Lecture 23

04/26/2021

Last time: state-space conditions for passivity

implications of passivity

Special case of linear systems

Today: Positive realness and its implications

KYP Lemma + Passivity Theorem

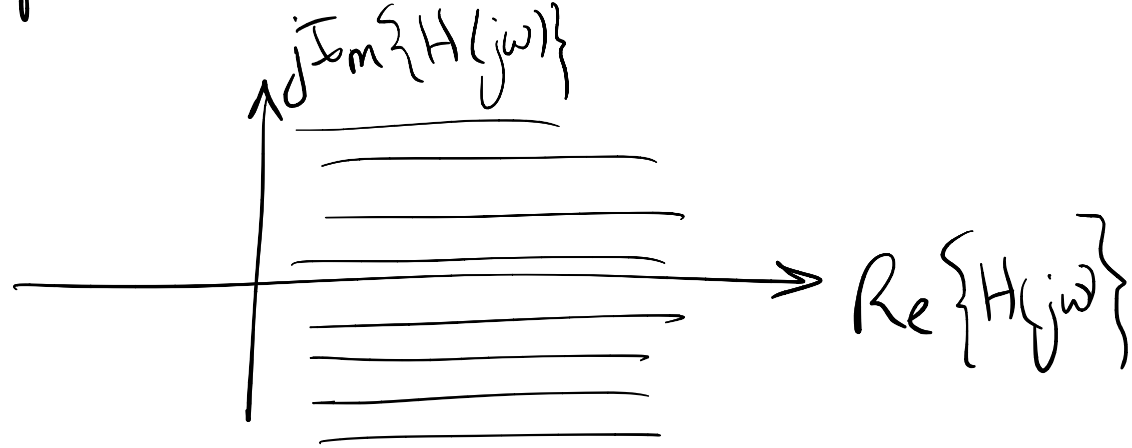
Implication of positive realness:

1) If $H(s)$ is PR $\Rightarrow$ stable

If $H(s)$ is SPR $\Rightarrow$ asymptotically stable

2) $|\measuredangle H(j\omega)| < 90^\circ$ (Bode plot)

Nyquist plot lies in the RHP



3) Relative degree of H is either 0 or 1.

$$H(s) = \frac{p(s)}{q(s)}$$; 'order of q' − 'order of p' = (# of poles) − (# zeros)

= relative degree

KYP Lemma (Kalman, Yakubovich, Popov)

(Positive real lemma)

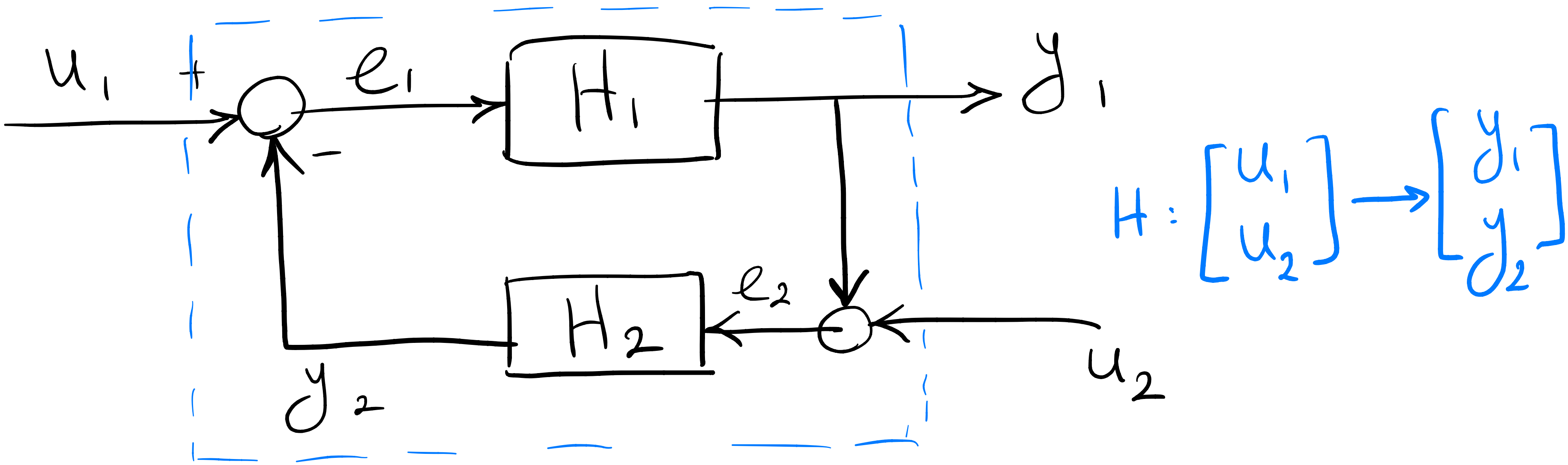
let $H(s) = C(sI - A)^{-1}B$ with $\text{Re}\{\lambda_i(A)\} \leq 0$

and (A, B) controllable then:

$\rightarrow$ $H(s)$ is PR **iff** $\exists\, P = P^T > 0$ s.t. $A^TP + PA \leq 0$ and $PB = C^T$

$\rightarrow$ $H(s)$ is SPR **iff** $\exists\, P = P^T > 0$ s.t. $A^TP + PA < 0$ and $PB = C^T$

Passivity thm:



a) H_i : passive with storage functions $V_{1,2}$

$$H_1: \quad \dot{V}_1 \leq e_1^T y_1 = (u_1 - y_2)^T y_1 = u_1^T y_1 - y_2^T y_1 \qquad (1)$$

$$H_2: \quad \dot{V}_2 \leq e_2^T y_2 = (u_2 + y_1)^T y_2 = u_2^T y_2 + y_1^T y_2 \qquad (2)$$

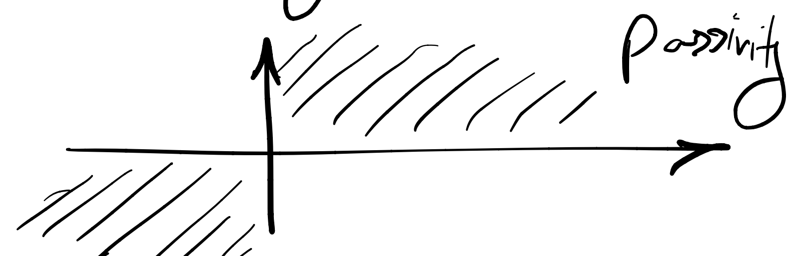
(1)+(2) with $V := V_1 + V_2$

$$\dot{V} = \dot{V}_1 + \dot{V}_2 \leq u_1^T y_1 + u_2^T y_2 = \begin{bmatrix} u_1^T & u_2^T \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = u^T \cdot y$$

Note This characterization is very general and the transfer functions H_i can be anything; even infinite dimensional

b) H_1 : same as in (a) but now let H_2 be memoryless sector bounded nonlinearity with

H_2 : $y_2^T e_2 \geq 0$



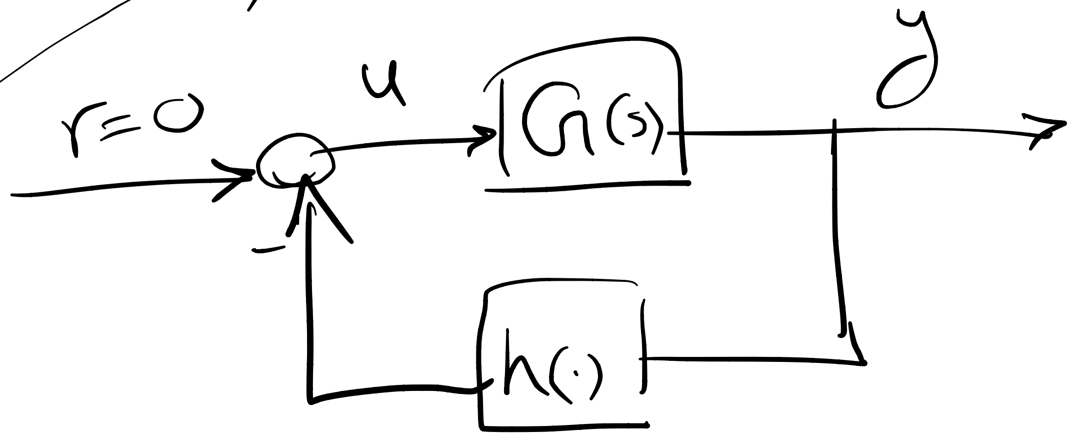
$$y_2^T(y_1+u_2) \geq 0$$

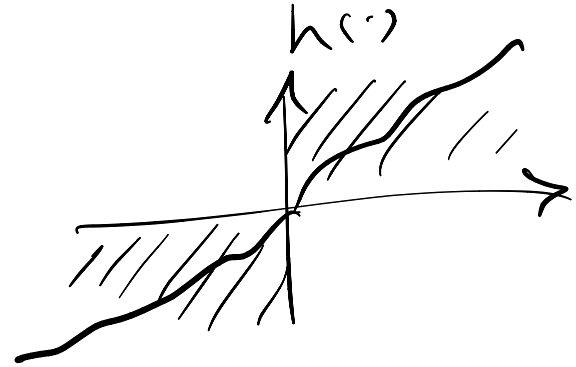
$$-y_2^T y_1 \leq y_2^T u_2 \qquad (3)$$

$(3) \rightarrow (1)$: $\dot{V}_1 \leq u_1^T y_1 + u_2^T y_2 = u^T y$

$\Rightarrow$ feedback interconnection passive with storage function V_1.

aside Note



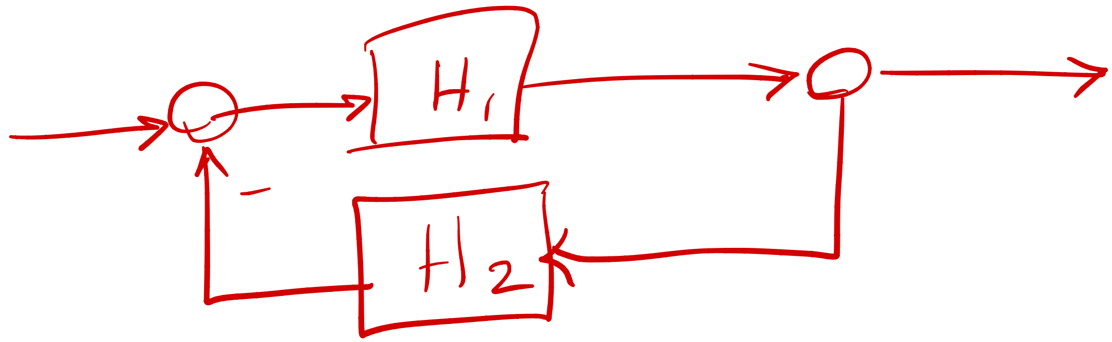


$h(\cdot)$ memoryless sector bounded

Passivity thm

- if $G(s)$ is PR and $h(\cdot)$ is some sector bounded nonlinearity then the equilibrium pt. of the fbk interconnection is stable
- if $G(s)$ is SPR and $h(\cdot)$ " " " " " is GAS.

Linear systems:



Small gain thm: $\quad |H_1(j\omega)||H_1(j\omega)| < 1 \; ; \; \forall \omega$

Passivity: info regarding phase characteristics become relevant, i.e.,

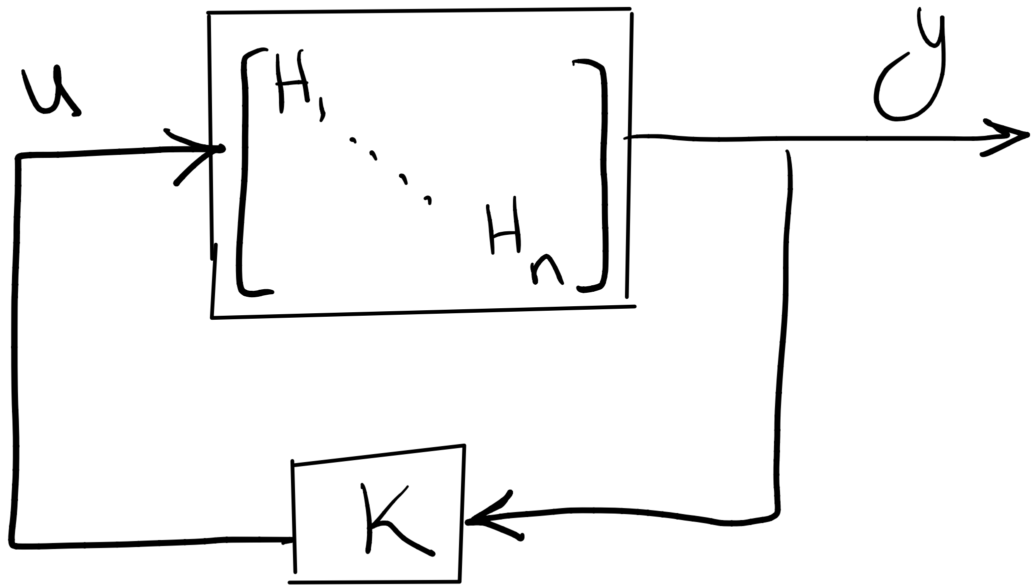
H_i : PR $\quad |\measuredangle H_i(j\omega)| < 90^\circ$

$\Downarrow$

$|\measuredangle H_1(j\omega) H_2(j\omega)| < 180^\circ$

Nyquist plot never crosses real line, i.e. doesn't encircle -1.

A generalization to (potentially) large-scale interconnections:



$$H_i: \quad \dot{x}_i = f(x_i) + g(x_i)u_i$$
$$y_i = h(x_i)$$

n uncoupled systems

SISO

each H_i output strictly passive

$$\dot{V}_i \leq -\varepsilon_i y_i^2 + y_i u_i$$

$$u := \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \qquad y := \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad u = Ky$$

K → interconnection matrix provides coupling between H_i

let $$\dot{V} = \sum_{i=1}^{n} d_i V_i \quad ; \; d_i \geq 0$$

$$\leq \sum d_i \left(-\varepsilon_i y_i^2 + y_i u_i\right)$$

Goal: we want to derive sufficient conditions for K so that the interconnection is <u>stable</u>.

$$D_d := \mathrm{diag}\{d_i\} = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

$$D_q := \mathrm{diag}\{\varepsilon_i\} = \begin{bmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_n \end{bmatrix}$$

$$\dot{V} \leq y^T(-D_d \cdot D_\varepsilon)y + y^T D_d \underbrace{K y}_{u} = y^T(-D_d D_\varepsilon + D_d K)y$$

$$= y^T D_d\left(-D_\varepsilon + K\right)y$$

$$= \frac{1}{2} y^T \left\{ (-D_\varepsilon + K)^T D_d + D_d(-D_\varepsilon + K) \right\} y$$

sufficient conditions for stability of the feedback interconnection

$$\Downarrow$$

existence of diagonal matrix D_d which is positive definite as the sol'n to

$$(-D_\varepsilon + K)^T D_d + D_d(-D_\varepsilon + K) < 0$$

$$\Downarrow$$

Stability

In Matlab:

$$P := \mathrm{diag}\{d_i\}$$

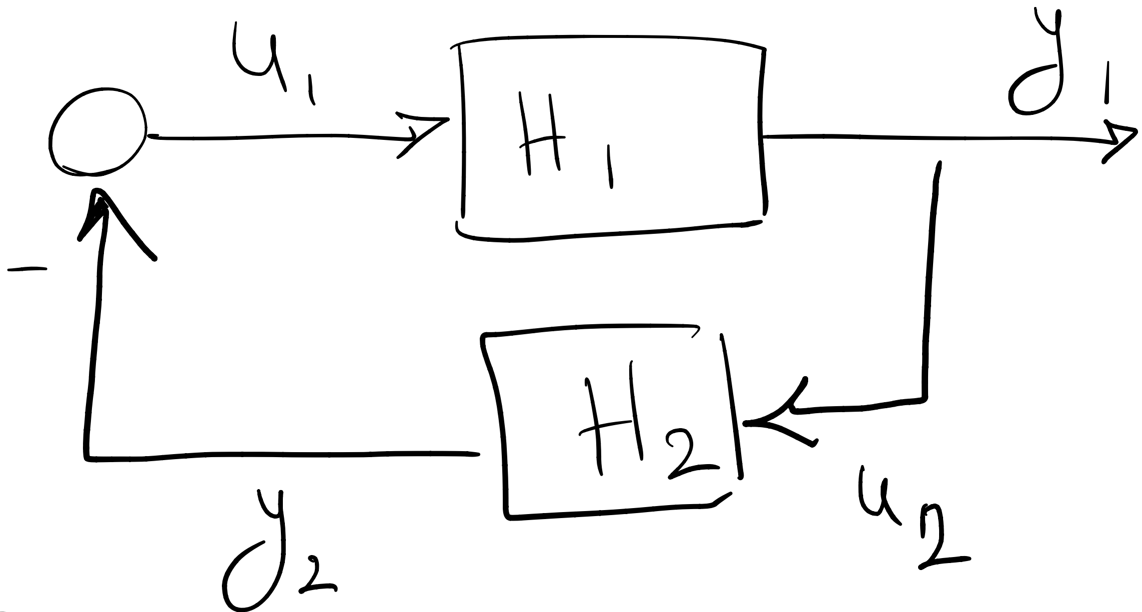
$$A := -\mathrm{diag}\{\varepsilon_i\} + K$$

and use CVX to solve a feasibility problem

$$A^T P + PA < 0$$

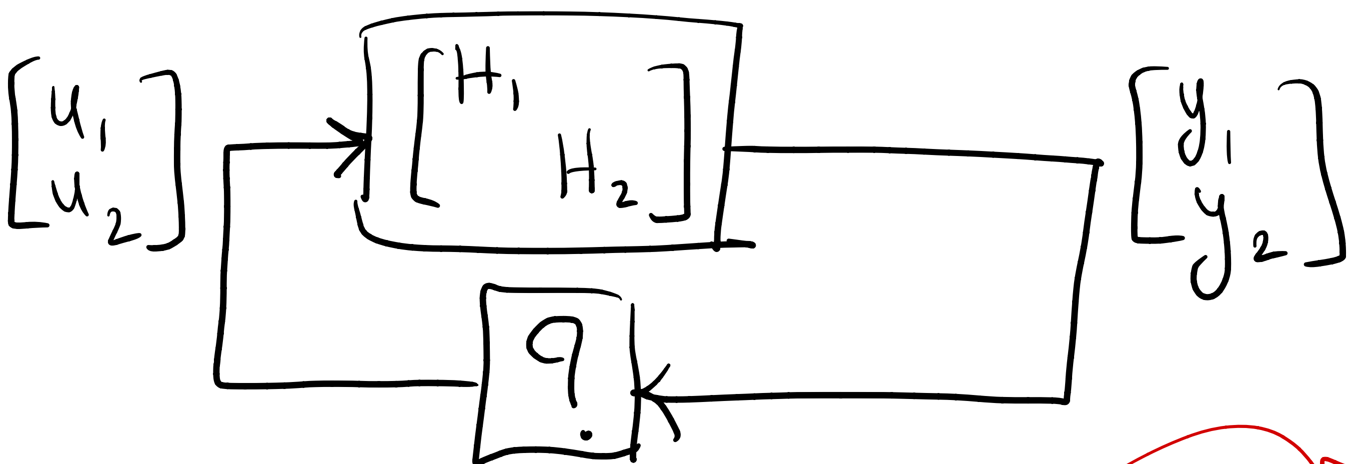
Google: "CVX Stephen Boyd"

Ex



H_i nonlinear

$\equiv$



$u_1 = -y_2$

$u_2 = y_1$

$$\Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} ? \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

skew symmetric

things we know:

H_i : $\dot{V}_i \leq -\varepsilon_i y_i^2 + y_i u_i$

and there is a general feedback matrix K that is skew-symmetric ($K + K^T = 0$)

Need to check existence of a diagonal matrix P s.t.

$$\left(-\begin{bmatrix}\varepsilon_1 & 0\\ 0 & \varepsilon_2\end{bmatrix} + K^T\right)P + P\left(-\begin{bmatrix}\varepsilon_1 & 0\\ 0 & \varepsilon_2\end{bmatrix} + K\right) < 0 \qquad (I)$$

earlier we showed that $V = V_1 + V_2$ works!

$$P = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad \cdots (\mathrm{VI})$$

$$(\mathrm{VI}) \rightarrow (\mathrm{I}) \Rightarrow \underbrace{-\begin{bmatrix} +2\varepsilon_1 & 0 \\ 0 & +2\varepsilon_2 \end{bmatrix}}_{<0} + \underbrace{\cancel{K^T + K}}_{0}$$

Summary

If H_i's were nonlinearities (static) you would be able to find linear combinations of V_i's as storage functions provided that the nonlinear systems are passive.