Lecture 07

02/10/2021

Last time: Poincare-Bendixon thm

Hopf bifurcation

Scaling /nondimensionalization

Today: Center manifold theory → (Khalil Chap. 8)

Existence/uniqueness of sol'ns

Center manifold theory:

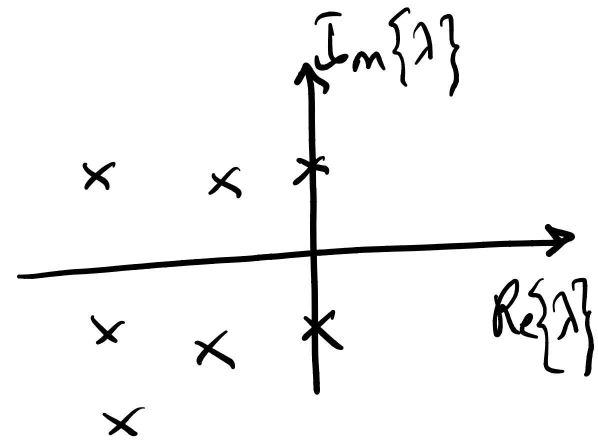
$$\dot{x} = f(x) \qquad x(t) \in \mathbb{R}^n \qquad (1)$$

$f(0) = 0 \Rightarrow \bar{x} = 0$ is an e.p.

and assume the linearization around $\bar{x} = 0$ has

K eigenvalues on $j\omega$-axis

n-K eigenvalues in the LHP



$$A = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0} \Rightarrow \dot{\tilde{x}} = A\tilde{x}$$

cannot be used to assess local stability properties of $\bar{x} = 0$.

Ex

$\dot{x} = x^3$ → unstable

$\dot{x} = -x^3$ → GAS

$\Big\}$ → both $A = 0$

$\bar{x} = 0$

Lets rewrite (1) as:

$$\dot{x} = Ax + \tilde{f}(x) \qquad \text{where } \tilde{f} = f(x) - \left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0} \cdot x$$

Taylor series of f around $\bar{x}=0$:

$$f(x) = \underbrace{\cancel{f(0)}}_{0} + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0}}_{A}(x-0) + \underbrace{\boxed{H.O.T.}}_{\tilde{f}}$$

everything but linear

Properties of $\tilde{f}(x)$: $\tilde{f}(x) = f(x) - \left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0} x \Rightarrow \boxed{\tilde{f}(0) = 0}$

$$\frac{\partial \tilde{f}(x)}{\partial x} = \frac{\partial f(x)}{\partial x} - \frac{\partial f}{\partial x}\bigg|_{\bar{x}=0} \rightarrow \boxed{\frac{\partial \tilde{f}}{\partial x}\bigg|_{\bar{x}=0} = 0}$$

So we could rewrite (1) as:

$$\boxed{\begin{gathered} \dot{x} = Ax + \tilde{f}(x) \qquad (2) \\ \text{where} \\ \tilde{f}(0) = 0 \quad ; \quad \frac{\partial \tilde{f}}{\partial x}(0) = 0 \end{gathered}}$$

Introduce a change of coordinates

$$\begin{bmatrix} y \\ z \end{bmatrix} = T^{-1} x \quad ; \quad \begin{array}{l} y(t) \in \mathbb{R}^{k} \\ z(t) \in \mathbb{R}^{n-k} \end{array}$$

$\rightarrow$ bring A into block diagonal form

$$\bar{A} = T^{-1} A T = \left[\begin{array}{c|c} A_1 & \\ \hline & A_2 \end{array}\right]$$

to bring system (2) into the following form:

$$\dot{y} = A_1 y + g_1(y, z)$$

$$\dot{z} = A_2 z + g_2(y, z)$$

where (a) A_1 contains all e-values on $j\omega$ axis and A_2 contains e-values in LHP.

In other words, coupling btwn y and z dynamics only enters at nonlinear level.

(b) $g_i(0,0) = 0 \quad ; i=1,2$

$$\left.\frac{\partial g_i}{\partial y}\right|_{(0,0)} = 0 \qquad \left.\frac{\partial g_i}{\partial z}\right|_{(0,0)} = 0$$

Theorem 1: There is an invariant manifold $z = h(y)$ in the neighborhood of the origin that satisfies

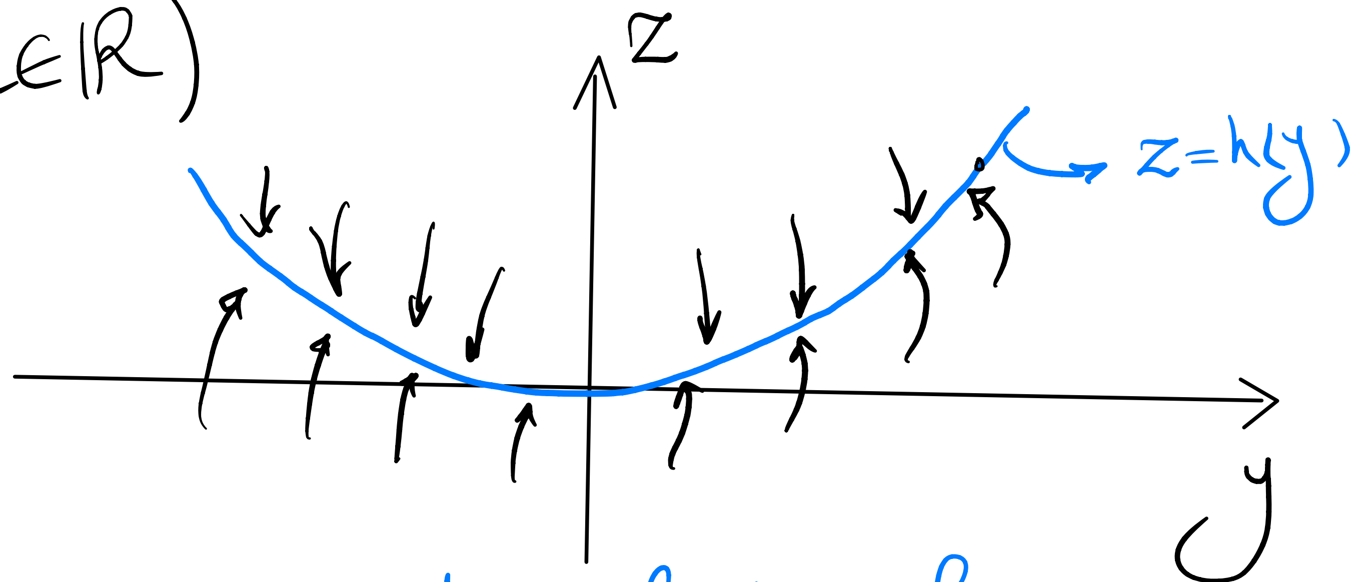
$$h(0) = 0$$

$$\left.\frac{\partial h}{\partial y}\right|_{0} = 0$$

$z(0) = h(y(0))$

$\forall t \geq 0 \quad z(t) = h(y(t))$

Ex In $\mathbb{R}^2$ $(y \in \mathbb{R}, z \in \mathbb{R})$



when you start on an invariant manifold surface you stay on it forever.

Main result

Theorem 2: If the origin of the reduced system

$(\bar{y} = 0)$ $$\dot{y} = A_1 y + g_1(y, h(y))$$

is asymptotically stable (unstable) then the origin of

$$\dot{y} = A_1 y + g_1(y, z)$$
$$\dot{z} = A_2 z + g_2(y, z)$$

is asymptotically stable (unstable) and so is $\bar{x} = 0$ for $\dot{x} = f(x)$.

Characterization of the center manifold:
(i.e. how to find $h(y)$?)

Let $w := z - h(y)$. Since $z = h(y)$ is invariant

then $w = 0 \Rightarrow \dot{w} = 0$

$$\boxed{\dot{w} = \dot{z} - \frac{\partial h}{\partial y}\dot{y} = A_2 h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y}\left[A_1 y + g_1(y, h(y))\right] = 0} \quad (*)$$

This equation characterizes the center manifold.

solve for $h(y)$ given the fact $h(0)=0$ $\left.\frac{\partial h}{\partial y}\right|_0 = 0$

Center manifold can be found by solving (*), but, in general, this is a nontrivial exercise!

Ex.

$$\dot{y} = 0y + \underbrace{y \cdot z}_{g_1(y,z)}$$

$$\dot{z} = -z + \underbrace{a y^2}_{g_2(y,z)}$$

$A_1 = 0$

$A_2 = -1$

$$-h(y) + ay^2 = \frac{\partial h}{\partial y}\left(0 \cdot y + y \cdot h(y)\right)$$

$$\Rightarrow \quad y\, h(y) \frac{\partial h}{\partial y} + h(y) = ay^2$$

No idea how to solve explicitly

$$h(0) = \left.\frac{\partial h}{\partial y}\right|_0 = 0$$

Approach: Taylor series of $h(y)$ around origin:

$$h(y) = \underbrace{\cancel{h(0)}}_{0} + \underbrace{\cancel{\frac{\partial h}{\partial y}}}_{0}\Big|_0 y + \underbrace{h_2}_{\frac{\partial^2 h}{\partial y^2}\big|_0} y^2 + \underbrace{h_3}_{\frac{\partial^3 h}{\partial y^3}\big|_0} y^3 + O(y^4)$$

$$h(y) = h_2 y^2 + h_3 y^3 + O(y^4) \qquad (\mathrm{I})$$

where h_i are constants. Thus, $h(y)$ contains quadratic and higher order terms. Plug (I) into equation (*) for $h(y)$ and equate like-orders of y on both side....

Ex.

$$\dot{y} = 0 \cdot y + y \cdot z \quad \to g_1$$

$$\dot{z} = -z + a y^2 \quad \to g_2 \qquad a \neq 0$$

$A_1 = 0$

$A_2 = -1$

$$-h(y) + a y^2 - \frac{\partial h}{\partial y}\left(0 \cdot y + y h(y)\right) = 0 \qquad (\mathrm{II})$$

plug (I) into (II) :

$$-h_2 y^2 - h_3 y^3 - O(y^4) + a y^2 - \left[2h_2 y + 3h_3 y^2 + O(y^3)\right] \cdot \left(h_2 y^3 + h_3 y^4 + O(y^5)\right)$$

$$= 0$$

$y^2: \Rightarrow h_2 = a$

$$\Rightarrow h(y) = ay^2 + O(y^3)$$

We then have,

$$\dot{y} = 0y + y\,h(y)$$

$$\dot{y} = 0 \cdot y + ay^3 + O(y^4)$$

We can now determine the stability of the system by studying the sign of a in $\dot{y} = ay^3$

$a > 0 \rightarrow$ unstable ; $a < 0 \rightarrow$ locally asymptotically stable

Existence & Uniqueness of Solutions:

Mathematical preliminaries (backround) → (Chap. 3 of Khalil)

$\dot{x} = f(x,t)$ → time-varying

$\dot{x} = f(x)$ → time-invariant

Linear Algebra : $Ax = b$

$A \in \mathbb{R}^{m \times n}$ $\quad x \in \mathbb{R}^n$ $\quad b \in \mathbb{R}^m$

If $b \in \text{Range}(A)$ then there exists a solution to $Ax = b$.

If $\text{Null}(A) = \{\ \}$ then the solution is unique.

Linear systems theory:

$$\dot{x} = Ax$$

$$x(t) = e^{At} x(0)$$

For LTI systems, matrix A is constant and the unique sol'n is given by $e^{At}x(0)$.

For LTV systems, $\dot{x} = \underbrace{A(t)}x$

$[a_{ij}(t)]$

piecewise continuous functions of time → existence + uniqueness of sol'n

All these questions make sense on both finite $[0,t_f]$ and infinite $[0,\infty)$ time intervals.

Question: What conditions do we need to impose in terms of dependence of x on t to have existence and uniqueness?

$$\dot{x} = f(x,t)$$

if f is piecewise continuous function of time.

Question: How about x?

Is continuity of f w.r.t. x enough?

No!

Ex. $\dot{x} = x^{1/3}$ $\quad x(t) \in \mathbb{R}$

solution $x(t) = 0$

Another sol'n is given by $x(t) = \left(\frac{2t}{3}\right)^{3/2}$

Multiple sol'ns!

So how do we avoid this?

The problem arises from the fact that the system is not continuously differentiable at the origin.

Fact! If f is a continuous function (C^0) then there is a solution on $[0, t_f)$ (but it may not be unique)