LECTURE 21 – MATH 6301

$(X, \mathfrak{S}, \mu)$, $E \in \mathfrak{S}$

DEFINITION: Let $(X, \mathfrak{S}, \mu)$ be a measure space, $E \in \mathfrak{S}$ and assume that $f_n : E \to \overline{\mathbb{R}}$ be a sequence of measurable functions (finite a.e.) and $f : E \to \overline{\mathbb{R}}$ a measurable function, such that

$$\forall_{\varepsilon > 0} \quad \lim_{n\to\infty} \mu\{x \in E : |f_n(x) - f(x)| \geq \varepsilon\} = 0$$

Then we say that the sequence f_n converges to f in measure μ, or μ-converges to f and we will denote it as

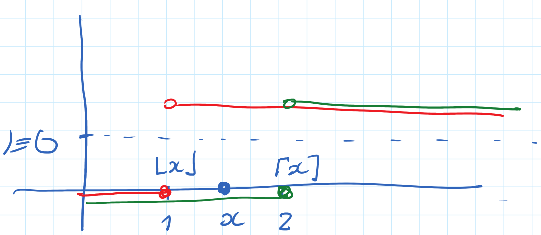
$$f_n \xrightarrow{\mu} f$$

THEOREM (Lebesgue) If $f_n : E \to \overline{\mathbb{R}}$ is a sequence of measurable (finite a.e) functions such that $f_n(x) \to f(x)$ a.e. (for some $f : E \to \overline{\mathbb{R}}$) and $\mu(E) < \infty$ Then f_n converges to f in measure μ, i.e

$$f_n(x) \to f(x) \text{ a.e} \implies f_n \xrightarrow{\mu} f$$

EXAMPLE. Notice that in the above theorem, the assumption $\mu(E) < \infty$ cannot be removed. Indeed, take the sequence $f_n : [0, \infty) \to \mathbb{R}$ given by

$$f_n(x) = \begin{cases} 1 & x \geq n \\ 0 & x < n \end{cases}$$



then $\forall_{x \in [0,\infty)}$ $\lim_{n\to\infty} f_n(x) = 0$ but $f(x) \equiv 0$

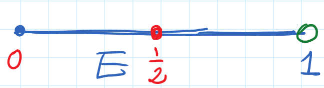
take $1 > \varepsilon > 0$

$$\mathfrak{m}\{x : |f_n(x) - 0| \geq \varepsilon\} = \mathfrak{m}\{[n, \infty)\} = \infty$$

so clearly it is not convergent in measure $\mathfrak{m}$.

EXAMPLE: It is possible that a given sequence f_n, $n = 1, 2, \dots$, is convergent in measure to f, but in the same time $f_n(x) \not\to f(x)$ for all $x \in E$.

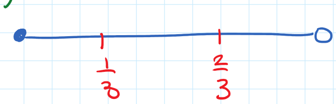
Indeed:



$\mathfrak{m}$ Lebesgue measure.

$$f_1(x) = \chi_{[0, \frac{1}{2})}$$

$$f_2(x) = \chi_{[\frac{1}{2}, 1)}$$

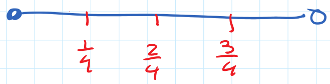


2 3 4 . . . . .

$f_3(x) = \chi_{[0,\frac{1}{3})}$

$f_4(x) = \chi_{[\frac{1}{3},\frac{2}{3})}$ 1

$f_5(x) = \chi_{[\frac{2}{3},1)}$ 2



$f_6(x) = \chi_{[0,\frac{1}{4})}$

$\vdots$

Notice that $\forall_{x\in[0,1)}$ $\lim_{n\to\infty} f_n(x)$ does not exist.

$\forall_N \ \exists_{\substack{n\geq N \\ m\geq N}}$ $f_n(x) = 0$, $f_m(x) = 1$

However, $f_n \xrightarrow{\mathcal{L}} 0$. Indeed, $\forall_{\varepsilon>0} \ \exists_N$ $\frac{1}{N} < \varepsilon$, and $\forall_{n \geq \frac{N(N+1)}{2}}$

$$\mathcal{L}\{x : |f_n(x)| \geq \varepsilon\} \leq \frac{1}{N} \quad \text{so}$$

$$\lim_{n\to\infty} \mathcal{L}\{x : |f_n(x) - 0| \geq \varepsilon\} = 0. \quad \square$$

THEOREM (Egorov) Suppose $f_n : E \to \overline{\mathbb{R}}$ are μ-measurable functions (finite a.e) and $f : E \to \overline{\mathbb{R}}$ a μ-measurable function, such that
$f_n(x) \to f(x)$ a.e. and $\mu(E) < \infty$.

Then
$\forall_{\varepsilon>0} \ \exists_{F\in\mathcal{S}}$ $F \subset E$ and $\mu(E\setminus F) < \varepsilon$ and $f_n \rightrightarrows f$ on F

f_n converges uniformly

$\forall_{\eta>0} \ \exists_N \ \forall_{n\geq N} \ \forall_{x\in F}$ $|f_n(x) - f(x)| < \eta$

(Moreover, if μ is a Lebesgue measure $\mathcal{L}$, then $F_0 = F$ can be made closed)

PROOF: Take $\varepsilon > 0$, $n \in \mathbb{N}$ and put

$$E_k\left(\tfrac{1}{n}\right) := \{x \in E : |f_k(x) - f(x)| \geq \tfrac{1}{n}\}$$

By Lemma, we have that

$$\lim_{k\to\infty} \mu\left(\bigcup_{m=k}^{\infty} E_m\left(\tfrac{1}{n}\right)\right) = 0, \quad \text{so}$$

$$\forall_{n\in\mathbb{N}} \ \exists_{N_n} \quad \mu\left(\bigcup_{k=N_n}^{\infty} E_k\left(\tfrac{1}{n}\right)\right) < \frac{\varepsilon}{2^{n+1}}$$

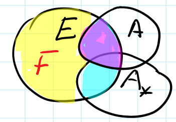
Notice that the set $A = \{x \in E : |f(x)| = \infty\}$ is of measure zero (since the functions f_n are a.e finite; indeed $A \subset \{x \in E : |f_n(x) - f(x)| \geq 1\} \cup A_n$. $\forall_n$ $\mu(A_n) = 0$
so $0 \leq \mu(A) \leq \mu(E_n(1) \cup A_n) \leq \mu(E_n(1)) \longrightarrow 0$ $E_n(1)$

the functions f_n are a.e finite; indeed $A \subset \underbrace{\{x\in E : |f_n(x)-f(x)| \geq 1\}}_{E_n(1)} \cup A_n$. $\forall_n$ $\mu(A_n)=0$

$A_n=\{x : |f_n(x)|=\infty\}$

so $0 \leq \mu(A) \leq \mu(E_n(1)\cup A_n) \leq \mu(E_n(1)) \xrightarrow[n\to\infty]{} 0$.

Put
$$F := (E\setminus A)\setminus \underbrace{\bigcup_{n=1}^{\infty}\bigcup_{k=N_n}^{\infty} E_k(\tfrac{1}{n})}_{A_*}$$



$E\setminus F \subset A\cup A_*$

$\mu(A)=0$

then
$$\mu(E\setminus F) \leq \mu\Big(\bigcup_{n=1}^{\infty}\bigcup_{k=N_n}^{\infty} E_k(\tfrac{1}{n})\Big) \leq \sum_{n=1}^{\infty}\mu\Big(\bigcup_{k=N_n}^{\infty} E_k(\tfrac{1}{n})\Big)$$
$$< \sum_{n=1}^{\infty}\frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2}.$$

We claim that F is the required set, i.e. $f_n \rightrightarrows f$ on F.
Indeed, for $\forall_{\eta>0}$ take n such that $\frac{1}{n}<\eta$ then
$$\forall_{x\in F} \quad x\in E\setminus \bigcup_{k=N_n}^{\infty} E_k(\tfrac{1}{n}) = \bigcap_{k=N_n}^{\infty}\big(E\setminus E_k(\tfrac{1}{n})\big)$$
$$= \bigcap_{k=N_n}^{\infty}\{x : |f_k(x)-f(x)| < \tfrac{1}{n}\}$$

which mean
$$\forall_{k\geq N_n}\ \forall_{x\in F} \quad |f_k(x)-f(x)| < \tfrac{1}{n} < \eta.$$

Suppose now that μ is the Lebesgue measure $\mathcal{M}$. Then, there exists a closed set $F_0\subset F$ such that
$$\mathcal{M}(F\setminus F_0) < \frac{\varepsilon}{2}$$
and hence
$$\mathcal{M}(E\setminus F_0) < \varepsilon$$
Moreover, $f_n \rightrightarrows f$ on F thus it also converges uniformly on F_0 $\square$

<u>DEFINITION:</u> Suppose $f_n : E\to\overline{\mathbb{R}}$, $n=1,2,\dots$, is a sequence of μ-measurable functions (finite a.e) and $f: E\to\overline{\mathbb{R}}$ a measurable function such that
$$\forall_{\varepsilon>0}\ \exists_{F\in\mathcal{S}}\ F\subset E \quad \mu(E\setminus F)<\varepsilon \text{ and } f_n\rightrightarrows f \text{ on } F.$$
Then we say that f_n converges almost uniformly to f.
(Notice that almost uniform convergence implies convergence a.e).

<u>THEOREM:</u> Suppose $f_n : E\to\overline{\mathbb{R}}$ are measurable finite a.e functions such that $f_n\xrightarrow{\mu} f$ and $f_n\xrightarrow{\mu} g$ then $f(x)=g(x)$ a.e.

THEOREM: suppose $f_n : E \to \mathbb{R}$ are measurable finite a.e. functions such that $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g$ then $f(x) = g(x)$ a.e.

PROOF Notice $\forall \varepsilon > 0$ $\forall n$

$$\{x : |f(x) - g(x)| \geq \varepsilon\} \subset \{x : |f_n(x) - g(x)| \geq \tfrac{1}{2}\varepsilon\} \cup \{x : |f_n(x) - f(x)| \geq \tfrac{1}{2}\varepsilon\}$$

$$\left[\varepsilon \leq |f(x) - g(x)| \leq \underbrace{|f_n(x) - g(x)|}_{\geq \frac{\varepsilon}{2}} + \underbrace{|f_n(x) - f(x)|}_{\geq \frac{\varepsilon}{2}} \quad \text{or} \qquad x \in A \cup B \iff x \in A \text{ or } x \in B \right]$$

so $\forall \varepsilon > 0$ $\forall n$

$$\mu\{x : |f(x) - g(x)| \geq \varepsilon\} \leq \underbrace{\mu\{x : |f_n(x) - g(x)| \geq \tfrac{\varepsilon}{2}\}}_{\xrightarrow{n \to \infty} 0} + \underbrace{\mu\{x : |f_n(x) - g(x)| \geq \tfrac{\varepsilon}{2}\}}_{\xrightarrow{n \to \infty} 0}$$

$$\mu\{x : |f(x) - g(x)| \geq \varepsilon\} = 0$$

which means

$$\{x : f(x) \neq g(x)\} = \bigcup_{n=1}^{\infty} \{x : |f(x) - g(x)| \geq \tfrac{1}{n}\}$$

and

$$\mu\{x : f(x) \neq g(x)\} = 0 \iff f(x) = g(x) \text{ a.e. } \square$$