MECH 6300 - HW 3

1)

a) $\begin{bmatrix} 1 & 0 & 0 \\ 10 & 2 & 0 \\ 0 & 5 & 4 \end{bmatrix}$

$$\Delta(s) = |sI - A| = \begin{vmatrix} s-1 & 0 & 0 \\ -10 & s-2 & 0 \\ 0 & -5 & s-4 \end{vmatrix}$$

$$\Delta(s) = (s-1)(s-2)(s-4)$$

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 4$$

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

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I)

i)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 2 & 0 \end{bmatrix}$$

$$\Delta(s) = |sI - A| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -4 & -2 & s \end{vmatrix}$$

$$= s(s^2-2) + (+1)(0-4)$$

$$= s^3 - 2s - 4$$

$$\Delta(s) = (s-2)(s-(-1+j))(s-(-1-j))$$

$$J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1+j & 0 \\ 0 & 0 & -1-j \end{bmatrix}$$

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1) a)

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & -75 & -45 \\ 0 & 100 & 60 \end{bmatrix}$$

$$\Delta(s) = |sI - A| = \begin{vmatrix} s & -2 & -1 \\ 0 & s+75 & 45 \\ 0 & -100 & s-60 \end{vmatrix}$$

$$\Delta(s) = s^2(s+15)$$

$\lambda_1 = 0,\ m_1 = 2$

$\lambda_2 = -15$

$(A - \lambda_i I)x_i = 0$

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -15 \end{bmatrix}$$

$0a\ \ 2b + c = 0$

$0\ \ -75b\ \ -45c = 0$

$0\ \ 100b + 60c = 0$

$\rho(A - 0I) = 2$

$\gamma = 1 \quad -1 \quad$ J.B.

MECH 6300 - HW 3

1) a)

$$A = \begin{bmatrix} 0 & 8 & 6 \\ 0 & 40 & 32 \\ 0 & -50 & -40 \end{bmatrix}$$

$$\Delta(s) = |sI - A| = \begin{vmatrix} s & -8 & -6 \\ 0 & s-40 & -32 \\ 0 & 50 & s+40 \end{vmatrix}$$

$$= s^3 \cancel{-160} \cancel{+160}$$

$$\boxed{\Delta(s) = s^3},\quad \lambda_1 = 0,\ m_1 = 3$$

$$\rho(A - \lambda_1 I) = \rho(A)$$

$$\rho(A) = 2$$

$$\gamma(A - \lambda_1 I) = 1$$

$$\boxed{J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}$$

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1) e)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9 & 18 & 8 & -2 \end{bmatrix}$$

$$\Delta(s) = s^4 + 2s^3 - 8s^2 - 18s - 9$$

$$\Delta(s) = (s+1)^2(s-3)(s+3)$$

$$\rho(A+1I) = \rho\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 9 & 18 & 8 & -1 \end{pmatrix}$$

$$\rho(A+I) = 3$$

$$\gamma(A-\lambda_1 I) = 1$$

$m_1 = 2$

$\lambda_1 = -1$

$\lambda_2 = 3$

$\lambda_3 = -3$

$$J = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

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1)

a)

$$A = \begin{bmatrix} \frac{7}{2} & \frac{21}{2} & 14 \\ -\frac{1}{2} & -\frac{3}{2} & -2 \\ -\frac{1}{2} & -\frac{3}{2} & -2 \end{bmatrix}$$

$$\Delta(s) = |sI - A| = \begin{vmatrix} s-\frac{7}{2} & -\frac{21}{2} & -14 \\ \frac{1}{2} & s+\frac{3}{2} & 2 \\ \frac{1}{2} & \frac{3}{2} & s+2 \end{vmatrix}$$

$$\boxed{\Delta(s) = s^3}$$

$$\boxed{\lambda_1 = 0, \ m_1 = 3}$$

$$\rho(A) = 2$$

$$\gamma(A) = 1$$

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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7) g)

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Delta(s) = s^5$ $\rho(A) = 4$

$\lambda_1 = 0, m_1 = 5$ $\gamma(A) = 1$

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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2)

a) $\Delta(s) = (s-\lambda_1)^3(s-\lambda_2)$

$\Psi(s) = \Delta(s) = (s-\lambda_1)^3(s-\lambda_2)$

b) $\Delta(s) = (s-\lambda_1)^4$

$\Psi(s) = (s-\lambda_1)^3$

c) $\Delta(s) = (s-\lambda_1)^4$

$\Psi(s) = (s-\lambda_1)^2$

d) $\Delta(s) = (s-\lambda_1)^4$

$\Psi(s) = (s-\lambda_1)^2$

MECH 6300-HW3

a)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 10 & 2 & 0 \\ 0 & 5 & 4 \end{bmatrix} \quad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$\lambda_1 = 1$
$\lambda_2 = 2$
$\lambda_3 = 4$

i) Cayley-Hamilton Therem Method

$$f(\lambda) = e^{\lambda t} \quad h(\lambda) = \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2$$

$$f(1) = e^{t} = h(1) = \beta_0 + \beta_1 + \beta_2$$

$$f(2) = e^{2t} = h(2) = \beta_0 + 2\beta_1 + 4\beta_2$$

$$f(4) = e^{4t} = h(4) = \beta_0 + 4\beta_1 + 16\beta_2$$

$$\beta_0 = \frac{8}{3}e^{t} - 2e^{2t} + \frac{1}{3}e^{4t}$$

$$\beta_1 = -2e^{t} + \frac{5}{2}e^{2t} - \frac{1}{2}e^{4t}$$

$$\beta_2 = \frac{1}{3}e^{t} - \frac{1}{2}e^{2t} + \frac{1}{6}e^{4t}$$

$$e^{At} = \beta_0 I + \beta_1 A + \beta_2 A^2$$

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3)

a)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 10 & 2 & 0 \\ 0 & 5 & 4 \end{bmatrix} \qquad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$\lambda_1 = 1$

$\lambda_2 = 2$

$\lambda_3 = 4$

i) Cayley - Hamilton Therem Method

$$\beta_0 I = \begin{bmatrix} \frac{8}{3}e^{t} - 2e^{2t} + \frac{1}{3}e^{4t} & 0 & 0 \\ 0 & \frac{8}{3}e^{t} - 2e^{2t} + \frac{1}{3}e^{4t} & 0 \\ 0 & 0 & \frac{8}{3}e^{t} - 2e^{2t} + \frac{1}{3}e^{4t} \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 30 & 4 & 0 \\ 50 & 30 & 16 \end{bmatrix}$$

$$\beta_1 A = \begin{bmatrix} -2e^{t} + \frac{5}{2}e^{2t} - \frac{1}{2}e^{4t} & 0 & 0 \\ -20e^{t} + 25e^{2t} - 5e^{4t} & -4e^{t} + 5e^{2t} - e^{4t} & 0 \\ 0 & -10e^{t} + \frac{25}{2}e^{2t} - \frac{5}{2}e^{4t} & -8e^{t} + 10e^{2t} - 2e^{4t} \end{bmatrix}$$

$$\beta_2 A^2 = \begin{bmatrix} \frac{1}{3}e^{t} - \frac{1}{2}e^{2t} + \frac{1}{6}e^{4t} & 0 & 0 \\ 10e^{t} - 15e^{2t} + 5e^{4t} & \frac{4}{3}e^{t} - 2e^{2t} + \frac{2}{3}e^{4t} & 0 \\ \frac{50}{3}e^{t} - 25e^{2t} + \frac{25}{3}e^{4t} & 10e^{t} - 15e^{2t} + 5e^{4t} & \frac{16}{3}e^{t} - 8e^{2t} + \frac{8}{3}e^{4t} \end{bmatrix}$$

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a)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 10 & 2 & 0 \\ 0 & 5 & 4 \end{bmatrix} \qquad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$\lambda_1 = 1$
$\lambda_2 = 2$
$\lambda_3 = 4$

i) Cayley-Hamilton Therem Method

$$e^{At} = \begin{bmatrix} \frac{2}{3}e^{t} + \frac{1}{2}e^{2t} - \frac{1}{6}e^{4t} & 0 & 0 \\ -10e^{t} + 10e^{2t} & -\frac{8}{3}e^{t} + 3e^{2t} - \frac{1}{3}e^{4t} & 0 \\ \frac{50}{3}e^{t} - 25e^{2t} + \frac{25}{3}e^{4t} & -\frac{5}{2}e^{2t} - \frac{5}{2}e^{4t} & -\frac{8}{3}e^{t} + 2e^{2t} + \frac{2}{3}e^{4t} \end{bmatrix}$$

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3)

i)
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 10 & 2 & 0 \\ 0 & 5 & 4 \end{bmatrix} \qquad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$\lambda_1 = 1$
$\lambda_2 = 2$
$\lambda_3 = 4$

ii) Jordan Form Method:

$(A - 1I)q_1 = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 10 & 1 & 0 \\ 0 & 5 & 3 \end{bmatrix} x_1 = 0$$

$0 + 0 + 0 = 0$
$10a + b + 0 = 0$
$0 + 5b + 3c = 0$

$$q_1 = \alpha \begin{bmatrix} \frac{1}{10} \\ -1 \\ \frac{5}{3} \end{bmatrix}$$

$(A - 2I)q_2 = 0$

$$\begin{bmatrix} -1 & 0 & 0 \\ 10 & 0 & 0 \\ 0 & 5 & 2 \end{bmatrix}$$

$a = 0$
$5b + 2c = 0$

$$q_2 = \alpha \begin{pmatrix} 0 \\ 1 \\ -\frac{5}{2} \end{pmatrix}$$

$(A - 4I)q_3 = 0$

$$\begin{bmatrix} -3 & 0 & 0 \\ 10 & -2 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

$a = 0$
$\cancel{10a}^{0} - \cancel{2b}^{0} = 0$
$b = 0$
$c = \sim$

$$q_3 = \alpha \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$Q = \left[q_1 \,\vdots\, q_2 \,\vdots\, q_3 \right] = \begin{bmatrix} \frac{1}{10} & 0 & 0 \\ -1 & 1 & 0 \\ \frac{5}{3} & -\frac{5}{2} & 1 \end{bmatrix}$$

MECH 6300-HW3

3)

i)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 10 & 2 & 0 \\ 0 & 5 & 4 \end{bmatrix} \qquad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$\lambda_1 = 1$
$\lambda_2 = 2$
$\lambda_3 = 4$

ii) Jordan Form Method:

$$e^{Jt} = \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix}$$

$$A = QJQ^{-1}$$

$$e^{At} = Q e^{Jt} Q^{-1}$$

$$e^{At} = \begin{bmatrix} \frac{1}{10} & 0 & 0 \\ -1 & 1 & 0 \\ \frac{5}{3} & \frac{-5}{2} & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 10 & 1 & 0 \\ \frac{25}{3} & \frac{5}{2} & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{t} & 0 & 0 \\ -10e^{t} + 10e^{2t} & e^{2t} & 0 \\ \frac{50}{3}e^{t} - 25e^{2t} + \frac{25}{3}e^{4t} & -\frac{5}{2}e^{2t} + \frac{5}{2}e^{4t} & e^{4t} \end{bmatrix}$$

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2)

i) $A = \begin{bmatrix} 1 & 0 & 0 \\ 10 & 2 & 0 \\ 0 & 5 & 4 \end{bmatrix}$ $J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$\lambda_1 = 1$
$\lambda_2 = 2$
$\lambda_3 = 4$

ii) Laplace Method:

$$(SI - A) = \begin{bmatrix} s-1 & 0 & 0 \\ -10 & s-2 & 0 \\ 0 & -5 & s-4 \end{bmatrix}$$

$$\Delta(s) = (s-1)(s-2)(s-4)$$

$$(SI - A)^{-1} = \begin{bmatrix} \frac{1}{s-1} & 0 & 0 \\ \frac{10}{(s-1)(s-2)} & \frac{1}{s-2} & 0 \\ \frac{50}{(s-1)(s-2)(s-3)} & \frac{5}{(s-2)(s-4)} & \frac{1}{s-4} \end{bmatrix}$$

$\downarrow$ expand for easy $\mathcal{L}^{-1}$

$$e^{At} = \mathcal{L}^{-1}\{(SI-A)^{-1}\}$$

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3) b)

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & -75 & -45 \\ 0 & 100 & 60 \end{bmatrix} \qquad \lambda_1 = 0,\ m_1 = 2 \qquad \lambda_2 = -15 \qquad J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -15 \end{bmatrix}$$

i) Cayley-Hamilton Method:

$$f(\lambda) = e^{\lambda t} \qquad h(\lambda) = \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2$$

$$f'(\lambda) = t e^{\lambda t} \qquad h'(\lambda) = \beta_1 + 2\beta_2 \lambda$$

$$f(0) = 1 = h(0) = \beta_0 \Rightarrow \boxed{\beta_0 = 1}$$

$$f'(0) = t = h'(0) = \beta_1 \Rightarrow \boxed{\beta_1 = t}$$

$$f(-15) = e^{-15t} = h(-15) = \cancel{\beta_0}^{1} - \cancel{15\beta_1}^{0} + 15^2 \beta_2$$

$$\boxed{\beta_2 = \frac{e^{-15t} - 1}{15^2}}$$

$$h(\lambda) = 1 + (t)\lambda + \frac{e^{-15t} - 1}{15^2}(\lambda)^2$$

$$\boxed{e^{At} = I + (e^{-15t} - 1)\frac{A^2}{15^2}} \qquad \frac{A^2}{15^2} = \begin{bmatrix} 0 & \frac{-2}{9} & \frac{-2}{15} \\ 0 & 5 & 3 \\ 0 & \frac{-20}{3} & -4 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & \frac{2(1-e^{-15t})}{9} & \frac{2(1-e^{-15t})}{15} \\ 0 & -4 + 5e^{-15t} & -3(1-e^{-15t}) \\ 0 & \frac{20(1-e^{-15t})}{3} & 5 - 4e^{-15t} \end{bmatrix}$$

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3) b) $A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & -75 & -45 \\ 0 & 100 & 60 \end{bmatrix}$ $\lambda_1 = 0, m_1 = 2$ $\lambda_2 = -15$ $J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -15 \end{bmatrix}$

i) Cayley-Hamilton Method:

$$h(\lambda) = 1 + (t)\lambda + (e^{-15t}-1)\frac{\lambda^2}{15^2}$$

$$h(A) = I + (t)A + (e^{-15t}-1)\frac{A^2}{15^2}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2t & t \\ 0 & -75t & -45t \\ 0 & 100t & 60t \end{bmatrix} + \begin{bmatrix} 0 & -\frac{2}{9} & -\frac{2}{15} \\ 0 & 5 & 3 \\ 0 & -\frac{20}{3} & -4 \end{bmatrix}(e^{-15t}-1)$$

$$e^{At} = \begin{bmatrix} 1 & \frac{2}{9} - \frac{4}{3}t + -\frac{2}{9}e^{-15t} & \frac{2}{15} - t + \frac{2}{15}e^{-15t} \\ 0 & -4 + 5e^{-15t} & -3 + 3e^{-15t} \\ 0 & \frac{20}{3} - \frac{20}{3}e^{-15t} & 5 - 4e^{-15t} \end{bmatrix}$$

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3) b)

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & -75 & -45 \\ 0 & 100 & 60 \end{bmatrix} \quad \lambda_1 = 0, m_1 = 2 \quad \lambda_2 = -15 \quad J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -15 \end{bmatrix}$$

ii) Jordan Form Method:

$(A - \cancel{\lambda_1 I}\, 0)\, q_1 = 0$

$$\boxed{q_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

$A\, q_2 = q_1$

$0 + 2b + c = 1$

$0 - 75b - 45c = 0$

$0 + 100b + 60c = 0$

$$\boxed{q_2 = \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix}}$$

$(A + 15I)\, q_3 = 0$

$$\begin{bmatrix} 15 & 2 & 1 \\ 0 & -60 & -45 \\ 0 & 100 & 75 \end{bmatrix} q_3 = 0$$

$15a + 2b + c = 0$

$0 - 60b - 45c = 0$

$0 + 100b + 75c = 0$ $\rightarrow$ $b = -45$, $c = 60$

$$\boxed{q_2 = \begin{pmatrix} 2 \\ -45 \\ 60 \end{pmatrix}} \quad a = 2$$

$$Q = [q_1 \vdots q_2 \vdots q_3] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -45 \\ 0 & -5 & 60 \end{bmatrix}$$

$\rho(Q) = 3$

All L.I.

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3) b) $A=\begin{bmatrix}0 & 2 & 1\\ 0 & -75 & -45\\ 0 & 100 & 60\end{bmatrix}$ $\lambda_1=0, m_1=2$ $\lambda_2=-15$ $J=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & -15\end{bmatrix}$

ii) Jordan Form Method: $Q=\begin{bmatrix}1 & 0 & 2\\ 0 & 3 & -45\\ 0 & -5 & 60\end{bmatrix}$

$$e^{Jt}=\begin{bmatrix}1 & t & 0\\ 0 & 1 & 0\\ 0 & 0 & e^{-15t}\end{bmatrix}$$

$$Q^{-1}=\begin{bmatrix}1 & \frac{2}{9} & \frac{2}{15}\\ 0 & \frac{-4}{3} & -1\\ 0 & \frac{-1}{9} & \frac{-1}{15}\end{bmatrix}$$

$$e^{At}=Q\,e^{JT}\,Q^{-1}$$

$$=\begin{bmatrix}1 & 0 & 2\\ 0 & 3 & -45\\ 0 & -5 & 60\end{bmatrix}\begin{bmatrix}1 & t & 0\\ 0 & 1 & 0\\ 0 & 0 & e^{-15t}\end{bmatrix}\begin{bmatrix}1 & \frac{2}{9} & \frac{2}{15}\\ 0 & -\frac{4}{3} & -1\\ 0 & \frac{-1}{9} & \frac{-1}{15}\end{bmatrix}$$

$$e^{At}=\begin{bmatrix}1 & \frac{2}{9}-\frac{4}{3}t-\frac{2}{9}e^{-15t} & \frac{2}{15}-t-\frac{2}{15}e^{-15t}\\ 0 & -4+5e^{-15t} & -3+3e^{-15t}\\ 0 & \frac{20}{3}-\frac{20}{3}e^{-15t} & 5-4e^{-15t}\end{bmatrix}$$