EESC 6349 / MECH 6312
Solutions to Homework 1

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Problem 1.8 (3rd Ed.), 1.8 (4th Ed.)
To shorten the notation, the event of drawing a ball with the number i will be denoted as i.
Then $\Omega = \{1, 2, \ldots, 10\}$, $E = \{1, 2, 3, 4, 5\}$, $F = \{4, 5, 6, 7, 8\}$.
Thus,

$$\begin{aligned}
&E^c = \{6, 7, 8, 9, 10\}, F^c = \{1, 2, 3, 9, 10\} \\
&EF = \{4, 5\}, E \cup F = \{1, 2, 3, 4, 5, 6, 7, 8\} \\
&EF^c = \{1, 2, 3\}, E^cF = \{6, 7, 8\}, E^c \cup F^c = \{1, 2, 3, 6, 7, 8, 9, 10\} \\
&(EF^c) \cup (E^cF) = \{1, 2, 3, 6, 7, 8\}, (EF) \cup (E^cF^c) = \{4, 5, 9, 10\} \\
&(E \cup F)^c = \{9, 10\}, (EF)^c = \{1, 2, 3, 6, 7, 8, 9, 10\} \\
&\text{where :} \\
&E^c = \{\text{drawing a ball greater than 5}\} \\
&F^c = \{\text{drawing a ball not in the range 4-8 inclusive}\} \\
&EF = \{\text{drawing a ball greater than 3 and no greater than 5}\}
\end{aligned}$$

etc.

Problem 1.20 (3rd Ed.), 1.32 (4th Ed.)

$$
\begin{aligned}
P[X=3] &= 3P[X=1]\ ,\ P[X=2] = 2P[X=1] \\
P[X=3] &+ P[X=2] + P[X=1] = 1 \\
P[X=1] &= \frac{1}{6}, P[X=2] = \frac{1}{3}, P[X=3] = \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
P[X=1|Y=1] &= \frac{P[Y=1|X=1]P[X=1]}{\sum_{i=1}^{3} P[Y=1|X=i]P[X=i]} \\
&= \frac{(1-\alpha)\frac{1}{6}}{(1-\alpha)\frac{1}{6} + \frac{\beta}{2}\frac{1}{3} + \frac{\gamma}{2}\frac{1}{2}} \\
&= \frac{1-\alpha}{1-\alpha+\beta+\frac{3}{2}\gamma}
\end{aligned}
$$

Problem 1.21 (3rd Ed.), 1.33 (4th Ed.)
Let $A \triangleq$ event that examinee knows,
$B \triangleq$ event that examinee guesses,
$C \triangleq$ event of getting right answer

$$
\begin{aligned}
& P[A] = p,\ P[B] = 1-p \\
& P[C|A] = 1, P[C|B] = \frac{1}{m} \\
& P[A|C] = \frac{P[C|A]P[A]}{P[C]} \\
& = \frac{1 * p}{P[C|A]P[A] + P[C|B]P[B]} \\
& = \frac{p}{p + \frac{1}{m}(1-p)} = \frac{mp}{mp+1-p}
\end{aligned}
$$

Problem 1.23 (3rd Ed.), 1.35 (4th Ed.)
Let $\tilde{A} \triangleq \{\text{event that rand. drawn chip is} \in A\}$,
$\tilde{B} \triangleq \{\text{event that rand. drawn chip is} \in B\}$,
$\tilde{C} \triangleq \{\text{event that rand. drawn chip is} \in C\}$,
$D \triangleq \{\text{event that rand. drawn chip is defective}\}$

$$\begin{aligned}
P[D] &= P[D|\tilde{A}]P[\tilde{A}] + P[D|\tilde{B}]P[\tilde{B}] + P[D|\tilde{C}]P[\tilde{C}] \\
&= 0.05 * 0.25 + 0.04 * 0.35 + 0.02 * 0.40 \\
&= 0.0345 \\
\text{Hence, } P[\tilde{A}|D] &= \frac{P[D|\tilde{A}]P[\tilde{A}]}{P[D]} = \frac{0.05 * 0.25}{0.0345} = 0.363 \\
P[\tilde{B}|D] &= \frac{P[D|\tilde{B}]P[\tilde{B}]}{P[D]} = \frac{0.04 * 0.35}{0.0345} = 0.406 \\
P[\tilde{C}|D] &= \frac{P[D|\tilde{C}]P[\tilde{C}]}{P[D]} = \frac{0.02 * 0.40}{0.0345} = 0.232
\end{aligned}$$

Problem 1.26 (3rd Ed.), 1.39 (4th Ed.)
The number of favorable ways is $r!$. However, the total number of ways is **not** n^r since cells can at most hold only one ball. For first ball there are n cells; for second ball there are $n-1$ cells ets. Thus
$N_T = n(n-1)\cdots(n-r+1) = n!/(n-r)!$
The desired probability is:

$$P = \frac{r!}{n!/(n-r)!} = \frac{r!(n-r)!}{n!} = \binom{n}{r}^{-1}$$

Problem 1.32 (3rd Ed.), 1.47 (4th Ed.)
Let $A \triangleq$ {event that a chip meets specs},
$B \triangleq$ {event that a chip needs rework},
$C \triangleq$ {event that a chip is discarded}
$P[A] = 0.85, P[B] = 0.10, P[C] = 0.05$
The Multinomial law applies here!

$$(a)\ P[\text{all 10 chips meet specs}] = \frac{10!}{10!0!0!}(0.85)^{10}(0.10)^0(0.05)^0 = 0.197$$

$$(b)\ P[\text{2 or more discards}] \triangleq P_2 = 1 - P[\text{no discards}] - P[\text{1 discard}]$$

$$P[\text{discard}] = P[C] = 0.05, P[C^c] = 0.95$$

$$P_2 = 1 - \begin{pmatrix} 10 \\ 0 \end{pmatrix} (0.95)^{10}(0.05)^0 - \begin{pmatrix} 10 \\ 1 \end{pmatrix} (0.95)^9(0.05)^1$$

$$= 1 - 0.6 - 0.315 = 0.085$$

$$(c)\ P[\text{ 8 meet specs, 1 needs work, 1 discard}]$$

$$= \frac{10!}{8!0!0!}(0.85)^8(0.10)^1(0.05)^1 = 0.122$$

Note on Problem 1.32 (3rd Ed.), 1.47 (4th Ed.)

The multinomial equation holds if all trials are independent and identically distributed(iid) (due to independent repetitions of the experiment). Note that the total number of chips produced is not 100. It just says "for every 100 chips produced,...". Hence, we can assume that the total number of chips produced is very large so that the 10 trials (selecting 10 chips) are iid. Then we can apply the multinomial law.
For the setup in problem 1.32 if the total number of chips produced were100, then applying the multinomial law would not yield correct solutions because different trials cannot be viewed as iid. In this case, the correct solution will be as follows:

$$(a)\frac{\binom{85}{10}}{\binom{100}{10}} = 0.1808$$

$$(b)1 - \frac{\binom{95}{10}}{\binom{100}{10}} - \frac{\binom{95}{9}\binom{5}{1}}{\binom{100}{10}} = 0.0769$$

$$(c)\frac{\binom{85}{8}\binom{10}{1}\binom{5}{1}}{\binom{100}{10}} = 0.139$$

If you proportionally increase 100,85,10 and 5 to very large numbers,the solutions will approach to multinomial solutions.

Problem 1.40 (3rd Ed.), 1.55 (4th Ed.)
$P_M = 0.1 = 2P_B$
P[patient dying without monitor system] $= P_B = 0.05$
P[Patient dying with monitor system] $= P_M P_B = (0.1)(0.05) = 0.005$
The above results show that Prof. X's argument is wrong.

Problem 1.41 (3rd Ed.), 1.56 (4th Ed.)
(a) At each attempt, the probability of successful transmission $= P^N$

$$\begin{aligned} P(m) &\triangleq P[\text{at least one successful transmission occurs in } m \text{ attempts}] \\ &= 1 - P[\text{no successful transmission occurs in } m \text{ attempts}] \\ &= 1 - \begin{pmatrix} m \\ 0 \end{pmatrix} (p^N)^0 (1-p^N)^m = 1 - (1-p^N)^m \end{aligned}$$

(b)

$$\begin{aligned} &P[\text{For each individual receiver, at least one successful transmission occurs in } m \text{ attempts}] \\ &= 1 - P[\text{For each individual receiver, no successful transmission occurs in } m \text{ attempts}] \\ &= 1 - \begin{pmatrix} m \\ 0 \end{pmatrix} (p)^0 (1-p)^m = 1 - (1-p)^m \\ &P_D(m) \triangleq P[\text{For all } N \text{ receivers, at least one successful transmission occurs in } m \text{ attempts}] \\ &= [1 - (1-p)^m]^N \end{aligned}$$

If $p = 0.9,\ N = 5,\ m = 2,$
$P(2) = 0.832\ <\ P_D(2) = 0.951$