Important Background

Imaginary Numbers

Euler's Identity

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Cos and Sin Identities

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{j2}$$

Partial Fraction Expansion

$H(s)$ must be strictly proper.
For Z-Inverse calculations: Divide by z to form $H(z) = \frac{Y(z)}{z}$ to ensure a simpler inverse calculation.

$$H(s) = \frac{f(s)}{(s-r)^L(s-\lambda_1)(s-\lambda_2)...(s-\lambda_{N-L})}$$
$$H(s) = \frac{C_1}{s-\lambda_1} + \frac{C_2}{s-\lambda_2} + ... + \frac{C_{N-L}}{s-\lambda_{N-L}} + \frac{K_{L-1}}{s-r} + \frac{K_{L-2}}{(s-r)^2} + ... + \frac{K_0}{(s-r)^L}$$

$$C_i = (s-\lambda_i)H(s)\Big|_{s=\lambda_i}$$
$$K_i = \frac{1}{i!}\frac{\mathrm{d}^i}{\mathrm{d}s^i}\big\{(s-r)^L H(s)\big\}\Big|_{s=r}$$

General Matrix Algebra Notes

* More info could be added to further explain even more background on Linear Algebra... such as (square) matrix properties... matrix operations/properties

Definitions:

1. Non-singular (Inverteble): A matrix is nonsingular if an inverse for the matrix exists.
   - Calculator Test: Take the inverse and ensure it exists.
   - An inverse exists if $\det(A) \neq 0$
2. Rank: The rank of the matrix is the number of linearly independent rows in the matrix.
   - Calculator Test: Find the Row Echelon form and the number of non-zero rows is the rank.
   - A matrix has full rank iff its determinant is nonzero.

Eigen Values and Vectors:
Eigen Values λ and Eigen Vectors x for the matrix A satisfy the following:

$$Ax = \lambda x$$

These can be found by solving for each in:

$$(A - \lambda I)x = 0$$

Characteristic Polynomial:
The characteristic polynomial is defined as:

$$\Delta(\lambda) = \det(A - \lambda I)$$

Linear Algebra Background

*Explain state-space equations and defintions... Linearization... Eigenvalue problem... ... Functions of a square matrix

Lyapnov Equation

At the moment info isn't here... check book page 85

Linear Systems Fundementals

System concepts/properties... LTI and LTV systems... transfer function...
jordan form/minimal polynomial

Simularity Transforms

Simularity Transforms(Model Decomp... Diagonilization... Equivelent Systems (conical forms)... Kalman Decomposition)

Model Decomposition (Jordan Form)

This is a simularity transform that splits everything into modal states... is the jordan form:
Procedure:

1. Find the eigen values and vectors of A:

$$
\begin{aligned}
A\zeta_i = \lambda_i \zeta_i \{\lambda_1, \cdots, \lambda_n\} \\
\{\zeta_1, \cdots, \zeta_n\}
\end{aligned}
$$

   If there are non-distinct eigenvectors the generalized eigenvalues need to be constructed and included.

2. Construct the simulatrity transform matrix M:

$$M = \begin{bmatrix} \zeta_1 & \zeta_2 & \cdots & \zeta_n \end{bmatrix}$$

3. The equivelent system is given as:

$$
\begin{aligned}
x &= Mq \\
\dot{q} &= M^{-1}AMq + M^{-1}Bu \\
y &= CMq + Du
\end{aligned}
$$

Controllable Conical Form

See notes I guess at this point...

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} \beta_n & \beta_{n-1} & \cdots & \beta_1 \end{bmatrix} \qquad D = 0$$

similar to this but this is $n = 3$ case:

$$\bar{\mathcal{C}}^{-1} = \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 & 1 \\ \alpha_2 & \alpha_1 & 1 & 0 \\ \alpha_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

System Realization... Info besides just draw it?

Solutions of State Equations

Need to define for both LTI and LTV systems...
LTV (general): State Transition Matrix and solutions to LTV systems

Stability

Defintions... (steal from DTC?)... BIBO... Marginal... Asymptotic...
LTV(general) notes... Lyapnov Direct Method
(Including Routh... Jury's Table/Schur-Cohn criterion?)

Controllability and Observability

Consider the following state-space system:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$
$$y = Cx + Du$$

where the vectors are of dimensions: $x_{x\times1},\ u_{p\times1},\ y_{q\times1}$
and the state matrices are of dimension: $A_{n\times n},\ B_{n\times p},\ C_{q\times n},\ D_{q\times p}$.

Definitons and Terms:

1. Controllability: A system is controllable if there exists a control sequence $u \in \mathbb{R}^m$ which steers the state x from x_0 to the origin in finite time.
2. Stabalizability: A system is stabalizable if the uncontrollable subsystem is asymptotically stable.

$$\text{Controllability} \implies \text{Stabalizability}$$

3. Observability: A system is observable if the initial condition x_0 can be determined from the knowledge of u and observation of the output y over a finite time interval.
4. Detectablility: A system is detectable if the unobservable subsystem is asymptotically stable.

$$\text{Observability} \implies \text{Detectability}$$

Contrllability and Obserabality Matrices:

Controllability Matrix: The controllability matrix is defined as:

$$\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}_{n\times np}$$

Observability Matrix: The observability matrix, $\mathcal{O}$, is defined as:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{np\times n}$$

Tests:

Controllability/Stabalizability: A system is controllable iff the controllability matrix is full rank:

$$\text{rank}(\mathcal{C}) = n$$

A system is stabalizable if (in the Kalman controllability decomposition) the eigenvalues of the uncontrollable subsystem are all in the LHP.
Controllability/Detectability: A system is controllable iff the observability matrix is full rank:

$$\text{rank}(\mathcal{O}) = n$$

A system is detectable if (in the Kalman observabilitty decomposition) the eigenvalues of the uncontrollable subsystem are all in the LHP.

Duality:

$$(A, B) \text{ is controllable} \iff (A^T, B^T) \text{ is observable}$$

Static State Feedback Control

Consider the following state-space system:

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0 \\ y &= Cx + Du \end{aligned}$$

where the vectors are of dimensions: $x_{x\times 1},\ u_{p\times 1},\ y_{q\times 1}$
and the state matrices are of dimension: $A_{n\times n},\ B_{n\times p},\ C_{q\times n},\ D_{q\times p}$. Let the control law u is defined as:

$$u = Kx + r$$

where r is a reference signal and $K_{1\times n}$ is a static gain matrix (noting that redefining this may introduce a negative K term).
The closed loop system is given as:

$$\dot{x} = (A + BK)x + Br, \quad x(0) = x_0$$

The closed-loop system is reachable iff the open-loop system is controllable. Additionally this means that the eigen values of $(A + BK)$ can be arbitrarily placed.

Pole Plaement Theorem

SISO Case

1. Verify that the system is reachable: $\mathcal{C}_n$ is non-singular.

$$\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}_{n\times n}$$

2. Find the characteristic polynomial of the lopen loop system.

$$\Delta(s) = |sI - A| = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

3. Construct the following matrices:

$$\bar{\mathcal{C}}^{-1} = \begin{bmatrix} \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 & 1 \\ \alpha_{n-2} & \ddots & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

4. For the desired set of closed loop eigenvalues: $\{\eta_1, \eta_2, \cdots, \eta_n\}$, define the ideal closed loop characteristic polynomial as:

$$\Delta_{cls}(s) = \prod_{i=1}^{n}(s - \eta_i) = s^n + \bar{\alpha}_{n-1}s^{n-1} + \cdots + \bar{\alpha}_1 s + \bar{\alpha}_0$$

5. The static feedback gain, F, can then be found using the Ackerman's Formula:

$$K = \begin{bmatrix} \alpha_0 - \bar{\alpha}_0 & \alpha_1 - \bar{\alpha}_1 & \cdots & \alpha_{n-1} - \bar{\alpha}_{n-1} \end{bmatrix} P^{-1}$$

MIMO Case

This can be solved in many ways, each detailed in the book. If you have to do in exam, this is the one that you know best:

Lyapnov-Eq Method

1. Select an arbitrary matrix F with the desired eigen values that are not shared with A.
2. Select an arbitrary $\bar{K}$ sch that $(F, \bar{K})$ is controllable
3. Solve for T in the lyapnov equation:
$$AT - TF = B\bar{K}$$
4. If T is singlular ($\det(T) = 0$) then go back to step 1 or 2. If T is nonsingular, the gain K is given as:
$$K = \bar{K}T^{-1}$$

Obsevers

Consider the following state-space system:

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0 \\ y &= Cx + Du \end{aligned}$$

where the vectors are of dimensions: $x_{x\times 1},\ u_{p\times 1},\ y_{q\times 1}$
and the state matrices are of dimension: $A_{n\times n},\ B_{n\times p},\ C_{q\times n},\ D_{q\times p}$.
A Luemberger Observer is defined by the following state equation:

$$\begin{aligned} \hat{x} &= A\hat{x} + Du + L(y - C\hat{x}) \\ &= (A - LC)\hat{x} + Ly + Bu \end{aligned} \tag{1}$$

where L is the Luemberger gain matrix and $(A - LC)$ defines the observer dynamics.

Full-order

SISO Case

First, test observability and construct $\mathcal{O}$:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{np\times n}$$

If $\text{rank}(\mathcal{O}) = n$ then the observer can be constructed. An observer can be designed similarly to state feedback controller using the following matrix:
$$A^T - C^T K$$
where $L = K^T$. So the pole-placement method can be used to do this for the SISO case.
Quick note to jump to:
$$L_T = \begin{bmatrix} \alpha_n - \bar{\alpha}_n & \alpha_{n-1} - \bar{\alpha}_{n-1} & \cdots & \alpha_1 - \bar{\alpha}_1 \end{bmatrix} P$$

Lyapnov-Eq Method

First, test observability and construct $\mathcal{O}$:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{np\times n}$$

If $\text{rank}(\mathcal{O}) = n$ then the observer can be constructed. Let, an observer be constructed as:
$$\dot{z} = Fz + Gy + Hu$$

1. Select an arbitrary matrix F with the desired eigen values that are not shared with A.
2. Select an arbitrary G such that (F, G) is controllable
3. Solve for T in the lyapnov equation:
$$-FT + TA = GC$$
4. If T is singlular ($\det(T) = 0$) then go back to step 1 or 2. If T is nonsingular:
$$H = TB$$
$$\hat{x} = T^{-1}z$$

Reduced-order

Two main methods... Split into measured and unmeasured and lyapnov based

Split into measured and unmeasured

Notes in notebook are ok...

Lyapnov Based

First, test observability and construct $\mathcal{O}$:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{np \times n}$$

If $\text{rank}(\mathcal{O}) = n$ then the observer can be constructed.
Let, an observer be constructed as:

$$\dot{z} = Fz + Gy + Hu$$

where z is the unmeasured states.

1. Select an arbitrary matrix F with the desired eigen values that are not shared with A.
2. Select an arbitrary $G_{n \times q}$ such that (F, G) is controllable
3. Solve for T in the lyapnov equation:
$$-FT + TA = GC$$
4. Construct P:
$$P = \begin{bmatrix} C \\ T \end{bmatrix}$$
5. If P is singlular ($\det(T) = 0$) then go back to step 1 or 2. If T is nonsingular then the observer can be constructed as:
$$\dot{z} = Fz + Gy + Hu$$
$$\hat{x} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1} \begin{bmatrix} y \\ z \end{bmatrix}$$

Dynamic Output Feedback

See notes I guess... its just state feedback using an observer...
Note from Design Problems...
The compensator TF is given as:

$$G(s) = K(sI - (A + BK + LC))^{-1}L$$

LQR

Probably not needed for this exam... might be useful to put in though