Lecture 17

03/31/2021

Class Exam : April 30th (Friday)

Today: sort out convergence problem for parameter estimation example!

LTV system

$$\boxed{\dot{\tilde{\theta}} = -\psi(t)\psi^T(t)\tilde{\theta}(t)}$$

conditions for UAS?

$$\dot{\tilde{\theta}}(t) = A(t)\tilde{\theta}(t) \quad ; \quad A(t) = -\psi(t)\psi^T(t)$$

Note $\dot{\tilde{\theta}} = -\psi(t)e(t)$ $\quad$ $e(t) = \psi^T(t)\tilde{\theta}(t)$

Good for analysis

implementation becomes tricky...

unknown

$$\boxed{\tilde{\theta}(0)} = \theta - \hat{\theta}(0)$$

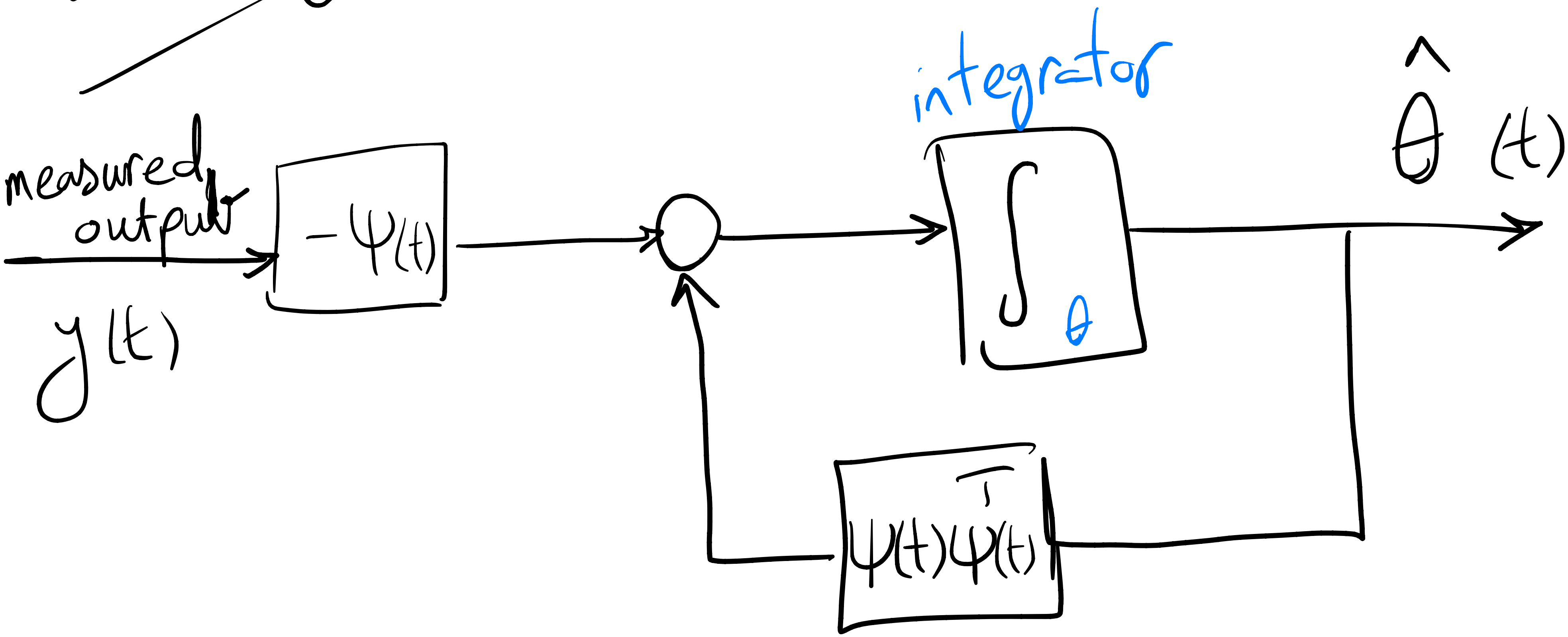
controlled by us

Implementation : diff eq. for $\hat{\theta}(t)$

$$\dot{\hat{\theta}}(t) = -\dot{\tilde{\theta}}(t) = -\psi(t)\psi^T(t)\underbrace{\left(\theta - \hat{\theta}(t)\right)}_{\tilde{\theta}(t)}$$

$$\begin{cases} \dot{\hat{\theta}}(t) = \psi(t)\psi^T(t)\hat{\theta}(t) - \psi(t)\underbrace{y(t)}_{\psi^T(t)\theta} \\ \hat{\theta}(0) \text{ our choice} \end{cases}$$

Block diagram



Task

Determine conditions for

$$\hat{\theta}(t) \xrightarrow{t\to\infty} \theta\ ?$$

One last interpretation for those who have taken estimation class last semester:

Model $\theta = \text{const}$ $\qquad y(t) = \varphi^T(t)\,\theta$

$$\left.\begin{aligned} \dot{\theta} &= 0\,\theta + 0\,u \\ y &= \varphi^T(t)\,\theta \end{aligned}\right\} \quad \begin{aligned} A &= 0\,; \quad B = 0 \\ C(t) &= \varphi^T(t) \end{aligned}$$

aside

$$\left.\begin{aligned} \dot{\theta} &= A\theta + Bu \\ y &= C(t)\,\theta \end{aligned}\right\}$$

system

$$\dot{\hat{\theta}} = A\hat{\theta} + Bu + L(t)(y - \hat{y}) \Rightarrow$$

$$\dot{\hat{\theta}} = (A - L(t)C(t))\,\hat{\theta} + Bu + L(t)\,y(t)$$

observer

In our case:

$$\boxed{\dot{\hat{\theta}} = \left(0 - L(t)\varphi^T(t) \right)\hat{\theta} + 0u + L(t)\,y(t)}$$

↳ observer for

$$\dot{\theta} = 0\theta + 0u$$
$$y = \varphi^T(t)\theta \qquad (*)$$

If $L(t) = -\varphi(t) \Rightarrow$ gradient update law!!!

Q. How to study properties of

$$\dot{\tilde{\theta}} = -\psi(t)\psi^T(t)\tilde{\theta} \quad ???$$

A. Need to study U.O. of $(*)$!

Ex 1 $\quad \psi(t) = \psi = \text{Const.} \quad ; \quad p > 1$

e.g. $\psi = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\left.\begin{aligned} \dot{\tilde{\theta}} &= \underbrace{-\psi\psi^T}_{A = \text{Const} \in \mathbb{R}^{p\times p}} \tilde{\theta} \end{aligned}\right\} \text{ LTI system}$$

$$\tilde{\theta}(t) \xrightarrow{t\to\infty} 0 \iff \operatorname{Re}\{\lambda_i(A)\} < 0$$

$$\forall i = 1, \dots, p$$

Q: Is $A = -\psi\psi^T$ a Hurwitz matrix?

Note: $A = -\psi\psi^T$

$\Rightarrow \text{rank}(A) = 1$

$$\lambda_i(A) = \{-\|\psi\|_2^2,\ \underbrace{0, 0, 0, \dots, 0}_{p-1}\}$$

Not Hurwitz!!!

Important

Thus, for $p > 1$ (more than on unknown parameter corresponding to more than one "feature")

we need $\psi(t)$ to be a function of time for

$$\hat{\theta}(t) \xrightarrow{t \to \infty} \theta.$$

[i.e. with $\psi(t) = \text{const}$ we can Never have $\hat{\theta}(t) \xrightarrow{t \to \infty} \theta$)

For $\Psi(t) \neq$ const., we can use Thm from the last lecture:

key concept: UNIFORM OBSERVABILITY

of system $(*)$

$$\begin{cases} (A(t), C(t)) \text{ U.O.} \\ \Updownarrow \\ (A(t)+L(t)C(t), C(t)) \text{ U.O.} \end{cases}$$

Ex. $\psi(t) = \begin{bmatrix} 1 \\ \sin(t) \end{bmatrix}$

Need to check U.O. of system with

$A(t) = 0 \quad ; \quad C(t) = \psi^T(t)$

$\Downarrow$

$\Phi(\tau, t_0) = I$ (verify on your own)

$$\int_{t_0}^{t_0+\delta} \psi(\tau)\psi^T(\tau)\, d\tau = \int_{t_0}^{t_0+\delta} \begin{bmatrix} 1 & \sin(\tau) \\ \sin(\tau) & \sin^2(\tau) \end{bmatrix} d\tau \underset{\text{if } \delta = 2\pi}{=}$$

$$= \begin{bmatrix} 2\pi & 0 \\ 0 & \pi \end{bmatrix} \succeq \pi \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\delta = 2\pi \quad \alpha = \pi$ (independent of t_0)

$$\Downarrow$$

U.O. !

Thm $\Longrightarrow$ Global exp. stability of $\dot{\tilde{\theta}} = -\psi(t)\psi^T(t)\tilde{\theta}$.

$\blacksquare$

In the language of system identification and Adaptive Control, regressor vector in Ex.2 is Persistently exciting, while in Ex.1 it is NOT.

Ex. $\psi(t) = e^{-t}$

($p=1$ one unknown parameter)

Q. Can we find $\underbrace{\alpha \text{ and } \delta}_{\text{independent of } t_0}$ s.t.

$$\int_{t_0}^{t_0+\delta} e^{-2\tau}\, d\tau \geq \alpha \quad \text{for all } t_0 \text{?}$$

A. Not U.O. because we cannot find α that does not depend on t_0. (please check on your own)

Aside: Thm tells us that

$\dot{\tilde{\theta}} = -\psi(t)\psi^T(t)\tilde{\theta}$ are GES if

$(-\psi(t)\psi^T(t),\ \psi^T(t))$ is U.O.

Challenge: how to find state-transition matrix of $A(t) = -\psi(t)\psi^T(t)$

Good luck!!!

Alternatively, can check U.O. of

$$\left(\underbrace{-\psi(t)\psi^T(t)}_{A(t)} + \underbrace{L(t)}_{+\psi(t)}\underbrace{\psi^T(t)}_{c(t)},\ \psi^T(t)\right)$$

Note: with $L(t)=\psi(t) \Rightarrow (0, \psi^T(t))$: system (*)

Thus, U.O. of system (*) $\Rightarrow$

GES of $\dot{\tilde\theta} = -\psi(t)\psi^T(t)\tilde\theta$ / of $\dot{\tilde\theta} = 0\cdot\tilde\theta$ $\Rightarrow$ Not GES.

Q. How to bring

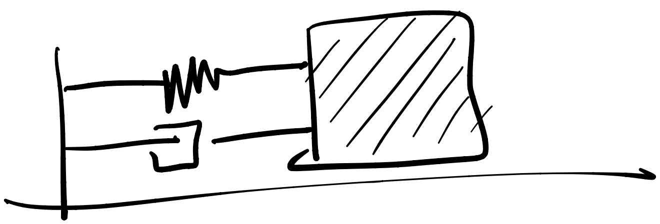
$$\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{i=0}^{m} b_i u^{(i)}(t)$$

$$a_n = 1 \qquad n > m$$

into the form of

$$y(t) = \psi^T(t)\,\theta$$

Ex.



$$m\ddot{y} + d\dot{y} + ky = u$$

$$\ddot{y} + \underbrace{d/m}_{a_1}\dot{y} + \underbrace{k/m}_{a_0} y = \underbrace{\frac{1}{m}}_{b_0} u$$

$$s^n Y(s) = -a_{n-1} s^{n-1} Y(s) - \dots - a_0 Y(s) + b_m s^m U(s) + \dots + b_0 U(s)$$

Naive approach: multiply by $\frac{1}{s^n}$ and proceed....

this is bad from the practical point of view because integrators are marginally stable and proceeding with such an approach can lead to instability.

Instead multiply with:

$$\frac{1}{\Lambda(s)} \; ; \quad \Lambda(s) := s^n + \lambda_{n-1} s^{n-1} + \dots + \lambda_1 s + \lambda_0$$

("stable polynomial")

LHS

$$\frac{s^n Y(s)}{\Lambda(s)} = \left(1 - \frac{\lambda_{n-1} s^{n-1} + \dots + \lambda_0}{\Lambda(s)} \right) Y(s)$$

$$= Y(s) - \frac{\lambda_{n-1} s^{n-1} + \dots + \lambda_0}{\Lambda(s)} Y(s)$$

Ex. $\ddot{y} + a_1 \dot{y} + a_0 y = b_1 u$

$$s^2 Y = -a_1 s Y - a_0 Y + b_1 U$$

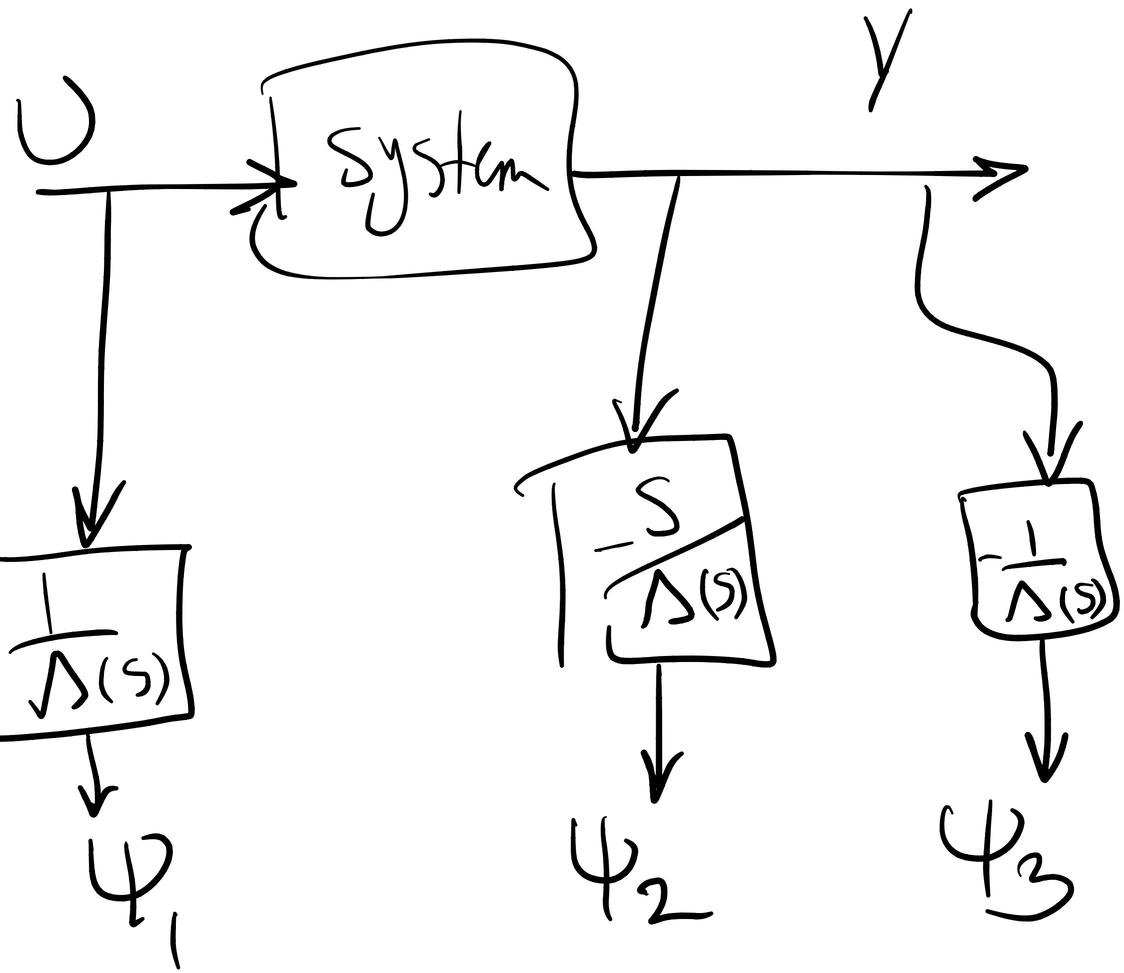
$$\frac{1}{s^2 + \lambda_1 s + \lambda_0}$$

$$Y(s) = b_1 \frac{1}{\Lambda(s)} U(s) - \underbrace{(a_1 - \lambda_1) \frac{s}{\Lambda(s)} Y(s) - \frac{(a - \lambda_0)}{\Lambda(s)} Y(s)}$$

you will be measuring $Y(s)$ and passing it through a stable filter therefore everything is fine now.

$$\theta := \begin{bmatrix} b_1 \\ a_1 - \lambda_1 \\ a_0 - \lambda_0 \end{bmatrix} = \begin{bmatrix} b_1 \\ \overline{a_1} \\ \overline{a_0} \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$\psi(t) = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$$



In order to get rich enough input to achieve P.E. you can shake the system with 2 sine functions (e.g., $\sin(t) + \sin(10t)$)

Rule of thumb: one sine per 2 input parameters. This is just a rule of thumb and if this does not achieve U.O. you can add another sine.