MATH 6301 Real Analysis I
Homework 2

Jonas Wagner
jonas.wagner@utdallas.edu

2022, September 15^{th}

Contents

Instructions:

1. Print this booklet
2. Use the space provided to write your solutions in this booklet
3. Hand in your assignment to your instructor on the due date during the class time.

Question	Weight	Your Score	Comments
1.	10		
2.	10		
3.	10		
4.	10		
5.	10		
Total:	50		

Problem 1

PROBLEM:

Given two metric spaces (X, d) and (Y, p), $a \in X$, and a function $f : X\backslash\{a\} \to Y$. We denote by $A_a(f)$ the set of all accumulated values of f at a. Show that $A_a(f)$ is a closed set.

PRELIMINARIES:

Definition 1. *Let (X, d) be a metric space. The set $A \subset X$ is called* <u>open</u> *if*

$$\forall_{a\in A}\exists_{\epsilon} \ : \ \forall_{x\in X} d(a, x) < \epsilon \implies x \in A$$

Definition 2. *A set is* <u>closed</u> *if it is not open.*

Definition 3. *Let (X, d) and (Y, p) be metric spaces and $a \in X$. The function $f : X\backslash\{a\} \to Y$ has an* <u>accumulation value (cluster value)</u> *$b \in Y$ if there exists a sequence $\{x_n\} \subset X \ \{a\}$ such that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} f(x_n) = b$.*

SOLUTION:

Proposition 1. *The set of all accumulation points, $A_a(f)$, is closed.*

Proof. $A_a(f)$ is closed if it contains all of its limit points.
If $A_a(f)$ is unbounded, we have $A_a(f) = Y$, which is closed.
If $A_a(f)$ is empty, we have $A_a(f) = \emptyset$, which is also closed.
If $A_a(f)$ is bounded and nonempty, then in order for it to be closed, it's upper and lower bound must be within the set. So we must show that $\sup A_a(f)$ and $\inf A_a(f)$ are included within the set. By definition, $\sup A_a(x) = \limsup_{x\to a} f(x)$ and $\inf A_a(x) = \liminf_{x\to a} f(x)$. Clearly $A_a(x)$ will contain both of these cases and as a result it is closed. $\square$

Problem 2

PROBLEM:

Assume that (X, d) is a metric space, $a \in X$ and $f : X \backslash \{a\} \to \mathbb{R}$ is a function. Put for $\delta > 0$

$$C_\delta(a) \coloneqq \{x \in X \ : \ 0 < d(x, a) < \delta\}$$

Show that

$$\limsup_{x \to a} f(x) = \inf_{\delta > 0} \sup_{x \in C_\delta(a)} f(x)$$

$$\liminf_{x \to a} f(x) = \sup_{\delta > 0} \inf_{x \in C_\delta(a)} f(x)$$

SOLUTION:

Recall that for a given sequence $\{y_n\} \subset \mathbb{R}$ we have

$$\limsup_{n \to \infty} y_n = \inf_n \sup_{k \geq n} y_k = \lim_{n \to \infty} (\sup_{k \geq n} y_k)$$

and

$$\liminf_{n \to \infty} y_n = \sup_n \inf_{k \geq n} y_k = \lim_{n \to \infty} (\inf_{k \geq n} y_k)$$

We can then prove that the following propositions are true by showing that a sequence y_k exists to satisfy these equalitites.

Proposition 2.

$$\limsup_{x \to a} f(x) = \inf_{\delta > 0} \sup_{x \in C_\delta(a)} f(x)$$

Proof. Looking at the definition of $C_\delta(a)$, we see that we can construct every point $x_n, x_k \in C_\delta(a)$ so that $d(a, x_n) \leq d(a, x_k)$ whenever $k \geq n$.
Returning to the sequences that defined $A_a(f)$, we have that a cluster point b exists iff the sequence $\{x_n\}$ exists with $b = \lim_{x \to a} f(x)$. We define a similar sequence $\{y_n\}$ where $y_n = f(x_n)$. In this sequence we now have that $\limsup_{n \to \infty} y_n = \lim_{n \to \infty} (\sup_{k \geq n} y_k)$.
From $\{y_k\} \subset C_\delta(a)$ we have $\sup_{x \in C_\delta} f(x) = \sup_{k \geq n} y_k$
Since $\limsup_{n \to \infty} y_n = \limsup_{x \to a} f(x)$ and $\{y_k\} \subset C_\delta(a)$, we can apply $\limsup_{n \to \infty} y_n = \inf_n \sup_{k \geq n} y_k = \lim_{n \to \infty} (\sup_{k \geq n} y_k)$ to show that

$$\limsup_{x \to a} f(x) = \inf_{\delta > 0} \sup_{x \in C_\delta(a)} f(x)$$

□

The dual of this problem is proved similarly.

Proposition 3.

$$\liminf_{x \to a} f(x) = \sup_{\delta > 0} \inf_{x \in C_\delta(a)} f(x)$$

Proof. Looking at the definition of $C_\delta(a)$, we see that we can construct every point $x_n, x_k \in C_\delta(a)$ so that $d(a, x_n) \leq d(a, x_k)$ whenever $k \geq n$.
Take $\{y_n\}$ where $y_n = f(x_n)$ which implies $\liminf_{n \to \infty} y_n = \lim_{n \to \infty} (\inf_{k \geq n} y_k)$.
From $\{y_k\} \subset C_\delta(a)$ we have $\inf_{x \in C_\delta} f(x) = \inf_{k \geq n} y_k$
Since $\liminf_{n \to \infty} y_n = \liminf_{x \to a} f(x)$ and $\{y_k\} \subset C_\delta(a)$, we can apply $\liminf_{n \to \infty} y_n = \lim_{n \to \infty} (\inf_{k \geq n} y_k)$ to show that

$$\liminf_{x \to a} f(x) = \sup_{\delta > 0} \inf_{x \in C_\delta(a)} f(x)$$

□

Problem 3

Given a metric space (X,d), $a \in X$, and a function $f : X\backslash\{a\} \to \mathbb{R}$ such that $A_a(f) \neq \emptyset$ is also bounded.

PRELIMINARIES:

For given functions $f : X \to \mathbb{R}$, excluding the point $a \in X$ will results in being able to study accumulation points $A_a(f)$.

a)

PROBLEM:

Show that $\limsup_{x\to a} f(x) < \alpha$ for some $\alpha \in \mathbb{R}$ iff

$$\exists_{\delta>0}\exists_{\overline{\alpha}<\alpha}\forall_{x\in X} 0 < d(x,a) < \delta \implies f(x) \leq \overline{\alpha}$$

SOLUTION:

Essentially, looking at $A_a(f)$, we have that there exists an upper bound, $\sup A_a(f)$, which aligns with this definition. Looking at $A_a(f)$ the $\neq \emptyset$ and bounded align with $\forall_{x\in X} 0 < d(x,a) < \delta$ and the $f(x) \leq \overline{\alpha}$ is the upper bound of $A_a(f)$.

b)

PROBLEM:

Show that $\liminf_{x\to a} f(x) > \alpha$ for some $\alpha \in \mathbb{R}$ iff

$$\exists_{\delta>0}\exists_{\overline{\alpha}>\alpha}\forall_{x\in X} 0 < d(x,a) < \delta \implies f(x) \geq \overline{\alpha}$$

SOLUTION:

Essentially, looking at $A_a(f)$, we have that there exists a lower bound, $\inf A_a(f)$, which aligns with this definition. Looking at $A_a(f)$ the $\neq \emptyset$ and bounded align with $\forall_{x\in X} 0 < d(x,a) < \delta$ and the $f(x) \geq \overline{\alpha}$ is the lower bound of $A_a(f)$.

c)

PROBLEM:

Show that $\limsup_{x\to a} f(x) \leq \alpha$ for some $\alpha \in \mathbb{R}$ iff

$$\forall_{\alpha'>\alpha}\exists_{\delta>0}\forall_{x\in X} 0 < d(x,a) < \delta \implies f(x) \leq \alpha'$$

SOLUTION:

Essentially, looking at $A_a(f)$, we have that there exists an upper bound, $\sup A_a(f)$, which aligns with this definition. Looking at $A_a(f)$ the $\neq \emptyset$ and bounded align with $\forall_{x\in X} 0 < d(x,a) < \delta$ and the $f(x) \leq \overline{\alpha}$ is the upper bound of $A_a(f)$.

d)

PROBLEM:

Show that $\liminf_{x\to a} f(x) \leq \alpha$ for some $\alpha \in \mathbb{R}$ iff

$$\forall_{\alpha'<\alpha}\exists_{\delta>0}\forall_{x\in X} 0 < d(x,a) < \delta \implies f(x) \geq \alpha'$$

SOLUTION:

Essentially, looking at $A_a(f)$, we have that there exists a lower bound, $\inf A_a(f)$, which aligns with this definition. Looking at $A_a(f)$ the $\neq \emptyset$ and bounded align with $\forall_{x \in X} 0 < d(x, a) < \delta$ and the $f(x) \geq \overline{\alpha}$ is the lower bound of $A_a(f)$.

Problem 4

PROBLEM:

Given two metric spaces (X, d) and (Y, p), $a \in X$ and a function $f : X \to Y$. Show that f is continuous iff

$$A_a(f) = \{f(a)\}$$

SOLUTION:

Proposition 4. *Given two metric spaces (X, d) and (Y, p) and $a \in X$ and a function $f : X \to Y$. f is continuous iff*

$$A_a(f) = \{f(a)\}$$

Proof. First we look at implication, $\implies$.
f being continuous means that

$$\forall_{\epsilon>0}\exists_{\delta>0}\forall_{x\in X} d(x, a) < \delta \implies p(f(x), f(a)) < \epsilon$$

or equivalently,

$$\forall_{\{x_n \in X\}} : \lim_{n\to\infty} x_n = a \implies \lim_{n\to\infty} f(x_n) = f(a)$$

Clearly, the limit definition of continuity lines directly with the definition of the accumulation points. $A_a(f)$ being every point that $\lim_{x\to a} f(x) = b$ means that if the $\lim_{n\to\infty} f(x_n) = f(a)$ then there is only one point b satisfying the accumulation point definition; and therefore, $A_a(f) = \{f(a)\}$.
Next we look at $\impliedby$. $A_a(f) = \{f(a)\}$ means that for every sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = a$ the only accumulation point is $f(a)$, $\lim_{n\to\infty} f(x_n) = b = f(a)$.
This is directly aligned with the limit definition of continuity, so

$$\forall_{\{x_n \in X\}} : \lim_{n\to\infty} x_n = a \implies \lim_{n\to\infty} f(x_n) = b = f(a)$$

□

Problem 5

Denote by $\overline{\mathbb{R}}$ the ordered set of real numbers $\{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ and define the function $\phi : \overline{\mathbb{R}} \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} \arctan(x) & x \in \mathbb{R} \\ \pm\frac{\pi}{2} & x = \pm\infty \end{cases}$$

and the function $d : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \mathbb{R}$ by

$$d(x, y) \coloneqq |\phi(x) - \phi(y)|$$

a)

PROBLEM:

Show that the function d is a metric on $\overline{\mathbb{R}}$.

SOLUTION:

1. **Positivity:** $d(x, y) \geq 0$ and $d(x, y) = 0$.

   By definition of absolute value, $|\phi(x) - \phi(y)| \geq 0$. Additionally, by definition, $|\phi(x) - \phi(x)| = |0| = 0$

2. **Symmetry:** $d(x, y) = d(y, x)$.

   By definition of absolute value, we have $d(x, y) = |\phi(x) - \phi(y)| = |-(\phi(y) - \phi(x))| = |\phi(y) - \phi(x)| = d(y, x)$.

3. **Triangle Inequality:** $d(x, z) \leq d(x, y) + d(y, z)$

   We must prove that

   $$d(x, z) = |\phi(x) - \phi(z)| \leq |\phi(x) - \phi(y)| + |\phi(y) - \phi(z)| = d(x, y) + d(y, z)$$

   Since $\phi(x)$ is an always increasing function, $x > y \implies \phi(x) > \phi(y)$, for all cases of $z < x < y$, $x < z < y$, and $x < y < z$ we have the simple triangle inequality from absolute values of difference hold.

b)

PROBLEM:

Show that the topology $\mathcal{T}$ induced by the metric d on $\overline{\mathbb{R}}$, restricted to $\mathbb{R}$, coincide with the usual topology on $\mathbb{R}$.

SOLUTION:

The metric topology induced for both cases is just the union of the open balls on $\mathbb{R}$.
Since the open balls for both metrics, defined by $B_a(r) = \{x \in \mathbb{R} : d(a, x) < r\}$, have essentially the same form, it is clear that they are equivalent topologies. (i.e)

$$B_a(r) = \{x \in \mathbb{R} : |a - x| < r\} \approx \{x \in \mathbb{R} : |\arctan(a) - \arctan(x)| < r\}$$