MECH6325 - Final Exam TLDR | Jonas Wagner | 2020-12-05 1

Topics:

- Probability Theory + Random Variables (RVs)
- Functions of RV
- Expetations + Law of Large Numbers (LLN)
- Random Vectors
- Parameter Estimation
- Statistical Tests

Probability Theory:

- Basics

H - Random Experiment

E_i - Event $\quad \Omega$ - Sample space (All outcomes)

$E_i \subset \Omega$

$P[E] \geq 0$

$P[\Omega] = 1$

$P[E+F] = P[E] + P[F]$ if $P[EF] = 0$

$P[\emptyset] = 0$

if $E \subset F \rightarrow P[E] \leq P[F]$

$P[EF'] = P[E] - P[EF]$

$P[E] = 1 - P[E']$

$P[EF] = P[E] + P[F] - P[EF]$

$P[E+F] = P[E] + P[F]$ if $P[EF] = 0$

$P[E+F] \leq P[E] + P[F]$

Set Notes

OR

$E+F \sim E \cup F$

AND

$EF \sim E \cap F$

NOT

$E' \sim E^c$

XOR

$E \oplus F \sim$ in $E+F$ but not EF

$A+B = B+A$

$AB = BA$

$A(B+C) = AB + AC$

$A + BC = (A+B)(A+C)$

$(A+B)^c = A^c B^c$

$(AB)' = A' + B'$

Probability Theory: cont.

Conditional probability

$$P[A|B] = \frac{P[AB]}{P[B]}, P[B] > 0$$

Independence:

A, B Independent iff $P[AB] = P[A]P[B]$

$A_i \ \forall i = 1, \dots, n$ Independent iff $P[A_i A_j] = P[A_i]P[A_j] \ \forall i \neq j$

$P[A_i A_j A_k] = P[A_i]P[A_j]P[A_k] \ \forall i \neq j \neq k$

$\vdots$

$$\text{AND} \rightarrow P\left[\bigcap_{i=1}^{n} A_i\right] = \prod_{i=1}^{n} P[A_i]$$

Union of Events: Multiple ORs

$$P\left[\bigcup_{i=1}^{n} E_i\right] = \sum_{i=1}^{n} (-1)^{i+1} S_i$$

where, $S_K = P[E_{i_1} E_{i_2} \cdots E_{i_K}]$

4-case:

$$P\left[\bigcup_{i=1}^{4} E_i\right] = S_1 - S_2 + S_3 - S_4$$

$$S_1 = P[E_1] + P[E_2] + \dots + P[E_4] = P[E_i] \ \forall i$$

$$S_2 = P[E_1 E_2] + P[E_1 E_3] + \dots + P[E_3 E_4] = P[E_i E_j] \ \forall i \neq j$$

$$S_3 = P[E_1 E_2 E_3] + \dots + P[E_2 E_3 E_4] = P[E_i E_j E_k] \ \forall i \neq j \neq k$$

$$S_4 = P[E_1 E_2 E_3 E_4]$$

Probability Theory: cont.

Law of Total Probability:

$$P[B] = \sum_{i=1}^{n} P[B|A_i] P[A_i]$$

Baye's Theorem:

$$P[A|B] = \frac{P[B|A] P[A]}{P[B]} \quad P[A_i|B] = \frac{P[B|A_i] P[A_i]}{\underbrace{\sum_{i=1}^{n} P[B|A_i] P[A_i]}_{\text{Total Probability: } P[B]}}$$

Combinatronics:

of ordered samples: n^{r} ← samples, n ← pop-size
w/ replacement

" w/out replacement: $(n)_r = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$

Binomial: $C_r^n \triangleq \binom{n}{r} = \frac{(n)_r}{r!} = \frac{n!}{(n-r)!\,r!}$
(unordered w/out replament)

Note: $\binom{n}{r} = \binom{n}{n-r}$

Repeated Independent trials:

Binomial probability Law: A_K = K successes

PDF: $P[A_K] \triangleq b(K; n, P) = \binom{n}{K} P^K (1-P)^{n-K}$ w/ $P[\text{success}] = P$

CDF: $B(K; n, P) = \sum_{i=0}^{K} b(i; n, P)$

successes $\leq K$

Ocupancy Problem: r balls into n cells
non-distinguishable

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

Reliability: N-independent components & $P[\text{success}] = P_i$

Series

$$P\begin{bmatrix}\text{success}\\\text{system}\end{bmatrix} = \prod_{i=1}^{N} P_i$$

Parallel

$$P\begin{bmatrix}\text{Failure}\\\text{system}\end{bmatrix} = \prod_{i=1}^{N} (1-P_i) \quad P\begin{bmatrix}\text{success}\\\text{system}\end{bmatrix} = 1 - \prod_{i=1}^{N} (1-P_i)$$

Random Varibles:

Cumulinative Distribution Function (CDF)

$F_X(x) = P[X \leq x]$

$F_X(\infty) = 1$

$F_X(-\infty) = 0$

(Always increases)

$x_1 \leq x_2 \rightarrow F_X(x_1) \leq F_X(x_2)$

(Contious from the right)

$0 \leq F_X(x) \leq 1$

PMF: (Mass Function)

DT

$P_X(x) = P[X = x]$

pdf: Probability density function

CT: $f_X(x) = \dfrac{d F_X(x)}{dx}$

$f_X(x) \geq 0 \qquad \int_{-\infty}^{\infty} f_X(x)\,dx = 1$

DT: $f_X(x) = \sum_{i=1}^{K} \underbrace{P_i}_{\text{From PMF}} \delta(x - x_i)$

pdf $\rightarrow$ ~~C~~ CDFs:

$F_X(x) = P[X \leq x] = \int_{-\infty}^{x} f_X(\tau)\,d\tau$

For CT ~~pdf~~ pdf: $P[X = x] = 0$

DT

$F_X(x) = P[X \leq x] = \sum_{x_i \leq x} P_X(x_i)$

$P_X[x_i] = F_X(x_i) - F_X(x_i^-)$

Expectations:

Mean:

$\bar{X} = \mu_X = \mu = E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

DT

$E[X] = \sum_i x_i P_X(x_i)$

Mode: (Max point)

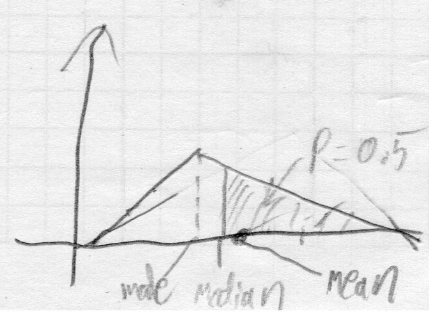
$f_X(x_{mode}) \geq f_X(x), \forall x$

$P_X(x_{mode}) \geq P_X(x)$

Median

$P[X < x_{med}] = P[X > x_{med}]$

or, $F_X(x_{med}) = 0.5$



Random Variables: cont.

Variance: $\sigma_x^2 = E[(X-\mu_x)^2] = E[X^2] - \mu_x^2$

Expectation (General): $E[g(x)] = \int_{-\infty}^{\infty} g(x) f_x(x) dx$

Joint Dists + Densities:

CDF $F_{XY}(x,y) = P[X \leq x, Y \leq y]$

pdf $f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$

DT: $P_{XY}(x_i, y_i) = P[X = x_i, Y = y_i]$

Properties

$F_{XY}(\infty,\infty) = 1$, $F_{XY}(-\infty,y) = F_{XY}(x,-\infty) = 0$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y) dx dy = 1$

For $x_1 \leq x_2$, $y_1 \leq y_2$

$F_{XY}(x_1, y_1) \leq F_{XY}(x_2, y_2)$

$F_{XY}(x,\infty) = F_X(x)$, $F_{XY}(\infty, y) = F_Y(y)$

$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y) dy$, $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y) dx$

$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y) dx dy$

$E[g(X)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x) f_{XY}(x,y) dx dy = \int_{-\infty}^{\infty} g(x) f_X(x) dx$

Random Varibles: cont.

Ortnogonal + Correlated

Correllation: $R_{XY} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\, dx\, dy$

$R_{XY} = 0 \implies$ Ortnogonal!

Covariance: $\mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$= R_{XY} - \mu_X \mu_Y$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) f_{XY}(x,y)\, dx\, dy$

$\mathrm{Cov}(X,Y) = 0$

$\Downarrow$

uncorrellated!

Correllation Coeficent: $\rho_{XY} = \dfrac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$

$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\,\mathrm{Cov}(X,Y)$

Conditional

$F_{X|B}(x|B) = \int_{-\infty}^{x} f_{X|B}(u|B)\, du = \dfrac{P[X \le x | B]}{P[B]}$

$f_{X|B}(x|B) = \begin{cases} \dfrac{f_X(x)}{P[B]}, & x \in (B \cap \Omega_X) \\ 0 & \text{otherwise} \end{cases}$

$F_{X|Y}(x|y) = \int_{-\infty}^{x} f_{X|Y}(u|y)\, du$ $\quad E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x,y)\, dx$

$f_{X|Y}(x,y) = \dfrac{f_{XY}(x,y)}{f_Y(y)}$ $\quad E[g(x)|Y=y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x,y)\, dx$

$f_{X_1 X_4}(x_1,x_4) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_X(x_1|x_2,x_3,x_4)\, f(x_2,x_3|x_4)\, dx_2\, dx_3$

Random Varibles: Cont.

Independence

$\{X_i, i=1,\cdots,n\}$ are independent iff, $(\forall x_1,\cdots,x_n)$

$$F_{X_1\cdots X_n}(x_1,\cdots,x_n) = \prod_{i=1}^{n} F_{X_i}(x_i)$$

$$f_{X_1\cdots X_n}(x_1,\cdots,x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

$\Downarrow$

$$f_{X_1\cdots X_k|X_{k+1}\cdots X_n}(x_1\cdots x_k|x_{k+1}\cdots x_n) = f_{X_1\cdots X_k}(x_1,\cdots,x_k)$$

Independence $\Rightarrow$ uncorrellated
($\Leftrightarrow$ if gaussian)

If $E[X_i X_j]=0 \rightarrow X_i, X_j$ are orthogonal

Baye's Formula; $P[B|X=x] = \dfrac{f_X(x|B)\,P[B]}{f_X(x)}$

Jointly Gaussian:

$$f_{XY}(x,y) = \left(2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}\right)^{-1} \exp\left(\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right)$$

$f_{XY}(X,Y)_{\text{jointy gaussian}} \longrightarrow f_X(x) = \mathcal{N}(\mu_x,\sigma_x^2)$, $f_Y = \mathcal{N}(\mu_y,\sigma_y^2)$

if $\rho=0 \longleftrightarrow$ independent, uncorrelated

if $\rho\neq 0 \longleftrightarrow$ not-independent, correllated

Random Variables: cont.

Failure Rates: X - lifetime

$$P[t < X \leq t+dt \mid X \geq t] = \frac{f_X(t)\,dt}{1-F_X(t)} = \alpha(t)\,dt$$

$$F_X(t) = 1 - e^{-\int_0^t \alpha(t)\,dt}$$

$$f_X(t) = \alpha(t)\, e^{-\int_0^t \alpha(t)\,dt}$$

$$\boxed{\alpha(t) = \frac{f_X(t)}{1-F_X(t)}}$$

if $\alpha(t)$ is constant $\Longleftrightarrow$ X is exponential

clarification:

$$f_X(x \mid X \geq t) = \begin{cases} 0, & x < t \\ \frac{f_X(x)}{1-F_X(t)}, & x \geq t \end{cases}$$

$$f_X(t \mid X \geq t) = \alpha(t)$$

Binomial → Poisson
Binomial → Gaussian
Poisson → Gaussian

Asymptotic Relationship:

Binomial → Poisson $\quad n \gg 1,\ p \ll 1$

$$\boxed{np = a} \qquad b(k; n, p) \simeq \frac{(np)^k}{k!} e^{-np}$$

Binomial → Gaussian

$P[\text{success}] = p$

S_n = # of success in n trials $\Rightarrow$ $X \sim \mathcal{N}(np, npq)$ if $n \gg 1$

$S_n - \mu = np$

$\sigma^2 = np(1-p) = npq$

$$P[\alpha \leq S_n \leq \beta] \approx P[\alpha - 0.5 \leq X \leq \beta + 0.5]$$

Discrete → Cont.

$$\simeq \Phi\left(\frac{\beta + 0.5 - np}{\sqrt{npq}}\right) - \Phi\left(\frac{\alpha - 0.5 - np}{\sqrt{npq}}\right)$$

$$P[S_n = k] = b(k; n, p)$$

$$\simeq (1)\, f_X(k) = \frac{1}{\sqrt{npq}}\, \underbrace{f_{SN}}_{\text{z-table...}}\left(\frac{k-np}{\sqrt{npq}}\right)$$

Random Variables: cont.

Asymptotic Relationships:

Poisson → Gaussian   Poisson Y w/ $(\lambda\tau, \lambda\tau)$ $\lambda\tau \gg 1 \to X \sim N(\lambda\tau, \lambda\tau)$

$$P[\alpha \le Y \le \beta] \approx P[\alpha - 0.5 \le X \le \beta + 0.5]$$

$$\sum_{K=\alpha}^{\beta} e^{-\lambda\tau} \frac{(\lambda\tau)^K}{K!} \approx \Phi\left(\frac{\beta + 0.5 - \lambda\tau}{\sqrt{\lambda\tau}}\right) - \Phi\left(\frac{\alpha - 0.5 - \lambda\tau}{\sqrt{\lambda\tau}}\right)$$

$$e^{-\lambda\tau} \frac{(\lambda\tau)^K}{K!} \approx \Phi\left(\frac{K + 0.5 - \lambda\tau}{\sqrt{\lambda\tau}}\right) - \Phi\left(\frac{\alpha - 0.5 - \lambda\tau}{\sqrt{\lambda\tau}}\right)$$

Functions of RVs:

→ CDF → pdf: $Y = g(X)$

$$F_Y(y) = P[g(X) \le y] = P[X \in C_y]$$

C_y: Region where $g(x) \le y$

$$f_Y(y) = \frac{dF_Y(y)}{dy}$$

Common: $Y = aX + b$

$E[Y] = aE[X] + b$ For $a \neq 0$: $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

$Var[Y] = a^2 Var[X]$

Direct Computation: $Y = g(X)$ check S_Y first (S_Y: Range of Y)

$$f_Y(y) = \sum_{i=1}^{n} f_X(x_i) \left| g'(x_i) \right|^{-1} \qquad \boxed{x_i = g^{-1}(y)}$$

Roots (Solutions)

$$= \sum_{i=1}^{n} f_X(x_i) \left| \frac{dx_i}{dy} \right|$$

if $y - g(x) = 0$ has no roots → $f_Y(y) = 0$

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Functions of RV:

$$V = g(X,Y) \quad , \quad W = h(X,Y)$$

Indirect: $P[V \le v, W \le w] \triangleq F_{VW}(v,w)$

$$= \iint_{(x,y)\in C_{vw}} f_{XY}(x,y)\,dx\,dy$$

$C_{vw} = \{(x,y) : g(x,y) \le v, h(x,y) \le w\}$

Direct: $f_{XY} \rightarrow f_{VW}$

For each Root

$$\begin{bmatrix} x_i = \phi_i(v,w) \\ y_i = \psi_i(v,w) \end{bmatrix}$$

$$|J_i| = \begin{vmatrix} \frac{\partial g}{\partial x_i} & \frac{\partial g}{\partial y_i} \\ \frac{\partial h}{\partial x_i} & \frac{\partial h}{\partial y_i} \end{vmatrix} \qquad |\tilde{J}_i| = |J_i|^{-1} = \begin{vmatrix} \frac{\partial \phi_i}{\partial v} & \frac{\partial \phi_i}{\partial w} \\ \frac{\partial \psi_i}{\partial v} & \frac{\partial \psi_i}{\partial w} \end{vmatrix}$$

$$f_{VW}(v,w) = \sum_{i=1}^{n} f_{XY}(x_i,y_i)|\tilde{J}_i| = \sum_{i=1}^{n} \frac{f_{XY}(x_i,y_i)}{|J_i|}$$

Method:

1) Check S_{VW} } Regions of VW

2) Find Roots $\forall$ region

3) $|J|$ or $|\tilde{J}|$

4) $f_{VW}(v,w) = \sum f(\,)|\tilde{J}_i| = \sum f(\,)|J|^{-1}$

Functions of RVs: Cont

$Z = g(X,Y)$

$$F_Z(z) = \iint_{(x,y)\in C_z} f_{XY}(x,y)\, dx\, dy$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} \qquad \{Z \le z\} = \{(X,Y) \in C_z\}$$

Special cases:

Sums

Poisson

$a + b \rightarrow (a+b)$

Binomial

$2\, b(k; n, p) \rightarrow b(k; 2n, p)$

χ^2

$2\,(\text{dof } n) \rightarrow 2n \text{ dof}$

Cauchy

$(\alpha_1, \beta_1) + (\alpha_2, \beta_2) \rightarrow (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$

Gaussian

$(\mu_1, \sigma_1^2) + (\mu_2, \sigma_2^2) \rightarrow (\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

polar transform Note:

$$\left.\begin{aligned} X &= r\cos\theta \\ Y &= r\sin\theta \end{aligned}\right] \rightarrow \begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \tan^{-1}(y/x) \end{aligned}$$

Expectations of Functions:

$Y = g(x) \qquad E[Y] = \int_{-\infty}^{\infty} g(x) f_x(x) dx$

$Z = g(x,y) \qquad E[Z] = \int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y) dx dy$

$$E\left[\sum_{i=1}^{N} g_i(x)\right] = \sum_{i=1}^{N} E[g_i(x)]$$ Linearity of Sum of Expectations

$Y = g(x)$

$$E[Y] = E\left[E[Y|X]\right] = \int_{-\infty}^{\infty} E[Y|X=x] f_X(x) dx$$

$Z = g(x,y)$

$$E[Z] = E\left[E[Z|X,Y]\right] = \iiint_{-\infty}^{\infty} z f_{Z|XY}(z|xy) f_{XY}(x,y) dx dy dz$$

$$Var[X] = E\left[Var[X|Y]\right] + Var\left[E[X|Y]\right]$$

Moments:

$m_r \triangleq E[X^r], \; r = 0, 1, \dots \qquad E[X] = \mu$

$c_r \triangleq E[(X-\mu)^r], \; r = 0, 1, \dots$

$\quad = \sum_{i=0}^{r} \binom{r}{i} (-1)^i \mu^i m^{r-i}$

More in notes

Joint:

$m_{ij} = E[X^i Y^j]$

$c_{ij} = E[(X-\bar{X})^i (Y-\bar{Y})^j]$

$$\rho \triangleq \frac{c_{11}}{\underbrace{\sqrt{c_{20} c_{02}}}_{\text{central moments}}}$$

Moments; cont.

MGR

$t \in \mathbb{C}$

$$\theta(t) \triangleq E\left[e^{tX}\right]$$

$$= \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dy$$

$$m_k = \theta^{(k)}(0), \quad k = 0, 1, \dots \qquad \theta^{(k)}(0) = \left[\frac{\partial^k}{\partial t^k} \theta(t)\right]_{t=0}$$

$$\theta_{XY}(t_1, t_2) = E\left[e^{(t_1 X + t_2 Y)}\right]$$

$$m_{ln} = \theta^{(l,n)}(0,0), \qquad \theta_{XY}^{(l,n)}(0,0) = \left[\frac{\partial^{l+n} \theta_{XY}(t_1, t_2)}{\partial t_1^l \, \partial t_2^n}\right]_{t_1 = t_2 = 0}$$

Charictoristic Function!

$$\Phi_X(\omega) \triangleq E\left[e^{j\omega X}\right] = \int_{-\infty}^{\infty} f_x(x) e^{j\omega x} dx$$

$$m_r \triangleq E[X^r] = (-j)^r \Phi_X^{(r)}(0) = (-j)^r \Phi_{X,Y}^{(r,0)}(0,0)$$

$$m_{rk} \triangleq E[X^r Y^k] = (-j)^{r+k} \Phi_{XY}^{(r,k)}(0,0)$$

$$\Phi_X^{(r)}(0) = \left[\frac{\partial^r \Phi_X(\omega)}{\partial \omega^r}\right]_{\omega=0}$$

$$\Phi_{X,Y}^{(r,k)}(0,0) = \left[\frac{\partial^{r+k} \Phi_{XY}(\omega_1, \omega_2)}{\partial \omega_1^r \, \partial \omega_2^k}\right]_{\omega_1 = \omega_2 = 0}$$

"MGF"/Char Function;

$Z = X + Y$, Independent RVs

$P_Z[k] = P_X[k] * P_Y[k]$

$\updownarrow$

$\Phi_Z(\omega) = \Phi_X(\omega)\,\Phi_Y(\omega)$

LLN: Law of Large Numbers

$P\left[\lim_{n\to\infty} \hat{\mu}_n = \mu_x\right] = 1$

Central Limit Theorem;

$X_1, \ldots, X_n$ are independent... $n \gg 1$ ($\gtrsim 10$)

$\hat{X}_k = 0$, $F_{X_1}(x_1) \cdots F_{X_n}(x_n)$, $Var[X_k] = \sigma_k^2$

$S_n^2 = \sigma_1^2 + \cdots + \sigma_n^2$

$Z_n \triangleq (X_1 + \cdots + X_n)/S_n$

$\lim_{n\to\infty} F_{Z_n}(z) = \Phi(z) = 1 - Q(z)$

$Z_n = \dfrac{S_n - E[S_n]}{\sqrt{Var[S_n]}}$; $S_n = \Sigma X_i$ $\quad \hat{X} = \frac{1}{n} \Sigma X_i$

CDF approx; $P[a \le Z_n \le b] = Q(a) - Q(b)$

$Q(-c) = 1 - Q(c)$

Random Vectors!

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$F_X(x) \triangleq P[X \leq x]$ CDF of x

$f_X(x) \triangleq \frac{d^n F_X(x)}{dx_1 \cdots dx_n}$ PDF of x

similar other rules

Expectation Vector

$\mu = \mu_X \triangleq E[X] = [\mu_1 \cdots \mu_n]^T$

$\mu_i = \int_{-\infty}^{\infty} x_i f_{x_i}(x_i)$ after worrying about conditional probabilites

Correllation Matrix

$R \triangleq E[X X^T]$ — $R_{ij} = E[x_i x_j]$

Covariance

$K \triangleq E[(x-\mu)(x-\mu)^T]$ — $K_{ii} = \sigma_i^2$

$K = R - \mu\mu^T$

$K_{ij} = \mathrm{COV}(x_i, x_j) = K_{ji}$

R_X = Diagonal $\rightarrow$ uncorrellated

K_X = Diagonal $\rightarrow$ orthogonal

$Z = X + Y$:

$\mu_Z = \mu_X + \mu_Y$

$R_Z = R_X + R_{XY} + R_{YX} + R_Y$

$\mu_Z \mu_Z^T = \mu_X \mu_X^T + \mu_X \mu_Y^T + \mu_Y \mu_X^T - \mu_Y \mu_Y^T$

$K_Z = K_X + K_{XY} + K_{YX} + K_Y$

If uncorrellated:

$R_Z = R_X + R_{XY} + R_{YX} + R_Y$

$K_Z = K_X + K_Y$

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Random Vectors; cont.

Transformed RVs → same as RVs ... see other notes

$\underline{Y = AX + b}$; A ← invertable, deterministic

$$\mu_Y = A\mu_X + b$$

$$R_Y = A R_X A^T + (A\mu_X) b^T + b(A\mu_X)^T + b b^T$$

$$K_Y = A K_X A^T$$

$$f_Y(y) = \frac{1}{|\det(A)|} f_X\left(A^{-1}(y-b)\right)$$

Eigen Decomp:

$$K = K^T \geq 0$$

if $\rho(K) = n \rightarrow K = K^T > 0$

$$\boxed{K = U \Lambda U^T}$$

1) $\{\lambda_i\} \leftarrow |K - \lambda I| = 0$ ← eigVl(·)

2) $\{\phi_i\} \leftarrow (K - \lambda_i I)\phi_i = 0$ ← eigVc(·) and $\|\phi_i\| = 1$

TI-nspire

3) $\Lambda = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$

$$U = [\phi_1 \cdots \phi_n]$$

Random Vectors:

Whitening Transform: $K_Y = I$

$$Y = \Lambda_X^{-1/2} U_X^H X$$

1) Eigen Decomp $\rightarrow K_X = U_X \Lambda_X U_X^H$
2) $\Lambda_X^{-1/2} = \text{diag}\{\lambda_1^{-1/2}, \dots, \lambda_n^{-1/2}\}$
3) $Y = \Lambda_X^{-1/2} U_X^H X$

Correllated Random Vector: $Y \sim (\mu_Y = b, K_Y = Q)$

$X \sim (0, I) \rightarrow Y \sim (b, Q)$

$$\boxed{Y = Ax + b}$$

1) Eigen Decomp $\rightarrow Q = U_Y \Lambda_Y U_Y^T$
2) $\Lambda_Y^{1/2} = \text{diag}\{\lambda_1^{1/2} \dots \lambda_Q^{1/2}\}$
3) $A = U_Y \Lambda_Y^{1/2}$

Simultanious Diagonilization: Diagonalize P and Q

$P = P_{n\times n}^T \geq 0 \qquad Q = Q_{n\times n}^T \geq 0$

$V^T P V = I$

$V^T Q V = \text{diag}\{\lambda_1, \dots, \lambda_n\}$

$Q v_i = \lambda_i P v_i$

1) $\{\lambda_i\} \leftarrow (P^{-1} Q - \lambda I) = 0$
2) $\{v_i'\} \leftarrow (P^{-1} Q - \lambda_i I) v_i' = 0$
3) $\{K_i\} \leftarrow v_i \hat{=} K_i v_i'$ w/ $v_i^T P v_i = 1$

   $V^T P V = I$
4) $V = [v_1 \cdots v_n] \rightarrow V^T Q V = \text{diag}\{\lambda_1 \dots \lambda_n\}$

see notes for: Gassian Random Vectors

Parameter Estimation:

$$\hat{\mu}_x(n) \triangleq \frac{1}{n}\sum_{i=1}^{n} X_i$$ Mean (unbiased + Consistant)

$$\hat{\sigma}_x^2(n) \triangleq \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\mu}_x(n))^2$$ (unbiased + Consistant)

Variance

$$\hat{\sigma}_x^2(n) \triangleq \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu}_x(n))^2$$ (Biased)

Confidence Intervals: (step by step in other TLDR or see printout) for mean

Non-Gaussian: w/ large n...

$$\hat{\mu}(n) = \frac{1}{n}\sum_{i=1}^{n} X_i \leftarrow N(\mu, \sigma^2/n)$$

$$F_{SN}(a) = (1+\delta)/2 \rightarrow a = z_{[(1+\delta)/2]}$$

$$[\hat{\mu} - (z_{(1+\delta)/2}\,\sigma/\sqrt{n}),\ \hat{\mu} + z_{[(1+\delta)/2]}\,\sigma/\sqrt{n}]$$

Non-Parametric Estimation:

Median $$\hat{X}_{[0.5]} = \begin{cases} Y_{K+1}, & \text{if } n = 2K+1 \\ 0.5(Y_K + Y_{K+1}), & \text{if } n = 2K \end{cases}$$

Percentile $$\hat{x}_{[u]} = Y_i + \frac{(Y_{i+1} - Y_i)\left(u - \frac{i}{n+1}\right)}{1/(n+1)}, \quad i = (n+1)/u$$

CI:

$$P[Y_K \leq X_{[u]} < Y_{K+r}] = \sum_{i=K}^{K+r-1}\binom{n}{i} u^i (1-u)^{n-i}$$

median CI if $N \gg 1$

$$r = (n - \sqrt{n}\, z_{[(1+\delta)/2]} + 1)/2$$

$$CI: [Y_r,\ Y_{n-r+1})$$