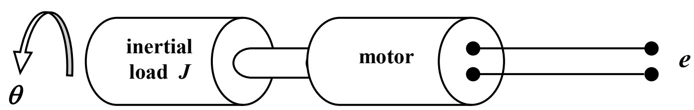
**LINEAR SYSTEMS (MECH 6300 / EECS 6331 / SYSM 6307)**
**Design Application #1**

## DC Motor with Load

A common application of control concepts is position control of an inertial load utilizing a DC motor. In this problem we describe the dynamics of such a system to be used in subsequent design applications problems and exercises.

Consider the setup depicted in the Figure below. The load, described by inertia *J*, could be a large rotating machine, or a small cassette tape head.




Employing laws of physics, and assuming ideal circumstances, the torque developed at the shaft is proportional to the motor input current $i$, whereas the (induced) back emf $v$ is proportional to the speed of rotation , or

$$\tau = K_1 i \quad \text{and} \qquad v = K_2\omega \quad .$$

To see the relationship between these motor constants, we note that the electrical power input to the motor, $p_e$, is just

$$p_e = vi = K_2\omega\left(\frac{\tau}{K_1}\right) \quad ,$$

whereas the mechanical power delivered is

$$p_m = \omega\tau \quad .$$

Thus,

$$p_e = \left(\frac{K_2}{K_1}\right) p_m$$

so that at 100% efficiency

$$K_1 = K_2 \quad ,$$

while at less than ideal efficiency,

$$K_2 > K_1 \quad .$$

What remains now is to relate the above to the terminal voltage $e$, armature resistance $R$, and angular velocity $\omega$ of the shaft. To this end, note that

$$e - v = Ri \quad ,$$

from Ohm's Law, and

$$\tau = J\dot{\omega} \quad .$$

From our previous expression for torque,

$$J\dot{\omega} = K_1 i = \frac{K_1}{R}(e - v) = \frac{K_1}{R}e - \frac{K_1 K_2}{R}\omega \quad ,$$

or, finally,

$$\dot{\omega} = \frac{K_1}{JR}e - \frac{K_1 K_2}{JR}\omega \quad .$$

The above expression represents the dynamics for state variable (angular velocity) and control input *e*.

In order to model and control the shaft speed *and* shaft (load) position, the first order model above is not sufficient. The position and speed are related by

$$\dot{\theta} = \omega \quad ,$$

which, when combined with the above, may be expressed in state-variable form as

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{K_1 K_2}{JR} \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K_1}{JR} \end{bmatrix} e \quad .$$

These equations will be used to model a motor-driven apparatus in future problems.