Lecture 3

01/27/2021

Last time: Essentially nonlinear phenomena:

1) finite escape time
2) multiple isolated equilibria
3) limit cycles

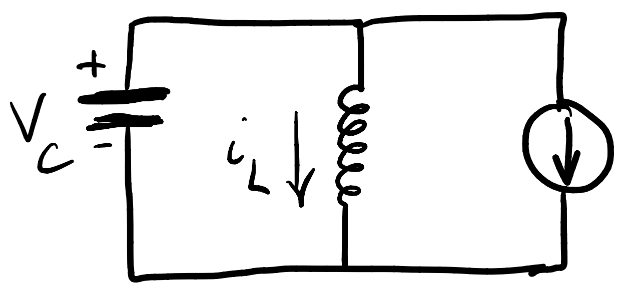
Today:

4) Chaos
5) subharmonic, harmonic or almost harmonic oscillations
6) Multiple modes of behavior

Bifurcations

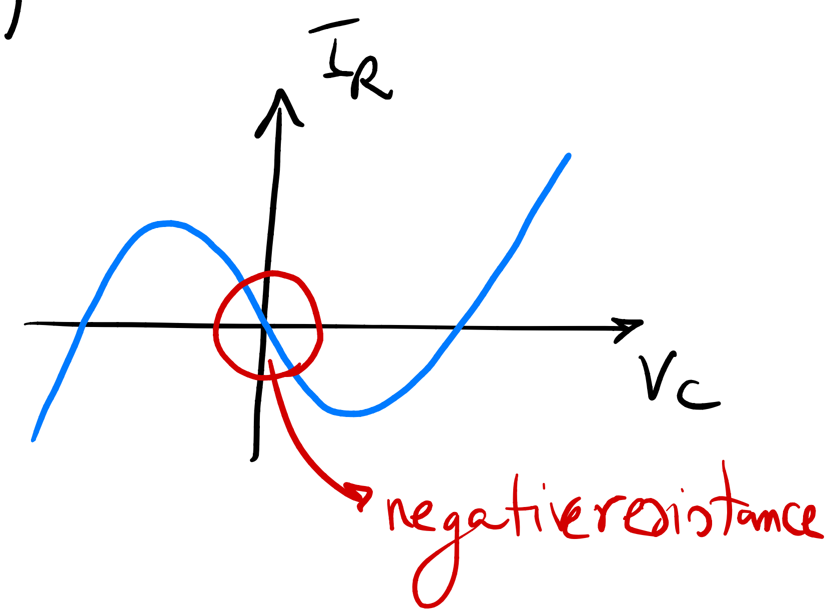
3) limit cycles:

ex. Van der pol oscillator



$I_r = I_r(V_c)$

$$I_r(V_c) = -V_c + V_c^3$$

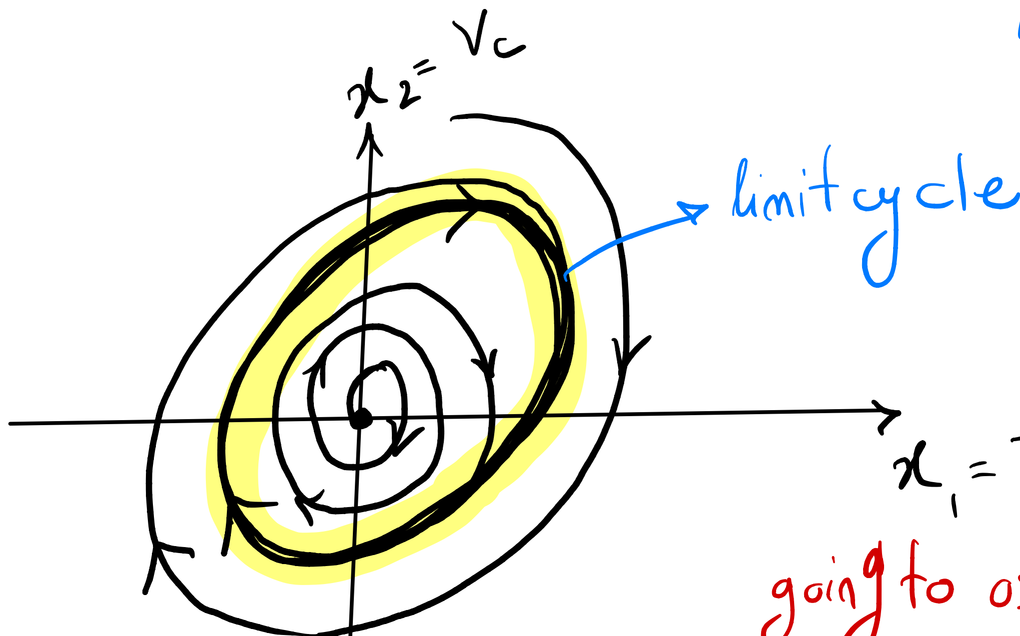


$$\left.\begin{aligned} \dot{I}_L &= \frac{1}{L} V_c \\ \dot{V}_c &= -\frac{1}{C} I_L + \frac{1}{C}\left(V_c - V_c^3\right) \end{aligned}\right\} \text{Van der pol oscillator}$$

e.p. $\begin{bmatrix}\bar{I}_L \\ \bar{V}_c\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$ unique

linearization around $\begin{pmatrix}0 \\ 0\end{pmatrix} \longrightarrow A = \begin{bmatrix}0 & 1/L \\ -1/C & +1/C\end{bmatrix}$

$+$ sign indicates that e.p. $\begin{pmatrix}0 \\ 0\end{pmatrix}$ is unstable.



structurally robust oscillator; no matter where you start you are going to oscillate on the limit cycle

4) Chaos (no precise definition)

Ex Lorenz system (attractor)

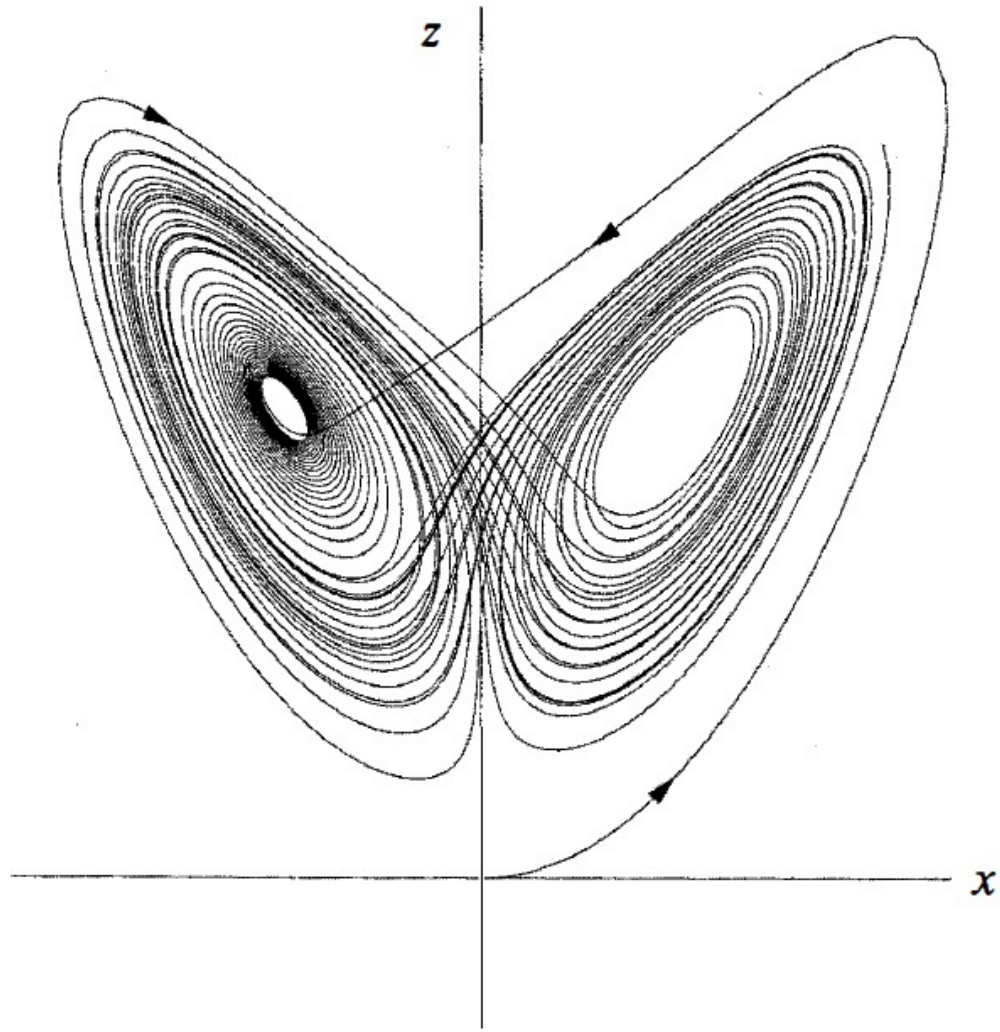
$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= x(b-z)-y \\ \dot{z} &= xy-\tau z \end{aligned} \quad \Bigg\}$$ simplified model of convective rolls in the atmosphere

3-order system (3 states x, y, z)

3 parameters a, b, τ

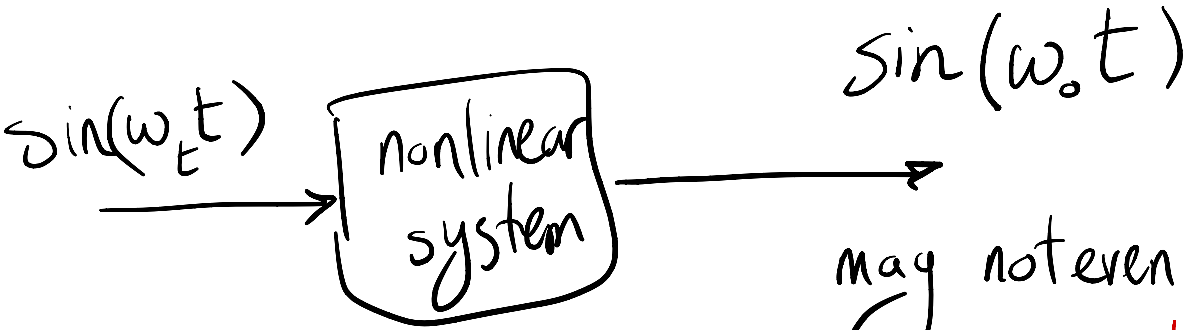
$a = 10$, $b = 28$, $\tau = 8/3$ huge sensitivity to initial condition

no simple characterization of asymtotic behavior



z
x

5) subharmonics, harmonics or almost harmonic oscillation



$\omega_o = \omega_t$

$= \omega_t / 2$

$= 2\omega_t$

$\vdots$

may not even be perfectly harmonic

↓

cannot be expressed as $\sin(\omega_o t)$

6) Multiple modes of behavior

unforced nonlinear system can have more than one limit cycle & can experience harmonic or asymptotic expansion or shrinkage depending on parameter changes.

Bifurcations: "splitting into two branches"

translation (real meaning): $\underset{\text{sudden}}{\text{abrupt}}$ change in qualitative behavior as parameters are varied

creation (or death) of equilibrium points (or limit cycles) and/or changes in the stability properties of e.p.'s.

* In presence of parameters that can change, even 1st-order systems (i.e. scalar state) can have interesting properties.

3 types bifurcations:

1. Fold (saddle-node ; blue sky)

$$\dot{x} = \alpha \pm x^2$$

↳ parameter $\alpha \in \mathbb{R}$

2. Transcritical

$$\dot{x} = \alpha x \mp x^2$$

3. Pitch Fork

$$\dot{x} = \alpha x \mp x^3$$

Note: These can appear in higher-order systems but are essentially one-dimensional!

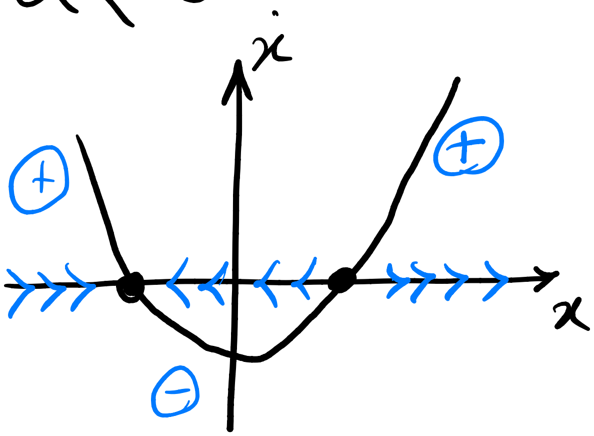
1 Fold

$$\dot{x} = \alpha + x^2$$

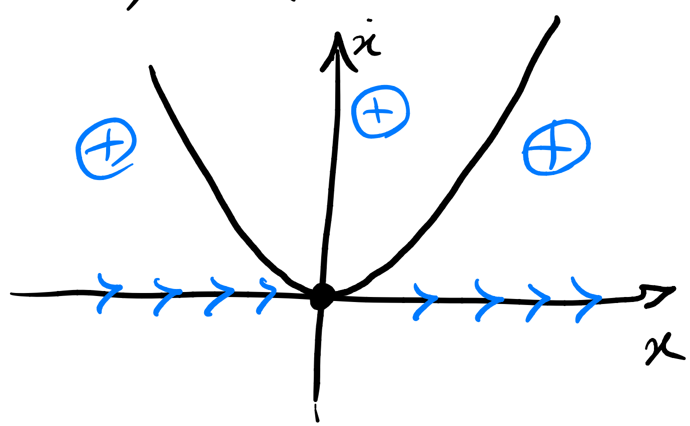
a mechanism by which fixed points are created or destroyed. How?

As a parameter in the system varies two e.p. move toward each other, collide, and eventually mutually annihilate.

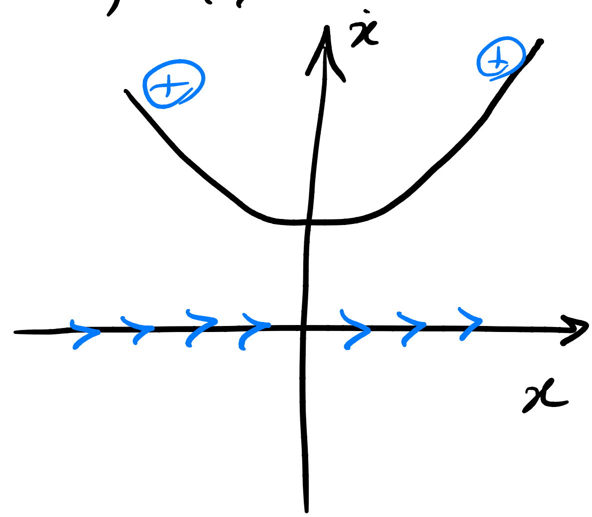
a) $\alpha < 0$



b) $\alpha = 0$



c) $\alpha > 0$



Equilibrium pts:

$$\bar{x} = \begin{cases} \pm\sqrt{|\alpha|} & \alpha < 0 \\ 0 & \alpha = 0 \\ \text{none} & \alpha > 0 \end{cases}$$

true for transcritical & pitchfork as well

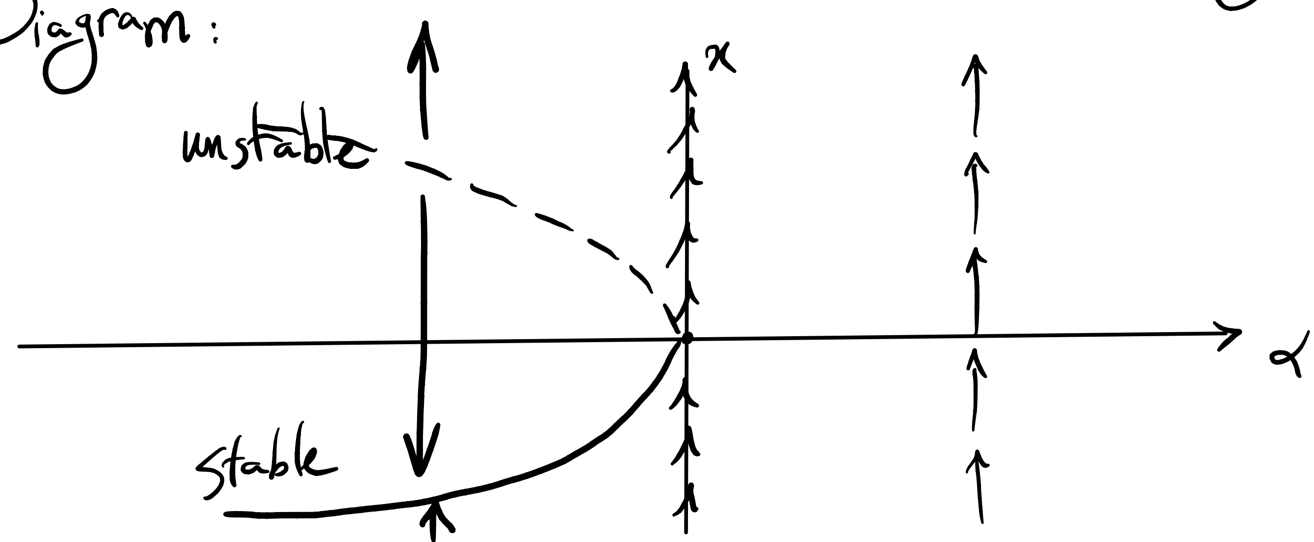
Critical value α : $\alpha_c = 0$ (linearization vanished)

linearization $\left.\partial f/\partial x\right|_{\bar{x}} = 2\bar{x} = \pm\sqrt{|\alpha|}$ $\alpha < 0$

unstable

locally asy. stable

Bifurcation Diagram:



2 Transcritical

↳ situations when a fixed point always exists even if a change in parameters changes its stability properties.

$$\dot{x} = \alpha x - x^2 \qquad x(t) \in \mathbb{R}$$

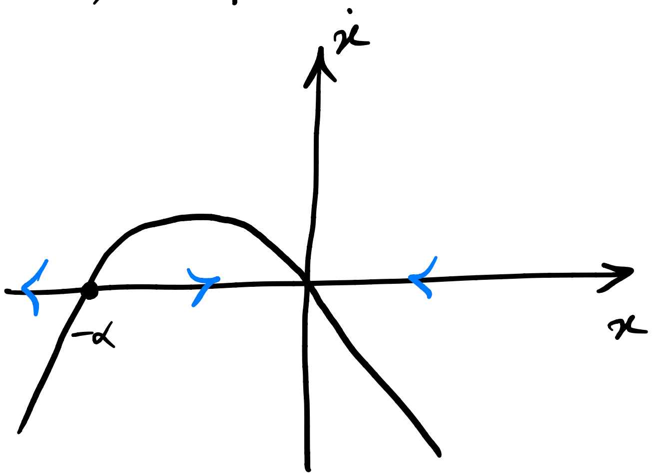
equilibrium points: $\bar{x}(\alpha - \bar{x}) = 0 \Rightarrow \begin{cases} \bar{x} = 0 \\ \bar{x} = \alpha \end{cases}$

linearization $\left.\frac{\partial f}{\partial x}\right|_{\bar{x}} = \alpha - 2\bar{x}$ $\xrightarrow{\bar{x}=0} \alpha$

$\xrightarrow{\bar{x}=\alpha} -\alpha$

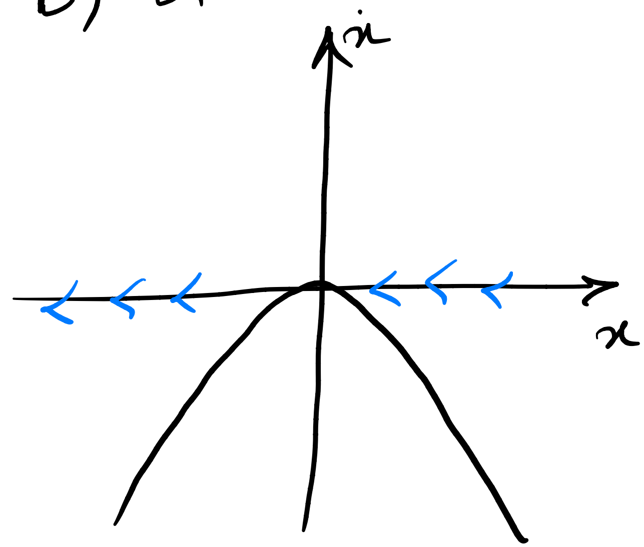
$\alpha<0$	$\alpha>0$
stable	unstable
unstable	stable

a) $\alpha < 0$



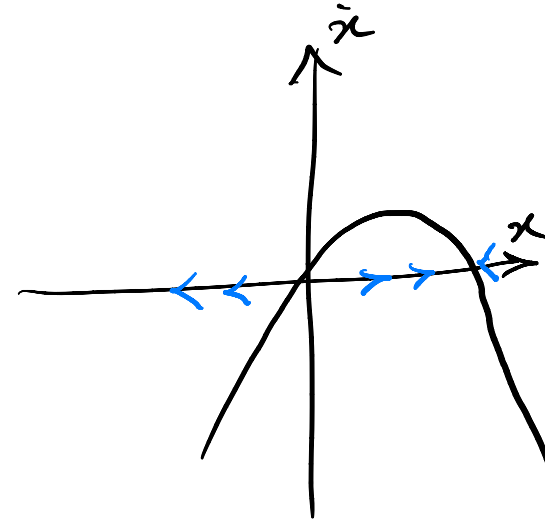
origin stable e.p.

b) $\alpha = 0$



$\alpha_c = 0$

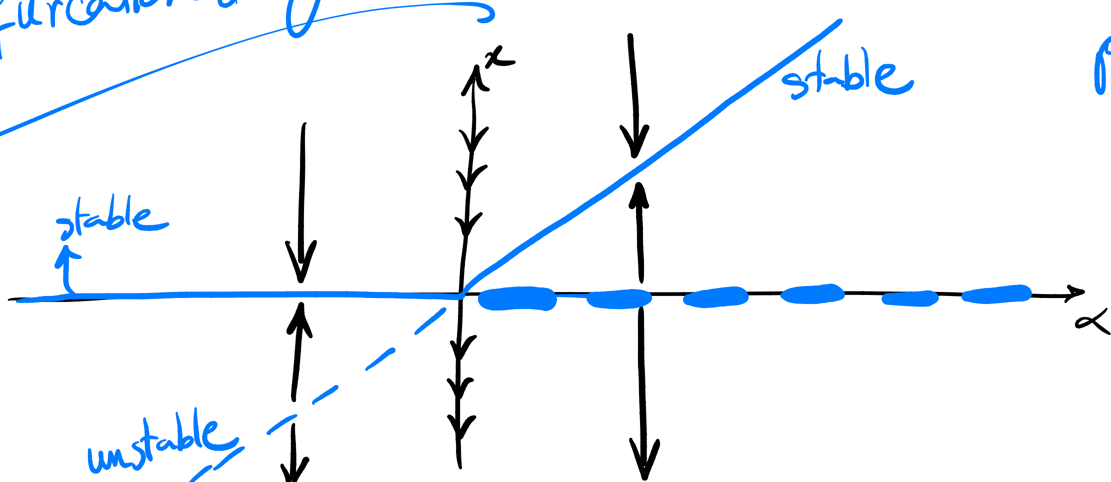
c) $\alpha > 0$



$\alpha > 0$

as α increases origin loses stability but another e.p. was formed which was positive and was stable.

bifurcation diagram



Summary: In contrast to fold bifurcation where e.p. dissapears or emerges, transcritical bifurcation is characterized by changes in the stability properties of a single e.p.