Lecture 16 03/29/2021

Last time: LTV systems
Differential Lyapunov eqn.

Today: Extension's of LaSalle's Invariance Principle to T.V. systems

Estimation of const. unknown parameters

Q. What happens if in the Lyapunov analysis of LTV systems $W_3(x)$ is NOT positive-definite but is ONLY p.s.d.

In time invariant case :

Ex.

$$\dot{x}_1 = x$$
$$\dot{x}_2 = -\sin x_1 - x_2$$

$\Rightarrow x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} \pi \\ 0 \end{bmatrix}$

down

energy-like Lyap. function gives

$$\dot{V} = 0 \cdot x_1^2 - x_2^2 \leq 0 \longrightarrow \text{semi-definite}$$

Largest invariant set where $\dot{V} \equiv 0$

$\Downarrow$

$-\sin x_1 - \cancel{x_2} = 0 \xLeftarrow{} \dot{x}_2 = 0 \xLeftarrow{\text{invariance}} x_2 \equiv 0$

0

$\sin x_1 = 0 \implies x_1 = 0$

The largest invariant set where $\dot{V} \equiv 0$ is given by $\bar{x}_{down} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow$ LAS

LaSalle's Invariance Princ.

Q. In time-varying case, does La'Salle hold?

A. In general, no!

CAN ONLY SAY:

$$\lim_{t \to \infty} W_3(x(t)) = 0$$

Q. What are possible additional assumptions/conditions that are needed for asymptotic stability ???

Ex.

$$\left.\begin{aligned} \dot{x}_1 &= -x_1 + W(t)\,x_2 \\ \dot{x}_2 &= -W(t)\,x_1 \end{aligned}\right\} \quad \begin{aligned} &x_i(t) \in \mathbb{R} \\ &i = 1, 2 \\ &W(t) \in \mathbb{R} \end{aligned}$$

$$A(t) = \begin{bmatrix} -1 & W(t) \\ -W(t) & 0 \end{bmatrix}$$

if $W(t) = \text{const.}$

$\Downarrow$

LTI system!

propose $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 > 0$

$\forall x \in \mathbb{R}^2 \setminus \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$

$$\dot V = x_1 \dot x_1 + x_2 \dot x_2$$

$$= x_1\left(-x_1 + \cancel{w(t)x_2}\right) + x_2\left(-\cancel{w(t)x_1}\right)$$

$$= -x_1^2 - \underbrace{0 \cdot x_2^2}_{\text{No } x_2 \text{ in } \dot V} \leq 0$$

negative semi-definite!

recap.

$$V(x) = \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{V}(x) = -\begin{bmatrix} x_1 & x_2 \end{bmatrix}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{Q = Q^T \geq 0}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 0$$

So we can immediately conclude U.S.

Q: UAS?

Note: In the LTI case: $w(t) = \text{Const.}$, we could invoke La'Salle to show A.S. of the origin $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$\dot{V} = -x_1^2 = 0 \Rightarrow x_1 \equiv 0 \Rightarrow \dot{x}_1 = 0$$

$$\Downarrow$$

largest invariant set is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and it is A.S. $\leftarrow$ $x_2 = 0$ $\Leftarrow$ $-x_1 + w(t)x_2 = 0$

can be problematic in TV. case

In general, for LTI systems La'Salle implies that we can factor $Q = C^T C$

We showed a while ago that

(A, C) observable $\Rightarrow$ A.S. of $\bar{x} = 0$

In the previous example, $w(t) = \text{const.}$

$$A = \begin{bmatrix} -1 & w \\ -w & 0 \end{bmatrix} \quad ; \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C}$$

Observability matrix:

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & w \end{bmatrix}$$

$$\det(W_o) = 1 \cdot w - 0(-1) = w \neq 0$$

observability

$$\Downarrow$$

A.S.

Q. How can we generalize observability to time varying case?

For LTV: $\dot{x} = A(t)x$

Diff Lyap. equation (DLE)

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + Q(t) = 0$$

assume $Q(t) = C^T(t)C(t) \geq 0$ → semi-definite

$$\dot{V}(x,t)\Big|_{\dot{x}=A(t)x} = -[C(t)x]^T \underbrace{C(t)x}$$

output for which we want to examine observability for $\dot{x} = A(t)x$

back to

Ex:

$$A(t) = \begin{bmatrix} -1 & w(t) \\ -w(t) & 0 \end{bmatrix}$$

$$C(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$Q(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

In this example $C = [1 \quad 0] = \text{const.}$ but we still need additional conditions of A.S.

Uniform Observability

(U.O.)

Thm. If $(A(t), C(t))$ is Uniformly observable

$\dot{x} = A(t)x$

$Q(t) = C^T(t)C(t)$: Matrix $Q(t)$ from DLE

then $\bar{x} = 0$ of $\dot{x} = A(t)x$ is Uniformly Exponentially stable.

This theorem allows us to conclude UES even when $\cancel{Q(t) \geq \alpha I}$ provided $(A(t), C(t))$ is U.O.

Def. Pair $(A(t), C(t))$ is Uniformly Observable if for all t_0, there are $\alpha > 0$ and $\delta > 0$ (no dependence on t_0)

such that:

$$W_0 = \int_{t_0}^{t_0+\delta} \underbrace{\phi^T(\tau,t_0)}_{\text{state-transition}} \underbrace{C^T(\tau)C(\tau)}_{Q(\tau)} \phi(\tau,t_0)\, d\tau \succeq \alpha I$$

Note: $\dfrac{\partial \phi(\tau,t_0)}{\partial \tau} = A(\tau)\phi(\tau,t_0)$; $\phi(t_0,t_0) = I$

In LTI case Observability Gramian on interval

$[t_0, t_0+\delta]$

$$\int_{t_0}^{t_0+\delta} e^{A^T(\tau - t_0)} C^T C \, e^{A(\tau - t_0)} \, d\tau \geq \alpha I$$

w.l.o.g. we can set $t_0 = 0$ (in LTI case)

Fact: $(A(t), C(t))$ is U.O.

$$\Updownarrow$$

$\big(A(t)+L(t)C(t),\ C(t)\big)$ is U.O.

Similarly for controllability $(A(t), B(t))$ is uniformly controllable (U.C.)

$$\Updownarrow$$

$\big(A(t)+B(t)K(t),\ B(t)\big)$ is U.C.

Application: Estimation of constant unknown parameters

Ex. $y(t) = \underbrace{\psi^T(t)}_{\text{known regressor vector } \psi(t)\in\mathbb{R}^p}\ \theta$

$y(t)$: scalar output (measured)

θ: vector of constant params $\theta \in \mathbb{R}^p$

Key feature: model: output is a linear function of unknown parameters.

Ex problem $\ddot{y} + a_1 \dot{y} + a_0 y = b_0 u$ (a bit later)

$\hat{\theta}(t)$: an estimate of θ

$\tilde{\theta}(t) := \theta - \hat{\theta}(t)$: an estimation error

Objective Design algorithm or diff. equation for $\hat{\theta}(t)$ st. $\lim_{t\to\infty} \hat{\theta}(t) = \theta$

(i.e., $\lim_{t\to\infty} \tilde{\theta}(t) = 0$)

Q. Given a diff. e.g. for $\hat{\theta}(t)$, under what conditions $\hat{\theta}(t)$ converges to $\theta = \text{const} \in \mathbb{R}^p$?

$$\tilde{\theta}(t) = \theta - \hat{\theta}(t) \Rightarrow \dot{\tilde{\theta}}(t) = \cancel{\dot{\theta}} - \dot{\hat{\theta}}(t)$$

$$\cancel{\dot{\theta}} \to 0$$

$$\boxed{\dot{\tilde{\theta}}(t) = -\dot{\hat{\theta}}(t)}$$

let: $e(t) = y(t) - \hat{y}(t)$

$y(t)$ → measured output

$\hat{y}(t)$ → estimated output

$$e(t) = \varphi^T(t).\theta - \varphi^T(t)\hat{\theta}(t)$$

$$= \varphi^T(t)\left[\theta - \hat{\theta}(t)\right] = \varphi^T(t)\tilde{\theta}(t)$$

Consider

A gradient descent algorithm (diff. eqn) to minimize:

$$\frac{1}{2} e^2(t) = \frac{1}{2} \underbrace{\tilde{\theta}^T(t)\,\psi(t)}_{e^T(t)}\,\underbrace{\psi^T(t)\,\tilde{\theta}(t)}_{e(t)}$$

$$\dot{\tilde{\theta}}(t) = -\nabla_{\tilde{\theta}}\left[\frac{1}{2} e^2(t)\right]$$

Gradient Descent on $\frac{1}{2} e^2(t)$:

$$\boxed{\dot{\tilde{\theta}}(t) = -\psi(t)\,\psi^T(t)\,\tilde{\theta}(t)} \rightarrow \text{LTV system}$$

Examine conditions for the U.A.S. of this gradient descent algorithm.

Next time