Lecture 6

02/08/2021

Last time: Bendixon's thm + examples

Invariant sets

Today: Poincare-Bendixon thm

Hopf bifurcations

Nondimensionalization

Poincare-Bendixon Thm:

Given a 2nd-order system $\dot{x} = f(x)$; $x \in \mathbb{R}^2$, for a compact set M (compact set $\rightarrow$ closed + bounded) which is connected, if

a) there are no e.p. in M;

b) M is positively invariant,

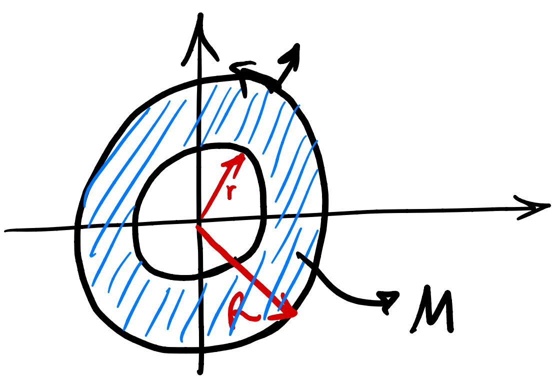
then M contains a periodic orbit.

Ex. Harmonic oscillator

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 \end{cases} \qquad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

unique e.p. is the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$M = \{ x \in \mathbb{R}^2 \,/\, r^2 \leq x_1^2 + x_2^2 \leq R^2 \}$$



$$V(x) = x_1^2 + x_2^2$$

$$\nabla V(x) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$$

on the boundary of M (∂M)

$$f^T(x) \cdot \nabla V(x) = (-x_2 \quad x_1) \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = -2x_2 x_1 + 2x_1 x_2 = 0$$

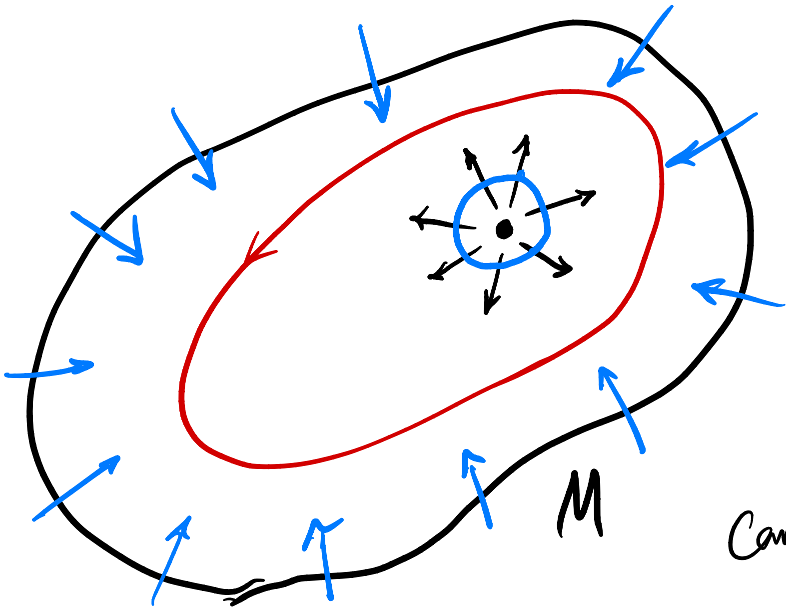
M positively invariant $\oplus$ doesn't contain e.p.

$\Rightarrow$ there is a periodic orbit in M

(in fact, there are infinitely many of them)

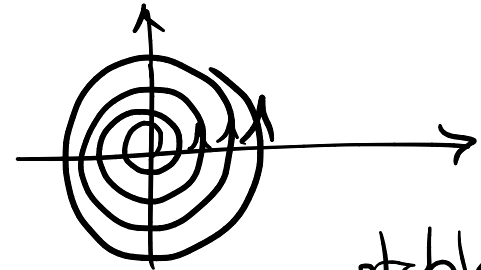
Note! It can be shown that the thm also holds if M contains a single equilibrium point which is either an unstable node or unstable focus.

No e.p. condition can be relaxed

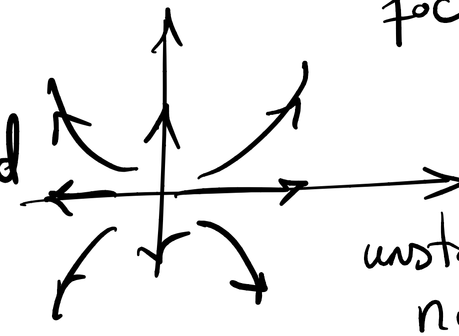


explanation

you take out the e.p. by carrying out a neighborhood around the e.p.



unstable focus



unstable node

Ex

$$\begin{cases} \dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2) = f_1 \\ \dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2) = f_2 \end{cases}$$

only e.p. $\rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$A = \frac{\partial f}{\partial x}\bigg|_{x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$\lambda_{1,2} = 1 \pm j\sqrt{2}$$

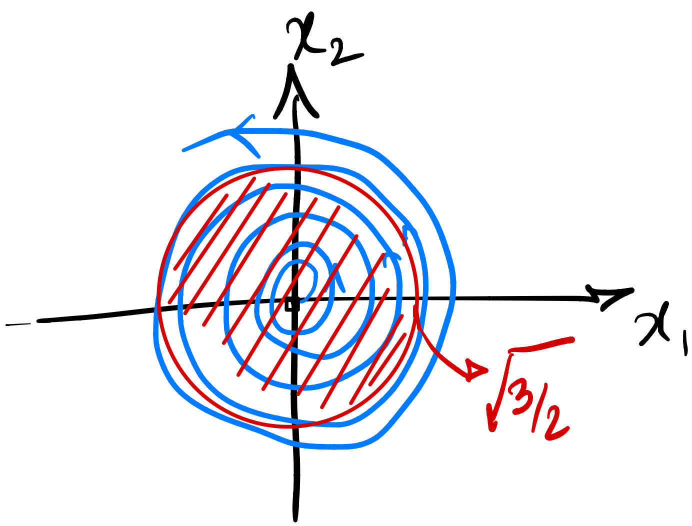
unstable focus

For this example, last time we showed that

a ball of radius $\geq \sqrt{3/2}$

is positively invariant

$$\Downarrow$$



$\exists$ a periodic orbit within the ball of radius $\geq \sqrt{3/2}$

We've corered 3 types of bifurcations:

→ Fold $\dot{x} = \alpha \pm x^2$

→ Transcritical $\dot{x} = \alpha x \pm x^2$

→ Pitchfork $\dot{x} = \alpha x \pm x^3$ → Super, → Sub

→ essentially 1D phenomena

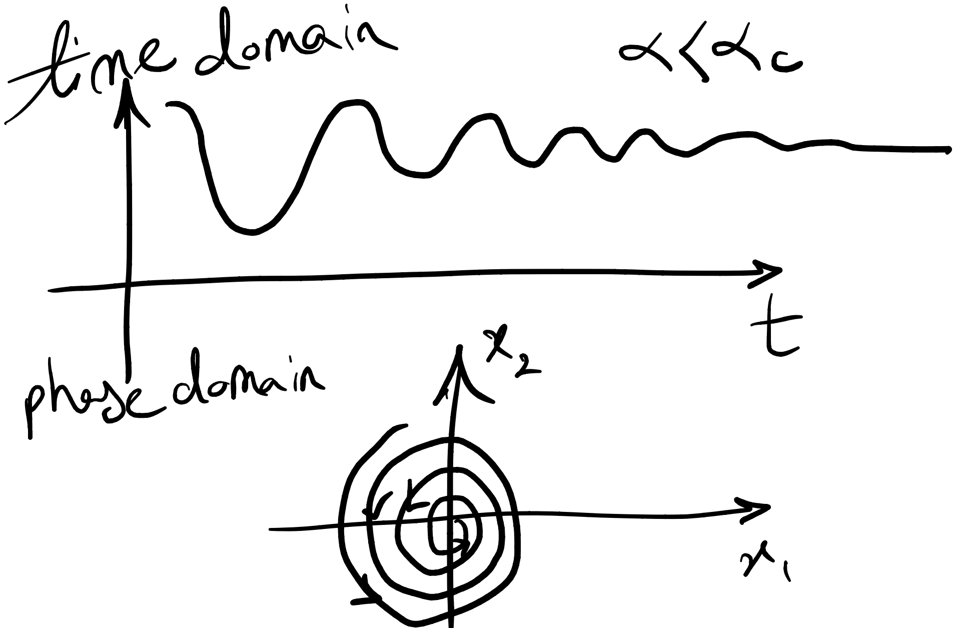
In all in all 3 cases, linearization @ the critical value of α vanishes:

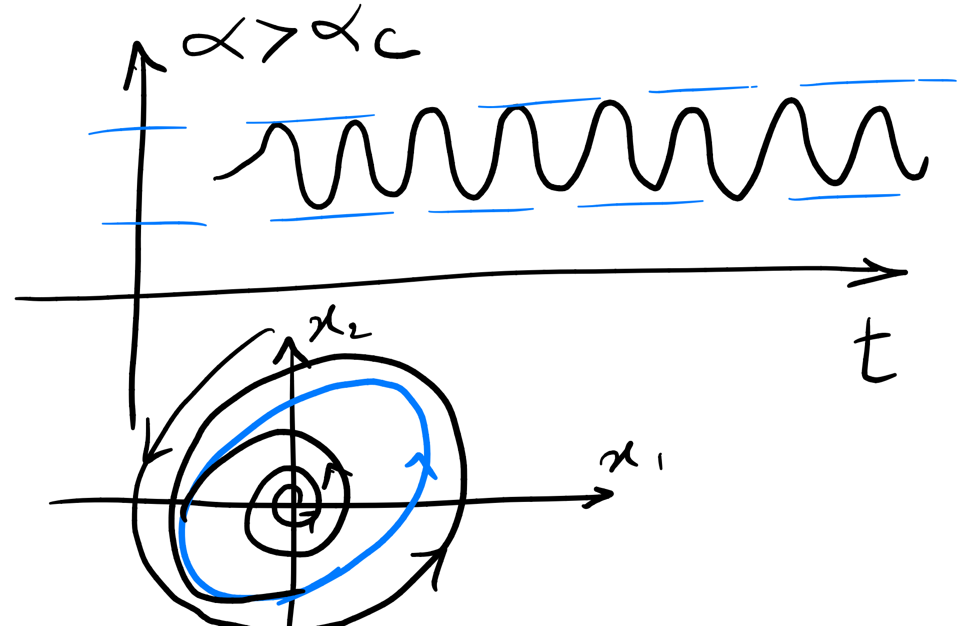
$$A = \left.\partial f / \partial x\right|_{\bar{x};\alpha_c} = 0$$

Hopf bifurcation

- supercritical
- subcritical

* Supercritical hopf bifurcation involve the loss of stability of an e.p. which is stable focus and the formation of a stable limit cycle.





Ex.

$$\dot{x}_1 = x_1(\alpha - x_1^2 - x_2^2) - x_2$$

$$\dot{x}_2 = x_2(\alpha - x_1^2 - x_2^2) + x_1$$

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polar coordinates:

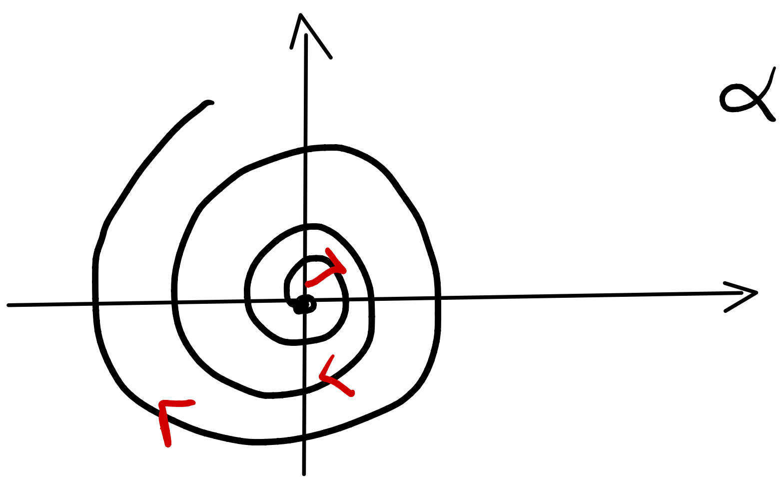
$$x_1 = r\cos\theta$$
$$x_2 = r\sin\theta$$

$$\begin{cases} \dot{r} = \alpha r - r^3 \\ \dot{\theta} = 1 \end{cases}$$

e.p.: $\bar{r} = 0$

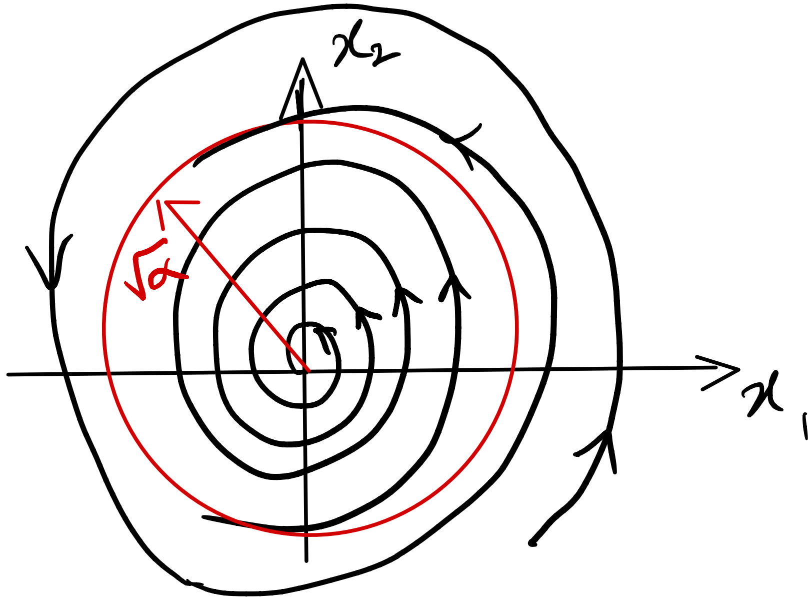
$\alpha - r^2 = 0 \Rightarrow \bar{r} = \sqrt{\alpha}$

$\alpha > 0$



$\alpha > 0$

$\hookrightarrow$ in vicinity of $\bar{r} = 0$, $f(r) \approx \alpha r$

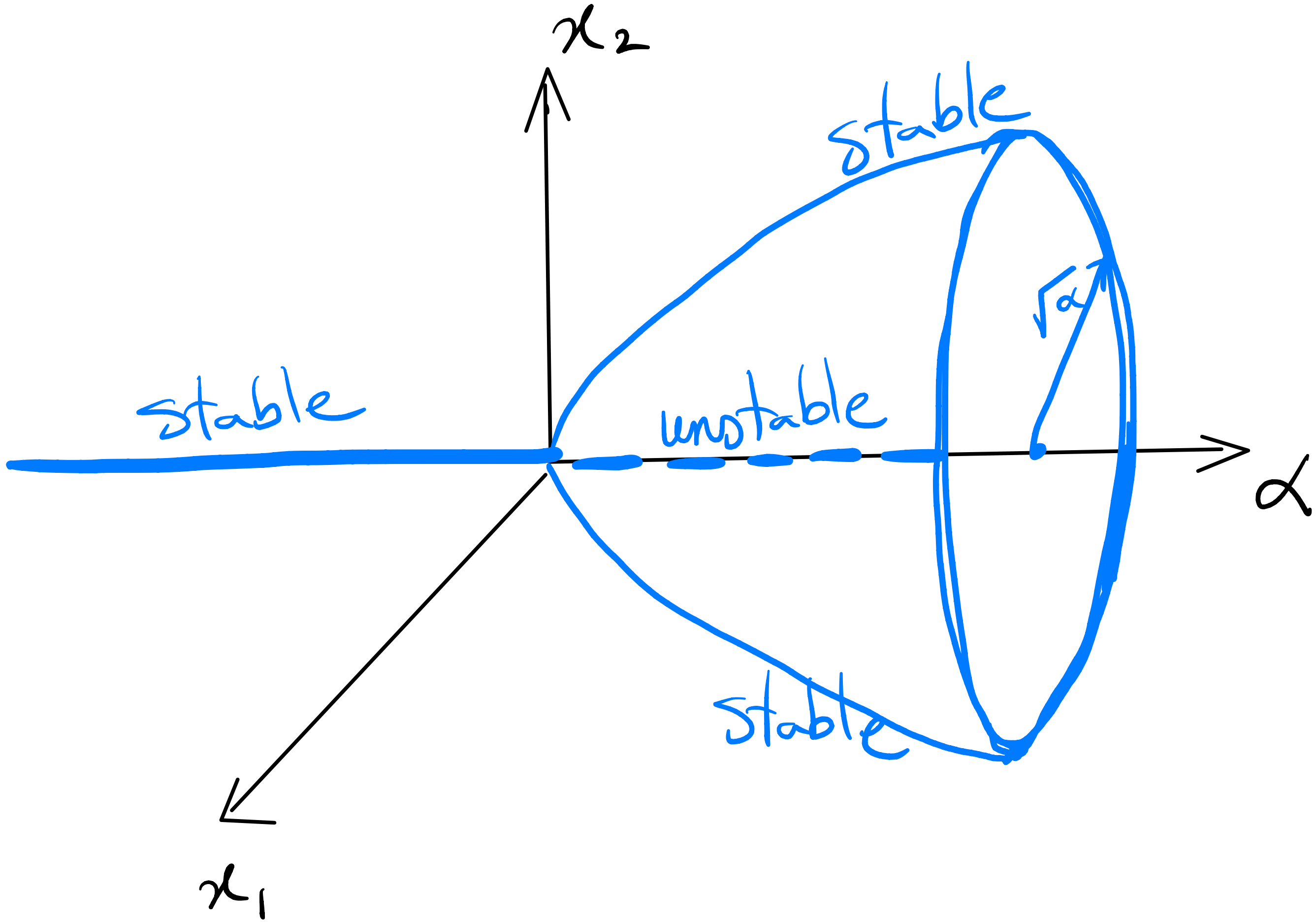


$\alpha < 0$ $\qquad f(r) \approx \alpha r$

$\dot{r} = \alpha r$

Summary :

$$\begin{cases} \alpha < 0 \Rightarrow \bar{r} = 0 \quad (\text{unique e.p.}) \quad \text{stable focus} \\ \alpha > 0 \Rightarrow \bar{r} = 0 \quad (\text{e.p.}) \quad \bar{r} = \sqrt{\alpha} \quad \text{limit cycle} \end{cases}$$



Q. Why is a super critical hopf bifurcation not considered as a dramatic change?

A. even though we lost stability of the origin if α is small positive number then departure from $\begin{pmatrix}0\\0\end{pmatrix}$ will be small as well.

subcritical Hopf bifurcation

Ex from Khalil $\rightarrow$
$$\begin{cases} \dot{x}_1 = x_1\left(\mu + (x_1^2+x_2^2) - (x_1^2+x_2^2)^2\right) - x_2 \\ \dot{x}_2 = x_2\left(\mu + (x_1^2+x_2^2) - (x_1^2+x_2^2)^2\right) - x_1 \end{cases}$$

in polar coord.:

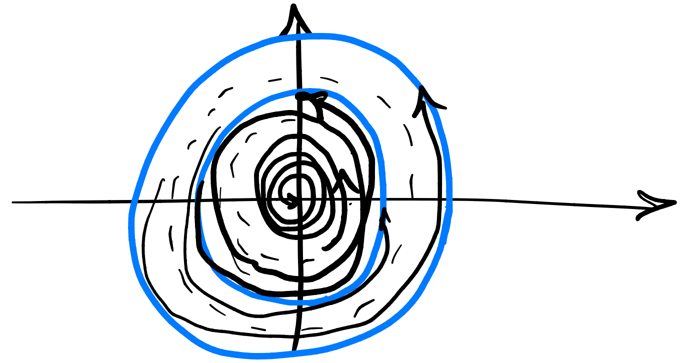
$$\begin{cases} \dot{r} = \alpha r + r^3 - r^5 \\ \dot{\theta} = 1 \end{cases}$$

$\bar{r} = 0$ $\longrightarrow$ $\alpha > 0$ unstable focus
$\longrightarrow$ $\alpha < 0$ stable focus

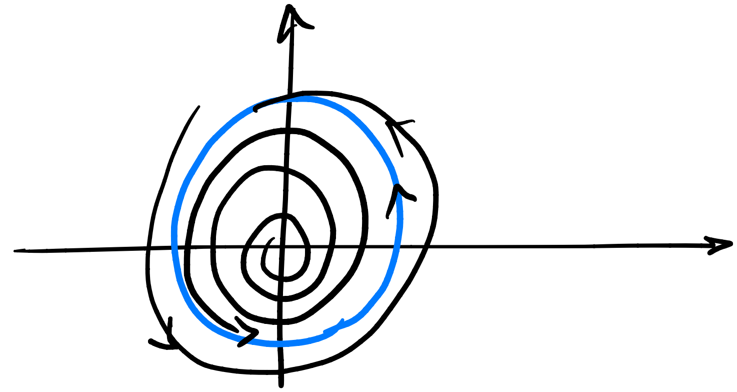
$$\bar{r}(\underbrace{\alpha + \bar{r}^2 - \bar{r}^4}) = 0 \implies \bar{q}^2 - \bar{q} - \alpha = 0 \quad (\bar{q} := \bar{r}^2)$$

$$\bar{q}_{1,2} = \frac{1 \pm \sqrt{1+4\alpha}}{2} \implies \alpha > -1/4$$

$-1/4 < \alpha < 0$



$\alpha > 0$



Big departure from origin for $\alpha > 0$!

departure from $\binom{0}{0}$ is of order 1α rather than $\sqrt{\alpha}$

Non dimensionalization

$$\begin{cases} \dot{x}_1 = -\alpha x_1 + \beta x_2 \\ \dot{x}_2 = \dfrac{\gamma x_1}{\delta + x_1^2} - \eta x_2 \end{cases}$$

Greek letters : parameters

Objective : introduce scaling

scale x_1, x_2 & time (t) in order to reduce # of parameters.

$z_1 := x_1 / X_1$

$z_2 := x_2 / X_2$

$\tau := t / T$

X_1, X_2, T to be determine

$$\frac{\partial x_1}{\partial t} = \partial \tau / \partial t \, \frac{\partial x_1}{\partial \tau} = \frac{1}{T} \frac{\partial x_1}{\partial \tau}$$

$$= \frac{X_1}{T} \, \partial z_1 / \partial \tau$$

$$\Rightarrow \quad \frac{dz_1}{d\tau} = \frac{T}{X_1}\left[-\alpha X_1 z_1 + \beta X_2 z_2\right]$$

$$\frac{dz_2}{d\tau} = \frac{T}{X_2}\left[\frac{\gamma X_1 z_1}{\delta + X_1^2 z_1^2}\right] - \frac{T}{X_2}\eta X_2 z_2$$

We can bring it to the following form:

$$\begin{cases} \dfrac{dz_1}{d\tau} = -a z_1 + z_2 \\[2ex] \dfrac{dz_2}{d\tau} = \dfrac{z_1}{1+z_1^2} - b z_2 \end{cases}$$

; a, b related to scaling parameters & original problem parameters.

proper choice of X_1, X_2, T.