LECTURE 10 — MATH 6301

σ-ALGEBRAS IN CARTESIAN PRODUCTS

Notation: Given two spaces X_1 and X_2 and two families of sets
$$\mathcal{K}_1 \subset \mathcal{P}(X_1) \quad \text{and} \quad \mathcal{K}_2 \subset \mathcal{P}(X_2).$$

Then we will use the notation:

$$\times \qquad \mathcal{K}_1 \overline{\times} \mathcal{K}_2 := \{E_1 \times E_2 : E_1 \in \mathcal{K}_1 \text{ and } E_2 \in \mathcal{K}_2\}$$

Then, if $\mathcal{S}_1 \subset \mathcal{P}(X_1)$ and $\mathcal{S}_2 \subset \mathcal{P}(X_2)$ are two σ-algebras, then we put

$$\times \qquad \mathcal{S}_1 \times \mathcal{S}_2 := \mathcal{S}(\mathcal{S}_1 \overline{\times} \mathcal{S}_2)$$

and call it the *Cartesian product* of σ-Algebras $\mathcal{S}_1$ and $\mathcal{S}_2$.

DEFINITION: For a class $\mathcal{S}' \subset \mathcal{P}(Y)$, we say that $\mathcal{S}'$ *satisfies condition (C)*, iff

$$\text{(C)} \quad \forall_{A,B \in \mathcal{S}'} \quad A \cap B = \emptyset \implies A \cup B \in \mathcal{S}'$$

THEOREM 2: Let $\mathcal{S}_1 \subset \mathcal{P}(X_1)$ and $\mathcal{S}_2 \subset \mathcal{P}(X_2)$ be two σ-algebras. Then the σ-algebra $\mathcal{S}_1 \times \mathcal{S}_2$ satisfies

$$\mathcal{S}_1 \times \mathcal{S}_2 \underset{\subset}{\overset{\supset}{=}} \bigcap \left\{ \mathcal{S}' \subset \mathcal{P}(X_1 \times X_2) : \begin{cases} 1) \ \mathcal{S}' \text{ is monotone} \\ 2) \ \mathcal{S}_1 \overset{\mathcal{K}}{\overline{\times}} \mathcal{S}_2 \subset \mathcal{S}' \\ 3) \ \mathcal{S}' \text{ satisfies (C)} \end{cases} \right\}$$

PROOF: Put $\mathcal{K} = \mathcal{S}_1 \overline{\times} \mathcal{S}_2$.

"$\supset$" Notice that $\mathcal{S}_1 \times \mathcal{S}_2 = \mathcal{S}(\mathcal{K})$ is a σ-algebra, so it is monotone, contains $\mathcal{K}$ and satisfies (C). So $\mathcal{S}' := \mathcal{S}_1 \times \mathcal{S}_2$ satisfies 1)—3) and consequently

$$\mathcal{S}_1 \times \mathcal{S}_2 \supset \mathcal{S}' \supset \bigcap \left\{ \mathcal{S}'' : \begin{array}{l} 1) \ \mathcal{S}'' \text{ monotone} \\ 2) \ \mathcal{K} \subset \mathcal{S}'' \\ 3) \ \mathcal{S}'' \text{ satisfies (C)} \end{array} \right\}$$

"$\subset$" We need to show that $\mathcal{S}_1 \times \mathcal{S}_2 \subset \mathcal{S}'$ for every $\mathcal{S}' \subset \mathcal{P}(X_1 \times X_2)$ satisfying (1)—(3).

By condition (C), $\mathcal{S}'$ contains the class

$$\mathcal{R} := \left\{ \bigcup_{k=1}^{n} B_k : \forall_{k=1,\dots,n} \ B_k \in \mathcal{K} \text{ and } B_k \cap B_\ell = \emptyset \text{ for } k \neq \ell \right\}$$

Then, by Theorem 1, if $\mathcal{R}$ is an algebra, then the conclusion follows.

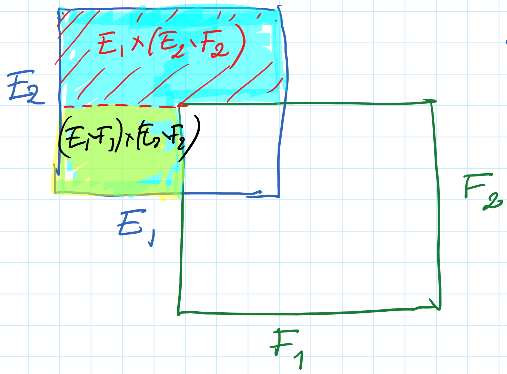
<u>Indeed</u>:
$$\mathcal{S}' \supset \mathcal{K} \underset{(C)}{\implies} \mathcal{S}' \underset{\text{algebra}}{\supset \mathcal{R}} \implies \mathcal{S}' \underset{(\text{by } (1))}{\supset} \mathcal{S}(\mathcal{R}) \supset \mathcal{S}(\mathcal{K}) = \mathcal{S}_1 \times \mathcal{S}_2$$

We show that (A1) is algebra: Indeed, notice that $X, \emptyset \in \mathcal{R}$ $\qquad X = X_1 \times X_2$
and if $A, B \in \mathcal{R}$ then $A \setminus B \in \mathcal{R}$: this fact

| (A'1) $X, \emptyset \in \mathcal{R}$

We show that (A1) is algebra: Indeed, notice that $X, \emptyset \in \mathcal{R}$ $\qquad X = X_1 \times X_2$
and if $A, B \in \mathcal{R}$ then $A \setminus B \in \mathcal{R}$: this fact follows from simple observation

(A'1) $X, \emptyset \in \mathcal{R}$
(A'2) $A, B \in \mathcal{R} \Rightarrow A \setminus B \in \mathcal{R}$
(A*3) $A, B \in \mathcal{R}, A \cap B = \emptyset \Rightarrow A \cup B \in \mathcal{R}$



$A = E_1 \times E_2$
$B = F_1 \times F_2$

$$A \setminus B = \underbrace{E_1 \times (E_2 \setminus F_2)}_{① \in \mathcal{K}} \cup \underbrace{(E_1 \setminus F_1) \times (E_2 \setminus F_2)}_{② \in \mathcal{K}} \in \mathcal{R}$$

Then if $A, B \in \mathcal{R}$ then we can write that

$$A = \bigcup_{i=1}^{m} A_i \qquad A_i \in \mathcal{K} \qquad A_i \cap A_j = \emptyset$$

$$B = \bigcup_{k=1}^{n} B_k \qquad B_k \in \mathcal{K} \qquad B_k \cap B_\ell = \emptyset$$

Then we have

$$A \setminus B = \left(\bigcup_{i=1}^{m} A_i\right) \setminus B = \bigcup_{i=1}^{m} (A_i \setminus B)$$

on the other hand

$$A_i \setminus B = (((A_i \setminus B_1) \setminus B_2) \dots) \setminus B_n$$

$A_i \in \mathcal{K}$ $\qquad A_i \setminus B_j \in \mathcal{R}$
$B_j \in \mathcal{K}$

$\square$

THEOREM 3: Let $\mathcal{K}_1 \subset \mathcal{P}(X_1)$, $\mathcal{K}_2 \subset \mathcal{P}(X_2)$. Then

$$\mathcal{S}(\mathcal{K}_1) \times \mathcal{S}(\mathcal{K}_2) = \boxed{\mathcal{S}(\mathcal{K}_1 \bar{\times} \mathcal{K}_2)} =: \mathcal{S} \quad \text{(notation for the proof)}$$

PROOF "$\supset$" Since $\mathcal{K}_1 \bar{\times} \mathcal{K}_2 \subset \mathcal{S}(\mathcal{K}_1) \times \mathcal{S}(\mathcal{K}_2)$, then clearly $\mathcal{S}(\mathcal{K}_1 \times \mathcal{K}_2) \subset \mathcal{S}(\mathcal{K}_1) \times \mathcal{S}(\mathcal{K}_2)$

"$\subset$" Since $\mathcal{K}_1 \bar{\times} \mathcal{K}_2 \subset \mathcal{S}$ it is sufficient to show the following implication

$\mathcal{L}_1 \subset \mathcal{P}(X_1)$
$\mathcal{L}_2 \in \mathcal{P}(X_2)$

$$\mathcal{L}_1 \bar{\times} \mathcal{L}_2 \subset \mathcal{S} \overset{?}{\Rightarrow} \mathcal{S}(\mathcal{L}_1) \bar{\times} \mathcal{L}_2 \in \mathcal{S} \quad ①$$
$$\mathcal{L}_1 \bar{\times} \mathcal{S}(\mathcal{L}_2) \in \mathcal{S} \quad ②$$

1) $\mathcal{L}_1 = \mathcal{K}_1 \quad \mathcal{L}_2 = \mathcal{K}_2$
$\Downarrow$
$\mathcal{S}(\mathcal{K}_1) \bar{\times} \mathcal{K}_2 \in \mathcal{S}$
2) $\mathcal{L}_1 = \mathcal{S}(\mathcal{K}_1) \quad \mathcal{L}_2 = \mathcal{K}_2$
$\Downarrow$
$\mathcal{S}(\mathcal{K}_1) \bar{\times} \mathcal{S}(\mathcal{K}_2) \in \mathcal{S}$
$\Downarrow$ $\qquad$ Ford.
$\mathcal{S}(\mathcal{K}_1) \times \mathcal{S}(\mathcal{K}_2) \in \mathcal{S}$

PROOF: Notice that for given $F \in \mathcal{L}_2$ the family of sets
$\{E \in \mathcal{P}(X_1) : E \times F \in \mathcal{S}\}$ is a σ-algebra containing $\mathcal{L}_1$, and consequently the conclusion follows.

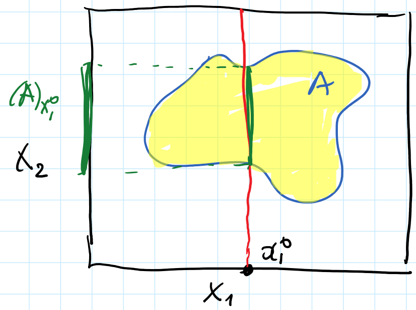
containing $\mathcal{I}_1$, and consequently the conclusion follows. | $\mathcal{S}(\mathcal{K}_1) \times \mathcal{S}(\mathcal{K}_2) \in \mathcal{S}$

i.e. $\mathcal{S}(\mathcal{L}_1) \times \{F\} \subset \mathcal{S} \quad \forall F \in \mathcal{L}_2$ so $\mathcal{S}(\mathcal{L}_1) \times \mathcal{L}_2 \subset \mathcal{S}$ $\square$

NOTATION: Take $A \subset X_1 \times X_2$ and $x_1^\circ \in X_1$, $x_2^\circ \in X_2$, then we write:

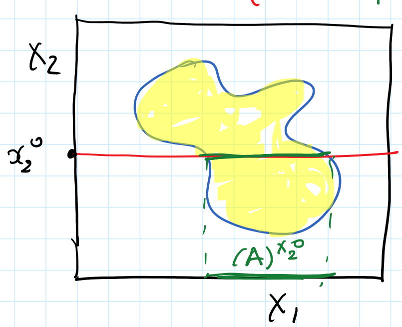
$$(A)_{x_1^\circ} := \{x_2 \in X_2 : (x_1^\circ, x_2) \in A\} \subset X_2$$

section of A through $x_1^\circ \in X_1$



$x_1^\circ \times X_2 \times A$

$$(A)^{x_2^\circ} := \{x_1 \in X_1 : (x_1, x_2^\circ) \in A\} \subset X_1$$



section of A through $x_2^\circ \in X_2$

Then we define
$$\varphi_{x_1^\circ} : X_2 \longrightarrow X_1 \times X_2 \qquad \varphi_{x_1^\circ}(x_2) = (x_1^\circ, x_2)$$
$$\varphi^{x_2^\circ} : X_1 \longrightarrow X_1 \times X_2 \qquad \varphi^{x_2^\circ}(x_1) = (x_1, x_2^\circ)$$

Then we have:

(1) $\varphi_{x_1^\circ}^{-1}(A) = (A)_{x_1^\circ}$

(2) $(A \setminus B)_{x_1^\circ} = (A)_{x_1^\circ} \setminus (B)_{x_1^\circ}$

(3) $\left(\bigcup_e A_e\right)_{x_1^\circ} = \bigcup_e (A_e)_{x_1^\circ}$

(4) $\left(\bigcap_e A_e\right)_{x_1^\circ} = \bigcap_e (A_e)_{x_1^\circ}$

} left as an exercise

THEOREM: Let $\mathcal{S}_1 \subset \mathcal{P}(X_1)$ and $\mathcal{S}_2 \subset \mathcal{P}(X_2)$ be two σ-algebras
If $A \in \mathcal{S}_1 \times \mathcal{S}_2$ then

(*) $\forall \quad (A)_{x_1} \in \mathcal{S}_2$ and $\forall \quad (A)^{x_2} \in \mathcal{S}_1$

If $A \in \mathcal{S}_1 \times \mathcal{S}_2$ then,

$$(*) \quad \forall_{x_1 \in X} \ (A)_{x_1} \in \mathcal{S}_2 \quad \text{and} \quad \forall_{x_2 \in X} \ (A)^{x_2} \in \mathcal{S}_1$$

PROOF: Put

$$\mathcal{S}^* := \{ A \subset X_1 \times X_2 : A \text{ satisfies condition } (*) \}$$

We claim that $\mathcal{S}^*$ is a σ-algebra.

(σA_1) $\emptyset, X \in \mathcal{S}^*$

(σA_2) $A \in \mathcal{S}^* \overset{(*)}{\Longrightarrow} \forall_{x_1 \in X_1} \ (A)_{x_1} \in \mathcal{S}_2$, $\forall_{x_2 \in X_2} \ (A)^{x_2} \in \mathcal{S}_1$

But

$(A^c)_{x_1} = X_2 \setminus (A)_{x_1} \Rightarrow (A^c)_{x_1} \in \mathcal{S}_2$

$(A^c)^{x_2} = X_1 \setminus (A)^{x_2} \Rightarrow (A^c)^{x_2} \in \mathcal{S}_1$

$\Downarrow \quad A^c \in \mathcal{S}^*$

(σA_3) $\{A_k\}_{k=1}^{\infty} \subset \mathcal{S}^* \Rightarrow \forall_{x_1 \in X_1} \forall_k \ (A_k)_{x_1} \in \mathcal{S}_2$

Then $\left(\bigcup_{k=1}^{\infty} A_k \right)_{x_1} = \bigcup_{k=1}^{\infty} (A_k)_{x_1} \in \mathcal{S}_2 \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{S}^*$

Then since $\mathcal{S}_1 \ast \mathcal{S}_2 \subset \mathcal{S}^* \Rightarrow \mathcal{S}_1 \times \mathcal{S}_2 \subset \mathcal{S}^*$, so the property $(*)$ is satisfied for $A \in \mathcal{S}_1 \times \mathcal{S}_2$. $\square$