Random Vectors:

$X = [X_1, X_2, \cdots, X_n]^T \quad x = [x_1, x_2, \cdots, x_n]^T$

$F_X(x) \triangleq P[X \leq x]$ CDF of X

$f_X(x) \triangleq \frac{\partial^n F_X(x)}{\partial x_1 \partial x_2 \cdots \partial x_n}$ PDF of X

$P[B] = \int_{x \in B} f_X(x)\, dx \qquad F_X(x) = \int_{-\infty}^{x} f_X(x')\, dx'$

$F_{X|B}(x|B) = P[X \leq x | B] = \frac{P[X \leq x, B]}{P[B]}, \quad P[B] \neq 0$

$F_X(x) = \sum_{i=1}^{n} F_{X|B_i}(x|B_i) P[B_i]$

$f_X(x) = \sum_{i=1}^{n} f_{X|B_i}(x|B_i) P[B_i]$

$F_{XY}(x, y) = P[X \leq x, Y \leq y] \qquad Y = [Y_1, \cdots, Y_m]$

$F_X(x) = F_{XY}(x, \infty) \qquad y = [y_1, \cdots, y_m]$

$f_{XY}(x, y) = \frac{\partial^{n+m} F_{XY}(x, y)}{\partial x_1 \cdots \partial x_n \partial y_1 \cdots \partial y_m}$

$f_X(x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy_1 \cdots dy_m$

Random Vectors: cont.

Expectation Vector

$$\mu = \mu_x = E[X] = [\mu_1, \cdots, \mu_n]^T$$

$$\mu_i = \int_{-\infty}^{\infty} x_i f_{X_i}(x_i)\,dx_i = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_i f_X(x_1, \cdots x_n)\,dx_1 \cdots dx_n$$

Correlation Matrix

$$R \triangleq E[XX^T]$$

$$R_{ij} = E[X_i X_j]$$

R_X = Diagonal → uncorrellated

K_X = Diagonal → orthogonal

Covariance Matrix

$$K \triangleq E[(X-\mu)(X-\mu)^T] = R - \mu\mu^T$$

$$K_{ii} = \sigma_i^2 \qquad K_{ij} = Cov(X_i, X_j) = K_{ji}$$

Vector Cross-Correlation

$$R_{XY} = E[XY^T] \qquad R_{XY} = K_{XY} + \mu_x \mu_y^T$$

Vector Cross-Covariance

$$K_{XY} = E[(X-\mu_x)(Y-\mu_y)^T]$$

$K_{XY} = 0$ or $E[XY^T] = \mu_x \mu_y^T$ → Uncorrellated

$E[XY^T] = 0$ → Orthogonal

$f_{XY}(x,y) = f_X(x) f_Y(x)$, $\forall (x,y)$ → Independent

Independence → uncorrellated

$\cancel{\Leftarrow}$ unless Gaussian

Random Vectors: cont

Let X, Y be $n \times 1$ Random Vectors

If $Z = X + Y$:

$$\mu_Z = \mu_X + \mu_Y$$

$$R_Z = R_X + R_{XY} + R_{YX} + R_Y$$

$$\mu_Z \mu_Z^T = \mu_X \mu_X^T + \mu_X \mu_Y^T + \mu_Y \mu_X^T + \mu_Y \mu_Y^T$$

$$K_Z = K_X + K_{XY} + K_{YX} + K_Y$$

If $Z = X + Y$ and X, Y are uncorrellated:

$$R_Z = R_X + R_{XY} + R_{YX} + R_Y$$

$$K_Z = K_X + K_Y$$

Transformed RV: $Y_i = g_i(x_1, x_2, \dots, x_n)$ Independent Y_i

1) Find $x_i = \phi_i(y_1, \dots, y_n) \; \forall i = 1, \dots, n$

2) Find Jacobian

$$\tilde{J} = \begin{vmatrix} \frac{\partial \phi_1}{\partial y_1} & \cdots & \frac{\partial \phi_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial y_1} & \cdots & \frac{\partial \phi_n}{\partial y_n} \end{vmatrix} \quad \text{or} \quad J = \left.\begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{vmatrix}\right|_X$$

3) $f_Y(y) = f_X(x)|\tilde{J}| = f_X(x)|J|^{-1}$

$$X = [x_1, \dots, x_n] = [\phi_1, \dots, \phi_n]$$

4) Include Valid Regions of Y

5) if multiple Roots... $f_Y(y) = \sum_{i=1}^{r} f_X(x_i)|\tilde{J}_i| = \sum_{i=1}^{r} f_X(x_i)|J_i|^{-1}$

Random Vectors:

Let $Y \triangleq AX + b$ A is invertable

A, b are deterministic

$\mu_Y = A\mu_X + b$

$R_Y = A R_X A^T + (A\mu_X) b^T + b (A\mu_X)^T + b b^T$

$K_Y = A K_X A^T$

$f_Y(y) = \frac{1}{|\det(A)|} f_X(A^{-1}(y-b))$

Linear Algebra:

Positive (semi-)definite: Leading Minors $>(\geq) 0$

if $z^T M z > (\geq) 0 \quad \forall z$ or if $Re\{\lambda_i\} > (\geq) 0 \; \forall i$

Eigen Values/Vectors: $\lambda \equiv$ Eigen Values

$M\phi = \lambda\phi, \quad \phi \neq 0$ ($[\phi_1, \phi_n]^T$

$|M - \lambda I| = 0 \rightarrow \lambda$ $\phi =$ Eigen vectors

Similar Matrices:

$B = T_1^{-1} A T$

$\Lambda \triangleq \text{diag}\{\lambda_1, \cdots, \lambda_n\}$ $U = [\phi_1, \cdots, \phi_n]$

$\boxed{U^T M U = \Lambda}$

Covariance Matrices:

$K = K^T \geq 0$, if $\rho(K) = n$ the $K_i = K^T > 0$

Eigen-Decomposition:

1) Find $\{\lambda_i\}$ from $|K - \lambda I| = 0$

2) Find $\{\phi_i\}$ from $(K - \lambda_i I)\phi_i = 0$ and $\|\phi_i\| = 1$

3) $\Lambda = \operatorname{diag}\{\lambda_1, \cdots, \lambda_n\}$

$U = [\phi_1, \cdots, \phi_n]$ $\qquad K = U \Lambda U^T$

Whitening Transform: $K_Y = I$

$Y = \Lambda_X^{-1/2} U_X^H X$

1) Eigen Decomposition $\rightarrow K_X = U_X \Lambda_X U_X^H$

2) $\Lambda_X^{-1/2} = \operatorname{diag}\{\lambda_1^{-1/2}, \cdots, \lambda_n^{-1/2}\}$

3) $Y = \Lambda_X^{-1/2} U_X^H X$

Corrrllated Random Vector: $\mu_X = 0 \quad K_X = I$

$Y = Ax + b$

1) Eigen Decomp $\rightarrow Q = U_Y \Lambda_Y U_Y^T$

2) $\Lambda_Y^{1/2} = \operatorname{diag}\{\lambda_1^{1/2}, \cdots, \lambda_n^{1/2}\}$

3) $A = U_Y \Lambda_Y^{1/2}$

$\mu_Y = b \quad K_Y = Q$

Simultanious Diagonilization:

$P_{n\times n} = P_{n\times n}^T > 0$, $Q_{n\times n} = Q_{n\times n}^T$

$V^T P V = I$

$V^T Q V = diag\{\lambda_1, \dots, \lambda_n\}$

1) Find $\{\lambda_i\}$ of $P^{-1}Q$ w/
$|P^{-1}Q - \lambda I| = 0$

2) Find $\{v_i'\}$ w/ $(P^{-1}Q - \lambda_i)v_i' = 0$

3) Find $\{K_i\}$ w/ $v_i = K_i v_i'$ such that $v_i^T P v_i = 1 \ \forall i = 1, \dots, n$

4) $V = [v_1, \dots, v_n]$

Muti-dimensional Gaussian: See Notes

Characteristic Functions: See Notes

Complex Random Vectors: See Notes