MATH 6301 Real Analysis I
Homework 2

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Instructions:

1. Print this booklet
2. Use the space provided to write your solutions in this booklet
3. Hand in your assignment to your instructor on the due date during the class time.

Question	Weight	Your Score	Comments
1.	10		
2.	10		
3.	10		
4.	10		
5.	10		
Total:	50		

Problem 1

PROBLEM:

Given two metric spaces (X, d) and (Y, p), $a \in X$, and a function $f : X \backslash \{a\} \to Y$. We denote by $A_a(f)$ the set of all accumulated values of f at a. Show that $A_a(f)$ is a closed set.

SOLUTION:

Problem 2

PROBLEM:

Assume that (X, d) is a metric space, $a \in X$ and $f : X\backslash\{a\} \to \mathbb{R}$ is a function. Put for $\delta > 0$

$$C_\delta(a) \coloneqq \{x \in X \ : \ 0 < d(x, a) < \delta\}$$

Show that

$$\limsup_{x\to a} = \inf_{\delta>0} \sup_{x\in C_\delta(a)} f(x)$$

$$\liminf_{x\to a} = \sup_{\delta>0} \inf_{x\in C_\delta(a)} f(x)$$

SOLUTION:

Problem 3

Given a metric space (X, d), $a \in X$, and a function $f : X\backslash\{a\} \to \mathbb{R}$ such that $A_a(f) \neq \emptyset$ is also bounded.

a)

PROBLEM:

Show that $\limsup_{x\to a} f(x) < \alpha$ for some $\alpha \in \mathbb{R}$ iff

$$\exists_{\delta>0}\exists_{\overline{\alpha}<\alpha}\forall_{x\in X} 0 < d(x,a) < \delta \implies f(x) \leq \overline{\alpha}$$

SOLUTION:

b)

PROBLEM:

Show that $\liminf_{x\to a} f(x) > \alpha$ for some $\alpha \in \mathbb{R}$ iff

$$\exists_{\delta>0}\exists_{\overline{\alpha}>\alpha}\forall_{x\in X} 0 < d(x,a) < \delta \implies f(x) \geq \overline{\alpha}$$

SOLUTION:

c)

PROBLEM:

Show that $\limsup_{x\to a} f(x) \leq \alpha$ for some $\alpha \in \mathbb{R}$ iff

$$\forall_{\alpha'>\alpha}\exists_{\delta>0}\forall_{x\in X} 0 < d(x,a) < \delta \implies f(x) \leq \alpha'$$

SOLUTION:

d)

PROBLEM:

Show that $\liminf_{x\to a} f(x) \leq \alpha$ for some $\alpha \in \mathbb{R}$ iff

$$\forall_{\alpha'<\alpha}\exists_{\delta>0}\forall_{x\in X} 0 < d(x,a) < \delta \implies f(x) \geq \alpha'$$

SOLUTION:

Problem 4

PROBLEM:

Given two metric spaces (X, d) and (Y, p), $a \in X$ and a function $f : X \to Y$. Show that f is continuous iff

$$A_\alpha(f) = \{f(a)\}$$

SOLUTION:

Problem 5

Denote by $\overline{\mathbb{R}}$ the ordered set of real numbers $\{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ and define the function $\phi : \overline{\mathbb{R}} \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} \arctan(x) & x \in \mathbb{R} \\ \pm\frac{\pi}{2} & x = \pm\infty \end{cases}$$

and the function $d : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \mathbb{R}$ by

$$d(x, y) := |\phi(x) - \phi(y)|$$

a)

PROBLEM:

Show that the function d is a metric on $\overline{\mathbb{R}}$.

SOLUTION:

b)

PROBLEM:

Show that the topology $\mathcal{T}$ induced by the metric d on $\overline{\mathbb{R}}$, restricted to $\mathbb{R}$, coincide with the usual topology on $\mathbb{R}$.

SOLUTION: