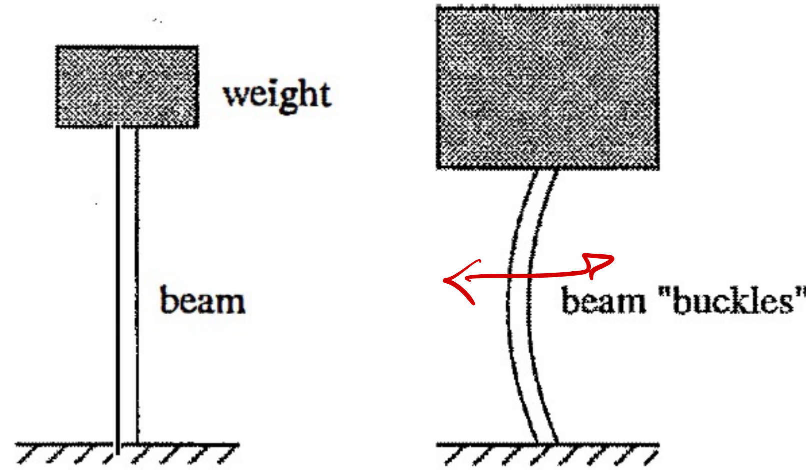
Nonlinear Sys. Quiz 1 Feb. 03, 2021

Problem 1: The picture below shows a weight placed on top of a metal beam. As the weight is increased, the beam cannot remain vertical and buckles. Consider the weight as a parameter and the deflection of the beam from vertical as the dynamical variable x. What kind of bifurcation best describes the nonlinear dynamics of x?



left-right physical symmetry $\longrightarrow$ pitchfork bifurcation

Problem 2: The first-order system below undergoes a saddle-node bifurcation as the parameter r is varied. Find the critical value of r at which the bifurcation happens.

$$\dot{x} = r - x - e^{-x} \longrightarrow \text{nonlinear system}$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}} = 0 \longleftarrow \text{for critical } r = r_c \qquad \partial f/\partial x = -1 + e^{-x} = 0 \Rightarrow \bar{x} = 0$$

$$f(\bar{x}) = 0 \Rightarrow r - \bar{x} - e^{-\bar{x}} = r - 1 = 0 \Rightarrow r_c = 1$$

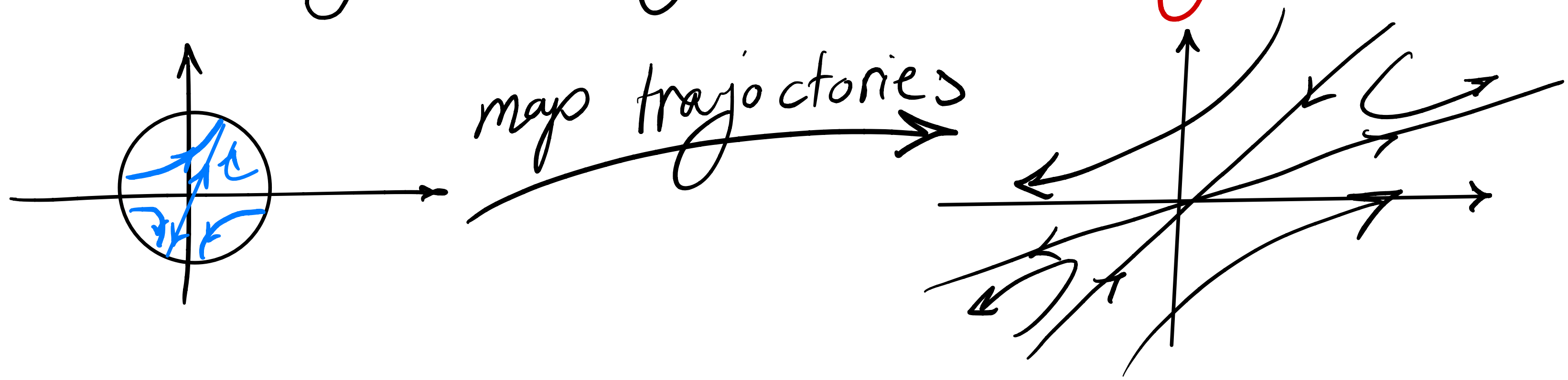
Lecture 5 02/03/2021

Last time: Pitchfork bifurcation
Phase portrait of 2nd-order systems
Hartman - Grobman Thm.

Today: Bendixon's criterion (absence of periodic orbits in 2nd-order sys.)
Application's of Bendixon's Thm.
Poincare - Bendixon Thm (existence of periodic orbits in 2nd-order sys)

Hartman-Grobman Thm

For $\dot{x}=f(x)$, given a neighborhood of a hyperbolic e.p.



linearization

Note! Requirement about $\bar{x}$ being $\underbrace{\text{hyperbolic}}_{\text{not on imag. axis}}$ is of essence.

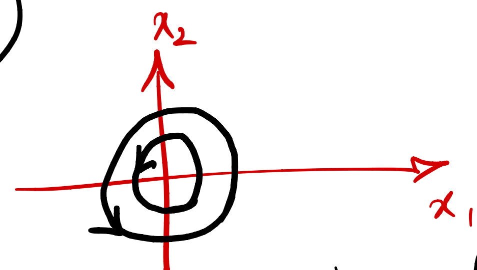
Ex.

$$\dot{x}_1 = -x_2 + a x_1 (x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + a x_2 (x_1^2 + x_2^2)$$

polar coord. $\longrightarrow$ $\dot{r} = a r^3$, $\dot{\theta} = 1$

$$\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

linearization around $\bar{x}$ $\longrightarrow$ $A = \left.\dfrac{\partial f}{\partial x}\right|_{\bar{x}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

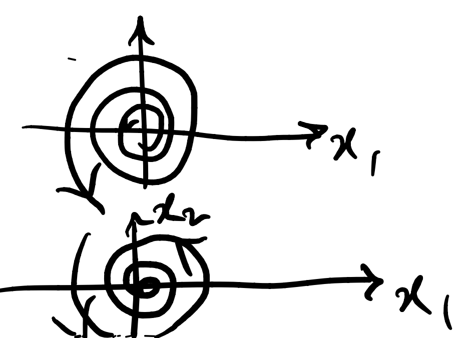


harmonic oscilator

center type of e.p. (not hyperbolic)

for nonlinear system: $a > 0$ unstable focus

$a < 0$ stable focus



Mapping does not exist

Bendixon's Criterion:

absence of periodic orbits
- → limit cycles (Van der pol ...)
- → neutrally stable cycles (pendulum)

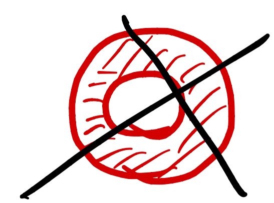
$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

$$f := \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\text{div}(f) = \nabla \cdot f = \begin{bmatrix} \partial/\partial x_1 & \partial/\partial x_2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

If div(f) is not identically equal to zero and does not change sign in a simply connected region D, then there are no periodic orbits in D.

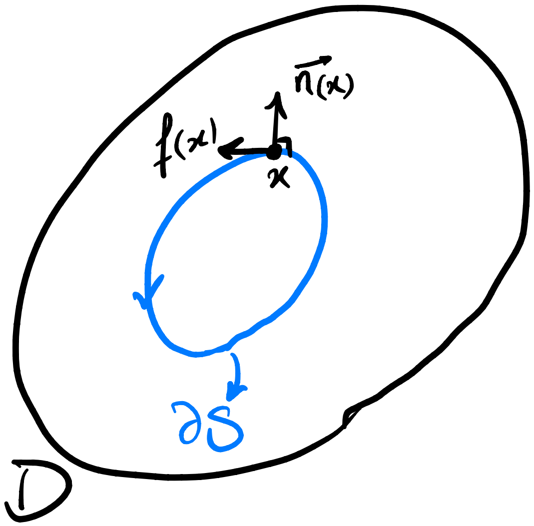
region with no holes



sufficient condition (not necessary)

$\Longrightarrow$ $\quad\quad$ $\not\Longleftarrow$

proof → Assume there is a periodic orbit (closed trajectory) with D



Following Green's thm:

$$\underbrace{\int_{\partial S} f(x)\cdot\vec{n}\,dl}_{0} = \iint_{S} \nabla\cdot f(x)\,dx$$

If $\nabla\cdot f(x)$ does not equal zero or change sign the RHS will never be zero

Thus, there cannot be such a periodic orbit in D ($\#$ contradiction)

Ex1. Given $A \in \mathbb{R}^{2\times 2}$ unless $\text{trace}(A) = 0 \Rightarrow$ no periodic orbits

$$\dot{x}_1 = a x_1 + b x_2$$
$$\dot{x}_2 = c x_1 + d x_2 \iff A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\text{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = a + d = \text{trace}(A)$$

Note! Only way a linear system can have periodic orbits is if $\text{trace}(A) = 0$

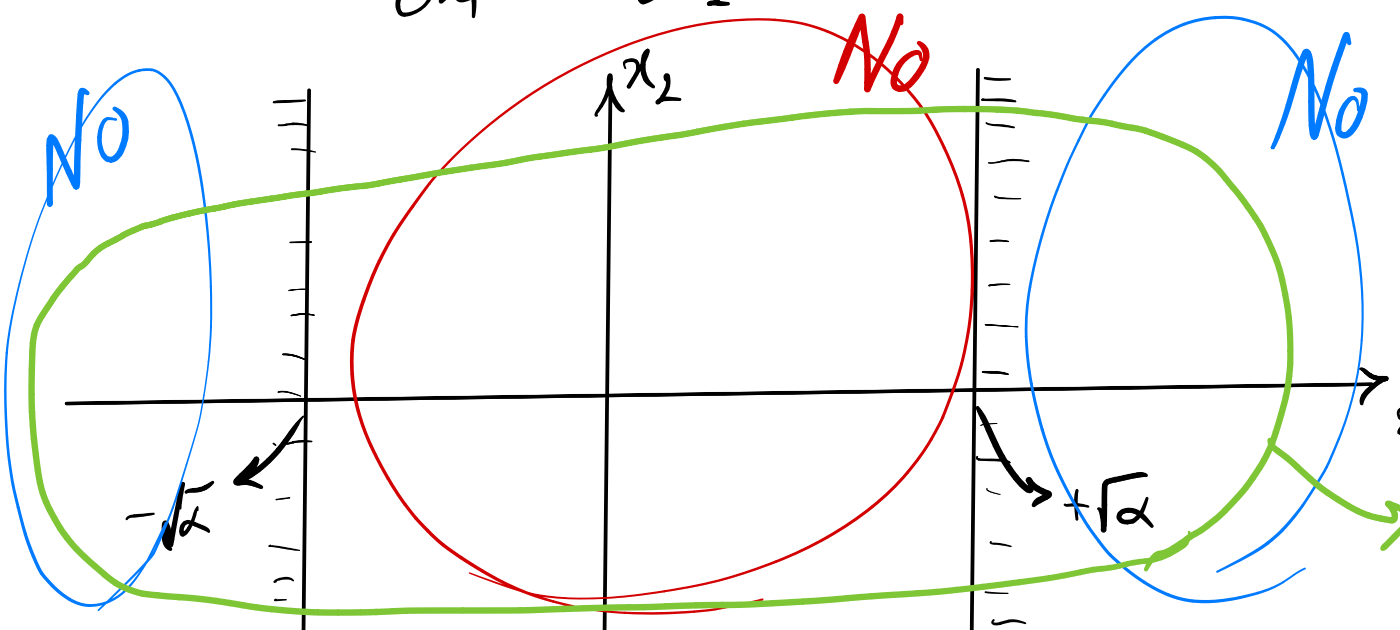
Note! In general Globally Asymptotically Stable (GAS) unique e.p. rule out the presence of periodic orbits.

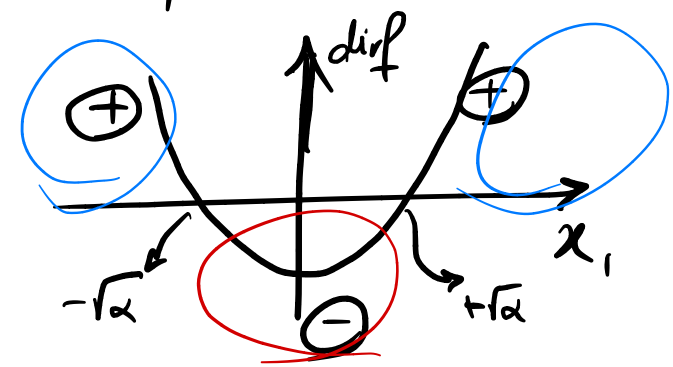
Note! This Thm. is only providing a sufficient condition.
$\underbrace{\qquad\qquad\qquad}_{\text{(conservative)}}$

Ex.

$$\dot{x}_1 = x_2 = f_1$$
$$\dot{x}_2 = -\alpha x_2 + x_1 - x_1^3 + x_1^2 x_2 = f_2 \quad ; \quad \alpha > 0$$

$$\mathrm{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0 - \alpha + x_1^2 \longrightarrow \mathrm{div} f = x_1^2 - \alpha$$





The thm in this region is inconclusive!

* Bendixon's Criterion can only tell you about the absense of periodic orbits. It cannot tell you if you have them; it can only rule out their presence.

Invariant sets

$$\dot{x} = f(x), \quad x(0) = x_0$$

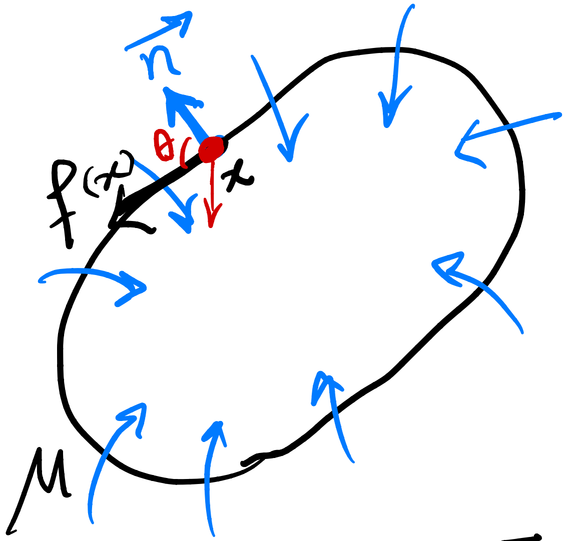
$\Phi(t, x_0)$: trajectory starting at x_0 describing system's evolution over time.

A set M is positively invariant if $x_0 \in M \Rightarrow \Phi(t, x_0) \in M \quad \forall t \geq 0$

(If you start in the set M, you will stay in the set as time progresses)

What condition should be satisfied for this to happen?

$f(x)$ should always be pointing into the set.



$f(x)$ is tangential to the trajectory at point x.

$$\underbrace{f^T(x) \cdot \vec{n}}_{} = 0$$ on the boundary of set.

$$|\vec{f}||\vec{n}|\cos\theta = \langle \vec{f}, \vec{n} \rangle$$

$\Downarrow$

positively invariant set

Ex. Predator-prey model

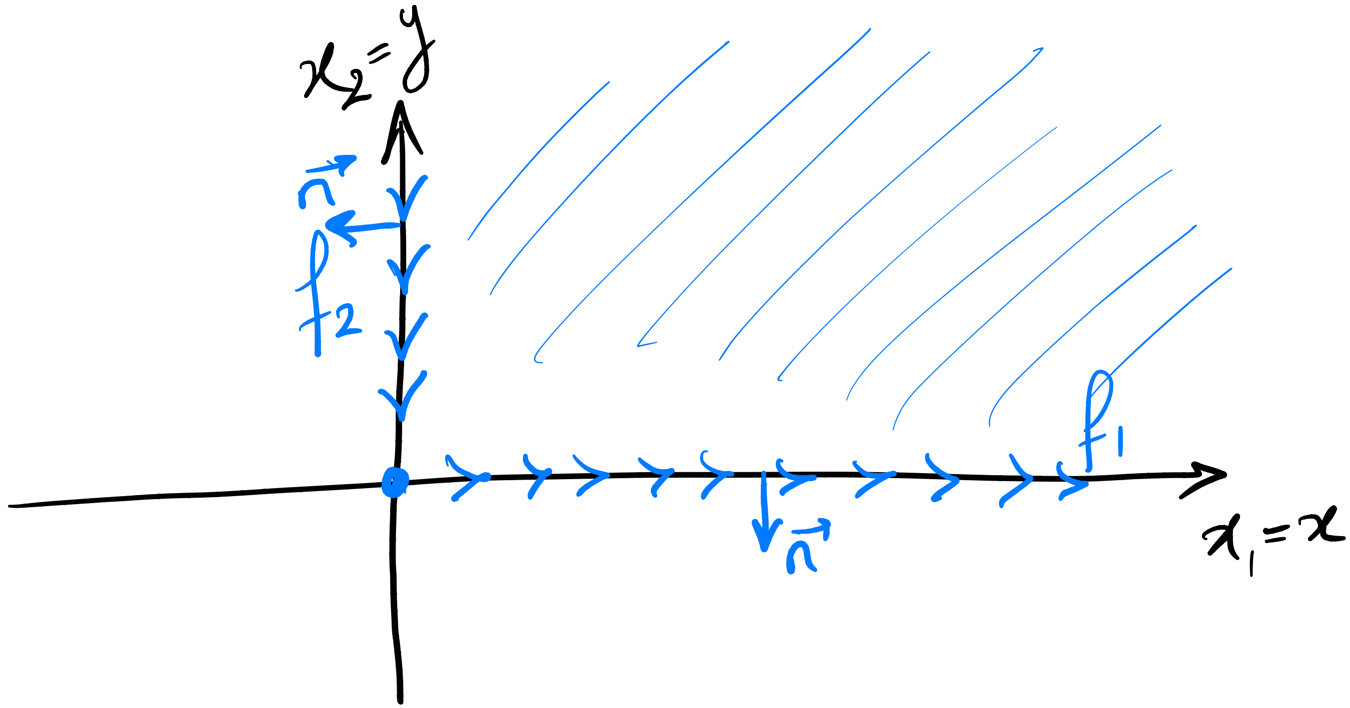
$$\begin{cases} \dot{x} = (a - by)x & \text{prey} \\ \dot{y} = (cx - d)y & \text{predator} \end{cases}$$

a, b, c, d positive constants

Sanity Check!

if $y = 0 \Rightarrow \dot{x} = ax$

if $x = 0 \Rightarrow \dot{y} = -dy$



$$\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\bar{x} = \begin{pmatrix} d/c \\ a/b \end{pmatrix}$$

$$f_1(x, y=0) = ax$$

$$f_2(x=0, y) = -dy$$

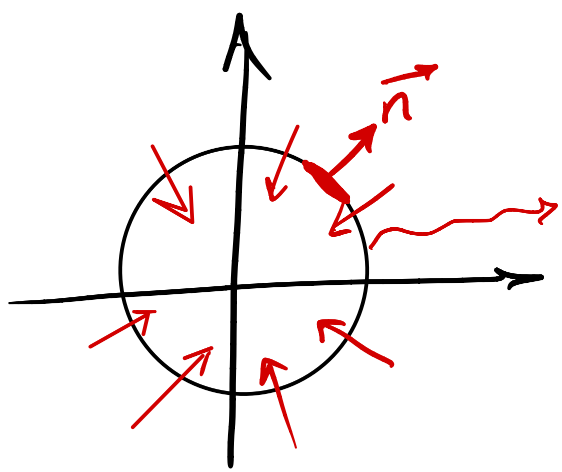
clearly $\vec{\underline{f}}^{\,T} \vec{n} = \vec{f} \cdot \vec{n} = 0$ along boundary of M which means set M is positively invariant.

Ex. $\dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2)$ $\qquad \bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2)$

We'll show that $B_r := \{ x \in \mathbb{R}^2 / \ x_1^2 + x_2^2 \leq r^2 \}$

is positively invariant for sufficiently large r (to be determined)



$V(x) = x_1^2 + x_2^2 = r^2$

$$\nabla V(x) = \begin{bmatrix} \partial V / \partial x_1 \\ \partial V / \partial x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$f(x)\cdot \nabla V(x) = f_1 \, \partial V/\partial x_1 + f_2 \, \partial V/\partial x_2$$

$$= 2x_1\left(x_1 + x_2 - x_1(x_1^2+x_2^2)\right) + 2x_2\left(-2x_1 + x_2 - x_2(x_1^2+x_2^2)\right)$$

$$= -2(x_1^2+x_2^2)^2 + 2x_1^2 + 2x_2^2 - 2x_1x_2$$

aside

$$(x_1^2+x_2^2)^2 \geq 0$$

$$\Downarrow$$

$$x_1^2+x_2^2 \geq -2x_1x_2$$

$$\leq -2(x_1^2+x_2^2)^2 + 2(x_1^2+x_2^2) + x_1^2 + x_2^2$$

$$= -2(x_1^2+x_2^2)^2 + 3(x_1^2+x_2^2)$$

$$= -2r^4 + 3r^2 = -2r^2(r^2 - 3/2) \overset{?}{\leq} 0$$

Yes if $r^2 \geq 3/2$ (or $r \geq \sqrt{3/2}$)