Due Saturday May 1st at noon (12 pm CST).

Your name:

Your UTD ID:

Important points:

- This is a take-home, open lecture/notes exam.
- You are not allowed to talk to anybody about this exam until you submit it.
- There are 6 problems.
- Think before you start solving the problems! If you are spending too much time on any of the problems, you're probably on the wrong track.
- You **can** use MATLAB but all codes and plots should be included in your submission.
- **PLEASE** write your answered in a clean and organized manner and **box** (or somehow highlight) your final answers to the problems.
- I will not be responding to emails during the exam time. Explain your solutions clearly and provide all necessary details.
- Your answers should be combined into a single PDF file and submitted via eLearning. To submit, click on the Term Exam assignment in the **Exams** folder and use the ASSIGNMENT SUBMISSION feature to browse your computer and upload your file. If you are not able to submit your answers via eLearning, send them to me via email (armin.zare@utdallas.edu). Note that after the deadline, I will not be accepting emails and submissions will be blocked on eLearning.

Good luck!

1. [20 points] Consider the system:

$$\tau\,\dot{x} = x - \frac{1}{3}x^3 - y$$
$$\dot{y} = x + \mu$$

where $\tau > 0$ and $\mu \geq 0$ are constants.

(a) Determine the equilibrium points and classify their stability properties depending on the values of parameter μ.

(b) At which value of μ does a bifurcation occur and what type of bifurcation is it?

(c) Assume $\tau \ll 1$ and sketch the phase portrait for two values of μ, one just below and one just above the bifurcation value.

2. [25 points] Consider the system:

$$\begin{aligned}\dot{x}_1 &= -\frac{1}{2}\tan\left(\frac{\pi x_1}{2}\right) + x_2 \\ \dot{x}_2 &= x_1 - \frac{1}{2}\tan\left(\frac{\pi x_2}{2}\right)\end{aligned}$$

(a) Find all equilibrium points of this system.

(b) Use linearization to study the stability of each equilibrium point.

(c) Using quadratic Lyapunov functions, estimate the region of attraction of each asymptotically stable equilibrium point. Try to make your estimates as large as possible.

(d) Plot the phase portrait of the system and show on it the exact regions of attraction as well as your estimates.

3. [20 points] Prove that the origin is the globally asymptotically stable equilibrium point of the system

$$\begin{aligned}
\dot{x}_1 &= -x_1 - \operatorname{sat}(x_3) \\
\dot{x}_2 &= -x_2 - \operatorname{sat}(x_1) \\
\dot{x}_3 &= -x_3 - \operatorname{sat}(x_2)
\end{aligned}$$

where

$$\operatorname{sat}(x) := \operatorname{sign}(x) \min\{1, |x|\}.$$

4. [15 points] Comment on the existence/uniqueness of solutions for the systems below. Provide your reasons.

   (a) $\dot{x} = x^2$

   (b) $\dot{x} = \sqrt{x}$

   (c) $\dot{x} = 1 + \dfrac{1 + x^3}{1 + x^4}$

5. [10 points] Show that the following system contains no closed orbits.

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2^3 + 1\\ \dot{x}_2 &= -4\,x_1^2 + 3\,x_2\end{aligned}$$

6. [10 points] Prove that the origin is the globally asymptotically stable equilibrium of the following system.

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -(\sin(x_1) + 2)(x_1 + x_2)\end{aligned}$$