Thursday, September 29, 2022 5:21 PM

LECTURE 12 - MATH 630 I

MEASURABLE FUNCTIONS (CONTINUATION)

RECALL: Algebraic Operations in $\overline{\mathbb{R}}$ (convention for $\pm\infty$)

$\forall a \in \mathbb{R}$

$a + (\pm\infty) = \pm\infty$ $\quad -\infty + (-\infty) = -\infty$

$\frac{a}{\infty} = 0$ $\quad \infty \cdot \infty = \infty$ $\quad (-\infty)\cdot(\infty) = -\infty$ $\quad +\infty + (+\infty) = +\infty$

$a > 0 \quad a\cdot(\pm\infty) = \pm\infty$

$a < 0 \quad a\cdot(\pm\infty) = \mp\infty$

UNDETERMINED EXPRESSIONS

$0\cdot\infty$ $\quad \frac{\infty}{\infty}$ $\quad \frac{0}{0}$

$\infty - \infty$

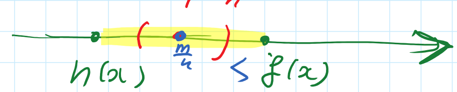
CONVENTION $\boxed{0\cdot\infty = 0}$

PROPOSITION 1: Let $\mathcal{S} \subset \mathcal{P}(X)$ be a σ-algebra, $E \in \mathcal{S}$, and $f, g : E \to \overline{\mathbb{R}}$ two measurable functions. Then for every $a \in \mathbb{R}$ we have
$$\{x \in E : f(x) - g(x) > a\} \in \mathcal{S},$$
i.e. $f(x) - g(x)$ is measurable.

PROOF:

Notice:
$$\{x \in E : f(x) > h(x)\} = \underbrace{\bigcup_{\substack{m,n\\ n\neq 0}}}_{\text{countable union}} \underbrace{\overbrace{\{x : f(x) > \tfrac{m}{n}\}}^{\text{measurable}} \cap \overbrace{\{x : h(x) < \tfrac{m}{n}\}}^{\text{measurable}}}_{\text{measurable}}$$

$f(x) > h(x)$ $\quad q = \frac{m}{n}$ $\quad n \neq 0$



$h(x) < \frac{m}{n} < f(x)$

therefore $\{x \in E : f(x) > h(x)\}$ is measurable.

Moreover, we have $\{x \in E : f(x) - g(x) > a\} = \{x \in E : f(x) > \underbrace{g(x) + a}_{h(x) \text{ measurable}}\}$

so, indeed, this set is measurable.

PROPOSITION 2. Let $\mathcal{S} \subset \mathcal{P}(X)$ be a σ-algebra, $E \in \mathcal{S}$, $f, g : E \to \overline{\mathbb{R}}$ two $\mathcal{S}$-measurable functions on E. Then

(a) $\alpha f + \beta g$ is $\mathcal{S}$-measurable on E $\quad \alpha, \beta \in \mathbb{R}$ (if $\alpha f + \beta g$ is well-defined)

(b) f^2 is $\mathcal{S}$-measurable on E

(c) $f \cdot g$ is $\mathcal{S}$-measurable on E

PROOF (a) f $\mathcal{S}$-measurable then αf is $\mathcal{S}$-measurable

Indeed, $\forall \alpha \neq 0$ $\quad \{x : \alpha f(x) > a\} = \{x : \begin{cases} f(x) > \frac{a}{\alpha} & \text{if } \alpha > 0 \\ f(x) < \frac{a}{\alpha} & \text{if } \alpha < 0 \end{cases}\}$ $\mathcal{S}$-measurable.

Therefore, by Proposition 1, $\alpha f + \beta g$ is $\mathcal{S}$-measurable.

(b) Notice, $\forall a \in \mathbb{R}$ the set
$$\{x \in E : f^2(x) > a\} = \{x \in E : f(x) > \sqrt{a}\} \quad a \geq 0$$

$t^2 > a \iff t > \sqrt{a}$, $t < -\sqrt{a}$

measurable

(b) Notice, $\forall a \in \mathbb{R}$ ...

$$\{x \in E : f^2(x) > a\} = \{x \in E : \underbrace{f(x) > \sqrt{a}}_{\text{measurable}}\} \cup \{x \in E : \underbrace{f(x) < -\sqrt{a}}_{\text{measurable}}\} \quad a \geq 0$$

$t^2 > a \iff t > \sqrt{a}$ or $t < -\sqrt{a}$

is measurable.

(c) Since $f(x)g(x) = \frac{1}{4}\left[(f(x)+g(x))^2 - (f(x)-g(x))^2\right]$, and measurability of f and g implies that (by (a)) $f(x)+g(x)$, $f(x)-g(x)$ are measurable, and (by (b)) $(f(x)+g(x))^2$, $(f(x)-g(x))^2$ are measurable, so

$$\frac{1}{4}\left[(f(x)+g(x))^2 - (f(x)-g(x))^2\right] \text{ is measurable. } \square$$

THEOREM: Assume that $\mathfrak{S} \subset \mathcal{P}(X)$ is a σ-algebra, $E \in \mathfrak{S}$ and $f_n : E \to \overline{\mathbb{R}}$ is a sequence of $\mathfrak{S}$-measurable functions. Put

$$\varphi(x) = \inf_n f_n(x) \qquad x \in E$$

$$\psi(x) = \sup_n f_n(x). \qquad x \in E$$

Then the functions $\varphi, \psi : E \to \overline{\mathbb{R}}$ are $\mathfrak{S}$-measurable.

PROOF: Notice that

$$\{x \in E : \varphi(x) < a\} = \{x \in E : \inf_n f_n(x) < a\}$$
$$= \{x \in E : \exists_n \; f_n(x) < a\}$$
$$= \bigcup_{n=1}^{\infty} \{x \in E : \underbrace{f_n(x) < a}_{\text{measurable}}\}$$

$\alpha = \inf_n f_n(x) < a \iff$ 1) $\forall_n \; \alpha \leq f_n(x)$ 2) $\forall_{\varepsilon > 0} \exists_n \; \alpha + \varepsilon > f_n(x)$

take. $a = \alpha + \varepsilon$

If $\inf_n f_n(x) < a$, then by 2) $\exists_n \; a > f_n(x)$ i.e. $x \in \bigcup_{n=1}^{\infty} \{x \in E : f_n(x) < a\}$

$$\{x : \forall_{i \in I} \; \varphi_i(x) > a\} = \bigcap_{i \in I} \{x : \varphi_i(x) > a\}$$

$$\{x : \exists_{i \in I} \; \varphi_i(x) > a\} = \bigcup_{i \in I} \{x : \varphi_i(x) > a\}$$

$$\{x \in E : \psi(x) > a\} = \{x \in E : \sup_n f_n(x) > a\}$$
$$= \{x \in E : \exists_n \; f_n(x) > a\}.$$
$$= \bigcup_{n=1}^{\infty} \{x \in E : f_n(x) > a\}. \quad \square$$

$\beta = \sup_n f_n(x) \iff \forall_n \; \beta \geq f_n(x)$, $\forall_{\varepsilon > 0} \exists_n \; \underset{a}{\beta - \varepsilon} < f_n(x)$

COROLLARY: If $\mathfrak{S} \subset \mathcal{P}(X)$ is σ-algebra, $E \in \mathfrak{S}$ and $f_n : E \to \overline{\mathbb{R}}$

is a sequence of $\mathcal{S}$-measurable functions, then

$$\varphi(x) = \liminf_{n\to\infty} f_n(x)$$
and
$$\psi(x) = \limsup_{n\to\infty} f_n(x)$$

are measurable.

PROOF Observe that

$$\psi(x) = \limsup_{n\to\infty} f_n(x) = \underbrace{\inf_n \underbrace{\sup_{k\geq n} f_k(x)}_{\text{measurable}}}_{\text{measurable}} \Rightarrow \text{measurable}$$

$$\varphi(x) = \liminf_{n\to\infty} f_n(x) = \underbrace{\sup_n \underbrace{\inf_{k\geq n} f_k(x)}}\ \Rightarrow \text{measurable.}$$

$\square$

THEOREM: Suppose there is a sequence of $\mathcal{S}$-measurable functions $f_n : E \to \mathbb{R}$ (here $\mathcal{S} \subset \mathcal{P}(X)$ σ-algebra, $E \in \mathcal{S}$) such that

$$\forall_{x\in E} \quad f(x) = \lim_{n\to\infty} f_n(x)$$

Then the function $f : E \to \overline{\mathbb{R}}$ is $\mathcal{S}$-measurable

PROOF: Notice that $f(x) = \lim_{n\to\infty} f_n(x) \iff f(x) = \liminf_{n\to\infty} f_n(x)$
$= \limsup_{n\to\infty} f_n(x)$ $\square$

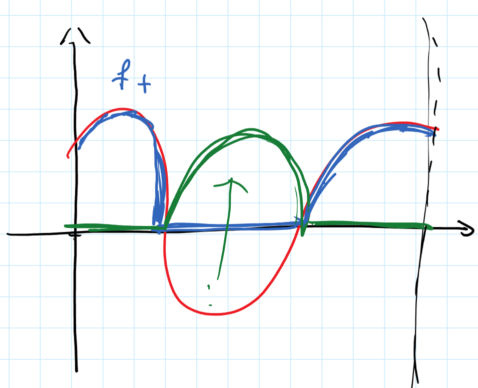
COROLLARY: For a σ-algebra $\mathcal{S} \subset \mathcal{P}(X)$, $E \in \mathcal{S}$ and given $\mathcal{S}$-measurable function $f_1, f_2, \dots, f_n : E \to \overline{\mathbb{R}}$, the functions

$$\varphi(x) = \min\{f_1(x), f_2(x), \dots, f_n(x)\}$$
and
$$\psi(x) = \max\{f_1(x), f_2(x), \dots, f_n(x)\}$$

are $\mathcal{S}$-measurable. $\square$

Let $f : E \to \overline{\mathbb{R}}$ be a function, $E \subset X$, then put

$$\forall_{x\in E} \quad f_+(x) := \max\{f(x), 0\}$$
$$f_-(x) := \max\{-f(x), 0\}$$



Then, clearly

1) $f(x) = f_+(x) - f_-(x)$

2) $|f(x)| = f_+(x) + f_-(x)$

2) $|f(x)| = f_+(x) + f_-(x)$

COROLLARY: If $\mathfrak{S} \subset \mathcal{P}(X)$ is a σ-algebra, $E \in \mathfrak{S}$, $f : E \to \overline{\mathbb{R}}$ is $\mathfrak{S}$-measurable then: $f_+, f_-, |f| : E \to \overline{\mathbb{R}}$ are measurable $\square$

Let $E \subset X$ be a given set. We define the so-called characteristic function $\chi_E : X \to \mathbb{R}$ by

\chi

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

By a simple function from X to $\mathbb{R}$ we mean a function $\sigma : X \to \mathbb{R}$ which has only finitely many values, i.e

$$\sigma(X) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \mathbb{R}$$

Then, put

$$E_i := \sigma^{-1}(\alpha_i), \quad i = 1, 2, \ldots, n$$

and notice that

$$\forall_{x \in X} \quad \sigma(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}(x), \qquad (*)$$

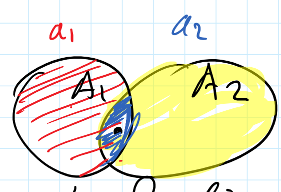
i.e every simple function can be expressed as a linear combination of characteristic functions. Conversely, notice that every linear combination of characteristic functions is a simple function.

Indeed, put

$$\xi(x) = \sum_{i=1}^{m} a_i \chi_{A_i}(x) \qquad A_i \subset X$$

and observe that if $\{i : x \in A_i\} =: I_i \subset \{1, 2, \ldots, m\}$, then

$$\xi(x) = \sum_{i \in I_i} a_i$$



1). The representation $(*)$ of a simple function as a linear combination of characteristic functions is unique $\iff$ the sets $E_i \cap E_j \neq \emptyset$ for $i \neq j$

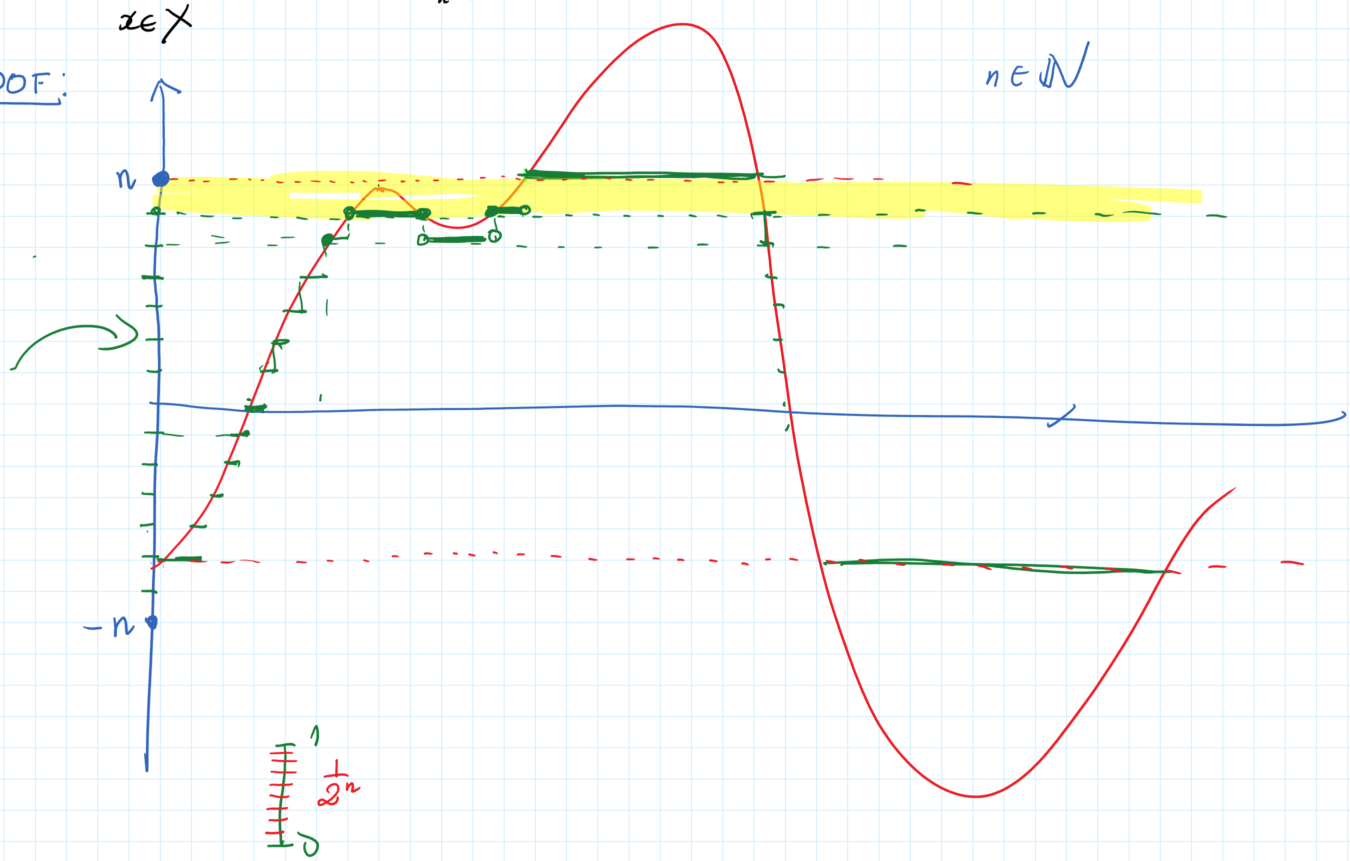
2) The simple function σ (given by $(*)$) is $\mathfrak{S}$-measurable (for some σ-algebra $\mathfrak{S} \subset \mathcal{P}(X)$) $\iff$ $E_i \in \mathfrak{S}$ for $i = 1, \ldots, n$.

THEOREM (SIMPLE FUNTIONS APPROXIMATION THEOREM)

Let $\mathcal{S} \subset \mathcal{P}(X)$ be a σ-algebra and $f: X \to \overline{\mathbb{R}}$ be an $\mathcal{S}$-measurable function. Then there exists a sequence of $\mathcal{S}$-measurable simple functions $\sigma_n : X \to \overline{\mathbb{R}}$ such that

$$\forall_{x \in X} \qquad f(x) = \lim_{n\to\infty} \sigma_n(x) .$$

PROOF:



Then the simple function σ_n is defined by

$$\sigma_n(x) = \begin{cases} n & \text{if } f(x) \geq n \\ \frac{k}{2^n} & \text{if } -n \leq \underset{a}{\frac{k}{2^n}} \leq f(x) < \underset{b}{\frac{k+1}{2^n}} \leq n \\ -n & \text{if } f(x) \leq n \end{cases} \qquad E_k = \{x : a \leq f(x) < b\}$$

$k \in \{0, \dots, 2n2^n\}$