LECTURE 24 - MATH 6301

LEBESGUE INTEGRATION OF POSITIVE FUNCTIONS

Let $(X, \mathfrak{S}, \mu)$ be a measure space, $E \in \mathfrak{S}$ and $f : E \to [0,\infty]$ a μ-measurable function. Then we put

$$\int_E f d\mu \overset{\text{def.}}{:=} \sup_{\substack{E = \bigcup_{n=1}^{\infty} E_n \\ E_n \cap E_k = \emptyset \text{ for } n \neq k \\ E_n \in \mathfrak{S}}} \sum_{n=1}^{\infty} \inf_{x \in E_n} f(x) \cdot \mu(E_n)$$

1) Some properties: for two μ-measurable functions $f, g : E \to [0,\infty]$, such that $\forall_{x\in E}\ f(x) \le g(x)$ we have $\int_E f d\mu \le \int_E g\, d\mu$

2) if $\mu(E) = 0$ $\int_E f d\mu = 0$

This implies that if $f(x) \le g(x)$ a.e $\int_E f d\mu \le \int_E g\, d\mu$

3) if $\mu(E) < \infty$ and $\int_E f d\mu < \infty$ then $0 \le f(x) < \infty$ a.e.

<u>Lebesgue Monotone Convergence Theorem</u>: Let $f_n : E \to [0,\infty]$ be a sequence of μ-measurable functions such that

1) $0 \le f_1(x) \le \dots \le f_n(x) \le f_{n+1}(x) \le \dots$ for a.e $x \in E$

2) $f(x) := \lim_{n\to\infty} f_n(x)$ for a.e. $x \in E$ for some $f : E \to [0,\infty)$ μ-measurable.

Then

$$\lim_{n\to\infty} \int_E f_n\, d\mu = \int_E f d\mu$$

<u>Dominated Convergence Theorem</u>: Let $g_n : E \to [0,\infty]$, $n = 1, 2, \dots$, $f : E \to [0,\infty]$ be μ-measurable functions such that

(a) $0 \le g_n(x) \le f(x)$ for a.e $x \in E$

(b) $\lim_{n\to\infty} g_n(x) = f(x)$ for a.e. $x \in E$

Then

$$\lim_{n\to\infty} \int_E g_n d\mu = \int_E f d\mu .$$

<u>PROPOSITION</u>: Let $f : E \to [0,\infty]$ be a μ-measurable function then we have

$$\int_E f d\mu = \sup \left\{ \int_E \sigma\, d\mu : \begin{array}{l} 1)\ \sigma \text{ is a simple } \mu\text{-measurable function } \sigma = \sum_{k=1}^{m} a_k \chi_{E_k} \\ 2)\ 0 \le \sigma(x) \le f(x) \quad \forall_{x\in E} \end{array} \right\}$$

<u>PROOF</u>: For μ-measurable simple function $\sigma = \sum_{k=1}^{m} a_k \chi_{E_k}$, $E = E_1 \cup \dots \cup E_m$, $E_k \in \mathfrak{S}$, $E_j \cap E_k = \emptyset$, $k \neq j$ we have

$$\lambda(E) := \int_E \sigma\, d\mu = \int_{\bigcup_{k=1}^{m} E_k} \sigma\, d\mu = \sum_{k=1}^{m} \int_{E_k} \sigma\, d\mu = \sum_{k=1}^{m} a_k\, \mu(E_k)$$

measure on $\mathfrak{S}$

Recall
$\int_E c\, d\mu = c\cdot\mu(E)$

On the other hand, if $\sigma(x) \le f(x)$ for all $x \in E$ then we have that

$$\int_E \sigma\, d\mu \le \int_E f\, d\mu \qquad \text{(for every such a simple function } \sigma)$$

thus

$$\alpha := \sup\left\{ \int_E \sigma\, d\mu : \begin{array}{l} (1)\ \sigma\ \mu\text{-measurable simple} \\ (2)\ 0 \le \sigma(x) \le f(x) \end{array} \right\} \le \int_E f\, d\mu$$

On the other hand, by Simple Function Approximation Theorem, there exists a sequence of simple μ-measurable functions $\sigma_n : E \to [0,\infty)$ such that

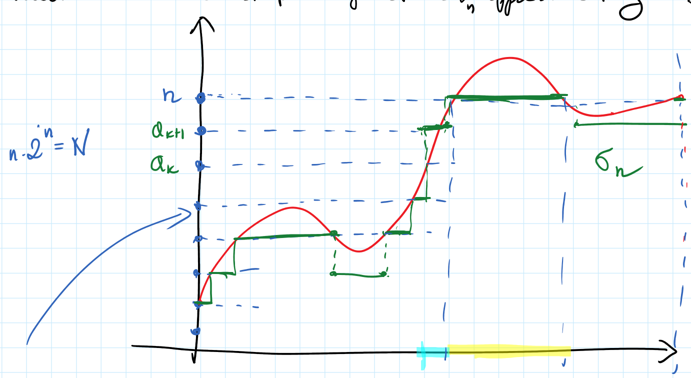
(a) $0 \le \sigma_1(x) \le \dots \le \sigma_n(x) \le \sigma_{n+1}(x) \le \dots$

(b) $\lim_{n\to\infty} \sigma_n(x) = f(x)$ for all $x \in E$.

So by LMCT (above) we have that

$$\alpha \ge \lim_{n\to\infty} \int_E \sigma_n\, d\mu = \int_E f\, d\mu$$ and the conclusion follows. $\square$

Recall how the simple functions σ_n approximating f (from below) are constructed



$$E_k = f^{-1}([a_k, a_{k+1}))$$

$$\sigma(x) = \sum_k a_k \chi_{E_k}(x)$$

COROLLARY: For a μ-measurable function $f : E \to [0,\infty]$ we have

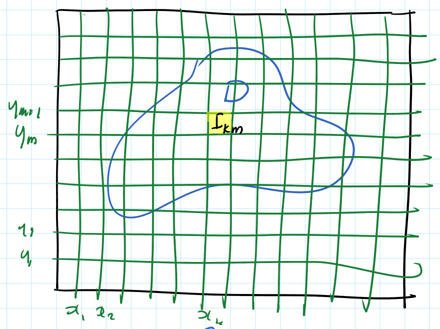
$$\int_E f\, d\mu = \sup\left\{ \sum_{k=1}^{N} a_k \mu(E_k) : \begin{array}{l} (1)\ 0 \le a_1 < \dots < a_N \\ (b)\ E_k = f^{-1}([a_k, a_{k+1})) \end{array} \right\}$$

COROLLARY: Assume that D is a closed bounded set in $\mathbb{R}^n$ and $f : D \to \mathbb{R}$ a Riemann-integrable function. Then f is also Lebesgue $\mathcal{M}_n$-measurable function and the Riemann integral of f coincides with the Lebesgue integral of f.

PROOF: Recall Riemann integral: For a closed bounded set $D \subset \mathbb{R}^n$

Lebesgue integral of f.

PROOF: Recall Riemann integral: For a closed bounded set $D \subset \mathbb{R}^n$ first we choose $R = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$ containing D.



I_{km}

$I_{i_1 i_2 \dots i_n} = [x_1^{i_1}, x_1^{i_1+1}] \times \dots \times [x_n^{i_n}, x_n^{i_n+1}]$

$\alpha = (i_1, i_2, \dots, i_n)$

Then we extend f to $\tilde{f} : R \to \mathbb{R}$ by $\tilde{f}(x) = \begin{cases} f(x) & x \in D \\ 0 & x \notin D \end{cases}$

and then we say that f is Riemann integrable $\overset{\text{def}}{\iff}$ $\tilde{f}$ is Riemann integrable which means,

$$\sup_{P \in \mathcal{P}} s(\tilde{f}, P) = \inf_{P \in \mathcal{P}} S(\tilde{f}, P)$$

$P = R = \bigcup_{\alpha} I_\alpha$

$\alpha = (i_1, i_2, \dots, i_n)$

$$s(\tilde{f}, P) = \sum_{\alpha} \underbrace{\inf_{x \in I_\alpha} \tilde{f}(x)}_{} \cdot |I_\alpha| \quad m_n(I_\alpha)$$

lower Darboux sum

$$S(\tilde{f}, P) = \sum_{\alpha} \sup_{x \in I_\alpha} \tilde{f}(x) \cdot |I_\alpha| \quad m_n(I_\alpha)$$

upper Darboux sum

$$\sigma(\tilde{f}, P, \{x^\alpha\}) = \sum_{\alpha} \tilde{f}(x^\alpha)\, |I_\alpha|$$

Riemann sum

$\{x^\alpha\}_\alpha$, $x^\alpha \in I_\alpha$

$$s(\tilde{f}, P) \le \sigma(\tilde{f}, P, \{x^\alpha\}) \le S(\tilde{f}, P)$$

THEOREM: $\tilde{f}$ is Riemann integrable if one of the following conditions is satisfied

(a) $\lim_{\|P\| \to 0} \left(S(\tilde{f}, P) - s(\tilde{f}, P) \right) = 0$

$\|P\| = \max_{\alpha} \operatorname{diam}(I_\alpha)$

(b) $\lim_{\|P\| \to 0} \sigma(\tilde{f}, P, \{x^\alpha\}) =: \int_R \tilde{f}(x)\,dx < \infty$ exists.

$\Updownarrow$

$$\forall_{\varepsilon > 0} \; \exists_{\delta > 0} \; \forall_{P = \{I_\alpha\}} \; \forall_{\{x_\alpha\},\, x_\alpha \in I_\alpha} \quad \|P\| < \delta \implies \left| \int_R \tilde{f}(x)\,dx - \sigma(\tilde{f}, P, \{x^\alpha\}) \right| < \varepsilon$$

Then by (a) we have

$$\int_R \tilde{f}(x)\,dx = \lim_{\|P\| \to 0} S(\tilde{f}, P) = \lim_{\|P\| \to 0} s(\tilde{f}, P)$$

$$\int_R \tilde{f}(x)\,dx = \lim_{\|P\|\to 0} S(\tilde{f}, P) = \lim_{\|P\|\to 0} s(\tilde{f}, P)$$

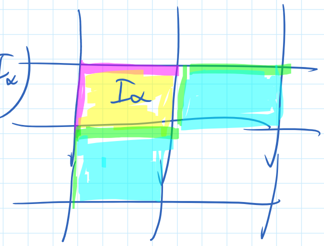
Notice that for a given partition $P = \{I_\alpha\}$ of R the function

$$\sigma(x) = \sum_\alpha \inf_{x \in I_\alpha} f(x)\, \chi_{I_\alpha}(x)$$

(we need to make small manipulation on boundaries of I_α)

and we obtain

$$s(\tilde{f}, P) = \int_R \sigma\, d\mu_n = \sum_\alpha \inf_{x\in I_\alpha} f(x)\, \mu_n(I_\alpha)$$



So, we can see that by taking a limit of simple functions $\sigma_n(x)$ (obtained by partitioning R) such that $\sigma_n(x) \le f(x)$ and we obtain that.

$\sigma_n(x) \le \sigma_{n+1}(x) \le \dots$

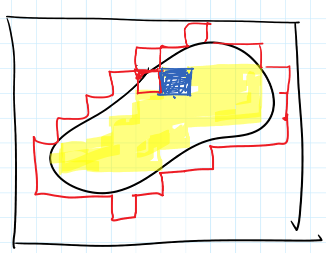
$$\int_R f\, d\mu \le \int_R f(x)\,dx$$

$\sigma'_n(x) \ge \sigma'_{n+1}(x) \ge \dots \ge f(x)$, so by a similar argument

$$\lim_{n\to\infty} S(f, P_n) = \int_R f(x)\,dx \ge \int_R f\, d\mu. \qquad \square$$

JORDAN MEASURE OF $D \subset \mathbb{R}^n$.

In order to be able to integrate over set D we need to assure that χ_D is integrable



$$S(\chi_D, P) = \sum_{I_\alpha \cap D \neq \emptyset} |I_\alpha|$$

$$s(\chi_D, P) = \sum_{I_\alpha \subset D} |I_\alpha|$$

Put $\nu_*(D) := \sup_P s(\chi_D, P)$ Lower Jordan measure

$\nu^*(D) := \inf_P S(\chi_D, P)$ Upper Jordan measure.

We say that D is Jordan measurable iff $\nu_*(D) = \nu^*(D) =: \nu(D)$.