Functions of RVs

- CDF and pdf of $Y = g(X)$
- Direct computation of pdf of $Y = g(X)$
- CDF and pdf of $Z = g(X, Y)$
- Direct computation of pdf of $Z = g(X, Y)$
- Joint CDF and pdf of $V = g(X, Y)$ and $W = h(X, Y)$
- Direct computation of joint pdf of $V = g(X, Y)$ and $W = h(X, Y)$

CDF and pdf of $Y = g(X)$

- Finding CDF and pdf of $Y = g(X)$ with known $f_X(x)$:

$$\boxed{F_Y(y) = P[g(X) \leq y] = P[X \in C_y]}$$

$$\boxed{f_Y(y) = \frac{dF_Y(y)}{dy}}$$

- need to find C_y (of X) such that $\boxed{\{\zeta : Y \leq y\} = \{\zeta : X \in C_y\}}$

- Good practice: Check S_Y, the support (or range) of Y.
- If C_y expressions are different for different intervals of y, develop $F_Y(y)$ and $f_Y(y)$ for each of such intervals.

- Example: Find pdf of $Y = aX + b$ where X is a continuous RV with pdf $f_X(x)$ while a and b are constants.

$$\text{For } a > 0,\ F_Y(y) = P[aX + b \leq y] = P[X \leq \frac{y-b}{a}] = F_X(\frac{y-b}{a})$$

$$\text{For } a < 0,\ F_Y(y) = P[aX + b \leq y] = P[X \geq \frac{y-b}{a}] = 1 - F_X(\frac{y-b}{a})$$

Note: $C_y = \{X \leq \frac{y-b}{a}\}$ for $a > 0$ and $C_y = \{X \geq \frac{y-b}{a}\}$ for $a < 0$.

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} \frac{1}{a} f_X(\frac{y-b}{a}), & a > 0 \\ \frac{-1}{a} f_X(\frac{y-b}{a}), & a < 0 \end{cases}$$

$$\boxed{\text{So, for } a \neq 0,\ f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$$

- If $Y = aX + b$, then $E[Y] = aE[X] + b$ and $\text{Var}[Y] = a^2\ \text{Var}[X]$.

- Example: Find CDF and PMF of $Y = aX + b$ where X is a discrete RV with PMF $P_X[x]$ while a and b are constants.

$$F_Y(y) = P[aX + b \le y] = \begin{cases} P[X \le \frac{y-b}{a}], & a > 0 \\ P[X \ge \frac{y-b}{a}], & a < 0 \end{cases}$$

$$= \begin{cases} F_X(\frac{y-b}{a}), & a > 0 \\ P_X[\frac{y-b}{a}] + 1 - F_X(\frac{y-b}{a}), & a < 0 \end{cases}$$

$$P_Y[y] = F_Y(y) - F_Y(y^-)$$

$$= \begin{cases} F_X(\frac{y-b}{a}) - F_X(\frac{y^- - b}{a}) = P_X[\frac{y-b}{a}], & a > 0 \\ P_X[\frac{y-b}{a}] - F_X(\frac{y-b}{a}) - P_X[\frac{y^- - b}{a}] + F_X(\frac{y^- - b}{a}) = P_X[\frac{y-b}{a}], & a < 0 \end{cases}$$

$$= P_X\left[\frac{y-b}{a}\right], \; a \neq 0.$$

Note: (1) For $a < 0$, $F_X(\frac{y^- - b}{a}) - F_X(\frac{y-b}{a}) = P_X[\frac{y^- - b}{a}]$.

Direct computation of $P_Y[y]$:
$P_Y[y] = P[Y = y] = P[aX + b = y] = P[X = \frac{y-b}{a}] = P_X[\frac{y-b}{a}]$.

- Example: $Y \triangleq F_X(x)$ will always be a uniform r.v. Conversely, given a uniform rv Y, the transformation $X \triangleq F_X^{-1}(Y)$ will generate a rv with CDF $F_X(x)$. (Transformation of PDF's)

  Let X have a continuous CDF $F_X(x)$. Let $Y \triangleq F_X(X)$.

  $$S_Y = [0,\ 1] \quad \Rightarrow \quad F_Y(y) = \begin{cases} 0, & y < 0 \\ 1, & y > 1 \end{cases}$$

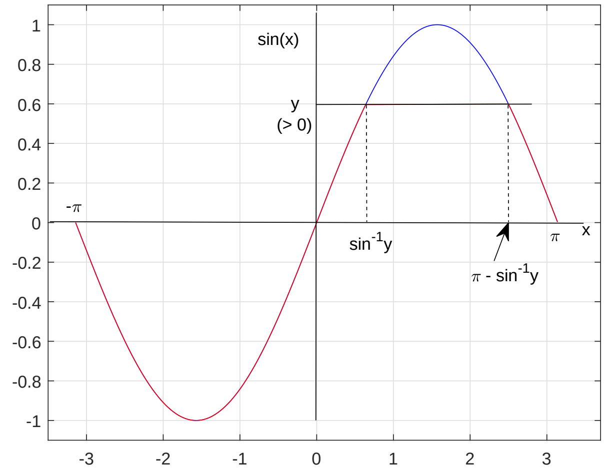
  For $0 \le y \le 1$,

  $$\begin{aligned} F_Y(y) &= P[Y \le y] = P\{F_X(X) \le y\} = P[X \le F_X^{-1}(y)] \\ &= \int_{-\infty}^{F_X^{-1}(y)} f_X(x)dx = F_X(F_X^{-1}(y)) = y. \end{aligned}$$

  $$\text{Hence,} \quad F_Y(y) = \begin{cases} 0, & y < 0 \\ y, & 0 \le y \le 1 \\ 1, & y > 1. \end{cases} \qquad \text{(CDF of a uniform RV)}$$

  $$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & \text{else.} \end{cases} \qquad \text{(pdf of a uniform RV)}$$

Example: Find pdf of $Y = \sin X$ where $f_X(x) = \begin{cases} \frac{1}{2\pi}, & -\pi < x \le \pi \\ 0, & \text{otherwise.} \end{cases}$



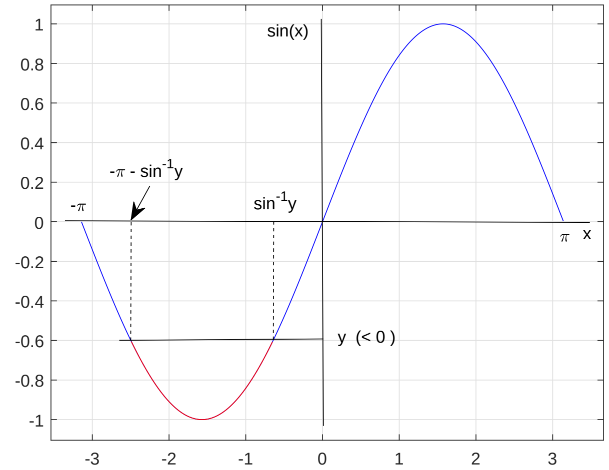


Fig. The roots of $y = \sin(x)$ when $0 \le y \le 1$ (Left figure) and $-1 < y < 0$ (Right figure).

$S_Y = (-1,\ 1]$.

$$\text{For } 0 \le y \le 1,\ \{Y \le y\} = \{\pi - \sin^{-1} y < X \le \pi\} \cup \{-\pi < X \le \sin^{-1} y\}$$

$$F_Y(y) = F_X(\pi) - F_X(\pi - \sin^{-1} y) + F_X(\sin^{-1} y) - F_X(-\pi).$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{f_X(\pi - \sin^{-1} y)}{\sqrt{1-y^2}} + \frac{f_X(\sin^{-1} y)}{\sqrt{1-y^2}} = \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}}.$$

If $-1 < y < 0$, $F_Y(y) = F_X(\sin^{-1} y) - F_X(-\pi - \sin^{-1} y)$; Same $f_Y(y)$. Thus,

$$f_Y(y) = \begin{cases} \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}}, & |y| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Direct Computation of the pdf of $Y = g(X)$

- The event $\{y < Y \leq y + dy\}$ = a union of disjoint elementary events $\{E_i\}$.

  If $\boxed{y = g(x) \text{ has } n \text{ real roots } x_1, \ldots, x_n}$, then

  $E_i = \{x_i - |dx_i| < X < x_i\}$ if $g'(x_i) < 0$, or

  $E_i = \{x_i < X < x_i + |dx_i|\}$ if $g'(x_i) > 0$,

  $P[E_i] = f_X(x_i)|dx_i|$ and

  $P[y < Y \leq y + dy] = f_Y(y)|dy| = \sum_{i=1}^{n} f_X(x_i)|dx_i|$

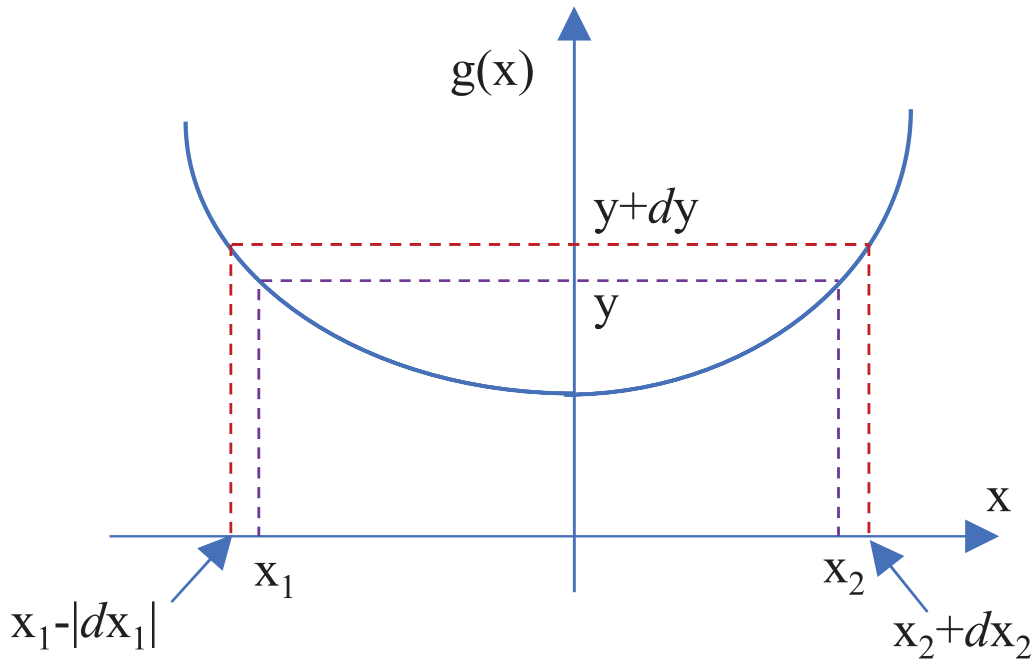


Fig. The event $\{y < Y \leq y + dy\}$ corresponds to the union of $n = 2$ disjoint events $\{x_1 - |dx_1| \leq X < x_1\}$ and $\{x_2 < X \leq x_2 + dx_2\}$.

Direct Computation of the pdf of $Y = g(X)$

- $P[y < Y \leq y + dy] = f_Y(y)|dy| = \sum_{i=1}^{n} f_X(x_i)|dx_i| \quad \Rightarrow$

$$\boxed{f_Y(y) = \sum_{i=1}^{n} f_X(x_i) \left| \frac{dx_i}{dy} \right| \quad \text{or} \quad f_Y(y) = \sum_{i=1}^{n} f_X(x_i) \left| \frac{dy}{dx_i} \right|^{-1}}$$

$$\text{where } x_i = g^{-1}(y) \ \text{ and } \ \frac{dy}{dx_i} \triangleq g'(x_i) \neq 0$$

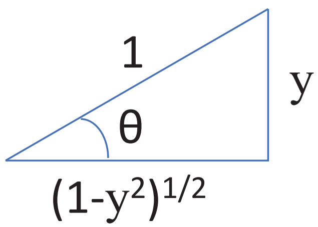
- If the roots and their number are different for different intervals of Y, develop pdf for each of such intervals.
- The final expression of $f_Y(y)$ should be a function of y only. Include the range of y.
- Good practice: Check S_Y first.
- If, for a given y, $y - g(x) = 0$ has no real roots, then $f_Y(y) = 0$.

Example: Alternative approach for pdf of $Y = \sin(X)$ with $f_X(x) = \frac{1}{2\pi}$, $-\pi < x \le \pi$.

$S_Y = (-1,\ 1]$. $g(x) = \sin x$ and $g'(x) \triangleq \frac{dg(x)}{dx} = \cos x$.

For $0 \le y \le 1$, the roots are $x_1 = \sin^{-1} y$ and $x_2 = (\pi - \sin^{-1} y)$.

$$\begin{aligned} g'(x_1) &= \cos(\sin^{-1} y), \\ g'(x_2) &= \cos(\pi - \sin^{-1} y) \\ &= \cos\pi\cos(\sin^{-1} y) + \sin\pi\sin(\sin^{-1} y) \\ &= -\cos(\sin^{-1} y). \end{aligned}$$



$$\cos(\sin^{-1} y) = \sqrt{1 - y^2}$$

$$|g'(x_1)| = |g'(x_2)| = \sqrt{1 - y^2}.$$

$$f_X(\sin^{-1} y) = f_X(\pi - \sin^{-1} y) = 1/(2\pi)$$

$$\text{Hence,} \qquad f_Y(y) = f_X(x_1)/\left|g'(x_1)\right| + f_X(x_2)/\left|g'(x_2)\right|$$

$$f_Y(y) = \begin{cases} \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}}, & 0 \le |y| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(The same result for $-1 < y < 0$ (DIY). Hence, $0 \le |y| < 1$ is used above.)

CDF and pdf of $Z = g(X, Y)$

- Computing CDF/pdf of $Z = g(X, Y)$ with known joint pdf $f_{XY}(x, y)$
- $$\boxed{\begin{aligned} F_Z(z) &= \int\int_{(x,y)\in C_z} f_{XY}(x, y)\ dx\ dy \\ f_Z(z) &= \frac{dF_Z(z)}{dz}. \end{aligned}}$$
- need to find C_z s.t. $\{\zeta : Z(\zeta) \leq z\} = \{\zeta : X(\zeta), Y(\zeta) \in C_z\}$, or simply, $\boxed{\{Z \leq z\} = \{(X, Y) \in C_z\}}$.
- Good practice: Check S_{XY}.
- If C_z's are different for different regions of z, develop CDF/pdf for each region.

  Note:

$$\frac{d}{dy}\int_{a(y)}^{b(y)} f(x, y)dx = f(b(y), y)\frac{db(y)}{dy} - f(a(y), y)\frac{da(y)}{dy} + \int_{a(y)}^{b(y)} \frac{\partial f(x, y)}{\partial y}dx$$

Example: Find the pdf of $Z = XY$.

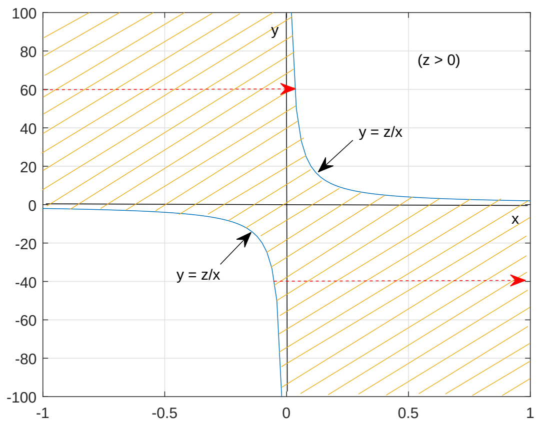


Fig. The region $xy \le z$ for $z > 0$

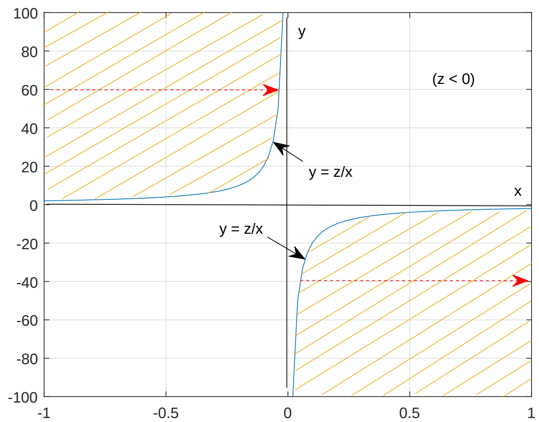


Fig. The region $xy \le z$ for $z < 0$

Based on the region $g(x,y) \triangleq \{xy \le z\}$, for $z \ge 0$,

$$F_Z(z) = \int_0^{\infty} \Big(\int_{-\infty}^{z/y} f_{XY}(x,y)dx\Big)dy + \int_{-\infty}^{0} \Big(\int_{z/y}^{\infty} f_{XY}(x,y)dx\Big)dy$$

$$= \int_0^{\infty} G_{XY}(z/y, y)dy + \int_{-\infty}^{0} [G_{XY}(\infty, y) - G_{XY}(z/y, y)]dy$$

$$\text{where } \; G_{XY}(x,y) \triangleq \int_{-\infty}^{x} f_{XY}(t,y)dt.$$

$f_Z(z) = \frac{dF_Z(z)}{dz}$ gives the following result (Same for $z < 0$ (DIY)).

$$\boxed{\text{If } Z = XY \;\; \Rightarrow \;\; f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|y|} f_{XY}(z/y, y)dy.}$$

Special case: $Z = XY$ where X and Y are independent, identically distributed (i.i.d.) Cauchy rvs with $f_X(x) = f_Y(x) \triangleq \frac{\alpha/\pi}{\alpha^2+x^2}$.

$f_{XY}(x,y) = f_X(x)f_Y(y)$ (independence) and the evenness of the integrand $\Rightarrow$

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} \frac{1}{|y|} f_{XY}(z/y, y)dy \\
&= \int_{-\infty}^{\infty} \frac{1}{|x|} \frac{\alpha/\pi}{\alpha^2 + (z/x)^2} \frac{\alpha/\pi}{\alpha^2 + (x)^2} dx = 2\int_0^{\infty} \frac{1}{x} \frac{x^2}{\alpha^2 x^2 + z^2} \frac{(\alpha/\pi)^2}{\alpha^2 + x^2} dx \\
&= \frac{\alpha^2}{\pi^2} \int_0^{\infty} \frac{1}{z^2 + \alpha^2 t} \cdot \frac{1}{\alpha^2 + t} dt, \ \Leftarrow (x^2 \to t) \\
&= \frac{\alpha^2}{\pi^2} \int_0^{\infty} \left(\frac{\alpha^2/(\alpha^4 - z^2)}{z^2 + \alpha^2 t} + \frac{1/(z^2 - \alpha^4)}{\alpha^2 + t}\right) dt, \quad \left(\int \frac{1}{a + bx} dx = \frac{1}{b} \ln|a + bx|\right) \\
&= \frac{\alpha^2}{\pi^2} \left[\frac{1}{\alpha^4 - z^2} \ln(|z^2 + \alpha^2 t|) + \frac{1}{z^2 - \alpha^4} \ln(|\alpha^2 + t|)\right]_{t=0}^{\infty} \\
&= \frac{\alpha^2}{\pi^2} \left[\frac{1}{z^2 - \alpha^4} \ln\left(\left|\frac{\alpha^2 + t}{z^2 + \alpha^2 t}\right|\right)\right]_{t=0}^{\infty} = \frac{\alpha^2}{\pi^2} \frac{1}{z^2 - \alpha^4} \left[\ln\left(\frac{1}{\alpha^2}\right) - \ln\left(\frac{\alpha^2}{z^2}\right)\right] \\
&= \frac{\alpha^2}{\pi^2} \frac{1}{z^2 - \alpha^4} \ln \frac{z^2}{\alpha^4}
\end{aligned}$$

Example: (Parallel operation) Compute the pdf of $Z = \max(X, Y)$ if X and Y are independent r.v.'s.

Since $\{\max(X, Y) \leq z\} = \{X \leq z, Y \leq z\}$,

$$F_Z(z) = P[Z \leq z] = P[X \leq z, Y \leq z] = F_X(z)F_Y(z)$$
$$f_Z(z) = \frac{dF_Z(z)}{dz} = f_Y(z)F_X(z) + f_X(z)F_Y(z).$$

Special case: $f_X(x) = f_Y(x)$ be the uniform pdf over $[0, 1]$. Then

$$f_Z(z) = 2z[u(z) - u(z-1)]$$

Example: If $Z = \min(X, Y)$ with independent X and Y, find $f_Z(z)$.

$$\begin{aligned} F_Z(z) &= 1 - P[\min(X, Y) > z] \\ &= 1 - P[X > z,\ Y > z] = 1 - (1 - F_X(z))(1 - F_Y(z)) \\ f_Z(z) &= f_Y(z)(1 - F_X(z)) + f_X(z)(1 - F_Y(z)) \end{aligned}$$

- **Example: pdf of $Z = X + Y$**

$C_z = \{X + Y \leq z\}$

$$F_Z(z) = \int\int_{x+y\leq z} f_{XY}(x,y)dxdy = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{z-y} f_{XY}(x,y)dx\right)dy$$

$$= \int_{-\infty}^{\infty}[G_{XY}(z-y,y) - G_{XY}(-\infty,y)]dy \text{ where } G_{XY}(x,y) \triangleq \int_{-\infty}^{x} f_{XY}(t,y)dt$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{-\infty}^{\infty}\frac{d}{dz}[G_{XY}(z-y,y)]dy \text{ which yields that}$$

$$\boxed{\text{if } Z = X + Y, \quad f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(z-y,y)dy.}$$

- If X and Y are independent, then the pdf of $Z = X + Y$ is

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)dy = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx = f_X(z) * f_Y(z)}$$

(*convolution integral* or *convolution* of f_X with f_Y)

- The pdf of the sum of independent RVs is the convolution of individual pdfs. For discrete RVs, use a discrete convolution.

- Example: pdf of $Z = aX + bY$:

  Let $a > 0$ and $b > 0$. Then $C_z = g(X,Y) \triangleq aX + bY \leq z$ is to the left of the line $y = z/b - ax/b$. Hence,

$$F_Z(z) = \int\int_{g(x,y)\leq z} f_{XY}(x,y)dxdy = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z/a-by/a} f_{XY}(x,y)dx \right) dy.$$

$$f_Z(z) = \frac{1}{a} \int_{-\infty}^{\infty} f_{XY}\left(\frac{z}{a} - \frac{by}{a}, y\right) dy$$

$$\text{If } X \text{ and } Y \text{ are independent, } f_Z(z) = \frac{1}{a} \int_{-\infty}^{\infty} f_X\left(\frac{z}{a} - \frac{by}{a}\right) f_Y(y)\, dy.$$

- Example: pdf of $Z = X + Y$ where X and Y are iid with $P_X[k] = P_Y[k] = (1-p)\ \delta[k] + p\ \delta[k-1]$

  (discrete convolution)

$$P_Z[m] = P_X[m] * P_Y[m] = \sum_{k=-\infty}^{\infty} P_X[k]P_Y[m-k] = \sum_{k=0}^{1} P_X[k]P_Y[m-k]$$
$$= (1-p)^2\ \delta[m] + 2p(1-p)\ \delta[m-1] + p^2\ \delta[m-2]$$

Special Cases of the Sum of Two RVs

- Sum of two independent Poisson RVs with parameters a and b (their mean values) is a Poisson RV with parameter $(a+b)$.
- Sum of two iid binomial RVs with PMF given by $b(k;n,p)$ is a binomial RV with PMF given as $b(k;2n,p)$.
- Sum of two iid Central Chi-square RVs with each with n DoF is a Central Chi-square RV with $2n$ DoF.
- Sum of two independent Cauchy RVs with parameters (α_1,β_1) and (α_2,β_2) is a Cauchy RV with parameters $(\alpha_1+\alpha_2,\beta_1+\beta_2)$.
- Sum of two independent Gaussian RVs with parameters (μ_1,σ_1^2) and (μ_2,σ_2^2) is a Gaussian RV with parameters $(\mu_1+\mu_2,\sigma_1^2+\sigma_2^2)$.

- Example: Let X and Y be i.i.d. $\mathcal{N}(0,\sigma^2)$. What is the pdf of $Z \triangleq X^2+Y^2$?

$S_Z = [0,\infty)$.

$$F_Z(z) = \int\int_{(x,y)\in C_z} f_{XY}(x,y)dxdy, \quad z \geq 0$$
$$= \frac{1}{2\pi\sigma^2}\int\int_{x^2+y^2\leq z} e^{-(\frac{1}{2\sigma^2})(x^2+y^2)}dxdy.$$

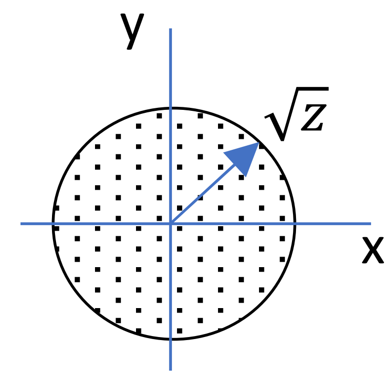


Fig. The region C_z for $\{X^2+Y^2 \leq z\}$ for $z \geq 0$.

Using polar coordinates,
$x = r\cos\theta,\ y = r\sin\theta,\ dxdy \to rdrd\theta$,
we have $x^2+y^2 \leq z \to r \leq \sqrt{z}$, and

$$F_Z(z) = \frac{1}{2\pi\sigma^2}\int_0^{2\pi} d\theta \int_0^{\sqrt{z}} r\ \exp[-\frac{1}{2\sigma^2}r^2]dr = [1-e^{-\frac{z}{2\sigma^2}}]u(z)$$
$$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{1}{2\sigma^2}e^{-\frac{z}{2\sigma^2}}u(z).$$

Thus, $Z = X^2+Y^2$ is an exponential (chi-square with 2 DoF) r.v. if X and Y are i.i.d. zero-mean Gaussian.

- Example (Rayleigh): Find the pdf of $Z \triangleq \sqrt{X^2+Y^2}$ with X and Y being i.i.d. $\mathcal{N}(0,\sigma^2)$.

$S_Z = [0,\infty)$.

$$F_Z(z) = \int\int_{(x,y)\in C_z} f_{XY}(x,y)dxdy, \quad z \geq 0$$
$$= \frac{1}{2\pi\sigma^2}\int\int_{\sqrt{x^2+y^2}\leq z} e^{-(\frac{1}{2\sigma^2})(x^2+y^2)}dxdy.$$

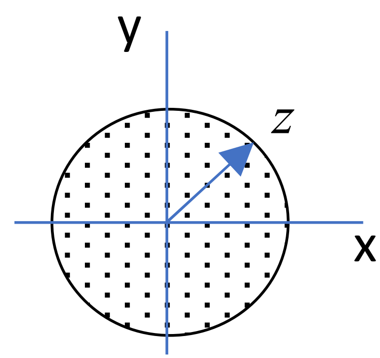


Fig. The region C_z for $\{\sqrt{X^2+Y^2} \leq z\}$ for $z \geq 0$.

Changing to polar coordinates,
$x = r\cos\theta,\ y = r\sin\theta,\ dxdy \to rdrd\theta$; we have $\sqrt{x^2+y^2} \leq z \to r \leq z$, and

$$F_Z(z) = \frac{1}{2\pi\sigma^2}\int_0^{2\pi} d\theta \int_0^z r\ \exp[-\frac{1}{2\sigma^2}r^2]dr$$
$$= (1-e^{-\frac{z^2}{2\sigma^2}})u(z)$$
$$f_Z(z) = \frac{z}{\sigma^2}e^{-\frac{z^2}{2\sigma^2}}u(z)$$

which is the Rayleigh density function (a.k.a. χ ("chi") distribution with two degrees of freedom).

- Example (The Rician density): Find the pdf of $Z \triangleq \sqrt{X^2+Y^2}$ for independent X and Y with $f_X(x) = \mathcal{N}(P, \sigma^2)$ and $f_Y(y) = \mathcal{N}(0, \sigma^2)$.

$$F_Z(z) = \begin{cases} \frac{1}{2\pi\sigma^2} \int\int_{\sqrt{(x^2+y^2)} \leq z} \exp\left[-\frac{1}{2}\left(\left[\frac{x-P}{\sigma}\right]^2 + \left(\frac{y}{\sigma}\right)^2\right)\right] dxdy, & z > 0 \\ 0, \quad z < 0 \end{cases}$$

Changing to polar coordinate, $x = r\cos\theta$, $y = r\sin\theta$, $r = x^2 + y^2$, $\theta = \tan^{-1}(y/x)$, and

$$\begin{aligned} F_Z(z) &= \frac{e^{-\frac{1}{2}\left(\frac{P}{\sigma}\right)^2}}{2\pi\sigma^2} \int_0^z e^{-\frac{1}{2}(r/\sigma)^2} \left(\int_0^{2\pi} e^{rP\cos\theta/\sigma^2} d\theta\right) rdr \cdot u(z) \\ &= \frac{e^{-\frac{1}{2}\left(\frac{P}{\sigma}\right)^2}}{\sigma^2} \int_0^z rI_0\left(\frac{rP}{\sigma^2}\right) e^{-\frac{r^2}{2\sigma^2}} dr \cdot u(z). \end{aligned}$$

where $I_0(x) \triangleq \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta} d\theta$ is the zero-order modified Bssel function of the first kind (monotonically increasing like e^x).

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{z}{\sigma^2}\exp\left[-\frac{1}{2}\left(\frac{P^2+z^2}{\sigma^2}\right)\right] I_0\left(\frac{zP}{\sigma^2}\right) \cdot u(z), \quad \text{Rician pdf}$$

(When $P = 0$, since $I_0(0) = 1$, we obtain the Rayleigh law.)
The Rician pdf is the pdf of the *envelope* of the sum of a strong sine wave and weak narrow-band Gaussian noise.

Joint CDF/pdf of $V = g(X,Y)$ and $W = h(X,Y)$

- Finding joint CDF/pdf of $V = g(X,Y)$ and $W = h(X,Y)$ from known $f_{XY}(x,y)$.
- Find C_{vw} such that $\{V \le v, W \le w\} = \{(X,Y) \in C_{vw}\}$, i.e., $\boxed{C_{vw} = \{(x,y) : g(x,y) \le v, h(x,y) \le w\}}$.
- Next,

$$P[V \le v, W \le w] \triangleq \boxed{F_{VW}(v,w) = \int\int_{(x,y)\in C_{vw}} f_{XY}(x,y)\, dx\, dy}$$

$$\boxed{f_{VW}(v,w) = \frac{\partial^2 F_{VW}(v,w)}{\partial v\, \partial w}}$$

- If different regions of (v,w) give different expressions of C_{vw}, find $F_{VW}(v,w)$ and $f_{VW}(v,w)$ for each region.
- Good practice: Check S_{VW}.

- Example: Find $f_{VW}(v,w)$ from $f_{XY}(x,y)$ where $V \triangleq X+Y$ and $W \triangleq X-Y$.

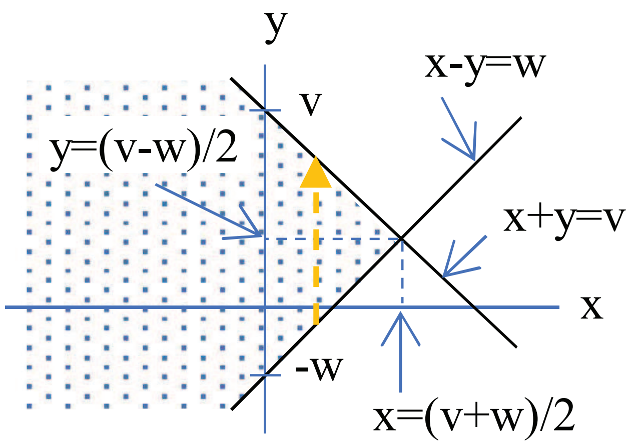


Fig. The point set $C_{vw} = \{g(x,y) \triangleq x+y \le v,\ h(x,y) \triangleq x-y \le w\}$ (shaded region) for $v>0$ and $w>0$.
(Note: The expression of C_{vw} is the same for other ranges of v and w.)

$$F_{VW}(v,w) = \int_{-\infty}^{\frac{(v+w)}{2}} \left(\int_{x-w}^{v-x} f_{XY}(x,y)dy \right) dx$$

$$\frac{\partial F_{VW}(v)}{\partial v} = \frac{1}{2}\int_{\frac{v+w}{2}-w}^{v-\frac{v+w}{2}} f_{XY}\left(\frac{v+w}{2}, y\right)dy + \int_{-\infty}^{\frac{v+w}{2}} \frac{d\int_{x-w}^{v-x} f_{XY}(x,y)dy}{dv}dx$$

$$= 0 + \int_{-\infty}^{\frac{v+w}{2}} f_{XY}(x, v-x)dx$$

$$f_{VW}(v,w) = \frac{\partial^2 F_{VW}(v,w)}{\partial v \partial w} = \frac{1}{2} f_{XY}\left(\frac{v+w}{2}, \frac{v-w}{2}\right)$$

Note:

$$\frac{d}{dy}\int_{a(y)}^{b(y)} f(x,y)dx = f(b(y),y)\frac{db(y)}{dy} - f(a(y),y)\frac{da(y)}{dy} + \int_{a(y)}^{b(y)} \frac{\partial f(x,y)}{\partial y}dx$$

Obtaining $f_{VW}(v,w)$ Directly from $f_{XY}(x,y)$

- Consider the elementary event $\{v < V \leq v+dv,\ w < W \leq w+dw\}$, the one-to-one differentiable functions $v = g(x,y)$, $w = h(x,y)$, and their inverse mappings $x = \phi(v,w)$, $y = \psi(v,w)$.
- An infinitesimal rectangle region $\mathcal{R}$ in the (v,w) plane maps to an infinitesimal parallelogram region $\mathcal{P}$ in the (x,y) plane. Their virtices are

$$\begin{array}{ll}
(v,w) & \rightarrow P_1 = (x,y) \\
(v+dv,w) & \rightarrow P_2 = (x + \frac{\partial\phi}{\partial v}dv,\ y + \frac{\partial\psi}{\partial v}dv) \\
(v,w+dw) & \rightarrow P_3 = (x + \frac{\partial\phi}{\partial w}dw,\ y + \frac{\partial\psi}{\partial w}dw) \\
(v+dv,w+dw) & \rightarrow P_4 = (x + \frac{\partial\phi}{\partial v}dv + \frac{\partial\phi}{\partial w}dw,\ y + \frac{\partial\psi}{\partial v}dv + \frac{\partial\psi}{\partial w}dw)
\end{array}$$

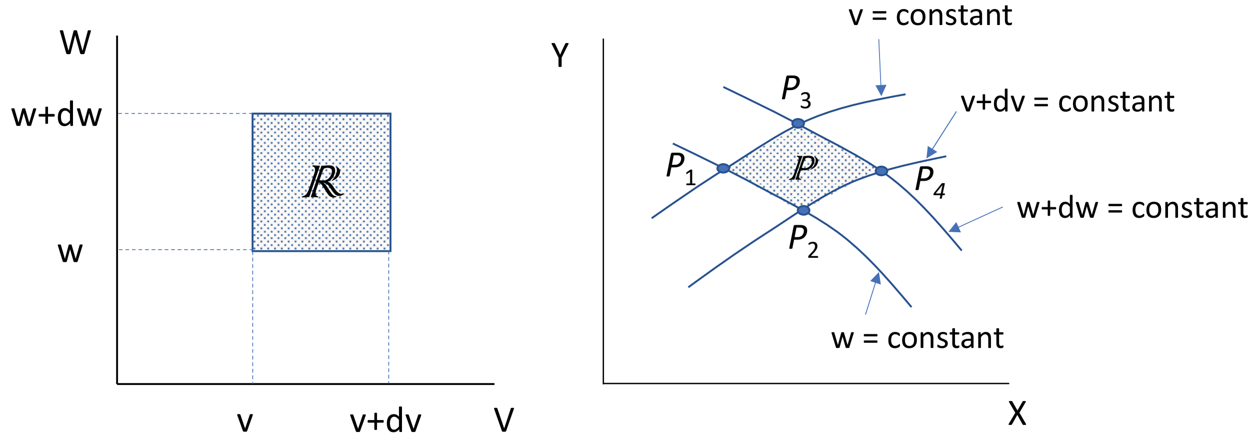


Fig. A mapping from an infinitesimal rectangle in the (v,w) plane to an infinitesimal parallelogram in the (x,y) plane.

- $A(\mathcal{R})$ and $A(\mathcal{P})$ denote the areas of $\mathcal{R}$ and $\mathcal{P}$. Then,

$$\begin{aligned} & P[(v,w) \in \mathcal{R}] = P[(x,y) \in \mathcal{P}] \\ & \Rightarrow \quad f_{VW}(v,w)A(\mathcal{R}) = f_{XY}(x,y)A(\mathcal{P}) \text{ where } x = \phi(v,w),\ y = \psi(v,w). \end{aligned}$$

- $\frac{A(\mathcal{P})}{A(\mathcal{R})} = |\tilde{J}| = 1/|J|$ where

$$|\tilde{J}| = \text{mag} \begin{vmatrix} \frac{\partial \phi}{\partial v} & \frac{\partial \phi}{\partial w} \\ \frac{\partial \psi}{\partial v} & \frac{\partial \psi}{\partial w} \end{vmatrix} = \left| \frac{\partial \phi}{\partial v}\frac{\partial \psi}{\partial w} - \frac{\partial \psi}{\partial v}\frac{\partial \phi}{\partial w} \right|$$

$$|J| = \text{mag} \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix}_{x,y} = \left| \frac{\partial g}{\partial x}\frac{\partial h}{\partial y} - \frac{\partial h}{\partial x}\frac{\partial g}{\partial y} \right|_{x,y}.$$

- $$\boxed{\begin{aligned} & f_{VW}(v,w) = |\tilde{J}|\ f_{XY}(x,y) \ \text{ or } \ f_{VW}(v,w) = f_{XY}(x,y)/|J|, \\ & \qquad \text{where } x = \phi(v,w),\ y = \psi(v,w) \end{aligned}}$$

- If the solution $(x,y)'_s$ are different for different regions of (v,w), find pdf for each.
- The final expression of $f_{VW}(v,w)$ should be a function of v and w only. Include the ranges of v and w in $f_{VW}(v,w)$.
- Good practice: Check S_{VW} first.

- If there are more than one solution to the equations $v = g(x, y)$, $w = h(x, y)$, say, $\boxed{x_1 = \phi_1(v, w),\ y_1 = \psi_1(v, w),\ \ldots,\ x_n = \phi_n(v, w),\ y_n = \psi_n(v, w)}$, then $\mathcal{R}$ maps into multiple disjoint infinitesimal regions $\mathcal{P}_1$, $\mathcal{P}_2$, ... , $\mathcal{P}_n$
- $A(\mathcal{P}_i)/A(\mathcal{R}) = |\mathcal{J}_i| = |J_i^{-1}|$, $i = 1, \ldots, n$, where

$$|\tilde{J}_i| = \mathrm{mag}\begin{vmatrix} \frac{\partial\phi_i}{\partial v} & \frac{\partial\phi_i}{\partial w} \\ \frac{\partial\psi_i}{\partial v} & \frac{\partial\psi_i}{\partial w} \end{vmatrix} = \left|\frac{\partial\phi_i}{\partial v}\frac{\partial\psi_i}{\partial w} - \frac{\partial\psi_i}{\partial v}\frac{\partial\phi_i}{\partial w}\right|$$

$$|J_i| = \mathrm{mag}\begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix}_{x_i, y_i} = \left|\frac{\partial g}{\partial x}\frac{\partial h}{\partial y} - \frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\right|_{x_i, y_i}$$

- $$\boxed{f_{VW}(v, w) = \sum_{i=1}^{n} f_{XY}(x_i, y_i)|\tilde{J}_i| \quad \text{or} \quad f_{VW}(v, w) = \sum_{i=1}^{n} f_{XY}(x_i, y_i)/|J_i|}$$
- If solutions $\{(x_i, y_i)\}$ are different for different regions of (v, w), find pdf for each.
- $f_{VW}(v, w)$ should be a function of v and w only. Include the ranges of v and w in $f_{VW}(v, w)$.
- Good practice: Check S_{VW} first.

- General steps in obtaining $f_{VW}(v, w)$ directly from $f_{XY}(x, y)$:
  - Check S_{VW}.
  - Find the roots.
  - Compute the Jacobian.
  - Substitute in the correct equation for $f_{VW}(v, w)$.

    Make sure it is a function of v and w and the valid ranges of (V, W) are included.

- Example: Find $f_{VW}(v,w)$ from $f_{XY}(x,y)$ where $V \triangleq X+Y$ and $W \triangleq X-Y$.

The only root to $v-(x+y)=0$ and $w-(x-y)=0$ is $x=(v+w)/2$ and $y=(v-w)/2$.

$$f_{VW}(v,w) = |\tilde{J}|\ f_{XY}(x,y)$$

$$\text{where} \quad |\tilde{J}| = \text{mag}\begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \end{vmatrix} = \text{mag}\begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = 1/2$$

Hence,

$$f_{VW}(v,w) = \frac{1}{2}\ f_{XY}\left(\frac{v+w}{2}, \frac{v-w}{2}\right)$$

(Easier than the indirect approach via CDF)

- Example: Let $V \triangleq \sqrt{X^2+Y^2}$ and $W \triangleq \frac{Y}{X}$ where X and Y are iid $N(0,\sigma^2)$. Find $f_{VW}(v,w)$.

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2}\exp\Big[-\frac{(x^2+y^2)}{2\sigma^2}\Big], \quad -\infty < x < \infty, -\infty < y < \infty$$

$S_V = [0,\infty)$, $S_W = (-\infty,\infty)$, $S_{VW} = \{v \geq 0, \ -\infty < w < \infty\}$

With $g(x,y) = \sqrt{x^2+y^2}$ and $h(x,y) = \frac{y}{x}$, the equations

$$w - h(x,y) = 0 \quad \Rightarrow \quad w - y/x = 0 \quad \Rightarrow \quad y = wx$$

$$v - g(x,y) = 0 \quad \Rightarrow \quad v - \sqrt{x^2+y^2} = 0 \quad \Rightarrow \quad x = \pm v(1+w^2)^{-1/2}$$

give 2 solutions for $-\infty < w < \infty$ and $v > 0$:
$(x_1 = v(1+w^2)^{-1/2}, \ y_1 = wx_1)$ and $(x_2 = -v(1+w^2)^{-1/2}, \ y_2 = wx_2)$.
Hence, $f_{VW}(v,w) = \frac{1}{|J_1|}f_{XY}(x_1,y_1) + \frac{1}{|J_2|}f_{XY}(x_2,y_2)$ where

$$|J_i| = \mathrm{mag}\begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix}_{x_i,y_i} = \mathrm{mag}\begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}}, & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2}, & \frac{1}{x} \end{vmatrix}_{x_i,y_i}$$

$$\Rightarrow \left|\frac{\partial g}{\partial x}\frac{\partial h}{\partial y} - \frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\right|_{x_i,y_i} \quad \Rightarrow \quad |J_1| = |J_2| = \frac{1+\omega^2}{v}.$$

$$f_{VW}(v,w) = \begin{cases} \frac{v}{\sigma^2}e^{-\frac{v^2}{2\sigma^2}}\frac{1/\pi}{1+w^2}, & v > 0, \ -\infty < w < \infty \\ 0, & \text{else} \end{cases}$$

Note: $f_{VW}(v,w) = f_V(v)f_W(w)$ with $f_V(v) = \frac{v}{\sigma^2}e^{-\frac{v^2}{2\sigma^2}}u(v)$ (Rayleigh) and $f_W(w) = \frac{1/\pi}{1+w^2}, -\infty < w < \infty$ (Cauchy).

- Example (Magnitude and angle): Joint pdf of

$$V = \sqrt{X^2 + Y^2} \quad \text{and} \quad \Theta = \begin{cases} \tan^{-1}(Y/X), & X > 0 \\ \tan^{-1}(Y/X) + \pi, & X < 0 \end{cases}$$

where $Z = X + jY$ with known $f_{XY}(x, y)$.

$S_V = [0, \infty)$, $S_\Theta = [-\pi/2, 3\pi/2)$.

From the Cartesian-Polar coordinate relationship, we obtain a single solution $(x = v\ \cos(\theta),\ \ y = v\ \sin(\theta))$.

$$\text{The Jacobian is} \quad \tilde{J} = \begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -v\ \sin(\theta) \\ \sin(\theta) & v\ \cos(\theta) \end{vmatrix} = v$$

$$\text{The joint pdf is} \quad f_{V\Theta}(v, \theta) = \tilde{J}\ f_{XY}(x, y) = v\ f_{XY}(v\ \cos(\theta), v\ \sin(\theta))$$

$$\text{If } X \text{ and } Y \text{ are iid } \mathcal{N}(0, \sigma^2), \quad i.e., \quad f_{XY}(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}},$$

$$\text{then,} \quad f_{V\Theta}(v, \theta) = \begin{cases} \left(\frac{v}{\sigma^2} e^{-\frac{v^2}{2\sigma^2}} \right) \frac{1}{2\pi}, & v > 0,\ -0.5\pi \le \theta < 1.5\pi \\ 0, & \text{else} \end{cases}$$

Note: $f_{V\Theta}(v, \theta) = f_V(v) f_\Theta(\theta)$ with Rayleigh V and uniform Θ.