The University of Texas at Dallas
Linear Systems, Second Examination: October 21, 2020

Covenants

- You must work alone and have no communication during the exam allotted time window with any other person, website or source other than course materials, as explained in the announcement and information provided for this exam.
- The use of Matlab is not allowed during the exam period.
- Present your work neatly and place a box around your final answer if appropriate.
- Read each problem carefully. Answer the question completely, showing all your work. Points will likely be deducted if you simply state an answer and do not show your work.

Problem 1:

> *On my honor as a student at UT Dallas, I adhered to all University policies regarding academic honesty in the completion of this exam. I pledge that I worked alone and had no communication during the exam allotted time window with any other person. I pledge that I used only the course materials allowed for use in this exam; that is, I did not use Matlab, any website, or any other source.*

If the following statement applies, write it on your exam paper and sign below it. If you choose not to write this and sign it, you must state your reason why.

I am in agreement with the pledge stated in Problem 1 on my exam.
(signature follows on your paper)

Also note that you have the responsibility of reporting misconduct in violation of this rule by anyone in the class, should you become aware of it. Such misconduct if unreported and undetected skews the overall grades of the class.

Problem 2:

Consider a LTI system S with $\{A, B, C, D\}$ realization, where $A = \begin{bmatrix} 0 & 1 \\ -\alpha & -5 \end{bmatrix}$, $x(t) = [x_1(t), x_2(t)]^T$ is the system state, and where $\alpha \in \mathbb{R}$ is a positive scalar.

(a) Using the following Lyapunov function candidate, what can you say about stability in the sense of Lyapunov and asymptotic stability for the system S? Your answer might be in terms of α.

$$V_1 = \alpha x_1^2 + x_2^2$$

(b) Now suppose $\alpha = 6$. Using the following Lyapunov function candidate, what can you say about stability in the sense of Lyapunov and asymptotic stability for the system S?

$$V_2 = x_1^2 + x_2^2 - x_1 x_2$$

Problem 3:

Consider the LTI system $\{A, B, C\}$ with state $x \in \mathbb{R}^3$, input $u \in \mathbb{R}^2$, and output $y \in \mathbb{R}$:

$$\dot{x} = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} x \quad .$$

There exists a decoupling similarity transformation P (where $\bar{x} = Px$) that transforms the above system into an equivalent system $\{\bar{A}, \bar{B}, \bar{C}\}$ in the state variable $\bar{x}$, where $\bar{A}$ is diagonal. Showing all your work, calculate the adjoint of $(\lambda I - A)$ (where I is the identity matrix and λ is a scalar variable) and use it to find $\{\bar{A}, \bar{B}, \bar{C}\}$. **Note:** You are <u>not</u> required to calculate the inverse of a matrix in this problem; thus, you may state your answer for one of the matrices $\{\bar{A}, \bar{B}, \bar{C}\}$ in terms of P or its inverse.

Problem 4:

In the following LTI system S, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ are positive scalars and $x \in \mathbb{R}^2$ is the system state:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\alpha x_1 - \beta x_2$$

In the following you must use the familiar Lyapunov matrix equation $A^T M + MA = -N$.

(a) Considering the case where $M = I$ (where I is the identity matrix), find N. What can you conclude about asymptotic stability of the system S? Your answer can be in terms of the arbitrary α and , but only if necessary.

(b) Now suppose you are told that $N = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ results in $M = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$. Find α and β. What can you conclude about asymptotic stability of the system S?

Problem 5:

Consider the following second-order, SISO linear system with no input, described in terms of its output $y(t)$ for $t > 0$:

$$\ddot{y} + \frac{4}{t}\dot{y} + \frac{2}{t^2}y = 0$$

You are told that $y(t) = t^{-1}$ and $y(t) = -t^{-2}$ are independent solutions of this differential equation.

(a) Using state variables defined by $x_1 = \dot{y}$ and $x_2 = y$, find the usual standard state variable description (state and output equations) for this system.

(b) Using your result in part **(a)** and the given information, find the fundamental matrix.

(c) Find the state transition matrix.

Problem 6:

In the following, for each sub-problem 6.1 through 6.5, specify every statement that is always true (there may be more than one correct answer for each sub-problem). Your individual responses for each sub-problem should give the sub-problem number indicating which statements (a), (b), (c), (d), (e), (f) are true. There is no need to show any calculations for this problem, and there is no need to indicate which statements are false.

For the following sub-problems, $\{A, B, C, D\}$ is a realization of a system S (assumed to be LTI, unless noted otherwise) that has n states, p inputs, q outputs, and transfer function $G(s)$ (it is assumed that $G(s)$ has coprime numerator and denominator).

Sub-Problem 6.1

(a) The fundamental matrix associated with A is not unique.
(b) The zero-state response of S must be computed using $x(t) = 0$ for $t > 0$.
(c) Suppose S is time-varying. The state transition matrix associated with A is not unique.
(d) The statements (a), (b), (c) are all false.
(e) The statements (a), (b), (c) are all true.
(f) Two of the statements (a), (b), (c) are true.

Sub-Problem 6.2

(a) If S has a single pole at $s = 0$ and all other poles are in the open left-half s-plane (not on the imaginary axis), then S is not BIBO stable.
(b) If $V(x)$ is a Lyapunov function candidate for S and $\dot{V} = 0$ only when $x = 0$, then S is asymptotically stable.
(c) If S is time-varying and at some time t the eigenvalues of A are in the open left-half s-plane (not on the imaginary axis), then S is asymptotically stable.
(d) The statements (a), (b), (c) are all false.
(e) The statements (a), (b), (c) are all true.
(f) Two of the statements (a), (b), (c) are true.

Sub-Problem 6.3

(a) For S to be BIBO stable, the n eigenvalues of A must be in the open left-half-plane.
(b) If some of the eigenvalues of A have zero real parts but those that do are simple roots of the minimal polynomial of A, then S is stable i.s.L.
(c) S may be BIBO stable if an eigenvalue of A is in the right-half-plane.
(d) The statements (a), (b), (c) are all false.
(e) The statements (a), (b), (c) are all true.
(f) Two of the statements (a), (b), (c) are true.

Sub-Problem 6.4

(a) If A is similar to a diagonal matrix, then it must have distinct eigenvalues.
(b) If A is a diagonal matrix, then its minimal polynomial must be of order n.
(c) Suppose $u \equiv 0$ and C has all nonzero elements. If the output is zero for $t > 0$, then $x(t) = 0$ for $t > 0$.

(d) The statements (a), (b), (c) are all false.
(e) The statements (a), (b), (c) are all true.
(f) Two of the statements (a), (b), (c) are true.

Sub-Problem 6.5

(a) $G(s)$ is said to be strictly proper if it has any finite value as $s \to \infty$.
(b) $G(s)$ must be rational, but not necessarily proper, to be realizable with $\{A, B, C, D\}$.
(c) In the SISO case, the eigenvalues of A are equal to the set of system poles.
(d) The statements (a), (b), (c) are all false.
(e) The statements (a), (b), (c) are all true.
(f) Two of the statements (a), (b), (c) are true.