1) $\dot{x} = Ax + Bu$

$y = Cx \quad , t \geq 0$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \quad x(0) = \begin{bmatrix} 1 \\ 8 \end{bmatrix} \quad u(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(sI - A) = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 2 & 4 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s+1)(s^2+2s+2)} \begin{bmatrix} s^2+3s+4 & s+3 & 1 \\ -2 & s(s+3) & s \\ -2s & -2(2s+1) & s^2 \end{bmatrix}$$

$$G(s) = C(sI - A)^{-1}B + D$$

$$= \frac{1}{(s+1)(s^2+2s+2)} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} \underline{s^2+3s+3} & s+4 \\ -(s+2) & s(s+4) \\ -s(s+2) & s^2-4s-2 \end{bmatrix} + 0$$

$$G(s) = \begin{bmatrix} \frac{s^2+s-2}{(s+1)(s^2+2s+2)} & \frac{8s+2}{(s+1)(s^2+2s+2)} \\ \frac{s^2+s-1}{(s+1)(s^2+2s+2)} & \frac{2s^2+9s+4}{(s+1)(s^2+2s+2)} \end{bmatrix}$$

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1) cont.

A in companion Form

$$e^{At} = \begin{bmatrix} 1 & e^{t} & 1 \\ 1 & 1 & e^{t} \\ e^{-2t} & e^{-4t} & e^{-3t} \end{bmatrix} \qquad x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 1 \\ 1 \\ e^{-2t} \end{bmatrix} \qquad C e^{At} = \begin{bmatrix} 1 - e^{-2t} \\ 3 \end{bmatrix}$$

$$Y(s) = G(s)\,U(s)$$

$$= \begin{bmatrix} \dfrac{s^2 + s - 2 + 8s + 2}{(s+1)(s^2+2s+2)} \\ \dfrac{s^2 + s - 1 + 2s^2 + 9s + 4}{(s+1)(s^2+2s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s(s+9)}{(s+1)(s^2+2s+2)} \\ \dfrac{3s^2+10s+3}{(s+1)(s^2+2s+2)} \end{bmatrix} = \begin{bmatrix} \dfrac{9s+16}{s^2+2s+2} - \dfrac{8}{s+1} \\ \dfrac{7s+11}{s^2+2s+2} - \dfrac{4}{s+1} \end{bmatrix}$$

$$\{\mathcal{L}^{-1}\{\ \}\}$$

$$y(t) = \begin{bmatrix} \cancel{1 - e^{-2t}} \\ 3 \end{bmatrix} + \begin{bmatrix} \cancel{e^{-2t}} + e^{-4t} \cancel{-1} \\ -e^{t} - 2 \end{bmatrix}$$

$$\boxed{y(t) = \begin{bmatrix} e^{-4t} \\ 1 - e^{t} \end{bmatrix}}$$

2)

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -16 & -2 & 7 \\ 0 & -1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

$$Ab = \begin{bmatrix} 10 \\ -9 \\ -2 \end{bmatrix} \quad A^2 b = \begin{bmatrix} 2 \\ 4 \\ 13 \end{bmatrix} \quad A^3 b = \begin{bmatrix} -11 \\ 51 \\ -30 \end{bmatrix}$$

$$|sI - A| = \begin{vmatrix} s-1 & 0 & 1 \\ 16 & s+2 & -7 \\ 0 & 1 & s+2 \end{vmatrix} = s^3 + 3s^2 + 7s + 5$$
$$= (s+1)(s+(1-j2))(s+(1+j2))$$

$$\boxed{\bar{A} = \begin{bmatrix} 0 & 0 & -5 \\ 1 & 0 & -7 \\ 0 & 1 & -3 \end{bmatrix}}$$

$$A^3 b = -5b - 7Ab - 3A^2 b$$

$$\begin{bmatrix} -11 \\ 51 \\ -39 \end{bmatrix} = \begin{bmatrix} -5 + 0 - 6 \\ 0 + 63 - 12 \\ -5 + 6 - 39 \end{bmatrix} \checkmark$$

3)

$$\lambda_1 = -1 \qquad q_1 = \alpha \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

$$\lambda_{2,3} = -1 \pm j2 \qquad q_{2,3} = \alpha \begin{bmatrix} -3 \pm j1 \\ 20 \\ -4 \pm j8 \end{bmatrix}$$

$$\hat{A} = PAP^{-1}$$

$$\boxed{\hat{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & -1 \end{bmatrix}}$$

$$P^{-1} = \begin{bmatrix} q_1 & Re(q_2) & Im(q_2) \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -1 & -3 & 1 \\ 2 & 20 & 0 \\ -2 & -4 & 8 \end{bmatrix}$$

$$\hat{B} = PB$$

$$\boxed{\hat{B} = \begin{bmatrix} -1.75 \\ 0.175 \\ -0.225 \end{bmatrix}}$$

$$P = \frac{1}{-80} \begin{bmatrix} 160 & 20 & -20 \\ -16 & -6 & 2 \\ 32 & 2 & -14 \end{bmatrix}$$

$$\hat{C} = C P^{-1}$$

$$P = \begin{bmatrix} -2 & -0.25 & 0.25 \\ 0.2 & 0.075 & -0.025 \\ -0.4 & -0.025 & 0.175 \end{bmatrix}$$

$$\boxed{\hat{C} = \begin{bmatrix} -3 & -23 & 1 \end{bmatrix}}$$

4)

$$\Delta(s) = (s-1)^4$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(sI - A) = \begin{bmatrix} s-1 & 0 & -1 & -1 \\ 0 & s-1 & 0 & 0 \\ 0 & 0 & s-1 & 1 \\ 0 & 0 & 0 & s-1 \end{bmatrix}$$

$$\text{adj}(sI-A) = \begin{bmatrix} (s-1)^3 & 0 & (s-1)^2 & -(s-1)(s-2) \\ 0 & (s-1)^3 & 0 & 0 \\ 0 & 0 & (s-1)^3 & -(s-1)^2 \\ 0 & 0 & 0 & (s-1)^3 \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{(s-1)^4} \text{adj}(sI-A)$$

$$(sI-A)^{-1} = \begin{bmatrix} \frac{1}{s-1} & 0 & \frac{1}{(s-1)^2} & \frac{(s-2)}{(s-1)^3} \\ 0 & \frac{1}{s-1} & 0 & 0 \\ 0 & 0 & \frac{1}{s-1} & \frac{-1}{(s-1)^2} \\ 0 & 0 & 0 & \frac{1}{s-1} \end{bmatrix}$$

4) cont.

$$\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^{-t}$$

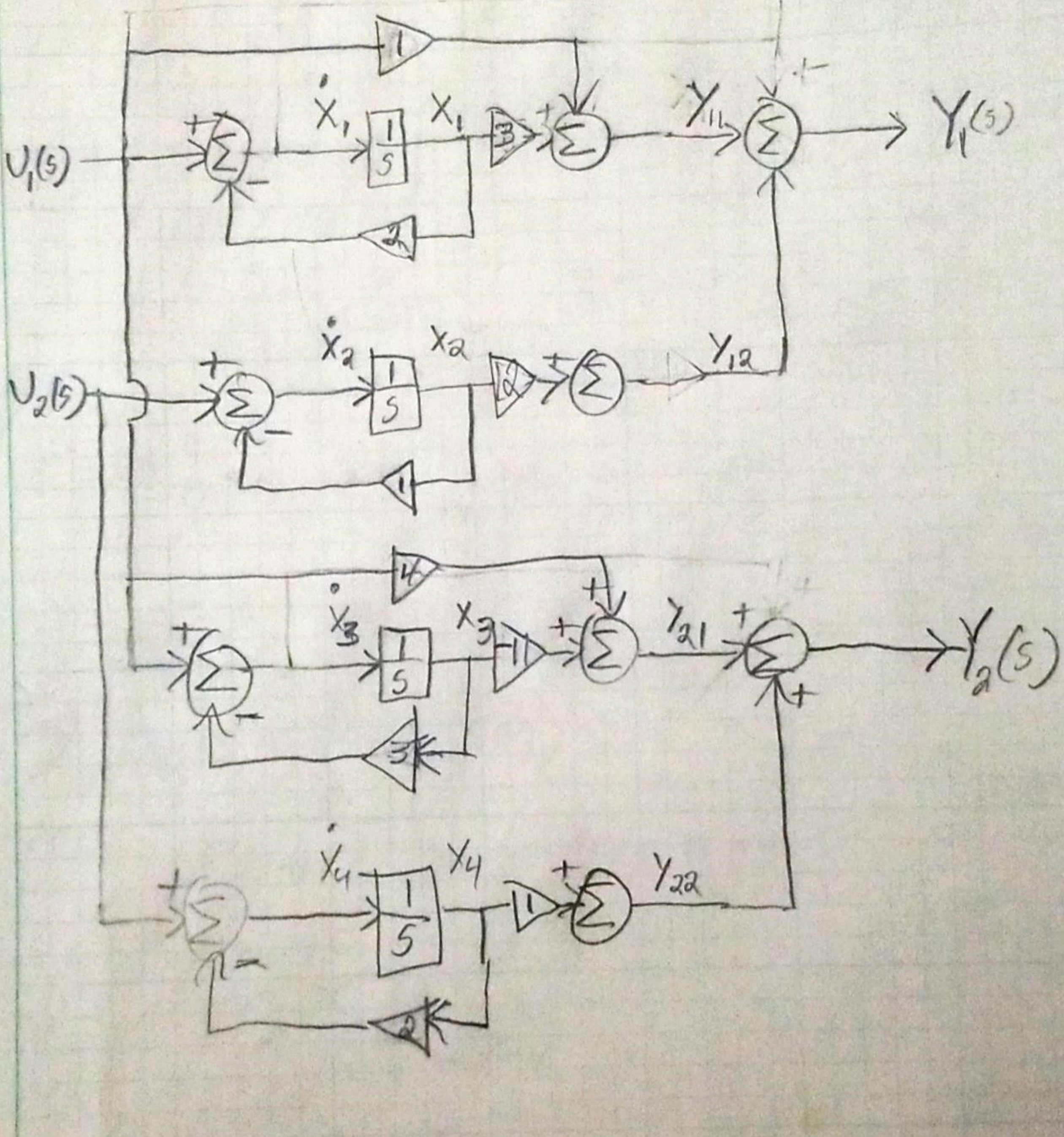
$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = te^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{s-2}{(s-1)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3} - \frac{1}{(s-1)^2}\right\} = \frac{t^2}{2}e^{-t} - te^{-t}$$

$$e^{At} = \begin{bmatrix} e^{-t} & 0 & te^{-t} & \frac{t^2}{2}e^{-t} - te^{-t} \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{-t} & -te^{-t} \\ 0 & 0 & 0 & e^{-t} \end{bmatrix}$$

5)

$$\hat{G}(s) = \begin{bmatrix} \frac{s+5}{s+2} & \frac{2}{s+1} \\ \frac{4s+1}{s+3} & \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} 1 + \frac{3}{s+2} & \frac{2}{s+1} \\ 4 - \frac{11}{s+3} & \frac{1}{s+2} \end{bmatrix}$$



5) cont.

$$\dot{x} = Ax + Bu \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$y = Cx + Du$$

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 0 & 0 & -11 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}$$

6)

a) $$A_1(s) = (s-2)(s+1)(s-1)$$

$$A_2(s) = (s-2)(s+1)^2$$

They are not equivent!

b) $$G_1(s) = \frac{2}{(s-2)(s+1)}$$

$$G_2(s) = \frac{6}{(s-2)(s+1)}$$

They are not zero-state equivelent!

7) $A(t) = \begin{bmatrix} 0 & 1 \\ 0 & t \end{bmatrix}$ $\quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = t x_2 \longrightarrow \boxed{x_2(t) = x_2(0) + \frac{t^2}{2} x_2(t)}$$

$$\frac{dx_1}{dt} = x_2(0) + \frac{t^2}{2} x_2(t) \qquad x_2(0) = \frac{t^2+2}{2} x_2(0)$$

$$\int dx_1 = \int_0^t \frac{\tau^2}{2} x_2(0)\, d\tau + x_1(0)$$

$$\boxed{x_1(t) = \frac{t^3}{6} x_2(0) + x_1(0)}$$

$$\boxed{\Psi(t) = \begin{bmatrix} 1 & \frac{t^3}{6} \\ 0 & \frac{t^2+2}{2} \end{bmatrix}}$$

$$\Psi^{-1}(t) = \frac{2}{t^2+2} \begin{bmatrix} \frac{t^2+2}{2} & \frac{-t^3}{6} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-t^3}{3(t^2+2)} \\ 0 & \frac{2}{t^2+2} \end{bmatrix}$$

$$\Phi(t,\tau) = \Psi(t)\, \Psi^{-1}(\tau)$$

$$= \begin{bmatrix} 1 & \frac{t^3}{6} \\ 0 & \frac{t^2+2}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{-\tau^3}{3(\tau^2+2)} \\ 0 & \frac{2}{\tau^2+2} \end{bmatrix}$$

$$\boxed{\Phi(t,\tau) = \begin{bmatrix} 1 & \frac{t^3-\tau^3}{3(\tau^2+2)} \\ 0 & \frac{t^2+2}{\tau^2+2} \end{bmatrix}}$$

8) $A(t) = \begin{bmatrix} -2 & e^{-t} \\ 0 & -1 \end{bmatrix}$ $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\dot{x}_1 = -2x_1 + e^{-t}x_2$

$\dot{x}_2 = -x_2 \longrightarrow \boxed{x_2(t) = x_2(0)e^{-t}}$

$\dot{x}_1 = -2x_1 + e^{-t}(x_2(0)e^{-t})$

$\dot{x}_1 + 2x_1 = x_2(0)e^{-2t}$

$\dot{x}_1 e^{2t} + 2x_1 e^{2t} = x_2(0)e^{-2t}e^{2t}$

$\frac{d}{dt}(x_1 e^{2t}) = x_2(0)$

$\int x_1 e^{2t} = \int_0^t x_2(0)\,d\tau$

$x_1 e^{2t} = t\,x_2(0) + \cancel{c}^{\,x_1(0)}$

$\boxed{x_1(t) = x_1(0)e^{-2t} + x_2(0) + e^{-2t}}$

$\Psi(t) = \begin{bmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-t} \end{bmatrix}$

$\Psi^{-1}(t) = \frac{1}{e^{-3t}}\begin{bmatrix} e^{-t} & -te^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$

$\Psi^{-1}(t) = \begin{bmatrix} e^{2t} & -te^{-t} \\ 0 & e^{t} \end{bmatrix}$

$\Phi(t,\tau) = \Psi(t)\Psi^{-1}(\tau)$

$\Phi(t,\tau) = \begin{bmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} e^{2\tau} & -te^{-\tau} \\ 0 & e^{\tau} \end{bmatrix} = \begin{bmatrix} e^{-2(t-\tau)} & -te^{-2t-\tau} + te^{-2t+\tau} \\ 0 & e^{-(t-\tau)} \end{bmatrix}$

$$\boxed{\Phi(t,\tau) = \begin{bmatrix} e^{-2(t-\tau)} & (t)\left(e^{-(2t-\tau)} - e^{-(2t+\tau)}\right) \\ 0 & e^{-(t-\tau)} \end{bmatrix}}$$