MECH 6300 Exam 2 | Jonas Wagner | 2020-10-27

1)
I am in agreement with the pledge as stated in problem 4 on this exam.

[signature]

2) $A = \begin{bmatrix} 0 & 1 \\ -\alpha & -5 \end{bmatrix}$ $\quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

a) $\underbrace{V_1 = \alpha x_1^2 + x_2^2}$ $\Big] \to V_1 > 0$ (P.d.)

$|sI - A| = \begin{vmatrix} s & -1 \\ \alpha & s+5 \end{vmatrix} = s(s+5) + \alpha$

for checking later...

$= s^2 + 5s + \alpha$

$\lambda_{1,2} = \frac{-5}{2} \pm \sqrt{\frac{25}{4} - \alpha} \Big] \to \alpha > 0 \to \max \lambda_2$

$-\frac{5}{2} + \frac{5}{2} - \sqrt{\alpha} < 0$

$\therefore$ A is always stable

$V_1 = x^T \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} x \leftarrow P_1$

$-x^T Q_1 x = \dot{V}_1 = x^T (A^T P_1 + P_1 A) x = x^T \left(\begin{bmatrix} 0 & -\alpha \\ 1 & -5 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\alpha & -5 \end{bmatrix} \right) x$

$-Q_1 = \begin{bmatrix} 0 & -\alpha \\ \alpha & -5 \end{bmatrix} + \begin{bmatrix} 0 & \alpha \\ -\alpha & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

~~$Q_1 = \begin{bmatrix} 0 & -\alpha^2 - 1 \\ \alpha^2 + \alpha & 0 \end{bmatrix}$ $|sI - Q_1| = s^2 + (-\alpha - 1)(\alpha^2 + \alpha)$~~

~~$s^2 - \alpha^3 - \alpha^2 - \alpha^2 - 1$~~

~~$\lambda_{1,2} = \pm \alpha + 2\alpha^2 + 1$~~

Q) cont.

Messy From Before: Diagonal Form

$\lambda_{1,2}(Q) = 0$

Note: Not Asymptotically stable as $Q=0$

Since $Re\{\lambda_i\} \geq 0 \quad \forall \alpha, \forall x$

It can be said that S is Marginally Stable.

b) $P_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $\quad V_2 = x_1^2 + x_2^2 - x_1 x_2$

$V_2 = X^T P X$

$$-Q_2 = (A^T P_2 + P_2 A) = \begin{bmatrix} 0 & -\alpha \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -\alpha & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\alpha \\ 1 & -6 \end{bmatrix} + \begin{bmatrix} \alpha & 6 \\ -\alpha & -5 \end{bmatrix} = \begin{bmatrix} \alpha & -\alpha+6 \\ 1-\alpha & -11 \end{bmatrix}$$

$\underline{\alpha = 6}$

$Q_2 = \begin{bmatrix} -6 & 0 \\ +5 & +11 \end{bmatrix}$ $\xrightarrow{\text{Lower Triangular}}$ $\lambda_1 = -6$, $\lambda_2 = 11$

Since $Re\{\lambda_i\} \not> 0$,

Nothing can be concluded from this test.

3) $A = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix}$ $D = 0$

$$(\lambda I - A) = \begin{bmatrix} \lambda+2 & 2 & 0 \\ 0 & \lambda & -1 \\ 0 & 3 & \lambda+4 \end{bmatrix}$$

$$\text{cofactors}(\lambda I - A)^{+} = \begin{bmatrix} \lambda^2+4\lambda+3 & 0 & 0 \\ 2(\lambda+4) & (\lambda+2)(\lambda+4) & 3(\lambda+2) \\ -2 & -(\lambda+2) & \lambda(\lambda+2) \end{bmatrix}$$

$$(\lambda I - A)^{+} = \begin{bmatrix} \lambda^2+4\lambda+3 & -2(\lambda+4) & -2 \\ 0 & (\lambda+2)(\lambda+4) & \lambda+2 \\ 0 & -3(\lambda+2) & \lambda(\lambda+2) \end{bmatrix}$$

$$\boxed{d(\lambda) = 1}$$

$$|sI - A| = \begin{vmatrix} s+2 & 2 & 0 \\ 0 & s & -1 \\ 0 & 3 & s+4 \end{vmatrix} = (s+2)(s(s+4)+3) + (2)(0) + 0$$

$$= (s+2)(s^2+4s+3)$$

$$\frac{-4 \pm \sqrt{16-12}}{2} \quad \sqrt{4}$$

$$\frac{-2 \pm \sqrt{4}}{2}$$

$$-1 \pm 1$$

Problems w/ quadratic formula...
I feel Dumb...

3) cont.

$$P = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \end{bmatrix} \leftarrow \text{found from roots}$$

$$\bar{A} = P A P^{-1}$$

$$\bar{B} = P B$$

$$\bar{C} = C P^{-1}$$

$$\bar{D} = 0$$

4) $A = \begin{bmatrix} 0 & 1 \\ -\alpha & -\beta \end{bmatrix}$

Let $M = I$ $\qquad A^T M + M A = -N$

$$-N = \begin{bmatrix} 0 & -\alpha \\ 1 & -\beta \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -\alpha & -\beta \end{bmatrix} = \begin{bmatrix} 0 & 1-\alpha \\ 1-\alpha & \beta^2 \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & \alpha-1 \\ \alpha-1 & -\beta^2 \end{bmatrix} \qquad |sI - N| = \begin{vmatrix} s & 1-\alpha \\ 1-\alpha & s+\beta^2 \end{vmatrix} = s(s+\beta^2) - (1-\alpha)(1-\alpha)$$

$$= s^2 + \beta^2 s - (\alpha^2 - 2\alpha + 1)$$

when $\lambda_{1,2} \geq 0 \Leftarrow \left[\lambda_{1,2} = \frac{-\beta^2 \pm \sqrt{\beta^4 + \alpha^2 - 2\alpha + 1}}{2} \right.$

The system is Marginally stable

If $\alpha = 1$ and $\beta = 0$, the system is not asymptotically stable as N would be $= 0 \; \forall x$...

4) cont.

b)

$$N = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \qquad M = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A^T M + M A = -N$$

$$\begin{bmatrix} 0 & -\alpha \\ 1 & -\beta \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -\alpha & -\beta \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\begin{bmatrix} -\alpha & -\alpha \\ 5-\beta & 1-\beta \end{bmatrix} + \begin{bmatrix} -\alpha & 5-\beta \\ -\alpha & 1-\beta \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\begin{bmatrix} -2\alpha & 5-\beta-\alpha \\ 5-\beta-\alpha & 2-2\beta \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$$

$$5 = \alpha + \beta \longrightarrow \beta = 3$$

$$-2\alpha = -4 \longrightarrow \alpha = 2$$

$$\boxed{\begin{aligned} \alpha &= 2 \\ \beta &= 3 \end{aligned}}$$

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5) $\ddot{y} + \frac{4}{t}\dot{y} + \frac{2}{t^2}y = 0$

$\psi_1 = \frac{1}{t} \quad \psi_2 = \frac{-1}{t^2}$

Let $x_1 = \dot{y}$
$x_2 = y$ $\rightarrow$ $x = \begin{bmatrix} \dot{y} \\ y \end{bmatrix}$

a) $\dot{x} = Ax$

$$\begin{bmatrix} \ddot{y} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{-4}{t} & \frac{-2}{t^2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{y} \\ y \end{bmatrix} \rightarrow \boxed{A(t) = \begin{bmatrix} \frac{-4}{t} & \frac{-2}{t^2} \\ 1 & 0 \end{bmatrix}}$$

$B(t) = 0$
$C(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}$
$D(t) = 0$

b) $\psi(t) = \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & \frac{-1}{t^2} \end{bmatrix}$

$$\boxed{\psi^{-1}(t) = \frac{1}{\frac{-1}{t^3}} \begin{bmatrix} \frac{-1}{t^2} & 0 \\ 0 & \frac{1}{t} \end{bmatrix} = \begin{bmatrix} t & 0 \\ 0 & -t^2 \end{bmatrix}}$$

c)
$$\Phi(t) = \psi(t)\,\psi^{-1}(t_0) = \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & \frac{-1}{t^2} \end{bmatrix} \begin{bmatrix} t_0 & 0 \\ 0 & -t_0^2 \end{bmatrix}$$

$$\boxed{\Phi(t) = \begin{bmatrix} \frac{t_0}{t} & 0 \\ 0 & \frac{t_0^2}{t^2} \end{bmatrix}}$$

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6)

6.1) a, c, f

6.2) a

6.3) b, c, f

6.4) d

6.5) c

6.2) b... only if $Q \geq 0$ as well

6.5) c... assuming cancellations don't count... because doing so loses stability info