LECTURE 25 - MATH 6301

Integration of Functions of Arbitrary Sign:

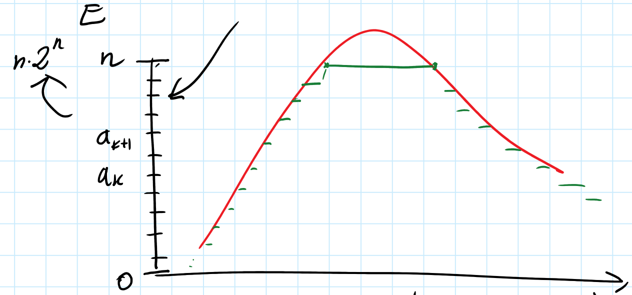
Suppose $(X, \mathfrak{S}, \mu)$ be a measure space, $E \in \mathfrak{S}$, and $f: E \to \overline{\mathbb{R}}$ be a μ-measurable function. How to define

$$\int_E f\,d\mu \quad ?$$

In the case $f(x) \geq 0$ we have a definition of $\int_E f\,d\mu$, namely we can use

$$(*) \qquad \int_E f\,d\mu = \lim_{n\to\infty} \sum_{k=0}^{n2^n} k2^{n}\cdot\mu(E_k), \quad \overset{\infty}{\underset{\text{for}}{k = n2^n}}$$

$$s_n(x) = \sum_{k=0}^{n2^n} k2^{n} \chi_{E_k}(x)$$



$$E_k = \begin{cases} f^{-1}[k2^{n}, (k+1)2^{n}) & k < n2^n \\ f^{-1}[n2^n, \infty) & k = n2^n \end{cases}$$

(1) $\quad 0 \leq s_n(x) \leq s_{n+1}(x) \leq \dots$

(2) $\quad \lim s_n(x) = f(x)$

By Lebesgue Monotone Convergence Theorem

$$\int_E f\,d\mu = \lim_{n\to\infty} \int_E s_n\,d\mu = \lim_{n\to\infty} \sum_{k=0}^{n2^n} k2^{n}\mu(E_k) \quad \begin{cases} < \infty \\ = \infty \end{cases}$$

For an arbitrary function $f: E \to \overline{\mathbb{R}}$, we put

$$f_+(x) = \max\{0, f(x)\}$$
$$f_-(x) = \max\{0, -f(x)\}$$

then $f(x) = f_+(x) - f_-(x)$

By <u>definition</u>: we put

$$\int_E f\,d\mu := \int_E f_+\,d\mu - \int_E f_-\,d\mu$$

whenever one of these integrals is finite.

disaster: $\infty - \infty$

In such a case, we say that $f: E \to \overline{\mathbb{R}}$ is ~~integrable~~ μ-integrable

In such a case, we say that $f: E \to \mathbb{R}$ is integrable
μ-integrable

If $\int_E f\,d\mu$ is a finite number, then we say that f is summable

REMARK: A μ-measurable function $f: E \to \overline{\mathbb{R}}$ is summable iff

$$\int_E |f|\,d\mu = \int_E f_+\,d\mu + \int_E f_-\,d\mu < \infty$$

In the case the measure μ stands for the Lebesgue measure $\mathcal{M}_n$ (in $\mathbb{R}^n$)
we will write

$$\text{instead of } \int_E f\,d\mathcal{M}_n, \quad \int_E f(x)\,dx = \int_E f(x_1, x_2, \dots, x_n)\,dx_1 dx_2 \dots dx_n$$

PROPERTIES:

① $\int_C c\,d\mu = c\,\mu(E)$ $\quad c \in \mathbb{R}$

and if $\mu(E) = 0$ then

$$\int_E f\,d\mu = 0$$

Moreover, if $f(x) = g(x)$ a.e on E then

$$\int_E f\,d\mu = \int_E g\,d\mu$$

Problem

$f(x) = g(x)$ a.e
$\Updownarrow$
$f_-(x) = g_-(x)$ a.e and
$f_+(x) = g_+(x)$ a.e

PROPOSITION: Let $f: E \to \overline{\mathbb{R}}$ be a μ-measurable function. Then
f is summable $\iff$ $\exists\, \varphi: E \to [0,\infty]$ i) φ μ-measurable ii) φ summable $\quad \forall x \in E \quad |f(x)| \le \varphi(x)$

PROOF $\Rightarrow$ then $|f(x)|$ is summable so we take $\varphi(x) := |f(x)|$

$\Leftarrow$ if $\int_E \varphi\,d\mu < \infty$ and $|f(x)| \le \varphi(x)$

then $0 \le \int_E |f|\,d\mu \le \int_E \varphi\,d\mu \implies \int_E |f|\,d\mu < \infty$ so f is summable.

$\square$

PROPOSITION: (a) If $f: E \to \overline{\mathbb{R}}$ is summable then for every $F \subset E$, $F \in \mathcal{S}$, f is also summable on F, i.e $\int_F |f|\,d\mu < \infty$.

PROOF: Put

$$g(x) = \begin{cases} |f(x)| & \text{if } x \in F \\ 0 & \text{if } x \in E \setminus F \end{cases}$$

then $g(x) \leq |f(x)| \quad \forall_{x \in E}$

and

$$\int_F |f| \, d\mu = \int_E g \, d\mu \leq \int_E |f| \, d\mu < \infty$$

and the conclusion follows.

(b) Assume that $E_n \subset E$, $E_n \in \mathfrak{S}$, $n = 1, 2, \dots, N$ are such that

$$E = \bigcup_{n=1}^{N} E_n$$

If f is summable on E_n for all $n = 1, 2, \dots, N$ then f is also summable on E.

PROOF We can assume that the sets E_n are disjoint and since

$$\lambda(A) := \int_A |f| \, d\mu \quad \text{is a measure on } \mathfrak{S}_E$$

thus we have (by σ-additivity) that

$$\int_E |f| \, d\mu = \lambda(E) = \lambda\Big(\bigcup_{n=1}^{N} E_n\Big)$$

$$= \sum_{n=1}^{N} \lambda(E_n) = \int_{E_1} |f| \, d\mu + \dots + \int_{E_N} |f| \, d\mu < \infty$$

so it is summable.

(c) if $E_n \subset E$, $E_n \in \mathfrak{S}$, $n = 1, 2, \dots$, is such that

(a) $\bigcup_{n=1}^{\infty} E_n = E$

(b) $\sum_{n=1}^{\infty} \int_{E_n} |f| \, d\mu < \infty$

then f is summable on E

PROPOSITION: If $f : E \to \overline{\mathbb{R}}$ is summable then we have

$$\mu\{x \in E : |f(x)| = \infty\} = 0$$

PROOF: f summable then $\int_E f_+ \, d\mu$, $\int_E f_- \, d\mu < \infty$ then $f_+(x) < \infty$ a.e E
$f_-(x) < \infty$ a.e E
$\Downarrow$
$|f(x)| = f_+(x) + f_-(x) < \infty$ a.e E

③ If for two μ-measurable (integrable) functions $f, g : E \to \overline{\mathbb{R}}$

we have $f(x) \le g(x)$ on E then

$$\int_E f\,d\mu \le \int_E g\,d\mu$$

PROOF: If $f(x) \le g(x)$ then $\underset{\max\{0, f(x)\}}{\underset{\shortparallel}{f_+(x)}} \le \underset{\max\{0, g(x)\}}{\underset{\shortparallel}{g_+(x)}}$

$\underset{\max\{0, -f(x)\}}{\underset{\shortparallel}{f_-(x)}} \geq \underset{\max\{0, -g(x)\}}{\underset{\shortparallel}{g_-(x)}}$

$$\int_E f\,d\mu = \int_E f_+\,d\mu - \int_E f_-\,d\mu \le \int_E g_+\,d\mu - \int_E g_-\,d\mu = \int_E g\,d\mu \quad \square$$

(4) Mean Value Theorem: For an integrable function $f: E \to \overline{\mathbb{R}}$

$$\left(\inf_{x\in E} f(x)\right)\cdot \mu(E) \le \int_E f\,d\mu \le \left(\sup_{x \in E} f(x)\right)\cdot \mu(E)$$

(5) $$\left|\int_E f\,d\mu\right| \le \int_E |f|\,d\mu$$ for any integrable function.

(6) $E = \bigcup_{n=1}^{\infty} E_n$, $E_n \in \mathcal{S}$, $\underset{n \neq m}{E_n \cap E_m} = \emptyset$

$$\int_E f\,d\mu = \sum_{n=1}^{\infty} \int_{E_n} f\,d\mu$$

summable on E $\qquad$ summable on E_n

PROOF:

$\lambda^+(A) = \int_A f_+\,d\mu$ is a measure

$\lambda^-(A) = \int_A f_-\,d\mu$ is also a measure

in particular they satisfy σ-additivity property

then we have

$$\int_E f\,d\mu = \int_E f_+\,d\mu - \int_E f_-\,d\mu = \lambda^+\left(\bigcup_{n=1}^{\infty} E_n\right) - \lambda^-\left(\bigcup_{n=1}^{\infty} E_n\right)$$

$$= \sum_{n=1}^{\infty} \lambda^+(E_n) - \sum_{n=1}^{\infty} \lambda^-(E_n)$$

$$n=1 \qquad n=1$$

$$= \sum_{n=1}^{\infty} \left(\lambda^{+}(E_n) - \lambda^{-}(E_n) \right) = \sum_{n=1}^{\infty} \left(\int_{E_n} f_+ \, d\mu - \int_{E_n} f_- \, d\mu \right)$$

$$= \sum_{n=1}^{\infty} \int_{E_n} f \, d\mu \qquad \square$$

(7) Assume that $f, g : E \to \overline{\mathbb{R}}$ are summable then $\forall \alpha, \beta \in \mathbb{R}$ we have that $\alpha f + \beta g$ is summable and

$$\int_E (\alpha f + \beta g) \, d\mu = \alpha \int_E f \, d\mu + \beta \int_E g \, d\mu$$

PROOF: (i) $\alpha f + \beta g$ is summable; indeed

$$\int_E |\alpha f + \beta g| \, d\mu \leq \int_E \left(|\alpha| |f| + |\beta| |g| \right) d\mu \leq |\alpha| \underbrace{\int_E |f| \, d\mu}_{<\infty} + |\beta| \underbrace{\int_E |g| \, d\mu}_{<\infty} < \infty$$

is summable

(ii) Step 1: First we will show that:

$$\int_E \alpha f \, d\mu = \alpha \int_E f \, d\mu$$

$\alpha \geq 0$: $(\alpha f)_+ = \alpha f_+$, $(\alpha f)_- = \alpha f_-$ $\qquad$ $\alpha \leq 0$: $(\alpha f)_+ = -\alpha f_-$, $(\alpha f)_- = -\alpha f_+$

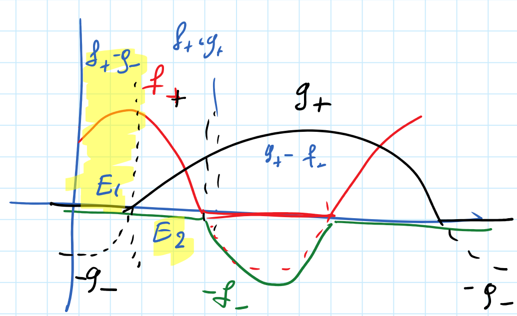
then $\alpha \leq 0$

$$\int_E \alpha f \, d\mu = \int_E (\alpha f)_+ \, d\mu - \int_E (\alpha f)_- \, d\mu = \int_E (-\alpha) f_- \, d\mu - \int_E (-\alpha) f_+ \, d\mu$$

$$= -\alpha \int_E f_- \, d\mu + \alpha \int_E f_+ \, d\mu = \alpha \int_E f \, d\mu$$

Step 2 We also have

$$\int_E (f+g) \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu$$



Notice that

$$(f+g)_+ - (f+g)_- = f_+ - f_- + g_+ - g_-,$$

$$\int_E (f+g) \, d\mu = \int_E (f+g)_+ - \int_E (f+g)_- \, d\mu$$

$$= \int f_+ \, d\mu - f_- \, d\mu + \int g_+ \, d\mu - \int g_- \, d\mu = \int f \, d\mu + \int g \, d\mu$$

$$= \int_E f_+ d\mu - \int_E f_- d\mu + \int_E g_+ d\mu - \int_E g_- d\mu = \int_E f d\mu + \int_E g d\mu$$

THEOREM (Absolute Continuity of Integral)

Assume $f: E \to \overline{\mathbb{R}}$ is summable. Then

$$\lim_{\mu(A)\to 0} \int_A f d\mu = 0 \iff \forall_{\varepsilon>0} \exists_{\delta>0} \forall_{\substack{A\subset E \\ A\in \mathfrak{S}}} \mu(A)<\delta \Rightarrow \left| \int_A f d\mu \right| < \varepsilon$$

$A \subset E$ $A \in \mathfrak{S}$

PROOF: Put $E_n := \{x \in E : |f(x)| \geq n\}$

$E_n \supset E_{n+1} \supset \ldots$ $\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n =: E_\infty = \{x : |f(x)| = \infty\}$

Since f is summable, ie $\int_E |f| d\mu < \infty$, thus $\mu(E_\infty) = 0$, and $\int_{E_\infty} |f| d\mu = 0$

and since

$$\lambda(A) = \int_A |f| d\mu \quad \text{is a measure on } \mathfrak{S}_E$$

thus

$$0 = \lambda(E_\infty) = \lim_{n\to\infty} \lambda(E_n).$$

Let $\varepsilon > 0$ be an arbitrary, then $\exists_m$ $\lambda(E_m) < \frac{\varepsilon}{2}$. if $\mu(A) < \frac{\varepsilon}{2m} =: \delta$

so $\left| \int_A f d\mu \right| \leq \int_A |f| d\mu = \lambda(A) = \lambda(A \cap E_m) + \lambda(A \setminus E_m)$

$$\leq \lambda(E_m) + \int_{A\setminus E_m} |f| d\mu \leq \frac{\varepsilon}{2} + m \int_{A\setminus E_m} d\mu$$

$$\leq \frac{\varepsilon}{2} + m\,\mu(A \setminus E_m) \leq \frac{\varepsilon}{2} + m\,\mu(A)$$

$$\leq \frac{\varepsilon}{2} + m \frac{\varepsilon}{2m} = \varepsilon. \quad \square$$