LECTURE 14 — MATH 630 J

DEFINITION: Let $(X,\mathfrak{S},\mu)$ be a measure space. We say that the measure μ is complete if

$$\forall_{A\in\mathfrak{S}} \left(\mu(A)=0 \Rightarrow \forall_{B\subset A} B\in\mathfrak{S} \right)$$

THEOREM (Completion of a measure)
Let $(X,\mathfrak{S},\mu)$ be a measure space and put

$$\mathcal{N} := \{ A\subset X : \exists_{F\in\mathfrak{S}} \; A\subset F \text{ and } \mu(F)=0 \}$$

and

$$\overline{\mathfrak{S}} \overset{\text{def}}{:=} \mathfrak{S}(\mathfrak{S}\cup\mathcal{N})$$

be the σ-algebra generated by $\mathfrak{S}\cup\mathcal{N}$.
Then there exists exactly one measure $\overline{\mu}:\overline{\mathfrak{S}}\to\overline{\mathbb{R}}$ such that $\forall_{F\in\mathfrak{S}} \; \overline{\mu}(F)=\mu(F)$, i.e. $\overline{\mu}$ is a unique extension of μ to $\overline{\mathfrak{S}}$.

COMMENT: The measure $\overline{\mu}$ is called the completion of μ to $\overline{\mathfrak{S}}$ and $\overline{\mathfrak{S}}$ is called completion of σ-algebra $\mathfrak{S}$.

PROOF: Notice that

(a) if $A\in\mathcal{N}$ and $B\subset A$ then $B\in\mathcal{N}$

(b) if $\{A_n\}_{n=1}^{\infty}\subset\mathcal{N} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in\mathcal{N}$. Indeed, $A_n\in\mathcal{N}$ means $\exists F_n\in\mathfrak{S}$ s.t $\mu(F_n)=0$ $A_n\subset F_n$, $n=1,2,\dots$. Put $F=\bigcup_{n=1}^{\infty}F_n$, $\bigcup_{n=1}^{\infty}A_n\subset F$ and $\mu(F)=\mu(\bigcup_{n=1}^{\infty}F_n)\le\sum_{n=1}^{\infty}\mu(F_n)=0$

We define the following equivalence relation for $E,F\subset X$, by

$$E \overset{\bullet}{\approx} F \overset{\text{def}}{\iff} E\setminus F\in\mathcal{N} \text{ and } F\setminus E\in\mathcal{N}$$

| transitivity? Please check it.

and observe that

(i) $E_1\overset{\bullet}{\approx}F_1$ and $E_2\overset{\bullet}{\approx}F_2$ then $E_1\setminus E_2\overset{\bullet}{\approx}F_1\setminus F_2$

(ii) $E_n\overset{\bullet}{\approx}F_n$, $n=1,2,\dots$ then $\bigcup_{n=1}^{\infty}E_n\overset{\bullet}{\approx}\bigcup_{n=1}^{\infty}F_n$ and $\bigcap_{n=1}^{\infty}E_n\overset{\bullet}{\approx}\bigcap_{n=1}^{\infty}F_n$

(ii) $E_n \dot\approx F_n$, $n=1,2,\dots$ then $\bigcup_{n=1} E_n \dot\approx \bigcup_{n=1} F_n$ and $\bigcap_{n=1} E_n \dot\approx \bigcap_{n=1} F_n$

Since $E_n \setminus F_n \in \mathcal{N}$ and $F_n \setminus E_n \in \mathcal{N}$ then we have (b)

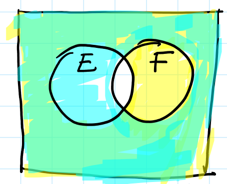
$$\bigcup_{n=1}^{\infty} E_n \setminus \bigcup_{k=1}^{\infty} F_k = \bigcup_{n=1}^{\infty}\Big(E_n \setminus \bigcup_{k=1}^{\infty} F_k\Big) \subset \bigcup_{n=1}^{\infty} \underset{\overset{\cap}{\mathcal{N}}}{(E_n \setminus F_n)} \in \mathcal{N}$$

Define

$$\mathfrak{S}' := \{E \subset X : \exists_{F \in \mathfrak{S}} \; E \dot\approx F\}$$

Notice that $\mathfrak{S}'$ is a σ-algebra. Indeed, $\mathfrak{S} \subset \mathfrak{S}'$ so $\emptyset, X \in \mathfrak{S} \subset \mathfrak{S}'$, (A_1)
and $E \in \mathfrak{S}' \iff \exists_{F \in \mathfrak{S}} \; E \setminus F$ and $F \setminus E \in \mathcal{N}$

$E^c \setminus F^c = F \setminus E$

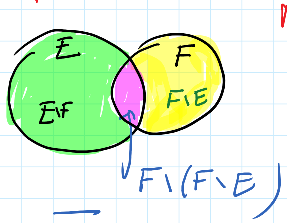


E^c yellow

F^c blue

Since $\mathfrak{S} \subset \mathfrak{S}'$ and $\mathcal{N} \subset \mathfrak{S}'$ thus $\overline{\mathfrak{S}} = \mathfrak{S}(\mathfrak{S} \cup \mathcal{N}) \subset \mathfrak{S}'$, thus we need to show that $\mathfrak{S}' \subset \overline{\mathfrak{S}}$. Take $E \in \mathfrak{S}'$, so (by definition) $\exists F \in \mathfrak{S}$ s.t. $E \setminus F, F \setminus E \in \mathcal{N}$. Then we have

$$E = (F \setminus \underset{?}{(F \setminus E)}) \cup \underset{\overset{\shortparallel}{N_2}}{(E \setminus F)} \quad \text{so} \quad \underset{\overset{\shortparallel}{N_1}}{F \setminus E},\ \underset{\overset{\shortparallel}{N_2}}{E \setminus F} \in \mathcal{N}$$



$F \in \mathfrak{S}$

$$F \setminus N_1 = F \cap N_1^c = (\overset{\overset{\mathfrak{S}}{\cup\!\!\!\shortmid}}{F^c} \underset{\overset{\cap}{\mathfrak{S}}}{\cup} \overset{\overset{\mathcal{N}}{\cup\!\!\!\shortmid}}{N_1})^c \in \overline{\mathfrak{S}}$$

therefore $E \in \overline{\mathfrak{S}} = \mathfrak{S}(\overset{\mathfrak{S} \cup \mathcal{N}}{\mathfrak{S} \cup \mathcal{N}})$.

Now, we can write the extension $\bar{\mu} : \overline{\mathfrak{S}} = \mathfrak{S}' \longrightarrow \overline{\mathbb{R}}$ by the formula

$$(*) \qquad \bar{\mu}(E) = \mu(F) \quad \text{where } F \in \mathfrak{S} \text{ and } F \dot\approx E$$

Notice that $(*)$ doesn't depend on the choice of representative $F \in \mathfrak{S}$ of $E \dot\approx F$ and $E \dot\approx F' \implies \mu(F) = \mu(F')$ $\square$

Assignment #4,

We need to show that $\bar{\mu}$ is indeed a measure, so

$$\bar{\mu}\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \mu(E_n) \quad \text{for } \{E_n\} \subset \mathfrak{S}' \quad E_n \cap E_m = \emptyset$$

For this we need a lemma:

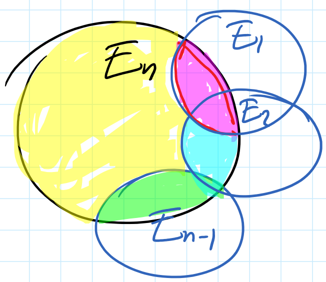
LEMMA: If $\{E_n\} \subset \mathcal{S}$ is such that $\mu(E_n \cap E_m) = 0$ for $m \neq n$ then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

PROOF:

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n, \qquad F_n := E_n \setminus (E_1 \cup \dots \cup E_{n-1})$$

$$= \left(\dots\left(E_n \setminus \overset{\in \mathcal{N}}{\underset{=N_1}{(E_n \cap E_1)}}\right) \setminus \dots \setminus \overset{\in\mathcal{N}}{\underset{N_2}{(E_n \cap E_{n-1})}}\right)$$



$$F_n \cap F_m = \emptyset$$

$$F_n = E_n \setminus \underbrace{(N_1 \cup \dots \cup N_{n-1})}_{= N \in \mathcal{N}}$$

so $$\mu\left(\bigcup_n E_n\right) = \mu\left(\bigcup_{n=1} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) = \sum_{n=1}^{\infty} \mu(E_n) \qquad \square$$

To conclude $\quad F_n \approx E_n \quad F_n \in \mathcal{S} \quad \mu(F_n \cap F_m) = 0 \quad n \neq m$

$$\bar{\mu}\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) = \sum_{n=1}^{\infty} \bar{\mu}(E_n) \qquad \square$$

OUTER MEASURE

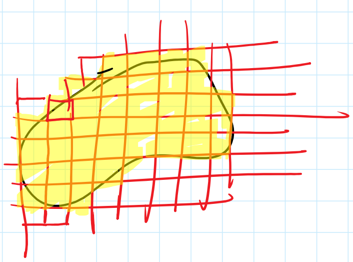
Example: For a given set $A \subset \mathbb{R}^n$ one can define the number

$$\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} |I_n| : A \subset \bigcup_{n=1}^{\infty} I_n\right\}$$

interval

$$I = (a_1, b_1) \times \dots \times (a_n, b_n)$$

$$|I| = \prod_{i=1}^{n} (b_i - a_i)$$



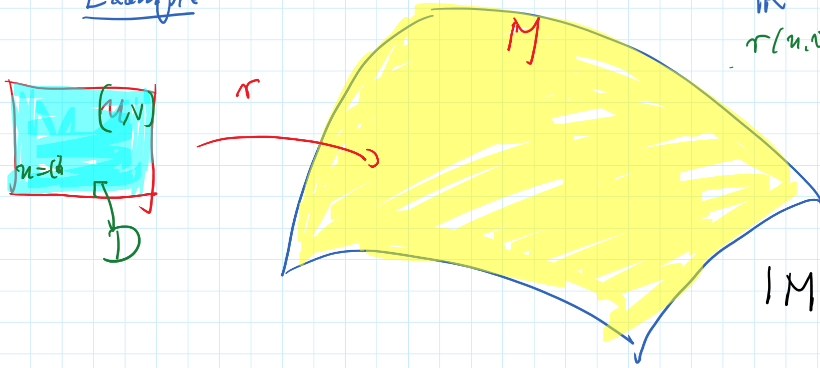
The idea of taking finitely many intervals I_n covering A leads to the so-called Jordan (outer) measure of A;
on the other hand countable covers by I_n lead to Lebesgue (outer) measure of A.

Example

$\mathbb{R}^3$

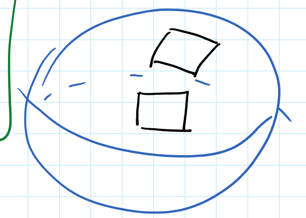
$r(u,v) = \begin{bmatrix} r_1(u,v) \\ r_2(u,v) \end{bmatrix}$

Example



$\mathbb{R}^3$

$$r(u,v) = \begin{bmatrix} r_1(u,v) \\ r_2(u,v) \\ r_3(u,v) \end{bmatrix}$$



$$|M| = \int_D \|\mathring{r}_u \times \mathring{r}_v\| \, du\, dv$$

Next time!