Lecture 10 03/01/2021

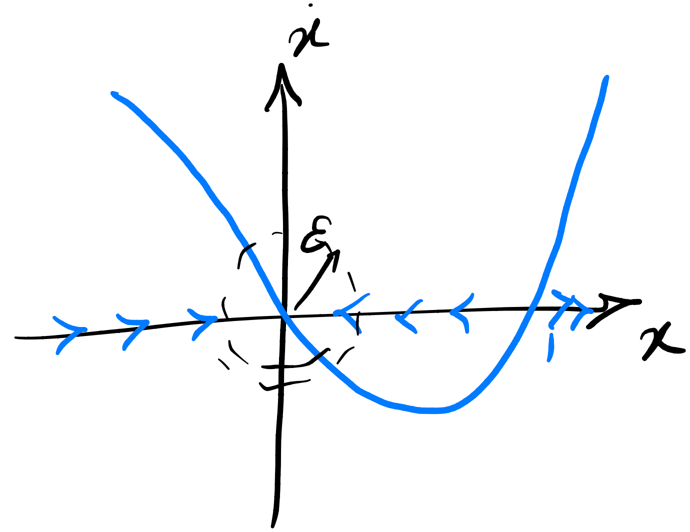
Last time: Sensitivity equations

Stability of e.p.'s

Today: Lyapunov's indirect method for stability analysis

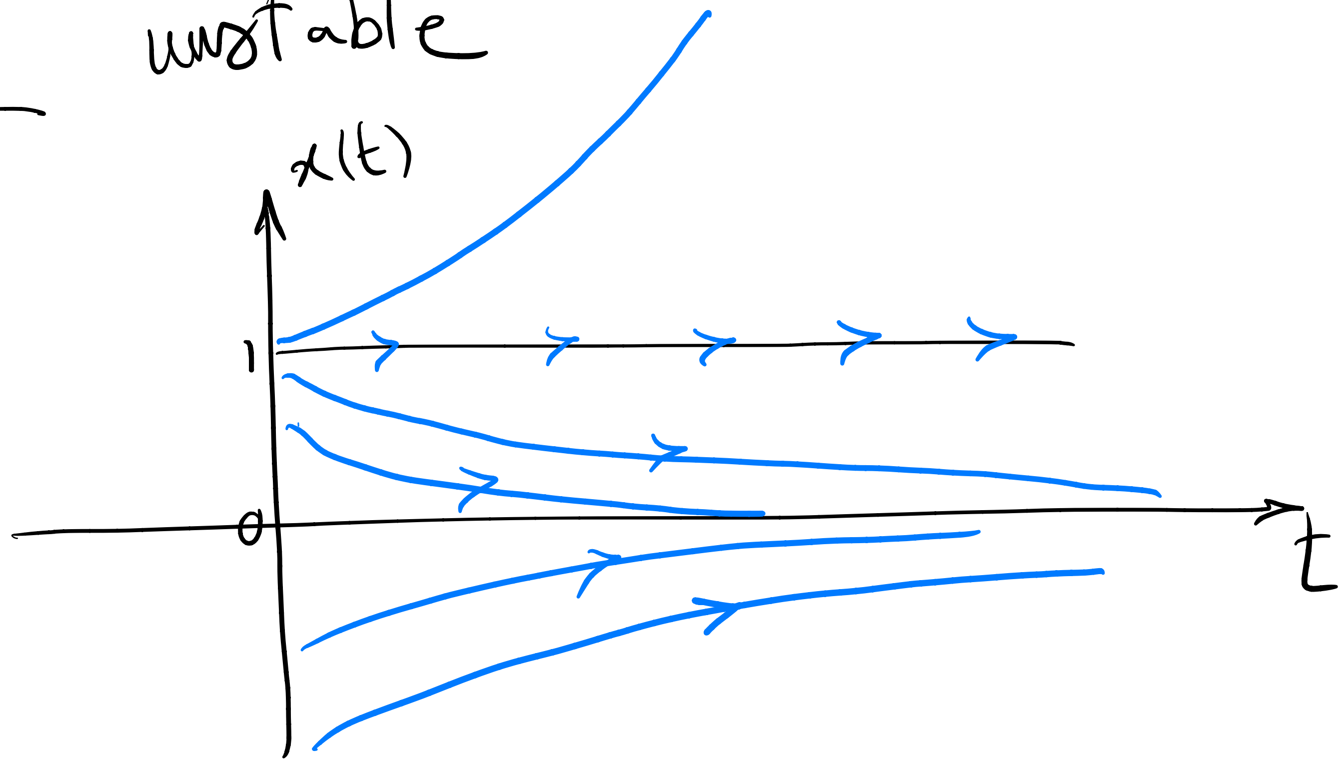
Ex $\dot{x} = x(x-1)$

e.p.: $\bar{x} = 1$ $\bar{x} = 0$



$\bar{x} = 0$ locally asymptotically stable

$\bar{x} = 1$ unstable

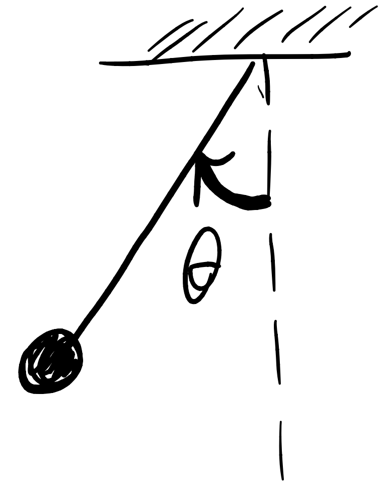


We can use similar arguments for second-order systems in mathematically characterizing the stability of their equilibria, but this would require knowledge of phase portraits.

Q. Can we check stability properties without finding sol'ns to $\dot{x} = f(x)$?

A. Yes. Lyapunov's indirect method

Ex. Pendulum



$$\ddot{\theta} + b\dot{\theta} + a\sin(\theta) = 0$$

b: damping factor

$$x_1 = \theta \qquad x_2 = \dot{\theta} \quad \Rightarrow \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a\sin x_1 - b x_2 \end{cases}$$

Equilibrium points:

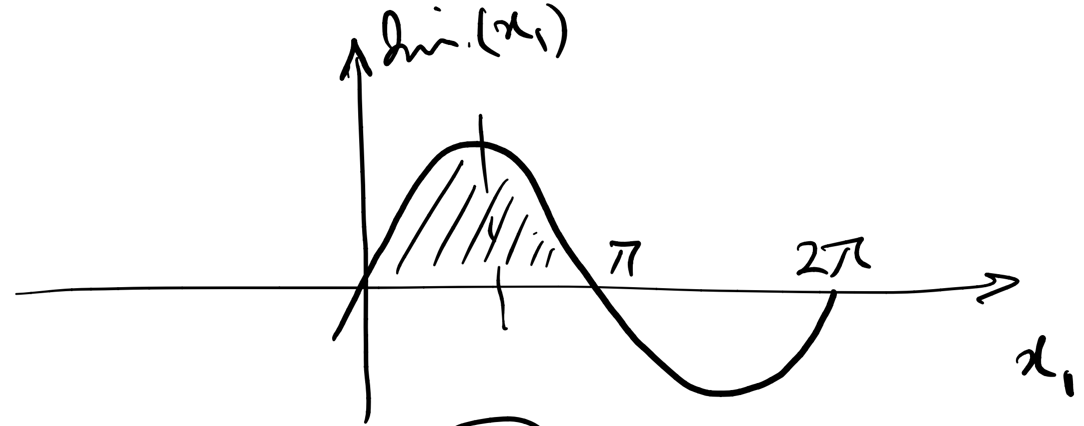
$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \left\{ \overset{\text{down}}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}, \overset{\text{up}}{\begin{bmatrix} \pi \\ 0 \end{bmatrix}} \right\}$$

Energy: $E(t) =$ potential + kinetic

$$E(x_1, x_2, t) = c_1 \int_0^{x_1} \sin(\xi)\, d\xi + \frac{1}{2} c_2 x_2^2$$

How does energy change along the solutions of the system (i.e. the trajectories)?

1. $E(0,0) = 0$
2. $E(x_1, x_2) > 0 \quad \forall (x_1, x_2) \in D \setminus \{0\}$



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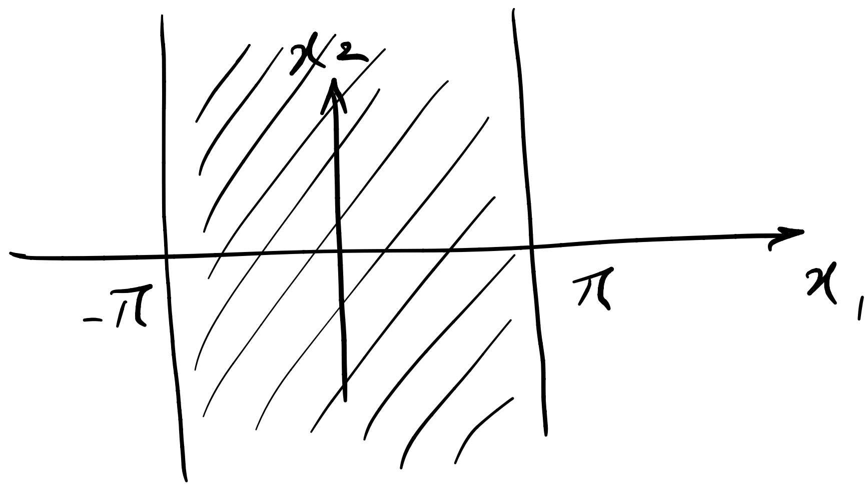


$$\frac{dE}{dt} = \left[\nabla_x E\right]^T \frac{dx}{dt} = \underbrace{\begin{bmatrix} \frac{\partial E}{\partial x_1} & \frac{\partial E}{\partial x_2} \end{bmatrix}}_{\text{only dependent on the form of energy we consider}} \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\text{state-space model}}$$

$$\frac{dE}{dt} = \begin{bmatrix} c_1 \sin(x_1) & c_2 x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -a \sin(x_1) - b x_2 \end{bmatrix}$$

$$= c_1 \sin(x_1)\, x_2 - c_2 x_2\, a \sin(x_1) - c_2 b x_2^2$$

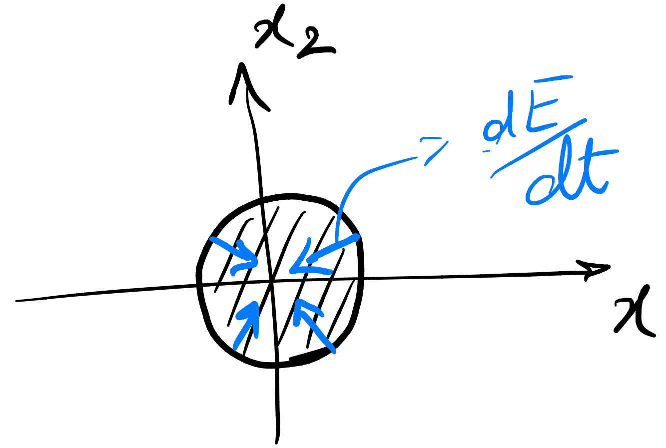
$$= \underbrace{(c_1 - a c_2)\, x_2 \sin(x_1)}_{\text{sign indefinite term}} \underbrace{- c_2 b x_2^2}_{\text{always negative } \leq 0}$$

can be either + or −

easiest way is to conside $c_1 = a c_2$

e.g., $c_2 = 1 \quad c_1 = a$

$$\frac{dE}{dt} = -b x_2^2 = 0\, x_1^2 - b x_2^2$$



$\frac{dE}{dt} \leq 0$

energy is positively invariant

$E(t)$: non-increasing function of time (on D)

We'll see that this implies stability (in the sense of Lyapunov) of $\bar{x} = 0$, but not local asymptotic stability!

This does not mean that $\bar{x}=0$ is not locally asymptotically stable. It just means that we cannot conclude this with this particular Lyapunov function $\longrightarrow$ energy.

Small detail: if $b=0$ (i.e. no damping)

$$\frac{dE}{dt}=0 \Rightarrow E(t)=\text{Const.}=E(t_0)$$

$$=a(1-\cos x_1(0))+\frac{1}{2}x_2(0)^2$$

(conservative system)

Lyapunov function: Energy-like functions for checking stability properties of $\bar{x}=0$ for $\dot{x}=f(x)$

$V(x): \mathbb{R}^n \longrightarrow \mathbb{R}_+$

$\underbrace{V(X)}_{\geq 0} \in \mathbb{R}_+$

all we need for local A.S.

with properties: (1) $V(0)=0$

(2) $V(X)>0 \quad \forall X \in D \setminus \{0\}$

radial unboundedness (3) $V(X) \rightarrow +\infty$ as $\|X\| \rightarrow \infty$ (needed for global asymptotic stability)

$$\frac{dV(x)}{dt} = \left[\nabla_x V(x)\right]^T \dot{x} = \left[\nabla V(x)\right]^T f(x) \overset{?}{\leq} 0$$

~~Big~~ **Theorem**: 1) Let D be an open connected subset of $\mathbb{R}^n$ that contains e.p. $\bar{x}=0$ of $\dot{x}=f(x)$. Then,

if there is a continuously differentiable function $V: D \to \mathbb{R}$ such that

a) $V(0)=0$

b) $V(x)>0 \quad \forall x \in D \setminus \{0\}$

$\Big\}$ $V(x)$ is locally positive definite

domain of system

c) $\frac{dV}{dt} = (\nabla V(x))^T f(x) \leq 0 \quad \forall x \in D \setminus \{0\}$

($\frac{dV}{dt}$ is locally negative semi-definite)

$\Rightarrow \bar{x} = 0$ is stable (in the sense of Lyapunov)

(start close to $\bar{x} = 0$; stay close to it)

2) For local asymptotic stability need

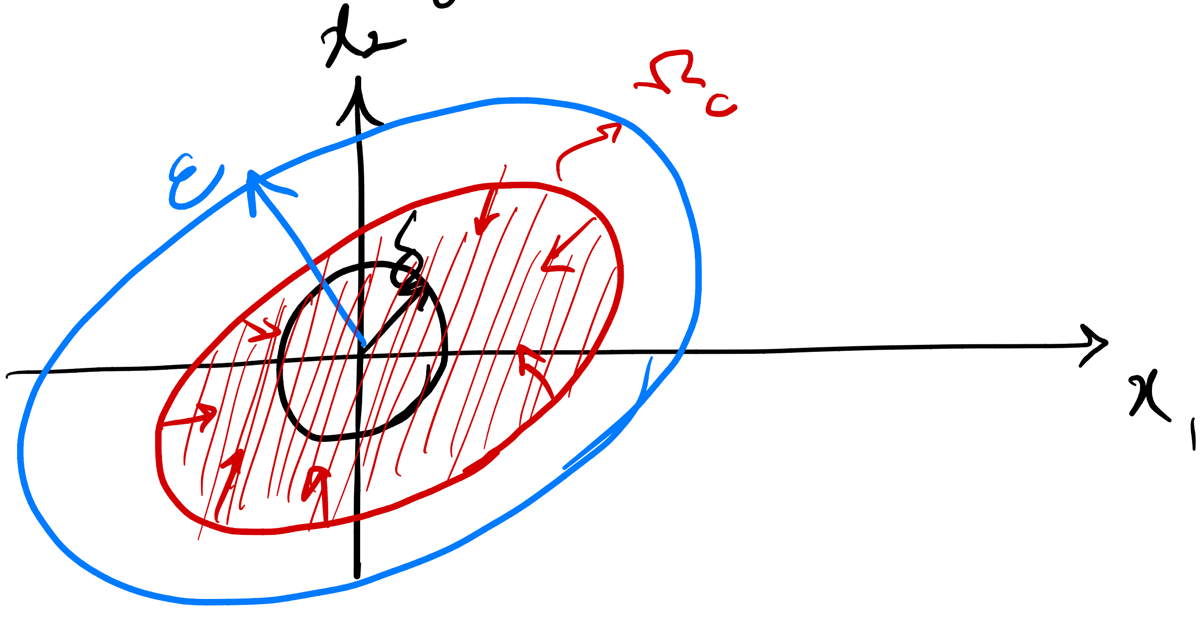
a), b) $\oplus$ c) $\frac{dV}{dt} < 0 \quad \forall x \in D \setminus \{0\}$

3) For global asymptotic stability need 2) to hold for $D = \mathbb{R}^n$ $\oplus$ d) $\lim_{\|x\| \to \infty} V(x) = \infty$

Sketch of proof

1) Stability follows from positive invariance of Ω_c, where $\Omega_c := \{x \,/\, V(x) \leq c\}$ are called level sets of $V(x)$.

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \nabla V(x) f(x) \leq 0 \quad \text{(positive invariance)}$$

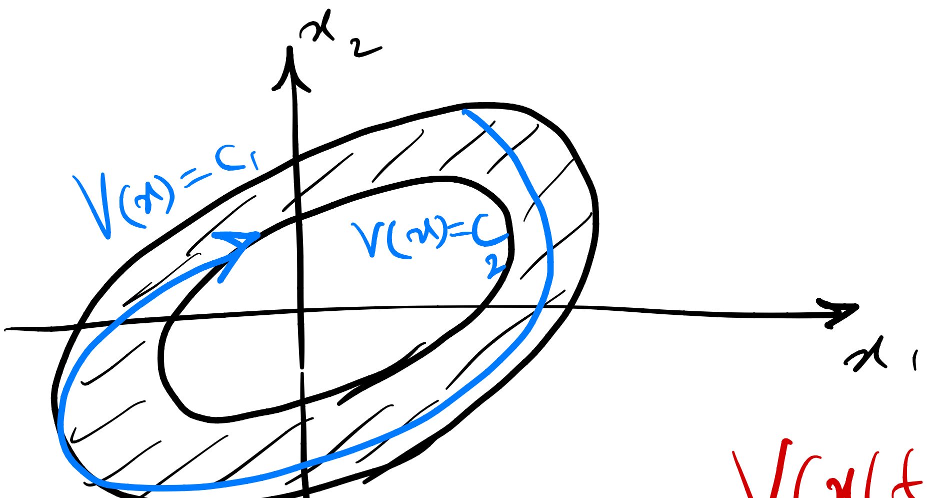


2. $\dot{V}(x) < 0 \quad \forall x \in D \setminus \{0\}$

$dV/dt < 0 \Rightarrow$ $V(x)$ is decreasing function bounded from below by zero

$\Rightarrow \exists\, c$ st. $\lim_{t \to \infty} V(x) = c$

assume $c > 0$ (proof by contradiction)



$\max \dot{V} = -\gamma < 0$ $\gamma > 0$

$$V(x(t)) = V(x_0) + \int_0^t \underbrace{\dot{V}(x(\tau))}_{-\gamma}\, d\tau$$

$V(x(t)) \leq V(x_0) - \gamma t$

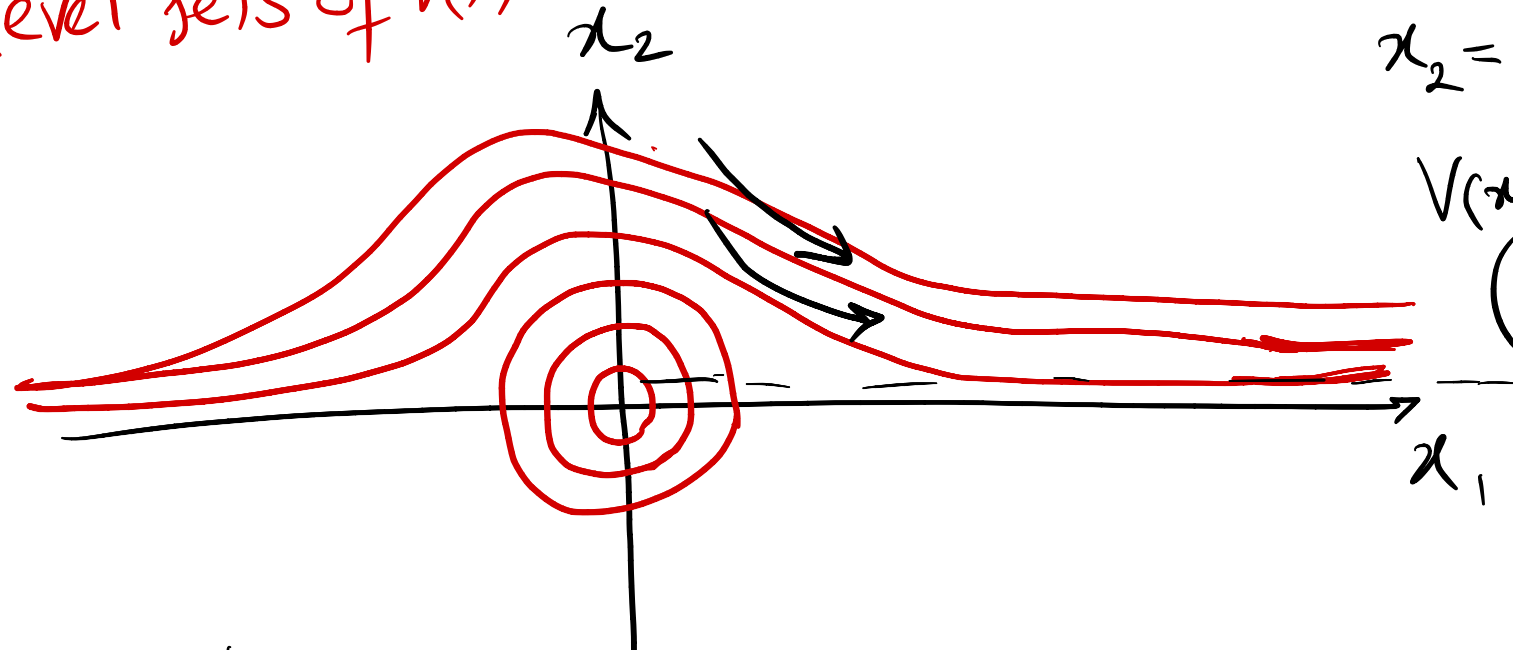
For t large enough $V(x(t))$ will become negative $\#$ contradiction with assumption of $c > 0$

So c has to be zero!

3) Why do we need radial unboundedness of $V(x)$ for GAS?

Ex. $V(x) = \dfrac{x_1^2}{x_1^2+1} + x_2^2$

look at level sets of $V(x)$



$x_2 = 0$, let $x_1 \to \infty$

$V(x) \to 1$

(Not radially unbounded)

Then Ω_c is not a bounded set for $c \geq 1$

Therefore, x_1 may grow unbounded while $V(x(t))$ is decreasing. $V(x)$ is not representative of the growth in the state x_1.

Q. How do we construct Lyapunov functions?

Ex. Scalar case

$$\dot{x} = -g(x) \qquad x(t) \in \mathbb{R}$$

unless we know about g, then we cannot say much.

aside ~~Application~~ $\min_{x} J(x)$

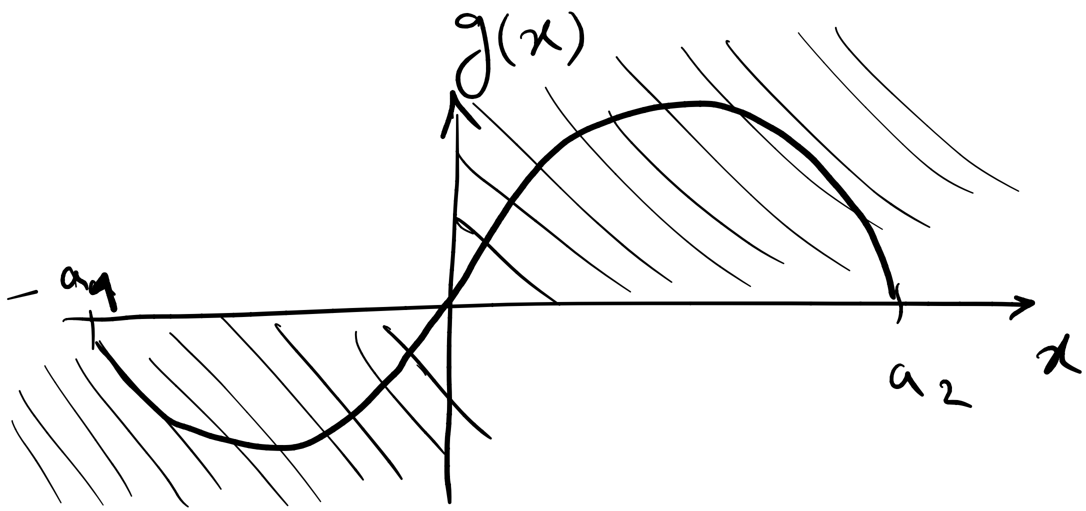
Gradient descent $\dot{x} = -\nabla J(x)$

$$\frac{x_{k+1} - x_k}{\alpha} \cdots > \quad x_{k+1} = x_k - \alpha \nabla J^T(x)$$

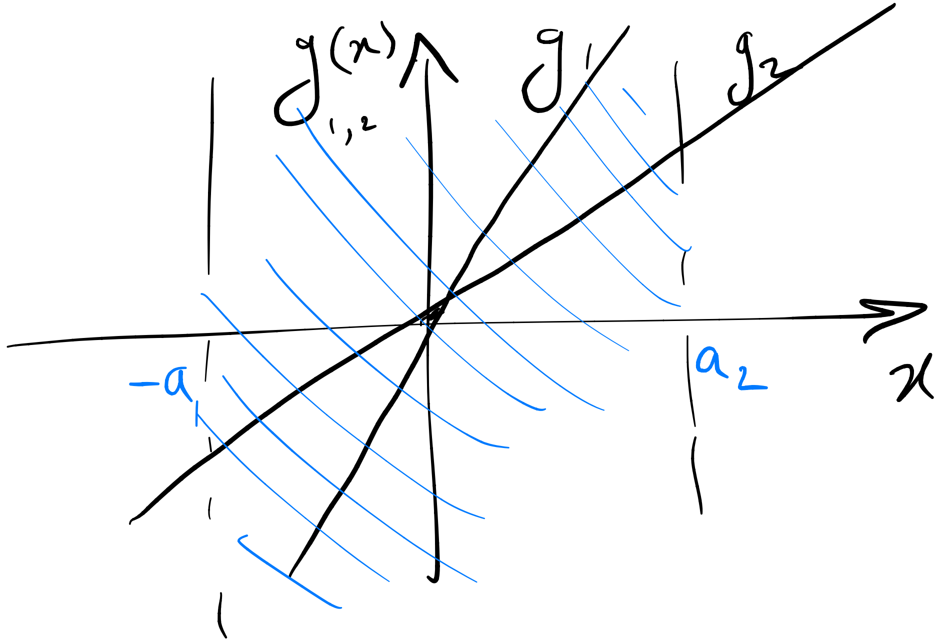
We need to restrict class of functions g to make some progress.

1) $g(0) = 0$ $\quad \bar{x}=0$ e.p.

2) g : locally Lipschitz continuous (for existence & uniqueness of sol'ns)

3) 

'g' lives in 1st and 3rd quadrants.



key feature: $\quad x\,g(x) \geq 0 \qquad x \in [-a_1, a_2]$