LECTURE #16 & 17 - MATH 6301

Let (X,d) be a metric space. A function $\mu^*:\mathcal{P}(X)\to\overline{\mathbb{R}}$ is called a metric outer measure if

outer measure
$$\begin{cases} (\mu^*1) & \mu^*(A)\geq 0, \quad A\subset X \\ (\mu^*2) & \mu^*(\emptyset)=0 \\ (\mu^*3) & A\subset B\subset X \Rightarrow \mu^*(A)\leq\mu^*(B) \\ (\mu^*4) & \{A_n\}_{n=1}^\infty\subset\mathcal{P}(X) \Rightarrow \mu^*\left(\bigcup_{n=1}^\infty A_n\right)\leq\sum_{n=1}^\infty\mu^*(A_n) \end{cases}$$

(μ^*5) $A,B\subset X$ and $\underset{\inf\{d(x,y):\,x\in A,\,y\in B\}}{\underset{\parallel}{\varrho(A,B)}}>0 \Rightarrow \mu^*(A\cup B)=\mu^*(A)+\mu^*(B)$

Caratheodory Condition

(C) $\forall_{A\subset X}$ $\mu^*(A)=\mu^*(A\cap E)+\mu^*(A\setminus E)$ then E satisfies (C)

$\mathfrak{S}_C:=\{E\in\mathcal{P}(X):E \text{ satisfies } (C)\}$ μ^*-measurable sets

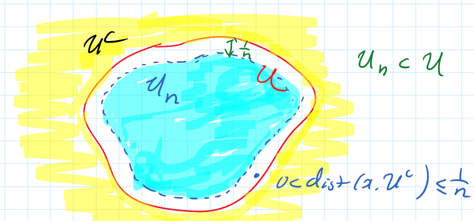
Then, we proved it that $\mathfrak{S}_C$ is a σ-algebra and μ^* restricted to $\mathfrak{S}_C$ is a complete measure.

THEOREM: Let (X,d) be a metric space and $\mu^*:\mathcal{P}(X)\to\overline{\mathbb{R}}$ a metric outer measure on X. Then all the Borel sets in X are μ^*-measurable, i.e. $\mathcal{B}(X)\subset\mathfrak{S}_C$

PROOF: Since $\mathfrak{S}_C$ is a σ-algebra, in order to show that $\mathcal{B}(X)\subset\mathfrak{S}_C$, it is sufficient to show that $\mathcal{T}\subset\mathfrak{S}_C$ (i.e. every open set $\mathcal{U}$ is μ^* measurable)

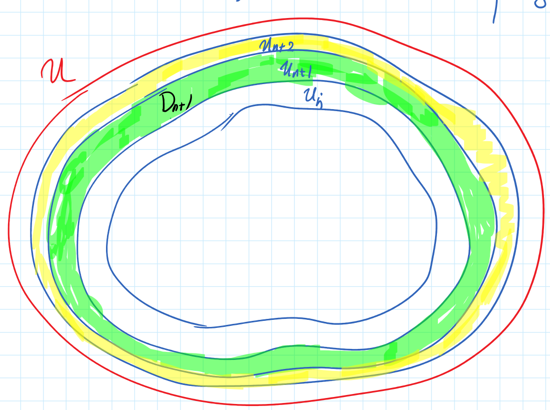
Take an open set $\mathcal{U}\subset X$ and put

$\forall_{n\in\mathbb{N}}$ $\mathcal{U}_n:=\{x\in X: dist(x,\mathcal{U}^c)>\frac{1}{n}\}$



Notice that $\mathcal{U}=\bigcup_{n=1}^\infty\mathcal{U}_n$. and also

$\varrho(\mathcal{U}_n,\mathcal{U}^c)\geq\frac{1}{n}>0$ | $\varrho(\mathcal{U}_n,\mathcal{U}^c)=\inf_{x\in\mathcal{U}_n}dist(x,\mathcal{U}^c)\geq\frac{1}{n}$



Define

$D_n:=\{x\in X:\frac{1}{n+1}<dist(x,\mathcal{U}^c)\leq\frac{1}{n}\}$

$\varrho(D_i,D_j)\geq\frac{1}{i+1}-\frac{1}{j}>0$ if $i+2\leq j$

and notice that

$\mathcal{U}\setminus\mathcal{U}_n=\bigcup_{i=n}^\infty D_i$

We need to show that $\mathcal{U}\subset\mathcal{T}$ is μ^*-measurable

We have (by (μ^*4))

$\mu^*(A)\leq\mu^*(A\cap\mathcal{U})+\mu^*(A\setminus\mathcal{U})$

so in order to show that $\leq \mu^*(A) = \mu^*(A \cap \mathcal{U}) + \mu^*(A \setminus \mathcal{U}) \leq$

it suffice to show that

$$(*) \quad \mu^*(A) \geq \mu^*(A \cap \mathcal{U}) + \mu^*(A \setminus \mathcal{U}) \quad ?$$

Notice that if $\mu^*(A) = \infty$, then $(*)$ is obvious, so assume that $\mu^*(A) < \infty$. Then we have by $(\mu^* 5)$

$$(1) \quad \mu^*(A \cap D_1) + \mu^*(A \cap D_3) + \dots + \mu^*(A \cap D_{2n-1}) = \mu^*(A \cap (D_1 \cup D_3 \cup \dots \cup D_{2n-1})) \underset{\text{by } (\mu^* 3)}{\leq} \mu^*(A)$$

and

$$(2) \quad \mu^*(A \cap D_2) + \mu^*(A \cap D_4) + \dots + \mu^*(A \cap D_{2n}) = \mu^*(A \cap (D_2 \cup D_4 \cup \dots \cup D_{2n})) \leq \mu^*(A)$$

Therefore $\displaystyle\sum_{i=1}^{2n} \mu^*(A \cap D_i) \underset{(1)+(2)}{\leq} 2\mu^*(A) \quad \forall_{n \in \mathbb{N}} \Rightarrow \sum_{i=1}^{\infty} \mu^*(A \cap D_i) < \infty$

so the series converges, i.e. $\displaystyle\sum_{i=n}^{\infty} \mu^*(A \cap D_i) \xrightarrow[n \to \infty]{} 0$

$$\sum_{i=n}^{\infty} \mu^*(A \cap D_i) \geq \mu^*\left(\bigcup_{i=n}^{\infty} (A \cap D_i)\right) = \mu^*\left(A \cap \left(\bigcup_{i=n}^{\infty} D_i\right)\right) = \mu^*(A \cap (\mathcal{U} \setminus \mathcal{U}_n))$$

i.e.

$$\mu^*(A \cap (\mathcal{U} \setminus \mathcal{U}_n)) \longrightarrow 0 \quad \text{as } n \to \infty$$

$A \cap \mathcal{U}_n$

$A \cap \mathcal{U}^c = A \setminus \mathcal{U}$

$\varrho(\mathcal{U}_n, \mathcal{U}^c) > 0$

$\varrho(A \cap \mathcal{U}_n, A \setminus \mathcal{U}) > 0$

$(*)$

Therefore we have

$$\mu^*(A \cap \mathcal{U}_n) + \mu^*(A \setminus \mathcal{U}) \overset{(*)}{=}$$

$$= \mu^*((A \cap \mathcal{U}_n) \cup (A \setminus \mathcal{U})) \leq \mu^*(A)$$

Thus

$$\begin{aligned}
\mu^*(A \cap \mathcal{U}) + \mu^*(A \setminus \mathcal{U}) &\leq \mu^*((A \cap \mathcal{U}_n) \cup A \cap (\mathcal{U} \setminus \mathcal{U}_n)) + \mu^*(A \setminus \mathcal{U}) \\
&\overset{(\mu^* 4)}{\leq} \mu^*(A \cap \mathcal{U}_n) + \mu^*(A \cap (\mathcal{U} \setminus \mathcal{U}_n)) + \mu^*(A \setminus \mathcal{U}) \\
&= \mu^*((A \cap \mathcal{U}_n) \cup (A \setminus \mathcal{U})) + \mu^*(A \cap (\mathcal{U} \setminus \mathcal{U}_n)) \qquad \forall_n \\
&\leq \mu^*(A) + \mu^*(A \cap (\mathcal{U} \setminus \mathcal{U}_n)) \xrightarrow[n \to \infty]{} \mu^*(A)
\end{aligned}$$

$\square$

THEOREM: Let (X, d) be a metric space with topology $\mathcal{T}$, and suppose $\lambda : \mathcal{T} \to [0, \infty)$ is a function satisfying conditions

(1) $\lambda(\mathcal{U}) \geq 0$

(2) $\lambda(\emptyset) = 0$

(3) $\mathcal{U} \subset \mathcal{V} \Rightarrow \lambda(\mathcal{U}) \leq \lambda(\mathcal{V})$
open sets

(4) $\displaystyle\lambda\left(\bigcup_{n=1}^{\infty} \mathcal{U}_n\right) \leq \sum_{n=1}^{\infty} \lambda(\mathcal{U}_n) \qquad \{\mathcal{U}_n\}_{n=1}^{\infty} \subset \mathcal{T}$

(5) $\varrho(\mathcal{U}, \mathcal{V}) > 0 \Rightarrow \lambda(\mathcal{U} \cup \mathcal{V}) = \lambda(\mathcal{U}) + \lambda(\mathcal{V})$

Then the function $\mu^*: \mathcal{P}(X) \to \overline{\mathbb{R}}$, defined by
$$\mu^*(A) := \inf_{\mathcal{U}} \{\lambda(\mathcal{U}) : A \subset \mathcal{U}, \mathcal{U} \in \mathcal{J}\}$$
is a metric outer measure.

PROOF: (1) $\Rightarrow \mu^*(A) \geq 0$

(2) $\Rightarrow \mu^*(\emptyset) = 0$

(3) $\Rightarrow A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$

We need to show $(\mu^* 4)$. Let $\{A_n\}_{n=1}^{\infty} \subset \mathcal{P}(X)$. and take arbitrary $\varepsilon > 0$,

By definition of infimum
$$\forall_n \ \exists_{\mathcal{U}_n \in \mathcal{J}} \quad \lambda(\mathcal{U}_n) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n} \qquad \bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \mathcal{U}_n$$

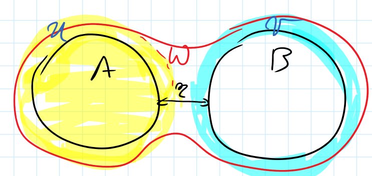
Then
$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \mu^*\left(\bigcup_{n=1}^{\infty} \mathcal{U}_n\right) \overset{(4)}{\leq} \sum_{n=1}^{\infty} \mu^*(\mathcal{U}_n)$$
$$\leq \sum_{n=1}^{\infty}\left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n}$$
$$= \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon$$

Thus
$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

To show $(\mu^* 5)$ take $A, B \subset X$ such that
$$\rho(A, B) = \eta > 0$$

Put
$$\mathcal{U} := \{x \in X : \operatorname{dist}(x, A) < \frac{\eta}{3}\}$$
$$\mathcal{V} := \{x \in X : \operatorname{dist}(x, B) < \frac{\eta}{3}\}$$

$\rho(\mathcal{U}, \mathcal{V}) \geq \frac{\eta}{3}$



So for any open set $W \subset \mathcal{J}$ s.t. $A \cup B \subset W$
$$\mu^*(A) + \mu^*(B) \leq \lambda(\mathcal{U} \cap W) + \lambda(\mathcal{V} \cap W)$$
$$= \lambda(\mathcal{U} \cap W \cup (\mathcal{V} \cap W))$$
$$= \lambda((\mathcal{U} \cup \mathcal{V}) \cap W) \leq \lambda(W)$$

$\mathcal{U} \cap W \supset A$

$\mathcal{V} \cap W \supset B$

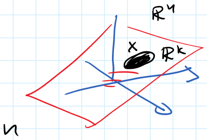
$\rho(\mathcal{U} \cap W, \mathcal{V} \cap W) \geq \frac{\eta}{3}$

which means, $\forall_{\substack{W \supset A \cup B \\ \text{open}}}$ we have
$$\mu^*(A) + \mu^*(B) \leq \lambda(W)$$

so
$$\mu^*(A \cup B) \underset{(\mu^* 4)}{\leq} \mu^*(A) + \mu^*(B) \leq \inf\{\lambda(W) : W \in \mathcal{J}, A \cup B \subset W\}$$
$$= \mu^*(A \cup B) \quad \square$$

DEFINITION: A set $I \subset \mathbb{R}^n$ is called an interval ~~in~~ $\mathbb{R}^n$ there exist $a_1 \le b_1,\ a_2 \le b_2,\ \dots,\ a_n \le b_n$ such that

$$\underbrace{(a_1,b_1)\times(a_2,b_2)\times\dots\times(a_n,b_n)}_{= \operatorname{int}(I)} \subset I \subset \underbrace{[a_1,b_1]\times[a_2,b_2]\times\dots\times[a_n,b_n]}_{= \overline{I}}$$



$k<n$

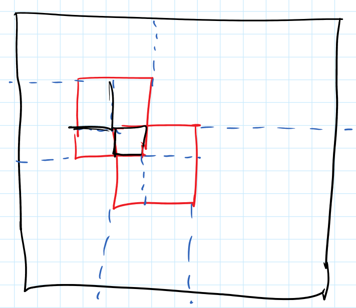
$\mathrm{vol}(X)=0$ in $\mathbb{R}^n$

$$|I| := \text{volume}(I) = \prod_{k=1}^{n} (b_k - a_k)$$

Denote by $\mathcal{J}$ the set of all intervals in $\mathbb{R}^n$. Notice that if $R = \overline{R} \in \mathcal{J}$ then the set

$$\mathcal{F}_R := \{ A \subset R : \exists_{I_1, I_2, \dots, I_N}\ A = \bigcup_{k=1}^{N} I_k ;\ I_k \in \mathcal{J} \}$$

Then $\mathcal{F}_R$ is an algebra of sets. $X = R$

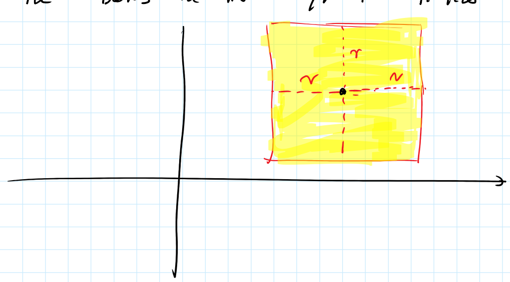


Put $\mathcal{F} := \{ A \subset \mathbb{R}^n : \exists_{I_1, I_2, \dots, I_N \in \mathcal{J}}\ A = \bigcup_{k=1}^{N} I_k ,\ \underset{k \ne j}{I_k \cap I_j = \emptyset} \}$

REMARK: Consider $\mathbb{R}^n$ equipped with the norm $\|\cdot\|_\infty : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|(x_1, x_2, \dots, x_n)\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\}$$

then the balls in $\mathbb{R}^n$ w/r to $\|\cdot\|_\infty$ are intervals



In addition, $A, B \in \mathcal{F}$ we have

(i) $A \cup B \in \mathcal{F}$

(ii) $A \setminus B \in \mathcal{F}$

(iii) $A \cap B \in \mathcal{F}$

PROPOSITION: Let $U \subset \mathbb{R}^n$ be an open set. Then there exists a sequence of intervals I_n, $n=1,2,\dots$ such that

(a) $\mathcal{U} = \bigcup_{n=1}^{\infty} I_n$

(b) $I_n \cap I_k = \emptyset \quad k \neq n$

<u>LEMMA:</u> Let (X,d) be a separable metric space. Then X can be represented as a countable union of open balls.

<u>PROOF:</u> Let $S = \{x_1, x_2, \dots\}$ dense set in X, $\overline{S} = X$ and consider $\{B_q(x_k) : q>0,\ q \in \mathbb{Q},\ k=1,2,\dots\}_{q,k}$ $\square$

<u>PROOF</u>

$$\mathcal{U} = \bigcup_{j=1}^{\infty} B_j \qquad B_j \text{ open ball w/r d.k.}$$

$$= \bigcup_{j=1}^{\infty} A_j \qquad A_1 = B_1, \quad A_j = B_j \setminus (B_1 \cup \dots \cup B_{j-1})$$

$$\lambda(\mathcal{U}) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} |I_n| \qquad \text{(a) \& (b) satisfied.}$$