LINEAR SYSTEMS (MECH 6300 / EECS 6331 / SYSM 6307)
Design Application #3

Coupled Carts

Consider the problem of a pair of masses connected by a spring as shown below. Each "cart" experiences an external force F_i in the "positive z_i direction" as indicated. Due to the lack of any rotational dynamics (as opposed to the pendulum on the cart problem), this system is very simple to model. The Lagrangian is given by the total kinetic energy, or

$$L = \frac{1}{2}\left(M_1 \dot{z}_1{}^2 + M_2 \dot{z}_2{}^2\right) \quad ,$$

where the generalized coordinates can be chosen as the positions z_1 and z_2. Note that the total external force experienced by each cart is the *sum* of the force F_i and the force exerted by the spring, F_{si}. But clearly

$$F_{s1} = -F_{s2} \quad ,$$

so that we simplify by assigning $F_s = F_{s1}$.

Applying Lagrange's equations to this problem, where

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_1}\right) - \frac{\partial L}{\partial z_1} &= F_1 + F_s \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_2}\right) - \frac{\partial L}{\partial z_2} &= F_2 - F_s \quad , \end{aligned}$$

and, from Hooke's law (for spring constant K)

$$F_s = K(z_2 - z_1) \quad ,$$

we have the dynamics described by the second order linear differential equations

$$M_1 \ddot{z}_1 + K(z_1 - z_2) = F_1 \quad ,$$

$$M_2 \ddot{z}_2 - K(z_1 - z_2) = F_2 \quad ,$$

where the forces F_1 and F_2 are determined by the torque generated at the wheel of radius r according to $F_i = \tau_i / r$.