\ 0 & I_i \not\subset D \end{cases}$$

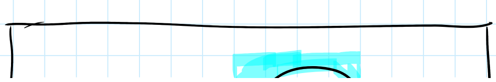
(2) $$S(\chi_D, P) = \sum_i M_i(\chi_D)\, |I_i| \qquad M_i(\chi_D) = \sup_{x \in I_i} \chi_D(x) = \begin{cases} 1 & I_i \cap D \neq \emptyset \\ 0 & I_i \subset R \setminus D \end{cases}$$

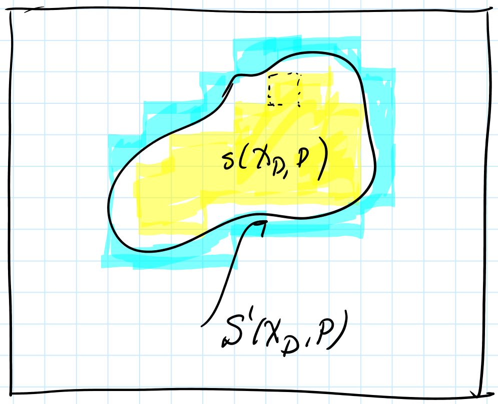
(1) $$\nu_*(D) = \sup_P s(\chi_D, P) := \sup_P \left\{ \sum |I_i| : I_i \subset D \right\}$$ Jordan inner measure

(2) $$\nu^*(D) = \inf_P S(\chi_D, P) := \inf_P \left\{ \sum |I_i| : I_i \cap D \neq \emptyset \right\}$$ Jordan upper measure.

Then, the set $D \subset R$ is Jordan measurable iff

$$\nu_*(D) = \nu^*(D) =: \nu(D).$$





In such a case

$$\lim_{\|P\|\to 0} \sum_{I_i \in D} |I_i| = \lim_{\|P\|\to 0} \sum_{I_i \cap D \neq \emptyset} |I_i|$$
$$= v(D)$$

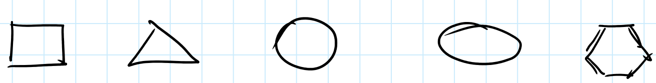
Then, for any Jordan measurable set D and a bounded function $f: D \to \mathbb{R}$ one can define the Riemann Integral

$$\underbrace{\lim_{\|P\|\to 0} \sum_{I_i \subset D} m_i(f)\, |I_i|}_{= \int_D f\,dx} = \underbrace{\lim_{\|P\|\to 0} \sum_{I_i \cap D \neq \emptyset} M_i(f)\, |I_i|}_{= \int_D f\,dx} = \int_D f\,dx\,.$$

<u>Example</u>: $n=1$, i.e. $\mathbb{R}$ $\quad D \subset \mathbb{R}$ $\quad D = [0,1]$ $\quad v(D) = 1$

$D = [0,1] \cap \mathbb{Q}$ $\quad v^*(D) = 1$, $\quad v_*(D) = 0$

$n=2$ i.e. $\mathbb{R}^2$



<u>THEOREM</u>: If $D \subset \mathbb{R}^n$ is a compact set such that ∂D is a submanifold of $\mathbb{R}^n$ then D is Jordan measurable. We say in such a case that D is a <u>regular</u> body. Moreover, if D is a finite intersection (or union) of regular bodies then it is Jordan measurable.

<u>THEOREM</u>: $D \subset \mathbb{R}^n$ (bounded set) is Jordan measurable $\Longleftrightarrow$ D is Lebesgue measurable and $m(\partial D) = 0$.