MECH 6300/EECS 6331/SYSM 6307 Linear Systems
Design Application Problem Set A:
State Variable Representations
Due: Monday, August 31, 2020 by 10:00pm (US Central time)

1. Consider **Design Application #2**, the inverted pendulum on a cart. If the translational motion is described by *z,* the Lagrangian function is

$$L = \frac{1}{2}(M + m)\dot{z}^2 + \dot{z}\dot{\theta}ml\cos\theta + \frac{1}{2}ml^2\,\dot{\theta}^2 - mgl\cos\theta$$

Define generalized coordinates z and θ and compute Lagrange's equations for this system, that is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = f \quad , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad .$$

2. Assume that the angle θ is kept small (that is, the system "operates in the linear region") so that the approximations

$$\cos\theta \approx 1 \quad , \quad \sin\theta \approx \theta$$

are valid, and that terms in $\dot{z}$ and $\dot{\theta}$ are negligible so that quadratic terms may be neglected. Derive the linearized model of this system under these conditions.

3. Define the state vector $x = [\, z \quad \theta \quad \dot{z} \quad \dot{\theta} \,]^T$ and put the system into standard state variable form; that is, formulate the equation

$$\dot{x} = Ax + Bu$$

where the input u is the external force f on the cart.

4. Suppose now we consider the realistic problem of having a DC motor (as discussed in **Design Application #1**) supply the "external" force. To this end, assume that the relation between the (linear) force f and (rotational) torque τ is given by $r = \frac{\tau}{f}$, where r is the wheel radius, and assume that the motor operates at 100% efficiency, with motor torque constant k. Working from the linearized dynamical equation derived above, show that the differential equations describing the overall system can be written as

$$\ddot{z} = -\frac{mg}{M}\theta + \frac{1}{M}\left(\frac{k}{Rr}e - \frac{k^2}{Rr^2}\dot{z}\right) \quad ,$$
$$\ddot{\theta} = \left(\frac{M+m}{Ml}\right)g\theta - \frac{1}{Ml}\left(\frac{k}{Rr}e - \frac{k^2}{Rr^2}\dot{z}\right) \quad .$$

5. Consider the coupled cart problem (**Design Application #3**), where now the wheels of each cart are independently driven by electric motors (effectively one motor for each cart). Assume that each cart motor operates at 100% efficiency with torque constant k_i, armature resistance R_i, torque-to-force relationship $r = \frac{\tau_i}{f_i}$, and applied voltage e_i, for $i = 1, 2$. Derive the state variable equations for the entire system using the following state, input, and output vectors:

$$x = \begin{bmatrix} z_1 \\ z_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} \quad , \quad u = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad , \quad y = \begin{bmatrix} z_1 \\ z_2 \\ \dot{z}_1/r \\ \dot{z}_2/r \end{bmatrix} \quad .$$