MECH 6313 - Homework 6

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Contents

1 Problem 1

Problem: Show that the parallel connection of two passive dynamical systems is passive. Can you claim the same for the series connection of two passive systems?

Solution: Let two passive systems be defined as a system taking an input u and generating an output y as

$$H_1 : y_1 = h_1(u), \text{ s.t. } \langle y_1|u\rangle \geq 0$$

and

$$H_2 : y_2 = h_2(u), \text{ s.t. } \langle y_2|u\rangle \geq 0$$

with $\langle y|u\rangle = \int_0^T y^T(t)u(t)\,\mathrm{d}t$

1.1 Parallel Connection of Passive System

The parallel system H_p can then be defined by

$$H_p : h_p(u) = y_p = y_1 + y_2 = h_1(u) + h_2(u)$$

whose passivity can be proven directly by testing $\langle y_p|u\rangle$ which is calculated as

$$\begin{aligned}
\langle y_p|u\rangle &= \int_0^T y_p^T u\,\mathrm{d}t && (1)\\
&= \int_0^T (y_1 + y_2)^T u\,\mathrm{d}t && (2)\\
&= \int_0^T y_1^T u + y_2^T u\,\mathrm{d}t && (3)\\
&= \int_0^T y_1^T u\,\mathrm{d}t + \int_0^T y_2^T u\,\mathrm{d}t && (4)\\
&= \langle y_1|u\rangle + \langle y_2|u\rangle && (5)
\end{aligned}$$

Since $\langle y_1|u\rangle \geq 0$ and $\langle y_2|u\rangle \geq 0$,

$$\langle y_p|u\rangle \geq 0 \quad (6)$$

which proves, by definition, that H_p is passive.

1.2 Series Connection of Passive System

The series system H_s can be defined by

$$H_s : h_s(u) = y_s = h_1(u) \circledast h_2(u) = h_2(h_1(u))$$

whose passivity can be tested using $\langle y_s|u\rangle$ which is calculated as:

$$\begin{aligned}
\langle y_s|u\rangle &= \int_0^T y_s^T u \,\mathrm{d}t && (7)\\
&= \int_0^T (h_1(u) \circledast h_2(u))^T u \,\mathrm{d}t && (8)\\
&= \int_0^T \left(\int_0^T h_1(t-\tau)h_2(\tau)\,\mathrm{d}\tau\right) \mathrm{d}t && (9)\\
&= \int_0^T h_2(\tau)\left(\int_0^T h_1(t-\tau)\,\mathrm{d}t\right) \mathrm{d}\tau && (10)
\end{aligned}$$

which is not explicitly ≥ 0 so this method cannot prove passivity.

A different method of analysis can be done to prove that this is not passive in general, but a counter example from MATLAB (Appendix A) can be shown to not be passive due to a loss of positive realness of the transfer functions when placed in series:

$$G_1(s) = \frac{5s^2 + 3s + 1}{s^2 + 2s + 1},\ G_2(s) = \frac{s^3 + s^2 + 5s + 0.1}{s^3 + 2s^2 + 3s + 4}$$

and when combined in series the system is no longer passive due to a loss of positive realness.

$$\frac{5s^5 + 8s^4 + 29s^3 + 16.5s^2 + 5.3s + 0.1}{s^5 + 4s^4 + 8s^3 + 12s^2 + 11s + 4}$$

2 Problem 2

Let

$$H(s) = \frac{s+\lambda}{s^2+as+b}$$

with $a > 0$ and $b > 0$.

2.1 Part a

Problem: For which values of λ is $H(s)$ Positive Real (PR)?

Solution: By definition, a transfer function must satisfy two conditions to be considered Postive Real:

1. $\Re\{\lambda(H(s))\} \leq 0$, any $j\omega$ roots are simple, and any residuals are non negative.
2. $\Re\{H(j\omega)\} \geq 0 \ \forall\omega \in \Re$

The transfer function for this problem will always satisfy the first condition, however, the second condition is violated under the following conditions:
Setting

$$s = j\omega$$

$$\begin{aligned}
H(j\omega) &= \frac{j\omega+\lambda}{(j\omega)^2+a(j\omega)+b} = \frac{j\omega+\lambda}{-\omega^2+ja\omega+b} &(11)\\
&= \frac{-\omega^2-ja\omega+b}{-\omega^2-ja\omega+b}\cdot\frac{j\omega+\lambda}{-\omega^2+ja\omega+b} &(12)\\
&= \frac{\big(a\omega^2+\lambda(\omega^2+b)\big)+j\big(\omega(\omega^2+b)-a\lambda\omega\big)}{a^2\omega^2+(\omega^2+b)^2} &(13)\\
&= \frac{a\omega^2+\lambda(\omega^2+b)}{a^2\omega^2+(\omega^2+b)^2}+j\frac{\omega(\omega^2+b)-a\lambda\omega}{a^2\omega^2+(\omega^2+b)^2} &(14)
\end{aligned}$$

The real component being nonnegative can then be seen to occur when

$$\begin{aligned}
a\omega^2+\lambda(\omega^2+b) &\geq 0 &(15)\\
\lambda(\omega^2+b) &\geq -a\omega^2 &(16)\\
\lambda &\geq \frac{-a\omega^2}{\omega^2+b} &(17)
\end{aligned}$$

Since this mus apply $\forall\omega \in \Re$, the following must be true

$$\lambda \geq 0 \tag{18}$$

2.2 Part b

Problem: Using the results from above, select λ_1, λ_2 such that

$$H_1(s) = \frac{s + \lambda_1}{s^2 + s + 1} \text{ is PR} \tag{19}$$

$$H_2(s) = \frac{s + \lambda_2}{s^2 + s + 1} \text{ is not PR} \tag{20}$$

Then verify the PR properties for each using the Nyquist plots of $H_1(s)$ and $H_2(s)$.

Solution: From the requirements set above, the zeros can be selected with $\lambda_1 = 1$ and $\lambda_2 = -1$ resulting in $H_1(s)$ and $H_2(s)$ being defined as

$$H_1(s) = \frac{s + 1}{s^2 + s + 1} \tag{21}$$

$$H_2(s) = \frac{s - 1}{s^2 + s + 1} \tag{22}$$

The nyquist plots, generated with the MATLAB code seen in Appendix A, can then be used to verify the PR properties. As can be seen in Figure 1, the nyquist diagram for $H_1(s)$ never crosses into the LHP and therefore is Positive Real. Conversely, in Figure 2, the nyquist diagram for $H_2(s)$ crosses into the LHP and therefore is not Positive Real.

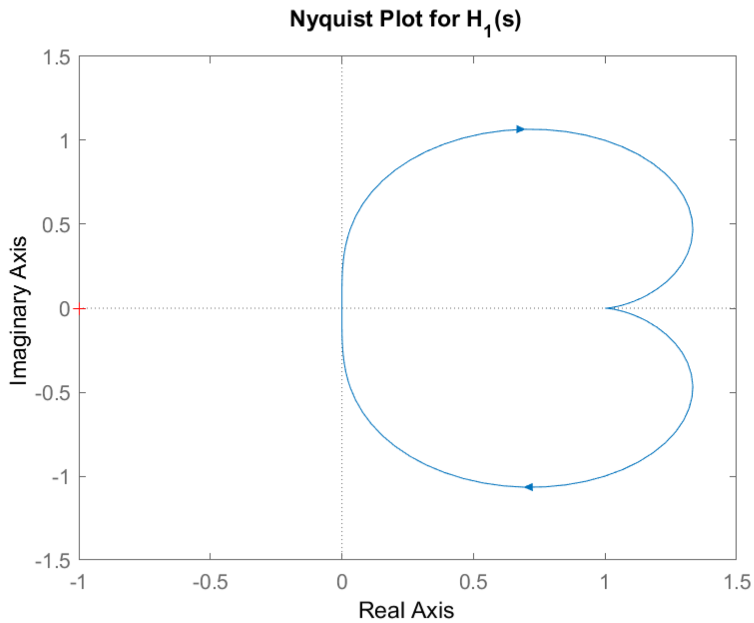

Figure 1: Nyquist Plot for the $H_1(s)$ transfer function.

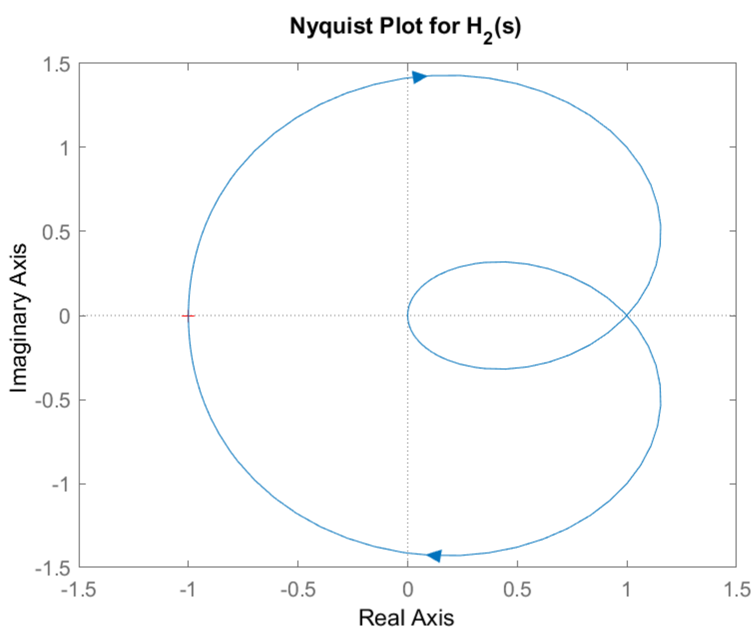

Figure 2: Nyquist Plot for the $H_2(s)$ transfer function.

2.3 Part c

Problem: For $H_1(s)$ and $H_2(s)$, write state-space realizations and solve for $P = P^T > 0$ in the PR lemma and explain why it fails for $H_2(s)$.

Solution: Given a second order transfer function

$$\frac{s + b_0}{s^2 + a_1 s + a_0}$$

a state space system can be defined by:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} & B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} b_0 & 1 \end{bmatrix} & D &= 0 \end{aligned} \tag{23}$$

This can be applied to the systems and results in the state space representations of H_1 as:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} & B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 1 \end{bmatrix} & D &= 0 \end{aligned} \tag{24}$$

and H_2 as

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} & B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} -1 & 1 \end{bmatrix} & D &= 0 \end{aligned} \tag{25}$$

Each of these systems can be tested for passivity by solving for a $P = P^T > 0$ s.t.

$$A^T P + PA < 0 \tag{26}$$

$$PB = C^T \tag{27}$$

Additionally, let

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$H_1(s)$ was found in (24)to be

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad D = 0$$

and can be proven to be passive by solving the following LMI:

$$A^T P + PA \leq 0 \tag{28}$$
$$PB - C^T = 0 \tag{29}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \leq 0 \tag{30}$$
$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \end{bmatrix}^T = 0 \tag{31}$$

$$\begin{bmatrix} -2p_{12} & p_{11} - p_{12} - p_{22} \\ p_{11} - p_{12} - p_{22} & 2p_{12} - 2p_{22} \end{bmatrix} \leq 0 \tag{32}$$
$$p_{12} - 1 = 0 \tag{33}$$
$$p_{22} - 1 = 0 \tag{34}$$

Two coefficients can then clearly be defined resulting

$$p_{12} = 1 \tag{35}$$
$$p_{22} = 1 \tag{36}$$
$$\begin{bmatrix} -2(1) & p_{11} - (1) - (1) \\ p_{11} - (1) - (1) & 2(1) - 2(1) \end{bmatrix} \leq 0 \tag{37}$$

$$\begin{bmatrix} -2 & p_{11} - 2 \\ p_{11} - 2 & 0 \end{bmatrix} \leq 0 \tag{38}$$

Which is equivalent to the standard PSD equation:

$$\begin{bmatrix} 2 & 2 - p_{11} \\ 2 - p_{11} & 0 \end{bmatrix} \geq 0 \tag{39}$$

And to ensure this is PSD, the principle minors can be analyzed:

$$a = 2 > 0 \tag{40}$$
$$\det \begin{bmatrix} 2 & 2 - p_{11} \\ 2 - p_{11} & 0 \end{bmatrix} \geq 0 \tag{41}$$
$$-(2 - p_{11})(2 - p_{11}) \geq 0 \tag{42}$$
$$-4 + 2p_{11} - p_{11}^2 \geq 0 \tag{43}$$

it is clear that a solution of $p_{11} = 0$ results in a PSD matrix and thus a Positive Real and passive system.

$H_2(s)$ was found in (25) to be

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} -1 & 1 \end{bmatrix} \qquad D = 0$$

and can be proven to be passive by solving the following LMI

$$A^T P + PA \leq 0 \tag{44}$$
$$PB - C^T = 0 \tag{45}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \leq 0 \tag{46}$$
$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \end{bmatrix}^T = 0 \tag{47}$$

$$\begin{bmatrix} -2p_{12} & p_{11} - p_{12} - p_{22} \\ p_{11} - p_{12} - p_{22} & 2p_{12} - 2p_{22} \end{bmatrix} \leq 0 \tag{48}$$
$$p_{12} + 1 = 0 \tag{49}$$
$$p_{22} - 1 = 0 \tag{50}$$

Two coefficients can then clearly be defined resulting

$$p_{12} = -1 \tag{51}$$
$$p_{22} = 1 \tag{52}$$
$$\begin{bmatrix} -2(-1) & p_{11} - (-1) - (1) \\ p_{11} - (-1) - (1) & 2(-1) - 2(1) \end{bmatrix} \leq 0 \tag{53}$$

$$\begin{bmatrix} 2 & p_{11} \\ p_{11} & -4 \end{bmatrix} \leq 0 \tag{54}$$

Which is equivalent to the standard PSD equation:

$$\begin{bmatrix} -2 & -p_{11} \\ -p_{11} & 4 \end{bmatrix} \geq 0 \tag{55}$$

And this is clearly not capable of being PSD due to the first principle minor being negative. Therefore, the system is not Positive Real and cannot be passive.

3 Problem 3

Consider the following 3-stage ring oscillator discussed in class:

$$\begin{aligned}
\tau_1 \dot{x}_1 &= -x_1 - \alpha_1 \tanh(\beta_1 x_3) \\
\tau_2 \dot{x}_2 &= -x_2 - \alpha_2 \tanh(\beta_2 x_1) \\
\tau_3 \dot{x}_3 &= -x_3 - \alpha_3 \tanh(\beta_3 x_2)
\end{aligned}$$

with $\tau_i, \alpha_i, \beta_i > 0$ and x_i represents a voltage for $i = 1, 2, 3$.

3.1 Part a

Problem: Suppose $\alpha_1\beta_1 = \alpha_2\beta_2 = \alpha_3\beta_3 = \mu$, prove the origin is GAS when $\mu < 2$.

Solution: The 3-stage ring oscillator can be rewritten in a coupling feedback system H and K defined as:

$$H = \begin{bmatrix} H_1 & & \\ & H_2 & \\ & & H_3 \end{bmatrix}, \qquad H_i = \begin{cases} \dot{x}_i = f(x_i) + g(x_i)u_i \\ y_i = h(x_i) \end{cases} \tag{56}$$

$$K = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \tag{57}$$

where K is the coupling matrix of the individual nonlinear subsystems with the following specific definitions:

$$H_i = \begin{cases} \dot{x}_i = \dfrac{-x_i + u_i}{\tau_i} \\ y_i = \alpha_j \tanh(\beta_j x_i) \end{cases} \tag{58}$$

with j being defined as either $i + 1$ or 1 if $i = n$, resulting in

$$H_1 = \begin{cases} \dot{x}_1 = \dfrac{-x_1 + u_1}{\tau_1} \\ y_1 = \alpha_2 \tanh(\beta_2 x_1) \end{cases} \tag{59}$$

$$H_2 = \begin{cases} \dot{x}_2 = \dfrac{-x_2 + u_2}{\tau_2} \\ y_2 = \alpha_3 \tanh(\beta_3 x_2) \end{cases} \tag{60}$$

$$H_3 = \begin{cases} \dot{x}_3 = \dfrac{-x_3 + u_3}{\tau_3} \\ y_3 = \alpha_1 \tanh(\beta_1 x_3) \end{cases} \tag{61}$$

From this model a storage function can be defined for each of the coupled systems as

$$V_i(x_i) = \tau_i \int_0^{x_i} h_i(\eta)\,\mathrm{d}\eta \tag{62}$$

and each individual subsystem storage functions are given as:

$$V_1(x_1) = \tau_1 \int_0^{x_1} \alpha_2 \tanh(\beta_2 x_1) \tag{63}$$

$$V_2(x_2) = \tau_2 \int_0^{x_2} \alpha_3 \tanh(\beta_3 x_2) \tag{64}$$

$$V_3(x_3) = \tau_3 \int_0^{x_3} \alpha_1 \tanh(\beta_1 x_3) \tag{65}$$

Taking a look at the change of the storage function over time, the output strict passivity can be proven by:

$$\dot{V}_i(x_i) = \frac{\mathrm{d}V_i}{\mathrm{d}x_i}\dot{x}_i \tag{66}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x_i}\tau_i \int_0^{x_i} h_i(\eta)\,\mathrm{d}\eta\,\dot{x}_i \tag{67}$$

$$= \tau_i h_i(x_i)\dot{x}_i \tag{68}$$

taking the definition for $\dot{x}_i$ and relating $h_i(x_i) = y_i$,

$$= \tau_i h_i(x_i)\frac{-x_i + u_i}{\tau_i} \tag{69}$$

$$= -x_i h_i(x_i) + u_i y_i \tag{70}$$

In order to guarantee Input passivity, the system must satisfy the following inequality:

$$xh_i(x) \leq \delta_i x^2 \tag{71}$$

In the case that input passivity is satisfied by $h_i(x_i)$, a δ_i will exist s.t.,

$$x_i h_i(x_i) \leq \delta_i x^2 \tag{72}$$

$$x_i(h_i(x_i) - \delta_i x_i) \leq 0 \tag{73}$$

clearly, $x_i h_i(x_i)$ can then be bounded from below by:

$$x_i h_i(x) \geq \frac{1}{\delta_i} h_i^2(x_i) \tag{74}$$

$$-x_i h_i(x) \geq -\frac{1}{\delta_i} h_i^2(x_i) \tag{75}$$

since $y_i = h_i(x_i)$,

$$-x_i h_i(x) \geq -\frac{1}{\delta_i} y_i^2(x_i) \tag{76}$$

Therefore, this demonstrates Output Strict Passivity:

$$\dot{V}_i \leq -\frac{1}{\delta_i} y_i^2 + y_i u_i \tag{77}$$

or with $d_i = \frac{1}{\delta_i}$ and

$$\dot{V}_i \leq d_i y_i^2 + y_i u_i \tag{78}$$

and the passivity theorem can then be applied.

Let

$$\epsilon_i = \frac{1}{\delta_i}$$

and then define

$$A = -\text{diag}\{\epsilon_i\} + K \tag{79}$$

$$P = \text{diag}\{d_i\} \tag{80}$$

which for this 3^{rd}-order system can be written as

$$A = -\begin{bmatrix} \epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & -1 \\ -1 & \epsilon_2 & 0 \\ 0 & -1 & \epsilon_3 \end{bmatrix} \tag{81}$$

$$P = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \tag{82}$$

From this, appropriate values for A and P can be found using CVX to prove stability of the full feedback interconnection.

3.2 Part b

Problem: Show that if $\tau_1 = \tau_2 = \tau_3 = \tau$, then $\mu < 2$ is necessary for asymptotic stability. What type of bifurcation occurs at $\mu = 2$?

Solution:

3.3 Part c

Problem: Investigate the dynamic behavior of this system for $\mu > 2$ with numerical simulations.

Solution:

4 Problem 4

A MATLAB Code:

All code I write in this course can be found on my GitHub repository:
https://github.com/jonaswagner2826/MECH6313

Script 1: MECH6313_HW6

```
% MECH 6313 - HW6
% Jonas Wagner
% 2021-04-27
%

clear
close all

pblm1 = false;
pblm2 = true;


if pblm1
%% Problem 1

G1 = tf([5 3 1], [1 2 1])
isPassive(G1)

G2 = tf([1 1 5 0.1], [1 2 3 4])
isPassive(G2)

Gp = G1 + G2
isPassive(Gp)

Gs = G1 * G2
isPassive(Gs)

end


if pblm2
%% Problem 2
pblm2a = false;
pblm2b = false;
pblm2c = true;

if pblm2a
% Part a
```

```
syms omega a b lambda
assume(a,'real'); assume(a > 0)
assume(b,'real'); assume(b > 0)
assume(omega,'real')
assume(lambda, 'real')

num = j*omega + lambda;
den = omega^2 + j*a*omega + b;

H_sym = num/den;
disp('H(s) = ')
pretty(H_sym)

H_real = real(H_sym);
disp('H_real = ')
pretty(H_real)

H_imag = imag(H_sym);
disp('H_imag = ')
pretty(imag(H_sym))
end

if pblm2b
% Part b
lambda1 = 1;
lambda2 = -1;

H1 = tf([1 lambda1], [1 1 1])
isPassive(H1)
figure
nyquist(H1)
title('Nyquist Plot for H_1(s)')
saveas(gcf, [pwd, '\Homework\HW6\fig\pblm2_H1.png'])

H2 = tf([1 lambda2], [1 1 1])
isPassive(H2)
figure
nyquist(H2)
title('Nyquist Plot for H_2(s)')
saveas(gcf, [pwd, '\Homework\HW6\fig\pblm2_H2.png'])
end

if pblm2c
```

```
% Part c
H1_sys = ss([0, 1; -1, -1], [0; 1], [1 1], 0)
tf (H1_sys)
[A,B,C,D] = ssdata(H1_sys)
syms p11 p12 p22
P = [p11, p12; p12, p22]
A'*P + P * A
P * B - C'

H2_sys = ss([0, 1; -1, -1], [0; 1], [-1 1], 0)
tf(H2_sys)
[A,B,C,D] = ssdata(H2_sys)
syms p11 p12 p22
P = [p11, p12; p12, p22]
A'*P + P * A
P * B - C'

end


end
```