Lecture 20

04/12/2021

Last time: Integrator backstepping

Today: Control Lyapunov Functions

Input-output stability (a start)

Ex.

$$\dot{x}_1 = x_1^2 + x_2$$

$$\dot{x}_2 = u$$

Objective: $x_1(t) \xrightarrow{t \to \infty} r(t)$

reference signal

$$z_1 := x_1 - r$$

$$\dot{z}_1 = \dot{x}_1 - \dot{r} = x_1^2 + x_2 - \dot{r}$$

$$= (z_1 + r)^2 + x_2 - \dot{r}$$

need to know r and its derivative

$$\vdots$$

$$V_1(z_1) = \frac{1}{2} z_1^2$$

... same as before

Control Lyapunov Function (CLF)

$$\dot{x} = f(x) + g(x)u$$

$u = \alpha(x)$: control law that satisfies

$$\underbrace{\frac{\partial V}{\partial x} f(x)}_{L_f V(x)} + \underbrace{\frac{\partial V}{\partial x}}_{L_g V(x)} g(x)\alpha(x) < 0$$

$$\Rightarrow \quad L_f V(x) + L_g V(x) \cdot \alpha(x) < 0$$

If there are states x st. $L_g V(x) = 0$ and $L_f V(x) < 0$ at these points then contro does not enter into the equations for the derivative of V.

Propose $V(x)$:

$$\dot{V} = \frac{\partial V}{\partial x}\left[f(x) + g(x)u\right] = L_f V(x) + \underbrace{L_g V(x)}_{\text{when } = 0} u$$

$\Rightarrow L_f V(x) < 0$

Def. A smooth positive definite radially unbounded function $V(x)$ is called **CLF** for system $\dot{x} = f(x) + g(x)u$ if for all $x \neq 0$, $L_g V(x) = 0 \Rightarrow L_f V(x) < 0$.

→ There are many ways to generate control laws from CLFs, e.g., Sontag's Formula

$$\alpha(x) = \begin{cases} 0 & L_g V(x) = 0 \\ -\dfrac{L_f V(x) + \sqrt{(L_f V(x))^2 + (L_g V(x))^2}}{L_g V(x)} & \text{o.w.} \end{cases}$$

Comments

This control law is continuous if the CLF satisfies the "small control property":

$\forall \varepsilon > 0 \;\; \exists \delta > 0$ st. for each $x \neq 0$ with $\|x\| \leq \delta$

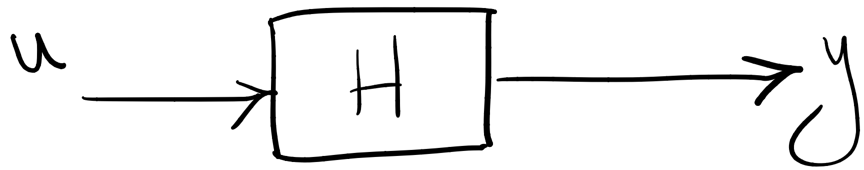
we can find u with $\|u\| \leq \varepsilon$ st.

$$L_f V(x) + L_g V(x)\, u < 0$$

Note * System $\dot{x} = f(x) + g(x)u$ is stabilizable by a continuous state-feedback contro <u>only if</u> it has a CLF.

* If V is a CLF for $\dot{x} = f(x) + g(x)u$ and we have $\partial V / \partial x (0) = 0$, Sontag's formula has a gain margin of $[1/2, \infty)$, i.e., $u = K\alpha(x)$ is stabilizing for $K \geq 1/2$.

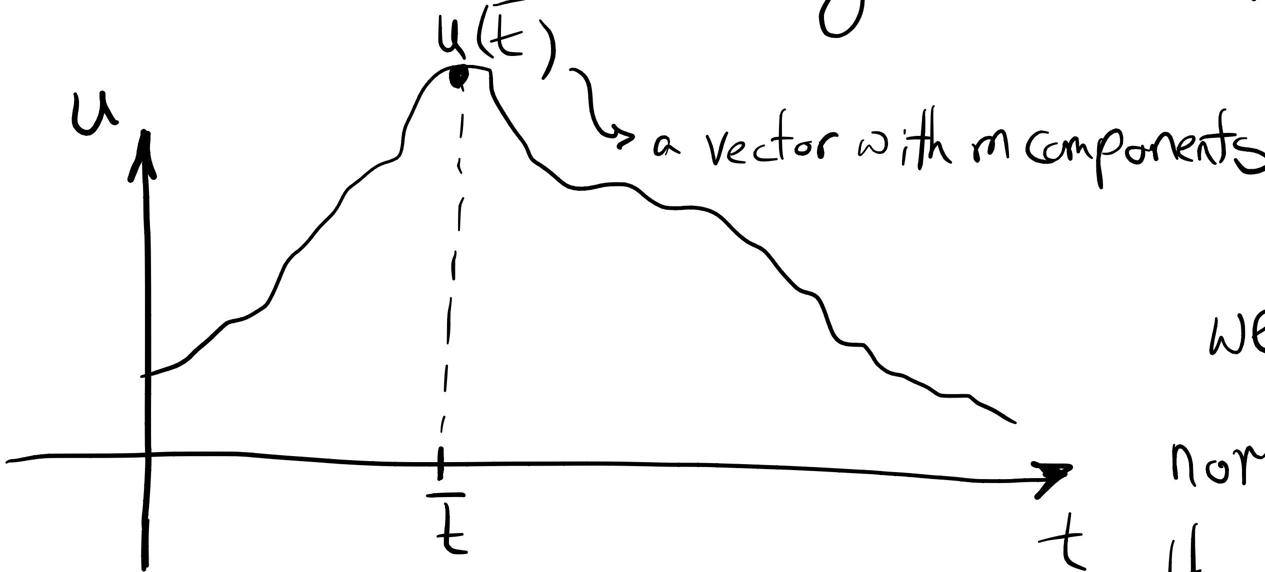
Input-output stability



u: input signal

y: output signal

H: system (mapping from inputs to outputs)



we can use traditional norms to get a measure of the signal at a specific time.

Lp-norm for SISO system

$$\|u\|_p^p = \int_0^\infty |u(t)|^p dt$$

Lp space: A signal $u \in L_p$; $p \in [1, +\infty)$

if

$$\int_0^\infty |u(t)|^p dt < +\infty$$

$$\sup_t |u(t)| < +\infty$$

Ex. $u(t) \in \mathbb{R}^m$; $p=2$

$$\|u\|_2^2 = \int_0^\infty u^*(t)\, u(t)\, dt$$

aside

$\|f\|_p$ signal norm

$\|f(t)\|_p$ vector norm

Important inequalities

$$\int_0^\infty |f(t) g(t)|\, dt \leq \|f\|_p \|g\|_q$$

Holder inequality

$$\frac{1}{p} + \frac{1}{q} = 1$$

special case: Cauchy Schwartz $p = q = 1$

Minkowski

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Ex. $u(t) = \dfrac{1}{1+t} \qquad t \geq 0$

$\|u\|_\infty = 1 \qquad$ (achieved @ $t=0$)

Is $u \in L_1$? $\qquad \displaystyle\int_0^\infty \frac{dt}{1+t} = \log(t+1)\Big|_0^\infty = +\infty$

No, but $u \in L_2$

Extended L_p space:

$$L_{p_e} = \left\{ u \; ; \; u_T \in L_p \; , \; \forall T \in [0, +\infty) \right\}$$

$$u_T(t) = \begin{cases} u(t) & 0 \leq t \leq T \\ 0 & \text{o.w.} \end{cases}$$

truncation operator

Ex. $\frac{1}{1+t} \in L_{1e}$ but not L_1.

Definition: An operator $H : L_{pe} \to L_{pe}$ is causal

if
$$[H(u)]_T = [H(u_T)]_T$$
$$\forall u \in L_{pe}$$
$$\forall T \in [0, +\infty)$$

Definition A causal operator $H : L_{pe} \to L_{pe}$

is ==L_p-stable== if

$$\|y_T\|_p \leq \underbrace{\alpha}_{\text{class } K \text{ function}}(\|u_T\|_p) + \underbrace{\beta}_{\text{const.}}$$

for all $u \in L_{pe}$ and $T \in [0, \infty)$

* The operator H is finite-gain L_p stable if:

$$\|y_T\|_p \leq K \|u_T\|_p + \beta$$

Const.

The smallest possible K is called L_p gain of H.

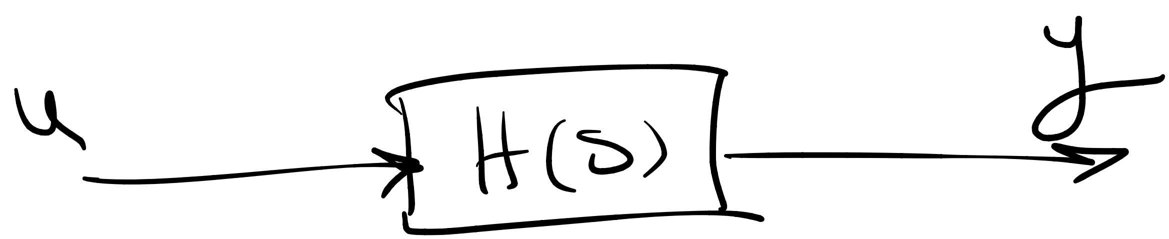
Note: If H is L_p stable,

$$u \in L_p \Longrightarrow y \in L_p$$

Question. When is stability in Lyapunov sense different than input-output sense?

Input-output stability $\longrightarrow$ all e-values in LHP

but there may be pole zero cancellations.



$$\left.\begin{array}{l} \text{If} \quad \mathrm{Re}\{\lambda_i(A)\} < 0 \\ \quad \forall\, i = 1, \dots, n \end{array}\right\} \Rightarrow L_p \text{ stability}$$

BIBO stability

$$y(t) = \int_0^\infty h(t-\tau)\,u(\tau)\,d\tau = \int_0^\infty h(\tau)\,u(t-\tau)\,d\tau$$

$$|y(t)| \leqslant \int_0^\infty |h(\tau)|\,d\tau \; \underbrace{\sup_t |u(t)|}_{L_\infty \text{ input signal}}$$

for linear systems with $h \in \mathcal{L}_1$ for any $p \in [1, \infty)$ we have $\|y_T\|_p \leq \|h\|_1 \|u_T\|_p$.

* What is the induced $\mathcal{L}_2$ gain for linear system?

H_∞ norm

aside
H_∞ norm: $\sup_\omega |H(j\omega)|$; SISO
$\sup_\omega \sigma_{max}(H(j\omega))$; MIMO

this is not the tightest bound but it is when considering input-output stability...

aside
induced norm of H
$\sup \dfrac{\|Hu\|_2}{\|u\|_2}$
self study...

an induced L_p gain $\leq \|h\|_1$

Tightest bound is obtained for $p = \infty$

Lyapunov-like conditions for L_p stability

(Khalil, Thm 5.1)

Consider the state-space model

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases} \qquad (*)$$

If the following two assumptions hold, then the system (*) is finite gain, L_p stable for any $p \in [1, \infty)$

A1) There is Lyapunov function $V(x)$ s.t.

$$\text{stability}\left\{\begin{array}{l} c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2 \\[2ex] \dfrac{\partial V}{\partial x} f(x, \underset{\substack{\downarrow \\ u}}{0}) \leq -c_3 \|x\|^2 \end{array}\right.$$

$$\Big\| \underbrace{\frac{\partial V}{\partial x}}_{\nabla V} \Big\| \leq c_4 \|x\|$$

A2) $\|f(x,u) - f(x,0)\| \leq c_5 \|u\|$

$$\|h(x,u)\| \leq c_6 \|x\| + c_7 \|u\|$$

for some constants $c_i > 0 \quad i = 1, \dots, 7$

Ex. $\begin{cases} \dot{x} = -x - x^3 + u \\ y = \tanh(x) + u \end{cases}$

if u is zero $\longrightarrow$ system is stable ?

(A1) holds with $V(x) = \frac{1}{2} x^2$

as a Lyap. function

A2) $\|f(x,u) - f(x,0)\| = \|u\|$

holds with $C_5 = 1$

$$h(x,u) = \tanh(x) + u$$

$|\tanh(x) + u|$

$$\|h(x,u)\| = \|\tanh(x) + u\| \leq |\tanh(x)| + |u|$$

$$\leq |x| + |u|$$

$$C_6 = 1 \quad C_7 = 1$$

Ex. $\dot{x} = -x + x^2 u$

$y = x$

A1) $u = 0 \longrightarrow$ stable ✓ $\begin{cases} \dot{x} = -x \\ y = x \end{cases}$

but what about input-output sense?

Choose $u = \text{const.} \Rightarrow$ finite escape time

$\Rightarrow$ not L_p stable!

A2) $|f(x,u) - f(x,0)| = |\cancel{-x} + x^2 u - (\cancel{-x}) - x^2 \cdot 0| = |x^2 u|$

$= |x^2||u|$

$c_5 = c_5(x)$

not uniform const.