LECTURE 9 — MATH 6301

Recall, a family $\mathcal{S} \subset \mathcal{P}(X)$ is a σ-algebra in X if

(σA1) $\emptyset \in \mathcal{S}$

(σA2) $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$

(σA3) $\{A_n\}_{n=1}^{\infty} \subset \mathcal{S} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$

PROPOSITION: If a family $\mathcal{S} \subset \mathcal{P}(X)$ satisfies the conditions

(σA'1) $\emptyset, X \in \mathcal{S}$

(σA'2) $A, B \in \mathcal{S} \Rightarrow A \setminus B \in \mathcal{S}$

(σA'3) $\{B_n\}_{n=1}^{\infty} \subset \mathcal{S}$ and $B_n \cap B_m = \emptyset$ for $n \neq m \Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathcal{S}$

Then $\mathcal{S}$ is a σ-algebra.

PROOF: (σA'1) $\Rightarrow$ (σA1)

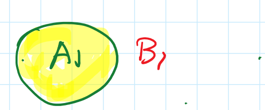
(σA'2) and (σA'1) $\Rightarrow$ $A \in \mathcal{S}$ then $A^c = \underbrace{\underset{\mathcal{S}}{\underset{\cap}{X}} \setminus \underset{\mathcal{S}}{\underset{\cap}{A}}}_{(\sigma A'2)} \in \mathcal{S}$.

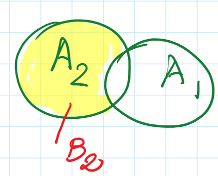
To show that (σA'1) — (σA'3) imply (σA3), notice that if $\{A_n\}_{n=1}^{\infty} \subset \mathcal{S}$ is an arbitrary family of sets, then the sets $\{B_n\}_{n=1}^{\infty}$ defined by

$$B_1 := A_1$$
$$B_2 := A_2 \setminus A_1$$
$$B_3 := A_3 \setminus (A_1 \cup A_2)$$
$$\vdots$$
$$B_n := A_n \setminus \bigcup_{k=1}^{n-1} A_k$$

satisfy the properties:

(a) $\forall_{n \in \mathbb{N}} \quad \bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{n} B_k$

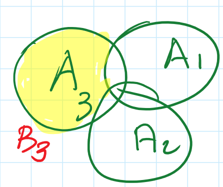




By induction, (a) is true for $n=1$, assume it is true for n, i.e.

$$\bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{n} B_k$$

then we have (by definition of B_{n+1})



$$B_{n+1} = A_{n+1} \setminus \bigcup^{n} A_k = A_{n+1} \setminus \bigcup^{n} B_k \qquad (*)$$

$$B_{n+1} = A_{n+1} \setminus \bigcup_{k=1}^{n} A_k = A_{n+1} \setminus \bigcup_{k=1}^{n} B_k \quad (*)$$

and therefore

$$\bigcup_{k=1}^{n+1} B_k = B_{n+1} \cup \bigcup_{k=1}^{n} B_k = \left(A_{n+1} \setminus \bigcup_{k=1}^{n} B_k\right) \cup \bigcup_{k=1}^{n} B_k$$

$$= \left(A_{n+1} \setminus \bigcup_{k=1}^{n} A_k\right) \cup \left(\bigcup_{k=1}^{n} A_k\right) = A_{n+1}$$

(b) $B_n \cap B_m = \emptyset$ for $n \neq m$. Indeed, suppose that $m < n$ then we have that

$$B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k = A_n \setminus \bigcup_{k=1}^{n} B_k \qquad \text{since } \boxed{B_m \subset \bigcup_{k=1}^{n} B_k}$$

thus $B_n \cap B_m = \emptyset$

(c) Notice $B_1 \in \mathcal{S}$ (indeed $A_1 = B_1$ and $A_1 \in \mathcal{S}$), and $B_2 \in \mathcal{S}$ (indeed $B_2 = \overset{\in \mathcal{S}}{A_2} \setminus \overset{\in \mathcal{S}}{B_1}$ and $A_2 \in \mathcal{S}$ so $B_2 \in \mathcal{S}$ by $(\sigma A'2)$)

Therefore, by induction we have that $\forall_{n \in \mathbb{N}}\ B_n \in \mathcal{S}$ and consequently, by $(\sigma A'3)$

$$\bigcup_{k=1}^{\infty} B_k \in \mathcal{S}.$$

But $\bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k \in \mathcal{S}$ so $(\sigma A3)$ is satisfied. $\square$

<u>NOTATION</u>. For a sequence of sets $\{A_n\} \subset \mathcal{P}(X)$, we say that

(a) $\{A_n\}$ is increasing iff $A_1 \subset A_2 \subset \dots \subset A_n \subset A_{n+1} \subset \dots$; $\lim_{n\to\infty} A_n := \bigcup_{n=1}^{\infty} A_n$

(b) $\{A_n\}$ is decreasing iff $A_1 \supset A_2 \supset \dots \supset A_n \supset A_{n+1} \supset \dots$; $\lim_{n\to\infty} A_n := \bigcap_{n=1}^{\infty} A_n$

Then increasing or decreasing sequence of sets is called monotone and, for such sequences we call $\lim_{n\to\infty} A_n$ the limit of $\{A_n\}$

<u>PROPOSITION</u>: An algebra $\mathcal{F} \subset \mathcal{P}(X)$ is a σ-algebra $\iff$ for any monotonic sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$ one has $\lim_{n\to\infty} A_n \in \mathcal{F}$.

<u>PROOF</u>: Notice that a sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$ is increasing, then $\{A_n^c\} \subset \mathcal{F}$ is decreasing. Therefore, it is sufficient to do the proof for $\overset{(A2)}{\text{an}}$ increasing sequence $\{A_n\}$

decreasing. Therefore, it is sufficient to do the proof for an increasing sequence $\{A_n\}$.

$$\bigcup_{n=1} A_n \in \mathcal{F} \quad (\text{by } (\sigma A3))$$

$\Leftarrow$ Assume that the algebra $\mathcal{F}$ satisfies the property: $\lim_{n\to\infty} B_n \in \mathcal{F}$ for every increasing sequence $\{B_n\} \subset \mathcal{F}$. Then, take an arbitrary sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$ and put

$$B_n := \bigcup_{k=1}^{n} A_k.$$

Since, by (A3), $B_n \in \mathcal{F}$, and $B_n \subset B_{n+1}$, we have that

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n = \lim_{n\to\infty} B_n \in \mathcal{F}$$

and $(\sigma A3)$ is satisfied. $\square$

RESTRICTION OF A σ-ALGEBRA TO A SUBSPACE

Given the space X and σ-algebra $\mathcal{S} \subset \mathcal{P}(X)$. Take $E \in \mathcal{S}$ (i.e. E is considered as a subspace of X). and define

$$\mathcal{S}_E := \{A \in \mathcal{S} : A \subset E\} \subset \mathcal{P}(E)$$

Then, clearly $\mathcal{S}_E$ is also a σ-algebra. Indeed: $(\sigma A1)$ satisfied $\emptyset \in \mathcal{S}_E$, $\emptyset \in \mathcal{S}^E$

$(\sigma A2)$ $A \in \mathcal{S}_E$ $(A \subset E,\ A \in \mathcal{S})$ then $\left.\begin{array}{l} E \setminus A = \underset{\mathcal{S}}{\underset{\cap}{E}} \cap \underset{\mathcal{S}}{\underset{\cap}{A}}^{c} \in \mathcal{S} \\ E \setminus A \subset A \end{array}\right\} \Rightarrow E \setminus A \in \mathcal{S}_E$

$(\sigma A3)$ $\{A_n\} \subset \mathcal{S}_E$ then $\{A_n\} \subset \mathcal{S}$ $\quad A_n \subset E \quad \forall_n \Rightarrow \bigcup_{n=1}^{\infty} A_n \subset E \quad (*)$

and by $(\sigma A3)$ $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S} \quad (**)$

so $(*)$ and $(**)$ implies

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}_E.$$

Then the σ-algebra $\mathcal{S}_E$ will be called restriction of $\mathcal{S}$ to E.

THEOREM: Let $f: X \to Y$ be a map and $\mathcal{S} \subset \mathcal{P}(X)$ be a σ-algebra. Then

$$\mathcal{E} := \{F \subset Y : f^{-1}(F) \in \mathcal{S}\}$$

(Induced by f σ-algebra on Y)

is a σ-algebra.

PROOF: (σA1) $\emptyset = f^{-1}(\emptyset) \in \mathcal{S} \Rightarrow \emptyset \in \mathcal{C}$

(σA2) $F \in \mathcal{C}$ then $F^C = Y \setminus F$ and since $f^{-1}(F) \in \mathcal{S}$ then $X \setminus f^{-1}(F) \in \mathcal{S}$ and

$$X \setminus f^{-1}(F) = f^{-1}(X \setminus F) = f^{-1}(F^C) \in \mathcal{S}$$

so $F^C \in \mathcal{C}$

(σA3) If $\forall_n \; f^{-1}(F_n) \in \mathcal{S} \Leftarrow F_n \in \mathcal{C}$, $n = 1, 2, \ldots$

Then

$$f^{-1}\left(\bigcup_{n=1}^{\infty} F_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(F_n) \in \mathcal{S} \quad \text{so} \quad \bigcup_{n=1}^{\infty} F_n \in \mathcal{C}$$

$\square$

DEFINITION: Let $\mathcal{K} \subset \mathcal{P}(X)$ be a given family of sets in X. Then the smallest σ-algebra containing $\mathcal{K}$ will be denoted by $\mathcal{S}(\mathcal{K})$, i.e

$$\mathcal{S}(\mathcal{K}) = \bigcap \left\{ \mathcal{S} : \begin{array}{l} (a)\ \mathcal{S} \subset \mathcal{P}(X)\ \sigma\text{-algebra} \\ (b)\ \mathcal{K} \subset \mathcal{S} \end{array} \right\}$$

and we will call $\mathcal{S}(\mathcal{K})$ the σ-algebra generated by $\mathcal{K}$.

Notice that we have the following obvious properties

(i) $\mathcal{K} \subset \mathcal{S}$ and $\mathcal{S}$ is σ-algebra then $\mathcal{S}(\mathcal{K}) \subset \mathcal{S}$

(ii) $\mathcal{K} \subset \mathcal{K}'$ then $\mathcal{S}(\mathcal{K}) \subset \mathcal{S}(\mathcal{K}')$

DEFINITION Let $\mathcal{M} \subset \mathcal{P}(X)$ be a family of sets in X. We say that $\mathcal{M}$ is monotone if for any monotone sequence $\{A_n\} \subset \mathcal{M}$ one has $\lim_{n\to\infty} A_n \in \mathcal{M}$.

Example: Every σ-algebra is monotone family

THEOREM 1: If $\mathcal{R} \subset \mathcal{P}(X)$ is an algebra then $\mathcal{S}(\mathcal{R})$ is the smallest monotone family containing $\mathcal{R}$, i.e

$$\mathcal{S}(\mathcal{R}) = \bigcap \left\{ \mathcal{M} : \begin{array}{l} (a)\ \mathcal{M} \subset \mathcal{P}(X) \text{ is monotone} \\ (b)\ \mathcal{R} \subset \mathcal{M} \end{array} \right\} =: \mathcal{N}$$

PROOF: Denote by $\mathcal{N}$ the smallest monotone family containing $\mathcal{R}$. Then clearly $\mathcal{N} \subset \mathcal{S}(\mathcal{R})$. So we need to show that $\mathcal{S}(\mathcal{R}) \subset \mathcal{N}$.

LEMMA: Let $\mathcal{L} \subset \mathcal{P}(X)$ be a family of sets. Then

$$\mathcal{J}(\mathcal{L}) := \{ E \in \mathcal{P}(X) : \forall_{F \in \mathcal{L}} \; \underset{(1)}{E \cup F}, \underset{(2)}{E \setminus F}, \underset{(3)}{F \setminus E} \in \mathcal{N} \}$$

$F \in \mathcal{L}$ ① ② ③

is monotone.

PROOF: Take $E_1 \subset E_2 \subset \dots \subset E_n \subset E_{n+1} \subset \dots$, $E_n \in \mathcal{J}(\mathcal{L})$ for all $n = 1, 2, \dots$

$\iff \forall_{F \in \mathcal{L}} \forall_{n \in \mathbb{N}} \quad E_n \cup F, E_n \setminus F, F \setminus E_n \in \mathcal{N}$.

Then, since $\mathcal{N}$ is monotone we have $E_n \cup F \subset E_{n+1} \cup F$

$\forall_{F \in \mathcal{L}}$ ① $\underbrace{\bigcup_{n=1}^{\infty}(\underset{\mathcal{N}}{\underset{\cap}{E_n \cup F}})}_{\overset{\cap}{\mathcal{N}}} = \left(\bigcup_{n=1}^{\infty} E_n\right) \cup F \in \mathcal{N}$

② $\bigcup_{n=1}^{\infty}(E_n \setminus F) = \bigcup_{n=1}^{\infty} E_n \setminus F \in \mathcal{N}$

③ $\bigcup_{n=1}^{\infty}(F \setminus E_n) = F \setminus \bigcap_{n=1}^{\infty} E_n \in \mathcal{N}$

$\Rightarrow \bigcup_{n=1}^{\infty} E_n \in \mathcal{J}(\mathcal{L})$

so $\mathcal{J}(\mathcal{L})$ is monotone. $\square$

Notice that for $\mathcal{L}, \mathcal{K} \subset \mathcal{P}(X)$ we have

$$\mathcal{K} \subset \mathcal{J}(\mathcal{L}) \iff \mathcal{L} \subset \mathcal{J}(\mathcal{K})$$

Indeed

$\mathcal{K} \subset \mathcal{J}(\mathcal{L}) \iff \forall_{E \in \mathcal{K}} \forall_{F \in \mathcal{L}} \quad E \cup F, E \setminus F, F \setminus E \in \mathcal{N} \iff \mathcal{L} \subset \mathcal{J}(\mathcal{K})$

Then, since $\mathcal{R} \subset \mathcal{J}(\mathcal{R})$ so $\mathcal{N} \subset \mathcal{J}(\mathcal{R}) \iff \mathcal{R} \subset \mathcal{J}(\mathcal{N})$

so $\mathcal{N} \subset \mathcal{J}(\mathcal{N})$ which means that
① $E \cup F \in \mathcal{N}$
② $E \setminus F \in \mathcal{N}$ $\quad \forall_{E, F \in \mathcal{N}}$
③ $F \setminus E \in \mathcal{N}$

Thus by the Proposition, $\mathcal{N}$ is a σ-algebra, thus $\mathcal{S}(\mathcal{R}) \subset \mathcal{N}$. $\square$