Random Processes

- Continuous-Time Random Process $X(t, \zeta)$ or simply $X(t)$:
  - At each time instant t, $X(t)$ is an RV.
  - In general, CDFs/PDFs/PMFs of $X(t_1)$ and $X(t_2)$, $t_1 \neq t_2$, can be different.

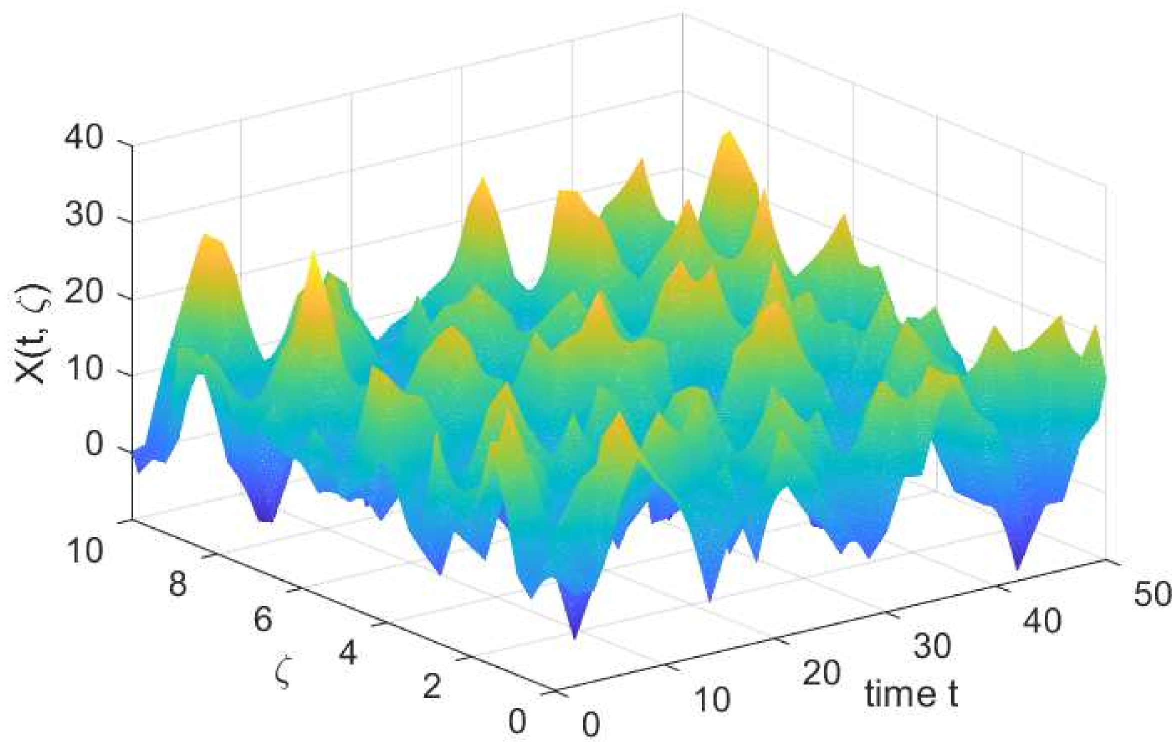


Fig. A random process for a continuous sample space $\Omega = [0, 10]$

- For a continuous-value process $X(t)$, PDF: $f_X(x;t)$
- For a discrete-value process $X(t)$, PMF: $P_X(x;t)$
- CDF: $F_X(x;t)$
- Mean function: $\mu_X(t) \triangleq E[X(t)], \quad -\infty < t < \infty$
- Correlation function:
  $$R_{XX}(t_1, t_2) \triangleq E[X(t_1)X^*(t_2)], \quad -\infty < t_1, t_2 < \infty$$
- Covariance function:

$$\begin{aligned} K_{XX}(t_1, t_2) &\triangleq E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))^*] \\ &= R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X^*(t_2). \end{aligned}$$

- Variance function: $\sigma_X^2(t) \triangleq K_{XX}(t, t)$
- Power function: $R_{XX}(t) = E[|X(t)|^2]$.
- n^{th} moment function: $E[X^n(t)]$

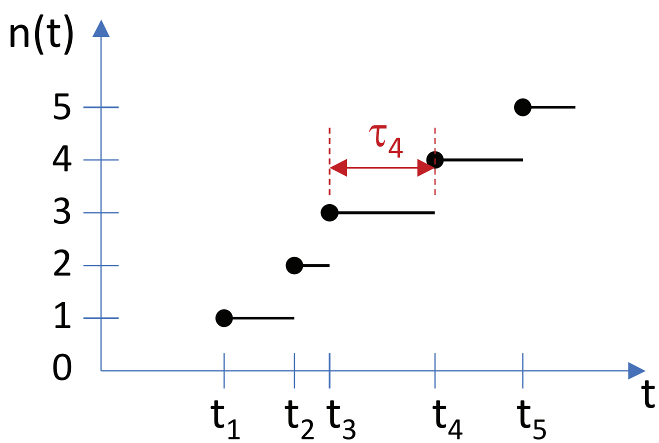
Poisson Counting Process $N(t)$

- $N(t) \triangleq$ the total number of counts (arrivals) up to time t:

$$N(t) \triangleq \sum_{n=1}^{\infty} u(t - T[n]),$$

where $u(t)$ is the unit-setp function, $T[n]$ is the time to the nth arrival, and the *interarrival times* $\tau[n] \triangleq T[n] - T[n-1]$ are jointly i.i.d. and having exponential pdf as $f_\tau(\tau) = \lambda e^{-\lambda \tau} u(\tau)$.

PMF of $N(t)$:



$$P[N(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t} u(t), n \geq 0$$

(PMF of a Poisson RV with mean λt)

$$\mu_N(t) = \lambda t$$

$$\sigma_N^2(t) = \lambda t$$

$$R_{NN}(t_1, t_2) = \lambda \min(t_1, t_2) + \lambda^2 t_1 t_2$$

$$K_{NN}(t_1, t_2) = \lambda \min(t_1, t_2)$$

Wiener Process $X(t)$ (aka *Wiener-Levy* or Brownian Motion)

- PDF of $X(t)$ is Gaussian

$$f_X(x;t) = \frac{1}{\sqrt{2\pi\alpha t}} \exp(-\frac{x^2}{2\alpha t}), \ t > 0.$$

with $\mu_X(t) = 0$, $\mathrm{Var}[X(t)] = \alpha t$, and

the PDF of the increment $\triangle \triangleq X(t) - X(\tau)$ for all $t > \tau$ is also Gaussian

$$f_\triangle(\delta; t-\tau) = \frac{1}{\sqrt{2\pi\alpha(t-\tau)}} \exp(-\frac{\delta^2}{2\alpha(t-\tau)})$$

with $E[\triangle] = 0$ and $\mathrm{Var}[\triangle] = \alpha(t-\tau)$.

- The covariance function of $X(t)$ is

$$K_{XX}(t_1, t_2) = \alpha \ \min(t_1, t_2), \ \alpha > 0.$$

Gaussian Random Process $X(t)$

- If for all positive integers n, the nth-order PDF's of a random process (i.e., the joint PDF of $X(t_1), \ldots, X(t_n)$) are all jointly Gaussian, then the process is called a *Gaussian Random process*.

- PDF of $X(t)$ is Gaussian with mean $\mu_X(t)$ and variance $\sigma_X^2(t)$:

$$f_X(x;t) = \frac{1}{\sqrt{2\pi\sigma_X^2(t)}} \exp\Big(-\frac{(x-\mu_X(t))^2}{2\sigma_X^2(t)} \Big).$$

- The Wiener process is an example of a Gaussian random process.

Markov Random Processes

- A *continuous-valued Markov process* $X(t)$ satisfies the conditional PDF expression

$$f_X(x_n|x_{n-1}, x_{n-2}, ..., x_1; t_n, ..., t_1) = f_X(x_n|x_{n-1}; t_n, t_{n-1}),$$

for all $x_1, x_2, ..., x_n$, for all $t_1 < ... < t_n$ and for all integers $n > 0$.

- A *discrete-valued Markov random process* satisfies the conditional PMF expression

$$P_X(x_n|x_{n-1}, x_{n-2}, ..., x_1; t_n, ..., t_1) = P_X(x_n|x_{n-1}; t_n, t_{n-1}),$$

for all $x_1, x_2, ..., x_n$, for all $t_1 < ... < t_n$ and for all integers $n > 0$.

- The values of the process $X(t)$ are called the *states of the process*, and the conditional probabilities are thought of as *transition probabilities* between the states.
- If only a finite or countable set of values x_i is allowed, the discrete-valued Markov process is called a *Markov chain*.

Stationarity and Wide-Sense Stationarity

- A random process $X(t)$ is *stationary* (strict-sense stationary (SSS)) if

$$F_X(x_1, ..., x_n; t_1, ..., t_n) = F_X(x_1, ..., x_n; t_1 + T, ..., t_n + T)$$
$$\text{or,} \quad f_X(x_1, ..., x_n; t_1, ..., t_n) = f_X(x_1, ..., x_n; t_1 + T, ..., t_n + T)$$

  for all T, for all positive integers n, and for all $t_1, ..., t_n$.

  - $f(x; t) = f(x; 0) \Rightarrow E[X(t)] = \mu_X(t) = \mu_X$.
  - $F(x_1, x_2; t_1, t_2) = F(x_1, x_2; t_1 - t_2, 0) \Rightarrow$
    $E[X(t_1)X^*(t_2)] = R_{XX}(t_1 - t_2, 0) \triangleq R_{XX}(t_1 - t_2) = R_{XX}(\tau)$ where $\tau = t_1 - t_2$.

- A random process $X(t)$ is *Wide-Sense Stationary* (WSS) if

  $\mu_X(t) = \mu_X$ and

  $R_{XX}(t + \tau, t) = R_{XX}(\tau)$ (also $K_{XX}(t + \tau, t) = K_{XX}(\tau)$)

  for all $-\infty < \tau < \infty$, independent of the time parameter t.

- SSS $\Rightarrow$ WSS
- For a Gaussian process, WSS $\Rightarrow$ SSS

Power Spectral Density (PSD)

- The PSD $S_{XX}(\omega)$ of a random process $X(t)$ is the Fourier Transform (if it exists) of $R_{XX}(\tau)$, i.e.,

$$S_{XX}(\omega) \triangleq \int_{-\infty}^{\infty} R_{XX}(\tau) \exp(-j\omega\tau) d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \exp(j\omega\tau) d\omega$$

i.e.,

$$S_{XX} = \mathrm{FT}\{R_{XX}\}$$

$$R_{XX} = \mathrm{IFT}\{S_{XX}\}$$

Wide-Sense Cyclo-Stationary Random Process

- A random process $X(t)$ is wide sense cyclo-stationary if there exists a positive integer T such that

$$\mu_X(t) = \mu_X(t+T) \text{ for all } t \text{ and}$$

$$K_{XX}(t_1, t_2) = K_{XX}(t_1 + T, t_2 + T) \text{ for all } t_1 \text{ and } t_2$$

(also $R_{XX}(t_1, t_2) = R_{XX}(t_1 + T, t_2 + T)$ for all t_1 and t_2).

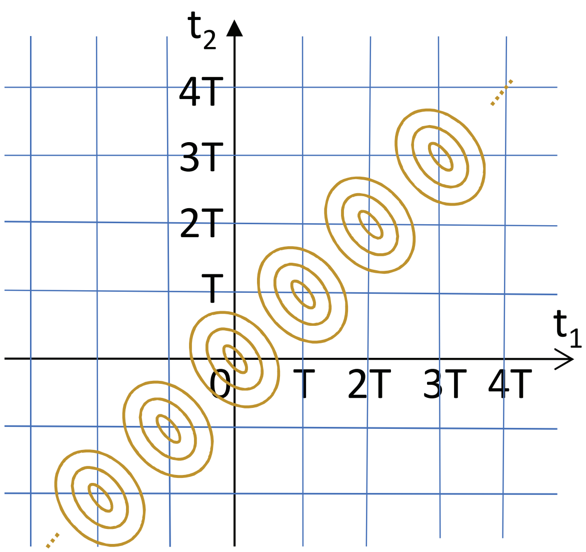


Fig. A contour plot of $K_{XX}(t_1, t_2)$ or $R_{XX}(t_1, t_2)$ of a WS cyclo-stationary random process $X(t)$

- As $K_{XX}(t+\tau, t)$ and $R_{XX}(t+\tau, t)$ are periodic in t with period T, the average correlation function is $\bar{R}_{XX}(\tau) \triangleq \frac{1}{T}\int_0^T R_{XX}(t+\tau, t)dt$.

- Average PSD can be given as the Fourier transform of $\bar{R}_{XX}(\tau)$.

Input-Output Relationship in a Linear Time-Invariant (LTI) System

- LTI system impulse response $= h(t)$; frequency response $= H(w)$
- For an input WSS process $X(t)$, the output $Y(t) = X(t) * h(t)$ is WSS
- Cross-correlations and Cross-PSD:

$$R_{XY}(\tau) = R_{XX}(\tau) * h^*(-\tau) \quad \Leftrightarrow \quad S_{XY}(w) = S_{XX}(w)\ H^*(w)$$
$$R_{YX}(\tau) = h(\tau) * R_{XX}(\tau) \quad \Leftrightarrow \quad S_{YX}(w) = H(w)\ S_{XX}(w)$$

- Auto-correlation and PSD:

$$R_{YY}(\tau) = R_{YX}(\tau) * h^*(-\tau) \quad \Leftrightarrow \quad S_{YY}(w) = S_{YX}(w)\ H^*(w)$$
$$R_{YY}(\tau) = h(\tau) * R_{XX}(\tau) * h^*(-\tau) \quad \Leftrightarrow \quad S_{YY}(w) = |H(w)|^2\ S_{XX}(w)$$
$$R_{YY}(\tau) = g(\tau) * R_{XX}(\tau) \quad \Leftrightarrow \quad S_{YY}(w) = G(w)\ S_{XX}(w)$$

  where $g(\tau) = h(\tau) * h^*(-\tau)$ and $G(w) = |H(w)|^2$.

- Output Mean: $\mu_Y = H(0)\ \mu_X$
- Output Power: $E[|Y(t)|^2] = R_{YY}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(w)\ dw$
- Output Variance: $\sigma_Y^2 = R_{YY}(0) - |\mu_Y|^2$

Random Sequences

- Random sequence: $X[n, \xi]$ or simply $X[n]$ (or X_n) where for each n, $X[n]$ is an RV.
- If n is related to time index, it is a discrete-time random process.
- Nth order CDF: $F_X(x_n, \ldots, x_{n+N-1}; n, \ldots, n+N-1)$ $\triangleq P[X[n] \leq x_n, \ldots, X[n+N-1] \leq x_{n+N-1}]$
- Nth order PDF or PMF: $f_X(x_n, \ldots, x_{n+N-1}; n, \ldots, n+N-1)$ or $P_X[x_n, \ldots, x_{n+N-1}; n, \ldots, n+N-1]$
- Mean function: $\mu_X[k] = \mu_{X_k} = E[X[k]]$
- Autocorrelation function: $R_{XX}[k, l] \triangleq E[X[k]X^*[l]]$
- Autocovariance function: $K_{XX}[k, l] \triangleq E[(X[k] - \mu_X[k])(X[l] - \mu_X[l])^*] = R_{XX}[k, l] - \mu_X[k]\mu_X^*[l]$
- Variance function: $\sigma_X^2[k] \triangleq K_{XX}[k, k] = \mathrm{Var}(X[k])$

SSS and WSS Random Sequences and Their PSDs

- The random sequence $X[n]$ is strict-sense stationary (SSS) if

$$
\begin{aligned}
&F_X(x_n, x_{n+1}, \dots, x_{n+N-1};\ n, n+1, \dots, n+N-1)\\
&\quad = F_X(x_n, x_{n+1}, \dots, x_{n+N-1};\ n+k, n+k+1, \dots, n+k+N-1)\\
&\text{for all } N > 1, \text{for all } -\infty < k < \infty, \text{and for all } x_n \text{ through } x_{n+N-1}.
\end{aligned}
$$

- The random sequence $X[n]$ is wide-sense stationary (WSS) if

$$
\begin{aligned}
&\mu_X[n] = \mu_X,\ \forall n,\\
&K_{XX}[k,l] = K_{XX}[k+n, l+n],\ \forall k, l, n \quad (\text{shift} - \text{invariant})\\
(\text{also}\ \ &R_{XX}[k,l] = R_{XX}[k+n, l+n],\ \forall k, l, n)
\end{aligned}
$$

- SSS $\Rightarrow$ WSS
- PSD $S_{XX}(\omega)$ of WSS sequence $X[n]$:

$$
S_{XX}(\omega) \triangleq \sum_{m=-\infty}^{\infty} R_{XX}[m]\, e^{-j\omega m}, \quad -\pi \le \omega \le \pi
$$

$$
R_{XX}[m] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{XX}(\omega)\, e^{j\omega m}\, d\omega
$$

Markov Random Sequences

- A Markov random sequence $X[n]$, defined for $n \geq 0$, satisfies $f_X(x_{n+k}|x_n, \ldots, x_0) = f_X(x_{n+k}|x_n)$ for a continuous sequence and $P_X(x_{n+k}|x_n, \ldots, x_0) = P_X(x_{n+k}|x_n)$ for a discrete sequence for all $x_0, \ldots, x_n, x_{n+k}$, for all $n > 0$, and for all integers $k \geq 1$. (Holding for just $k = 1$ is sufficient)

- Finite-State Markov Chain (FSMC): MC where $X[n]$ takes on values from a finite set of size K:
    - states at time n are $\{S_n = X[n]\}$; the number of states $M = K$
    - state probability vector at time n: $\boldsymbol{p}[n] = [p_1[n], p_2[n], \ldots, p_M[n]]$
    - state transition probability matrix from time $n-1$ to n: $\boldsymbol{P}[n]$ with $(i,j)^{\text{th}}$ element $= P[S_n = \text{state } j | S_{n-1} = \text{state } i]$

- Homogeneous MC $\Leftrightarrow$ $\boldsymbol{P}[n] = \boldsymbol{P}$ is independent of n, where $P[S_n = \text{state } j | S_{n-1} = \text{state } i] = p_{ij}$.

- For a homogeneous MC, $\boldsymbol{p}[n] = \boldsymbol{p}[n-1]\boldsymbol{P} = \boldsymbol{p}[0]\boldsymbol{P}^n$ and the steady-state probability vector $\boldsymbol{p}$, if exists, satisfies $\boldsymbol{p}(\boldsymbol{I} - \boldsymbol{P}) = \mathbf{0}$.

MC State Diagram and Trellis Diagram

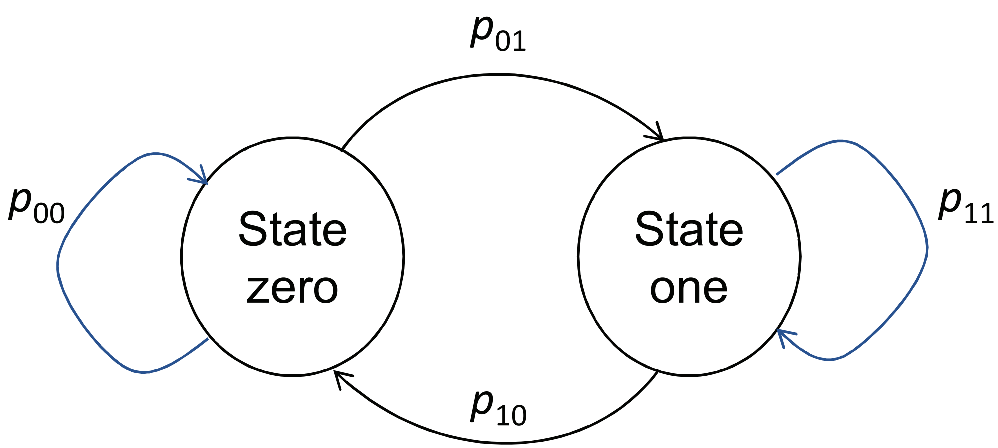


Fig. State transition diagram of a general MC

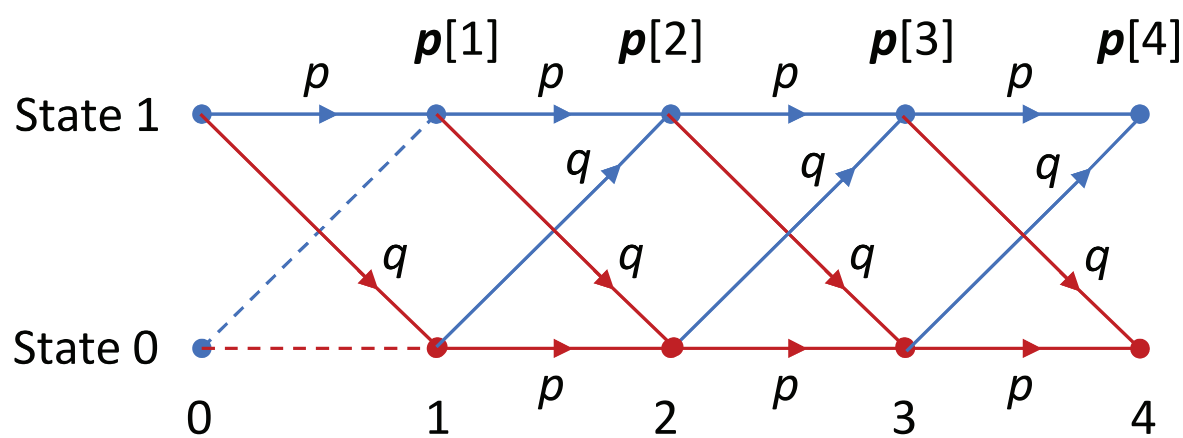


Fig. Trellis diagram of a two-state MC conditioned on $X[0] = 1$ (dashed lines are not needed) where $p_{11} = p_{00} = p$, $p_{10} = p_{01} = 1 - p = q$

Some Random Sequences

- **Gaussian Random Sequence**: if Nth order CDF's (pdf's) are jointly Gaussian for all $N \geq 1$; i.e., $X[1], \cdots, X[N]$ are jointly Gaussian.
  $f_X(x; n) = N(\mu_X[n], \sigma_X^2[n])$
- **The Random Walk Sequence**: the running sum of the number of successes minus the number of failures in n **independent** trials times a step size s:
  $X[n] = \sum_{k=1}^{n} W[k]$ with $X[0] = 0$, and
  $W[k] = s$ for success and $W[k] = -s$ for failure at kth trial.
  $P[X[n] = rs] = P[\frac{n+r}{2} \text{ successes}]$ for a non-negative integer $\frac{n+r}{2}$ and
  $P[X[n] = rs] = 0$ otherwise.
- **Independent Increments**: A random sequence is said to have independent increments if for all integer parameters $n_1 < n_2 < \ldots < n_N$, the increments $X[n_1], X[n_2] - X[n_1], \ldots, X[n_N] - X[n_{N-1}]$ are jointly independent for all integers $N > 1$. (e.g., random walk sequence)
- **Gauss Markov Random Sequence:** defined for $n \geq 0$,
  $f_X(x; 0) = N(0, \sigma_0^2)$,
  $f_X(x_n | x_{n-1}; n, n-1) = N(\rho x_{n-1}, \sigma_W^2)$ with $|\rho| < 1$, and
  $f_X(x; n) = N(0, \sigma_X^2[n])$ with $\sigma_X^2[n] = \left(\sum_{i=0}^{n-1} \rho^{2i}\right) \sigma_W^2 + \rho^{2n} \sigma_0^2$.

Input-Output Relationship in a Discrete-Time LTI System

- Discrete-time LTI system impulse response $= h[n]$;
  its frequency response $= H(w)$
- For an input WSS random sequence $X[n]$, the output random sequence $Y[n] = h[n] * X[n]$ is WSS
- Cross-correlations and Cross-PSD:

$$R_{XY}[m] = R_{XX}[m] \; * \; h^*[-m] \quad \Leftrightarrow \quad S_{XY}(w) = S_{XX}(w)\; H^*(w)$$

$$R_{YX}[m] = h[m] \; * \; R_{XX}[m] \quad \Leftrightarrow \quad S_{YX}(w) = H(w)\; S_{XX}(w)$$

- Auto-correlation and PSD:

$$R_{YY}[m] = R_{YX}[m] \; * \; h^*[-m] \quad \Leftrightarrow \quad S_{YY}(w) = S_{YX}(w)\; H^*(w)$$

$$R_{YY}[m] = h[m] * h^*[-m] \; * \; R_{XX}[m] \quad \Leftrightarrow \quad S_{YY}(w) = |H(w)|^2\; S_{XX}(w)$$

$$R_{YY}[m] = g[m] \; * \; R_{XX}[m] \quad \Leftrightarrow \quad S_{YY}(w) = G(w)\; S_{XX}(w)$$

  where $g[m] = h[m] * h^*[-m]$ and $G(w) = |H(w)|^2$.
- Output Mean: $\mu_Y = H(0)\; \mu_X$
- Output Power: $E[|Y[n]|^2] = R_{YY}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{YY}(w)\; dw$
- Output Variance: $\sigma_Y^2 = R_{YY}[0] - |\mu_Y|^2$