LINEAR SYSTEMS (MECH 6300 / EECS 6331 / SYSM 6307)
Design Application #2

Inverted Pendulum on a Cart

Consider the classic problem of an inverted pendulum (bob on a mass-less rod) pivoted at its base (frictionless hinge) which is mounted on a motorized cart. Following the notation in the diagram below, the horizontal displacement of the cart determines the total kinetic energy as

$$E_K = \frac{1}{2}M\dot{z}^2 + \frac{1}{2}m\left(\dot{\bar{z}}^2 + \dot{h}^2\right) \quad ,$$

which is nothing more than the sum of kinetic energies of the cart and bob, respectively. Simple rules of trigonometry, along with the fact that the rod is rigid, allow us to write

$$\bar{z} = z + l\sin\theta \quad ,$$
$$h = l\cos\theta \quad ,$$

so that

$$E_K = \frac{1}{2}M\dot{z}^2 + \frac{1}{2}m\left(\dot{z}^2 + 2\dot{z}\dot{\theta}l\cos\theta + l^2\dot{\theta}^2\right) \quad .$$

Given that the potential energy is (stored in the bob)

$$E_P = mg(l\cos\theta) \quad ,$$

where g is the acceleration due to gravity, the Lagrangian for the system, $L = E_K - E_P$, can be written down as

$$L = \frac{1}{2}(M+m)\dot{z}^2 + ml\,\dot{z}\dot{\theta}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta \quad .$$

From the Lagrangian, via Lagrange mechanics, the dynamics of the system may be defined and state equations may be derived.