Lecture 25

04/28/2021

Last time: Input-to-state stability

Today: Feedback linearization

important notions: input-output linearization

relative degree

zero dynamics

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

seek $\quad u = \alpha(x) + \beta(x)v$

motivating example: $\quad \dot{x}_1 = x_2$
$$\dot{x}_2 = -a[\sin(x_1+\delta) - \sin\delta] - bx_2 + cu$$

if I choose $\quad u = a/c\,[\sin(x_1+\delta) - \sin\delta] + \frac{v}{c}$

$$\Downarrow$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -bx_2 + v$$

now if I have $$v = -k_1 x_1 - k_2 x_2$$
$$= -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 x_1 - (k_2 + b) x_2 \end{aligned} \qquad (*)$$

Therefore, the overall state-feedback control law given by

$$u = \left(\frac{a}{c}\right)\left[\sin(x_1 + \delta) - \sin\delta\right] - \frac{1}{c}(k_1 x_1 + k_2 x_2)$$

will be able to stabilize the controllable linear dynamics $(*)$.

The ability for a feedback to convert the nonlinear system to a linear one depends on the nonlinear system having the form

$$\dot{x} = Ax + B\gamma(x)\left[u - \alpha(x)\right]$$

with (A,B) controllable and function $\gamma(x)$ being nonsingular in the neighborhood of the equilibrium of interest

Input-output Linearization

$$\dot{x} = f(x) + g(x)u \qquad (*)$$
$$y = h(x)$$

$$\left.\begin{array}{l} u(t) \in \mathbb{R} \\ y(t) \in \mathbb{R} \end{array}\right\} \text{scalars}$$

(SISO nonlinear system)

Relative degree:

Number of times that we need to differentiate the output to "see" the input (in the output eqn.)

$$y = h(x) \Rightarrow \dot{y} = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}\left(f(x) + g(x)u\right)$$

$$= \underbrace{\frac{\partial h}{\partial x} f(x)}_{L_f h(x)} + \underbrace{\frac{\partial h}{\partial x} g(x)}_{L_g h(x)} u$$

Lie derivative of function $h(x)$ in the direction of $f(x)$

→ evaluate change of h along the trajectories $\dot{x} = f(x)$

If $L_g h(x) \neq 0$ in an open set containing the equilibrium

then the relative degree (r.d.) $= 1$

if not keep differentiating ...

$$\ddot{y} = \frac{\partial L_f h(x)}{\partial x}\dot{x} = \underbrace{L_f L_f h(x)}_{L_f^2 h(x)} + \underbrace{L_g L_f h(x)}_{\text{Same story}} u$$

Def: system $\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$ has a relative degree r if

in a neighborhood of the equilibrium:

$$L_g L_f^{i-1} h(x) = 0 \qquad ; \; i = 1, 2, \dots, r-1$$

$$L_g L_f^{r-1} h(x) \neq 0$$

Ex

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^3 + u$$

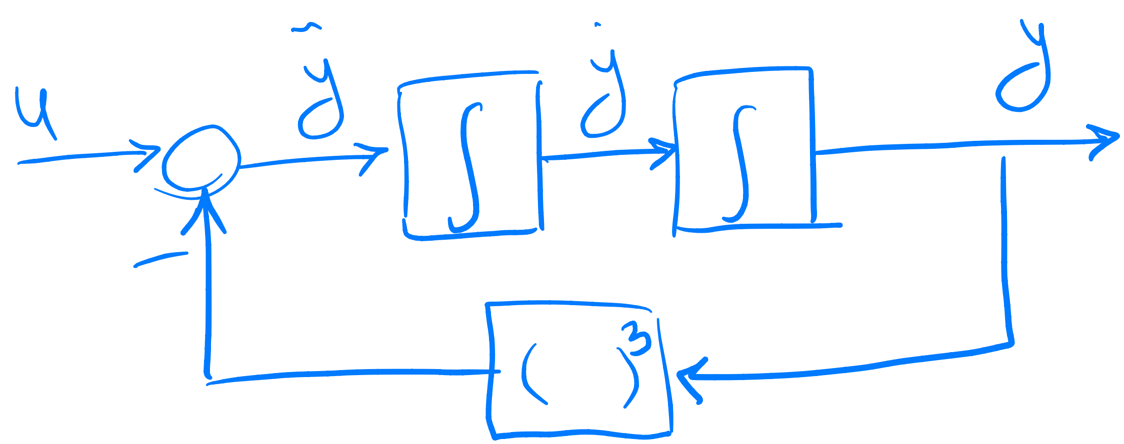
$$y = x_1$$

$\dot{y} = \dot{x}_1 = x_2$ no input $\longrightarrow$ keep differentiating

$\ddot{y} = \dot{x}_2 = -x_1^3 + \underline{\underline{u}}$

$$\boxed{r.d. = 2}$$

$$\ddot{y} = -y^3 + u$$



if we now choose $u = -k_1 \dot{y} - k_0 y + y^3$

Ex 2 Assume exact same nonlinear dynamics but this time with an interest in the second state, i.e.,

$$y = x_2$$

$$\dot{y} = \dot{x}_2 = -x_1^3 + u$$

$$r.d. = 1$$

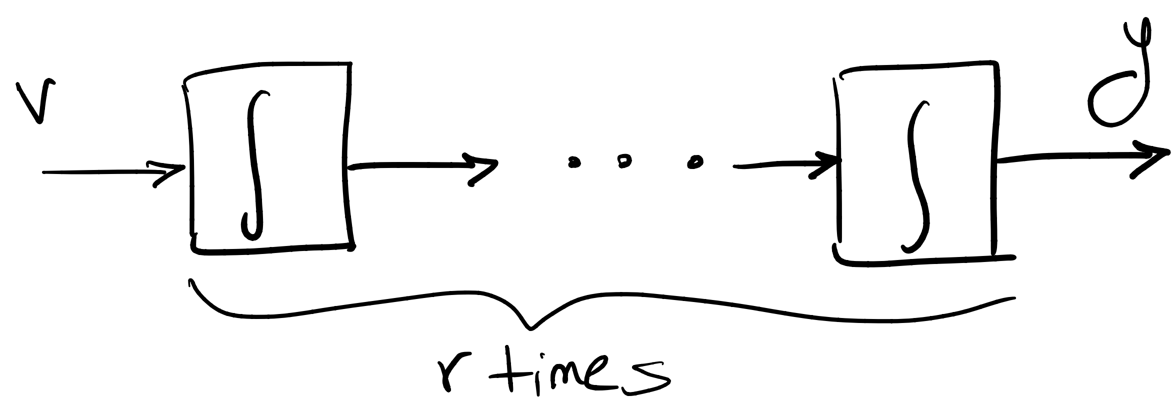
If a system has a well-defined relative degree r, then it is input-output linearizable.

derivative power

$$y^{(r)} = L_f^{r} h(x) + L_g L_f^{r-1} h(x) . u$$

$$u = \frac{1}{L_g L_f^{r-1} h(x)} \left(- L_f^{r} h(x) + v \right) \qquad (1)$$

$$y^{(r)} = v$$



$$v = -K_{r-1} y^{(r-1)} - \dots - K_0 y \qquad (2)$$

where $K_0, K_1, \dots, K_{r-1}$ are selected s.t.

$$f(s) = s^r + K_{r-1} s^{r-1} + \dots + K_1 s + K_0$$

is a Hurwitz polynomial, then the output dynamics will be "well behaved"

$$y^{(r)} + K_{r-1} y^{(r-1)} + \dots + K_1 \dot{y} + K_0 y = 0$$

- Equivalent question:

Does controller (1)-(2) provide asymptotic stability of (*)?

A. Not necessarily!

(1)-(2) renders a (n-r) dimensional manifold,

$$h(x) = L_f h(x) = \cdots = L_f^{r-1} h(x) = 0$$

invariant and attractive.

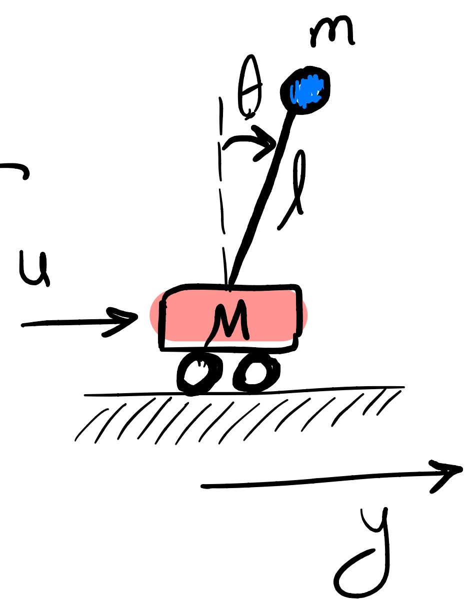
$$y = \dot{y} = \ddot{y} = \cdots = y^{(r-1)} = 0$$

The dynamics restricted to this manifold are called **ZERO DYNAMICS**. These dynamics determine whether $\bar{x} = 0$ of $(*)$ is stable.

original nonlinear system

If $\bar{x}=0$ of the zero dynamics is A.S., then the entire system is called <u>minimum phase</u>.

Ex.



(cart position)

4 states $\begin{pmatrix} y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{pmatrix}$

model: $$\ddot{y} = \frac{1}{M/m + \sin^2\theta}\left[\frac{1}{m}u + \dot{\theta} l \sin\theta - \frac{1}{2} g \sin 2\theta\right]$$

$$\ddot{\theta} = \frac{1}{l(M/m + \sin^2\theta)}\left[-\frac{1}{m}u\cos\theta - \dot{\theta}^2\frac{l}{2}\sin 2\theta + \frac{M+m}{m} g \sin\theta\right]$$

Note: input u

output y

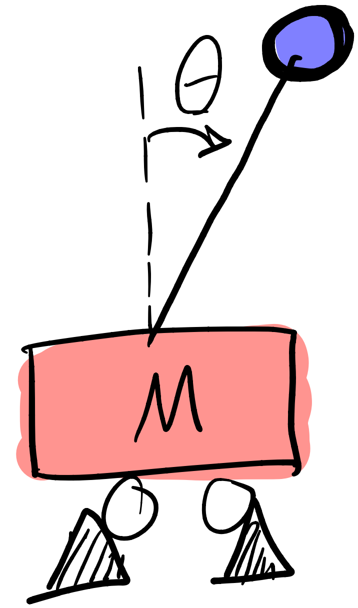
r.d. $= 2$

zero dynamics: $y(t) = \dot{y}(t) = 0$

$$u = m\left\{ \tfrac{1}{2} g \sin 2\theta - \dot{\theta} l \sin\theta \right\}$$

now by plugging u into $\ddot{\theta}$ equation + "exciting" algebra

$$\ddot{\theta} = g/l \sin\theta$$



inverted pendulum

falling like a rock, and we don't have any idea that this going on from the measured output

(θ dynamics is not observable from cart position)

Non-minimum phase system (because zerodynamic not stable)

Back to linear systems (SISO)

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

when $D \neq 0$ $\longrightarrow$ input can influence output at any frequency. We would never have roll off in the freq. response.

$$\dot{y} = C\dot{x} = C(Ax + Bu) = CAx + CBu$$

if $CB \neq 0$ $\longrightarrow$ relative degree $= 1$

if not keep differentiating ...

$$\ddot{y} = C\ddot{x} = CA\dot{x} = CA^2x + \underbrace{CAB}_{\neq 0}u \Rightarrow r.d = 2$$

$$\vdots$$

$$y^{(r)} = CA^r x + \underbrace{CA^{r-1}B}_{\neq 0}u \Rightarrow r.d = r$$

The physical interpretation of this is the coefficients of the discrete-time impulse response.

Important

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

positive real $\Longleftrightarrow$ r.d. $= 1$

proof

$\exists P = P^T > 0$ st $A^T P + PA < 0$

$PB = C^T$ (2)

P is positive definite matrix $\Rightarrow$ $CB \overset{(2)}{=} B^T P B > 0$

r.d $= 1$

If transfer function PR $\Longleftrightarrow$ r.d. $= 1$

difference btwn # poles and zeros $= 1$