EESC 6349 / MECH 6312
Probabilities, Random Variables, and Statistics

Dr. H. Minn
UT-Dallas

Email: hlaing.minn@utdallas.edu

Main Topics

- Probability Theory
- Random Variables
- Functions of Random Variables
- Expectation and LLNs
- Random Vectors
- Parameter Estimation
- Statistical Testing
- Intro to Random Processes

Probability Theory

- Set Algebra
- Axioms of Probability
- Theorems in Probability Theory
- Combinatorics/Counting/Bernoulli Trials and Probabilities
- Reliability

Applying Set Theory To Probability

- Random Experiment $\mathcal{H}$
  e.g., Consider an experiment of throwing a six sided die.
- Outcome [Element in Set theory]
  e.g., the outcome will be 1, 2, 3, 4, 5, or 6.
- Sample Space Ω (the set of all outcomes of $\mathcal{H}$) [Universal set]
  e.g., $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Event (a subset of Ω) [Set]
  e.g., E_1 = outcome is even = $\{2, 4, 6\}$

Set Algebra

- Union ($E \cup F$ or $E + F$)
  e.g., if $E = \{1, 2, 3\}$ and $F = \{3, 4\}$, then $E \cup F = \{1, 2, 3, 4\}$.
- Intersection ($E \cap F$ or EF)
  e.g., if $E = \{1, 2, 3\}$ and $F = \{3, 4\}$, then $EF = \{3\}$.
- Complement (E^c)
  e.g., if $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $E = \{1, 2, 3\}$, $E^c = \{4, 5, 6\}$.
- Difference ($E \setminus F$, $F \setminus E$, or $E - F$, $F - E$)
  e.g., if $E = \{1, 2, 3\}$ and $F = \{3, 4\}$, then $E \setminus F = \{1, 2\}$ and $F \setminus E = \{4\}$.
- Exclusive OR ($E \oplus F$: in E or F but not both)
  e.g., if $E = \{1, 2, 3\}$ and $F = \{3, 4\}$, then $E \oplus F = \{1, 2, 4\}$.
- Set Equality ($E = F$ if $E \subset F$ & $F \subset E$)
- Definition: Event Space ($\{E_i : \bigcup_i E_i = \Omega$ (exhaustive) & $E_i \cap E_j = \emptyset \ \forall i \neq j$ (mutually exclusive / disjoint)$\}$)
  e.g., $E_1 = \{2, 4, 6\}$ and $E_2 = \{1, 3, 5\}$ form an event space.
  Or, $E_1 = \{1, 2\}$, $E_2 = \{3, 4\}$, and $E_3 = \{5, 6\}$ form an event space.

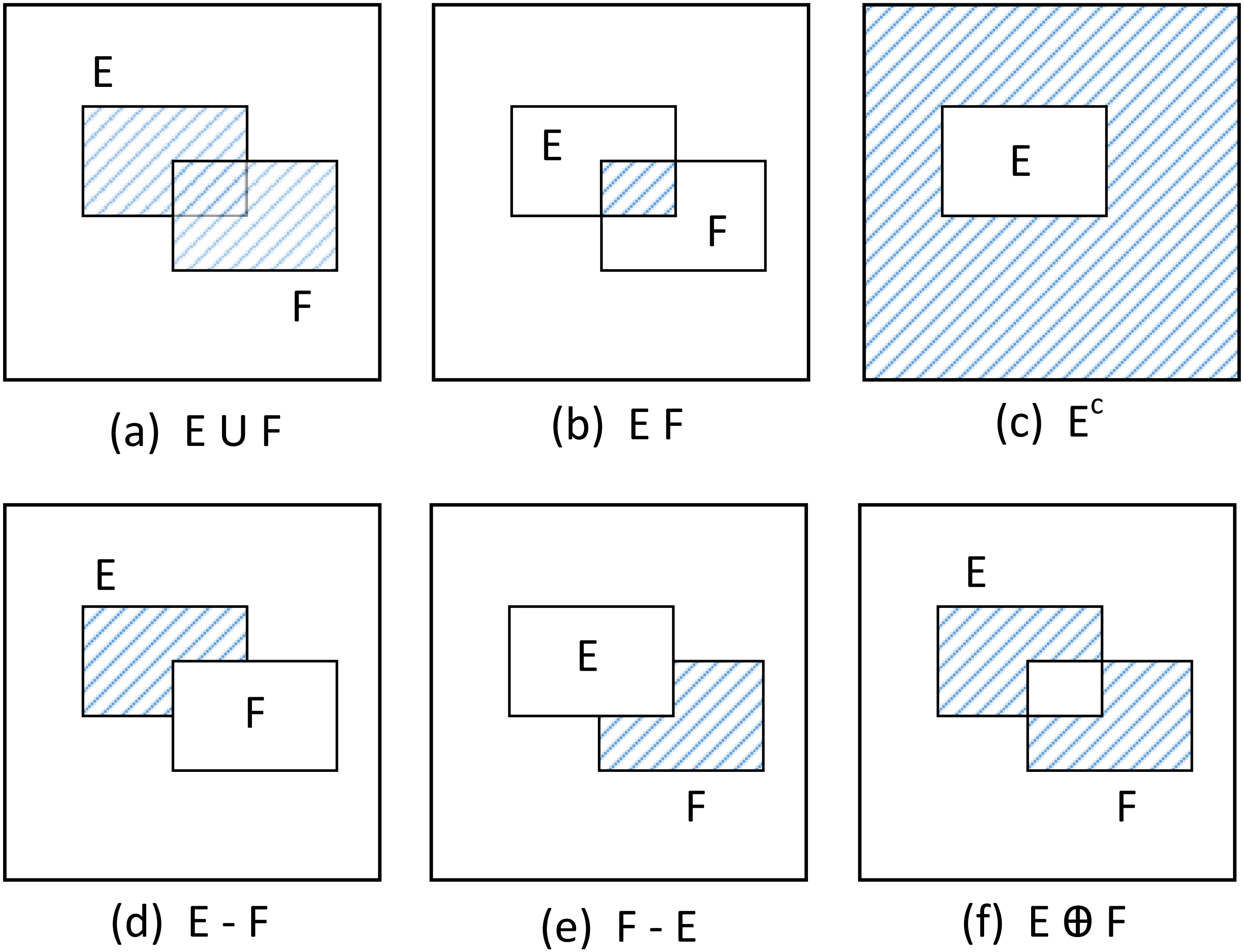


Figure: Venn Diagram Illustration of Set Algebra

Set Algebra

- Commutative Property:
  $A \cup B = B \cup A, \qquad A \cap B = B \cap A$
- Associative Property:
  $A \cup (B \cup C) = (A \cup B) \cup C, \qquad A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive Property:
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- De Morgan's Laws:
  $[\bigcup_{i=1}^{n} E_i]^c = \bigcap_{i=1}^{n} E_i^c, \qquad [\bigcap_{i=1}^{n} E_i]^c = \bigcup_{i=1}^{n} E_i^c$
  e.g., $\Omega = \{1,2,3,4,5,6\}$, $E_1 = \{1,2,3\}$, $E_2 = \{1,3,5\}$, $E_3 = \{2,4,5\}$.
  $\left[\bigcup_{i=1}^{3} E_i\right]^c = [\{1,2,3,4,5\}]^c = \{6\}$
  $\bigcap_{i=1}^{3} E_i^c = \{4,5,6\} \cap \{2,4,6\} \cap \{1,3,6\} = \{6\}$
  $\left[\bigcap_{i=1}^{3} E_i\right]^c = \Omega$
  $\bigcup_{i=1}^{3} E_i^c = \{4,5,6\} \cup \{2,4,6\} \cup \{1,3,6\} = \Omega$
- For an event space $\{E_i\}$, an arbitrary event $A = \bigcup_i AE_i$
  e.g., if $E_1 = \{1,2\}, E_2 = \{3,4\}, E_3 = \{5,6\}$, and $A = \{2,3,4\}$, then $AE_1 = \{2\}$, $AE_2 = \{3,4\}$, $AE_3 = \emptyset$ and $AE_1 \cup AE_2 \cup AE_3 = \{2,3,4\} = A$.

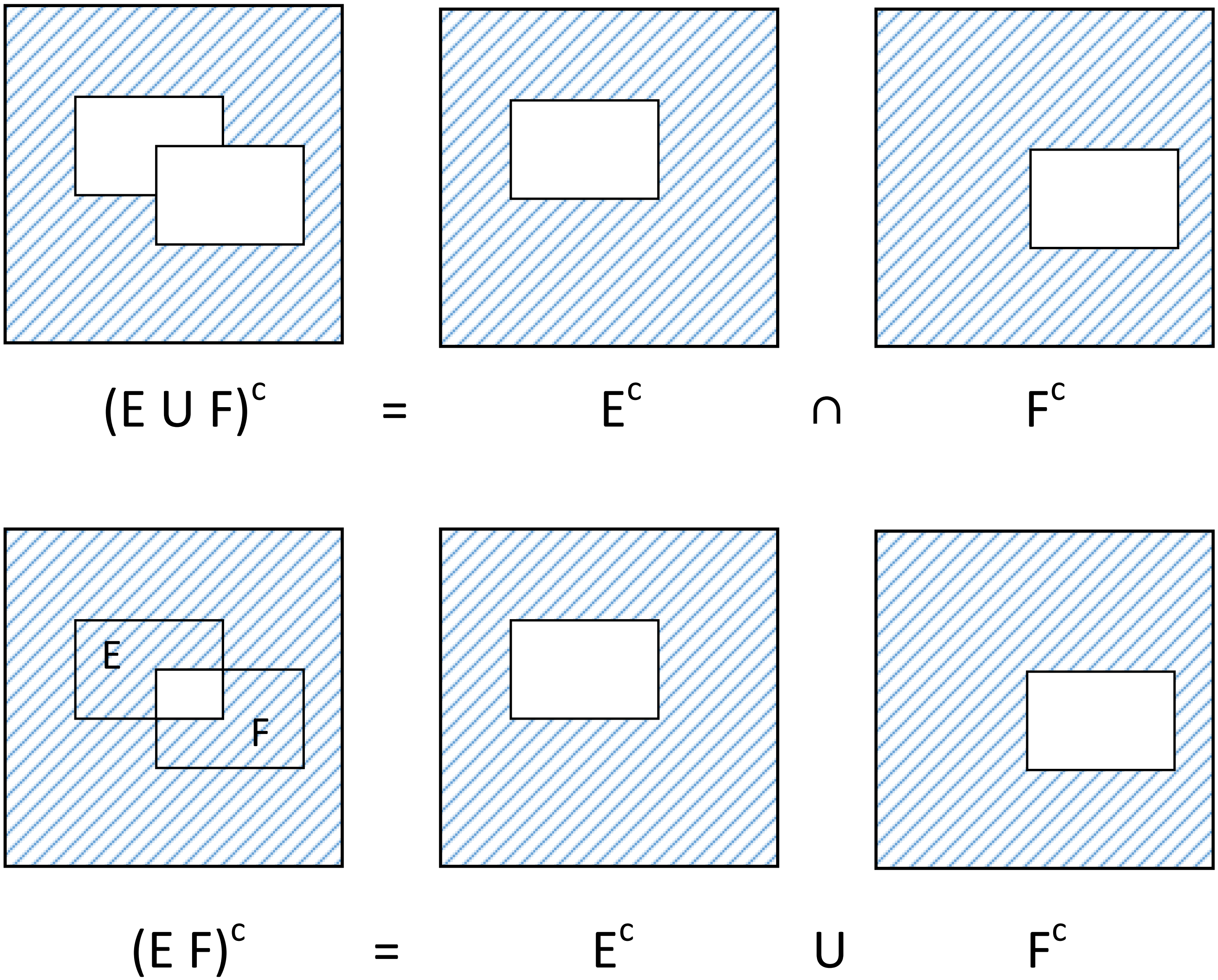


Figure: Venn Diagram Illustration of De Morgan's Laws

Probability Space

- Field $\mathcal{F}$: A collection of sets which satisfies (i) $\emptyset \in \mathcal{F}$, $\Omega \in \mathcal{F}$, (ii) if $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$, and (iii) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
- Sigma Field (σ-field $\mathcal{F}$): a field that is closed under any countable set of unions, intersections, and combinations, i.e., if $E_i \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{F}$, and $\bigcap_{i=1}^{\infty} E_i \in \mathcal{F}$.
  e.g., flipping a coin: $\mathcal{F}$ is $\emptyset, \Omega, \{H\}, \{T\}$
- Borel Field: a sigma field on the real line R (or rectangle R^2) (Ω is uncountable); (the sigma field generated by the countable unions, countable intersections, and complements of events of the form $(-\infty, b])$.
- Probability Space ($\mathcal{R}$): defined by a triplet $(\Omega, \mathcal{F}, P)$ where P is a probability measure.

Axioms of Probability

- Probability is a set function $P[\cdot]$ that assigns to every event $E \in \mathcal{F}$ a number $P[E]$ called the probability of E such that
  - Axiom 1: $P[E] \geq 0$
  - Axiom 2: $P[\Omega] = 1$
  - Axiom 3: $P[E \cup F] = P[E] + P[F]$ if $EF = \emptyset$.
  - Axiom 3A: $P[\cup_{i=1}^{\infty} E_i] = \sum_{i=1}^{\infty} P[E_i]$ if $E_i E_j = \emptyset, \ \forall i \neq j$.

Consequences of Axioms of Probability

- $P[\emptyset] = 0$.
- If $E \subset F$, then $P[E] \leq P[F]$.
- $P[EF^c] = P[E] - P[EF]$ where $E \in \mathcal{F}$, $F \in \mathcal{F}$.
- $P[E] = 1 - P[E^c]$.
- $P[E \cup F] = P[E] + P[F] - P[EF]$.
- $P\left[\bigcup_{i=1}^{n} E_i\right] = \sum_{i=1}^{n} P[E_i]$ if $E_i E_j = \emptyset \ \forall i \neq j$.
- Union Bound: $P\left[\bigcup_{i=1}^{n} E_i\right] \leq \sum_{i=1}^{n} P[E_i]$.

Conditional Probability

- $P[A \text{ given } B]$: $\boxed{P[A|B] = P[AB]/P[B]}$, $P[B] > 0$
- $P[B \text{ given } A]$: $\boxed{P[B|A] = P[AB]/P[A]}$, $P[A] > 0$
- $P[AB \text{ given } C]$: $\boxed{P[AB|C] = P[ABC]/P[C]}$, $P[C] > 0$
- $P[AB \text{ given } CD]$: $\boxed{P[AB|CD] = P[ABCD]/P[CD]}$, $P[CD] > 0$

  e.g., Equal probable outcomes with $\Omega = \{1, 2, 3, 4, 5, 6\}$.
  $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{1, 3, 5\}$, $D = \{1, 3, 4, 6\}$, $E = \{3, 4\}$, $F = \{2\}$.
  $P[A] = 1/3,\ P[B] = 1/3,\ P[C] = 1/2,\ P[D] = 2/3,\ P[E] = 1/3,\ P[F] = 1/6$
  $P[A|B] = P[AB]/P[B] = P[\{1\}]/P[B] = (1/6)/(1/3) = 1/2$
  $P[A|C] = P[AC]/P[C] = P[\{1\}]/P[C] = (1/6)/(1/2) = 1/3$
  $P[A|D] = P[AD]/P[D] = P[\{1\}]/P[D] = (1/6)/(2/3) = 1/4$
  $P[A|E] = P[AE]/P[E] = P[\emptyset]/P[E] = 0$
  $P[A|F] = P[AF]/P[F] = P[\{2\}]/P[F] = (1/6)/(1/6) = 1$
  $P[AB|C] = P[ABC]/P[C] = P[\{1\}]/P[C] = (1/6)/(1/2) = 1/3$
  $P[AB|CD] = P[ABCD]/P[CD] = P[\{1\}]/P[\{1, 3\}] = (1/6)/(1/3) = 1/2$

Independence

- $A \in \mathcal{F}$ and $B \in \mathcal{F}$ with $P[A] > 0$, $P[B] > 0$.

$$\boxed{A \text{ and } B \text{ are independent iff } P[AB] = P[A]P[B]}$$

$$\boxed{\Rightarrow P[A|B] = P[A],\ P[B|A] = P[B]}$$

- A, B, C defined on $\mathcal{P}$ and having nonzero probabilities are independent iff

$$\boxed{\begin{aligned} & P[ABC] = P[A]P[B]P[C], \\ \text{and} \quad & P[AB] = P[A]P[B], \quad P[AC] = P[A]P[C], \quad P[BC] = P[B]P[C] \end{aligned}}$$

- A_i, $i = 1, \ldots, n$ are independent iff

$$\begin{aligned} P[A_i A_j] &= P[A_i]P[A_j], \forall i \neq j \\ P[A_i A_j A_k] &= P[A_i]P[A_j]P[A_k], \forall i \neq j \neq k \\ \ldots &= \ldots \\ P\left[\bigcap_{i=1}^{n} A_i\right] &= \prod_{i=1}^{n} P[A_i] \end{aligned}$$

- **Example:** Consider a sample space $\Omega = \{1,2,3,4,5,6,7,8\}$ with equi-probable outcomes.
  i) Are the events $E_1 \triangleq \{1,2,3,4\}$ and $E_2 \triangleq \{5,6,7,8\}$ independent?
  $P[E_1 E_2] = P[\emptyset] = 0 \neq P[E_1]P[E_2] = 1/4.$
  Hence, E_1 and E_2 are not independent. (Disjoint events are not independent)

  ii) Are the events $A_1 \triangleq \{1,2,3,4\}$, $A_2 \triangleq \{2,3,4,5\}$, and $A_3 \triangleq \{4,6,7,8\}$ independent ?
  $P[A_1] = P[A_2] = P[A_3] = 1/2$, $P[A_1 A_2 A_3] = P[\{4\}] = 1/8 = P[A_1]P[A_2]P[A_3]$
  But $P[A_1 A_2] = P[\{2,3,4\}] = 3/8 \neq P[A_1]P[A_2]$
  Hence, A_1, A_2, A_3 are not independent.

  iii) Are the events $C_1 \triangleq \{1,2,3,4\}$, $C_2 \triangleq \{3,4,5,6\}$, and $C_3 \triangleq \{1,3,5,7\}$ independent ?
  $P[C_1] = P[C_2] = P[C_3] = 1/2$
  $P[C_1 C_2 C_3] = P[\{3\}] = 1/8 = P[C_1]P[C_2]P[C_3]$
  $P[C_1 C_2] = 1/4 = P[C_1]P[C_2]$; $P[C_1 C_3] = 1/4 = P[C_1]P[C_3]$;
  $P[C_2 C_3] = 1/4 = P[C_2]P[C_3]$;
  Hence, C_1, C_2, C_3 are independent.

Theorems in Probability Theory

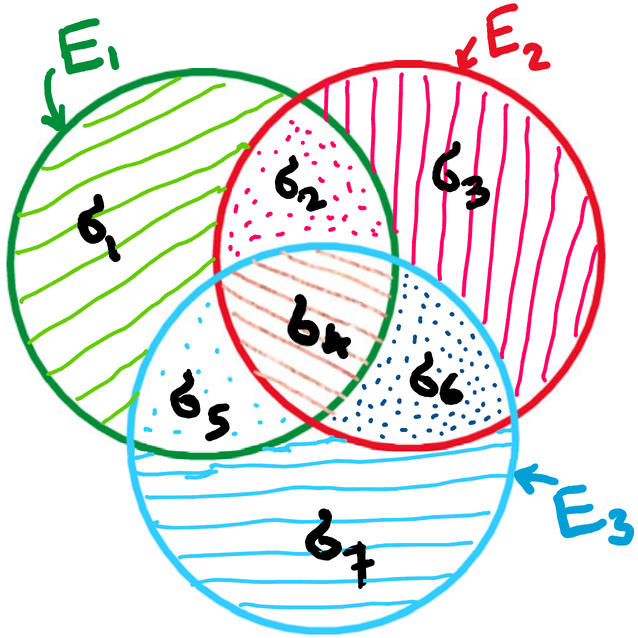
- **Probability of Union of Events**: (P[at least one among $E_1, \ldots, E_n$ occurs])

$$P[\bigcup_{i=1}^{n} E_i] = \sum_{i=1}^{n} (-1)^{i+1} S_i$$

where $S_k \triangleq \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} P_{i_1 i_2 \ldots i_k}$ and $P_{i_1 i_2 \ldots i_k} \triangleq P[E_{i_1} E_{i_2} \ldots E_{i_k}]$.

$$\boxed{\begin{aligned} & P\left[\bigcup_{i=1}^{4} E_i\right] = S_1 - S_2 + S_3 - S_4 \\ \text{where}\quad & S_1 = P[E_1] + P[E_2] + P[E_3] + P[E_4], \\ & S_2 = P[E_1E_2] + P[E_1E_3] + P[E_1E_4] + P[E_2E_3] \\ & \qquad + P[E_2E_4] + P[E_3E_4], \\ & S_3 = P[E_1E_2E_3] + P[E_1E_2E_4] + P[E_1E_3E_4] + P[E_2E_3E_4], \\ & S_4 = P[E_1E_2E_3E_4] \end{aligned}}$$

- **Example:** $\bigcup_{i=1}^{3} E_i$ is composed of 7 disjoint regions $\sigma_1 \ldots \sigma_7$



$$
\begin{aligned}
E_1 &= \sigma_1 \cup \sigma_2 \cup \sigma_4 \cup \sigma_5 \\
E_2 &= \sigma_2 \cup \sigma_3 \cup \sigma_4 \cup \sigma_6 \\
E_3 &= \sigma_4 \cup \sigma_5 \cup \sigma_6 \cup \sigma_7 \\
E_1E_2 &= \sigma_2 \cup \sigma_4 \\
E_1E_3 &= \sigma_4 \cup \sigma_5 \\
E_2E_3 &= \sigma_4 \cup \sigma_6 \\
E_1E_2E_3 &= \sigma_4
\end{aligned}
$$

Define $P[\sigma_i] = p_i$. Then,

$S_1 = P[E_1] + P[E_2] + P[E_3] = p_2 + 2p_4 + p_5 + p_6 + \sum_{i=1}^{7} p_i$

$S_2 = P[E_1E_2] + P[E_1E_3] + P[E_2E_3] = (p_2 + p_4) + (p_4 + p_5) + (p_4 + p_6)$

$S_3 = P[E_1E_2E_3] = p_4$

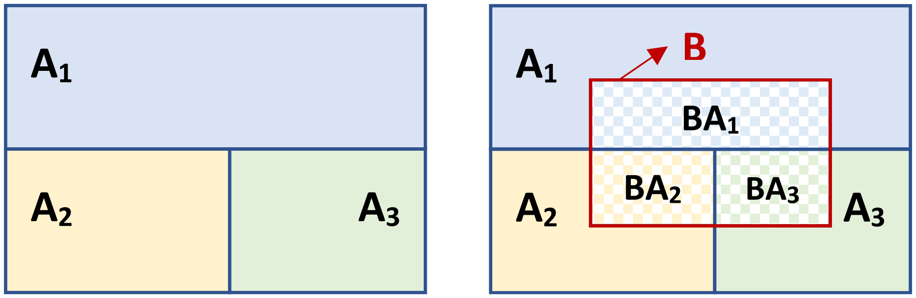
$P[E_1 \cup E_2 \cup E_3] = S_1 - S_2 + S_3 = \sum_{i=1}^{7} p_i$

$P[E_1 \cup E_2 \cup E_3] = P[\cup_{i=1}^{7} \sigma_i] = \sum_{i=1}^{7} p_i$

Theorems in Probability Theory

- **Law of Total Probability**: For an event space $\{A_i : i = 1, \ldots, n\}$ (i.e., $\bigcup_{i=1}^n A_i = \Omega$, $A_i A_j = \emptyset, \forall i \neq j$),
  $\boxed{P[B] = \sum_{i=1}^n P[BA_i] = \sum_{i=1}^n P[B|A_i]P[A_i]}$

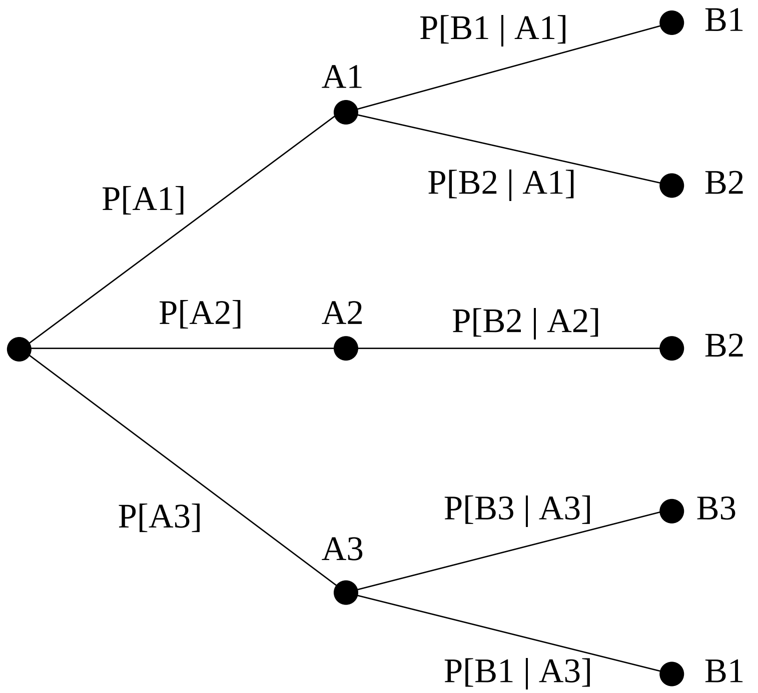
Note: $B = BA_1 \cup BA_2 \cup \ldots \cup BA_n$



- **Bayes' Theorem**: For an event space $\{A_i : i = 1, \ldots, n\}$ and any event B defined on $\mathcal{P}$ with $P[B] > 0$,
  $$\boxed{P[A_j \mid B] = \frac{P[A_j B]}{P[B]} = \frac{P[B|A_j]P[A_j]}{\sum_{i=1}^n P[B|A_i]P[A_i]}.}$$
  - $$\boxed{P[A \mid B] = \frac{P[B|A]P[A]}{P[B]} \quad \text{and} \quad P[B \mid A] = \frac{P[A|B]P[B]}{P[A]}, \quad \ldots}$$

Tree Diagram

- **Example:** Two-stage sequential experiment



P[A1 B1] = P[A1] P[B1 | A1]

P[A1 B2] = P[A1] P[B2 | A1]

P[A2 B2] = P[A2] P[B2 | A2]

P[A3 B3] = P[A3] P[B3 | A3]

P[A3 B1] = P[A3] P[B1 | A3]

$$P[B_i] = ?, \qquad P[A_i|B_k] = ?$$

$$P[B_1] = P[A_1B_1] + P[A_3B_1], \quad P[B_2] = P[A_1B_2] + P[A_2B_2], \quad P[B_3] = P[A_3B_3]$$

$$P[A_1|B_1] = \frac{P[A_1B_1]}{P[B_1]} = \frac{P[A_1B_1]}{P[A_1B_1] + P[A_3B_1]}$$

$$P[A_3|B_3] = \frac{P[A_3B_3]}{P[B_3]} = 1, \qquad P[A_2|B_3] = \frac{P[A_2B_3]}{P[B_3]} = 0$$

- **Example:** Binary communication channel subject to noise

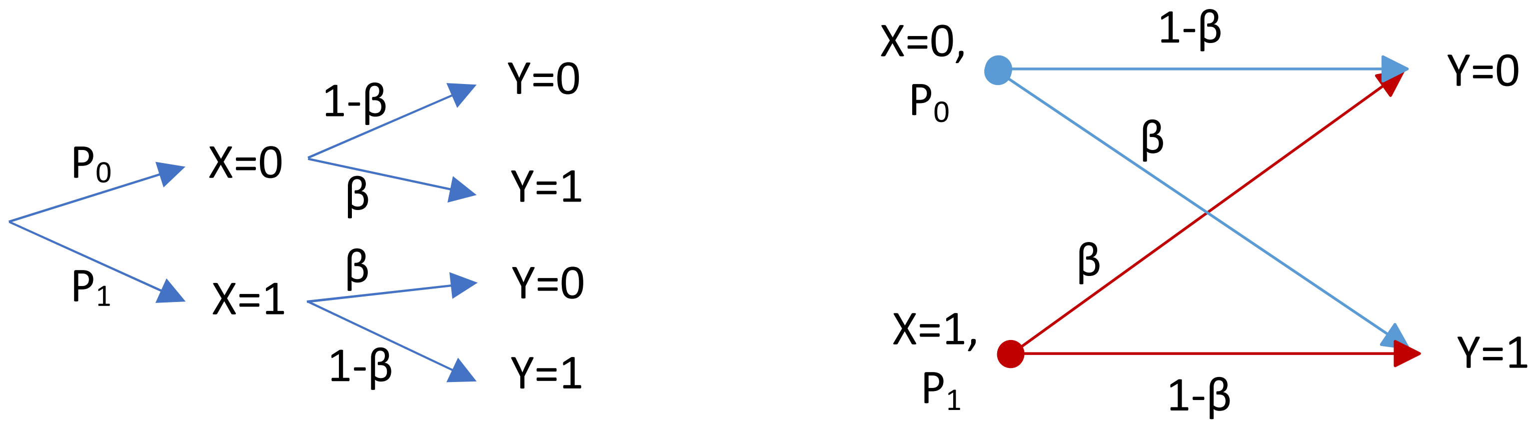


Fig. Tree diagrams of a binary communication channel.
(X = transmitted bit, Y = received bit)

$$P[Y=1] = P_0\beta + P_1(1-\beta), \quad P[Y=0] = P_0(1-\beta) + P_1\beta$$

$$P[X=1|Y=1] = \frac{P[Y=1|X=1]\ P[X=1]}{P[Y=1]} = \frac{P_1(1-\beta)}{P_0\beta + P_1(1-\beta)}$$

$$P[X=1|Y=0] = \frac{P[Y=0|X=1]\ P[X=1]}{P[Y=0]} = \frac{P_1\beta}{P_0(1-\beta) + P_1\beta}$$

$$P[X=0|Y=1] = \frac{P[Y=1, X=0]}{P[Y=1]} = \frac{P_0\beta}{P_0\beta + P_1(1-\beta)}$$

$$P[X=0|Y=1] = 1 - P[X=1|Y=1] = \frac{P_0\beta}{P_0\beta + P_1(1-\beta)}$$

- **Example:** Accuracy of a (fictitious) cancer test
  $A =$ test states that tested person has cancer.
  $B =$ person has cancer.
  $A^c =$ test states person is free from cancer.
  $B^c =$ person is free from cancer.
  $P[A|B] = P[A^c|B^c] = 0.95$ and $P[B] = 0.005$. Is it a good test? Or, what is the likelihood that a person actually has cancer if the test so states, that is, $P[B|A] =?$.

$$P[B|A] = \frac{P[AB]}{P[A]} = \frac{P[A|B]P[B]}{P[A|B]P[B] + P[A|B^c]P[B^c]}$$
$$= \frac{(0.005)(0.95)}{(0.95)(0.005) + (0.05)(0.995)} = 0.087$$

In only 8.7% of the cases where the tests are positive will the person actually have cancer.
(not a good test).

Combinatorics / Counting Methods

- From a population of n elements $a_1, \ldots, a_n$, # of groups (sub-populations) of size r=?
- # of diff. ordered samples $a_{k_1}, a_{k_2}, \ldots, a_{k_r}$ of size r for sampling with replacement = $\boxed{n^r}$

  e.g., For $n = 6$, $a_i = i$ for $i = 1, \ldots, 6$, and $r = 2$, we have $6^2 = 36$ different ordered samples,
  i.e., $(1,1), (1,2), \ldots, (1,6)$, $(2,1)$, $(2,2)$, $\ldots$, $(2,6)$, $\ldots$, $(6,1), (6,2), \ldots, (6,6)$

- # of diff. ordered samples of size r for sampling without replacement = $\boxed{(n)_r \triangleq n(n-1)(n-2)\ldots(n-r+1) = n!/(n-r)!}$

  e.g., For $n = 6$, $a_i = i$ for $i = 1, \ldots, 6$, and $r = 2$, we have $6!/(6-2)! = 30$ different ordered samples,
  i.e., $(1,2), (1,3), \ldots, (1,6)$, $(2,1)$, $(2,3)$, $\ldots$, $(2,6)$, $\ldots$, $(6,1)$, $(6,2)$, $\ldots$, $(6,5)$

- **Binomial Law**: $\#$ of sub-populations of size r in a population of size n for non-ordered sampling without replacement ($\#$ of Combinations) =

$$\boxed{C_r^n = \frac{(n)_r}{r!} = \frac{n!}{(n-r)!r!} \triangleq \left(\begin{array}{c} n \\ r \end{array} \right)}$$

e.g., For $n = 6$, $a_i = i$ for $i = 1, \ldots, 6$, and $r = 2$, we have $C_2^6 = 6!/(2!4!) = 15$ different ways, i.e., $(1,2), (1,3)$, $(1,4), (1,5)$, $(1,6)$, $(2,3)$, $(2,4)$, $(2,5)$, $(2,6)$, $(3,4)$, $(3,5)$, $(3,6)$, $(4,5)$, $(4,6)$, $(5,6)$

- Binomial coefficient: $C_r^n = C_{n-r}^n$
- **Multinomial Law**:
  Let $r_1, \ldots, r_k$ be a set of nonnegative integers such that $r_1 + r_2 + \ldots + r_k = n$. The number of ways in which a population of n elements can be partitioned into k subpopulations of which the first contains r_1 elements, the second r_2, and so forth, is

$$\left(\begin{array}{c} n \\ r_1 \end{array} \right) \left(\begin{array}{c} n - r_1 \\ r_2 \end{array} \right) \ldots \left(\begin{array}{c} n - r_1 - \ldots - r_{k-2} \\ r_{k-1} \end{array} \right)$$

$$\boxed{= \frac{n!}{r_1!r_2!\ldots r_k!} \qquad \text{(multinomial coefficient)}}$$

Independent Repeated Trials

- Sample space of n independent repeated trials: $\boxed{\Omega_n = \Omega \times \Omega \times \ldots \times \Omega}$ (Cartesian Product)
- If Ω consists of k outcomes, then Ω_n contains k^n outcomes, and corresponding $\mathcal{F}$ contains 2^{k^n} events.
  e.g., if $\Omega = \{s, f\}$, then $\Omega_2 = \{ss, sf, fs, ff\}$ and corresponding $\mathcal{F}$ contains 2^4 events.
- Bernoulli Trials: n independent repeated trials where Ω consists of two outcomes (e.g., $\{s, f\}$)
  Example: Suppose b_1, b_2, and b_3 are independent and identically distributed random bits with $P[b_i = 1] = p$ and $P[b_i = 0] = 1 - p$.
  The sample space is $\Omega_3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$.
  $$\begin{aligned} P[b_1 b_2 b_3 = 001] &= P[b_1 = 0]P[b_2 = 0]P[b_3 = 1] = (1-p)^2 p \\ &= P[b_1 b_2 b_3 = 010] = P[b_1 b_2 b_3 = 100] \end{aligned}$$
  The number of outcomes containing exactly one 1 is $\left(\begin{array}{c} 3 \\ 1 \end{array} \right) = 3$.
  $P[\text{the outcome containing exactly one 1}] = \left(\begin{array}{c} 3 \\ 1 \end{array} \right) (1-p)^2 p$

- **Binomial Probability Law**: Probability of A_k (k success in n independent trials with $P[s] = p$)

$$\boxed{P[A_k] \triangleq b(k;n,p) = \binom{n}{k} p^k (1-p)^{n-k}}$$

- Binomial (Cumulative) Distribution Function:
  (Probability of at most k successes in n independent trials)
  $B(k;n,p) = \sum_{i=0}^{k} b(i;n,p)$
- **Multinomial Probability Law**:
  Consider a generalized Bernoulli trial with k outcomes ζ_i, $i = 1, \ldots, k$, $P[\zeta_i] = p_i$, and $\sum_{i=1}^{k} r_i = n$.
  $P[\zeta_i$ occurs r_i times, $i = 1, \ldots, k] =$

$$\boxed{P(\boldsymbol{r}; n, \boldsymbol{p}) = \frac{n!}{r_1! r_2! \ldots r_k!} \, p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}}$$

Stirling's Formula: $n! \approx \sqrt{2\pi n} \; n^n e^{-n}$ (for large n)

- **Example:** A submarine attempts to sink an aircraft carrier. It will be successful only if two or more torpedoes hit the carrier. If the sub fires three torpedoes, and the probability of a hit is 0.4 for each torpedo, what is the probability that the carrier will be sunk ?

$$P[2 \text{ hits}] = \binom{3}{2}(0.4)^2(0.6)^1 = 0.288$$

$$P[3 \text{ hits}] = \binom{3}{3}(0.4)^3(0.6)^0 = 0.064$$

$$P[\text{carrier sunk}] = P[\text{two or more hits}] = P[2 \text{ hits}] + P[3 \text{ hits}] = 0.352$$

$$\text{OR,} \quad P[\text{no hits}] = \binom{3}{0}(0.4)^0(0.6)^3 = 0.216$$

$$P[1 \text{ hit}] = \binom{3}{1}(0.4)^1(0.6)^2 = 0.432$$

$$P[\text{carrier sunk}] = 1 - (P[\text{no hits}] + P[1 \text{ hit}]) = 0.352$$

(Sometimes, computing the probability of the complement event is faster.)

- **Example**: Suppose among the (911) emergency calls, 60% request the police, 25% request an ambulance, and 15% request the fire department. If we observe 10 emergency calls, P[6 calls for police, 3 for ambulances, 1 for fire dept.] $= p =$?

  $n = 10$, $r_1 = 6$, $r_2 = 3$, $r_3 = 1$, $p_1 = 0.6$, $p_2 = 0.25$, $p_3 = 0.15$,

  Multinomial probability:

  $p = \frac{n!}{r_1!r_2!r_3!} p_1^{r_1} p_2^{r_2} p_3^{r_3} = \frac{10!}{6!3!1!} (0.6)^6 (0.25)^3 (0.15)^1 \approx 0.092$

- **Example**: Consider r indistinguishable balls and n cells where $n > r$. The r balls are placed at random into the n cells (multiple occupancy is possible). What is the probability P that the r balls appear in r preselected cells (one to a cell)?

  Number of favorable ways $= r(r-1)(r-2)\cdots 1 = r!$

  Total number of ways $= n^r$

  $P =$ number of favorable ways/ total number of ways $= r!/n^r$

- **Example:** r indistinguishable balls are to be randomly distributed among n cells. P[all cells are occupied] $= ?$
Solution: $E_i \triangleq$ cell i is empty, $(i = 1, \ldots, n)$.
at least one cell is empty $\triangleq E = \bigcup_{i=1}^{n} E_i$
all cells are occupied $= E^c$.
If $r < n, P[E^c] = 0$.

$$\text{For } r \geq n, \quad P[E_i] = (n-1)^r/n^r = (1-1/n)^r \triangleq P_i$$

$$P[E_i E_j] \triangleq P_{ij} = (n-2)^r/n^r = (1-2/n)^r$$

$$P[E_i E_j E_k] \triangleq P_{ijk} = (1-3/n)^r$$

$$S_i \triangleq \sum_{1 \leq l_1 < l_2 < \ldots < l_i \leq n} P_{l_1 l_2 \ldots l_i} = \binom{n}{i} \left(1 - \frac{i}{n}\right)^r$$

$$P[E] = \sum_{i=1}^{n} (-1)^{i+1} S_i = \sum_{i=1}^{n} \binom{n}{i} (-1)^{i+1} \left(1 - \frac{i}{n}\right)^r$$

$$P[E^c] = 1 - P[E] = \sum_{i=0}^{n} \binom{n}{i} (-1)^{i} \left(1 - \frac{i}{n}\right)^r$$

- **Occupancy Problem**:(Random placement of r balls into n cells)
  - \# of distinguishable distributions of non-distinguishable balls:

$$\boxed{\binom{n+r-1}{r} = \binom{n+r-1}{n-1}}$$

n cells $\Rightarrow n+1$ bars, r balls $\Rightarrow r$ stars. $|*|**|\ \ |*|\ldots|***|*|$

The starting bar and ending bar are fixed; so $(n+r-1)$ null spaces to allocate r stars

- **Example:** R (indistinguishable) balls are placed in N cells where $N < R$.
$K = \#$ of distinguishable distributions where no cell remains empty $=$?

First, N balls must occupy N cells so that no cell remains empty.
The remaining $R-N$ balls can be placed arbitrarily in the N cells. Hence,
$K = \#$ of distinguishable ways of placing $R-N$ balls in N cells
$\quad =$ occupancy problem with $r = R-N$ balls and $n = N$ cells

$$= \binom{R-N+N-1}{R-N} = \binom{R-1}{R-N} = \binom{R-1}{N-1}$$

- **Example (Occupancy distributions versus probabilities)**

Random placement of 3 balls into 3 cells:

$n = 3$ cells and $r = 3$ balls
$\Rightarrow \quad n + 1 = 4$ bars and $r = 3$ stars

Out of 7 vacant slots, the first and last slots are reserved for 2 bars.

So, $n - 1 + r = 5$ vacant slots, each to be occupied by either a bar or a star.

$$\# \text{ of distributions} = \binom{n+r-1}{r} = 10.$$

Distributions	vacant slots						
	1	2	3	4	5	6	7
1	|	*	*	*	|	|	|
2	|	*	*	|	*	|	|
3	|	*	*	|	|	*	|
4	|	*	|	*	*	|	|
5	|	*	|	*	|	*	|
6	|	|	*	*	*	|	|
7	|	|	*	*	|	*	|
8	|	|	*	|	*	*	|
9	|	*	|	|	*	*	|
10	|	|	|	*	*	*	|

• Example (Occupancy distributions versus probabilities):

Example of inserting 3 balls into 3 cells randomly				
distribution with distinguishable balls	distribution with indistinguishable balls	cell-1	cell-2	cell-3
1	1	1 2 3		
2	2	1 2	3	
3	2	1 3	2	
4	2	2 3	1	
5	3	1 2		3
6	3	1 3		2
7	3	2 3		1
8	4	1	2 3	
9	4	2	1 3	
10	4	3	1 2	
11	5	1	2	3
12	5	2	1	3
13	5	2	3	1
14	5	1	3	2
15	5	3	1	2
16	5	3	2	1
17	6		1 2 3	
18	7		1 2	3
19	7		1 3	2
20	7		2 3	1
21	8		1	2 3
22	8		2	1 3
23	8		3	1 2
24	9	1		2 3
25	9	2		1 3
26	9	3		1 2
27	10			1 2 3

Random placement of $r = 3$ balls into $n = 3$ cells:

# of distributions of distinguishable balls $= n^r = 3^3 = 27$

P[a distribution of distinguishable balls] $= 1/27$

$P[| ** | * | |] = 3/27$

$P[| * | * | * |] = 6/27$

$P[| * * * | | |] = 1/27$

Reliability Problem

- System with N Independent Components in Series:
  P[success of component i] $= p_i$
  $\boxed{\text{P[success of the system]} = \prod_{i=1}^{N} p_i}$

- System with N Independent Components in Parallel:
  P[success of component i] $= p_i$
  P[failure of the system] $= \prod_{i=1}^{N}(1 - p_i)$
  $\boxed{\text{P[success of the system]} = 1 - \prod_{i=1}^{N}(1 - p_i)}$

- **Example:** Suppose a home is equipped with an old intrusion detection system which provides a correct detection probability of 0.7. The owner wants to upgrade the intrusion detection system, and finds out that there is a new system which is independent of the old system technology. The new system consists of two sequential independent sub-systems, each with a corresponding correct detection probability of 0.9. Compare the correct detection probabilities for the following two options.
  Option 1: Replace the old system with the new one
  Option 2: Keep the old system and add the new one.

P[correct detection (success) for old system] $= P[S_{\text{old}}] = 0.7$

P[success for new system] $= P[S_{\text{new}}] = 0.9 \times 0.9 = 0.81$ (Option 1)

$$P[\text{failure for the parallel system}] = (1 - P[S_{\text{old}}])(1 - P[S_{\text{new}}]) = 0.3 \times 0.19 = 0.057$$

P[success for the parallel system] $= 1 - 0.057 = 0.943$ (Option 2)

Redundancy (parallel system) improves reliability!