

$$\dot{x} = Ax + Bu$$

$$y = Cx, u \geq 0$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \quad x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad u(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(sI - A) = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 2 & 4 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s+1)(s^2+2s+2)} \begin{bmatrix} s^2+3s+4 & s+3 & 1 \\ -2 & s(s+3) & s \\ -2s & -2(2s+1) & s^2 \end{bmatrix}$$

$$G(s) = C(sI - A)^{-1}B + D$$

$$= \frac{1}{(s+1)(s^2+2s+2)} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{s^2+3s+3}{-(s+2)} & s+4 \\ -s(s+2) & s(s+4) \\ s^2-4s-2 \end{bmatrix} + 0$$

$$G(s) = \begin{bmatrix} \frac{s^2+s-2}{(s+1)(s^2+2s+2)} & \frac{8s+2}{(s+1)(s^2+2s+2)} \\ \frac{s^2+s-1}{(s+1)(s^2+2s+2)} & \frac{2s^2+9s+4}{(s+1)(s^2+2s+2)} \end{bmatrix}$$

1) cont.

A in
companion
form

$$e^{At} = \begin{bmatrix} 1 & e^t & 1 \\ 1 & 1 & e^t \\ e^{-2t} & e^{-4t} & e^{-3t} \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 1 \\ 1 \\ e^{-2t} \end{bmatrix}$$

$$C e^{At} = \begin{bmatrix} 1 - e^{-2t} \\ 3 \end{bmatrix}$$

$$Y(s) = G(s) U(s)$$

$$= \begin{bmatrix} \frac{s^2 + s - 2 + 8s + 2}{(s+1)(s^2 + 2s + 2)} \\ \frac{s^2 + s - 1 + 2s^2 + 9s + 4}{(s+1)(s^2 + 2s + 2)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s(s+9)}{(s+1)(s^2 + 2s + 2)} \\ \frac{3s^2 + 10s + 3}{(s+1)(s^2 + 2s + 2)} \end{bmatrix} = \begin{bmatrix} \frac{9s+16}{s^2 + 2s + 2} - \frac{8}{s+1} \\ \frac{7s+11}{s^2 + 2s + 2} - \frac{4}{s+1} \end{bmatrix}$$

$$\begin{bmatrix} 9s+16 \\ 7s+11 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 1 - e^{-2t} \\ 3 \end{bmatrix} + \begin{bmatrix} e^{-2t} + e^{-4t} - 1 \\ -e^t - 2 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} e^{-4t} \\ 1 - e^t \end{bmatrix}$$

$$2) \quad A = \begin{bmatrix} 1 & 0 & -1 \\ -16 & -2 & 7 \\ 0 & -1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

$$Ab = \begin{bmatrix} 10 \\ -9 \\ -2 \end{bmatrix} \quad A^2 b = \begin{bmatrix} 2 \\ 4 \\ 13 \end{bmatrix} \quad A^3 b = \begin{bmatrix} -11 \\ 51 \\ -30 \end{bmatrix}$$

$$\det(sI - A) = \begin{vmatrix} s-1 & 0 & 1 \\ 16 & s+2 & -7 \\ 0 & 1 & s+2 \end{vmatrix} = s^3 + 3s^2 + 7s + 5$$

$$= (s+1)(s+(1-j2))(s+(1+j2))$$

$$\bar{A} = \begin{bmatrix} 0 & 0 & -5 \\ 1 & 0 & -7 \\ 0 & 1 & -3 \end{bmatrix}$$

$$A^3 b = -5b - 7Ab - 3A^2 b$$

$$\begin{bmatrix} -11 \\ 51 \\ -30 \end{bmatrix} = \begin{bmatrix} -5 + 0 - 6 \\ 0 + 63 - 12 \\ 5 + 6 - 39 \end{bmatrix}$$

$$3) \quad \lambda_1 = -1$$

$$\lambda_{2,3} = -1 \pm j2$$

$$\hat{A} = P \Lambda P^{-1}$$

$$\hat{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & -1 \end{bmatrix}$$

$$\hat{B} = P B$$

$$\hat{B} = \begin{bmatrix} -1.75 \\ 0.175 \\ -0.225 \end{bmatrix}$$

$$\hat{C} = C P^{-1}$$

$$\hat{C} = \begin{bmatrix} -3 & -23 & 1 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \quad q_{2,3} = \alpha \begin{bmatrix} -3 \pm j1 \\ 20 \\ -4 \pm j8 \end{bmatrix}$$

$$P^{-1} = [q_1, \operatorname{Re}(q_2), \operatorname{Im}(q_2)]$$

$$P^{-1} = \begin{bmatrix} -1 & -3 & 1 \\ 2 & 20 & 0 \\ -2 & -4 & 8 \end{bmatrix}$$

$$P = \frac{1}{-80} \begin{bmatrix} 160 & 20 & -20 \\ -16 & -6 & 2 \\ 32 & 2 & -14 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 & -0.25 & 0.25 \\ 0.2 & 0.675 & -0.025 \\ -0.4 & -0.025 & 0.175 \end{bmatrix}$$

4)

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\chi(s) = (s-1)^4$$

$$(sI - A) = \begin{bmatrix} s-1 & 0 & -1 & -1 \\ 0 & s-1 & 0 & 0 \\ 0 & 0 & s-1 & 1 \\ 0 & 0 & 0 & s-1 \end{bmatrix}$$

$$\text{adj}(sI - A) = \begin{bmatrix} (s-1)^3 & 0 & (s-1)^3 - (s-1)(s-2) & 0 \\ 0 & (s-1)^3 & 0 & 0 \\ 0 & 0 & (s-1)^3 - (s-1)^2 & 0 \\ 0 & 0 & 0 & (s-1)^3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s-1)^4} \text{adj}(sI - A)$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s-1} & 0 & \frac{1}{(s-1)^2} \frac{(s-2)}{(s-1)^3} & 0 \\ 0 & \frac{1}{s-1} & 0 & 0 \\ 0 & 0 & \frac{1}{s-1} & \frac{-1}{(s-1)^2} \\ 0 & 0 & 0 & \frac{1}{s-1} \end{bmatrix}$$

4) cont.

$$\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^{-t}$$

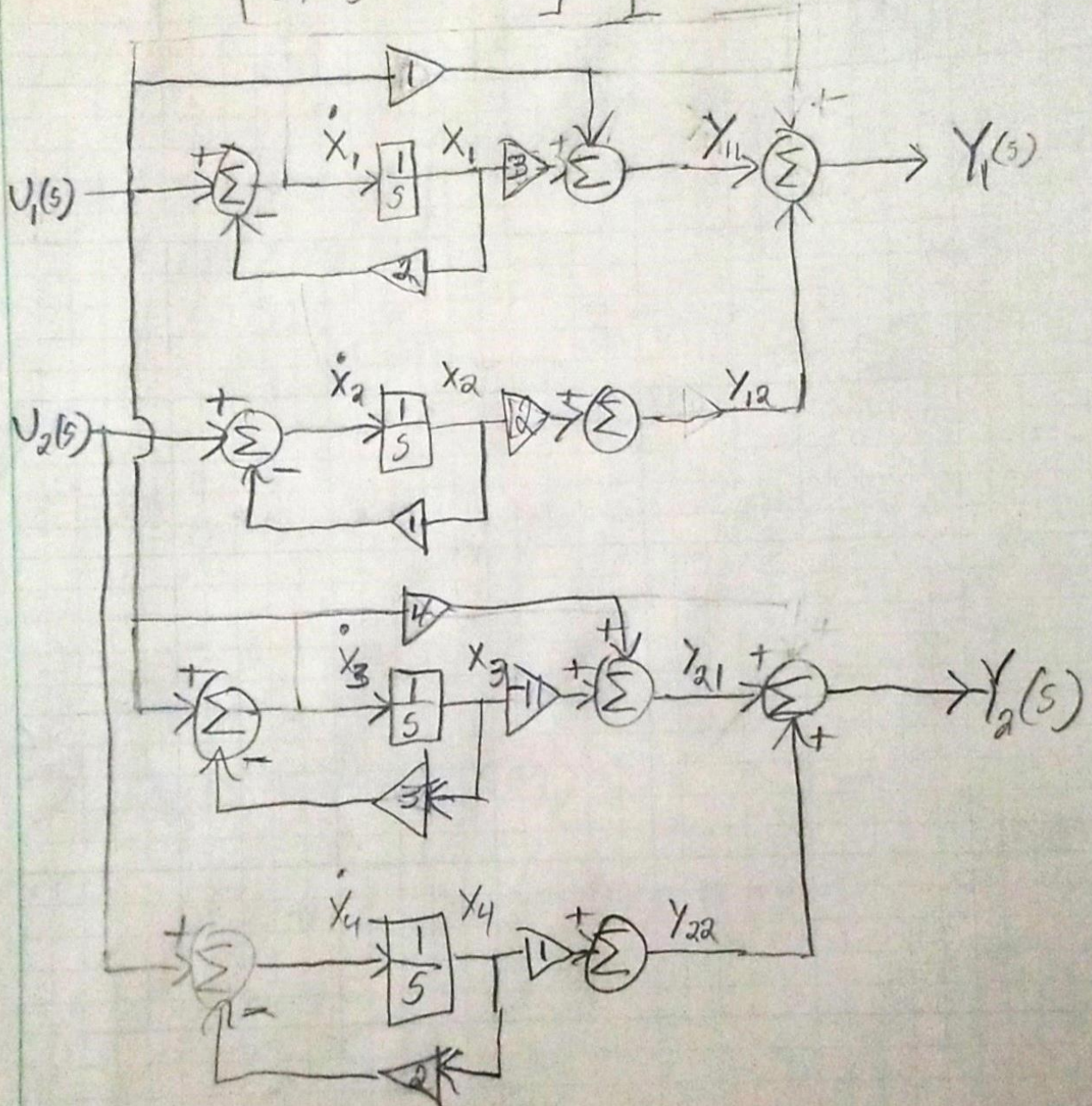
$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = te^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{s-2}{(s-1)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3} - \frac{1}{(s-1)^2}\right\} = \frac{t^2}{2}e^{-t} - te^{-t}$$

$$e^{At} = \begin{bmatrix} e^{-t} & 0 & te^{-t} & \frac{t^2}{2}e^{-t} - te^{-t} \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{-t} & -te^{-t} \\ 0 & 0 & 0 & e^{-t} \end{bmatrix}$$

5)

$$\hat{G}(s) = \begin{bmatrix} \frac{s+5}{s+2} & \frac{2}{s+1} \\ \frac{4s+1}{s+3} & \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} 1 + \frac{3}{s+2} & \frac{2}{s+1} \\ 4 - \frac{11}{s+3} & \frac{1}{s+2} \end{bmatrix}$$



5) cont. $\dot{X} = AX + Bu$

$$Y = CX + Du$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}$$

b) $A_1(s) = (s-2)(s+1)(s-1)$

a) $A_2(s) = (s-2)(s+1)^2$

They are not equivalent!

b) $G_1(s) = \frac{2}{(s-2)(s+1)}$

$$G_2(s) = \frac{6}{(s-2)(s+1)}$$

They are not zero-state equivalent!

$$7) \quad A(t) = \begin{bmatrix} 0 & 1 \\ 0 & + \end{bmatrix} \quad X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\dot{X}_1 = X_2 \rightarrow X_1 = X_1(0) + \int X_2(t) dt$$

$$\dot{X}_2 = +X_2 \rightarrow X_2(t) = X_2(0) + \frac{+^2}{2} X_2(0)$$

$$\frac{dX_1}{dt} = X_2(0) + \frac{+^2}{2} X_2(0) \quad X_2(0) = \frac{+^2 + 2}{2} X_2(0)$$

$$\int dX_1 = \int_0^+ \frac{+^2}{2} X_2(0) dt + X_1(0)$$

$$X_1(t) = \frac{+^3}{6} X_2(0) + X_1(0)$$

$$\Psi(t) = \begin{bmatrix} 1 & \frac{+^3}{6} \\ 0 & \frac{+^2+2}{2} \end{bmatrix}$$

$$\Psi^{-1}(t) = \frac{2}{+^2+2} \begin{bmatrix} \frac{+^2+2}{2} & -\frac{+^3}{6} \\ 0 & 1 \end{bmatrix}$$

$$\Phi(t, \tau) = \Psi(t) \Psi^{-1}(\tau)$$

$$= \begin{bmatrix} 1 & \frac{+^3}{6} \\ 0 & \frac{+^2+2}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{-\tau^3}{3(\tau^2+2)} \\ 0 & \frac{2}{\tau^2+2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{-+^3}{3(+^2+2)} \\ 0 & \frac{2}{+^2+2} \end{bmatrix}$$

$$\Phi(t, \tau) = \begin{bmatrix} 1 & \frac{+^3 - \tau^3}{3(+^2+2)} \\ 0 & \frac{+^2+2}{\tau^2+2} \end{bmatrix}$$

$$8) \quad A(t) = \begin{bmatrix} -2 & e^{-t} \\ 0 & -1 \end{bmatrix} \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\dot{x}_1 = -2x_1 + e^{-t}x_2$$

$$\dot{x}_2 = -x_2$$

$$\rightarrow x_2(t) = x_2(0)e^{-t}$$

$$\dot{x}_1 = -2x_1 + e^{-t}(x_2(0)e^{-t}) \quad \Psi(t) = \begin{bmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-t} \end{bmatrix}$$

$$\dot{x}_1 + 2x_1 = x_2(0)e^{-2t}$$

$$\dot{x}_1 e^{2t} + 2x_1 e^{2t} = x_2(0)e^{-2t} e^{2t} = x_2(0)$$

$$\frac{d}{dt}(x_1 e^{2t}) = x_2(0)$$

$$\int x_1 e^{2t} = \int_0^t x_2(0) d\tau$$

$$x_1 e^{2t} = t x_2(0) + x_1(0)$$

$$x_1(t) = x_1(0)e^{-2t} + x_2(0)t e^{-2t}$$

$$\Psi^{-1}(t) = \frac{1}{e^{-3t}} \begin{bmatrix} e^t - te^{-2t} & -te^{-2t} \\ 0 & e^{2t} \end{bmatrix}$$

$$\Psi^{-1}(t) = \begin{bmatrix} e^{2t} - te^{-t} & -te^{-t} \\ 0 & e^t \end{bmatrix}$$

$$\Phi(t, \tau) = \Psi(t) \Psi^{-1}(\tau)$$

$$\Phi(t, \tau) = \begin{bmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} e^{2\tau} - \tau e^{-\tau} & -\tau e^{-\tau} \\ 0 & e^{\tau} \end{bmatrix} = \begin{bmatrix} e^{-2(t-\tau)} & -te^{-2t} + \tau e^{-2\tau} \\ 0 & e^{-(t-\tau)} \end{bmatrix}$$

$$\Phi(t, \tau) = \begin{bmatrix} e^{-2(t-\tau)} (1 + \tau e^{-(2t-\tau)} - e^{-(2t+\tau)}) & \\ 0 & e^{-(t-\tau)} \end{bmatrix}$$

MECH 6300 - Homework 4

```
close all

% Problem 1

A = [0  1  0
      0  0  1
     -2 -4 -3];
B = [1  0
      0  1
     -1 1];
C = [0  1  -1
      1  2   0];
D = 0;

sys = ss(A,B,C,D)

% Matrix Inversions
syms s
sI_A = (s * eye(3) - A);
poles = factor(det(sI_A),s);
sI_A_inv = sI_A^-1

% Problem 2 and 3
A = [1  0  -1
     -16 -2  7
      0  -1 -2];
B = [1;0;1];
C = [1  -1  0];
D = 0;

sys = ss(A,B,C,D);
[csys,T] = canon(sys)

P_inv = [-1 -3  1
          2  20  0
          -2 -4  8];
P = inv(P_inv);

A_hat = P*A*P_inv;
B_hat = P*B;
C_hat = C*P_inv;
D_hat = D;

sys2 = ss(A_hat,B_hat,C_hat,D_hat)

figure()
step(sys)
hold on
step(csys)
```

```

step(sys2)
legend()
hold off

%Problem 4 Verification
A = [1  0  1  1
      0  1  0  0
      0  0  1 -1
      0  0  0  1];

s*eye(4)-A;
adjoint(s*eye(4)-A);
inv(s*eye(4)-A)
e_At = ilaplace(inv(s*eye(4)-A))
exp(A*t)

%Problem 6
A1 = [2 -1  2
       0 -1 -1
       0  0  1];
B1 = [1; 1; 0];
C1 = [1 -1 0];
D1 = 0;
sys1 = ss(A1,B1,C1,D1);
tf1 = tf(sys1)

A2 = [2 3  2
       0 -1 1
       0 0 -1];
B2 = [1; 1; 0];
C2 = [1 -1 0];
D2 = 0;
sys2 = ss(A2,B2,C2,D2);
tf2 = tf(sys2)

```

sys =

A =

	<i>x1</i>	<i>x2</i>	<i>x3</i>
<i>x1</i>	0	1	0
<i>x2</i>	0	0	1
<i>x3</i>	-2	-4	-3

B =

	<i>u1</i>	<i>u2</i>
<i>x1</i>	1	0
<i>x2</i>	0	1
<i>x3</i>	-1	1

C =

	x1	x2	x3
y1	0	1	-1
y2	1	2	0

D =

	u1	u2
y1	0	0
y2	0	0

Continuous-time state-space model.

sI_A_inv =

```
[ (s^2 + 3*s + 4)/(s^3 + 3*s^2 + 4*s + 2),      (s + 3)/(s^3 + 3*s^2
+ 4*s + 2),      1/(s^3 + 3*s^2 + 4*s + 2)]
[      -2/(s^3 + 3*s^2 + 4*s + 2),      (s*(s + 3))/(s^3 + 3*s^2
+ 4*s + 2),      s/(s^3 + 3*s^2 + 4*s + 2)]
[      -(2*s)/(s^3 + 3*s^2 + 4*s + 2),      -(2*(2*s + 1))/(s^3 + 3*s^2
+ 4*s + 2),      s^2/(s^3 + 3*s^2 + 4*s + 2)]
```

csys =

A =

	x1	x2	x3
x1	-1	0	0
x2	0	-1	2
x3	0	-2	-1

B =

	u1
x1	5.026
x2	0.3246
x3	1.484

C =

	x1	x2	x3
y1	1.044	2.617	-3.436

D =

	u1
y1	0

Continuous-time state-space model.

T =

5.7446	0.7181	-0.7181
0.9561	-0.1534	-0.6315
2.1836	0.3925	-0.6994

sys2 =

A =

	x1	x2	x3
x1	-1	0	0
x2	0	-1	2
x3	0	-2	-1

B =

	u1
x1	-1.75
x2	0.175
x3	-0.225

C =

	x1	x2	x3
y1	-3	-23	1

D =

	u1
y1	0

Continuous-time state-space model.

ans =

[1/(s - 1),	0,	1/(s - 1)^2,	(s - 2)/(s - 1)^3]
[0,	1/(s - 1),	0,
[0,	0,	1/(s - 1),
[0,	0,	0,

e_At =

[exp(t),	0,	t*exp(t),	t*exp(t) - (t^2*exp(t))/2]
[0,	exp(t),	0,
[0,	0,	exp(t),
[0,	0,	0,

ans =

[exp(t),	1,	exp(t),	exp(t)]
[1,	exp(t),	1,
[1,	1,	exp(t),
[1,	1,	1,

tf1 =

2

s^2 - s - 2

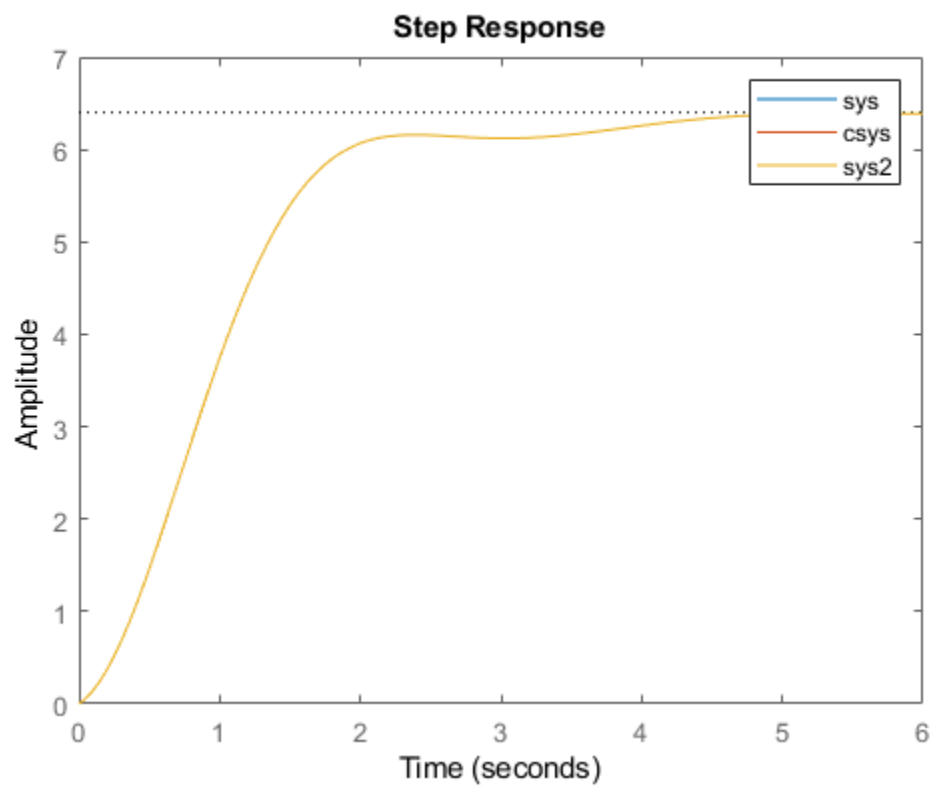
Continuous-time transfer function.

tf2 =

$$\frac{6}{s^2 - s - 2}$$

Continuous-time transfer function.

2/3



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