MATH 6301 Real Analysis I
Homework 4

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Contents

Instructions:

1. Print this booklet
2. Use the space provided to write your solutions in this booklet
3. Hand in your assignment to your instructor on the due date during the class time.

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Problem 1

PROBLEM:

Assume that $U \subset \mathbb{R}^n$ is an open set and $f : U \to \mathbb{R}$ is a differentiable function. Show that for every $k = 1, 2, \ldots, n$, the partial derivative

$$\frac{\partial f}{\partial x_k} : U \to \mathbb{R}$$

is $\mathcal{B}_n$-measurable (here $\mathcal{B}_n$ stands for the σ-algebra of Borel sets in $\mathbb{R}^n$).

PRELIMINARIES:

Definition 1. *Let $\mathcal{S} \subset P(X)$ is a σ-algebra and $E \in \mathcal{S}$. The function $f : E \to \overline{\mathbb{R}}$ is called* <u>measurable</u> *relative to $\mathcal{S}$ (i.e. $\mathcal{S}$-measurable) iff*

$$\forall_{a \in \mathbb{R}} f^{-1}(a, \infty] := \{x \in E \ : \ f(x) > a\} \in \mathcal{S}$$

Remark 1. *Assume that $f : E \to \overline{\mathbb{R}}$, $E \in \mathcal{S} \subset P(X)$ is $\mathcal{S}$-measurable. Then the following are also $\mathcal{S}$-measurable*

1. $f^2 : E \to \overline{R}$
2. $|f| : E \to \overline{R}$
3. $\frac{1}{f} : E \to \overline{R}$
4. $a \cdot f : E \to \overline{R}, \ a \in \mathbb{R}$

Definition 2. *Let $U \subset \mathbb{R}^n$ and $f : U \to \mathbb{R}$ be a differentiable function. The* <u>partial derivative</u> $\frac{\partial f}{\partial x_k}$ *is defined as follows*

$$\frac{\partial f}{\partial x_k} := \lim_{n \to \infty} \frac{f(x_1, \cdots, x_k + 1/n, \cdots, x_n) - f(x_1, \cdots, x_k, \cdots, x_n)}{1/n}$$

SOLUTION:

Let $U \subset \mathbb{R}^n$ and $f : U \to \mathbb{R}$ be a differentiable function. The partial derivative, $\frac{\partial f}{\partial x_k}$, can be increasingly estimated by the sequence of simple functions where for each borel-set region $(a, b) \in \mathcal{B}_n$ the simple function value of $[\frac{\partial f}{\partial x_k}\Big|_{(a,b)}]_i$ is defined by

$$\frac{f(a_1, \cdots, a_k + 1/i, \cdots, a_n) - f(a_1, \cdots, a_k, \cdots, a_n)}{1/i}$$

Since this simple function can approximate $\frac{\partial f}{\partial x_k} \forall_{k=1,\ldots,n}$, it is $\mathcal{B}_n$-measurable.

Problem 2

PROBLEM:

Let X be a space and $\mathcal{S} \subset \mathcal{P}(X)$ a σ-algebra in X. We say that the map $f : X \to \mathbb{R}^n$ is $\mathcal{S}$-measurable if and only if

$$\forall_{V \in \mathcal{B}_n} f^{-1}(V) \in \mathcal{S}$$

Assume that $f : X \to \mathbb{R}^n$ is a map that for all $v \in \mathbb{R}^n$ the function $\phi_y(x) := f(x) \bullet v$, $x \in X$, is $\mathcal{S}$-measurable. Show that the map f is $\mathcal{S}$-measurable.

SOLUTION:

In order for $\phi_y(x)$ to be measurable each dimension of the dot product must be measurable. (i.e)

$$\phi_y(x) = f_1(x) \cdot v_1 + \cdots + f_n(x) \cdot v_n \text{ measurable } \implies f_i(x) \text{ measurable } \forall_{i=1 \to n}$$

we now know that each dimension measurable in $\mathcal{B}$, therefore f is measurable in $\mathcal{B}_n$.

Problem 3

PROBLEM:

Let X be a bounded set in Banach space $\mathcal{E}$. We define the following function $\mu^* : \mathcal{P}(X) \to \mathbb{R}$ by

$$\mu^*(A) := \inf\left\{ r > 0 \ : \ \exists_{x_1, x_2, \ldots, x_k \in X} A \subset \bigcup_{j=1}^{k} B_r(x_j) \right\}, A \subset X$$

where $B_r(x_0) := \{x \in \mathcal{E} \ : \ \|x - x_0\| < r\}$. Verify if the function μ^* is an outer measure on X and if it is check if it is a metric outer measure.
(The function μ^* defined above is called a *measure of non-compactness*. Can you guess what would be μ^* if $\mathcal{E} = \mathbb{R}^n$?)

PRELIMINARIES:

Definition 3. *An* ***outer measure*** $\mu \ : \ \mathcal{P}(X) \to [0, \infty]$ *must satisfy the following:*

- *Null empty set:* $\mu(\emptyset) = 0$
- *Monotone:* $A, B \subset X \ : \ A \subseteq B \implies \mu(A) \leq \mu(B)$
- *For arbritrary subsets* $B_1, B_2, \ldots, \subset X$,

$$\mu\left(\bigcup_{j=1}^{\infty} B_j\right) \leq \sum_{j=1}^{\infty} \mu(B_j)$$

SOLUTION:

We show that μ^* is an outer measure as follows:

- Null empty set:

$$\mu * (\emptyset) = \inf\left\{ r > 0 \ : \ \exists_{x_1, x_2, \ldots, x_k \in X} \emptyset \subset \bigcup_{j=1}^{k} B_r(x_j) \right\} = \inf \emptyset = 0$$

- Monotone:

$$\begin{aligned} A_1, A_2 \subset X, A_1 \subseteq A_2 &\implies \mu(A_1) \leq \mu(A_2) \\ &\implies \inf\left\{ r > 0 \ : \ \exists_{x_1, x_2, \ldots, x_k \in X} A_1 \subset \bigcup_{j=1}^{k} B_r(x_j) \right\} \\ &\quad \leq \inf\left\{ r > 0 \ : \ \exists_{x_1, x_2, \ldots, x_k \in X} A_2 \subset \bigcup_{j=1}^{k} B_r(x_j) \right\} \end{aligned}$$

Since $A_1 \subseteq A_2$, the set of r for the first inf will allways be containted in the secound one. (i.e.)

$$\left\{ r > 0 \ : \ \exists_{x_1, x_2, \ldots, x_k \in X} A_1 \subset \bigcup_{j=1}^{k} B_r(x_j) \right\} \subseteq \left\{ r > 0 \ : \ \exists_{x_1, x_2, \ldots, x_k \in X} A_2 \subset \bigcup_{j=1}^{k} B_r(x_j) \right\}$$

therefore $\mu(A_1) \leq \mu(A_2)$

- Let $A_1, A_2, \ldots, \subset X$ be arbitrary,

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu(A_j)$$

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \inf\left\{r > 0 \ : \ \exists_{x_1, x_2, \ldots, x_k \in X} \bigcup_{j=1}^{\infty} A_j \subset \bigcup_{j=1}^{k} B_r(x_j)\right\}$$

$$\sum_{j=1}^{\infty} \mu(A_j) = \sum_{j=1}^{\infty} \inf\left\{r > 0 \ : \ \exists_{x_1, x_2, \ldots, x_k \in X} A_j \subset \bigcup_{j=1}^{k} B_r(x_j)\right\}$$

Clearly, $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu(A_j)$ since the common ball radius that contains the union of all subsets will be less then the sum of each individual subsets ball radius. It ends up like $\max r_1, r_2, \ldots, r_k \leq \sum_{i=1}^{k} r_k$.

This isn't much of a metric outer measure though

Within $\mathbb{R}^n$, μ^* doesn't mean much though as every bounded closed set is compact so it just becomes 0 or ∞ as a test of boundedness or not.

Problem 4

PROBLEM:

For two given spaces X and Y and assume that $\mu_1^* : \mathcal{P}(X) \to \overline{\mathbb{R}}$ and $\mu_2^* : \mathcal{P}(Y) \to \overline{\mathbb{R}}$ are two outer measures. Define the function $\nu^* : \mathcal{P}(X \times Y) \to \overline{\mathbb{R}}$ by

$$\nu^*(C) := \inf \left\{ \sum_{k=1}^{\infty} \mu_1^*(A_k)\mu_2^*(B_k) \ : \ C \subset \bigcup_{k=1}^{\infty} A_k \times B_k,\ A_k \subset X,\ B_k \subset Y \right\}$$

Check if the function ν^* is an outer measure on $X \times Y$.

SOLUTION:

We test ν^* as an outer measure on $X \times Y$ as follows:

- Null empty set:
$$\nu^*(\emptyset) = \inf \left\{ \sum_{k=1}^{\infty} \mu_1^*(\emptyset)\mu_1^*(\emptyset) \right\} = 0$$

- Monotone:
$$C_1 \subseteq C_2 \implies \inf \left\{ \sum_{k=1}^{\infty} \mu_1^*(A_k)\mu_2^*(B_k) \ : \ C_1 \subset \bigcup_{k=1}^{\infty} A_k \times B_k,\ A_k \subset X,\ B_k \subset Y \right\}$$
$$\leq \inf \left\{ \sum_{k=1}^{\infty} \mu_1^*(A_k)\mu_2^*(B_k) \ : \ C_2 \subset \bigcup_{k=1}^{\infty} A_k \times B_k,\ A_k \subset X,\ B_k \subset Y \right\}$$

  Since $C_1 \subseteq C_2$, every $C_1 \subseteq C_2 \subset \bigcup_{k=1}^{\infty} A_k \times B_k$ so therefore $\mu(A_1) \leq \mu(A_2)$

- Let $C_1, C_2, \ldots, \subset X$ be arbitrary,
$$\nu^*\left(\bigcup_{j=1}^{\infty} C_j\right) \leq \sum_{j=1}^{\infty} \nu^*(C_j)$$

  We've established that $C_1 \subseteq C_2 \subset \bigcup_{k=1}^{\infty} A_k \times B_k$, and we use this same logic to show that $\bigcup_{j=1}^{\infty} C_j \subset \bigcup_{j=1}^{\infty} \left(\bigcup_{k=1}^{\infty} A_k \times B_k\right)_j$ and therefore the $\sum_{k=1}^{\infty} \mu_1(A_k)\mu_2(B_k)$ of the union is less then $\sum_{j=1}^{k} \left(\sum_{k=1}^{\infty} \mu_1^*(A_k)\mu_2^*(B_k)\right)$.

Problem 5

PROBLEM:

A set $I \subset \mathbb{R}^n$ is called an *interval* in $\mathbb{R}^n$ if there exists $a_1 \leq b_1, a_2 \leq b_2, \ldots, a_n \leq b_n$ such that

$$(a_1, b_1) \times (a_2, b_2) \times \cdots, \times (a_n, b_n) \subset I \subset [a_1, b_2] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

We denote by $\mathcal{F}$ the family of all intervals in $\mathbb{R}^n$. Consider the set

$$X := [c_1, d_1] \times [c_2, d_2] \times \cdots \times [c_n, d_n], \quad c_k < d_k$$

Is the family $\mathcal{R} \subset \mathcal{P}(X)$, given by

$$\mathcal{R} := \left\{ A \subset X : \exists_{I_1, I_2, \ldots, I_N \in \mathcal{F}} A := \bigcup_{k=1}^{N} I_k, I_k \subset X \right\}$$

and algebra of sets in X? Justify your answer.

SOLUTION:

This does form an algebra. Similarly to the construction of the Borel Algebra, each individual interval can be combined to form an algebra.
This can be checked by the definition of an algebra as $\mathcal{R}$ satisfies each of the conditions.