University of Texas at Dallas

Exam 2 (sample)

Last Name:	**First Name and Initial:** ....
Course Name: Real Analysis 1	**Number:** MATH 6301
Instructor: Wieslaw Krawcewicz	**Due Date:** November 6, 2022
E-mail Address:	**Student's Signature:** ....

Instructions:

1. Print this booklet
2. Use the space provided to write your solutions in this booklet
3. Hand in your assignment to your instructor on the due date during the class time.

Question	**Weight**	**Your Score**	**Comments**
1.	10		
2.	10		
3.	10		
4.	10		
5.	10		
Total:	50		

Problem 1. Consider the measure space $(\mathbb{R}, \mathscr{B}_1, \mathfrak{m})$, where $\mathfrak{m}$ stands for the Lebesgues measure. $f : [a, b] \to \mathbb{R}$ be a C^1-differentiable function. Show that for every $A \subset [a, b]$ such that $\mathfrak{m}(A) = 0$ we have $\mathfrak{m}(f(A)) = 0$. What can you say about $\mathfrak{m}(f(A))$ when f is continuous but not differentiable.

Solution: Notice that since f is continuously differentiable, thus there is $L > \max\{|f'(x)| : x \in [a, b]\}$, such that the function f is L-Lipschitzian. Indeed, by Mean Value theorem

$$\forall_{x,y\in[a,b]}\ \exists_{c\in[a,b]} \quad |f(x) - f(y)| = |f'(c)||x - y| \le L|x, y|.$$

If $\mathfrak{m}(A) = 0$, then for every $\varepsilon > 0$ there exists a countable cover of A by subintervals I_k of $[a, b]$ such that

$$A \subset \bigcup_{k=1}^{\infty} I_k \quad \text{and} \quad \sum_{k=1}^{\infty} \mathfrak{m}(I_k) < \frac{\varepsilon}{L}.$$

then the image $f(I_k)$, $k = 1, 2, \dots,$ is an interval and $\mathfrak{m}(f(I_k) \le L\mathfrak{m}(I_k)$, therefore

$$\mathfrak{m}(f(A)) \le \mathfrak{m}\left(\bigcup_{k=1}^{\infty} f(I_k)\right) \le \sum_{k=1}^{\infty} \mathfrak{m}(f(I_k)) \le L \sum_{k=1}^{\infty} \mathfrak{m}(I_k) < L\frac{\varepsilon}{L} = \varepsilon.$$

Since $\varepsilon > 0$ can be arbitrary, it follows that $\mathfrak{m}(f(A)) = 0$.

Problem 2. Consider the Lebesgue measure $\mathfrak{m}$ on $\mathbb{R}^n$ and a set $A \in \mathscr{B}_n$ such that $0 < \mathfrak{m}(A) < \infty$. Show that for every $\varepsilon > 0$ there exists a continuous **surjective** function $\alpha : \mathbb{R}^n \to [0,1]$ such that $\alpha(x) > 0$ for all $x \in A$ and

$$\mathfrak{m}\Big(\alpha_\varepsilon^{-1}((0,1])\Big) - \mathfrak{m}(\alpha_\varepsilon^{-1}(\{1\})) < \varepsilon.$$

Solution: Since A is measurable and $\mathfrak{m}(A) < \infty$, then for every $\varepsilon > 0$ there exists an open set U such that $A \subset U$ and a compact set $K \subset A$ such that

$$\mu(U) - \mu(K) < \varepsilon.$$

Define the function $\alpha_\varepsilon : [a,b] \to [0,1]$ by the formula

$$\alpha(x) := \frac{\operatorname{dist}(x, U^c)}{\operatorname{dist}(x, U^c) + \operatorname{dist}(x, K)}, \quad x \in [a,b].$$

Then, clearly

$$\alpha^{-1}(\{0\}) = U^c \quad \text{and} \quad \alpha^{-1}(\{1\}) = K,$$

thus

$$\mathfrak{m}\Big(\alpha^{-1}((0,1])\Big) - \mathfrak{m}(\alpha^{-1}(\{1\})) = \mathfrak{m}(U) - \mathfrak{m}(K) < \varepsilon.$$

Problem 3. Let $(X, \mathscr{S}, \mu)$ be a measure space, $E \in \mathscr{S}$ be a measurable set such that $\mu(E) < \infty$ and $f_n : E \to \mathbb{R}$ a sequence of measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$, where $|f(x)| < \infty$ for almost all $x \in E$. Define for a fixed $\varepsilon > 0$ the sets

$$A_n := \{x \in E : |f_n(x) - f(x)| > \varepsilon\} \quad \text{and} \quad S_n := \bigcup_{k\geq n} A_k.$$

Show that:

(a): $\lim_{n\to\infty} \mu(A_n) = 0$.

(b): $\lim_{n\to\infty} \mu(S_n) = 0$.

Solution: Assume that $\varepsilon > 0$ is fixed. Since $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$ (and $|f(x)| < \infty$) it follows that

$$\forall_{x\in X}\ \exists_n\ \forall_{k\geq n}\quad |f_k(x) - f(x)| \leq \varepsilon \quad \Leftrightarrow \quad \forall_{x\in X}\ \exists_n\ \forall_{k\geq n}\quad x \in A_k^c := X \setminus A_k. \tag{1}$$

(b): Notice that $S_n \subset S_{n+1}$ and we have by (1) that

$$\begin{aligned} x \in \left(\bigcap_{n=1}^{\infty} S_n\right)^c \quad &\Leftrightarrow \ \sim \forall_n\ \ x \in \bigcup_{k\geq n} A_k \\ &\Leftrightarrow \ \sim \forall_n\ \exists k \geq n\ \ x \in A_k \\ &\Leftrightarrow \ \ \exists_n\ \forall k \geq n\ \ x \in A_k^c \end{aligned}$$

which by (1) implies that $\left(\bigcap_{n=1}^{\infty} S_n\right)^c = X$, i.e. $\bigcap_{n=1}^{\infty} S_n = \emptyset$ and therefore

$$0 = \mu(\emptyset) = \lim_{n\to\infty} \mu(S_n).$$

(a): On the other hand since for all $n \in \mathbb{N}$

$$\mu(S_n) = \mu(A_n) \bigcup_{k=n}^{\infty} A_k) \geq \mu(A_n) \geq 0,$$

therefore, by (1) and the squeeze property we obtain

$$\lim_{n\to\infty} \mu(A_n).$$

Problem 4. Let $\mathfrak{m}$ denotes the Lebesgue measure in $\mathbb{R}^n$. Show that for every Lebesgue measurable set $A \subset \mathbb{R}^n$ we have

(a) $\mathfrak{m}(x + A) = \mathfrak{m}(A)$ for all $x \in \mathbb{R}^n$,

(b) $\mathfrak{m}(rA) = |r|\mathfrak{m}(A)$ for all $r \in \mathbb{R}$,

(c) For every linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$, $\mathfrak{m}(T(A))_= |\det(T)|\mathfrak{m}(A)$.

Problem 5. Consider the space $\mathbb{R}^n$, $n \in \mathbb{N}$. Show that there exists a non-measurable (with respect to the Lebesgue measure $\mathfrak{m}$ in $\mathbb{R}^n$) set $M \subset \mathbb{R}^n$.

Solution: Follow the Vitali;s construction.