Lecture 04

02/01/2021

Last time: Fold, Transcritical } Bifurcations

Today: Pitchfork bifurcations

Phase portraits of 2nd-order systems

3. Pitchfork

$$\dot{x} = \begin{cases} \alpha x - x^3 \\ \alpha x + x^3 \end{cases}$$

$\alpha x - x^3 \rightarrow$ supercritical

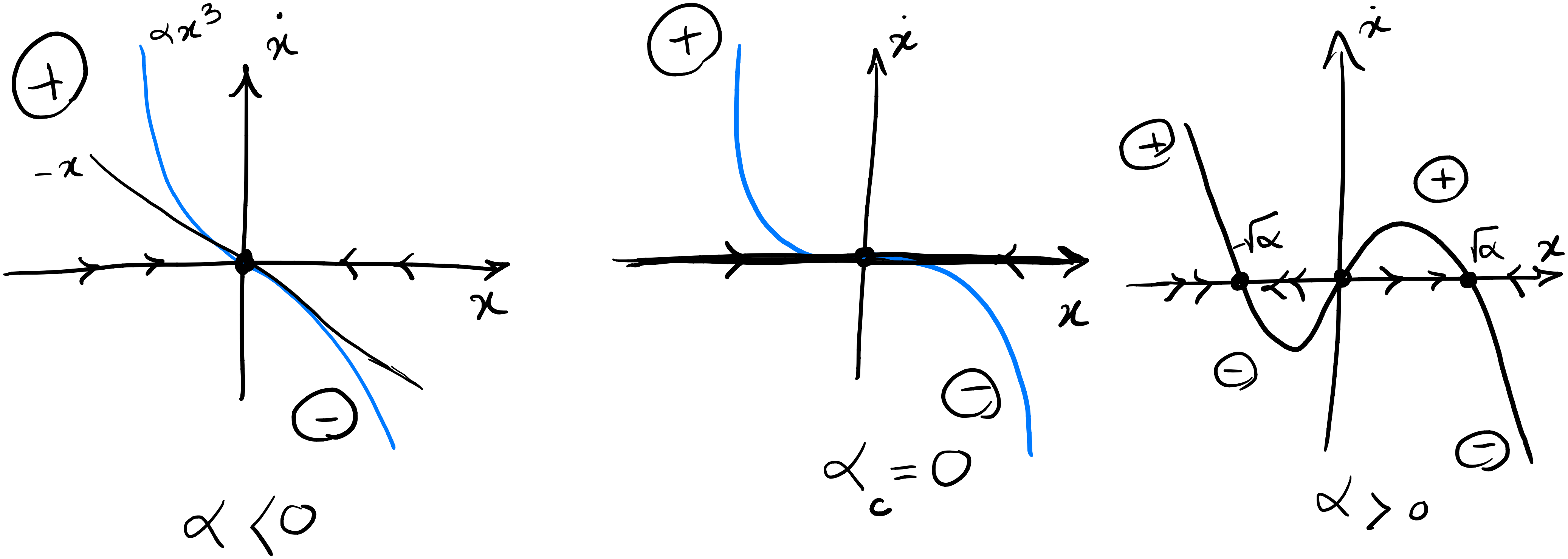
$\alpha x + x^3 \rightarrow$ subcritical

✓ usually happen in physical problems with some form of symmetry, e.g., problem that has a spatial symmetry between left/right.

$$\dot{x} = \underbrace{\alpha x - x^3}_{f(x)}$$

E.P. $\quad f(\bar{x}) = 0$

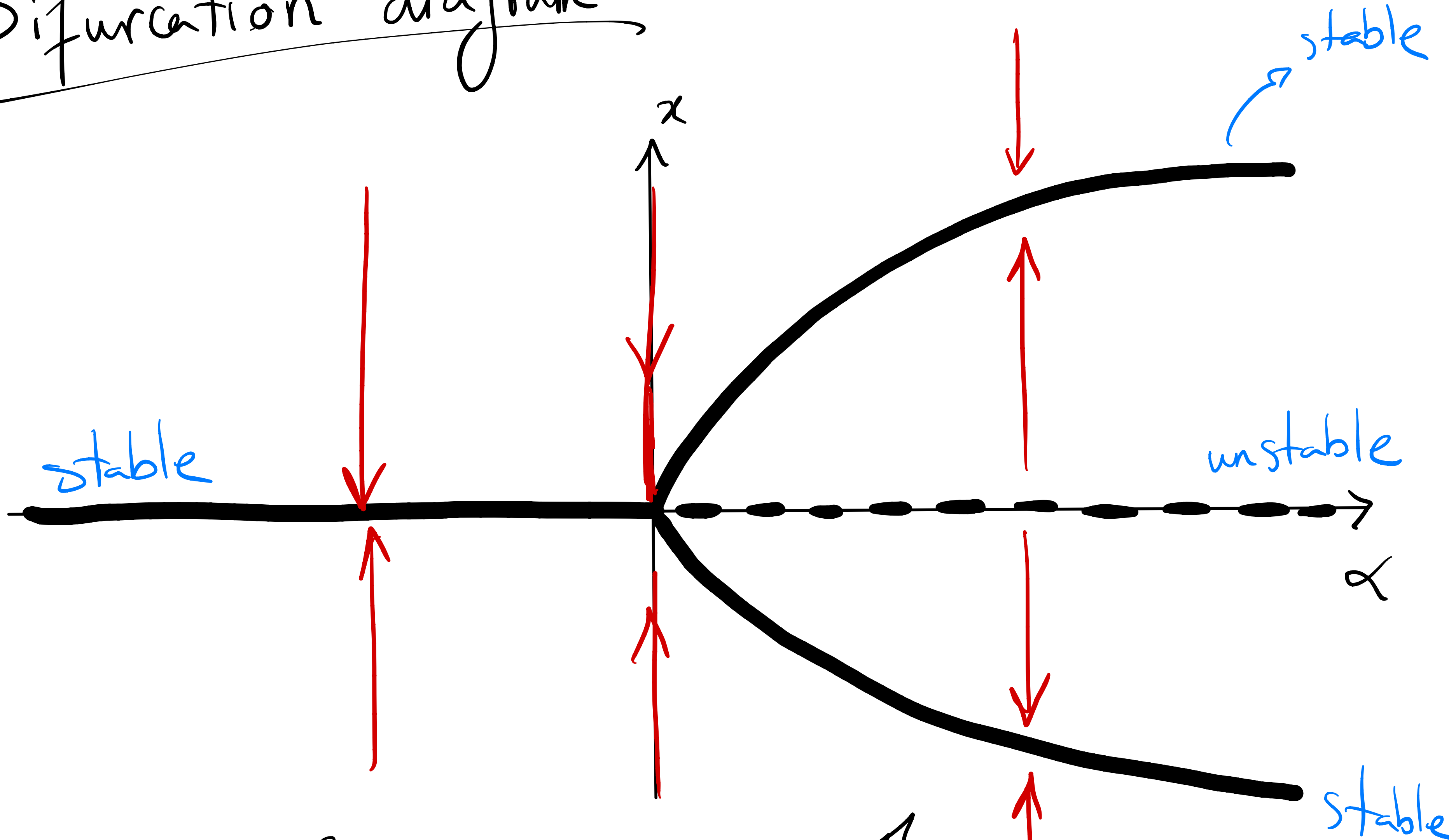
$$\bar{x}(\alpha - \bar{x}^2) = 0 \Rightarrow \bar{x} \begin{cases} 0 \\ \pm\sqrt{\alpha} & \alpha > 0 \end{cases}$$



* 2 e.p. emerge when we increase α.

As α increases equilibrium profile corresponding to e.p $\bar{x}=0$ lost its stability but no matter where we start we converge to a neighbourhood of origin.

Bifurcation diagram



$$f(x) = \alpha x - x^3$$

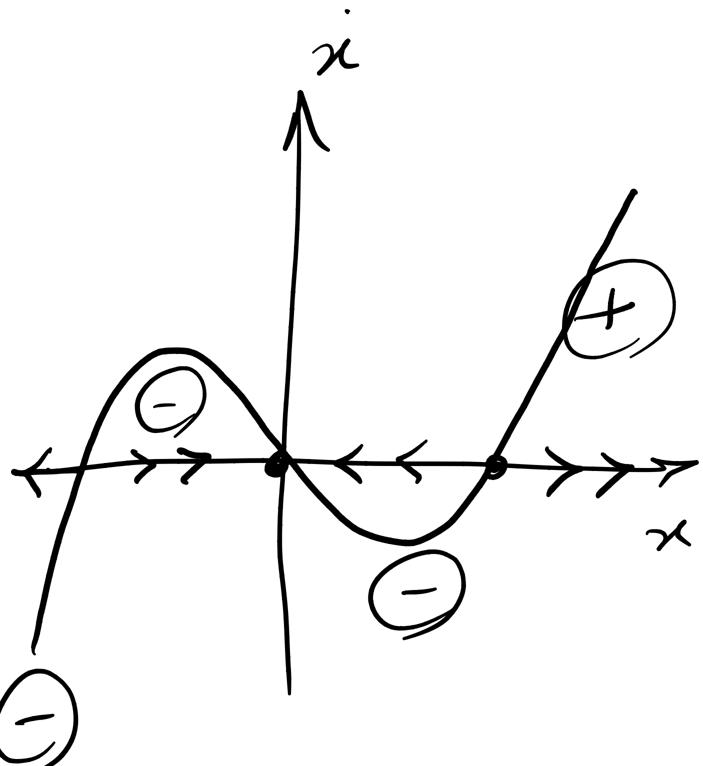
$$\underline{x}\,\underbrace{(\alpha - x^2)}$$

$$\dot{x} = \alpha x + x^3$$

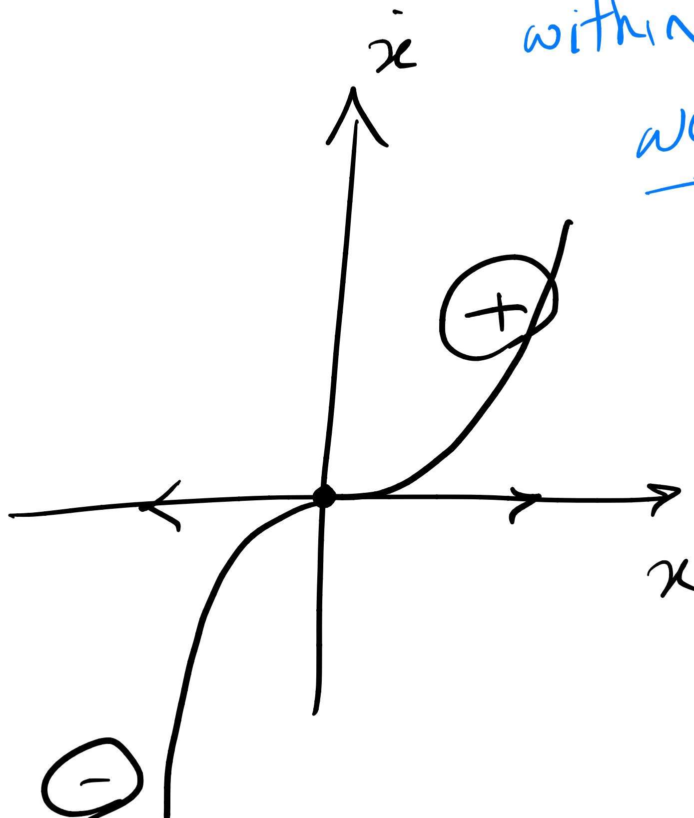
$$\bar{x} = \begin{cases} 0 \\ \pm\sqrt{\alpha} \end{cases} \qquad \alpha < 0$$

as $\alpha\uparrow$ not only do we lose e.p.'s but we lose stability. Origin is still e.p. but we cannot guarantee remaining within its neighbourhood.

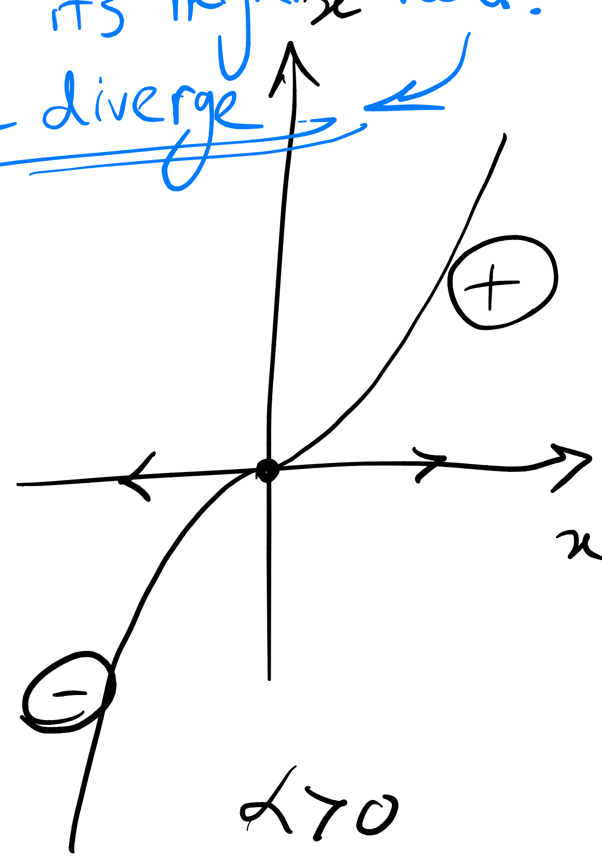
we diverge



$\alpha < 0$

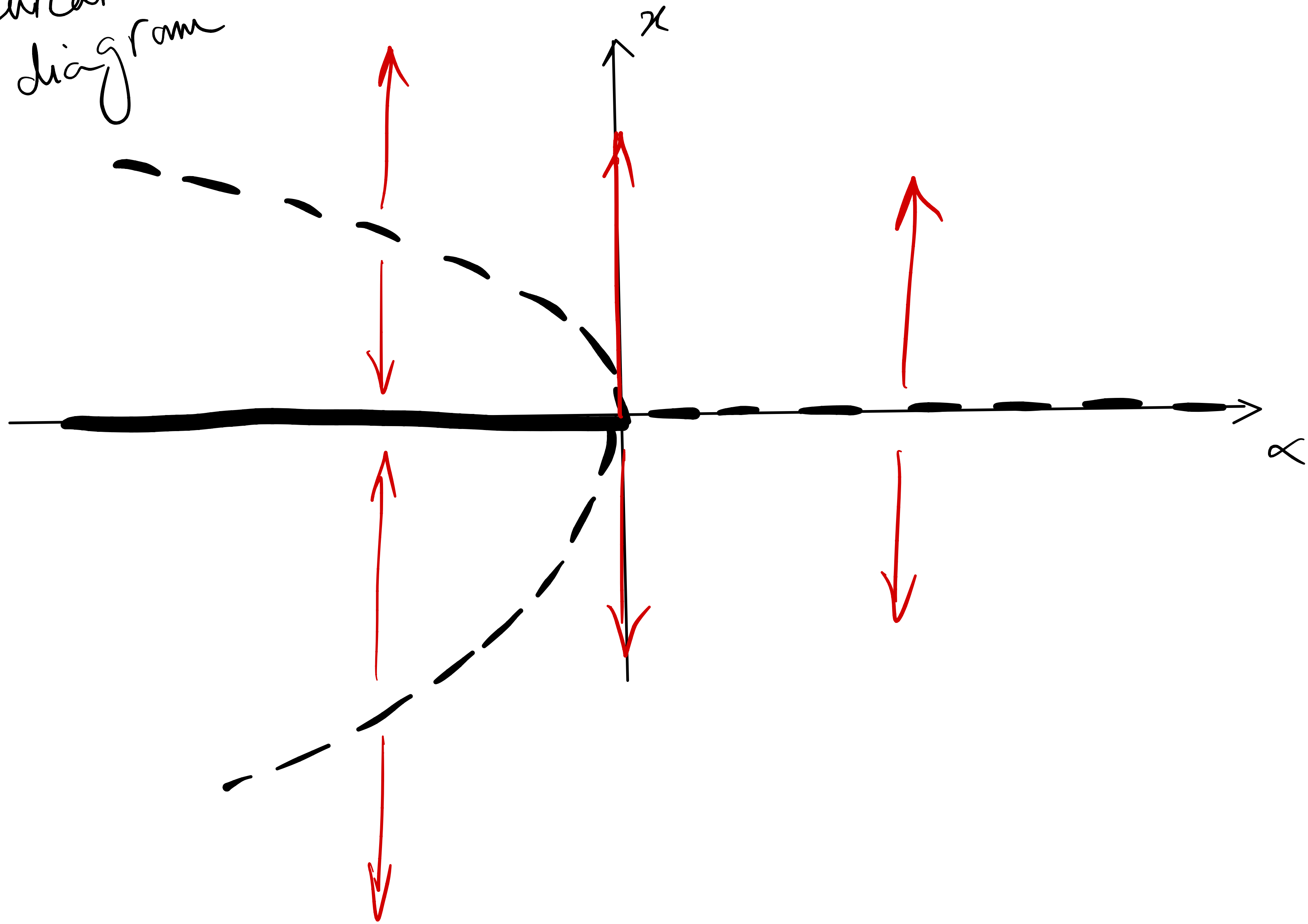


$\alpha_c = 0$



$\alpha > 0$

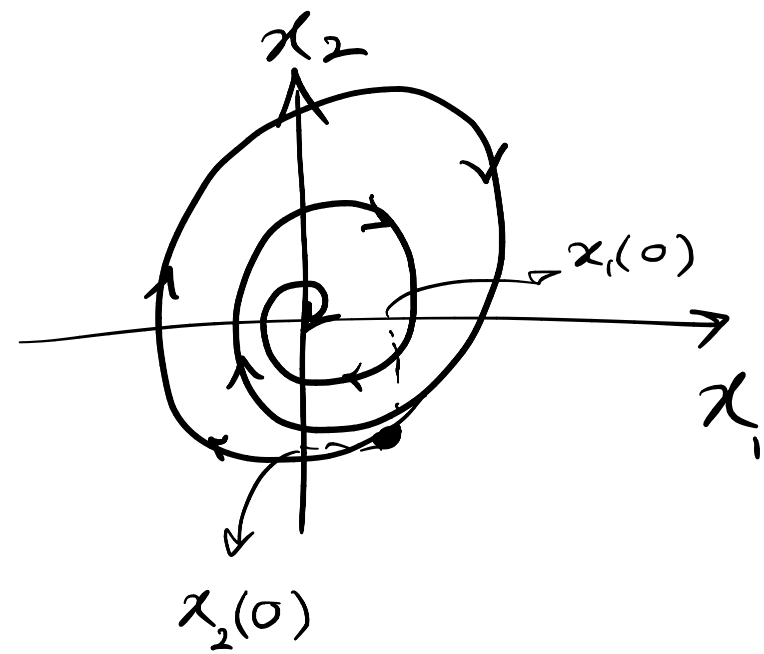
Bifurcation diagram



Second-order systems & phase plane analysis

$$\dot{x} = Ax$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} * & * \\ * & * \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



$$\bar{A} = T^{-1}AT \qquad z = T^{-1}x$$

three interesting cases:

case 1) $\bar{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $\lambda_1 \neq \lambda_2$
$\lambda_1, \lambda_2 \in \mathbb{R}$

case 2) $\bar{A} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ $\lambda \in \mathbb{R}$

case 3) $\bar{A} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ $\lambda_{1,2} = \alpha \pm j\beta$
$(\beta > 0)$

case 1)

$$\dot{z}(t) = \bar{A}\, z(t)$$

$$\dot{z}_1 = \lambda_1 z_1 \qquad \lambda_1 \neq 0 \Rightarrow z_1(t) = e^{\lambda_1 t} z_1(0)$$

(1)

$$\dot{z}_2 = \lambda_2 z_2 \qquad \lambda_2 \neq 0 \Rightarrow z_2(t) = e^{\lambda_2 t} z_2(0)$$

(2)

$$(1) \Rightarrow e^{\lambda_1 t} = \left.\frac{z_1(t)}{z_1(0)}\right|_{z_1(0) \neq 0}$$

$$\Downarrow$$

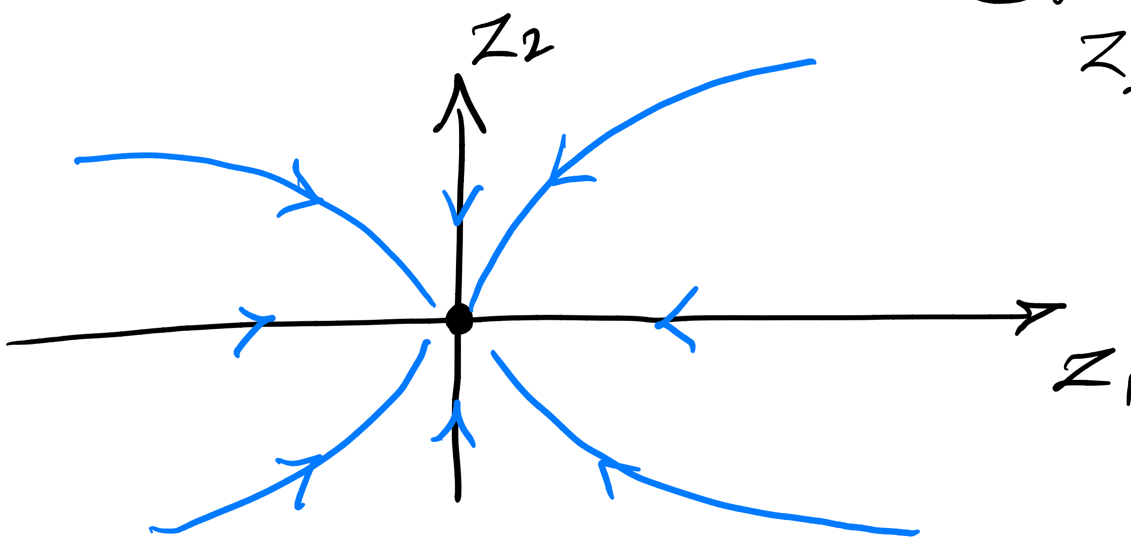
$$z_2(t) = z_2(0)\left[e^{\lambda_1 t}\right]^{\lambda_2/\lambda_1} = z_2(0)\left[\frac{z_1(t)}{z_1(0)}\right]^{\lambda_2/\lambda_1}$$

$$\Rightarrow \quad z_2(t) = c \; z_1(t)^{\lambda_2/\lambda_1}$$

$$c = \frac{z_2(0)}{[z_1(0)]^{\lambda_2/\lambda_1}}$$

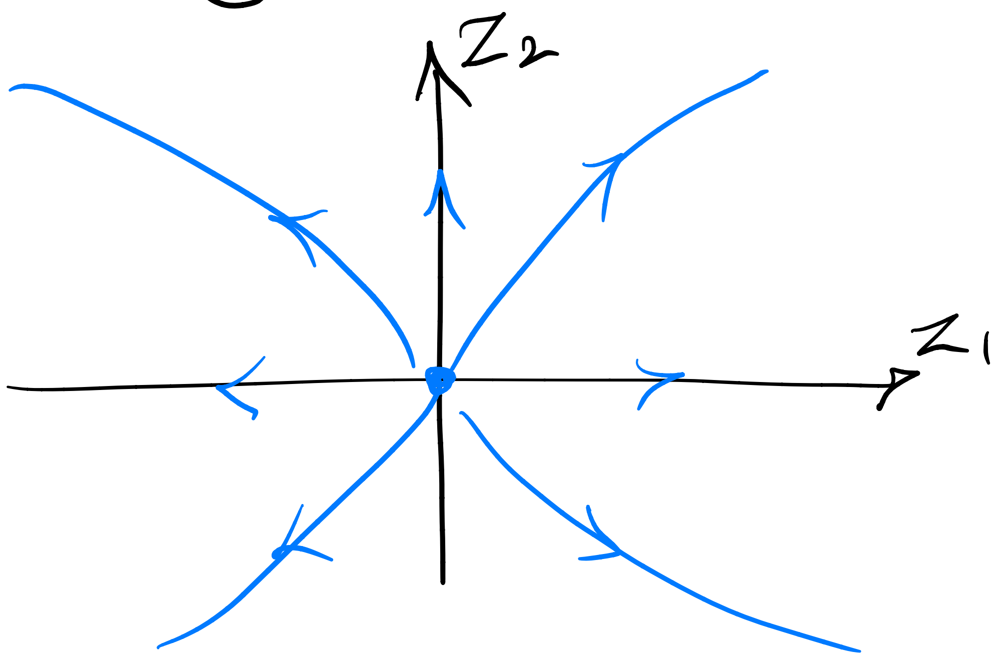
Assume $\lambda_1 < \lambda_2 < 0$ (e.g. $\lambda_1 = -2$, $\lambda_2 = -1$)

$$z_2 = c\, z_1^{1/2}$$

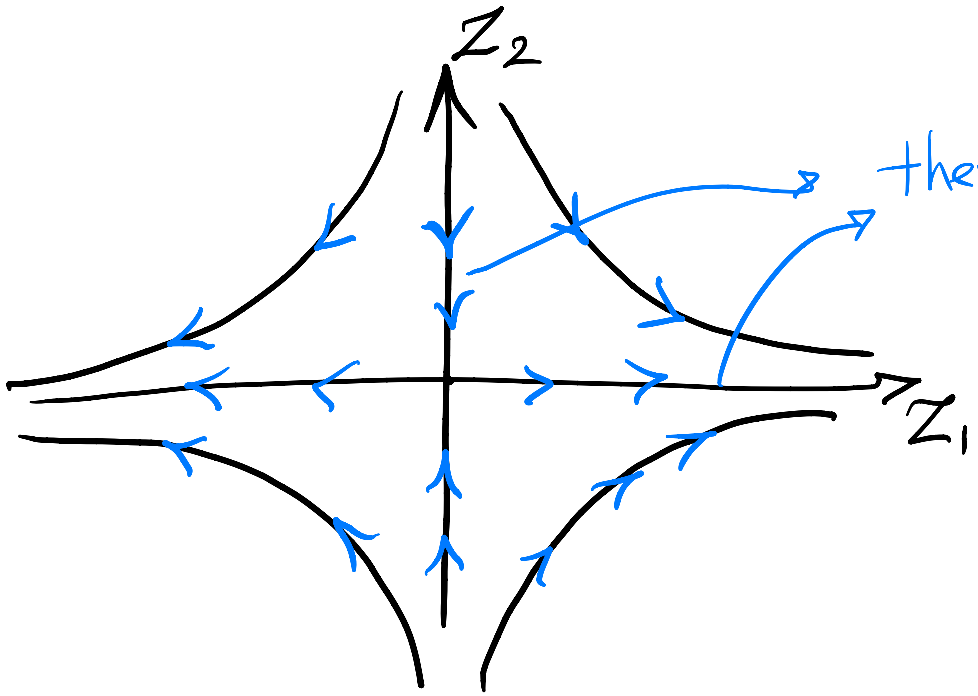


(stable node)

Now if $0 < \lambda_2 < \lambda_1$ we will have similar phase plane portrait but the direction of arrows will change and we will have an unstable node.

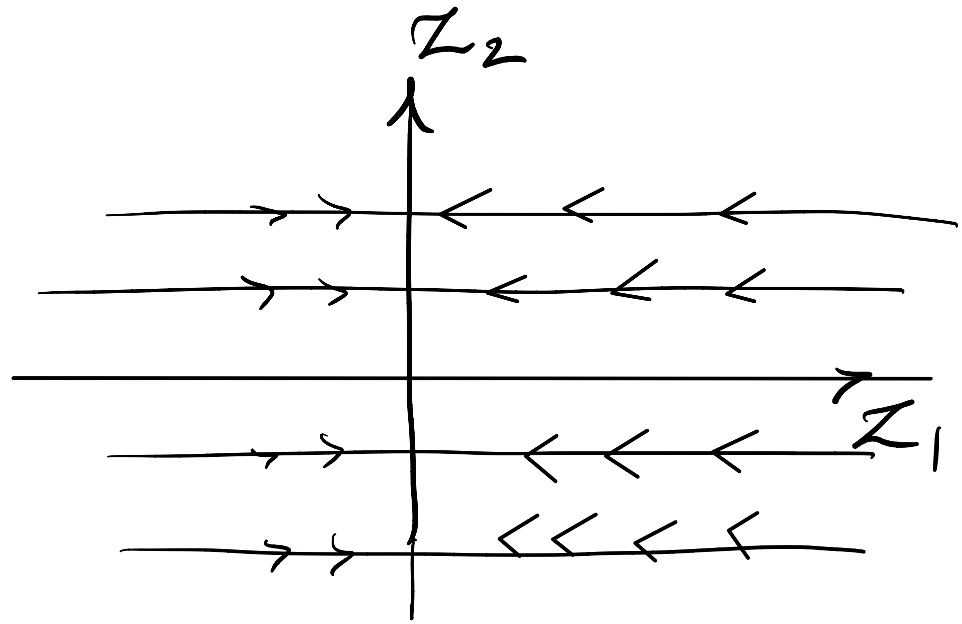


if we have $\lambda_2 < 0 < \lambda_1$ we will have saddle node.



these 2 directions are special because the corresponding e-vectors will be $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

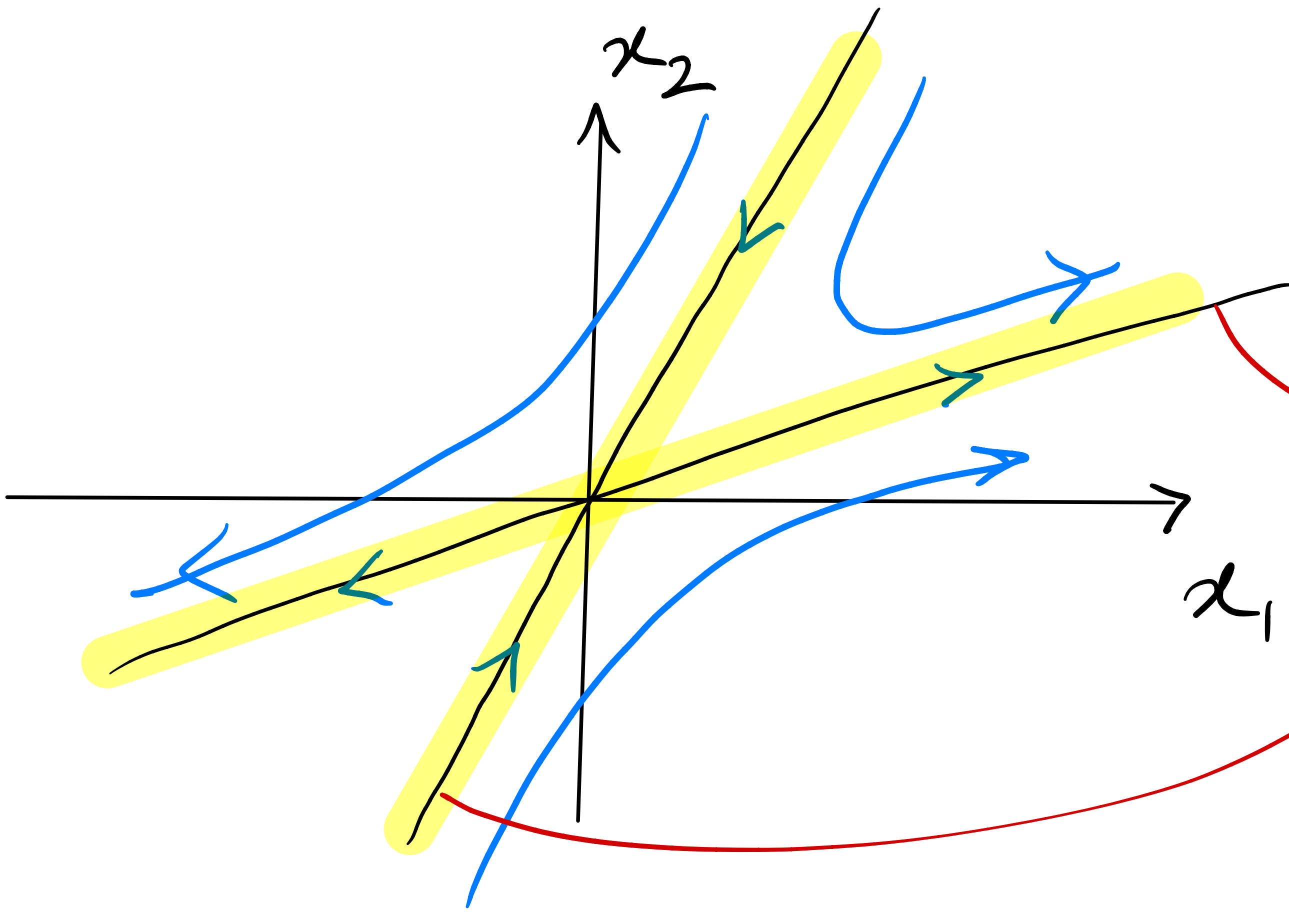
$\lambda_2 = 0$ $\lambda_1 < 0$



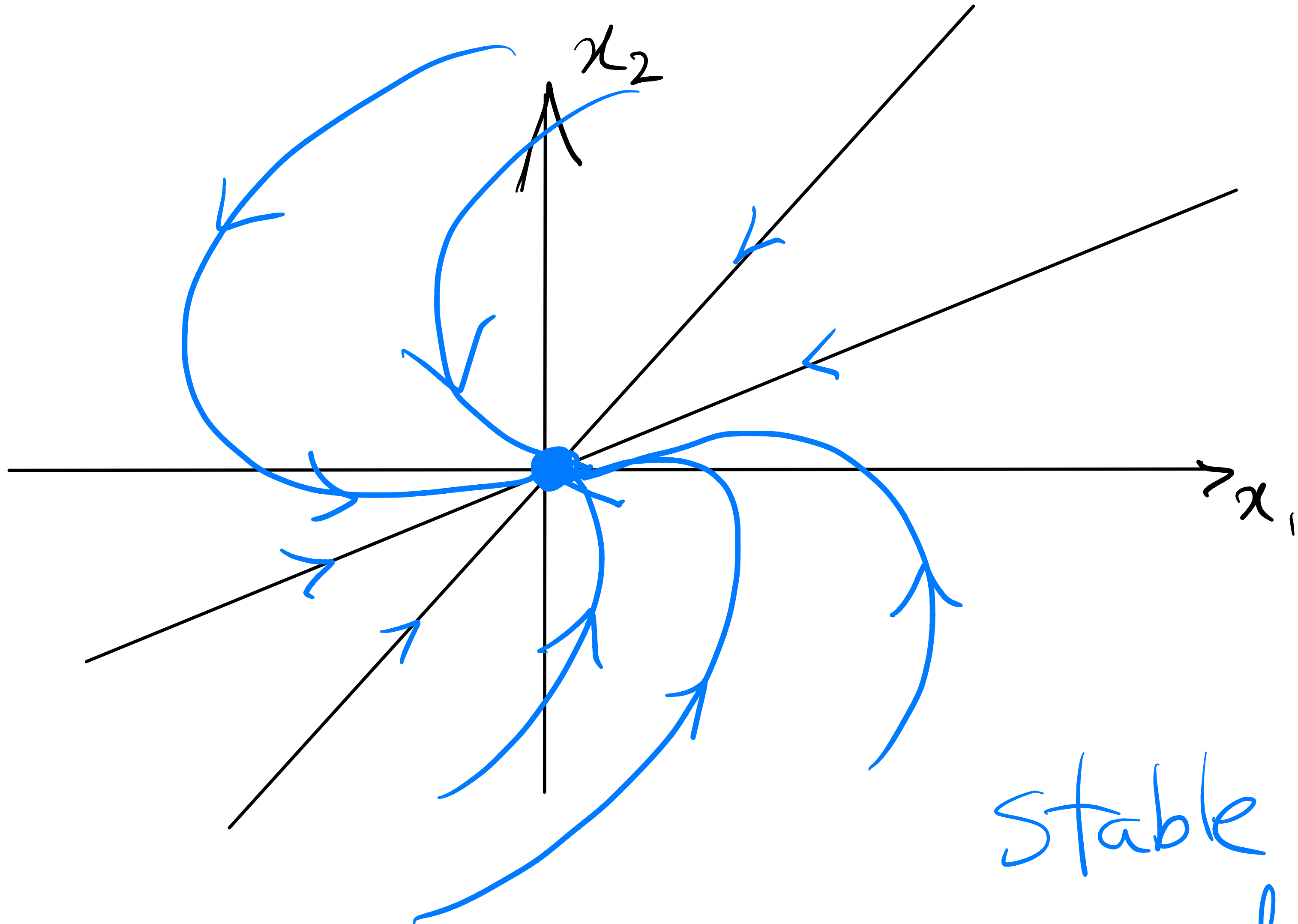
$$z(t) = T^{-1} x(t)$$

original coordinates

$$x(t) = T^{+1} z(t)$$



eigenvectors corresponding to $\lambda_1 = 0$ or $\lambda_2 = 0$



stable
node

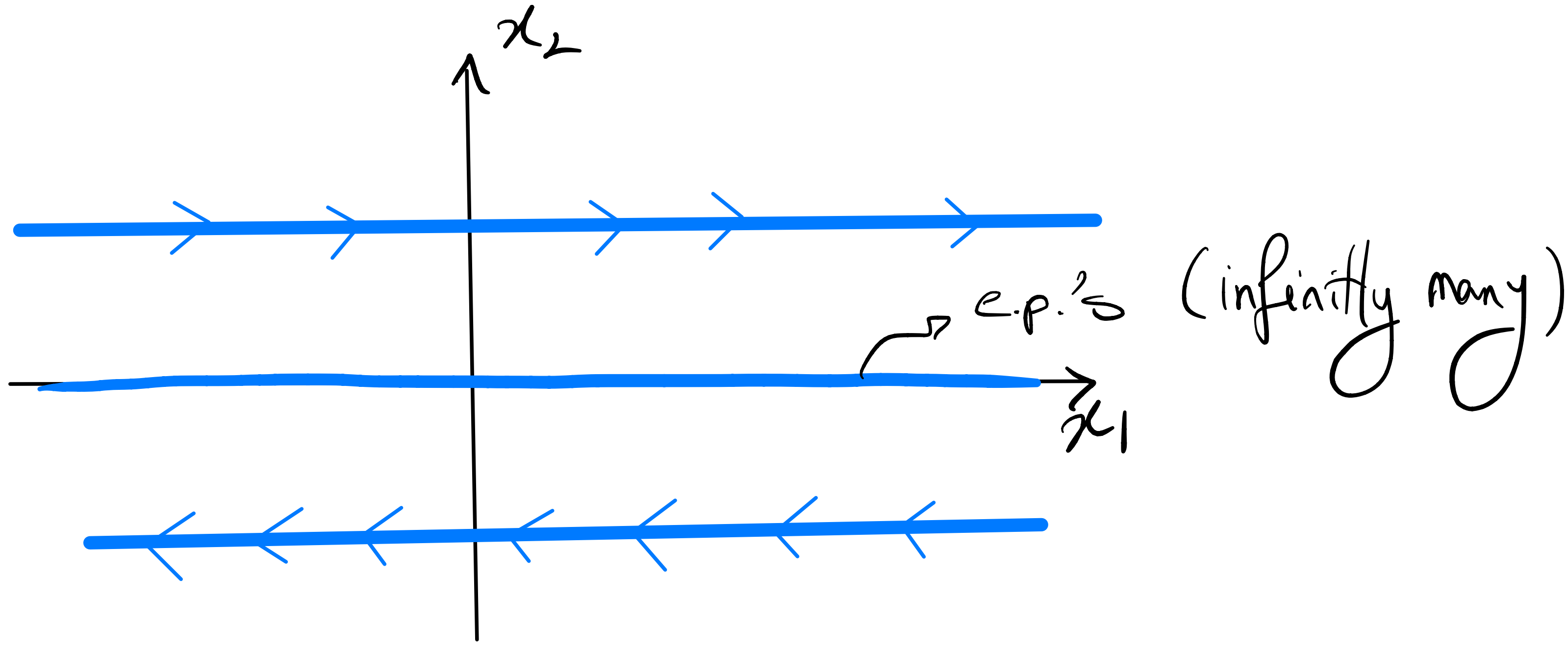
case 2) $\bar{A} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

An example with e-values at origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\lambda = 0 \quad \left(\bar{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$

$\dddot{y} = 0$ (double integrator) $\longrightarrow$ e.g. puck on frictionless surface

$$\left.\begin{array}{l} x_1 = y \\ x_2 = \dot{y} \end{array}\right\} \begin{array}{l} \dot{x}_1 = x_2 \longrightarrow x_1(t) = x_1(0) + x_2(0)t \\ \dot{x}_2 = 0 \longrightarrow x_2(t) = \text{Const} = x_2(0) \end{array}$$



x_2
x_1
e.p.'s (infinitly many)

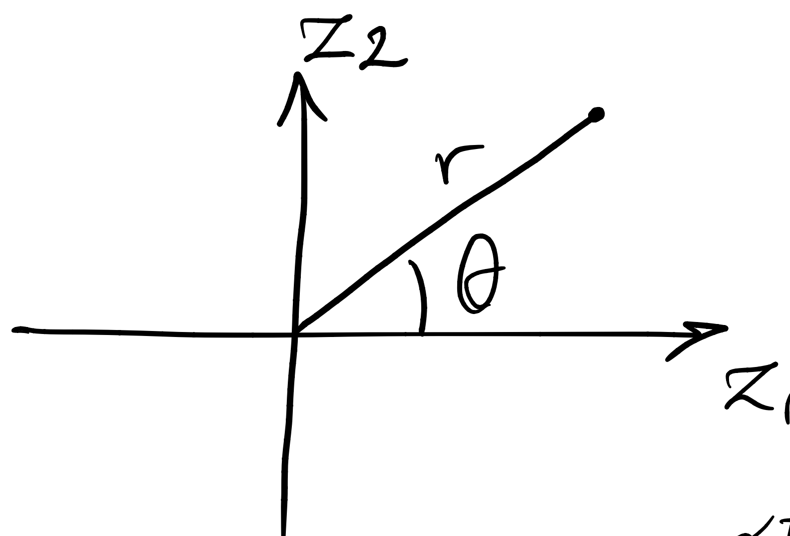
case 3) Complex-conjugate e-values

$$\left.\begin{aligned} \dot{z}_1(t) &= \alpha z_1(t) - \beta z_2(t) \\ \dot{z}_2(t) &= \beta z_1(t) + \alpha z_2(t) \end{aligned}\right\} \quad \lambda_{1,2} = \alpha \pm j\beta$$

polar coordinates:

$$z_1 = r\cos\theta$$

$$z_2 = r\sin\theta$$



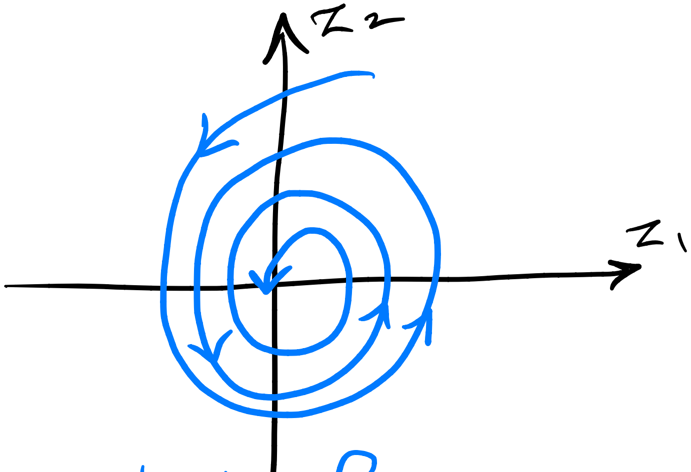
In polar coordinates

$$\dot{r} = \alpha r \longrightarrow r(t) = e^{\alpha t} r(0)$$

$$\dot{\theta} = \beta \longrightarrow \theta(t) = \theta_0 + \beta t$$

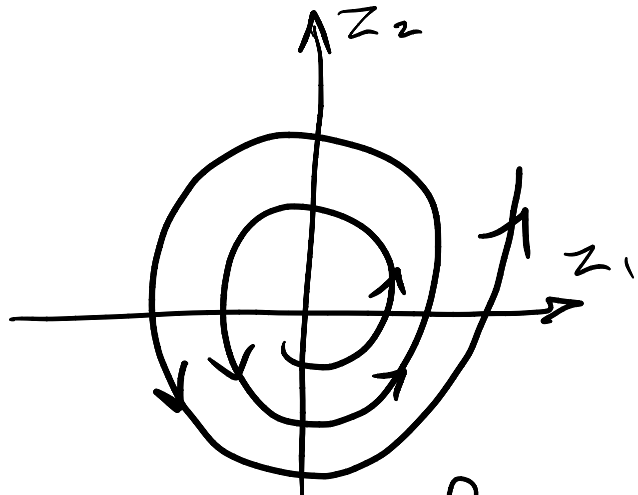
$\beta > 0$

a) $\alpha < 0$



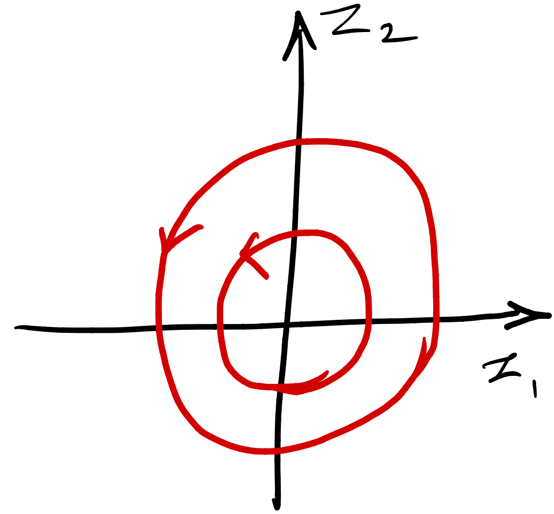
stable focus

b) $\alpha > 0$



unstable focus

c) $\alpha = 0$



center

not limit cycle

not an isolated closed trajectory

Why did we spend so much time talking about phase portraits of LTI systems in nonlinear systems class?

phase portraits of nonlinear systems with hyperbolic equilibrium points (linearization around e.p doesn't have e-values on $j\omega$-axis, i.e., if $\dot{x}=f(x)$, $f(\bar{x})=0$, $\mathrm{Re}\left\{\lambda_i\left(\frac{\partial f}{\partial x}\Big|_{\bar{x}}\right)\right\} \neq 0$)

can be related to phase portrait of

a corresponding linear system via Harman-Grobman theorem.

- Cases that are covered by this theorem:

  ✓ stable/unstable nodes

  ✓ saddle

  ✓ focus

Hartman-Grobman Thm:

If $\bar{x}$ is a hyperbolic e.p. of $\dot{x}=f(x)$, $x(t)\in\mathbb{R}^n$, then there is a homomorphism from a neighbourhood of $\bar{x} \to \mathbb{R}^n$ that maps trajectories of $\dot{x}=f(x)$ to those of corresponding linearization.

Aside

homomorphism: a cts map with cts inverse