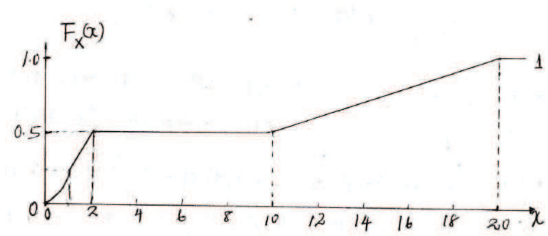
EESC 6349 / MECH 6312
Solutions to Homework 2

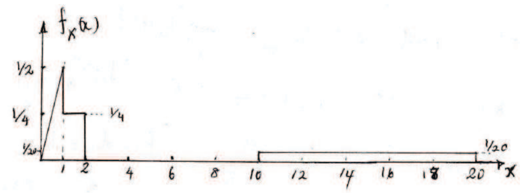
Instructor: Dr. Minn

Problem 2.2 (3rd Ed.), 2,3 (4th Ed.)
(a)



(b) By differentiating

$$f_X(x) = \begin{cases} x/2 & , \quad 0 \leq x \leq 1 \\ 1/4 & , \quad 1 \leq x \leq 2 \\ 0 & , \quad 2 \leq x \leq 10 \\ 1/20 & , \quad 10 \leq x \leq 20 \\ 0 & , \quad x \geq 20 \end{cases}$$



So, $$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{4}\,dx + \int_{10}^{20} \frac{1}{20}\,dx = 1$$

(c) Let X be the waiting time, Then

1. $P[X \geq 10] = 1 - P[X \leq 10] = 1 - F_X(10) = 1/2$
2. $P[X \leqq 5] = F_X(5) = 1/2$
3. $P[5 \leq X \leq 10] = \int_5^{10} f_X(x)\,dx = 0$
4. $P[X = 1] = 0.$

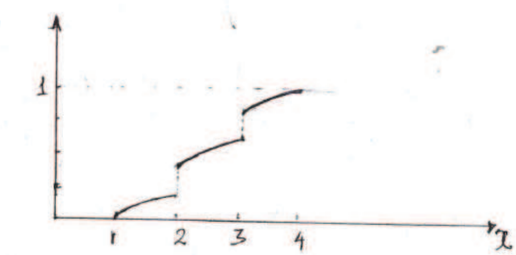
Problem 2.8 (3rd Ed.), 2.10 (4th Ed.)

(a) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$, Therefore

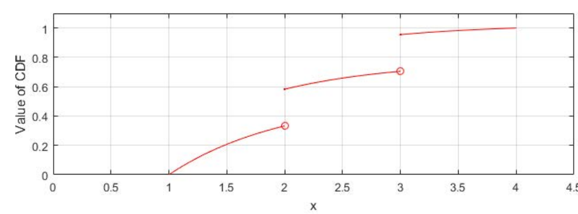
$$A\int_1^4 e^{-x}\,dx+\frac{1}{4}\int_{-\infty}^{\infty}\delta(x-2)\,dx+\frac{1}{4}\int_{-\infty}^{\infty}\delta(x-3)\,dx = A[e^{-1}-e^{-4}]+\frac{1}{4}+\frac{1}{4}=1$$

$$A = 1.43$$

(b) The sketch of CDF is



The accurate CDF is



The CDF of X is given by

$$F_X(x) = \begin{cases} 0, & x<1 \\ \frac{e^{-1}-e^{-x}}{2(e^{-1}-e^{-4})}, & 1\le x<2 \\ \frac{e^{-1}-e^{-x}}{2(e^{-1}-e^{-4})}+\frac{1}{4}, & 2\le x<3 \\ \frac{e^{-1}-e^{-x}}{2(e^{-1}-e^{-4})}+\frac{1}{2}, & 3\le x<4 \\ 1, & x\ge 4. \end{cases}$$

(c)

$$f_X(x) = 1.43e^{-x}[u(x-1)-u(x-4)]+\frac{1}{4}\delta(x-2)+\frac{1}{4}\delta(x-3)$$

$$P[2\le X<3] = \int_2^3 1.43e^{-x}\,dx+\frac{1}{4}\int_{2^-}^{3}\delta(x-2)\,dx = \frac{e^{-2}-e^{-3}}{2(e^{-1}-e^{-4})}+\frac{1}{4}.$$

(d)

$$P[2< X\le 3] = \int_2^3 1.43e^{-x}\,dx+\frac{1}{4}\int_{2^+}^{3^+}\delta(x-3)\,dx = \frac{e^{-2}-e^{-3}}{2(e^{-1}-e^{-4})}+\frac{1}{4}.$$

(e)

$$F_X(3) = \int_1^3 1.43e^{-x}\,dx+\frac{1}{4}\int_1^{3^+}\delta(x-2)\,dx+\frac{1}{4}\int_1^{3^+}\delta(x-3)\,dx = \frac{e^{-1}-e^{-3}}{2(e^{-1}-e^{-4})}+\frac{1}{4}+\frac{1}{4}.$$

Problem 2.11 (3rd Ed.), 2.17 (4th Ed.)

Let number of bulbs produced by B $= n_B$
number of bulbs produced by A $= n_A$.

$$n_B = 3n_A,\ n_A+3n_A = n,\ 4n_A = n,\ n_A = n/4,\ n_B = \frac{3n}{4}$$

$$P[A] = n_A/n = 1/4, \quad P[B] = 3/4$$

$$F_X(x|A) = (1-e^{-x/5})u(x), \quad F_X(x|B) = (1-e^{-x/2})u(x)$$

$$F_X(x) = F_X(x|A)P[A]+F_X(x|B)P[B] = (1-e^{-x/5})\frac{1}{4}u(x)+(1-e^{-x/2})\frac{3}{4}u(x)$$

$$F_X(2) = 0.56, \quad F_X(5) = 0.85, \quad F_X(7) = 0.92$$

$$P[\text{Burns at least in 2 months}] = 1-F_X(2) = 0.44$$

$$P[\text{Burns at least in 5 months}] = 1-F_X(5) = 0.15$$

$$P[\text{Burns at least in 5 months}] = 1-F_X(7) = 0.08$$

Problem 2.14 (3rd Ed.), 2.22 (4th Ed.)

It could be seen clearly that X and Y are independent. By observing decoupled random variables x and y in the joint pdf, we can write the marginal pdf's as

$f_X(x) = Ae^{-\frac{1}{2}(x/3)^2}u(x)$ and $f_Y(y) = Be^{-\frac{1}{2}(y/2)^2}u(y)$.

Then, solving $\int_0^\infty f_X(x)dx = 1$ gives $A = \frac{2}{3\sqrt{2\pi}}$.

Similarly, with $B = \frac{2}{2\sqrt{2\pi}}$, $\int_0^\infty f_Y(y)dy = 1$.

And with $AB = \frac{1}{3\pi}$, we have

$f_X(x)f_Y(y) = ABe^{-\frac{1}{2}(x/3)^2}e^{-\frac{1}{2}(y/2)^2}u(x)u(y) = f_{X,Y}(x,y)$. Hence, X and Y are independent.

Note that you can also obtain $f_X(x)$ and $f_Y(y)$ by $f_X(x) = \int_0^\infty f_{X,Y}(x,y)dy$, and $f_y(y) = \int_0^\infty f_{X,Y}(x,y)dx$.

$P[0 < Y \leq 2] = \frac{2}{2\sqrt{2\pi}}\int_0^2 e^{-\frac{1}{2}(y/2)^2}dy = 2\text{erf}(1)$

$P[0 < X \leq 3] = \frac{2}{3\sqrt{2\pi}}\int_0^3 e^{-\frac{1}{2}(x/3)^2}dx = 2\text{erf}(1)$

Hence, $P[0 < X \leq 3,\ 0 < Y \leq 2] = 4\text{erf}^2(1) = 0.466$.

Problem 2.16 (3rd Ed.), 2.25 (4th Ed.)

Use Baye's formula for conditional pdf's:

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)} \\
f_X(x) &= \frac{1}{2}\text{rect}(\frac{x}{2}) \\
f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_{-\infty}^{\infty} f_{Y|X}(y|x)f_X(x)\,dx \\
&= \int_{-1}^{1} \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(y-x)^2}{2\sigma^2}\right]dx \\
&\text{Let } \xi = \frac{x-y}{\sigma}, \quad d\xi = \frac{dx}{\sigma}, \text{ then} \\
f_Y(y) &= \frac{1}{2}\int_{\frac{-1-y}{\sigma}}^{\frac{1-y}{\sigma}} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\xi^2}\,d\xi = \frac{1}{2}\left[\text{erf}(\frac{1-y}{\sigma}) - \text{erf}(\frac{-1-y}{\sigma})\right] \\
\text{and} \quad \text{erf}(x) &= -\text{erf}(-x).\text{Hence,} \\
f_Y(y) &= \frac{1}{2}\left[\text{erf}(\frac{y+1}{\sigma}) - \text{erf}(\frac{y-1}{\sigma})\right] \\
\text{Then, } f_{X|Y}(x|y) &= \frac{\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(y-x)^2}{2\sigma^2}\right]\text{rect}(\frac{x}{2})}{\text{erf}(\frac{y+1}{\sigma}) - \text{erf}(\frac{y-1}{\sigma})}
\end{aligned}
$$

Problem 2.20 (3rd Ed.), 2.30 (4th Ed.)
(This is a rather classic problem in detection theory.)

$$P[A|M] = P[X \geq 0.5|M] = \frac{1}{\sqrt{2\pi}} \int_{0.5}^{\infty} e^{-\frac{1}{2}(x-1)^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-0.5}^{\infty} e^{-\frac{1}{2}(y)^2} dy = \frac{1}{2} + \text{erf}(0.5) = 0.69$$

$$P[A|M^C] = \frac{1}{\sqrt{2\pi}} \int_{0.5}^{\infty} e^{-\frac{1}{2}(x)^2} dx = \frac{1}{2} - \text{erf}(0.5) = 0.31$$

$$P[A^C|M^C] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.5} e^{-\frac{1}{2}(x)^2} dx = \frac{1}{2} + \text{erf}(0.5) = 0.69$$

$$P[A^C|M] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-0.5} e^{-\frac{1}{2}(y)^2} dy = \frac{1}{2} - \text{erf}(0.5) = 0.31$$

Problem 2.23 (3rd Ed.), 2.33 (4th Ed.)
From the data $\lambda = 9 \times 10^6$photons/second During the counting interval $\Delta t = 10^{-6}$ Second. Hence $\lambda \Delta t = 9$.

$$\begin{aligned} P[\text{False alarm in counting interval}] &= P[\text{0 photon in counting interval}] + P[\text{1 photon in counting interval}] \\ &= \frac{e^{-9}(9)^0}{0!} + \frac{e^{-9}(9)^1}{1!} = 10 \times e^{-9} \approx 1.235 \times 10^{-3} \\ P[\text{at least 1 false alarm in } 10^6 \text{ tries}] &= 1 - P[\text{0 false alarm in } 10^6 \text{ tries}] \\ &= 1 - \binom{10^6}{0} (1.235 \times 10^{-3})^0 (1 - 1.235 \times 10^{-3})^{10^6} = 1 - (0.99877)^{10^6} \\ &\approx 1 \end{aligned}$$

The following uses a different value of λ as $\lambda = 30 \times 10^6$photons/second.
During the counting interval $\Delta t = 10^{-6}$ Second. Hence $\lambda \Delta t = 30$.

$$\begin{aligned} P[\text{False alarm in counting interval}] &= P[\text{0 photon in counting interval}] + P[\text{1 photon in counting interval}] \\ &= \frac{e^{-30}(30)^0}{0!} + \frac{e^{-30}(30)^1}{1!} = 31 \times e^{-30} \approx 29.6 \times 10^{-13} \\ P[\text{at least 1 false alarm in } 10^6 \text{ tries}] &= 1 - P[\text{0 false alarm in } 10^6 \text{ tries}] \\ &= 1 - \binom{10^6}{0} (29.6 \times 10^{-13})^0 (1 - 29.6 \times 10^{-13})^{10^6} \\ &\approx 0.3 \times 10^{-5} \end{aligned}$$

Note: You can use Poisson approximation as follows:
$a = np = 10^6 \times 29.6 \times 10^{-13} = 2.96 \times 10^{-6}, \quad k = 0,$
P[0 false alarm in 10^6 tries] $= \frac{a^k}{k!} e^{-a} = e^{-2.96 \times 10^{-6}}$
P[at least 1 false alarm in 10^6 tries]$= 1 -$ P[0 false alarm in 10^6 tries]
$= 1 - e^{-2.96 \times 10^{-6}} \approx 3 \times 10^{-6}$.

Problem 2.27 (3rd Ed.), 2.38 (4th Ed.)

The pdf of the failure time X is $f_X(t) = \alpha(t) e^{-\int_0^t \alpha(\tau) d\tau} = \mu e^{-\mu t}$ in this case.
Assume μ is measured in $(\text{hours})^{-1}$.
If $A = \{\text{event of failure in 100 hours or less}\}$, then

$$P[A] = P[X \leq 100] = \int_0^{100} \mu e^{-\mu t} dt = 1 - e^{-\mu 100} \leq 0.05$$
$$\Rightarrow e^{-\mu 100} \geq 0.95$$

Taking log of both side yields $\mu \leq 5.13 \times 10^{-4}$.