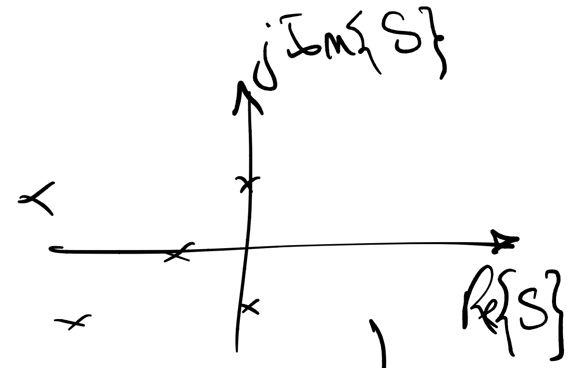
Lecture 08

02/22/2021

Last time: Center manifold theory



spectrum of linearization around e.p.

- k evalues on $j\omega$ axis
- $n-k$ evalues in LHS

$$\begin{cases} \dot{y} = A_1 y + g_1(y,z) \\ \dot{z} = A_2 z + g_2(y,z) \end{cases}$$

Then, stability properties characterized by stability properties of the reduced order system:

$$\dot{y} = A_1 y + g_1(y, h(y))$$

$h(y)$: center manifold

Today: existence + uniqueness of sol'n

continuous dependence on IC's/Parameters

Existence and uniqueness

$$\dot{x} = f(t, x) \quad , \quad x(t_0) = x_0$$

In MECH 6300 (linear systems) we restricted $f(t,x)$ to

$$f(t,x) = A(t)\,x \;\longrightarrow\; \text{time varying}$$

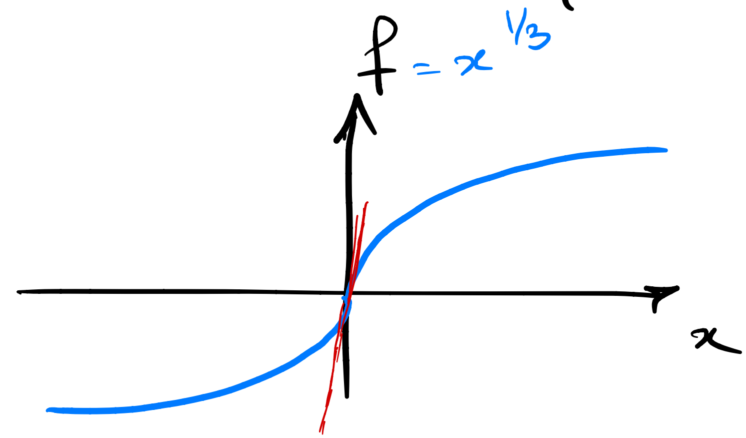
$$f(x) = Ax \;\longrightarrow\; \text{time invariant} \qquad \longrightarrow \quad x(t) = e^{At} x_0$$

piecewise continuity (→ overt) was all we needed for proving existence and uniqueness in the linear case.

But what can be said about nonlinear case?

We'll assume that $f(t,x)$ is piecewise cts function of time $\longrightarrow$ we will discuss conditions on the dependence on x that guarantee existence + uniquence.

Ex. $\dot{x} = x^{1/3}$ $\quad x(0) = x_0$



for $x_0 = 0$ we have 2 sol'ns!

$x(t) \equiv 0$ $\quad (1)$

$$x^{-1/3} dx = dt \Rightarrow \frac{3}{2} x^{2/3} \Big|_0^{x(t)} = t \Big|_0^t \Rightarrow x(t) = \left(\frac{2t}{3}\right)^{3/2} \quad (2)$$

$$\frac{\partial f}{\partial x}\Big|_{x=0} = x^{-2/3}\Big|_0 = \infty$$

Therefore, if f is a cts function on x

$$\not\Downarrow$$

existence & uniqueness

Issue: infinite slop of f at the origin

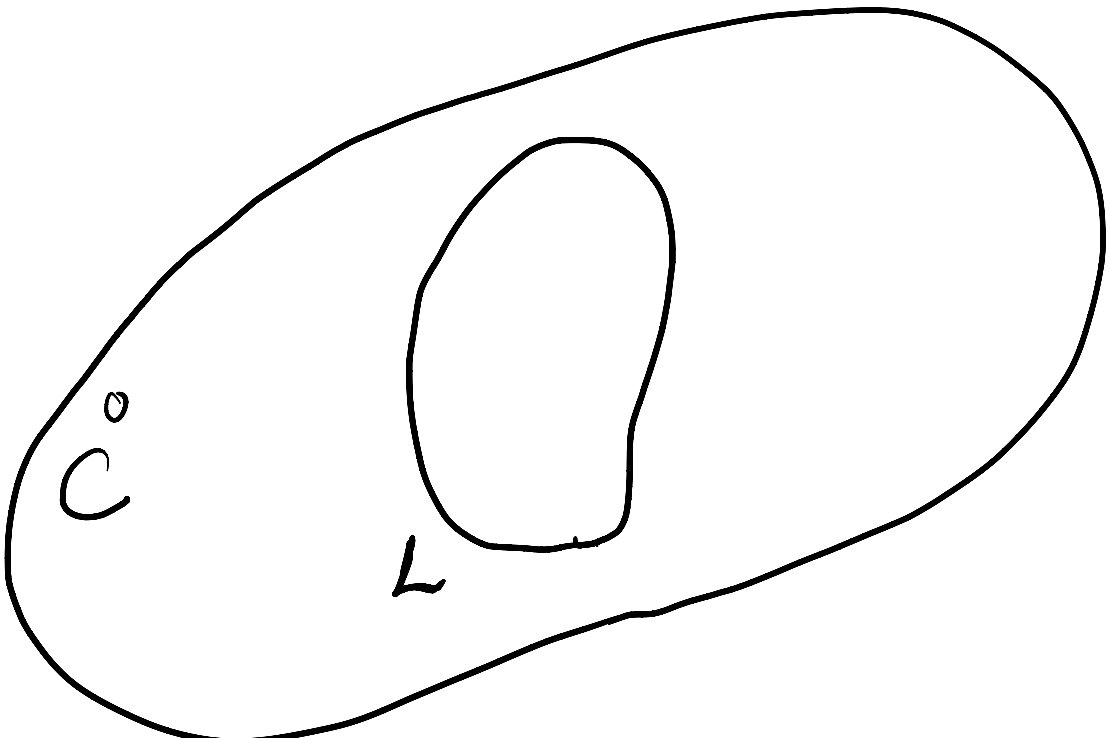
Fact: If $f(t,x)$ is cts in x $\Rightarrow$ there is a sol'n on $[0, t_f)$

but, it may not be unique!

We need to further restrict classes of admissible functions f in our statements.

$f \in C^0$

$f \in C^1$



Lipschitz continuity:

$$\|f(t,x) - f(t,y)\| \leq L \, \|x-y\|$$

* If this holds for all t and for all points in a certain neighborhood of an arbitrary point $\bar{x} \in \mathbb{R}^n$ for some L (Lipschitz constant) then f is locally Lipschitz (in x).

* if this holds for any $x, y \in \mathbb{R}^n$ then f is globally Lipschitz.

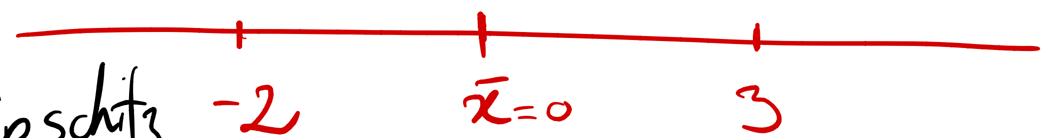
Ex 2

$$f(x) = x^2$$

$$f(x) - f(y) = x^2 - y^2 = (x+y)(x-y)$$

$$|f(x) - f(y)| = |(x+y)(x-y)| \leq \underbrace{|x+y|}_{L}\,|x-y|$$

$f(x) = x^2$ is locally Lipschitz

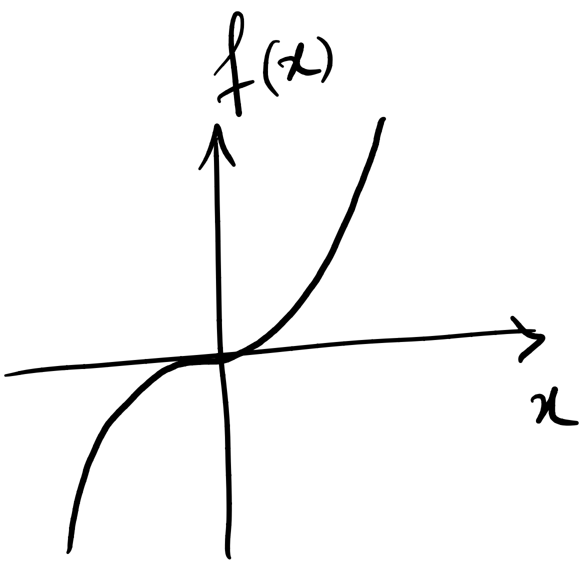


but <u>not</u> globally Lipschitz.

Ex 3 $\quad f(x) = x^3$

$$f(x) - f(y) = x^3 - y^3 = \underbrace{(x^2 + xy + y^2)}_{\text{Lipschitz constant}}(x-y)$$

Same conclusion as in Ex. 2 (locally but not globally)



Fact Any function that is continuously differentiable (differentiable and has cts 1st. derivative) is locally Lipschitz.

aside

$$\|f(x)-f(y)\| \leq L\|x-y\|$$

$$\frac{df}{dx} = \lim_{\|x-y\|\to 0} \frac{f(x)-f(y)}{x-y}$$

Ex 1 $\quad f(x) = x^{1/3}$

$\partial f/\partial x = x^{-2/3} \longrightarrow$ not cts diff'able

Ex2 $\quad f(x) = x^2$

$\partial f/\partial x = 2x$

Ex3 $f(x) = x^3$

$\partial f / \partial x = 3x^2$

the last 2 examples are cts diff'ble $\Rightarrow$ locally lipschitz

but $\partial f / \partial x$ is not uniformly bounded in x $\Rightarrow$ not globally Lipschitz

Ex $f(x) = 2x$

$\partial f / \partial x = 2 \longrightarrow$ uniformly bounded derivative

$\hookrightarrow$ globally Lipschitz

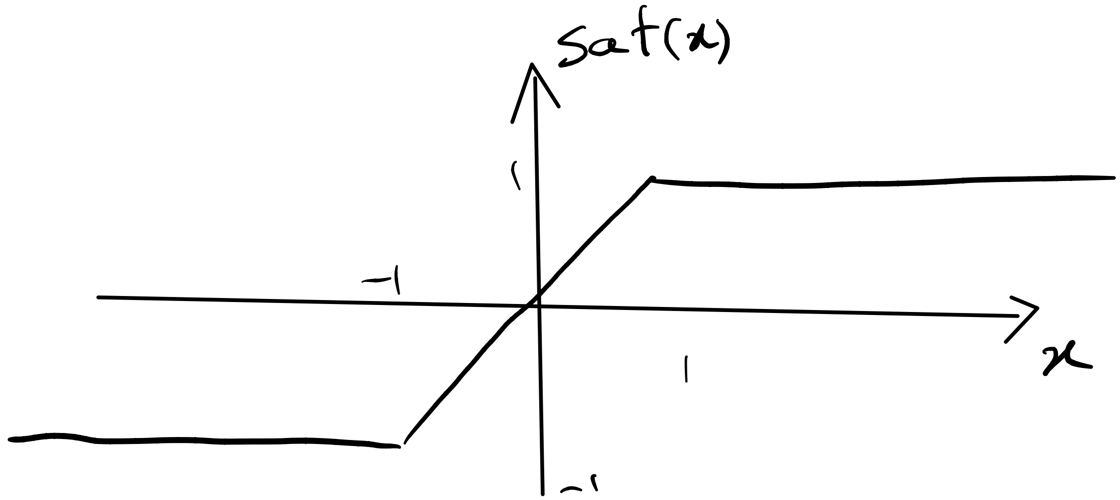
Summary: sufficient conditions for being Lipschitz is C^1.

$f \in C^1 \Rightarrow$ locally Lipschitz

$\left\|\frac{\partial f}{\partial x}\right\| \leq L$ for all $x \in \mathbb{R}^n \Rightarrow$ globally Lipschitz

* However, differentiability is not <u>necessary</u> for Lipschitz continuity

Ex



$f(x) = \text{sat}(x)$ is not differentiable (@ $x = \pm 1$ problematic)

yet globally Lipschitz (with $L = 1$)

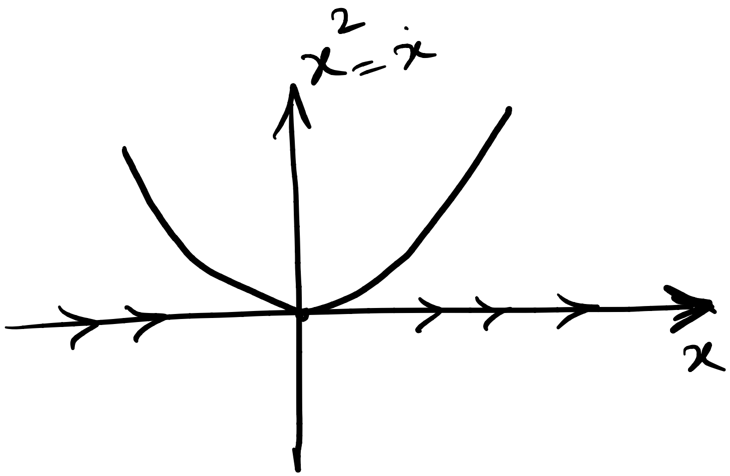
Back Ex 2

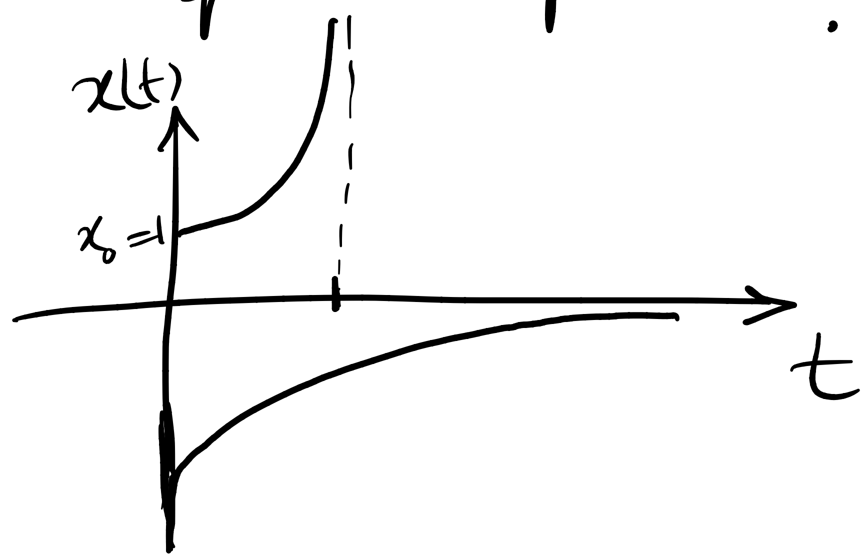
$$\dot{x} = f(x) = x^2 \longrightarrow x^{-2}\,dx = dt$$

$$-x^{-1}\Big]_{x_0}^{x(t)} = t - 0$$

$$\Rightarrow \quad x(t) = \frac{x_0}{1 - x_0 t}$$

So $x(t) = \dfrac{1}{1-t}$ when $x_0 = 1$





Here, we have existence and uniqueness on $[0, t_f)$

finite time interval

Note! Solutions blow up in finite time

Summary

of sufficient conditions

f is piecewis cts w.r.t. time

1) f cts w.r.t. x $\Rightarrow$ existence on $[0, t_f)$

2) f locally Lipschitz w.r.t. x $\Rightarrow$ existence & uniqueness on $[0, t_f)$

3) f globally Lipschitz w.r.t. x $\Rightarrow$ existense & uniqueness on $[0, \infty)$

proof $\to$ Khalil Chapt. 3.

Ex.5 $\quad \dot{x} = -x^3$

cts differentiable : $\partial f / \partial x = -3x^2$

but not globally Lipschitz. Yet, existence & uniqueness on $[0, +\infty)$

proof $\to$ solve differential eq'n

The conditions stated above are sufficient, but not necessary!

Back to linear system

Ex 6 $\dot{x} = A(t)x$

$$\|f(t,x) - f(t,y)\| = \|A(t)x - A(t)y\| = \|A(t)(x-y)\|$$

$$\leq \underbrace{\|A(t)\|}_{\text{induced norm}} \|x-y\|$$

A piecewise cts $\rightarrow$ globally Lipschitz

Fact If existence & uniqueness $\Rightarrow$ cts dependence on initial conditions on finite time interval

Given f: locally Lipschitz

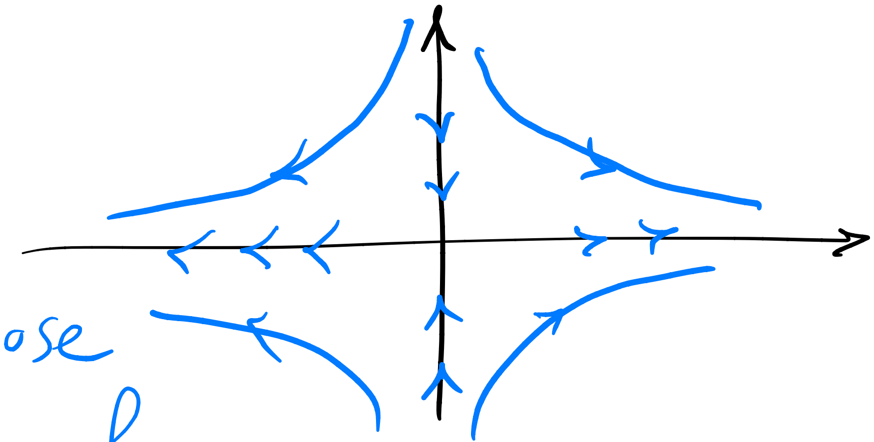
$\forall \varepsilon > 0$ and $T \in [0, t_f)$ there is $\delta = \delta(\varepsilon, T)$

such that $\|x_0 - y_0\| \leq \delta \Rightarrow \|\varphi(t, x_0) - \varphi(t, y_0)\| < \varepsilon$

for $t \in [0, T)$

<u>Moral</u> Even if 'f' is globally Lipschitz this <u>cannot</u> be extended on $[0, +\infty)$

Ex. Saddle point



even two pts that are close to each other get infinitely far from each other on $[0,\infty)$

$x_1(t), x_2(t)$ 2 sol'ns of $\dot{x} = f(t,x)$

starting from $x_{1,0}$ & $x_{2,0}$, and staying in a set with Lipschitz constant L for $t \in [0,T)$

$\forall \varepsilon > 0 \ \exists \ \delta(\varepsilon, T) > 0$ st. $\|x_{1_0} - x_{20}\| < \delta \Rightarrow \|x_1(t) - x_2(t)\| \le \varepsilon$ for $t \in [0,T)$

* How about cts dependence w.r.t. parameters?

$$\dot{x} = f(t, x, \mu) \qquad (1)$$

μ : constant parameter

Augment diffential eq'n (1) with $\dot{\mu} = 0$ (2)

and study system (1)+(2) with $z = \begin{pmatrix} x \\ \mu \end{pmatrix}$

$$\dot{z} = g(t, z)$$

$$g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} f(t, x, \mu) \\ 0 \end{bmatrix}$$

use cts dependence on IC's $\begin{pmatrix} x_0 \\ \mu \end{pmatrix} \Rightarrow$ cts dependence on parameters.