The University of Texas at Dallas
Linear Systems, First Examination: September 23, 2020

Covenants

- You must work alone and have no communication during the exam allotted time window with any other person, website or source other than course materials, as explained in the announcement and information provided for this exam.
- The use of Matlab is not allowed during the exam period.
- Present your work neatly and place a box around your final answer if appropriate.
- Read each problem carefully. Answer the question completely, showing all your work. Points will likely be deducted if you simply state an answer and do not show your work.

Problem 1:

On my honor as a student at UT Dallas, I adhered to all University policies regarding academic honesty in the completion of this exam. I pledge that I worked alone and had no communication during the exam allotted time window with any other person. I pledge that I used only the course materials allowed for use in this exam; that is, I did not use Matlab, any website, or any other source.

If the following statement applies, write it on your exam paper and sign below it. If you choose not to write this and sign it, you must state your reason why.

I am in agreement with the pledge stated in Problem 1 on my exam.
(signature follows on your paper)

Also note that you have the responsibility of reporting misconduct in violation of this rule by anyone in the class, should you become aware of it. Such misconduct if unreported and undetected skews the overall grades of the class.

Problem 2:

For $\alpha, \beta \in \mathbb{R}$, consider the following state variable representation for a LTI SISO system:

$$\dot{x}_1(t) = x_1(t) - x_1^2(\text{t}) + \alpha u^2(t)$$
$$\dot{x}_2(t) = x_1(t) + x_2^3(t)$$

where the $x_i(t)$ are state variables comprising the state $x(t)$ (a two-vector) and $u(t)$ is the system input. In the following, consider the point defined by $x_e = [-1 \quad \beta]^T$ and $u_e = 1$.

(a) For part **(a)** only, suppose $\alpha = 2$. Find a value of β such that (x_e, u_e) is an equilibrium point for this system.

(b) Now for α and β (nonzero but otherwise arbitrary), specify A and B in $\dot{x} = Ax + Bu$ for the above system linearized about the given point (x_e, u_e). Your answer may be in terms of α and β.

Problem 3:

For $\alpha, \beta \in \mathbb{R}$, consider the following sets of vectors:

$$w_1 = \begin{bmatrix} 1 \\ \alpha \\ 3 \end{bmatrix} \quad w_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad w_3 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \qquad\qquad v_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ \beta \end{bmatrix}$$

(a) Find all values of α for which the vectors w_1, w_2, w_3 are linearly dependent.

(b) Find all values of β for which the vectors v_1, v_2, v_3 are linearly independent.

(c) Form the matrix V whose columns are v_1, v_2, and v_3. Compute the adjoint of the matrix $(\lambda I - V)$ for the independent variable λ and the identity matrix I of appropriate dimension. Your answer may be in terms of β.

(d) Suppose $\beta = 1$. Use your answer in part **(c)** to find the minimal polynomial $\psi(\lambda)$ of V.

(e) Now suppose $\beta = 2$. Use your answer in part **(c)** to find the minimal polynomial $\psi(\lambda)$ of V.

Problem 4:

Consider the linear, time-invariant system with the following state variable description:

$$\dot{x}_1 = -3x_1 - 4x_2$$
$$\dot{x}_2 = \alpha x_1 + x_2$$

(a) For $\alpha, \beta \in \mathbb{R}$, compute the system eigenvalues in the general form $(-1 \pm \sqrt{\beta})$. You must separately specify β (your answer for β may be in terms of α).

(b) Suppose $\alpha = 1$. Find eigenvectors and generalized eigenvectors of the system matrix A. You must show all your work.

(c) Again, suppose $\alpha = 1$. You are told that the matrix $M = \begin{bmatrix} 2 & 0 \\ -1 & -\frac{1}{2} \end{bmatrix}$ transforms A into its Jordan form $\hat{A}$ by way of

$$\hat{A} = M^{-1}AM.$$

Use this information to find $e^{\hat{A}t}$.

(d) Using the value $\alpha = 1$ and your result in parts **(b)** and **(c)**, compute e^{At}.

<u>Problem 5</u>:

For $\alpha, \beta \in \mathbb{R}$, consider the following LTI system:

$$\dot{x}_1 = x_1 + \alpha x_2$$
$$\dot{x}_2 = -2x_1 + \beta x_2$$

The A matrix of this system has Jordan form given by $\hat{A} = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}$. Use this information to compute α and β.

<u>Problem 6</u>:

In the following, for each sub-problem 6.1 through 6.5, <u>specify every statement that is always true</u> (there may be more than one correct answer for each sub-problem). Your individual responses for each sub-problem should give the sub-problem number indicating which statements (a), (b), (c), (d), (e), (f) are true. There is no need to show any calculations for this problem.

For the following sub-problems, $\{A, B, C, D\}$ is a state variable representation of a system S that has n states, p inputs, q outputs, and impulse response matrix $G(t, \tau)$.

<u>Sub-Problem 6.1</u>

(a) The dimension of A is $n \times n$ and the dimension of B is $p \times n$.
(b) The rank of A is n, but D may have any rank.
(c) The dimension of D depends on p and q but not n.
(d) The statements (a), (b), (c) are all false.
(e) The statements (a), (b), (c) are all true.
(f) Two of the statements (a), (b), (c) are true.

<u>Sub-Problem 6.2</u>

(a) Suppose the elements of A are continuous functions of time; then the transfer function of S is the Laplace transform of $G(t, \tau)$.
(b) If S is LTI, all n initial conditions are required to compute $G(s)$.
(c) An initially relaxed system is causal if and only if $G(t, \tau) = 0$ for all $t > \tau$.
(d) The statements (a), (b), (c) are all false.
(e) The statements (a), (b), (c) are all true.
(f) Two of the statements (a), (b), (c) are true.

<u>Sub-Problem 6.3</u>

(a) If the eigenvalues of A are distinct (non-repeating), then the minimal polynomial of A is of degree less than n.
(b) Suppose A is similar to B and the characteristic polynomial of B is $\Delta_B(\gamma)$; then $\Delta_B(A) = 0$.
(c) The matrix A can always be transformed into a diagonal matrix.
(d) The statements (a), (b), (c) are all false.
(e) The statements (a), (b), (c) are all true.
(f) Two of the statements (a), (b), (c) are true.

<u>Sub-Problem 6.4</u>

(a) For a given matrix, the roots of the minimal polynomial are roots of the characteristic polynomial.
(b) If a matrix is singular, than its inverse is zero.
(c) The nullity of a matrix is always equal to the dimension of the null space of that matrix.
(d) The statements (a), (b), (c) are all false.
(e) The statements (a), (b), (c) are all true.
(f) Two of the statements (a), (b), (c) are true.

<u>Sub-Problem 6.5</u>

(a) For a square matrix, the Caley-Hamilton Theorem says that a matrix satisfies its own characteristic polynomial.
(b) For a square matrix M, the determinant of M is equal to the determinant of $-M$.
(c) For a square matrix M, the determinant of $-M$ is equal to the negative of the determinant of M.
(d) The statements (a), (b), (c) are all false.
(e) The statements (a), (b), (c) are all true.
(f) Two of the statements (a), (b), (c) are true.