# Lecture 12 — 03/08/2021

Last time: Constructing Lyapunov functions

LaSalle's invariance principle

Lyapunov theory for linear systems

Today: Finish discussion on Lyapunov's indirect method for linear systems

Lyapunov direct method

Theorem $\dot{x} = Ax$ is stable ($\bar{x}=0$ is GAS)

i.e., $A$ is Hurwitz ($\lambda_i(A) \in$ LHP)

$$\Updownarrow$$

$\forall Q = Q^T > 0$, there is $P = P^T > 0$ s.t.

$$A^T P + PA = -Q$$

$$\longleftarrow$$

If this holds, then $P = \int_0^\infty e^{A^T t} Q e^{At} \, dt$ is the unique sol'n to the Lyapunov equation and

$V(x) = x^T P x$ is the Lyapunov function

for $\dot{x} = Ax$.

aside

$$\left.\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}\right\} \quad A^T P + PA = -C^T C$$

$P$: observability Gramian

$$\dot{x} = Ax + Bu : \quad AX + XA^T = -BB^T$$

$X$: controllability Gramian

$\Bigg\}$ useful in the context of stability

Ex.
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\det(sI - A) = \det\begin{pmatrix} s & -1 \\ a & s+b \end{pmatrix} = s^2 + bs + a$$

$$a, b > 0 \iff A \text{ Hurwitz}$$

Question: Can $Q = Q^T \geq 0$ work here?

$$V(x) = \int_0^{x_1} \underbrace{g(\xi)}_{a\xi}\, d\xi + \frac{1}{2}x_2^2 = \frac{1}{2}a x_1^2 + \frac{1}{2}x_2^2$$

$$x^T P x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T P \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$P = \frac{1}{2}\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

from the Pendulum example

$$\dot{V}(x) = 0x_1^2 - b x_2^2$$

$$= -(x_1\ x_2)\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}}_{Q}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Note $Q = \underbrace{\begin{bmatrix} 0 \\ \sqrt{b} \end{bmatrix}}_{C^T} \underbrace{\begin{bmatrix} 0 & \sqrt{b} \end{bmatrix}}_{C}$

* Thus, $P$ is an observability Gramian for an LTI system

$$\left.\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}\right\} \quad A = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \sqrt{b} \end{bmatrix}$$

$\dot{V} = -bx_2^2 = -y^2(t) = 0$ → study these problematic cases for La Salle...

$y = Cx = \begin{bmatrix} 0 & \sqrt{b} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Given the observability of the system, the output can only be zero if the initial condition is zero.

$$x_2(t) = e^{At} \underline{x_2(0)}$$

$$x_2 = 0 \quad \Leftarrow \quad x_2(0) = 0$$

Fact

La Salle's Invariance Principle for LTI systems

with $A^T P + PA = -C^T C \preceq 0$

$$\Updownarrow$$

$\bar{x} = 0$ is GAS if pair $(A, C)$ is observable.

let's check the observability matrix for the example above.

Observability matrix

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{b} \\ -a\sqrt{b} & -b\sqrt{b} \end{bmatrix} = \sqrt{b}\begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$$

if $\det \begin{bmatrix} C \\ CA \end{bmatrix} = ab \neq 0$ then we have observability.

$b = 0 \longrightarrow$ conservative system & this analysis doesn't even apply.

$\Rightarrow$ thus $a \neq 0$

Recap:

for the example presented here

$$P := \frac{1}{2}\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow a > 0$$

$$\Updownarrow$$

$$P^T = P > 0$$

$$\dot{V} = -x^T Q x \; ; \; Q = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \Rightarrow b > 0$$

$$\Updownarrow$$

$$Q = Q^T \geq 0$$

Thus, theorem can be relaxed to :

$$\dot{x} = Ax \text{ stable}$$

$$\Updownarrow$$

$$\forall Q = Q^T \geq 0 \quad \oplus \quad \text{pair } (A, Q^{1/2}) \text{ observable}$$

there exists $P = P^T > 0$ st.

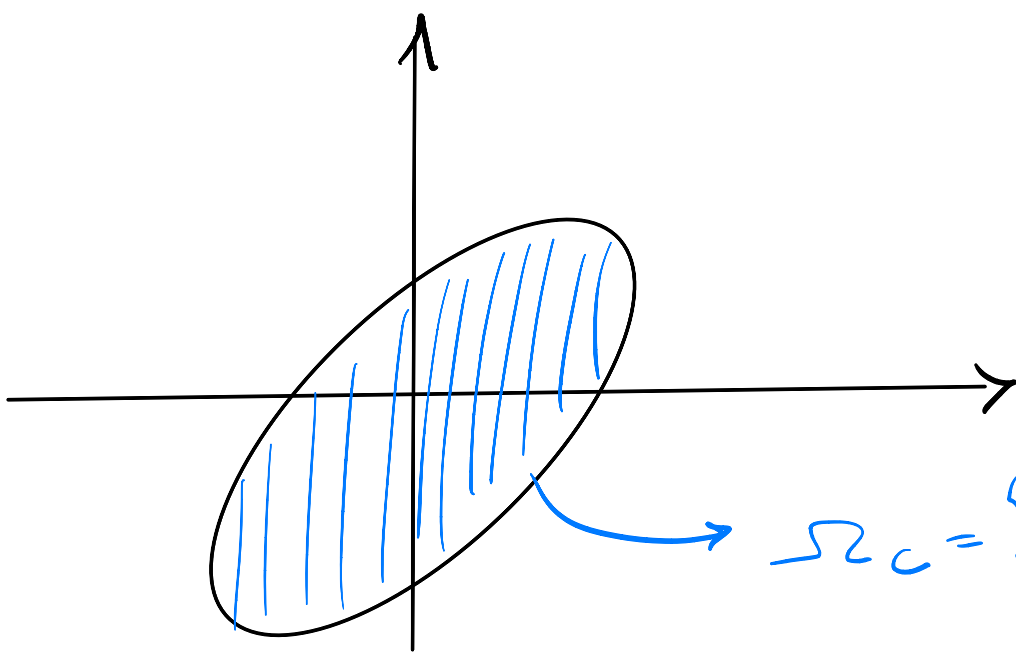
$$A^T P + PA = -Q$$

Moral: The "Big" Lyapunov theorem we learned about in linear systems theory was direct application of Lyapunov's indirect method with quadratic Lyap. function $V = x^T P x$.

La'Salles invariance principle allowed us to relax the second part of Lyapunov's theorem and account for $Q = Q^T \succeq 0$ when we have observability of $(A, Q^{1/2})$.

# Region of Attraction:

set of points s.t. $\{x_0, \underbrace{\phi(t, x_0)}_{x(t) \text{ s.t. } x(0)=x_0} \to 0\}$



$\Omega_c = \{x \,;\, V(x) \leq c\}$

$\dot{V}(x) < 0$ in $\Omega_c$

Q. How to estimate the region of attraction?

A possible approach: use Lyapunov based analysis

(caveat: may be conservative!)

for example: use of $V(x) = x^T P x$ where $P$ is $P^T = P > 0$

as a Lyapunov function for the corresponding linearization.

use of sum-of-squares (SOS) techniques can reduce conservatism (mainly based on work of P. Parrilo @ MIT)

(caveat: computationally expensive)

# Lyapunov direct method (linearization)

LDM

stability via linear approximation of nonlinear system.

$\dot{x} = f(x)$

$f(0) = 0 \Rightarrow \bar{x} = 0$ is an equilibrium point

linearization around $\bar{x} = 0$ ;

replace $x(t) = \bar{x} + \tilde{x}(t)$ and substitute into the dynamics

$$\dot{x} = f(x) = \cancelto{0}{\dot{\bar{x}}} + \dot{\tilde{x}} = f(\bar{x}+\tilde{x})$$

$$= \underbrace{\dot{\tilde{x}} = f(\bar{x}+\tilde{x})}_{\text{nonlinear equation for } \tilde{x}}$$

$$\boxed{\dot{\tilde{x}} = A\tilde{x} + g(\tilde{x})} \quad (I)$$

Taylor series expansion :

$$A = \partial f / \partial x \Big|_{x=\bar{x}}$$

$$g(\tilde{x}) = f(\tilde{x}) - A\tilde{x}$$

higher order term in Taylor series

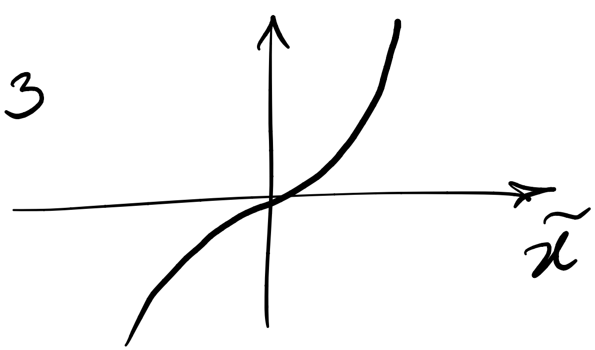
$$f(\tilde{x}) \approx \cancelto{0}{f(\bar{x})} + \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} \tilde{x} + \underbrace{H.O.T.}_{g(\tilde{x})}$$

Key assumption $\quad \dfrac{\|g(\tilde{x})\|}{\|\tilde{x}\|} \xrightarrow{\|\tilde{x}\| \to 0} 0$

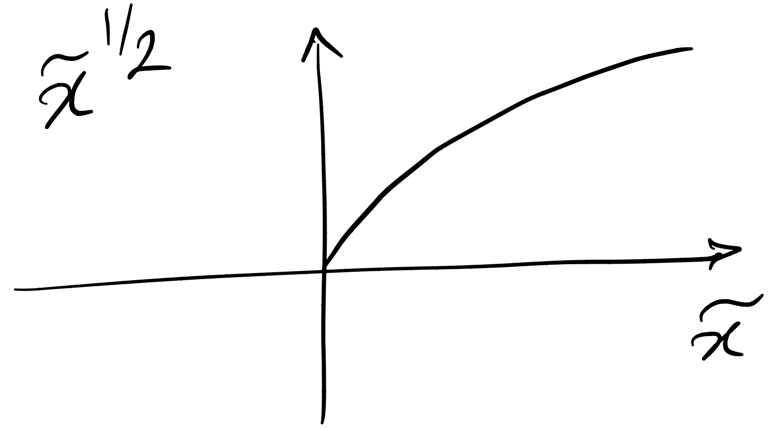
i.e.,

$$\lim_{\|x\| \to 0} \frac{\|g(x)\|}{\|x\|} = 0$$

Examples $\quad g(\tilde{x}) = \tilde{x}^3$



however, $g(\tilde{x}) = \tilde{x}^{1/2}$



bad!

doesn't meet assumption

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Let $\dot{\tilde{x}} = A\tilde{x}$ be $\underbrace{\text{stable}}_{\text{GAS at e.p. } \bar{x}=0}$ $\left(A \text{ Hurwitz } \operatorname{Re}\{\lambda_i(A)\} < 0\right)$

LTI

We know that $P = P^T > 0$ s.t. $V(\tilde{x}) = \tilde{x}^T P \tilde{x}$

is a Lyapunov function for LTI system

We are going to use the Lyapunov function to assess stability of the nonlinear system.

for nonlinear system (I), $V(\tilde{x})$ works as Lyapunov function

$$A^T P + PA = -Q$$

$$Q = Q^T \succ 0$$

Now, study $\frac{dV}{dt}$ along sol'ns of (I)

$V = \tilde{x}^T P \tilde{x}$

$$\dot{V} = \dot{\tilde{x}} P \tilde{x} + \tilde{x}^T P \dot{\tilde{x}}$$

$$= \left(A\tilde{x} + g(\tilde{x})\right)^T P \tilde{x} + \tilde{x}^T P \left(A\tilde{x} + g(\tilde{x})\right)$$

$$= \tilde{x}^T \underbrace{\left(A^T P + PA\right)}_{-Q} \tilde{x} + \underbrace{2\tilde{x} P g(\tilde{x})}_{\text{sign-indefinite}} \quad ? \lessgtr 0$$

We want to minimize damage by figuring out what the worst values of $2\tilde{x}^T P g(\tilde{x})$ could be.

$\Downarrow$ let's upperbound $\dot{V}$:

$$\dot{V} = -\tilde{x}^T Q \tilde{x} + 2\tilde{x}^T P g(\tilde{x})$$

$$\leq -\lambda_{min}(Q)\|\tilde{x}\|_2^2 + 2\|\tilde{x}\|_2 \|P\|_{2i} \cdot \|g(\tilde{x})\|_2$$

Aside: Note that

$$\lambda_{min}(Q)\|x\|_2^2 \leq x^T Q x \leq \lambda_{max}(Q)\|x\|_2^2$$

$$\Rightarrow \quad -x^T Q x \leq -\lambda_{min}(Q)\|x\|_2^2$$

$$x^T y \leq \|x\|_2 \|y\|_2$$

proof

$$x^T y = \|x\|_2 \|y\|_2 \cos(\theta)$$

Here, we consider $y = Pz \Rightarrow \|P\|_{2i} := \sup \dfrac{\|Pz\|_2}{\|z\|_2}$

$$\frac{\|Pz\|_2}{\|z\|_2} \leq \|P\|_{2i} \Rightarrow \|Pz\|_2 \leq \|P\|_{2i} \|z\|_2$$

Going back:

$$\dot{V} \leq -\|\tilde{x}\|_2 \underbrace{\left( \lambda_{min}(Q) \|\tilde{x}\|_2 - 2\|P\|_{2i} \|g(\tilde{x})\|_2 \right)}_{\text{need to be positive}}$$

we can find a region around the origin where this is true

$$\Rightarrow \quad \|\tilde{x}\|_2 \geq \frac{2\|P\|_{2i}}{\lambda_{min}(Q)} \|g(\tilde{x})\|$$

small enough $\tilde{x}$

$$\text{if} \quad \frac{\|g(\tilde{x})\|_2}{\|\tilde{x}\|_2} \xrightarrow{\|\tilde{x}\|_2 \to 0} 0$$

the quantity inside the parantheses on the right will be positive and we can conclude LAS of $\bar{x} = 0$ with ROA.

region of attraction