Nonlinear Sys. Quiz 2 Feb. 17, 2021

Problem 1: How many periodic orbits does the linear system below have?

$$\dot{x} = \begin{bmatrix} -1 & 3 \\ 1 & 0 \end{bmatrix} x; \qquad x \in \mathbb{R}^2$$

Problem 2: Comment on the stability of the following nonlinear system around the origin.

$$\begin{cases} \dot{x} = xy \\ \dot{y} = -2y + \cos(x) \end{cases} \qquad x, y \in \mathbb{R}$$

Hint: $\cos(x) \simeq 1 - \frac{x^2}{2!}$ around $x = 0$.

Solutions

Problem 1:

this is a linear system trace$(A) = -1 \Rightarrow$ No Periodic orbits

Problem 2:

linearization $A = \begin{bmatrix} y & x \\ \sin(x) & -2 \end{bmatrix}\Bigg|_{\binom{0}{0}} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$

$\lambda_1 = 0 \quad \lambda_2 = -2$ linearization inconclusive b/c of e-value on $j\omega$-axis!

use center manifold theory

$$\begin{cases} \dot{x} = xy \\ \dot{y} = -2y + 1 - x^2 \end{cases}$$

$\sim \cos(x)$

$g_1(x,y) = xy$

$g_2(x,y) = -x^2/2$

find $y = h(x)$ center manifold so that reduced order system

$\dot{x} = g_1(x, h(x))$

$$w = y - h(x) \Rightarrow \dot{w} = 0 = -2h(x) + g_2(x, h(x)) - \partial h/\partial x \, (g_1(x, h(x)))$$

$h(0) = \partial h/\partial x\big|_0 = 0$

$h(x) = h_2 x^2$

$$\Rightarrow \quad -2h_2x^2 - \frac{x^2}{2} - (2h_2x + 3h_3x^2)(h_2x^3) = 0 \quad \Rightarrow \quad h_2 = -1/4$$

$h(x) = -\frac{1}{4}x^2 \Rightarrow \quad \dot{x} = -\frac{1}{4}x^3$ Linearly asymptotically stable

Lecture 09 02/24/2021

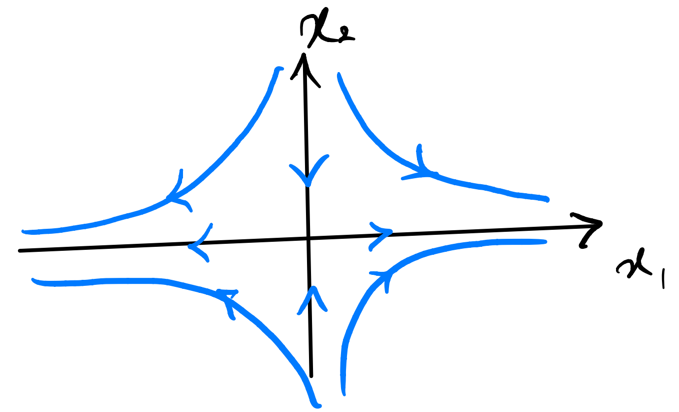
Last time: existence and uniqueness of sol'ns
cts dependence on initial conditions
and parameters

Recap

Q. Can we guarantee cts dependence on initial conditions over some time interval?

A. Yes!

Ex. 2nd-order LTI system with saddle type e.p.



We see that it is impossible to expect a small difference in trajectories for a small difference in IC's even for LTI systems.

Note

Lipschitz continuity of f $\Rightarrow$ continuity of sol'ns w.r.t. initial conditions over finite time interval

Q. How about continuity w.r.t. parameters?

A. Results for local Lipschitz cts f w.r.t. μ (parameter) extend to this case.

Today: Sensitivity equations

Stability of equilibrium pts.

Consider $\dot{x} = f(x, \mu, t)$

Let the sol'n for $\mu = \bar{\mu}$ be given by $\underbrace{x(\bar{\mu}, t)}_{\text{trajectory starting } x_0}$

I've suppressed dependence of x_0

Q. How much will the solution change when we perturb the parameter $\underbrace{\bar{\mu} + \tilde{\mu}}_{\mu}$?

$$x(\mu,t) = x(\bar{\mu},t) + \underbrace{\frac{\partial x}{\partial \mu}\Big|_{\bar{\mu}}}_{\text{Sensitivity function}} \cdot \overset{(\mu-\bar{\mu})}{\tilde{\mu}} + \underset{\text{in } \tilde{\mu}}{\text{H.O.T.}}$$

Sensitivity function

$$S(t) := \frac{\partial x(\mu,t)}{\partial \mu}\Big|_{\mu=\bar{\mu}} = x_{\mu}(\bar{\mu},t)$$

for small changes in parameter μ:

$$x(\mu,t) \approx x(\bar{\mu},t) + S(t)(\mu-\bar{\mu}) + O(\|\tilde{\mu}\|^2)$$

Objective: Determine equation that governs the evolution of $S(t)$.

Let's rewrite
$$\dot{x} = f(x, \mu, t)$$
$$\iff x(\mu,t) = x_0 + \int_0^t f(x(\mu,\tau), \mu, \tau)\, d\tau$$

Steps

1. We can differentiate $x(\mu,t)$ w.r.t. μ.
2. Evaluate at $\bar{\mu}$
3. Compute derivative w.r.t. time using Leibniz formula

$$1.\quad x_\mu(\mu,t) = \int_0^t \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial \mu} + \frac{\partial f}{\partial \mu} \right) d\tau$$

$$2.\quad x_\mu(\mu,t)\Big|_{\mu=\bar{\mu}} = S(\tau) = x_\mu(\bar{\mu},\tau)$$

$$S(t) = \int_0^t A(\tau) S(\tau) + B(\tau)\, d\tau \qquad (*)$$

$$A(\tau) = f_x(x(\bar{\mu},\tau), \bar{\mu}, \tau)$$

$$B(\tau) = f_\mu(x(\bar{\mu},\tau), \bar{\mu}, \tau)$$

3. $$\frac{d(*)}{dt} \xRightarrow{\text{leibniz formula}} \frac{d\delta(t)}{dt} = \underbrace{A(t)}_{\left.\frac{\partial f}{\partial x}\right|_{(x(\bar{\mu},t),\bar{\mu},t)}}\delta(t) + \underbrace{B(t)}_{\left.\frac{\partial f}{\partial \mu}\right|_{(x(\bar{\mu},t),\bar{\mu},t)}}$$

aside

Leibniz's formula

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = f(x,b(x))\cdot\frac{d}{dx}(b(x)) - f(x,a(x))\cdot\frac{d}{dx}(a(x))$$

$$+ \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,t)\,dt$$

Ex. Fold bifurcation

$$\dot{x} = x^2 + \mu = f(x,\mu)$$

$$\partial f/\partial x = f_x(x,\mu) = 2x \Big|_{x(\bar{\mu},t)} = 2x(\bar{\mu},t)$$

$$\partial f/\partial \mu = f_\mu(x,\mu) = 1$$

$$\frac{dS(t)}{dt} = \underbrace{2x(\bar{\mu},t)}_{\text{determines coefficient in the sensitivity eq'n}} S(t) + 1$$

Note Can simulate

$$\dot{x} = x^2 + \mu$$

$$\dot{S} = 2x(\bar{\mu},t)S + 1 \quad , \quad S(0) = 0$$

Ex (Khalil)

$$\begin{cases} \dot{x}_1 = x_2 = f_1(x,\mu) \\ \dot{x}_2 = -c\sin(x_1) - (a + b\cos(x_1))x_2 = f_2(\overset{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}{x},\mu) \end{cases}$$

$$\mu = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

determine evolution of $\delta(t)$ around $\bar{\mu} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$\partial f / \partial x = \left.\begin{bmatrix} 0 & 1 \\ -c\cos(x_1) + b\sin(x_1)x_2 & -(a + b\cos(x_1)) \end{bmatrix}\right|_{\bar{\mu}} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix}$$

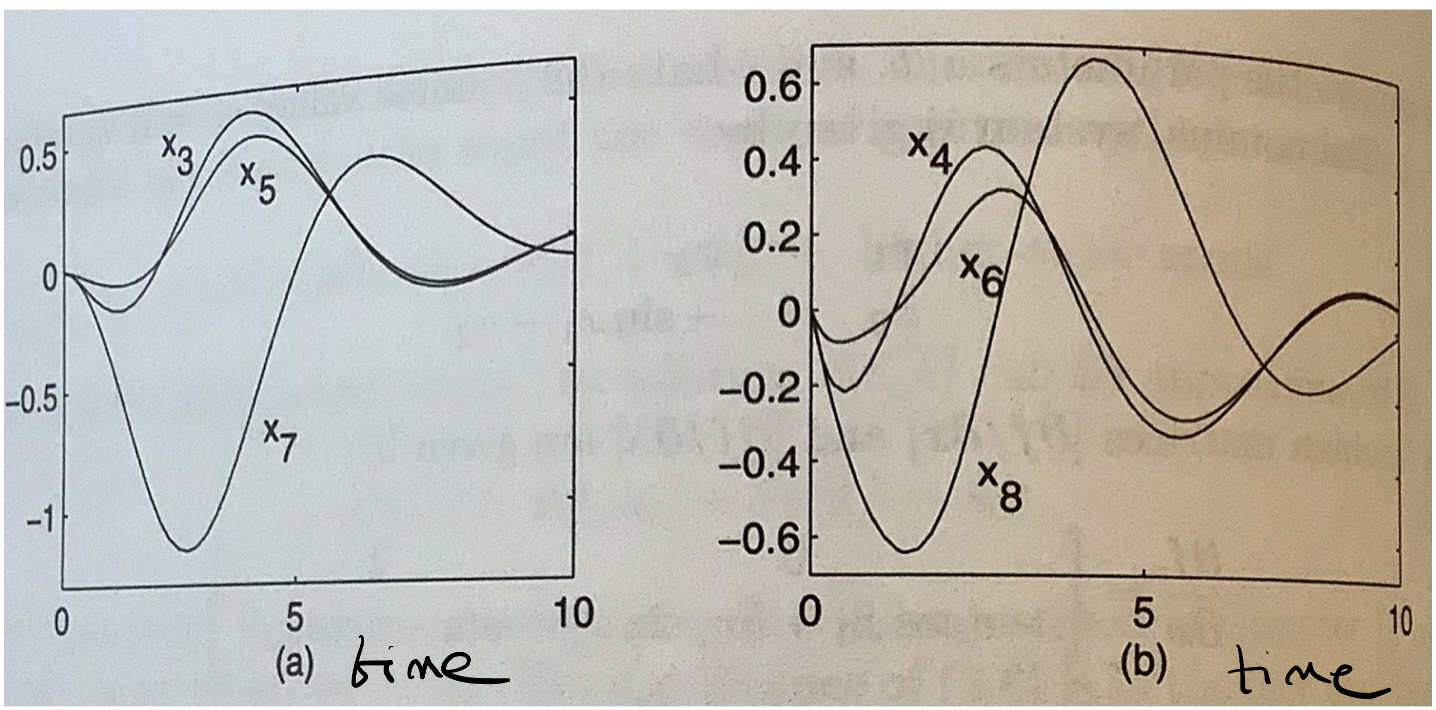
$$\partial f / \partial \mu = \begin{bmatrix} \partial f_1/\partial a & \partial f_1/\partial b & \partial f_1/\partial c \\ \partial f_2/\partial a & \partial f_2/\partial b & \partial f_2/\partial c \end{bmatrix} = \left.\begin{bmatrix} 0 & 0 & 0 \\ -x_2 & -\cos(x_1)x_2 & -\sin(x_1) \end{bmatrix}\right|_{\bar{\mu}}$$

In total we have $\underline{8}$ different equations to simulation.

Why?

$$\begin{matrix} x_1 \\ x_2 \end{matrix} \longrightarrow \text{states}$$

$$\delta(t) = \begin{bmatrix} x_3 & x_5 & x_7 \\ x_4 & x_6 & x_8 \end{bmatrix}$$



Stability

The type of stability we are going to be talking about is stability in the sense of Lyapunov w.r.t. initial conditions.

Time-invariant systems:

$\dot{x} = f(x)$ No external input; only initial conditions

Assume 1. f is locally Lipschitz cts, which means existence and uniqueness of solution

2. $f(0) = 0 \Rightarrow \bar{x} = 0$ is an e.p.

<u>Note</u> If $f(x)=0$ for $\bar{x}\neq 0$

Change of variables $Z(t)=x(t)-\bar{x}$

$$\boxed{\dot{Z}(t)} = \dot{x}(t) - \underbrace{\cancel{\dot{\bar{x}}}}_{0} = f(x)$$

$$\boxed{= f(x+\bar{Z})}$$

$f(\bar{x})=0 \Rightarrow \bar{Z}=0$ is an equilibrium point of f

<u>Ex</u> $\dot{x} = x(x-1) \Rightarrow \bar{x}=0$ or $\bar{x}=1$

essentially $\hookrightarrow$ $\dot{x}=f(x)$ (if not shift)

Ex $\dot{x} = x^2 + 1$ this system doesn't have a notion of stability because it has no e.p.

Fact $\bar{x} = 0$ is 1 stable (in the sense of Lyapunov)

if $\forall \varepsilon > 0 \quad \exists\, \delta_1(\varepsilon) > 0$ s.t.

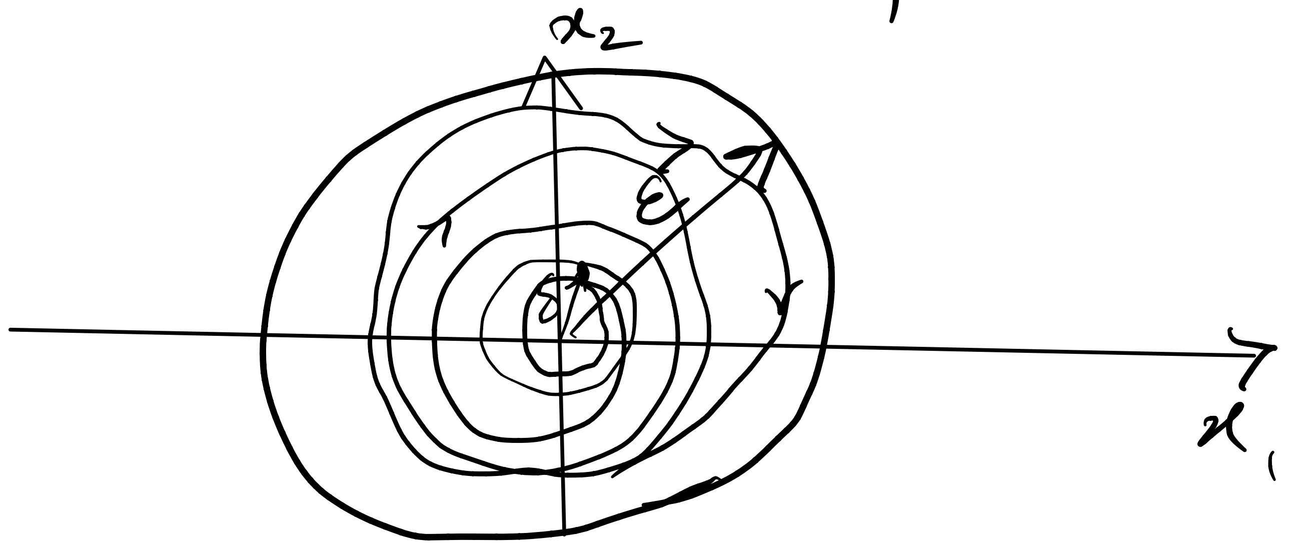
$$\|x_0\| < \delta_1 \Rightarrow \|x(t, x_0)\| < \varepsilon \quad \forall t$$

$x_0 - \bar{x}$ $\qquad$ $x_0 - \bar{x}$

$$\delta_1(\varepsilon) < \varepsilon$$

This is nothing but the definition of continuity;
"start close to $\bar{x}$ $\Rightarrow$ stay close to $\bar{x}$ for all t"

If we cannot do this (find a δ) then we say the system is generally unstable for $\bar{x}=0$ unless we start in the ball of radius δ.



(2) unstable if (1) doesn't hold.

(3) Locally asymptotically stable if (1) holds

and $\exists\ \delta_2 > 0$ st $\|x_0\| < \delta_2 \Rightarrow \lim_{t \to \infty} \|x(t, x_0)\| = 0$

(4) Globally asymptotically stable if (1) & (3) hold

for $\delta_2 = \infty$.