Random Vectors

- Definition/Characterization
- Transformed Random Vectors
- Properties and Diagonalization of Covariance Matrices
- Characteristic Function of Random Vectors
- Gaussian Random Vectors
- Complex Gaussian Random Vectors

Random Vectors

- Notations/Definitions/Relationships:

$$\boldsymbol{X} \triangleq [X_1, X_2, \ldots, X_n]^T, \quad \boldsymbol{x} \triangleq [x_1, x_2, \ldots, x_n]^T$$

$$\boxed{\{\boldsymbol{X} \le \boldsymbol{x}\} \triangleq \{X_1 \le x_1, \ldots, X_n \le x_n\}}$$

$$F_{\boldsymbol{X}}(\boldsymbol{x}) \triangleq P[\boldsymbol{X} \le \boldsymbol{x}], \quad \text{(simply joint CDF of } \{X_i\})$$

$$f_{\boldsymbol{X}}(\boldsymbol{x}) \triangleq \frac{\partial^n F_{\boldsymbol{X}}(\boldsymbol{x})}{\partial x_1 \ldots \partial x_n}, \quad \text{(simply joint pdf of } \{X_i\})$$

$$P[B] = \int_{\boldsymbol{X} \in B} f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x}$$

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = \int_{-\infty}^{\boldsymbol{x}} f_{\boldsymbol{X}}(\boldsymbol{x}') d\boldsymbol{x}'$$

$$F_{\boldsymbol{X}|B}(\boldsymbol{x}|B) \triangleq P[\boldsymbol{X} \le \boldsymbol{x}|B] = \frac{P[\boldsymbol{X} \le \boldsymbol{x}, B]}{P[B]}, \quad (P[B] \neq 0)$$

- For an event space $\{B_i : i = 1, \ldots, n\}$,

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = \sum_{i=1}^{n} F_{\boldsymbol{X}|B_i}(\boldsymbol{x}|B_i)\ P[B_i]$$

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \sum_{i=1}^{n} f_{\boldsymbol{X}|B_i}(\boldsymbol{x}|B_i)\ P[B_i], \quad \text{where} \quad f_{\boldsymbol{X}|B_i}(\boldsymbol{x}|B_i) \triangleq \frac{\partial^n F_{\boldsymbol{X}|B_i}(\boldsymbol{x}|B_i)}{\partial x_1 \ldots \partial x_n}$$

- Joint distributions/densities of $\boldsymbol{X} = [X_1, \ldots, X_n]^T$ and $\boldsymbol{Y} = [Y_1, \ldots, Y_m]^T$:

$$F_{\boldsymbol{XY}}(\boldsymbol{x}, \boldsymbol{y}) \triangleq P[\boldsymbol{X} \leq \boldsymbol{x}, \boldsymbol{Y} \leq \boldsymbol{y}]$$
$$\text{where} \quad \boldsymbol{x} = [x_1, \ldots, x_n]^T, \quad \boldsymbol{y} = [y_1, \ldots, y_m]^T$$

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = F_{\boldsymbol{XY}}(\boldsymbol{x}, \boldsymbol{\infty})$$

$$f_{\boldsymbol{XY}}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\partial^{n+m} F_{\boldsymbol{XY}}(\boldsymbol{x}, \boldsymbol{y})}{\partial x_1 \ldots \partial x_n\ \partial y_1 \ldots \partial y_m}$$

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\boldsymbol{XY}}(\boldsymbol{x}, \boldsymbol{y}) dy_1 \ldots dy_m$$

- Expectation Vector:

$$\boldsymbol{\mu} = \boldsymbol{\mu}_{\boldsymbol{X}} = E[\boldsymbol{X}] = [\mu_1, \ldots, \mu_n]^T$$
$$\mu_i \triangleq \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_i f_{\boldsymbol{X}}(x_1, \ldots, x_n) dx_1 \ldots dx_n = \int_{-\infty}^{\infty} x_i f_{X_i}(x_i) dx_i$$

- Correlation Matrix (real symmetric for real $\boldsymbol{X}$):

$$\boxed{\begin{aligned} \boldsymbol{R} &\triangleq E[\boldsymbol{X}\boldsymbol{X}^T] \\ \text{where} \quad R_{ij} &= E[X_i X_j] \end{aligned}}$$

- Covariance Matrix (real symmetric for real $\boldsymbol{X}$):

$$\boxed{\begin{aligned} &\boldsymbol{K} \triangleq E[(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^T] = \boldsymbol{R} - \boldsymbol{\mu}\boldsymbol{\mu}^T \\ &\text{where} \quad K_{ii} = \sigma_i^2, \quad K_{ij} = \mathrm{Cov}(X_i, X_j) = E[(X_i - \mu_i)(X_j - \mu_j)] = K_{ji} \end{aligned}}$$

- If $\boldsymbol{K}_{\boldsymbol{X}}$ is diagonal, $\{X_i\}$ are uncorrelated;

  If $\boldsymbol{R}_{\boldsymbol{X}}$ is diagonal, $\{X_i\}$ are orthogonal.

- Vector Cross Correlation: $\boldsymbol{R}_{\boldsymbol{XY}} = E[\boldsymbol{X}\boldsymbol{Y}^T]$
- Vector Cross-Covariance: $\boldsymbol{K}_{\boldsymbol{XY}} = E[(\boldsymbol{X} - \boldsymbol{\mu}_{\boldsymbol{X}})(\boldsymbol{Y} - \boldsymbol{\mu}_{\boldsymbol{Y}})^T]$
- $\boldsymbol{R}_{\boldsymbol{XY}} = \boldsymbol{K}_{\boldsymbol{XY}} + \boldsymbol{\mu}_{\boldsymbol{X}}\boldsymbol{\mu}_{\boldsymbol{Y}}{}^T$
- For real $n \times 1$ random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$,

$$\boldsymbol{K}_{\boldsymbol{XY}} = \boldsymbol{0} \;\text{ or }\; E[\boldsymbol{X}\boldsymbol{Y}^T] = \boldsymbol{\mu}_{\boldsymbol{X}}\boldsymbol{\mu}_{\boldsymbol{Y}}{}^T \;\Rightarrow\; \boldsymbol{X} \;\&\; \boldsymbol{Y} \text{ are uncorrelated}$$

$$E[\boldsymbol{X}\boldsymbol{Y}^T] = \boldsymbol{0} \;\Rightarrow\; \boldsymbol{X} \;\&\; \boldsymbol{Y} \text{ are orthogonal}$$

$$f_{\boldsymbol{XY}}(\boldsymbol{x}, \boldsymbol{y}) = f_{\boldsymbol{X}}(\boldsymbol{x}) f_{\boldsymbol{Y}}(\boldsymbol{y}), \;\; \forall (x, y) \;\Rightarrow\; \boldsymbol{X} \;\&\; \boldsymbol{Y} \text{ are independent}$$

- Independence always implies uncorrelatedness, but the converse is not generally true except multidimensional Gaussian.

- For real $n \times 1$ random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, if
  $\boldsymbol{Z} = \boldsymbol{X} + \boldsymbol{Y}$,
  then

$$\begin{aligned}
\boldsymbol{\mu}_{\boldsymbol{Z}} &= \boldsymbol{\mu}_{\boldsymbol{X}} + \boldsymbol{\mu}_{\boldsymbol{Y}} \\
\boldsymbol{R}_{\boldsymbol{Z}} &= \boldsymbol{R}_{\boldsymbol{X}} + \boldsymbol{R}_{\boldsymbol{XY}} + \boldsymbol{R}_{\boldsymbol{YX}} + \boldsymbol{R}_{\boldsymbol{Y}} \\
\boldsymbol{\mu}_{\boldsymbol{Z}}\boldsymbol{\mu}_{\boldsymbol{Z}}{}^{T} &= \boldsymbol{\mu}_{\boldsymbol{X}}\boldsymbol{\mu}_{\boldsymbol{X}}{}^{T} + \boldsymbol{\mu}_{\boldsymbol{X}}\boldsymbol{\mu}_{\boldsymbol{Y}}{}^{T} + \boldsymbol{\mu}_{\boldsymbol{Y}}\boldsymbol{\mu}_{\boldsymbol{X}}{}^{T} + \boldsymbol{\mu}_{\boldsymbol{Y}}\boldsymbol{\mu}_{\boldsymbol{Y}}{}^{T} \\
\boldsymbol{K}_{\boldsymbol{Z}} &= \boldsymbol{K}_{\boldsymbol{X}} + \boldsymbol{K}_{\boldsymbol{XY}} + \boldsymbol{K}_{\boldsymbol{YX}} + \boldsymbol{K}_{\boldsymbol{Y}}
\end{aligned}$$

- For $\boldsymbol{Z} = \boldsymbol{X} + \boldsymbol{Y}$ where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are uncorrelated,

$$\begin{aligned}
\boldsymbol{R}_{\boldsymbol{Z}} &= \boldsymbol{R}_{\boldsymbol{X}} + \boldsymbol{R}_{\boldsymbol{XY}} + \boldsymbol{R}_{\boldsymbol{YX}} + \boldsymbol{R}_{\boldsymbol{Y}} \\
\boldsymbol{K}_{\boldsymbol{Z}} &= \boldsymbol{K}_{\boldsymbol{X}} + \boldsymbol{K}_{\boldsymbol{Y}}
\end{aligned}$$

pdf of a Transformed Random Vector

- For n real functions $y_i = g_i(x_1, x_2, \ldots, x_n)$, $i = 1, \ldots, n$, with functionally independent $\{g_i(\cdot)\}$ (i.e., there exists no function $H(y_1, \ldots, y_n)$ that is identically zero), how to find $f_{\boldsymbol{Y}}(\boldsymbol{y})$?

  Suppose from $\{y_i = g_i(\cdot)\}$, we obtain a single solution vector **x** with $x_i = \phi_i(y_1, \ldots, y_n)$, $i = 1, \ldots, n$, then

$$\boxed{\begin{aligned} & f_{\boldsymbol{Y}}(\boldsymbol{y}) = f_{\boldsymbol{X}}(\boldsymbol{x})|\tilde{J}| = f_{\boldsymbol{X}}(\boldsymbol{x})/|J| \\ & \text{where} \quad \tilde{J} = \begin{vmatrix} \frac{\partial \phi_1}{\partial y_1} & \cdots & \frac{\partial \phi_1}{\partial y_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial \phi_n}{\partial y_1} & \cdots & \frac{\partial \phi_n}{\partial y_n} \end{vmatrix} = \left. \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{vmatrix}^{-1} \right|_{\mathbf{x}} = J^{-1}. \end{aligned}}$$

- Check and include valid ranges of $\{y_i\}$ in the pdf of **Y**.

- If there are r solutions (roots) $\{\boldsymbol{x}^{(i)} : \; i = 1, \ldots, r\}$, where $\boldsymbol{x}^{(i)} = [x_1^{(i)}, \ldots, x_n^{(i)}]^T$ with $x_k^{(i)} = \phi_k^{(i)}(y_1, \ldots, y_n)$, then

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = \sum_{i=1}^{r} f_{\boldsymbol{X}}(\boldsymbol{x}^{(i)})|\tilde{J}_i| = \sum_{i=1}^{r} f_{\boldsymbol{X}}(\boldsymbol{x}^{(i)})/|J_i|$$

$$\text{where} \qquad \tilde{J}_i = \begin{vmatrix} \frac{\partial \phi_1^{(i)}}{\partial y_1} & \cdots & \frac{\partial \phi_1^{(i)}}{\partial y_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial \phi_n^{(i)}}{\partial y_1} & \cdots & \frac{\partial \phi_n^{(i)}}{\partial y_n} \end{vmatrix} = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{vmatrix}_{\boldsymbol{x}^{(i)}}^{-1} = J_i^{-1}.$$

- Check and include valid ranges of $\{y_i\}$ in the pdf of **Y**.

- Example: $\boldsymbol{X} = [X_1, X_2, X_3]^T$, $\quad \boldsymbol{Y} = [Y_1, Y_2, Y_3]^T$
  $Y_1 = X_1^2 - X_2^2$, $\quad Y_2 = X_1^2 + X_2^2$, $\quad Y_3 = X_3$
  $f_{\boldsymbol{X}}(\boldsymbol{x}) = (2\pi)^{-3/2} \exp[-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)]$, $\quad f_{\boldsymbol{Y}}(\boldsymbol{y}) = ?$

  There are 4 roots, $\boldsymbol{x}^{(1)}$, $\boldsymbol{x}^{(2)}$, $\boldsymbol{x}^{(3)}$, $\boldsymbol{x}^{(4)}$, where

$$
\begin{aligned}
x_1^{(1)} &= \sqrt{(y_1+y_2)/2}, & x_2^{(1)} &= \sqrt{(y_2-y_1)/2}, & x_3^{(1)} &= y_3 \\
x_1^{(2)} &= \sqrt{(y_1+y_2)/2}, & x_2^{(2)} &= -\sqrt{(y_2-y_1)/2}, & x_3^{(2)} &= y_3 \\
x_1^{(3)} &= -\sqrt{(y_1+y_2)/2}, & x_2^{(3)} &= \sqrt{(y_2-y_1)/2}, & x_3^{(3)} &= y_3 \\
x_1^{(4)} &= -\sqrt{(y_1+y_2)/2}, & x_2^{(4)} &= -\sqrt{(y_2-y_1)/2}, & x_3^{(4)} &= y_3
\end{aligned}
$$

  For the roots to be real, $y_2 > 0$ and $y_2 > |y_1|$. The Jacobian is

$$
J = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} 2x_1 & -2x_2 & 0 \\ 2x_1 & 2x_2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 8x_1x_2
$$

  Substituting the solutions gives $|J_1| = |J_2| = |J_3| = |J_4| = 4\sqrt{y_2^2 - y_1^2}$. Thus,

$$
f_{\boldsymbol{Y}}(\boldsymbol{y}) = \sum_{i=1}^{4} f_{\boldsymbol{X}}(\boldsymbol{x}^{(i)})/|J_i| = \frac{(2\pi)^{-3/2}}{\sqrt{y_2^2 - y_1^2}} \exp[-\frac{1}{2}(y_2 + y_3^2)], \; y_2 > 0, \; y_2 > |y_1|
$$

  and $f_{\boldsymbol{Y}}(\boldsymbol{y}) = 0$ outside the above region.

- For $\boxed{\boldsymbol{Y} \triangleq \boldsymbol{A}\boldsymbol{X} + \boldsymbol{b}}$ where $\boldsymbol{X}$ is a continuous random vector, $\boldsymbol{A}$ is an invertible deterministic matrix and $\boldsymbol{b}$ is a deterministic vector,

$$\boxed{\begin{aligned}
\boldsymbol{\mu}_{\boldsymbol{Y}} &= \boldsymbol{A}\boldsymbol{\mu}_{\boldsymbol{X}} + \boldsymbol{b} \\
\boldsymbol{R}_{\boldsymbol{Y}} &= \boldsymbol{A}\boldsymbol{R}_{\boldsymbol{X}}\boldsymbol{A}^T + (\boldsymbol{A}\boldsymbol{\mu}_{\boldsymbol{X}})\boldsymbol{b}^T + \boldsymbol{b}(\boldsymbol{A}\boldsymbol{\mu}_{\boldsymbol{X}})^T + \boldsymbol{b}\boldsymbol{b}^T \\
\boldsymbol{K}_{\boldsymbol{Y}} &= \boldsymbol{A}\boldsymbol{K}_{\boldsymbol{X}}\boldsymbol{A}^T \\
f_{\boldsymbol{Y}}(\boldsymbol{y}) &= \frac{1}{|\det(\boldsymbol{A})|} f_{\boldsymbol{X}}\left(\boldsymbol{A}^{-1}(\boldsymbol{y} - \boldsymbol{b})\right)
\end{aligned}}$$

Linear Algebra

- An $n \times n$ real matrix $\boldsymbol{M}$ is positive semidefinite (p.s.d) if $\boldsymbol{z}^T \boldsymbol{M} \boldsymbol{z} \geq 0$, $\forall$ $\boldsymbol{z}$.
- If $\boldsymbol{z}^T \boldsymbol{M} \boldsymbol{z} > 0$ for all $\boldsymbol{z} \neq \mathbf{0}$, $\boldsymbol{M}$ is positive definite (p.d.).

- The eigenvalues of an $n \times n$ matrix $\boldsymbol{M}$ are those numbers λ for which the characteristic equation $\boldsymbol{M}\boldsymbol{\phi} = \lambda\boldsymbol{\phi}$ has a solution $\boldsymbol{\phi} \neq \mathbf{0}$. The column vector $\boldsymbol{\phi} = [\phi_1, \phi_2, \ldots, \phi_n]^T$ is called an eigenvector. Eigenvectors are often normalized to $\boldsymbol{\phi}^T \boldsymbol{\phi} = 1$.
- Theorem: λ is an eigenvalue of the square matrix $\boldsymbol{M}$ iff $\det(\boldsymbol{M} - \lambda \boldsymbol{I}) = 0$.

- Two $n \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are called *similar* if there exists an $n \times n$ matrix $\boldsymbol{T}$ with $\det(\boldsymbol{T}) \neq 0$ such that $\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} = \boldsymbol{B}$.
- Theorem: An $n \times n$ matrix $\boldsymbol{M}$ is similar to a diagonal matrix iff $\boldsymbol{M}$ has n linearly independent eigenvectors.

- Theorem: Let $\boldsymbol{M}$ be a real symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then $\boldsymbol{M}$ has n mutually orthogonal unit eigenvectors $\boldsymbol{\phi}_1$, ... , $\boldsymbol{\phi}_n$.

- Theorem: Let $\boldsymbol{M}$ be a real symmetric matrix with eigen values $\lambda_1, \ldots, \lambda_n$. Then $\boldsymbol{M}$ is similar to the diagonal matrix $\boldsymbol{\Lambda} \triangleq \text{diag}\{\lambda_1, \ldots, \lambda_n\}$ under the transformation $\boldsymbol{U}^{-1}\boldsymbol{M}\boldsymbol{U} = \boldsymbol{\Lambda}$ where $\boldsymbol{U}$ is a matrix whose columns are the ordered orthogonal unit eigenvectors $\boldsymbol{\phi}_i$, $i = 1, \ldots, n$ of $\boldsymbol{M}$, i.e., $\boldsymbol{U} = [\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n]$. Moreover, $\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{I}$ (thus $\boldsymbol{U}^{-1} = \boldsymbol{U}^T$), so $\boldsymbol{U}^T\boldsymbol{M}\boldsymbol{U} = \boldsymbol{\Lambda}$.

- Unitary matrix $\Leftrightarrow$ $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$.

- A real symmetric matrix $\boldsymbol{M}$ is p.d. iff all its eigenvalues are positive.

- Theorem: Let $\boldsymbol{M}$ be a real symmetric matrix with largest eigenvalue λ_1. The maximum of the quadratic form $\boldsymbol{u}^T\boldsymbol{M}\boldsymbol{u}$ subject to $\|\boldsymbol{u}\| = 1$ (unit sphere) is λ_1 and it occurs when $\boldsymbol{u} = \boldsymbol{\phi}_1$, the unit eigenvector associated with λ_1.

Properties and Eigen-Decomposition of Covariance Matrices

- A covariance matrix $\boldsymbol{K}$ is at least p.s.d. When $\boldsymbol{K}$ is full-rank, then $\boldsymbol{K}$ is p.d.

- Since $\boldsymbol{K}$ is real symmetric, it can be easily diagonalized by $\boldsymbol{U}$ whose columns are eigenvectors of $\boldsymbol{K}$.

- A p.d. covariance matrix $\boldsymbol{K}$ has all positive eigenvalues, and consequently det$(\boldsymbol{K}) = \prod_{i=1}^{n} \lambda_i > 0$. Note that p.s.d of $\boldsymbol{K}$ means $\lambda_i \geq 0$. Thus, when $\boldsymbol{K}$ is full-rank, it is p.d.

- Eigen decomposition: $\boldsymbol{K} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T$,

  $\boldsymbol{U} = [\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \cdots, \boldsymbol{\phi}_n]$, $\boldsymbol{\Lambda} =$diag$\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, $\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{I}$,

  where λ_i is eigen-value associated with the eigen-vector $\boldsymbol{\phi}_i$.

  - **Find $\{\lambda_i\}$ from det$(\boldsymbol{K} - \lambda\boldsymbol{I}) = 0$.**

  - **Using each λ_i, find $\boldsymbol{\phi}_i$ from $(\boldsymbol{K} - \lambda_i\boldsymbol{I})\boldsymbol{\phi}_i = \boldsymbol{0}$ and $\|\boldsymbol{\phi}_i\|^2 = 1$.**

Whitening Transformation

- For a $n \times 1$ random vector $\boldsymbol{X}$ with covariance matrix $\boldsymbol{K_X}$, its whitening transformation is given by

$$\boxed{\boldsymbol{Y} = \boldsymbol{\Lambda_X}^{-1/2} \boldsymbol{U_X}^H \boldsymbol{X}}$$

such that $\boldsymbol{K_Y} = \boldsymbol{I}$ where columns of $\boldsymbol{U_X}$ are eigen vectors of $\boldsymbol{K_X}$ and

$$\boxed{\boldsymbol{\Lambda_X}^{-1/2} \triangleq \mathrm{diag}(\lambda_1^{-1/2}, \lambda_2^{-1/2}, \cdots, \lambda_n^{-1/2})}$$

with λ_i being the eigen-value associated with the ith eigen-vector (ith column of $\boldsymbol{U_X}$).

- Steps:
  - **Eigen-decomposition of $\boldsymbol{K_X}$: $\boldsymbol{K_X} = \boldsymbol{U_X \Lambda_X U_X}^H$**
  - $\boldsymbol{Y} = \boldsymbol{AX} \quad \Rightarrow \quad \boldsymbol{K_Y} = \boldsymbol{A K_X A}^H = \boldsymbol{A U_X \Lambda_X U_X}^H \boldsymbol{A}^H$.
    So a solution for $\boldsymbol{A}$ is $\boldsymbol{A} = (\boldsymbol{U_X \Lambda_X}^{1/2})^{-1} = \boldsymbol{\Lambda_X}^{-1/2} \boldsymbol{U_X}^{-1}$ and $\boldsymbol{U_X}^{-1} = \boldsymbol{U_X}^H$.
- useful in signal processing algorithms and performance analysis

- Example: Find a transformation $\boldsymbol{Y} = \boldsymbol{D}\boldsymbol{X}$ such that $\boldsymbol{Y} = [Y_1,\ Y_2]^T$ is a Normal random vector with uncorrelated components of unity variance while $\boldsymbol{X} = [X_1, X_2]^T$ is a zero-mean Normal random vector with covariance matrix $\boldsymbol{K_X}$ given by

$$\boldsymbol{K_X} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

.Solution: $E[\boldsymbol{X}] = 0$ and $E[\boldsymbol{Y}] = \boldsymbol{D}E[\boldsymbol{X}] = 0$. Thus, $\boldsymbol{R_X} = \boldsymbol{K_X}$ and $\boldsymbol{R_Y} = \boldsymbol{K_Y}$.

Consider the eigen-decomposition $\boldsymbol{K_X} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T$ where $\boldsymbol{\Lambda} = \mathrm{diag}([\lambda_1,\ \lambda_2])$ and $\boldsymbol{U} = [\boldsymbol{\phi}_1,\ \boldsymbol{\phi}_2]$. Here, $\{\lambda_i\}$ are eigenvalues and $\{\boldsymbol{\phi}_i\}$ are the unit eigenvectors of $\boldsymbol{K_X}$.

Define $\boldsymbol{\Lambda}^{1/2} = \mathrm{diag}([\lambda_1^{1/2},\ \lambda_2^{1/2}])$. Then $\boldsymbol{\Lambda}^{-1/2} = \mathrm{diag}([\lambda_1^{-1/2},\ \lambda_2^{-1/2}])$. Next,

$$E[\boldsymbol{Y}\boldsymbol{Y}^T] = \boldsymbol{D}\boldsymbol{K_X}\boldsymbol{D}^T = \boldsymbol{D}\boldsymbol{U}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Lambda}^{1/2}\boldsymbol{U}^T\boldsymbol{D}^T = \boldsymbol{I}.$$

Then, $\boldsymbol{K_Y} = \boldsymbol{I}$ is achieved by $\boldsymbol{D} = (\boldsymbol{U}\boldsymbol{\Lambda}^{1/2})^{-1} = \boldsymbol{\Lambda}^{-1/2}\boldsymbol{U}^T$. (note: $\boldsymbol{U}^{-1} = \boldsymbol{U}^T$)

Next, from $\det(\boldsymbol{K_X} - \lambda\boldsymbol{I}) = 0$, we find $\lambda_1 = 4$ and $\lambda_2 = 2$. Hence,

$$\boldsymbol{\Lambda} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}, \quad \boldsymbol{Z} = \boldsymbol{\Lambda}^{-1/2} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$(\boldsymbol{K_X} - \lambda_1\boldsymbol{I})\boldsymbol{\phi}_1 = \boldsymbol{0} \text{ with } ||\boldsymbol{\phi}_1|| = 1 \quad \Rightarrow \quad \boldsymbol{\phi}_1 = [1/\sqrt{2},\ -1/\sqrt{2}]^T$$

$$(\boldsymbol{K_X} - \lambda_2\boldsymbol{I})\boldsymbol{\phi}_2 = \boldsymbol{0} \text{ with } ||\boldsymbol{\phi}_2|| = 1 \quad \Rightarrow \quad \boldsymbol{\phi}_2 = [1/\sqrt{2},\ 1/\sqrt{2}]^T$$

$$\boldsymbol{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \text{ Hence, } \boldsymbol{D} = \boldsymbol{\Lambda}^{-1/2}\boldsymbol{U}^T = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

(You can double-check if $\boldsymbol{D}\boldsymbol{K}\boldsymbol{D}^T = \boldsymbol{I}$.)

Correlated Random Vector

- Suppose random vector $\boldsymbol{X}$ has $\boldsymbol{\mu_X} = \mathbf{0}$ and $\boldsymbol{K_X} = \boldsymbol{I}$. How to generate random vector $\boldsymbol{Y}$ from $\boldsymbol{X}$ such that $\boldsymbol{\mu_Y} = \boldsymbol{b}$ and $\boldsymbol{K_Y} = \boldsymbol{Q}$?
- To obtain $\boldsymbol{\mu_Y} = \boldsymbol{b}$, consider $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{b}$.
- Apply eigen-decomposition to $\boldsymbol{K_Y}$:
  $\boldsymbol{Q} = \boldsymbol{U_Y}\boldsymbol{\Lambda_Y}\boldsymbol{U_Y}^T = \boldsymbol{U_Y}\boldsymbol{\Lambda_Y}^{1/2}\boldsymbol{\Lambda_Y}^{1/2}\boldsymbol{U_Y}^T$.
- As $\boldsymbol{K_Y} = \boldsymbol{A}\boldsymbol{K_X}\boldsymbol{A}^T = \boldsymbol{A}\boldsymbol{A}^T$, we can obtain $\boxed{\boldsymbol{A} = \boldsymbol{U_Y}\boldsymbol{\Lambda_Y}^{1/2}}$
- useful in generating correlated random vectors in computer simulation

- Example: Suppose $\boldsymbol{Z}$ is a 2×1 random vector with mean vector $\boldsymbol{\mu_Z} = [1, 1]^T$ and covariance matrix $\boldsymbol{K_Z} = \boldsymbol{I}$. Find a transformation from $\boldsymbol{Z}$ to $\boldsymbol{X}$ such that $\boldsymbol{X}$ has zero mean vector and covariance matrix $\boldsymbol{K_X}$ given by

$$\boldsymbol{K_X} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

.Solution:

We consider the transformation $\boldsymbol{X} = \boldsymbol{A}(\boldsymbol{Z} - \boldsymbol{\mu_Z})$, which yields $E[\boldsymbol{X}] = 0$.
Next, $\boldsymbol{K_X} = \boldsymbol{A}\boldsymbol{K_Z}\boldsymbol{A}^T = \boldsymbol{A}\boldsymbol{A}^T$.
Suppose $\{\lambda_1, \lambda_2\}$ are eigenvalues and $\{\boldsymbol{\phi}_1, \boldsymbol{\phi}_2\}$ are the unit eigenvectors of $\boldsymbol{K_X}$. Define $\boldsymbol{U} = [\boldsymbol{\phi}_1,\ \boldsymbol{\phi}_2]$, $\boldsymbol{\Lambda} = \mathrm{diag}([\lambda_1,\ \lambda_2])$, and $\boldsymbol{\Lambda}^{1/2} = \mathrm{diag}([\lambda_1^{1/2},\ \lambda_2^{1/2}])$. Then, we have $\boldsymbol{K_X} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T = \boldsymbol{U}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Lambda}^{1/2}\boldsymbol{U}^T$
By comparing the two equations of $\boldsymbol{K_X}$, we have $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}^{1/2}$.
Numerical details are given below.
From $\det(\boldsymbol{K_X} - \lambda\boldsymbol{I}) = 0$, we find $\lambda_1 = 4$ and $\lambda_2 = 2$. Hence, $\boldsymbol{\Lambda}^{1/2} = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{bmatrix}$

$$(\boldsymbol{K_X} - \lambda_1\boldsymbol{I})\boldsymbol{\phi}_1 = \boldsymbol{0} \text{ with } ||\boldsymbol{\phi}_1|| = 1 \quad \Rightarrow \quad \boldsymbol{\phi}_1 = [1/\sqrt{2},\ -1/\sqrt{2}]^T$$

$$(\boldsymbol{K_X} - \lambda_2\boldsymbol{I})\boldsymbol{\phi}_2 = \boldsymbol{0} \text{ with } ||\boldsymbol{\phi}_2|| = 1 \quad \Rightarrow \quad \boldsymbol{\phi}_2 = [1/\sqrt{2},\ 1/\sqrt{2}]^T$$

Thus, we obtain

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}^{1/2} = \begin{bmatrix} \sqrt{2} & 1 \\ -\sqrt{2} & 1 \end{bmatrix}.$$

(You can double-check if $\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{K_X}$.)

Simultaneous Diagonalization of Two Covariance Matrices

- Consider an $n \times 1$ real-valued random vector $\boldsymbol{X} = \boldsymbol{S} + \boldsymbol{N}$ and an $n \times n$ real-valued deterministic matrix $\boldsymbol{A}$.

  Let $\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}$.

  - Then, $\boldsymbol{X}' = \boldsymbol{S}' + \boldsymbol{N}'$ where $\boldsymbol{S}' = \boldsymbol{A}\boldsymbol{S}$ and $\boldsymbol{N}' = \boldsymbol{A}\boldsymbol{N}$.
  - $\boldsymbol{K}_{\boldsymbol{X}'} = \boldsymbol{K}_{\boldsymbol{S}'} + \boldsymbol{K}_{\boldsymbol{S}'\boldsymbol{N}'} + \boldsymbol{K}_{\boldsymbol{N}'\boldsymbol{S}'} + \boldsymbol{K}_{\boldsymbol{N}'}$

    where $\boldsymbol{K}_{\boldsymbol{X}'} = \boldsymbol{A}\boldsymbol{K}_{\boldsymbol{X}}\boldsymbol{A}^T$, $\quad \boldsymbol{K}_{\boldsymbol{S}'} = \boldsymbol{A}\boldsymbol{K}_{\boldsymbol{S}}\boldsymbol{A}^T$, $\quad \boldsymbol{K}_{\boldsymbol{N}'} = \boldsymbol{A}\boldsymbol{K}_{\boldsymbol{N}}\boldsymbol{A}^T$
  - $\boldsymbol{R}_{\boldsymbol{S}'\boldsymbol{N}'} = E[\boldsymbol{A}\boldsymbol{S}\boldsymbol{N}^T\boldsymbol{A}^T] = \boldsymbol{A}\boldsymbol{R}_{\boldsymbol{S}\boldsymbol{N}}\boldsymbol{A}^T$
  - $\boldsymbol{\mu}_{\boldsymbol{S}'}\boldsymbol{\mu}_{\boldsymbol{N}'}{}^T = \boldsymbol{A}\boldsymbol{\mu}_{\boldsymbol{S}}\boldsymbol{\mu}_{\boldsymbol{N}}{}^T\boldsymbol{A}^T$
  - $\boldsymbol{K}_{\boldsymbol{S}'\boldsymbol{N}'} = \boldsymbol{R}_{\boldsymbol{S}'\boldsymbol{N}'} - \boldsymbol{\mu}_{\boldsymbol{S}'}\boldsymbol{\mu}_{\boldsymbol{N}'}{}^T = \boldsymbol{A}\boldsymbol{K}_{\boldsymbol{S}\boldsymbol{N}}\boldsymbol{A}^T$
  - If $\boldsymbol{S}$ and $\boldsymbol{N}$ are uncorrelated, i.e., $\boldsymbol{K}_{\boldsymbol{S}\boldsymbol{N}} = \boldsymbol{0}$, then $\quad \boldsymbol{K}_{\boldsymbol{S}'\boldsymbol{N}'} = \boldsymbol{0}$.
  - With uncorrelated $\boldsymbol{S}$ and $\boldsymbol{N}$, $\quad \boldsymbol{K}_{\boldsymbol{X}'} = \boldsymbol{K}_{\boldsymbol{S}'} + \boldsymbol{K}_{\boldsymbol{N}'}$.
  - If $\boldsymbol{K}_{\boldsymbol{S}'}$ and $\boldsymbol{K}_{\boldsymbol{N}'}$ are desired to be diagonal matrices, then $\boldsymbol{A}$ must simultaneously diagonalize $\boldsymbol{K}_{\boldsymbol{S}}$ and $\boldsymbol{K}_{\boldsymbol{N}}$ through $\boldsymbol{A}\boldsymbol{K}_{\boldsymbol{S}}\boldsymbol{A}^T$ and $\boldsymbol{A}\boldsymbol{K}_{\boldsymbol{N}}\boldsymbol{A}^T$.

Simultaneous Diagonalization of Two Covariance Matrices

- **Theorem:** Let $\mathbf{P}$ and $\mathbf{Q}$ be $n \times n$ real symmetric matrices. If $\mathbf{P}$ is positive definite, then there exists a $n \times n$ matrix $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$ which achieves

$$\boxed{\mathbf{V}^T\mathbf{P}\mathbf{V} = \mathbf{I}}$$

and

$$\boxed{\mathbf{V}^T\mathbf{Q}\mathbf{V} = \mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)}.$$

The real number $\lambda_1, \ldots, \lambda_n$ satisfy the generalized eigenvalue equation

$$\boxed{\mathbf{Q}\mathbf{v}_i = \lambda_i \mathbf{P}\mathbf{v}_i}.$$

The numbers λ_i and vectors $\mathbf{v}_i$ for $i = 1, \ldots, n$ are sometimes called generalized eigenvalues and eigenvectors.

Procedure for diagonalizing two matrices P and Q simultaneously

- Calculate the eigenvalues $\{\lambda_i\}$ of $\mathbf{P}^{-1}\mathbf{Q}$ from $\mathbf{P}^{-1}\mathbf{Q}\mathbf{v}_i = \lambda_i \mathbf{v}_i$ by using $\det(\mathbf{P}^{-1}\mathbf{Q} - \lambda\mathbf{I}) = 0$.
- Calculate unnormalized eigenvectors $\mathbf{v}'_i$ for $i = 1, \ldots, n$ by solving

$$(\mathbf{P}^{-1}\mathbf{Q} - \lambda_i \mathbf{I})\mathbf{v}'_i = \mathbf{0};$$

- Find constants $\{K_i\}$ such that $\mathbf{v}_i \triangleq K_i \mathbf{v}'_i$ satisfies $\mathbf{v}_i^T \mathbf{P} \mathbf{v}_i = 1$, $i = 1, \ldots, n$. Then, $\mathbf{V} = [\mathbf{v}_1, \cdots, \mathbf{v}_n]$ which yields $\mathbf{V}^T\mathbf{P}\mathbf{V} = \mathbf{I}$ and $\mathbf{V}^T\mathbf{Q}\mathbf{V} = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}$.

- Note: Any real symmetric, positive definite matrix $\mathbf{P}$ can be factored as $\mathbf{P} = \mathbf{C}\mathbf{C}^T$, and $\mathbf{C}^{-1}\mathbf{P}[\mathbf{C}^T]^{-1} = \mathbf{I}$, $\mathbf{C}^T\mathbf{P}^{-1}\mathbf{C} = \mathbf{I}$.

- Example: Simultaneous diagonalization of two covariance matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$:

$$\boldsymbol{P} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \boldsymbol{Q} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\boldsymbol{P}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \boldsymbol{P}^{-1}\boldsymbol{Q} = \begin{bmatrix} \frac{7}{3} & \frac{-5}{3} \\ \frac{-5}{3} & \frac{7}{3} \end{bmatrix}.$$

The eigenvalues: $\det(\boldsymbol{P}^{-1}\boldsymbol{Q} - \lambda\boldsymbol{I}) = 0 \;\Rightarrow\; \lambda_1 = 4$ and $\lambda_2 = 2/3$.
Unnormalized eigenvectors: $(\boldsymbol{P}^{-1}\boldsymbol{Q} - \lambda_i\boldsymbol{I})\boldsymbol{v}_i' = 0, \quad i = 1, 2 \;\Rightarrow\; \boldsymbol{v}_1 = K_1[1,\ 1]^T$ and $\boldsymbol{v}_2 = K_2[1,\ -1]^T$ Next, we find K_1 and K_2 from $\boldsymbol{v}_i^T\boldsymbol{P}\boldsymbol{v}_i = 1, i = 1, 2$ as follows:

$$\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} K_1^2 = 1 \;\Rightarrow\; K_1 = \pm 1/\sqrt{6}$$

$$\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} K_2^2 = 1 \;\Rightarrow\; K_2 = \pm 1/\sqrt{2}$$

Thus, using $K_1 = 1/\sqrt{6}$ and $K_2 = 1/\sqrt{2}$, the matrix $\boldsymbol{V} = [\boldsymbol{v}_1, \boldsymbol{v}_2]$ is given by

$$\boldsymbol{V} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

It is easily verified that $\boldsymbol{V}^T\boldsymbol{P}\boldsymbol{V} = \boldsymbol{I}$ and $\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V} = \mathrm{diag}(4,\ 2/3)$.

Multidimensional Gaussian Law

- $\boldsymbol{X} = [X_1, \ldots, X_n]^T$ is a Gaussian random vector with $E[\mathbf{X}] = \mu$ and a p.d. covariance matrix $\boldsymbol{K}$ if its pdf is

$$\boxed{f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^n \ \det(\boldsymbol{K})}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{K}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]}$$

- Let $\boldsymbol{X} \sim N(\boldsymbol{\mu_X}, \boldsymbol{K_X})$ with p.d. $\boldsymbol{K_X}$. For a non-singular $n \times n$ transformation $\boldsymbol{A}$, $\boldsymbol{Y} \triangleq \boldsymbol{AX} \sim N(\boldsymbol{\mu_Y}, \boldsymbol{K_Y})$ where $\boldsymbol{\mu_Y} = \boldsymbol{A\mu_X}$ and $\boldsymbol{K_Y} = \boldsymbol{AK_XA^T}$.
- Let $\boldsymbol{X} \sim N(\boldsymbol{\mu_X}, \boldsymbol{K_X})$ with p.d. $\boldsymbol{K_X}$. For $\boldsymbol{Y} \triangleq \boldsymbol{AX} + \boldsymbol{b}$ where $\boldsymbol{A}$ is non-singular, $\boldsymbol{Y} \sim N(\boldsymbol{\mu_Y}, \boldsymbol{K_Y})$ where $\boldsymbol{\mu_Y} = \boldsymbol{A\mu_X} + \boldsymbol{b}$ and $\boldsymbol{K_Y} = \boldsymbol{AK_XA^T}$.
- For $\boldsymbol{X} \sim N(\boldsymbol{\mu_X}, \boldsymbol{K_X})$ with p.d. $\boldsymbol{K_X}$ and a rank m matrix $\boldsymbol{A}_{mn}$, $(m \leq n)$, $\boldsymbol{Y} \triangleq \boldsymbol{A}_{mn}\boldsymbol{X} + \boldsymbol{b} \sim N(\boldsymbol{\mu_Y}, \boldsymbol{K_Y})$ where $\boldsymbol{\mu_Y} = \boldsymbol{A}_{mn}\boldsymbol{\mu_X} + \boldsymbol{b}$ and $\boldsymbol{K_Y} = \boldsymbol{A}_{mn}\boldsymbol{K_X}\boldsymbol{A}_{mn}^T$.
- Let $\boldsymbol{X} \sim N(\boldsymbol{\mu_X}, \boldsymbol{K_X})$ with p.d. $\boldsymbol{K_X}$. There exists a nonsingular $n \times n$ matrix $\boldsymbol{C}$ (e.g., with $\boldsymbol{CC}^T = \boldsymbol{K}$) such that under the transformation $\boldsymbol{Y} = \boldsymbol{C}^{-1}\boldsymbol{X}$, $Y_1, \ldots, Y_n$ are independent.

Characteristic Functions of Random Vectors

- Define $\boldsymbol{w} = [w_1, \ldots, w_n]^T$. Then, C.F. of $\mathbf{X} = [X_1, \cdots, X_n]^T$ is

$$\boxed{\Psi_{\mathbf{X}}(\boldsymbol{w}) \triangleq E[e^{j\boldsymbol{w}^T\mathbf{X}}]} = \int_{-\infty}^{\infty} f_{\mathbf{X}}(\boldsymbol{x}) e^{j\boldsymbol{w}^T\boldsymbol{x}} d\boldsymbol{x} \quad \text{(simply joint C.F.)}$$

(n-dimensional Fourier transform of $f_{\mathbf{X}}(\boldsymbol{x})$, except a sign reversal)

- The pdf of $\boldsymbol{X}$ is

$$f_{\mathbf{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \Psi_{\mathbf{X}}(\boldsymbol{w})\ e^{-j\boldsymbol{w}^T\boldsymbol{x}} d\boldsymbol{w}.$$

- The C.F. for the normal random vector $\mathbf{X} = [X_1, \ldots, X_n]^T$ is:

$$\boxed{\Phi_{\mathbf{X}}(\boldsymbol{\omega}) = \exp(j\boldsymbol{\omega}^T\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{\omega}^T\mathbf{K}\boldsymbol{\omega})},$$

where $\boldsymbol{\omega} = [\omega_1, \ldots, \omega_n]^T$, $\boldsymbol{\mu}$ is the mean vector, and $\mathbf{K}$ is the covariance matrix.

Complex Random Variables

- $X = X_R + jX_I$,

  CDF of X: $\boxed{F_X(x) \triangleq F_{X_R,X_I}(x_R, x_I)} = P[X_R \leq x_R, X_I \leq x_I]$

  pdf of X: $\boxed{f_X(x) \triangleq f_{X_R,X_I}(x_R, x_I)} = \frac{\partial^2 F_X(x_R,x_I)}{\partial x_R \partial x_I}$

- **Mean**: $\mu_X = E[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_R + jx_I)\ f_X(x_R, x_I)\ dx_R\ dx_I$

- **Autocorrelation**: $\boxed{R_X \triangleq E[XX^*]}$

- **Autocovariance**: $\boxed{K_X \triangleq E[(X - \mu_X)(X - \mu_X)^*] = R_X - |\mu_X|^2}$

Complex Random Vectors

- $\mathbf{Z} = \mathbf{U} + j\mathbf{V}$ ($\mathbf{U}$ and $\mathbf{V}$ are real-valued $N \times 1$ random vectors)
- Mean vector: $E[\mathbf{Z}] = E[\mathbf{U}] + jE[\mathbf{V}]$
- Covariance Matrix: $\boxed{\mathbf{C_Z} = E[(\mathbf{Z} - E[\mathbf{Z}])(\mathbf{Z} - E[\mathbf{Z}])^H]}$ (Hermitian symmetric, i.e., $\mathbf{C_Z} = \mathbf{C_Z}^H$)
- Correlation Matrix: $\boxed{\mathbf{R_Z} = E[\mathbf{Z}\mathbf{Z}^H] = \mathbf{C_Z} + E[\mathbf{Z}]\ E[\mathbf{Z}]^H,}$ $(\mathbf{R_Z}{=}\mathbf{R_Z}^H)$
- Pseudo-Covariance Matrix: $\boxed{\tilde{\mathbf{C}}_\mathbf{Z} = E[(\mathbf{Z} - E[\mathbf{Z}])(\mathbf{Z} - E[\mathbf{Z}])^T]}$ (Pseudo-Covariance matrix is skew-Hermitian, i.e., $\mathbf{A}^H = -\mathbf{A}$)
- A complex random vector $\mathbf{Z}$ is called **proper** if $\tilde{\mathbf{C}}_\mathbf{Z} = \mathbf{0}$.

$$\tilde{\mathbf{C}}_\mathbf{Z} = \mathbf{0} \Rightarrow \mathbf{C_U} = \mathbf{C_V} \ \& \ \mathbf{C_{UV}} = -\mathbf{C_{VU}}$$

$$\mathbf{C_Z} = 2\mathbf{C_U} + j2\mathbf{C_{VU}}, \quad \mathbf{C_U} = \mathbf{C_V} = \frac{1}{2}\Re[\mathbf{C_Z}],$$

$$\mathbf{C_{VU}} = -\mathbf{C_{UV}} = \frac{1}{2}\Im[\mathbf{C_Z}]$$

$$\text{and for} \quad \mathbf{X} = \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}, \quad \text{we have} \quad \mathbf{C_X} = \begin{bmatrix} \mathbf{C_U} & \mathbf{C_{UV}} \\ -\mathbf{C_{UV}} & \mathbf{C_U} \end{bmatrix}.$$

Complex Gaussian Vectors

- Suppose a real random vector $\mathbf{x}$ of dimension $2N \times 1$ can be partitioned into two $N \times 1$ vectors as $\mathbf{x} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$ and $\mathbf{x}$ has the pdf

$$\mathbf{x} \sim N\left(\begin{bmatrix} \mu_{\mathbf{u}} \\ \mu_{\mathbf{v}} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{C}_{\mathbf{uu}} & \mathbf{C}_{\mathbf{uv}} \\ \mathbf{C}_{\mathbf{vu}} & \mathbf{C}_{\mathbf{vv}} \end{bmatrix} \right).$$

- Define $\mathbf{z} = \mathbf{u} + j\mathbf{v}$. Then $\boxed{\text{the pdf of } \mathbf{z} \text{ can be given by the pdf of } \mathbf{x}.}$
- If $\mathbf{C}_{\mathbf{uu}} = \mathbf{C}_{\mathbf{vv}}$, $\mathbf{C}_{\mathbf{uv}} = -\mathbf{C}_{\mathbf{vu}}$,

  then $\mathbf{z}$ has the complex multivariate Gaussian pdf, i.e., $\mathbf{z} \sim \mathrm{CN}(\mu_{\mathbf{z}}, \mathbf{C}_{\mathbf{z}})$ as

$$\boxed{\begin{aligned} p(\mathbf{z}) &= \frac{1}{\pi^N \det(\mathbf{C}_{\mathbf{z}})} \exp\left[-(\mathbf{z} - \mu_{\mathbf{z}})^H {\mathbf{C}_{\mathbf{z}}}^{-1} (\mathbf{z} - \mu_{\mathbf{z}})\right] \\ \text{where} \quad \mu_{\mathbf{z}} &= \mu_{\mathbf{u}} + j\mu_{\mathbf{v}} \\ \mathbf{C}_{\mathbf{z}} &= 2(\mathbf{C}_{\mathbf{uu}} + j\mathbf{C}_{\mathbf{vu}}) \end{aligned}}$$