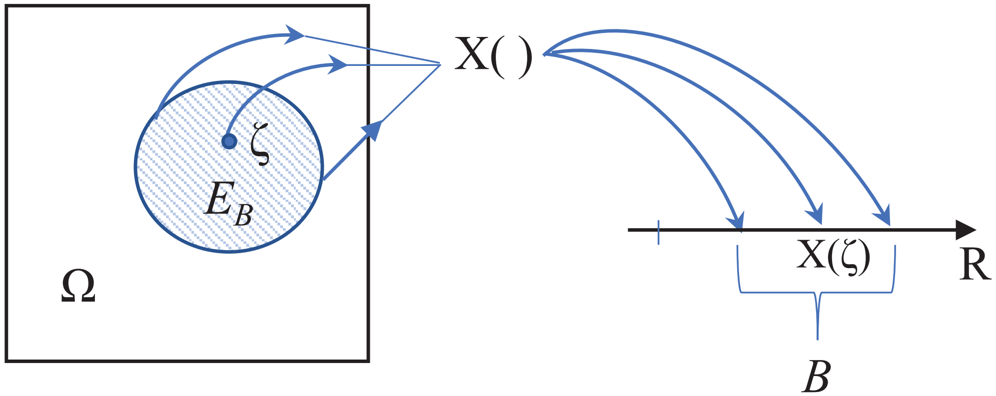
Random Variables

- Characterization of RVs (Types, CDF, PMF, pdf, Mean, Mode, Median, Variance, Expectation)
- Common Discrete RVs
- Common Continuous RVs
- Joint CDF, PMF, pdf, expectation
- Conditional CDF, PMF, pdf, Expectation; Bayes Formula
- Orthogonal RVs, Correlated RVs, and Independent RVs
- Jointly Gaussian RVs
- Failure Rate, Poisson Transform
- Asymptotic Relationship

Characterization of Random Variables

- Random Variable (RV) X is a function $X(\zeta)$ which maps $\zeta \in \Omega$ to a real number (in a Borel set B, say, $X \in S_X$).



The event $\{\zeta : X(\zeta) \leq x\}$ is abbreviated as $\{X \leq x\}$.

- Types of RV:
  - Continuous RV: takes value (number) from one or more continuous ranges (e.g., $S_X = \{[-2,-1],[2,4]\}$)
  - Discrete RV: takes value from discrete points (e.g., $S_X = \{0.5, 2, 10\}$ or $S_X = \{$ all positive integers$\}$)
  - Mixed RV: takes value from continuous range(s) as well as discrete point(s) with non-zero probability (e.g., $S_X = \{\{2\},[-1,1]\}$ with $P[X = 2] > 0$)

CDF, PMF, pdf

- Cumulative Distribution Function (CDF):
  $\boxed{F_X(x) = P[\{\zeta : X(\zeta) \leq x\}] = P[X \leq x]}$
  - $F_X(\infty) = 1, \ F_X(-\infty) = 0$
  - $x_1 \leq x_2 \rightarrow F_x(x_1) \leq F_X(x_2)$
  - $F_X(x)$ is continuous from the right, i.e., $F_X(x) = \lim_{\epsilon \to 0} \ F_X(x + \epsilon), \ \epsilon > 0.$
  - $0 \leq F_X(x) \leq 1.$

- Probability Mass Function (PMF) for Discrete RV: $\boxed{P_X(x) = P[X = x]}$
  where $0 \leq P_X(x) \leq 1$ and $\sum_x P_X(x) = 1$

- Probability Density Function (pdf): $\boxed{f_X(x) = \frac{dF_X(x)}{dx}}$
  where $f_X(x) \geq 0$ and $\int_{-\infty}^{\infty} f_X(x) dx = 1$

- pdf for discrete RV X with $P_X(x_i) = p_i$ for $i = 1, \cdots, K$ and 0 elsewhere:
  $\boxed{f_X(x) = \sum_{i=1}^{K} p_i \ \delta(x - x_i)}$

- For discrete RV X and $a \leq b$:
  - $\boxed{F_X(x) = P[X \leq x] = \sum_{x_i \leq x} P_X(x_i)}$
  - $P[a < X \leq b] = F_X(b) - F_X(a) = \sum_{a < x_i \leq b} P_X(x_i)$
  - $P[a \leq X \leq b] = F_X(b) - F_X(a) + P[X = a] = \sum_{a \leq x_i \leq b} P_X(x_i)$
  - $P[a \leq X < b] = F_X(b) - F_X(a) + P[X = a] - P[X = b] = \sum_{a \leq x_i < b} P_X(x_i)$
  - $\boxed{P_X[x_i] = F_X(x_i) - F_X(x_i^-), \ (\text{or } F_X(x_i) - F_X(x_{i-1}), \ \text{with } x_i > x_{i-1}, \forall i)}$
- For continuous RV or mixed RV X and $a \leq b$:
  - $\boxed{F_X(x) = P[X \leq x] = \int_{-\infty}^{x} f_X(t)dt}$
  - $P[a < X \leq b] = F_X(b) - F_X(a) = \int_a^b f_X(t)dt - P[X = a]$
  - $P[a \leq X \leq b] = F_X(b) - F_X(a) + P[X = a] = \int_a^b f_X(t)dt$
  - $P[a \leq X < b] = F_X(b) - F_X(a) + P[X = a] - P[X = b]$
    $= \int_a^b f_X(t)dt - P[X = b]$
  - For continuous RV, $P[X = x] = 0$

Expectation

- Expected value of an RV X: $E[X]$

$$\boxed{E[X] = \sum_i x_i P_X(x_i)} \quad \text{for Discrete RV}$$

$$\boxed{E[X] = \int_{-\infty}^{\infty} x f_X(x) dx} \quad \text{for Continuous/Discrete/Mixed RV}$$

- Expected value of a function of RV X: $E[g(X)]$

$$\boxed{E[g(X)] = \sum_i g(x_i) P_X(x_i)} \quad \text{for Discrete RV}$$

$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx} \quad \text{for Continuous/Discrete/Mixed RV}$$

Mean, Mode and Median

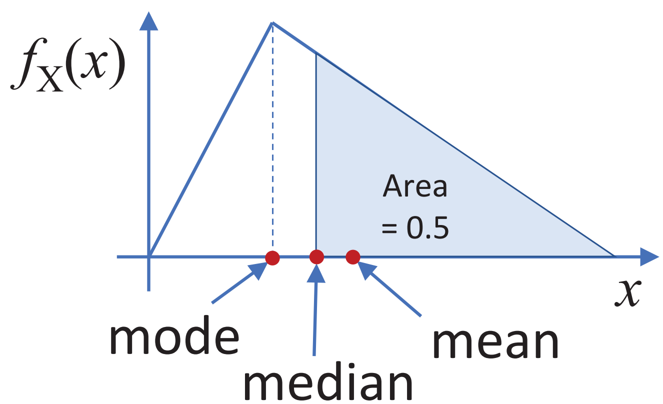
- $\boxed{\text{Mean of } X \triangleq E[X]}$, (also denoted by $\bar{X}$, μ_X or μ):

$$E[X] = \sum_i x_i P_X(x_i) \quad \text{for Discrete RV}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx \quad \text{for Continuous/Discrete/Mixed RV}$$

- Mode: x_{mode} satisfying $P_X(x_{\text{mode}}) \geq P_X(x)$ or $f_X(x_{\text{mode}}) \geq f_X(x)$, $\forall\ x$.
- Median: x_{med} satisfying $P[X < x_{\text{med}}] = P[X > x_{\text{med}}]$.

  Another definition: x_{med} satisfying $F_X(x_{\text{med}}) = 0.5$.



Variance

- Variance of X = $\boxed{\sigma_X^2 \triangleq E[(X-\mu_X)^2]}$:

$$\sigma_X^2 = \sum_{x_i}(x_i-\mu_X)^2 P_X(x_i) \ \text{ for Discrete RV}$$

$$\sigma_X^2 = \int_{-\infty}^{\infty}(x-\mu_X)^2 f_X(x)dx \ \text{ for Continuous/Discrete/Mixed RV}$$

$$\boxed{\sigma_X^2 = E[X^2]-\mu_X^2}$$

If $Y = g(X)$,

$$\boxed{E[Y] = E[g(X)] = \int_{-\infty}^{\infty} y f_Y(y)dy = \int_{-\infty}^{\infty} g(x) f_X(x)dx}$$

$$\sigma_X^2 = E[g(X)] \text{ with } g(X) = (X-\mu_X)^2$$

- **Example:** The random variable X takes the values 1 and 0 with probabilities p and $q = 1 - p$, respectively. $E[X] =?$, $\sigma^2 =?$

$$E[X] = \sum_i x_i P_X(x_i) = 1 \times p + 0 \times q = p$$

$$E[X^2] = \sum_i x_i^2 P_X(x_i) = 1^2 \times p + 0^2 \times q = p$$

$$\sigma^2 = E[X^2] - (E[X])^2 = p - p^2 = pq$$

- **Example:** Mean of Exponential R.V.

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^{\infty} x \lambda e^{-\lambda x} dx \\
&= \left[-x e^{-\lambda x} \right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx, \quad \text{(integration by parts)} \\
&= \lim_{x \to \infty} (-x e^{-\lambda x}) - 0 + \left[-\frac{e^{-\lambda x}}{\lambda} \right]_0^{\infty} \\
&= 0 - \lim_{x \to \infty} \frac{e^{-\lambda x}}{\lambda} + \frac{1}{\lambda} = \frac{1}{\lambda}
\end{aligned}$$

- **Example (pdf $\rightarrow$ CDF):** Find the CDF of X if its pdf is

$$f_X(x) = \begin{cases} 0.2, & -4 \leq x \leq -2 \\ 0.3, & 2 \leq x \leq 4 \\ 0, & \text{otherwise.} \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(u) du$$

Note that $S_X = \{[-4, -2], [2, 4]\}$.
For $x < -4$, $F_X(x) = 0$. For $x \geq 4$, $F_X(x) = 1$.
For $-4 \leq x \leq -2$, $F_X(x) = \int_{-4}^{x} 0.2 \; du = 0.2(x + 4)$.
For $-2 < x < 2$, $F_X(x) = \int_{-4}^{-2} 0.2 \; du = 0.4$.
For $2 \leq x < 4$, $F_X(x) = \int_{-4}^{-2} 0.2 \; du + \int_{2}^{x} 0.3 \; du = 0.4 + 0.3(x - 2)$.

$$F_X(x) = \begin{cases} 0, & x < -4 \\ 0.2(x+4), & -4 \leq x \leq -2 \\ 0.4, & -2 < x < 2 \\ 0.4 + 0.3(x-2), & 2 \leq x < 4 \\ 1, & x \geq 4 \end{cases}$$

- **Example (CDF $\rightarrow$ pdf):** Find the pdf of X if its CDF is

$$F_X(x) = \begin{cases} 0, & x < -4 \\ 0.2(x+4), & -4 \leq x \leq -2 \\ 0.4, & -2 < x < 2 \\ 0.4 + 0.3(x-2), & 2 \leq x < 4 \\ 1, & x \geq 4 \end{cases}$$

Applying $f_X(x) = \frac{dF_X(x)}{dx}$ for each interval of the CDF expression,

$$f_X(x) = \begin{cases} 0, & x < -4 \\ 0.2, & -4 \leq x \leq -2 \\ 0, & -2 < x < 2 \\ 0.3, & 2 \leq x < 4 \\ 0, & x \geq 4 \end{cases}$$

$$= \begin{cases} 0.2, & -4 \leq x \leq -2 \\ 0.3, & 2 \leq x \leq 4 \\ 0, & \text{otherwise.} \end{cases}$$

- **Example (PMF → CDF):** Find the CDF of Y if its PMF is

$$P_Y(y) = \begin{cases} \frac{1}{4}\left(\frac{3}{4}\right)^{y-1}, & y = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

We have $y_i = i$ for $i = 1, 2, \ldots$
For $y < y_1$, $F_Y(y) = 0$.
For $y_1 \leq y < y_2$, $F_Y(y) = P_Y(y_1)$.
For $y_2 \leq y < y_3$, $F_Y(y) = P_Y(y_1) + P_Y(y_2)$.
For $y_n \leq y < y_{n+1}$,
$F_Y(y) = \sum_{i=1}^{n} P_Y(y_i) = \sum_{i=1}^{n} \frac{1}{4}\,(3/4)^{i-1} = 1 - \left(\frac{3}{4}\right)^n$
where we have used the formula $\sum_{i=1}^{n} x^{i-1} = \frac{1-x^n}{(1-x)}$ with $x = \frac{3}{4}$.
Thus, we obtain

$$F_Y(y) = \begin{cases} 1 - \left(\frac{3}{4}\right)^n, & n \leq y < n+1; (n = \text{positive integer}) \\ 0, & y < 1 \end{cases}$$

$$\text{Alternatively,} \quad F_Y(y) = \begin{cases} 1 - \left(\frac{3}{4}\right)^{\lfloor y \rfloor}, & y \geq 1 \\ 0, & y < 1 \end{cases}$$

(Note: CDF is a continuous function.)

- **Example (CDF $\rightarrow$ PMF):** Find the PMF of Y if its CDF is

$$F_Y(y) = \begin{cases} 1 - \left(\frac{3}{4}\right)^{\lfloor y \rfloor}, & y \geq 1 \\ 0, & y < 1 \end{cases}$$

We can express $F_Y(y)$ as

$$F_Y(y) = \begin{cases} 1 - \left(\frac{3}{4}\right)^{n}, & n \leq y < n+1; (n = \text{positive integer}) \\ 0, & y < 1 \end{cases}$$

We have $y_i = i$ for $i = 1, 2, \ldots$
$P_Y(y_n) = F_Y(y_n) - F_Y(y_{n-1}) = \left(\frac{3}{4}\right)^{n-1} - \left(\frac{3}{4}\right)^{n}$. Thus,

$$P_Y(y) = \begin{cases} \left(\frac{3}{4}\right)^{y-1} - \left(\frac{3}{4}\right)^{y}, & y = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Alternatively, using the formula $(1-x)\sum_{i=1}^{n} x^{i-1} = 1 - x^n$ with $x = \frac{3}{4}$,
$F_Y(y_n) = \sum_{i=1}^{n} \frac{1}{4}\ (3/4)^{i-1}$ and hence,
$P_Y(y_n) = F_Y(y_n) - F_Y(y_{n-1}) = \frac{1}{4}\ (3/4)^{n-1}$.
Thus,

$$P_Y(y) = \begin{cases} \frac{1}{4}\left(\frac{3}{4}\right)^{y-1}, & y = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Common Discrete Random Variables

- Bernoulli
- Binomial
- Negative Binomial (Pascal)
- Poisson
- Discrete Uniform
- Geometric
- Hypergeometric
- Zeta (or Zipf)

- Bernoulli
  - PMF:

$$P_X(x) = \begin{cases} p, & x = 1 \\ q = 1 - p, & x = 0 \\ 0 & \text{else.} \end{cases}$$

  - Mean: p
  - Variance: $p(1-p)$
  - Characteristic function: $\Psi_X(\omega) = pe^{j\omega} + q$

- Discrete Uniform
  - PMF:

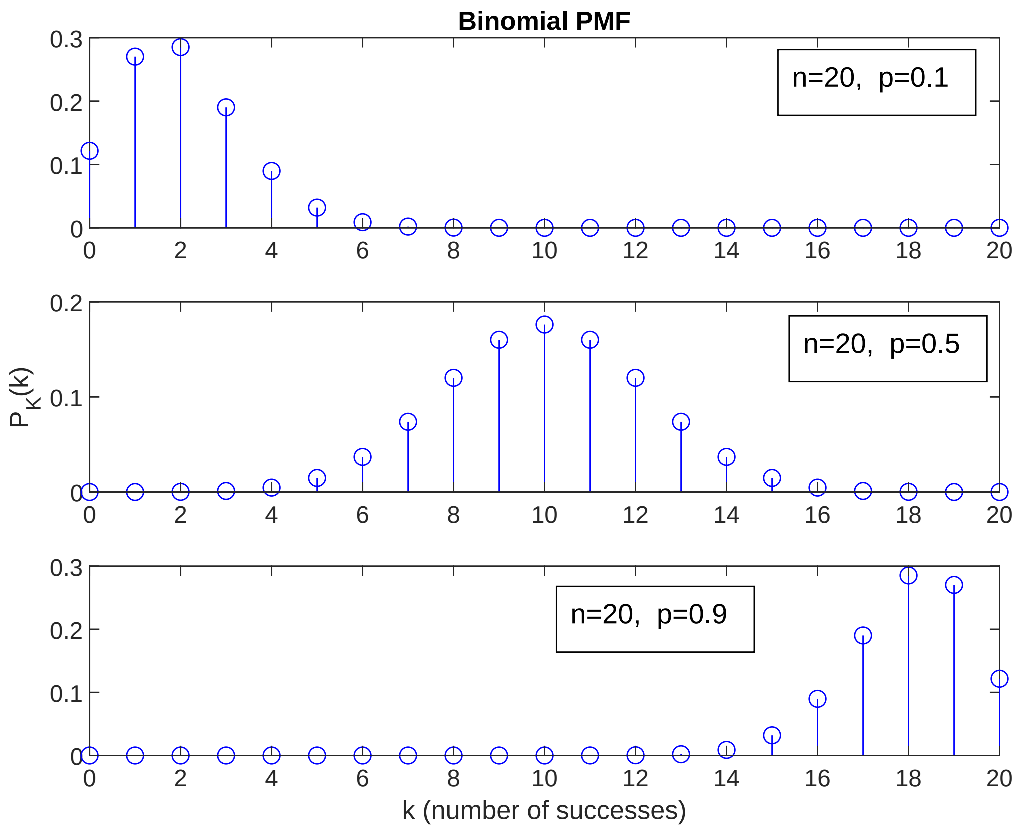
$$P_X(k) = \begin{cases} \frac{1}{N}, & k = 1, 2, \ldots, N \\ 0, & \text{else.} \end{cases}$$

  - Mean: $\frac{N+1}{2}$ (only for the above PMF)
  - Variance: $\frac{N^2-1}{12}$ (only for the above PMF)
  - Characteristic function: $\Psi_X(\omega) = e^{\frac{j(N+1)\omega}{2}} \frac{\sin(N\omega/2)}{\sin(\omega/2)}$ (only for the above PMF)

- Binomial (k successes out of n Bernoulli trials)
  - PMF:

$$P_X(k) = \begin{cases} \binom{n}{k} p^k q^{n-k}, & p+q=1;\ k=0,1,2,\ldots,n \\ 0, & \text{else.} \end{cases}$$

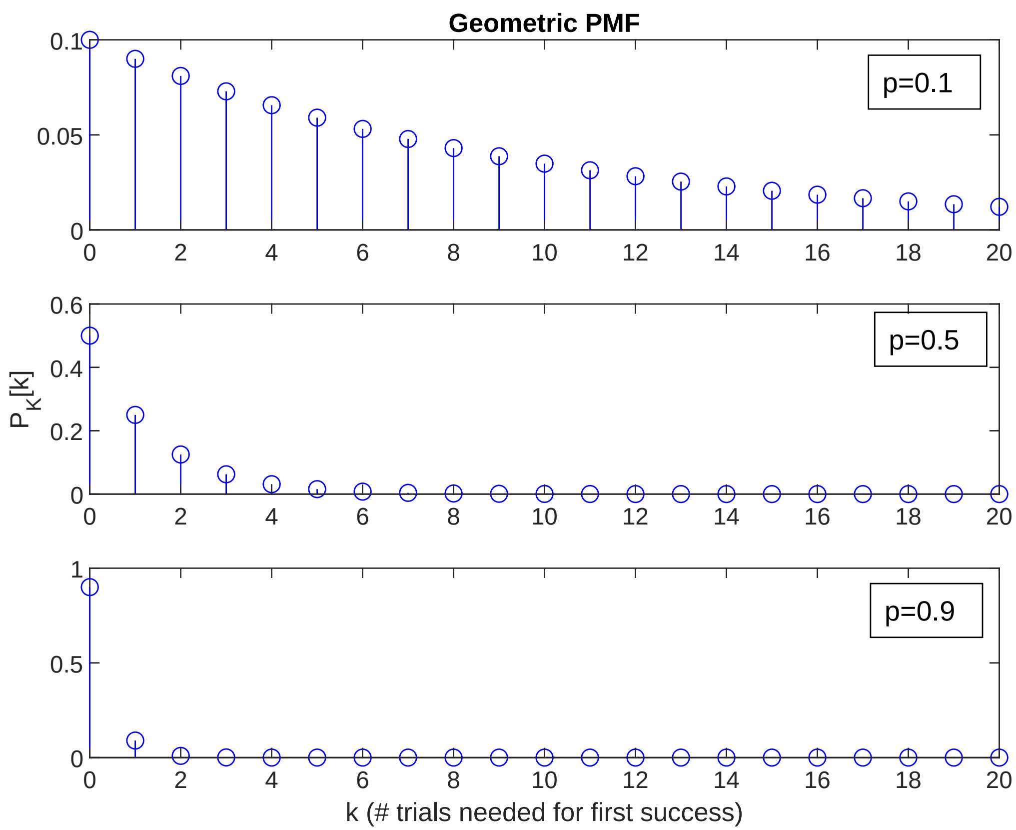
  - Mean: np
  - Variance: npq
  - Characteristic function: $\Psi_X(\omega) = (pe^{j\omega}+q)^n$



- Geometric (# trials needed for the first success in Bernoulli trials)
  - PMF:

$$P_X(k) = \begin{cases} pq^{k-1}, & k = 1, 2, \ldots, \infty;\ p + q = 1 \\ 0, & \text{else.} \end{cases}$$

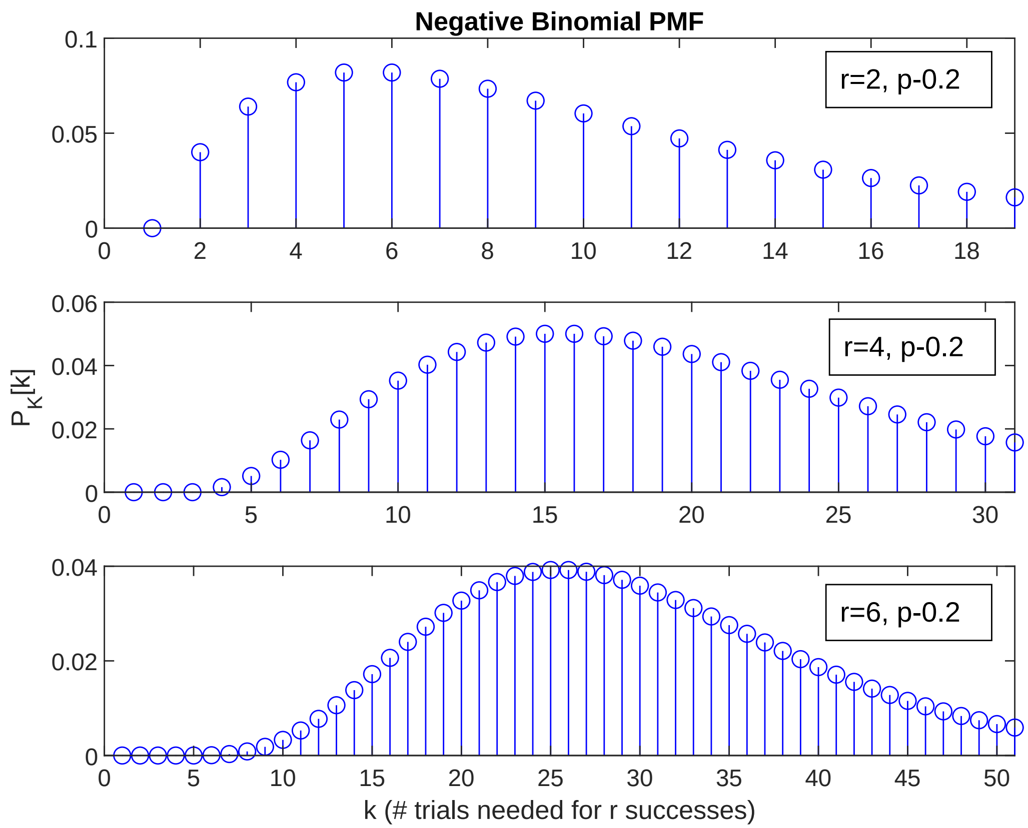
  - Mean: $\frac{1}{p}$
  - Variance: $\frac{q}{p^2}$
  - Characteristic function: $\Psi_X(\omega) = \frac{p}{e^{-j\omega} - q}$
  - Memoryless Property: $P[X > m + n | X > m] = P[X > n]$



- Negative Binomial (Pascal) (# Bernoulli trials needed for r successes)
  - PMF:

$$P_X(k) = \begin{cases} \binom{k-1}{r-1} p^r q^{k-r}, & k = r, r+1, \ldots, \infty \\ 0, & \text{else.} \end{cases}$$

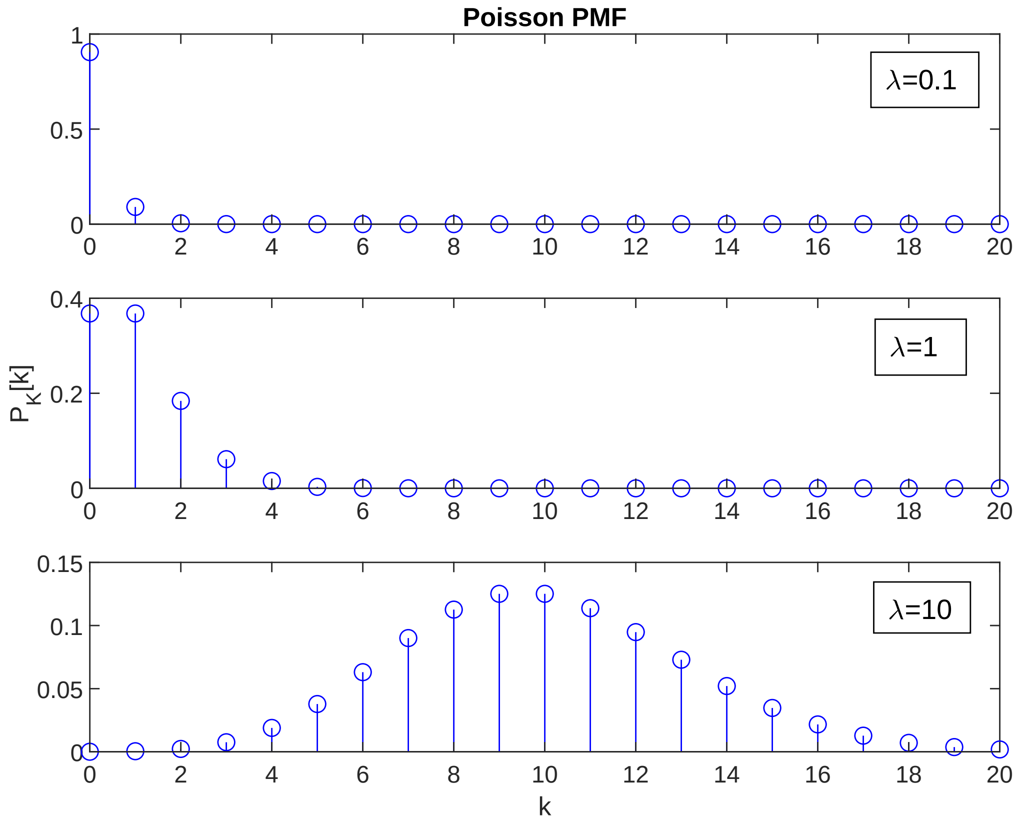
  - Mean: $\frac{r}{p}$
  - Variance: $\frac{rq}{p^2}$



- Poisson (# arrivals within an interval)
  - PMF:

$$P_X(k) = \begin{cases} e^{-\lambda}\frac{\lambda^k}{k!}, & k = 0, 1, 2, \ldots, \infty \\ 0, & \text{else.} \end{cases}$$

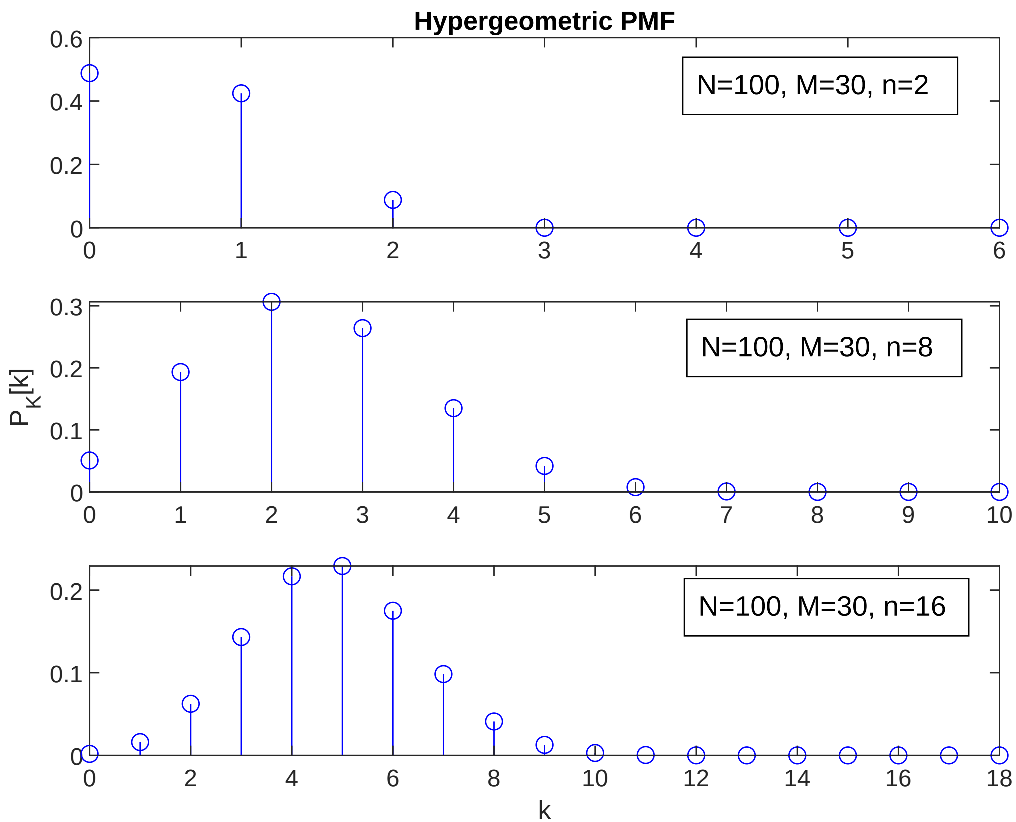
  - Mean: λ
  - Variance: λ
  - Characteristic function: $\Psi_X(\omega) = e^{-\lambda(1-e^{j\omega})}$



- Hypergeometric (# white balls obtained when selecting n balls from N balls composed of M white balls and $N-M$ non-white balls)
  - PMF:

$$P_X(k) = \begin{cases} \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}, & \max(0, M+n-N) \leq k \leq \min(M, n) \\ 0, & \text{else.} \end{cases}$$

  - Mean: $\frac{nM}{N}$
  - Variance: $n\frac{M}{N}(1-\frac{M}{N})(1-\frac{n-1}{N-1})$



- Zeta (or Zipf)
  - PMF:

$$P_X(k) = \begin{cases} \frac{C}{k^{\alpha+1}}, & k = 1, 2, \ldots, ;\ \alpha > 0, \\ 0, & \text{else.} \end{cases}$$

where

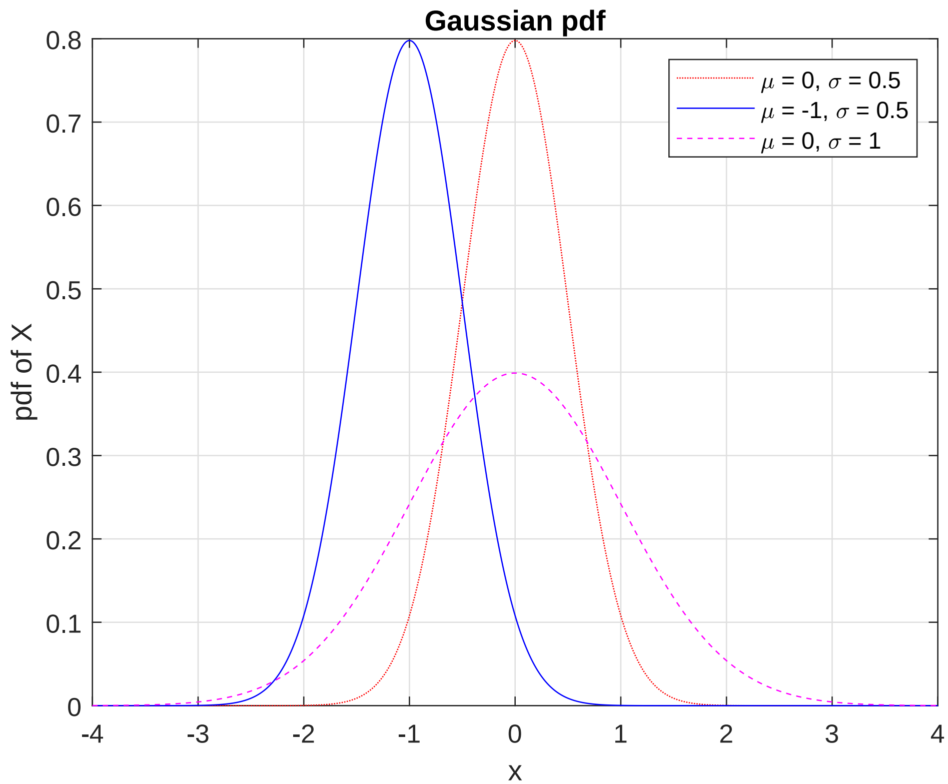
$$C = \left[\sum_{k=1}^{\infty} \left(\frac{1}{k} \right)^{\alpha+1} \right]^{-1}$$

Common Continuous Random Variables

Normal (Gaussian)	Lognormal
Exponential	Weibull
Central Chi-square	Non-central Chi-square
Rayleigh	Generalized Rayleigh
Rice	Generalized Rice
Erlang-k	Gamma
Nakagami m	Student-t
Uniform	Laplace
Pareto	Beta
Maxwell	Cauchy
F distribution	

- Normal (Gaussian)
  - $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$
  - $F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) = 1 - Q\left(\frac{x-\mu}{\sigma}\right)$, (no closed-form)

    where $\Phi(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$ and $Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2} dt$
  - Mean μ and Variance σ^2
  - Characteristic function: $\Psi_X(\omega) = e^{j\mu\omega - \frac{\sigma^2\omega^2}{2}}$



- $Q(0) = 0.5$ and $Q(\infty) = 0$. For $x > 0$, $Q(-x) = 1 - Q(x)$.
- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then
  for $a > \mu$, $P[X > a] = Q(\frac{a-\mu}{\sigma})$
  for $a < \mu$, $P[X > a] = 1 - P[X \leq a] = 1 - Q(\frac{\mu-a}{\sigma})$
  $P[a < X < b] = Q(\frac{a-\mu}{\sigma}) - Q(\frac{b-\mu}{\sigma})$
- Error function (this textbook):

$$\mathrm{erf}(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{1}{2}t^2} dt,$$

$$Q(x) = \frac{1}{2} - \mathrm{erf}(x), \quad x \geq 0; \quad Q(x) = \frac{1}{2} + \mathrm{erf}(-x), \quad x \leq 0.$$

  If $X \sim \mathcal{N}(\mu, \sigma^2)$, then
  $P[a < X \leq b] = \mathrm{erf}(\frac{b-\mu}{\sigma}) - \mathrm{erf}(\frac{a-\mu}{\sigma})$, $P[X > a] = 0.5 -$ $\mathrm{erf}(\frac{a-\mu}{\sigma})$
- Error function (Matlab and some other textbooks):

$$\mathrm{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

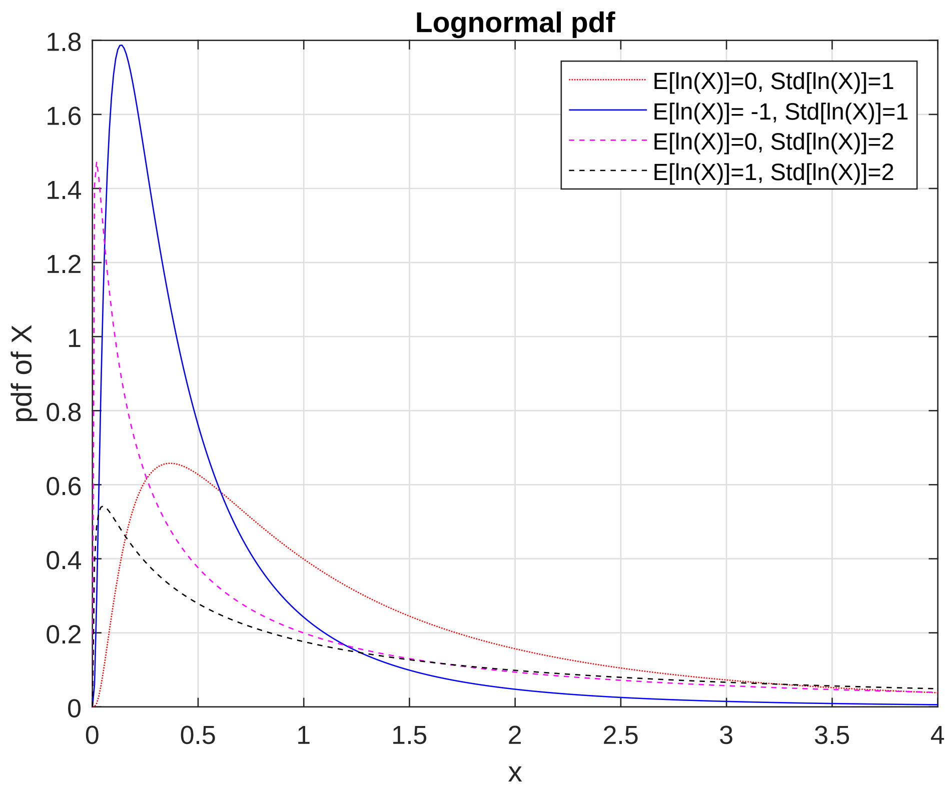
$$Q(x) = \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right), \quad x \geq 0; \quad Q(x) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{-x}{\sqrt{2}}\right), \quad x \leq 0.$$

- Denoting this textbook definition as $\mathrm{erf}_1(x)$ and Matlab definition as $\mathrm{erf}_2(x)$, we have $\mathrm{erf}_1(x) = 0.5\ \mathrm{erf}_2(x/\sqrt{2})$.

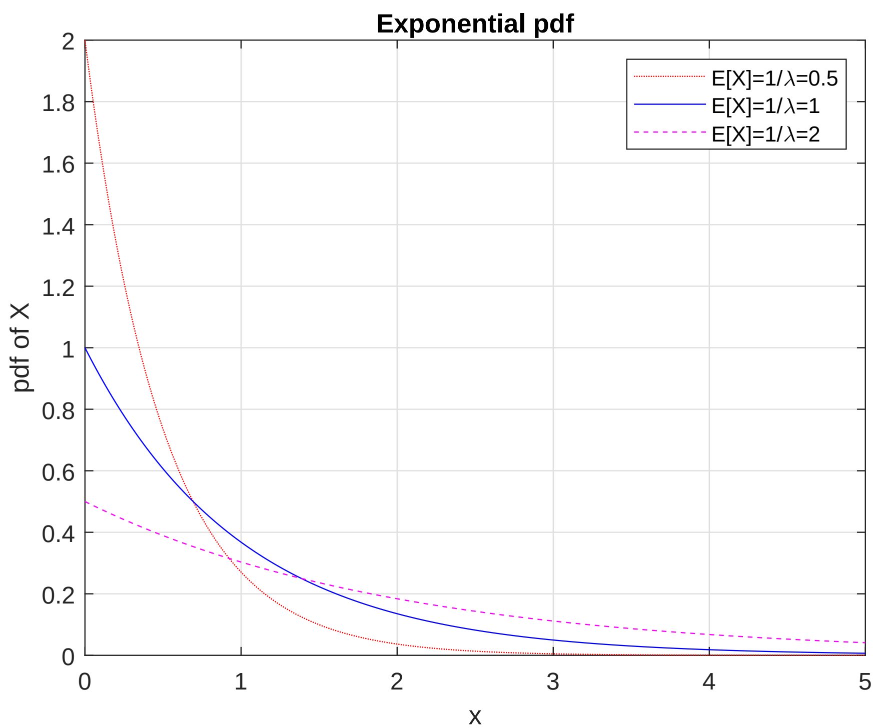
- Lognormal
  - If $X \sim N(\mu,\ \sigma^2)$ and $R = e^X$, then R has a lognormal pdf:

$$f_R(r) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma r} e^{-\frac{(\ln r-\mu)^2}{2\sigma^2}},\ r \geq 0 \\ 0,\ \ r < 0 \end{cases}$$

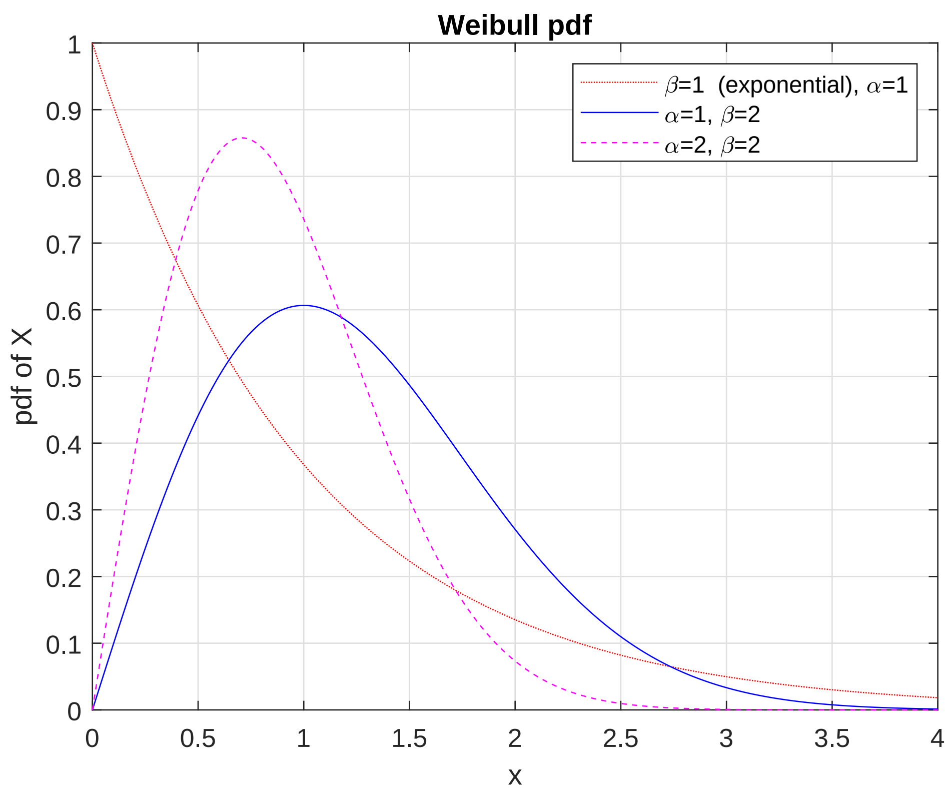
  - Mean: $e^{\mu+\frac{\sigma^2}{2}}$
  - Variance: $e^{\sigma^2+2\mu}(e^{\sigma^2} - 1)$



- Exponential
  - $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, \ x \geq 0, \ \lambda > 0 \\ 0, \ \text{otherwise} \end{cases}$
  - $F_X(x) = (1 - e^{-\lambda x})U(x)$
  - Mean $\frac{1}{\lambda}$ and Variance $\frac{1}{\lambda^2}$
  - Characteristic function: $\Psi_X(\omega) = \left(1 - \frac{j\omega}{\lambda}\right)^{-1}$
  - Memoryless property: $P[X > t + s | X > s] = P[X > t]$
    (For a continuous non-negative r.v. X, if the above memoryless property holds for all $s, t \geq 0$, then X must have an exponential distribution)
  - nth moment $E[X^n] = \frac{n!}{\lambda^n}$.



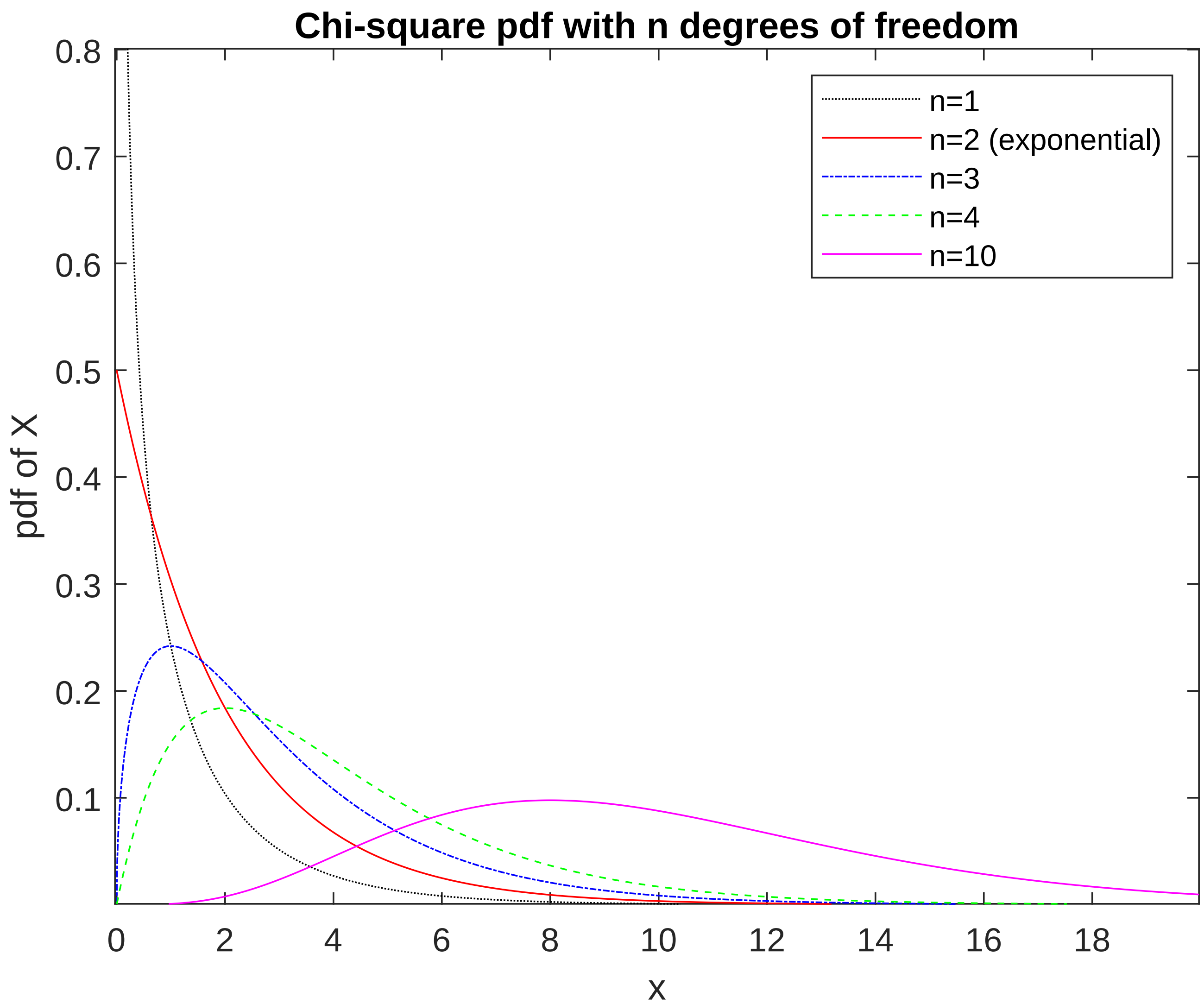
- Weibull
  - $f_X(x) = \begin{cases} \alpha x^{\beta-1} e^{\frac{-\alpha x^{\beta}}{\beta}}, \; x > 0 \\ 0, \text{otherwise} \end{cases}$
  - Mean: $\left(\frac{\beta}{\alpha}\right)^{\frac{1}{\beta}} \Gamma(1+\frac{1}{\beta})$
  - Variance: $\left(\frac{\beta}{\alpha}\right)^{\frac{2}{\beta}} [\Gamma(1+\frac{2}{\beta}) - \{\Gamma(1+\frac{1}{\beta})\}^2]$
  - $\beta = 1 \;\; \Rightarrow$ exponential



- Chi-square
  - If $X \sim N(0, \sigma^2) \Rightarrow Y = X^2$ is central Chi-square
  - If $X \sim N(\mu_x, \sigma^2) \Rightarrow Y = X^2$ is non-central Chi-square
  - Central Chi-square
    - $f_Y(y) = \frac{1}{\sqrt{2\pi y}\sigma} e^{-\frac{y}{2\sigma^2}}$
    - $F_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} \int_0^y \frac{1}{\sqrt{u}} e^{-\frac{u}{2\sigma^2}} du$ (no closed-form)
    - $\Psi_X(\omega) = \frac{1}{\sqrt{1 - j2\omega\sigma^2}}$

- Central Chi-square with n degrees of freedom (DoF): If $X_i, i = 1, 2, \ldots, n$ are iid $N(0, \sigma^2)$, then $Y = \sum_{i=1}^n X_i^2$ is central Chi-square (or Gamma) with n DoF
  - $f_Y(y) = \frac{1}{\sigma^n 2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} y^{\frac{n}{2}-1} e^{-\frac{y}{2\sigma^2}}, \; y \geq 0$
  - $F_Y(y) = \int_0^y \frac{u^{\frac{n}{2}-1} e^{-\frac{u}{2\sigma^2}}}{\sigma^n 2^{\frac{n}{2}} \Gamma(\frac{n}{2})} du, \; y \geq 0$
  - If $n = 2m$ $(m = \text{integer})$, $F_Y(y) = 1 - e^{-\frac{y}{2\sigma^2}} \sum_{k=0}^{m-1} \frac{1}{k!} \left(\frac{y}{2\sigma^2}\right)^k, \; y \geq 0$
  - $E[Y] = n\sigma^2$
  - $E[Y^2] = 2n\sigma^4 + n^2\sigma^4, \; \sigma_y^2 = 2n\sigma^4$
  - $\Psi_y(\omega) = \frac{1}{\left(1 - j2\omega\sigma^2\right)^{\frac{n}{2}}}$
  - $n = 2 \Rightarrow$ exponential

- Central Chi-square with n DoF



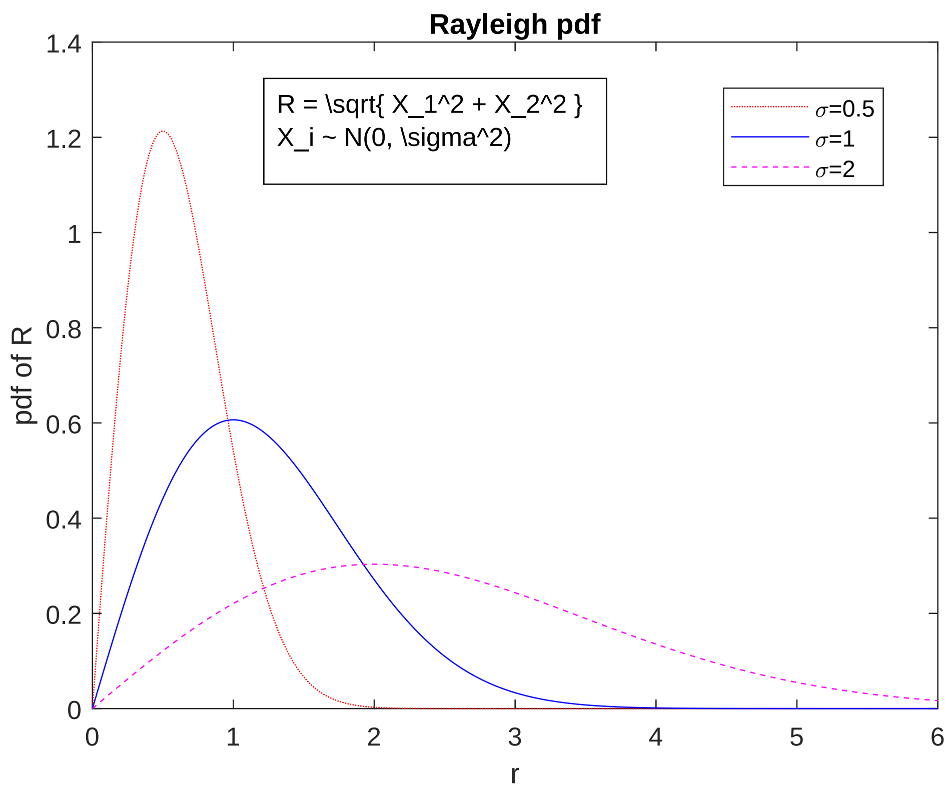
- Non-central Chi-square
  - If $X \sim N(\mu_x, \sigma^2) \Rightarrow Y = X^2$ is non-central Chi-square.
  - $f_Y(y) = \frac{1}{\sqrt{2\pi y}\sigma} e^{-\frac{y+\mu_x^2}{2\sigma^2}} \cosh\left(\frac{\sqrt{y}\mu_x}{\sigma^2}\right), \ y \geq 0$
  - $\Psi_X(\omega) = \frac{1}{\sqrt{1-j2\omega\sigma^2}} e^{\frac{j\mu_x^2\omega}{1-j2\omega\sigma^2}}$
- Non-central Chi-square with n DoF: If $X_i \sim N(\mu_i, \sigma^2)$ are independent, then $Y = \sum_{i=1}^n X_i^2$ is non-central Chi-square with n DoF.
  - $f_Y(y) = \frac{1}{2\sigma^2}\left(\frac{y}{s^2}\right)^{\frac{n-2}{4}} e^{-\frac{s^2+y}{2\sigma^2}} I_{\frac{n}{2}-1}\left(\frac{\sqrt{y}s}{\sigma^2}\right), \ y \geq 0$
  - $F_Y(y) = \int_0^y \frac{1}{2\sigma^2}\left(\frac{u}{s^2}\right)^{\frac{n-2}{4}} e^{-\frac{s^2+u}{2\sigma^2}} I_{\frac{n}{2}-1}\left(\frac{\sqrt{u}s}{\sigma^2}\right) du, \ y \geq 0$ (No closed-form)
  - $\Psi_X(\omega) = \frac{1}{\left(1-j2\omega\sigma^2\right)^{\frac{n}{2}}} \exp\left(\frac{j\omega\sum_{i=1}^n \mu_i^2}{1-j2\omega\sigma^2}\right)$

    where $I_\alpha(x) = \sum_{k=0}^\infty \frac{(\frac{x}{2})^{\alpha+2k}}{k!\Gamma(\alpha+k+1)}, x \geq 0$ (α^{th} order modified Bessel function of first kind) and $s^2 = \sum_{i=1}^n \mu_i^2$ (non-centrality parameter)
  - $E[Y] = n\sigma^2 + s^2$
  - $E[Y^2] = 2n\sigma^4 + 4\sigma^2 s^2 + (n\sigma^2 + s^2)^2$
  - $\sigma_y^2 = 2n\sigma^4 + 4\sigma^2 s^2$

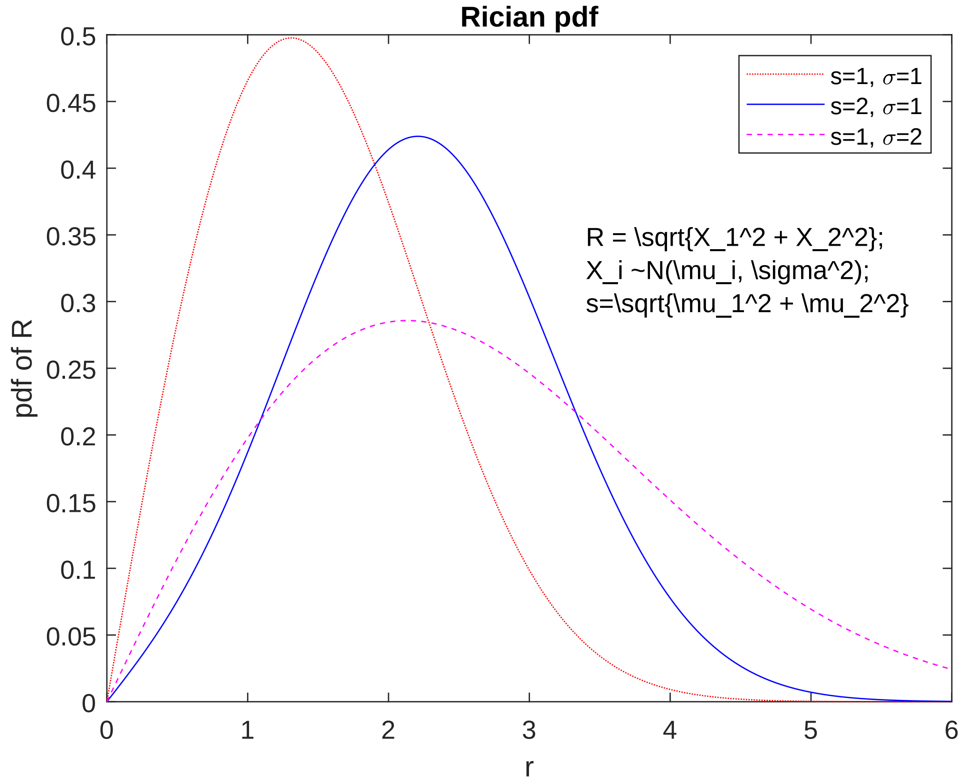
- Non-central Chi-square with DoF $n = 2m$ $(m = \text{integer})$:
  (substituting $x^2 = \frac{u}{\sigma^2}$, $\alpha^2 = \frac{s^2}{\sigma^2}$ in $F_Y(y)$ yields)
  - $F_Y(y) = 1 - Q_m\left(\frac{s}{\sigma},\ \frac{\sqrt{y}}{\sigma}\right),\ y \geq 0$
    where $Q_m(a, b)$ is the Generalized Marcum's Q function given by
    $Q_m(a,b) = \int_b^\infty x(\frac{x}{a})^{m-1} e^{-\frac{x^2+a^2}{2}} I_{m-1}(ax)dx$
    $= Q_1(a,b) + e^{-\frac{a^2+b^2}{2}} \sum_{k=1}^{m-1} (\frac{b}{a})^k I_k(ab)$
    with $Q_1(a,b) = e^{-\frac{a^2+b^2}{2}} \sum_{k=0}^{\infty} (\frac{a}{b})^k I_k(ab), b > a > 0.$

- Erlang-k
  - $f_X(x) = \frac{\lambda^k}{(k-1)!} x^{k-1} e^{-\lambda x}$
  - Mean: $\frac{k}{\lambda}$
  - Variance: $\frac{k}{\lambda^2}$
  - Characteristic function: $\left(1 - \frac{j\omega}{\lambda}\right)^{-k}$
  - $k = 1 \Rightarrow$ exponential

- Rayleigh
  - If X_1, X_2 are independent $N(0,\ \sigma^2)$, $R = \sqrt{X_1^2 + X_2^2}$ has a Rayleigh pdf: $f_R(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}},\ r \geq 0$
  - $F_R(r) = (1 - e^{-\frac{r^2}{2\sigma^2}})U(r)$
  - Mean: $\sqrt{\frac{\pi}{2}}\sigma$
  - Variance: $(2 - \frac{\pi}{2})\sigma^2$
  - Characteristic function: $\Psi_X(\omega) = \left(1 + j\sqrt{\frac{\pi}{2}}\sigma\omega\right) e^{-\frac{\sigma^2\omega^2}{2}}$



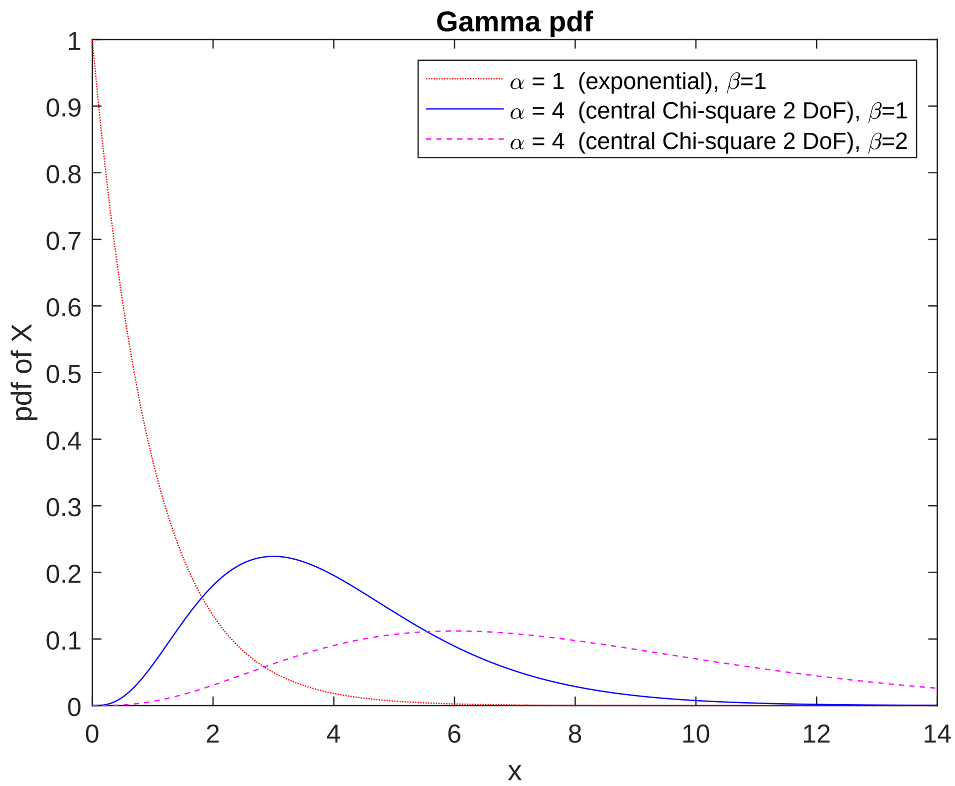
- Rice
  - If X_1, X_2 are independent $N(\mu_i,\ \sigma^2)$, $R = \sqrt{X_1^2 + X_2^2}$ has a Rice pdf:
    - $f_R(r) = \frac{r}{\sigma^2} e^{-\frac{r^2+s^2}{2\sigma^2}} I_0(\frac{rs}{\sigma^2}),\ r \geq 0$, where $s^2 = \sum_{i=1}^n \mu_i^2$
    - $F_R(r) = 1 - Q_1\left(\frac{s}{\sigma},\ \frac{r}{\sigma}\right),\ r \geq 0$



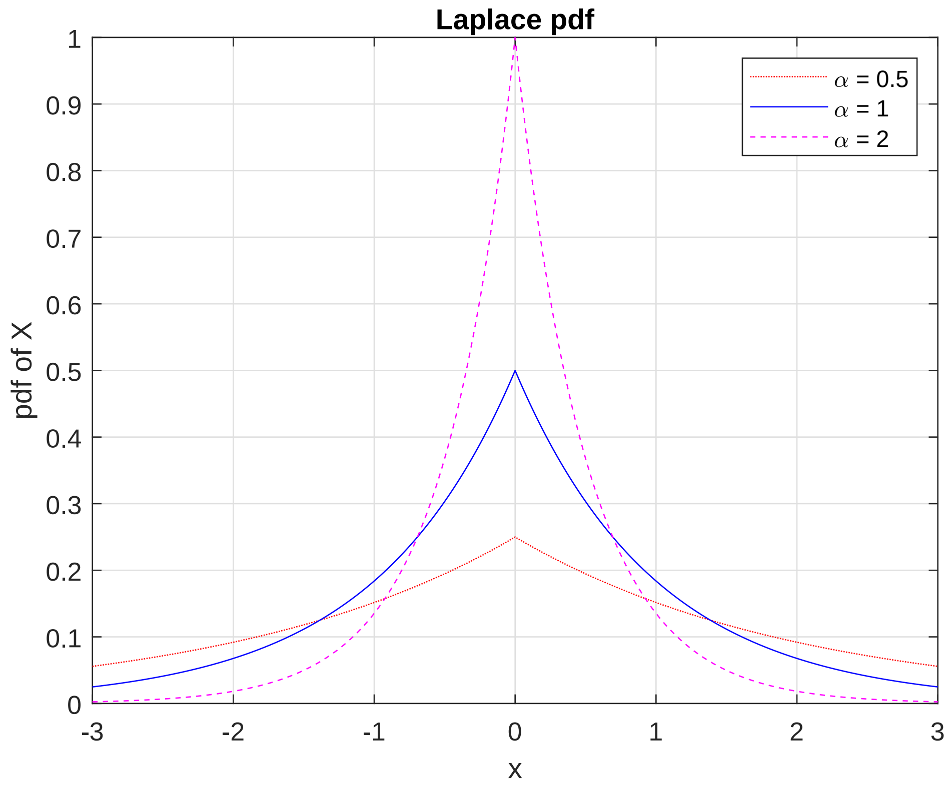
- Generalized Rayleigh
  - If X_i are iid $N(0,\ \sigma^2)$, $R = \sqrt{\sum_{i=1}^{n} X_i^2}$ has a generalized Rayleigh pdf:
    - $f_R(r) = \frac{r^{n-1}}{2^{\frac{n-2}{2}} \sigma^n \Gamma(\frac{n}{2})} e^{-\frac{r^2}{2\sigma^2}},\ r \geq 0$
  - If $n = 2m$ $(m = \text{integer})$,
    - $F_R(r) = 1 - e^{-\frac{r^2}{2\sigma^2}} \sum_{k=0}^{m-1} \frac{1}{k!} \left(\frac{r^2}{2\sigma^2}\right)^k,\ r \geq 0$
    - $E[R^k] = (2\sigma^2)^{\frac{k}{2}} \frac{\Gamma(\frac{n+k}{2})}{\Gamma(\frac{n}{2})},\ k \geq 0, n = \text{integer}$

- Generalized Rice
  - If X_i are iid $N(\mu_i,\ \sigma^2)$, $R = \sqrt{\sum_{i=1}^{n} X_i^2}$ has a generalized Rician pdf:
    - $f_R(r) = \frac{r^{\frac{n}{2}}}{\sigma^2 s^{\frac{n-2}{2}}} e^{-\frac{r^2+s^2}{2\sigma^2}} I_{\frac{n}{2}-1}\left(\frac{rs}{\sigma^2}\right),\ r \geq 0$
    - $F_R(r) = F_Y(r^2)$, Y is non-central chi-square, $Y = \sum_{i=1}^{n} X_i^2$
  - If $n = 2m$ $(m = \text{integer})$,
    - $F_R(r) = 1 - Q_m\left(\frac{s}{\sigma},\ \frac{r}{\sigma}\right),\ r \geq 0$
    - $E[R^k] = (2\sigma^2)^{\frac{k}{2}} e^{-\frac{s^2}{2\sigma^2}} \frac{\Gamma(\frac{n+k}{2})}{\Gamma(\frac{n}{2})} {}_1F_1\left(\frac{n+k}{2}, \frac{n}{2}; \frac{s^2}{2\sigma^2}\right),\ k \geq 0$, where ${}_1F_1(\alpha, \beta; x)$ is the confluent hypergeometric function.

- Gamma (with parameters $\alpha > 0, \beta > 0$):
  - $f_X(x) = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha)\ \beta^{\alpha}} e^{-x/\beta}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$
    where $\Gamma(\alpha) \triangleq \int_0^{\infty} x^{\alpha-1} e^{-x} dx$
  - Mean: $\alpha\beta$
  - Variance: $\alpha\beta^2$
  - $\alpha = 1 \Rightarrow$ exponential
    $\alpha = n/2 \Rightarrow$ central Chi-square
    $\alpha = k,\ \beta = 1/\lambda \Rightarrow$ Erlang-k



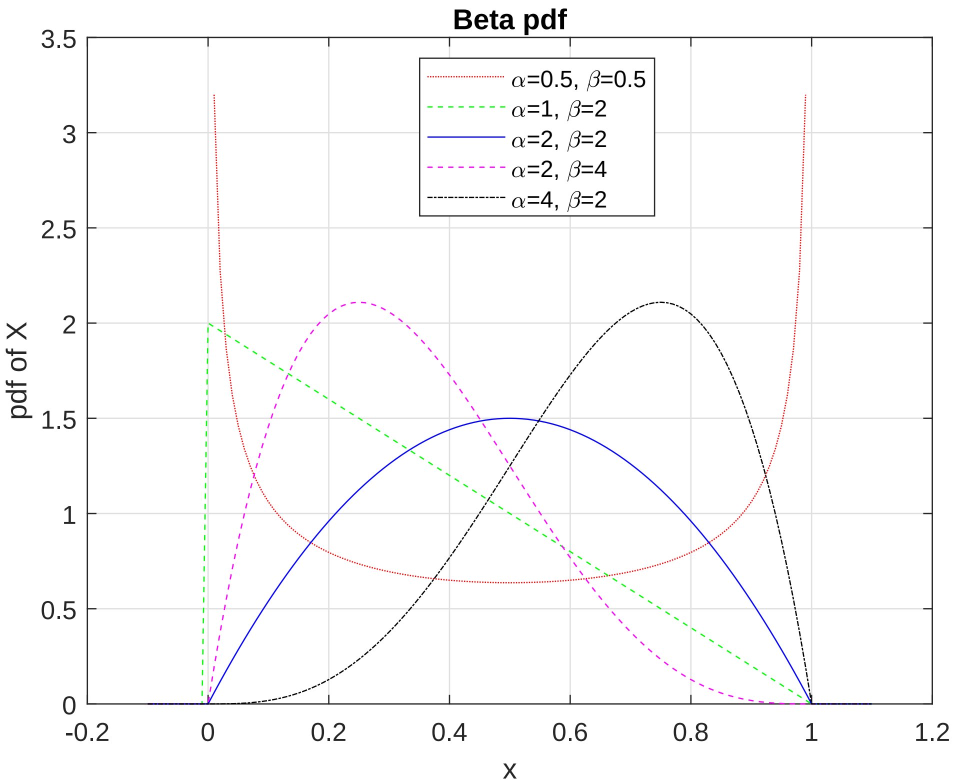
- Laplace
  - $f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}, -\infty < x < +\infty$ and $\alpha > 0$
  - Mean: 0
  - Variance: $\frac{2}{\alpha^2}$
  - Characteristic function: $\Psi_X(\omega) = \frac{\alpha^2}{\omega^2+\alpha^2}$



- Beta
  - For $\alpha > 0$ and $\beta > 0$,

$$f_X(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\ x^{\alpha-1}(1-x)^{\beta-1}, & 0 < x < 1 \\ 0, \ \text{otherwise} \end{cases}$$

  - Mean: $\frac{\alpha}{\alpha+\beta}$
  - Variance: $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
  - $\alpha = 1 \Rightarrow$ uniform



- Uniform
  - $f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x \le b, \ b > a \\ 0, \text{otherwise} \end{cases}$
  - $F_X(x) = \begin{cases} \frac{x-a}{b-a}, & a < x \le b, \ b > a \\ 1, & x > b \end{cases}$
  - Mean: $\frac{b+a}{2}$
  - Variance: $\frac{(b-a)^2}{12}$
  - Characteristic function: $\Psi_X(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$

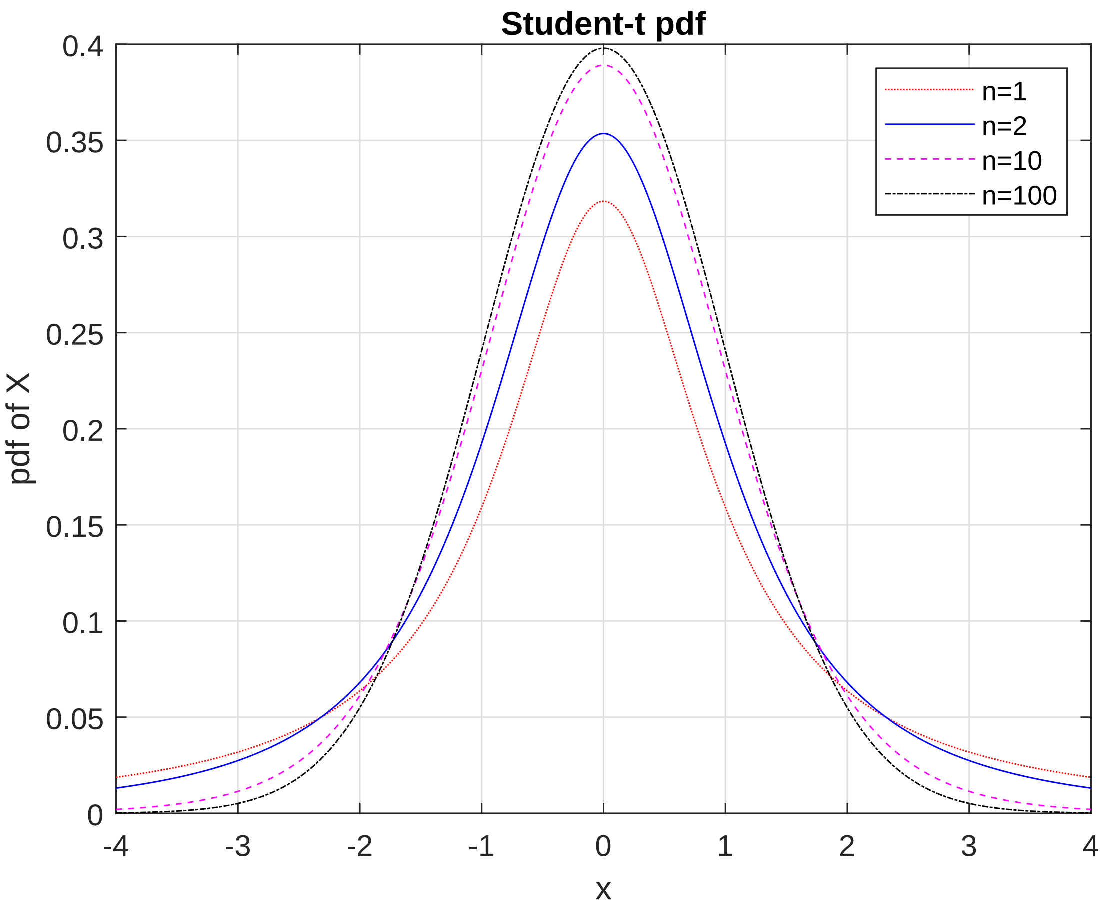
- Cauchy
  - For $-\infty < \alpha < \infty$ and $\beta > 0$,

$$f_X(x) = \frac{1}{\pi\beta(1 + (\frac{x-\alpha}{\beta})^2)} = \frac{\beta}{\pi(\beta^2 + (x-\alpha)^2)}, \quad -\infty < x < \infty$$

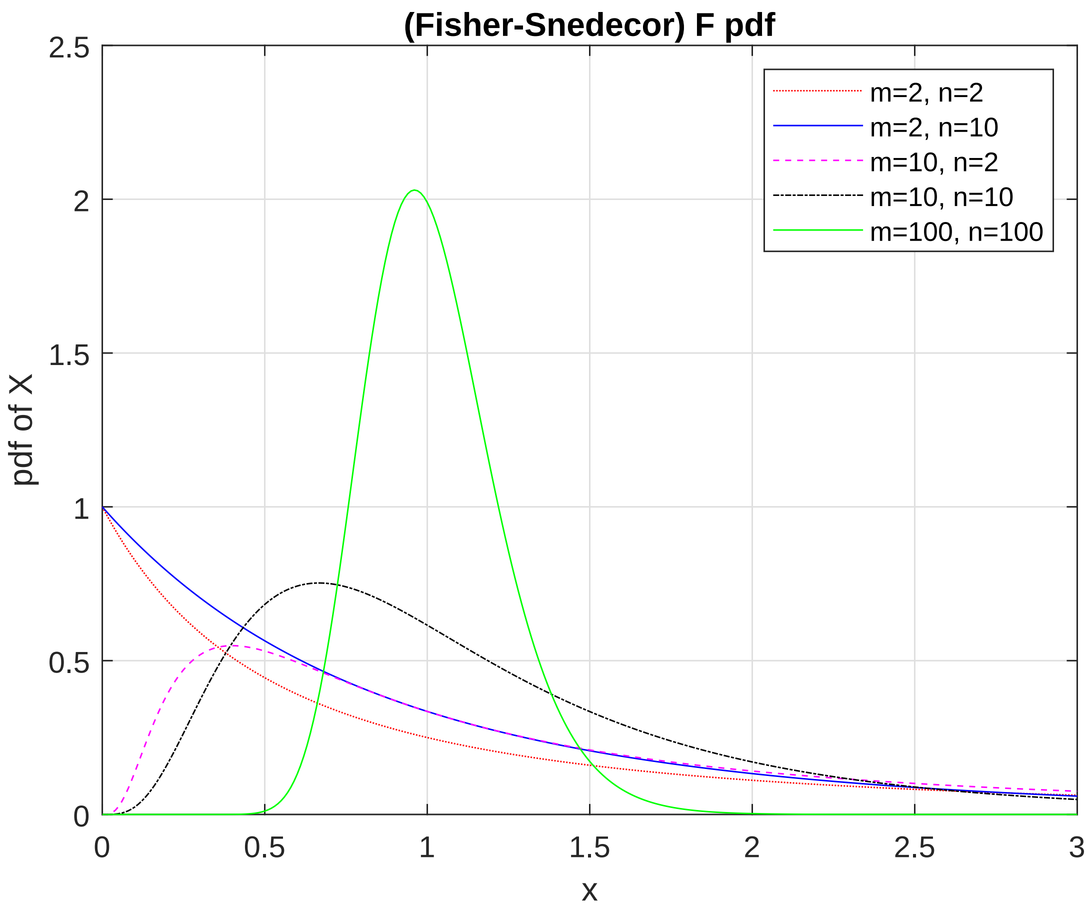
  - Mean: α
  - Variance: ∞
  - Characteristic function: $e^{j\alpha\omega} e^{-\beta|\omega|}$

- Nakagami m
  - $f_R(r) = \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m r^{2m-1} e^{-\frac{mr^2}{\Omega}}, \; r \geq 0$
    - $\Omega = E[R^2], m = \frac{\Omega^2}{E[(R^2-\Omega)^2]}, m \geq \frac{1}{2}$ fading figure
  - $E[R^n] = \frac{\Gamma(m+\frac{n}{2})}{\Gamma(m)} (\frac{\Omega}{m})^{\frac{n}{2}}$
  - If $m = 1, \Rightarrow$ Rayleigh
  - If $\frac{1}{2} \leq m < 1, \Rightarrow$ Larger tails than Rayleigh
  - If $m > 1, \Rightarrow$ pdf decays faster than Rayleigh
- Pareto
  - $f_X(x) = \begin{cases} \alpha \frac{x_m^\alpha}{x^{\alpha+1}}, \; x \geq x_m \\ 0, \; x < x_m \end{cases}$
  - $\text{Mean: } \frac{\alpha x_m}{\alpha-1} \text{ for} \alpha > 1$
  - $\text{Variance: } \frac{\alpha x_m^2}{(\alpha-2)(\alpha-1)^2} \text{ for } \alpha > 2$
  - Can be viewed as a continuous version of the Zipf discrete random variable.
- Maxwell
  - $f_X(x) = \begin{cases} \frac{4}{\alpha^3\sqrt{\pi}} x^2 e^{-x^2/\alpha^2}, \; x \geq 0 \\ 0, \; \text{otherwise} \end{cases}$
  - $\text{Mean: } 2\alpha\sqrt{2/\pi}$
  - $\text{Variance: } (3 - \frac{8}{\pi})\alpha^2$

- Student-t
    - $f_X(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})}(1+\frac{x^2}{n})^{-\frac{n+1}{2}}$
    - Mean: 0
    - Variance: $\frac{n}{n-2},\ n > 2$



- F Distribution with (m, n) degrees of freedom
  - $f_Z(z) = \begin{cases} \frac{\Gamma((m+n)/2)m^{m/2}n^{n/2}}{\Gamma(m/2)\Gamma(n/2)} \frac{z^{m/2-1}}{(n+mz)^{(m+n)/2}}, & z \geq 0 \\ 0, & \text{otherwise} \end{cases}$
  - Mean: $\frac{n}{n-2}, \; n > 2$
  - Variance: $\frac{n^2(2m+2n-4)}{m(n-2)^2(n-4)}, \; n > 4$



Joint Distributions and Densities

- Joint CDF: $\boxed{F_{XY}(x,y) \triangleq P[X \leq x, Y \leq y]}$
- Joint PMF: $\boxed{P_{XY}(x_i,y_k) \triangleq P[X = x_i, Y = y_k]}$
- Joint pdf: $\boxed{f_{XY}(x,y) \triangleq \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)}$
- Discrete RV: $\boxed{F_{XY}(x,y) = \sum_{x_i \leq x} \sum_{y_k \leq y} P_{XY}(x_i,y_k)}$
- Continuous/Discrete/Mixed RV: $\boxed{F_{XY}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(\alpha,\beta) d\alpha d\beta}$
- $F_{XY}(\infty,\infty) = 1, \quad F_{XY}(-\infty,y) = F_{XY}(x,-\infty) = 0$
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y) dx dy = 1$
- $\sum_i \sum_k P_{XY}(x_i,y_k) = 1$
- if $x_1 \leq x_2, y_1 \leq y_2$, then $F_{XY}(x_1,y_1) \leq F_{XY}(x_2,y_2)$
- $F_{XY}(x,y) = \lim_{\epsilon \to 0, \delta \to 0} F_{XY}(x+\epsilon, y+\delta), \quad \epsilon, \delta > 0$
  (value at discontinuity: immediately from the right & above)

Marginal Distributions/Densities and Expectation

- $F_{XY}(x,\infty) = F_X(x), \quad F_{XY}(\infty,y) = F_Y(y)$
- $\boxed{P_X(x) = \sum_k P_{XY}(x,y_k)}, \quad \boxed{P_Y(y) = \sum_i P_{XY}(x_i,y)}$
- $\boxed{f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy}, \quad \boxed{f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx}$
- $\boxed{E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)dxdy}$
- $\boxed{E[g(X,Y)] = \sum_i \sum_k g(x_i,y_k) P_{XY}(x_i,y_k)}$
- $\boxed{E[g(X)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x) f_{XY}(x,y)dxdy = \int_{-\infty}^{\infty} g(x) f_X(x)dx}$
- $\boxed{E[g(X)] = \sum_i \sum_k g(x_i) P_{XY}(x_i,y_k) = \sum_i g(x_i) P_X(x_i)}$

Orthogonal RVs & Correlated RVs

- Correlation of X and Y: $\boxed{R_{XY} \triangleq E[XY]}$

$$\text{Discrete}: R_{XY} = \sum_k \sum_i x_i y_k P_{XY}(x_i, y_k)$$

$$\text{Continuous or Discrete}: R_{XY} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \; f_{XY}(x,y) dx dy$$

- Covariance of X and Y: $\boxed{\text{Cov}(X,Y) \triangleq E[(X-\mu_X)(Y-\mu_Y)] = R_{XY} - \mu_X \mu_Y}$

$$\text{Discrete RV}: \text{Cov}(X,Y) = \sum_k \sum_i (x_i - \mu_X)(y_k - \mu_Y) P_{XY}(x_i, y_k)$$

$$\text{Continuous/Discrete}: \text{Cov}(X,Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) \; f_{XY}(x,y) dx dy$$

- Correlation Coefficient: $\boxed{\rho_{XY} = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}}$, $\quad (-1 \le \rho_{XY} \le 1)$
- $\boxed{\text{Cov}(X,\, Y) = 0 \text{ (i.e., } \rho_{XY} = 0) \Rightarrow \text{uncorrelated}}$
- $\boxed{R_{XY} = 0 \Rightarrow \text{orthogonal}}$
- $\boxed{\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\,\text{Cov}(X,Y)}$

Conditional CDF/pdf/PMF and Conditional Expectation

- Satisfy all properties of ordinary distribution/density function
- Conditional distribution $(P[B] \neq 0)$:

$$\boxed{F_{X|B}(x|B) = \frac{P[X \leq x,\ B]}{P[B]} = \int_{-\infty}^{x} f_{X|B}(u|B)du}$$

- Conditional pdf $(P[B] \neq 0)$:

$$\boxed{f_{X|B}(x|B) \triangleq \frac{dF_{X|B}(x|B)}{dx} = \begin{cases} \frac{f_X(x)}{P[B]}, & x \in (B \cap \Omega_X) \\ 0, & \text{else} \end{cases}}$$

- Conditional PMF $(P[B] \neq 0)$: $\boxed{P_{X|B}(x|B) \triangleq \begin{cases} \frac{P_X(x)}{P[B]}, & x \in (B \cap \Omega_X) \\ 0, & \text{else} \end{cases}}$

- For an event space $\{A_i : i = 1, \ldots, n\}$,

$$\boxed{\begin{aligned} F_X(x) &= \sum_{i=1}^{n} F_{X|A_i}(x|A_i)P[A_i]; \\ f_X(x) &= \sum_{i=1}^{n} f_{X|A_i}(x|A_i)P[A_i] \\ P_X(x) &= \sum_{i=1}^{n} P_{X|A_i}(x|A_i)P[A_i] \quad \text{(for discrete RV)} \end{aligned}}$$

- Conditional Expectation:

$$\boxed{\begin{aligned} E[X|B] &= \int_{x \in (B \cap \Omega_X)} x f_{X|B}(x|B)dx \\ E[X|B] &= \sum_{x_i \in (B \cap \Omega_X)} x_i P_{X|B}(x_i|B) \quad \text{(for discrete RV)} \\ E[g(X)|B] &= \int_{x \in (B \cap \Omega_X)} g(x) f_{X|B}(x|B)dx \\ E[g(X)|B] &= \sum_{x_i \in (B \cap \Omega_X)} g(x_i) P_{X|B}(x_i|B) \quad \text{(for discrete RV)} \end{aligned}}$$

- **Conditional pdf/PMF/CDF and Conditional Expectation (2 RVs)**
- $\boxed{f_{X|Y}(x|y) = f_{XY}(x,y)/f_Y(y)}$
- $\boxed{P_{X|Y}(x|y) = P_{XY}(x,y)/P_Y(y)}$ (for discrete RV)
- $\boxed{F_{X|Y}(x|y) = \int_{-\infty}^{x} f_{X|Y}(u|y)du}$
- $\boxed{F_{X|Y}(x|y) = \sum_{x_i \leq x} P_{X|Y}(x_i|y)}$ (for discrete RV)
- $\boxed{E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx}$
- $\boxed{E[X|Y=y] = \sum_{x_i} x_i P_{X|Y}(x_i|y)}$ (for discrete RV)
- $\boxed{E[g(X)|Y=y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)dx}$
- $\boxed{E[g(X)|Y=y] = \sum_{x_i} g(x_i) P_{X|Y}(x_i|y)}$ (for discrete RV)

- **Conversion from Higher to Smaller Order Conditional Joint pdf/PMF**

Chain rule:

$$f(x_1, \ldots, x_n) = f(x_n|x_{n-1}, \ldots, x_1) \ldots f(x_3|x_2, x_1) f(x_2|x_1) f(x_1)$$

$$f(x_1|x_3) = \int_{-\infty}^{\infty} f(x_1, x_2|x_3)\ dx_2$$

$$f(x_1|x_4) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1|x_2, x_3, x_4)\ f(x_2, x_3|x_4)\ dx_2\ dx_3$$

$$P[X_1 = a_i|X_3 = c_k] = \sum_j \left(P[X_1 = a_i|X_2 = b_j, X_3 = c_k]\ P[X_2 = b_j|X_3 = c_k]\right)$$

Independent RVs

- $\{X_i,\ i = 1, \ldots, n\}$ are statistically independent iff, for all $x_1, \ldots, x_n$,

$$\boxed{\begin{aligned} F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) &= \prod_{i=1}^{n} F_{X_i}(x_i) \\ \text{alternatively,}\quad f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) &= \prod_{i=1}^{n} f_{X_i}(x_i), \\ \text{(for discrete RVs)}\quad P_{X_1,\ldots,X_n}(x_1, \ldots, x_n) &= \prod_{i=1}^{n} P_{X_i}(x_i) \end{aligned}}$$

- If independent, for continuous RVs,
  $f_{X_1,\ldots,X_k|X_{k+1},\ldots,X_n}(x_1, \ldots, x_k|x_{k+1}, \ldots, x_n) = f_{X_1,\ldots,X_k}(x_1, \ldots, x_k)$
  and for discrete RVs,
  $P_{X_1,\ldots,X_k|X_{k+1},\ldots,X_n}(x_1, \ldots, x_k|x_{k+1}, \ldots, x_n) = P_{X_1,\ldots,X_k}(x_1, \ldots, x_k)$

- $\boxed{\text{Independence} \Rightarrow \text{Uncorrelated}}$; (converse is not true except Gaussian)

- Orthogonal: if $E[X_iX_j] = 0$.

- **Example (Joint PMF and Marginal PMFs):** Random variables X and Y have the joint PMF

$$P_{X,Y}(x,y) = \begin{cases} cxy & x = 1, 2, 4;\ y = 1, 3, \\ 0 & \text{otherwise.} \end{cases}$$

a) Find the marginal PMFs of X and Y.
b) Find the mean and variance of X and Y.
c) Are X and Y orthogonal or not?
d) Are X and Y uncorrelated or not?
e) Find the variance of $X + Y$, $\text{Var}[X + Y]$.
f) Find $E[Y/X]$.

- **Solution:**

  a) Using $\sum_i \sum_k P_{X,Y}(x_i, y_k) = 1$, we obtain $c = 1/28$.
  The marginal PMFs of X and Y are

$$P_X(x) = \sum_{y=1,3} P_{X,Y}(x,y) = \begin{cases} 4/28 & x = 1 \\ 8/28 & x = 2 \\ 16/28 & x = 4 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(y) = \sum_{x=1,2,4} P_{X,Y}(x,y) = \begin{cases} 7/28 & y = 1 \\ 21/28 & y = 3 \\ 0 & \text{otherwise} \end{cases}$$

  b) The expected values of X and Y are

$$E[X] = \sum_{x=1,2,4} xP_X(x) = (4/28) + 2(8/28) + 4(16/28) = 3$$

$$E[Y] = \sum_{y=1,3} yP_Y(y) = (7/28) + 3(21/28) = 5/2$$

Solution (continues)

The second moments are

$$E[X^2] = \sum_{x=1,2,4} x^2 P_X(x) = 1^2(4/28) + 2^2(8/28) + 4^2(16/28) = 73/7$$

$$E[Y^2] = \sum_{y=1,3} y^2 P_Y(y) = 1^2(7/28) + 3^2(21/28) = 7$$

The variances are

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2 = 10/7$$

$$\mathrm{Var}[Y] = E[Y^2] - (E[Y])^2 = 3/4$$

c) The correlation of X and Y is

$$R_{XY} = E[XY] = \sum_{x=1,2,4} \sum_{y=1,3} xy P_{X,Y}(x,y)$$

$$= \frac{1 \cdot 1 \cdot 1}{28} + \frac{1 \cdot 3 \cdot 3}{28} + \frac{2 \cdot 1 \cdot 2}{28} + \frac{2 \cdot 3 \cdot 6}{28} + \frac{4 \cdot 1 \cdot 4}{28} + \frac{4 \cdot 3 \cdot 12}{28} = 15/2$$

Since $E[XY] \neq 0$, X and Y are not orthogonal.

Solution (continues)

d) The covariance of X and Y is

$$\mathrm{Cov}[X,Y] = E[XY] - E[X]E[Y] = \frac{15}{2} - 3 \cdot \frac{5}{2} = 0$$

Since $\mathrm{Cov}[X,Y] = 0$, X and Y are uncorrelated.

Note: The correlation coefficient is

$$\rho_{X,Y} = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}} = 0.$$

e) The variance of $X+Y$ is

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathrm{Cov}[X,Y] = \frac{61}{28}.$$

f)

$$\begin{aligned} E[Y/X] &= \sum_{x=1,2,4} \sum_{y=1,3} \frac{y}{x} P_{X,Y}(x,y) \\ &= \frac{1}{1}\frac{1}{28} + \frac{3}{1}\frac{3}{28} + \frac{1}{2}\frac{2}{28} + \frac{3}{2}\frac{6}{28} + \frac{1}{4}\frac{4}{28} + \frac{3}{4}\frac{12}{28} = 15/14 \end{aligned}$$

- **Example (Joint and Marginal PDFs):** X and Y are random variables with the joint PDF given by

$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & -1 \le x \le 1; 0 \le y \le x^2, \\ 0 & \text{otherwise.} \end{cases}$$

a) PDF of X ?
b) PDF of Y ?
c) Mean and variance of X ?
d) Mean and variance of Y ?
e) Covariance of X and Y ?
f) Mean and variance of $X+Y$?

- Solution:

$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & -1 \le x \le 1; 0 \le y \le x^2, \\ 0 & \text{otherwise.} \end{cases}$$

a) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$.
First, $f_X(x) = 0$ if $x \notin [-1,1]$.
For $x \in [-1,1]$, $f_X(x) = \int_0^{x^2} \frac{5x^2}{2} dy = 5x^4/2$. Thus,

$$f_X(x) = \begin{cases} 5x^4/2 & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

b) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$.

First, $f_Y(y) = 0$ if $y \notin [0,1]$.
For $0 \le y \le 1$, $f_Y(y) = \int_{-1}^{-\sqrt{y}} \frac{5x^2}{2} dx + \int_{\sqrt{y}}^{1} \frac{5x^2}{2} dx = 5(1 - y^{3/2})/3$. Thus,

$$f_Y(y) = \begin{cases} 5(1 - y^{3/2})/3 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

c) $E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-1}^{1} \frac{5x^5}{2} dx = \left.\frac{5x^6}{12}\right|_{-1}^{1} = 0.$

$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_{-1}^{1} \frac{5x^6}{2} dx = \left.\frac{5x^7}{14}\right|_{-1}^{1} = \frac{10}{14}$

Thus, $\mathrm{Var}[X] = E[X^2] - (E[X])^2 = \frac{10}{14}$.

- Solution (continues):

d) The k^{th} moment of Y can be computed by $E[Y^k] = \int_{-\infty}^{\infty} y^k f_Y(y)dy$ or $E[Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^k f_{XY}(x,y)dxdy$.

$$E[Y] = \int_{-1}^{1}\int_{0}^{x^2} y\frac{5x^2}{2}dydx = \frac{5}{14}$$

$$E[Y^2] = \int_{-1}^{1}\int_{0}^{x^2} y^2\frac{5x^2}{2}dydx = \frac{5}{27}$$

Therefore, $\mathrm{Var}[Y] = E[Y^2] - (E[Y])^2 = 5/27 - (5/14)^2 = 0.0576.$

e) Since $E[X] = 0$, $\mathrm{Cov}[X,Y] = E[XY] - E[X]E[Y] = E[XY]$. Thus,

$$\mathrm{Cov}[X,Y] = E[XY] = \int_{-1}^{1}\int_{0}^{x^2} xy\frac{5x^2}{2}dydx = \int_{-1}^{1}\frac{5x^7}{4}dx = 0$$

f) $E[X+Y] = E[X] + E[Y] = 5/14$

$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X,Y] = 0.7719$

- **Example (Joint CDF):**

$$f_{X,Y}(x,y) = \begin{cases} 1, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases} \qquad F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u,v)\ du\ dv$$

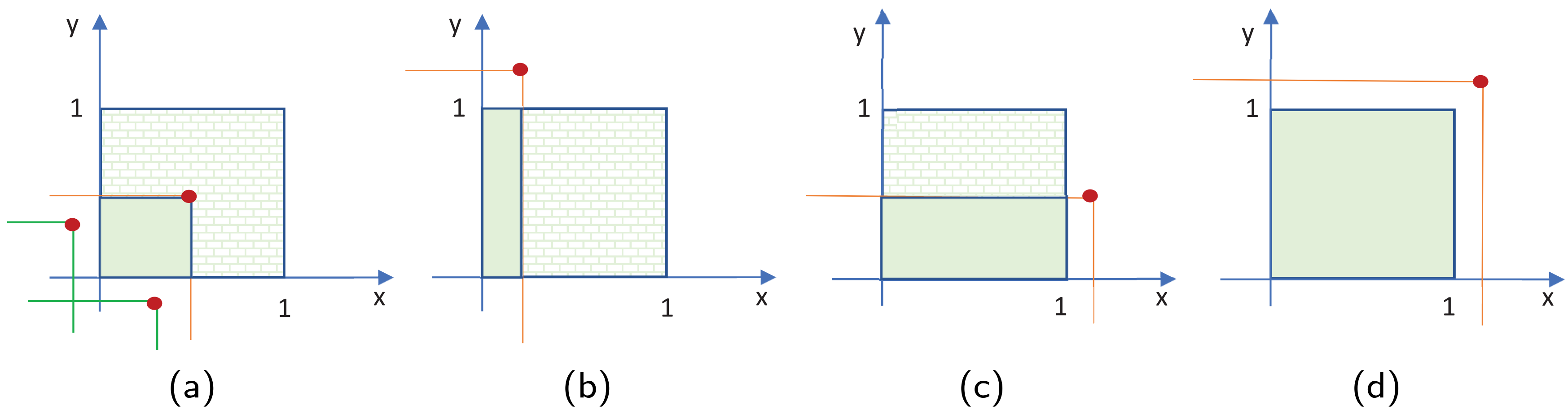


Fig. Integration areas shown by solid color for (a) $x < 0$; $y < 0$; $(0 < x < 1, 0 < y < 1)$; (b) $(0 < x < 1, y > 1)$; (c) $(x > 1, 0 < y < 1)$; (d) $(x > 1, y > 1)$

Solving for each interval gives

$$F_{X,Y}(x,y) = \begin{cases} 0, & x < 0, \text{ or } y < 0 \\ xy, & 0 \le x \le 1,\ 0 \le y \le 1 \\ x, & 0 \le x \le 1,\ y \ge 1 \\ y, & 0 \le y \le 1,\ x \ge 1 \\ 1, & x \ge 1,\ y \ge 1 \end{cases}$$

- **Example (A two-stage hyper-exponential random variable):**
  With a probability p, X is an exponential RV with mean $1/a$, and otherwise X is an exponential RV with mean $1/b$. X is called a two-stage hyper-exponential RV.
  a) What is the pdf of X ?
  b) Given that X is larger than 3, what is the conditional pdf of X ?

  Solution: (a)

$$f_X(x) = P[E_1]f_{X|E_1}(x) + P[E_2]f_{X|E_2}(x) = \left(pae^{-ax} + (1-p)be^{-bx}\right)U(x)$$

  ($U(x)$ = unit step function)

  (b)

$$A \triangleq \{X > 3\}$$

$$P[A] = \int_3^\infty f_X(x)dx = pe^{-3a} + (1-p)e^{-3b}$$

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{P[A]} = \frac{pae^{-ax}+(1-p)be^{-bx}}{pe^{-3a}+(1-p)e^{-3b}}, & x > 3 \\ 0, & \text{otherwise} \end{cases}$$

- **Example (Conditional Expectation):**
  The time between telephone calls at a telephone switch is exponential random variable T with expected value 0.01. Suppose $T > 0.02$ is given.
  a) What is $E[T|T > 0.02]$, the conditional expected value of T ?
  b) What is $\mathrm{Var}[T|T > 0.02]$, the conditional variance of T ?

  Solution: (a) We first find the conditional PDF of T. The PDF of T is

$$f_T(t) = \begin{cases} 100e^{-100t}, & t \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

  The conditioning event has probability

$$P[T > 0.02] = \int_{0.02}^{\infty} f_T(t)\ dt = -e^{-100t}|_{0.02}^{\infty} = e^{-2}$$

  The conditional PDF of T is

$$f_{T|T>0.02}(t) = \begin{cases} \frac{f_T(t)}{P[T>0.02]}, & t \geq 0.02 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 100e^{-100(t-0.02)}, & t \geq 0.02 \\ 0, & \text{otherwise} \end{cases}$$

Solution (continues)

The conditional expected value of T is

$$E[T|T>0.02] = \int_{-\infty}^{\infty} t f_{T|T>0.02}(t)dt = \int_{0.02}^{\infty} t(100)e^{-100(t-0.02)}dt$$
$$= \int_{0}^{\infty} (\tau+0.02)(100)e^{-100\tau}d\tau = \int_{0}^{\infty} (\tau+0.02)f_T(\tau)d\tau = E[T+0.02] = 0.03$$

(b) The conditional second moment of T is

$$E[T^2|T>0.02] = \int_{-\infty}^{\infty} t^2 f_{T|T>0.02}(t)dt = \int_{0.02}^{\infty} t^2(100)e^{-100(t-0.02)}dt$$
$$= \int_{0}^{\infty} (\tau+0.02)^2(100)e^{-100\tau}d\tau = \int_{0}^{\infty} (\tau+0.02)^2 f_T(\tau)d\tau = E[(T+0.02)^2]$$

Now, we can calculate the conditional variance as

$$\mathrm{Var}[T|T>0.02] = E[T^2|T>0.02] - (E[T|T>0.02])^2$$
$$= E[(T+0.02)^2] - (E[T+0.02])^2 = \mathrm{Var}[T+0.02] = \mathrm{Var}[T] = 0.01$$

Bayes' Formula

- For an event B with $P[B] \neq 0$,

$$\boxed{\begin{aligned}
&\text{Discrete X} : P[B|X=x] = \frac{P[B, X=x]}{P[X=x]} = \frac{P_X[x|B]P[B]}{P_X[x]} \\
&\text{Continuous X} : P[B|X=x] = \frac{f_X(x|B)P[B]}{f_X(x)} \\
&\text{Discrete X \& Y} : P_{Y|X}[y|x] = \frac{P_{XY}[x,y]}{P[X=x]} = \frac{P_{X|Y}[x|y]P_Y[y]}{P_X[x]} \\
&\text{Continuous X \& Y} : f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{f_{X|Y}(x|y)f_Y(y)}{f_X(x)} \\
&\text{Continuous X \& Discrete Y} : P_{Y|X}[y|x] = \frac{f_{X|Y}(x|y)P_Y[y]}{f_X(x)}
\end{aligned}}$$

- **Example for Bayes' formula:** A discrete-time signal r obtained from a monitoring system for the operation status of a device is given by $r = x + n$ where x is either 1 representing the good operation status of the target device or 0 representing the failure status of the device operation, and n is the Gaussian measurement noise with mean 0 and variance 0.01. The probability of good operation status of the device is 0.9.
  a) pdf of $r =$?
  b) If $r = 0.51$, what is the probability of x being 1?

Solution:

(a)

$$f_r(r) = f_{r|x=1}(r)P[x=1] + f_{r|x=0}(r)P[x=0]$$

$$\text{where} \quad f_{r|x=1}(r) = \mathcal{N}(1, 0.01) \quad \text{and} \quad f_{r|x=0}(r) = \mathcal{N}(0, 0.01)$$

$$\text{Thus,} \quad f_r(r) = \frac{0.9}{\sqrt{0.02\pi}} e^{-\frac{(r-1)^2}{0.02}} + \frac{0.1}{\sqrt{0.02\pi}} e^{-\frac{r^2}{0.02}}$$

(b) $P[x=1|r=0.51] =?$

$$P[x=1|r=0.51] = \frac{f_{r|x=1}(0.51)P[x=1]}{f_r(0.51)}$$

where

$$f_{r|x=1}(0.51)P[x=1] = \frac{0.9}{\sqrt{0.02\pi}} e^{-\frac{(0.51-1)^2}{0.02}} = 2.1951 \times 10^{-5}$$

$$f_r(0.51) = \frac{0.9}{\sqrt{0.02\pi}} e^{-\frac{(0.51-1)^2}{0.02}} + \frac{0.1}{\sqrt{0.02\pi}} e^{-\frac{0.51^2}{0.02}} = 2.2848 \times 10^{-5}$$

After substituting, $P[x=1|r=0.51] = 0.9607$

- **Example (Joint pdf of mixed RVs):**
  W = waiting time for a newly arriving customer at a restaurant (continuous RV)
  N = total # of customers (discrete integer RV)

  Suppose the joint CDF (for a continuous variable w and a discrete integer variable n) is given as

$$F_{W,N}(w,n) = \begin{cases} 0, & n < 0 \text{ or } w < 0, \\ (1-e^{-w/\mu_0})\frac{n}{10}, & 0 \le n < 5,\ w \ge 0, \\ (1-e^{-w/\mu_0})\frac{5}{10} + (1-e^{-w/\mu_1})\frac{n-5}{10}, & 5 \le n < 10,\ w \ge 0, \\ (1-e^{-w/\mu_0})\frac{5}{10} + (1-e^{-w/\mu_1})\frac{5}{10}, & n \ge 10,\ w \ge 0 \end{cases}$$

  Find the joint pdf.

Solution:

Joint mixed probability density - mass function:

$$f_{W,N}(w,n) \triangleq f_{W|N}(w|n)P_N(n) = \frac{\partial}{\partial w}\nabla_n F_{W,N}(w,n)$$

$$\text{where} \quad \frac{\partial}{\partial w}\nabla_n F_{W,N}(w,n) \triangleq \frac{\partial}{\partial w}\{F_{W,N}(w,n) - F_{W,N}(w,n-1)\}$$

$$\nabla_n F_{W,N}(w,n) = \begin{cases} (1 - e^{-w/\mu_0})\frac{1}{10}, & 1 \le n \le 5, w \ge 0, \\ (1 - e^{-w/\mu_1})\frac{1}{10}, & 6 \le n \le 10, w \ge 0, \\ 0, & \text{else.} \end{cases}$$

Note: n in $\nabla_n F_{W,N}(w,n)$ takes a discrete point and $\nabla_n F_{W,N}(w,0) = 0$.

$$f_{W,N}(w,n) = \begin{cases} \frac{1}{10}\frac{1}{\mu_0}e^{-w/\mu_0}, & 1 \le n \le 5, \ w \ge 0, \\ \frac{1}{10}\frac{1}{\mu_1}e^{-w/\mu_1}, & 6 \le n \le 10, \ w \ge 0, \\ 0, & \text{else.} \end{cases}$$

where n is a discrete (integer) variable and w is a continuous variable.

- **Example (Joint CDF of mixed RVs):**
Suppose the number of customers N is uniformly distributed in $[1, 10]$. The waiting time W is exponentially distributed with mean μ_0 for $1 \leq N \leq 5$ and mean μ_1 for $6 \leq N \leq 10$. The joint CDF = ?

$$f_{W|N}(w|n) = \begin{cases} \frac{1}{\mu_0} e^{-w/\mu_0}, & 1 \leq n \leq 5,\ w \geq 0, \\ \frac{1}{\mu_1} e^{-w/\mu_1}, & 6 \leq n \leq 10,\ w \geq 0, \\ 0, & \text{else.} \end{cases}$$

$$P_N[n] = \begin{cases} \frac{1}{10}, & n = 1, \ldots, 10 \\ 0, & \text{else} \end{cases}$$

$$F_{W|N}(w|k) = \int_{-\infty}^{w} f_{W|N}(u|k) du = \begin{cases} (1 - e^{-w/\mu_0}), & w \geq 0,\ 1 \leq k \leq 5 \\ (1 - e^{-w/\mu_1}), & w \geq 0,\ 6 \leq k \leq 10 \\ 0, & \text{else} \end{cases}$$

where k = integer.

Solution (continues):

First, consider positive integers for n.

$$F_{W,N}(w,n) = \sum_{k=1}^{n} F_{W|N}(w|k) P_N(k)$$
$$= \begin{cases} 0, & n \le 0 \text{ or } w < 0, \\ (1-e^{-w/\mu_0})\frac{n}{10}, & 1 \le n \le 5,\ w \ge 0, \\ (1-e^{-w/\mu_0})\frac{5}{10} + (1-e^{-w/\mu_1})\frac{n-5}{10}, & 6 \le n \le 10,\ w \ge 0, \\ (1-e^{-w/\mu_0})\frac{5}{10} + (1-e^{-w/\mu_1})\frac{5}{10}, & n \ge 11,\ w \ge 0 \end{cases}$$

Next, adjusting for a continuous variable n, we have

$$F_{W,N}(w,n) = \begin{cases} 0, & \lfloor n \rfloor \le 0 \text{ or } w < 0, \\ (1-e^{-w/\mu_0})\frac{\lfloor n \rfloor}{10}, & 1 \le \lfloor n \rfloor \le 5,\ w \ge 0, \\ (1-e^{-w/\mu_0})\frac{5}{10} + (1-e^{-w/\mu_1})\frac{\lfloor n \rfloor-5}{10}, & 6 \le \lfloor n \rfloor \le 10,\ w \ge 0, \\ (1-e^{-w/\mu_0})\frac{5}{10} + (1-e^{-w/\mu_1})\frac{5}{10}, & \lfloor n \rfloor \ge 11,\ w \ge 0 \end{cases}$$

Jointly Gaussian Random Variables

- X and Y are jointly Gaussian if

$$f_{XY}(x,y) = \frac{\exp\left(-\frac{1}{2(1-\rho^2)} \left\{ \left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 \right\} \right)}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}$$

($\rho =$ correlation coefficient)

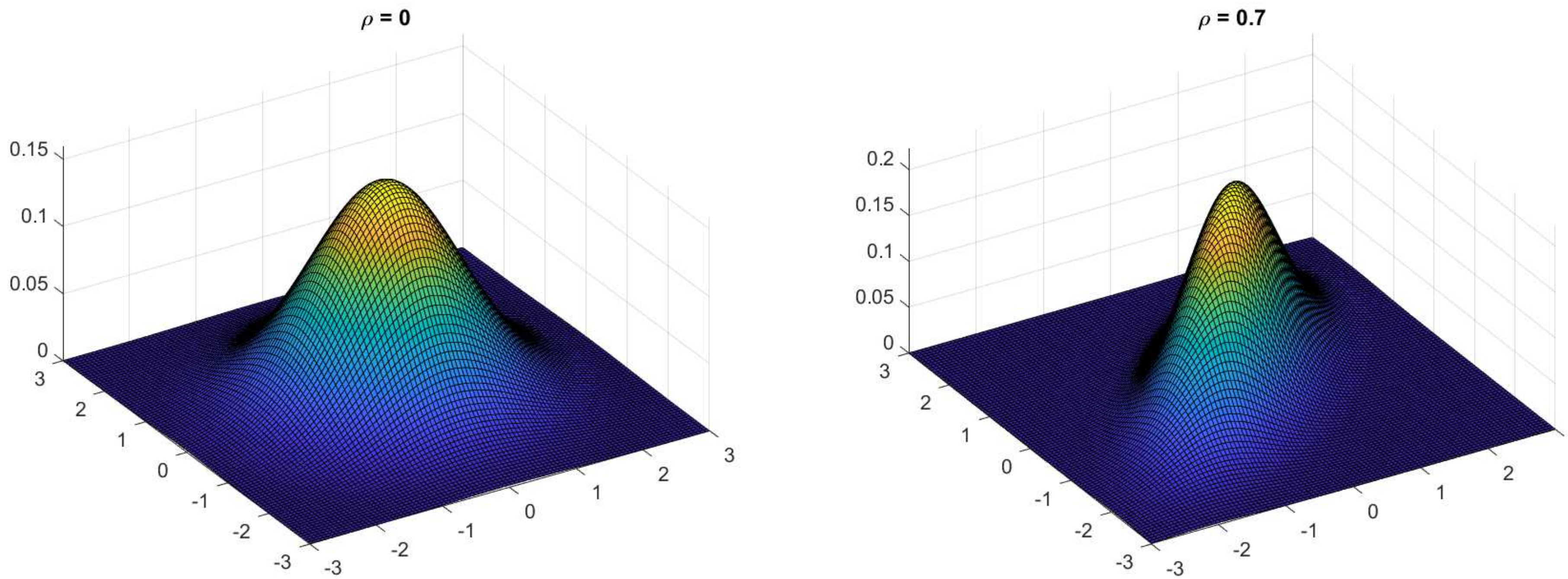


Fig. Jointly Gaussian pdf with $\sigma_x^2 = \sigma_y^2 = 1, \mu_x = \mu_y = 0$, $\rho = 0$ (Left) and $\rho = 0.7$ (Right)

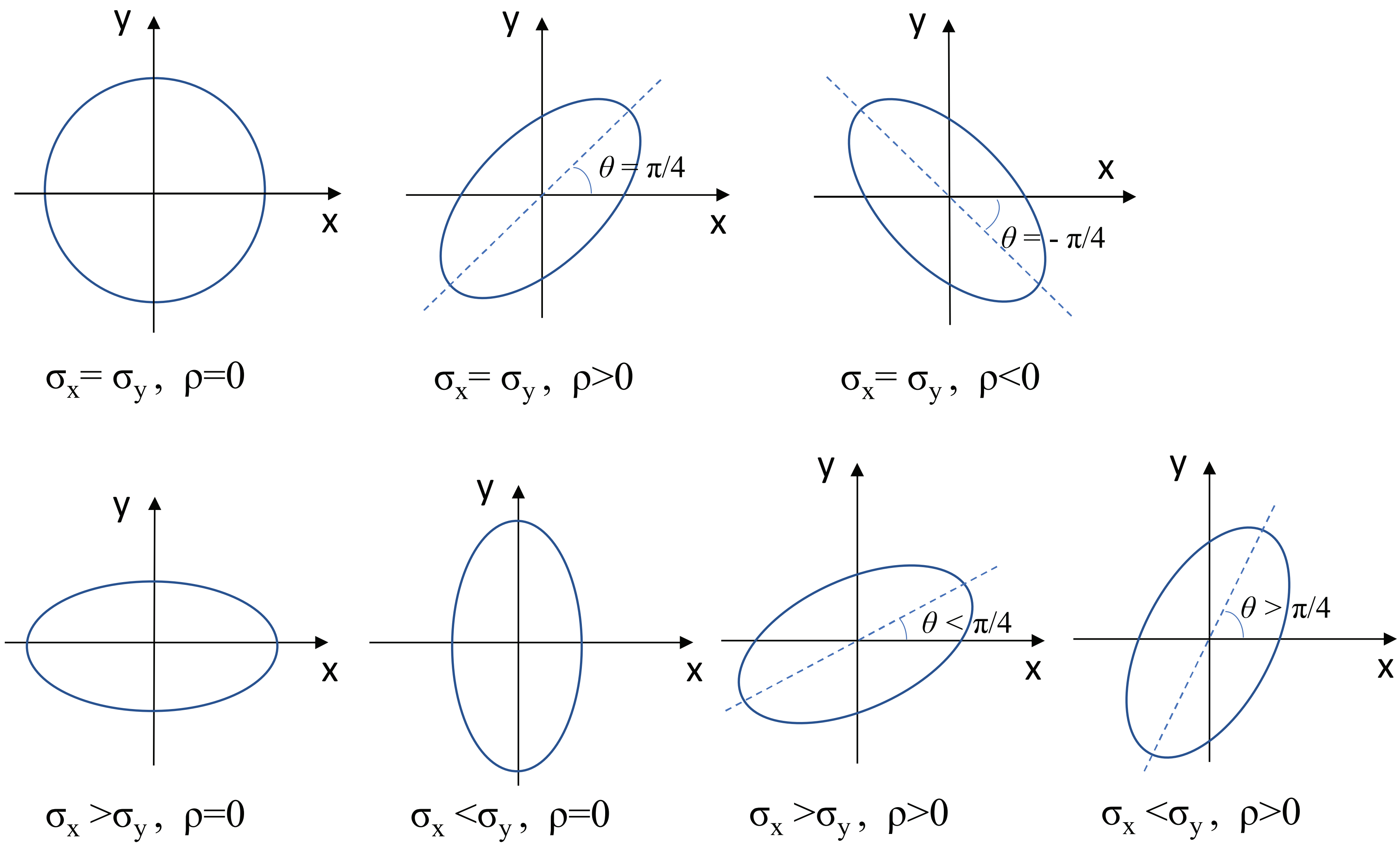


Fig. Contours of bi-variate jointly Gaussian pdf for various cases

Jointly Gaussian Random Variables (Continues)

- If X and Y are jointly Gaussian, then $f_X(x)$ and $f_Y(y)$ are Gaussian regardless of what ρ is, i.e., $f_X(x) = \mathcal{N}(\mu_x, \sigma_x^2)$ and $f_Y(y) = \mathcal{N}(\mu_y, \sigma_y^2)$. The converse does not always hold.
- If $\rho = 0$, X and Y are uncorrelated and independent, otherwise they are correlated and dependent.
- Conditional pdfs are also Gaussian:

$$f_{Y|X}(y|x) = \frac{1}{\tilde{\sigma}_y\sqrt{2\pi}} e^{-\frac{(y-\tilde{\mu}_y(x))^2}{2\tilde{\sigma}_y^2}}$$

$$f_{X|Y}(x|y) = \frac{1}{\tilde{\sigma}_x\sqrt{2\pi}} e^{-\frac{(x-\tilde{\mu}_x(y))^2}{2\tilde{\sigma}_x^2}}$$

$$\text{where} \quad \tilde{\mu}_y(x) = E[Y|X=x] = \mu_y + \rho\frac{\sigma_y}{\sigma_x}(x-\mu_x),$$

$$\tilde{\mu}_x(y) = E[X|Y=y] = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(x-\mu_y),$$

$$\tilde{\sigma}_y^2 = \mathrm{Var}[Y|X=x] = \sigma_y^2(1-\rho^2),$$

$$\tilde{\sigma}_x^2 = \mathrm{Var}[X|Y=y] = \sigma_x^2(1-\rho^2).$$

Failure Rate

- X = failure time (life time)

$$P[t < X \leq t+dt|X > t] = \frac{P[t < X \leq t+dt, \;\; X > t]}{P[X > t]}$$

$$= \frac{P[t < X \leq t+dt]}{P[X > t]} = \frac{F_X(t+dt) - F_X(t)}{1 - F_X(t)}$$

Using a Tailor expansion of $F_X(t+dt)$, i.e., $F_X(t+dt) = F_X(t) + f_X(t)dt$,

$$\boxed{P[t < X \leq t+dt|X > t] = \frac{f_X(t)\; dt}{1 - F_X(t)} = \alpha(t)\; dt}$$

$$\text{where } \boxed{\alpha(t) \triangleq \frac{f_X(t)}{1 - F_X(t)}}.$$

$\alpha(t)$ is called *conditional failure rate, hazard rate, force of mortality, intensity rate, instantaneous failure rate*, or simply *failure rate*.

- **CDF and pdf of failure time:**

With $F_X(t+dt) - F_X(t) = F'_X(t)dt = dF_X$ and

$$\int_a^b \frac{dy}{1-y} = -\int_{1-b}^{1-a} \frac{dy}{y} = \ln \frac{1-a}{1-b},$$

$$\text{we have} \quad \alpha(t)dt = \frac{F_X(t+dt) - F_X(t)}{1 - F_X(t)} = \frac{dF_X}{1-F_X} \quad \text{and}$$

$$\int_0^t \alpha(\tau)d\tau = \int_{F_X(0)=0}^{F_X(t)} \frac{dF_X}{1-F_X} = -\ln[1 - F_X(t)].$$

$$\text{Thus,} \quad \boxed{F_X(t) = 1 - e^{-\int_0^t \alpha(\tau)d\tau}}$$

$$\boxed{f_X(t) = \alpha(t)e^{-\int_0^t \alpha(\tau)d\tau}} \quad (\text{different } \alpha(t) \Rightarrow \text{different pdf})$$

$\alpha(t)$ is constant $\Leftrightarrow$ failure time X is exponential

$$\text{Note:} \quad f_X(x|X \geq t) = \begin{cases} 0, & x < t \\ \frac{f_X(x)}{1-F_X(t)}, & x \geq t \end{cases}$$

$$f_X(t|X \geq t) = \alpha(t)$$

Poisson Transform

- Poisson Transform ($f_X \to P_Y$): When the Poisson parameter (a constant in ordinary Poisson law) is a random variable (say X)), the PMF of Y, the number of arrivals within the observation interval, is

$$P_Y(k) = \int_0^\infty \frac{x^k}{k!} e^{-x} f_X(x) dx, \quad k \geq 0$$

- Inverse Poisson Transform ($P_Y \to f_X$):

$$F(w) \triangleq \frac{1}{2\pi} \int_0^\infty e^{jwx} e^{-x} f_X(x) dx, \quad \& \quad e^{jwx} = \sum_{k=0}^{\infty} (jwx)^k / k!$$

$$F(w) = \frac{1}{2\pi} \sum_{k=0}^{\infty} (jw)^k \int_0^\infty \frac{x^k}{k!} e^{-x} f_X(x) dx = \frac{1}{2\pi} \sum_{k=0}^{\infty} (jw)^k P_Y(k)$$

$$f_X(x) = e^x \int_{-\infty}^{\infty} F(w) e^{-jwx} dw$$

($f_X(x)$ can be computed from the experimental results of $P_Y(k)$)

Asymptotic relationship

- Binomial $\to$ Poisson
- Binomial $\to$ Gaussian
- Poisson $\to$ Gaussian

- **Asymptotic Behavior of Binomial Law: Poisson Law**
  For $b(k;n,p)$ with $n \gg 1, p \ll 1$, $\boxed{np = a}$, when $n \to \infty$, $p \to 0$, and $k \ll n$,

  $$\boxed{b(k;n,p) \simeq \frac{(np)^k}{k!} e^{-np}}$$

$$\text{Proof}: b(k;n,p) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} (\frac{a}{n})^k (1 - \frac{a}{n})^{n-k}$$

$$\simeq \frac{a^k}{k!} (1 - \frac{a}{n})^{n-k} \to \frac{a^k}{k!} e^{-a}. \quad (\Leftarrow \frac{n!}{(n-k)!\; n^k} \simeq 1;\ (1 - \frac{a}{n})^{n-k} \to e^{-a})$$

- Example: Suppose n independent points are placed at random in an interval $(0, T)$. For $\tau/T \ll 1$ and $n \gg 1$, P[observing exactly k points in an interval of τ] $= ?$

  P[a point appears in the interval of τ] $= \tau/T$
  P[k points appear in the interval of τ] $= p = \binom{n}{k} p^k (1-p)^{n-k}$
  Using above approx for $n \gg 1$,
  P[k points in the interval of τ] $\simeq (n\tau/T)^k \frac{e^{-(n\tau/T)}}{k!}$. (Poisson law)

- **Generalized Poisson Law:**

Poisson PMF with parameter a, $(a > 0)$ is

$$P[k \text{ points}] = e^{-a}\frac{a^k}{k!}, \quad k = 0, 1, 2, \ldots$$

With $a \triangleq \lambda\tau$, where λ is the average number of events per unit interval (e.g., time) and τ is the length of interval of interest $(t, t+\tau)$,

$$P[k \text{ events in } \tau] = P(k;\; t, t+\tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}$$

If λ depends on t,

$$P(k; t, t+\tau) = \exp\left[-\int_t^{t+\tau}\lambda(\xi)d\xi\right]\frac{1}{k!}\left[\int_t^{t+\tau}\lambda(\xi)d\xi\right]^k$$

- **Normal Approximation to Binomial Law** $(n \gg 1)$:

$S_n = \#$ of successes in n Bernoulli trials; P[success in each trial]$=p$

S_n is Binomial with mean np and variance npq, $(q = 1 - p)$

$f_{\mathrm{SN}}(x) =$ standard Normal probability density function

$\Phi(x) =$ standard Normal CDF

$X \sim \mathcal{N}(np, npq)$ (Gaussian with mean np and variance npq)

Note: If $Y = aX + b$, $f_Y(y) = \frac{1}{|a|} f_X(\frac{y-b}{a})$ and $F_Y(y) = F_X(\frac{y-b}{a})$.

For $X \sim \mathcal{N}(\mu, \sigma^2)$, $F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$.

When $n \gg 1$,

$$P[\alpha \leq S_n \leq \beta] \simeq P[\alpha - 0.5 < X \leq \beta + 0.5],$$

$$\simeq \Phi\left(\frac{\beta - np + 0.5}{\sqrt{npq}}\right) - \Phi\left(\frac{\alpha - np - 0.5}{\sqrt{npq}}\right)$$

$$P[S_n = k] = b(k; n, p) \simeq P[k - 0.5 < X \leq k + 0.5]$$

$$\simeq (1) f_X(k) = \frac{1}{\sqrt{npq}} f_{\mathrm{SN}}\left(\frac{k - np}{\sqrt{npq}}\right)$$

- **Normal Approximation to Poisson Law ($\lambda\tau \gg 1$):**
  (extension of the normal approx to Binomial law)

  For Poisson RV Y with mean $\lambda\tau$ and variance $\lambda\tau$, and $X \sim \mathcal{N}(\lambda\tau, \lambda\tau)$,

$$P[\alpha \le Y \le \beta] \simeq P[\alpha - 0.5 < X \le \beta + 0.5]$$

$$\sum_{k=\alpha}^{\beta} e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!} \simeq \Phi\left(\frac{\beta - \lambda\tau + 0.5}{\sqrt{\lambda\tau}}\right) - \Phi\left(\frac{\alpha - \lambda\tau - 0.5}{\sqrt{\lambda\tau}}\right)$$

$$e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!} \simeq \Phi\left(\frac{k - \lambda\tau + 0.5}{\sqrt{\lambda\tau}}\right) - \Phi\left(\frac{k - \lambda\tau - 0.5}{\sqrt{\lambda\tau}}\right)$$