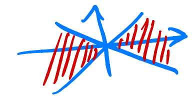
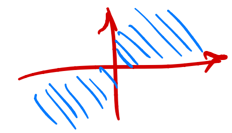
Nonlinear Sys. Quiz 6 — Apr 19, 2021

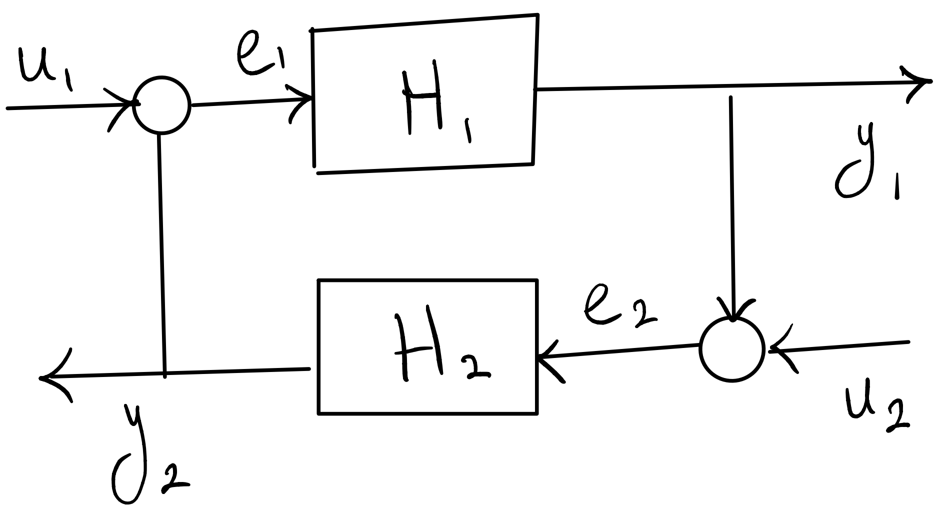
Problem 1: True or False? When a memory less system has an Lp gain that is bounded from above by a constant scalar b for all p, this system is passive.

False

$|y| \leq b|u| \Rightarrow$  bow tie relation

passivity $\langle y, u \rangle \geq -\beta \Rightarrow$ 

Problem 2: Suppose system H_1 has an Lp gain of 2 for all p. Find constant K so that the feedback interconnection is Lp stable.



finite gain Lp stable

$$\|y_{2,T}\|_p \leq K \|e_{2,T}\|_p + \beta$$

$\gamma_1 = 2 \quad \gamma_2 = K \quad \gamma_1\gamma_2 < 1 \Rightarrow K < \frac{1}{2}$ is acceptable

Lecture 22 04/19/2021

Last time: L_2 stability of nonlinear systems

Bounded real lemma

Small-gain thm

Passivity

Today: State-space conditions for passivity

Implications of passivity

Special case of LTI systems

Standard notion of passivity: $\quad u \longrightarrow \boxed{h(\cdot)} \longrightarrow \quad y = h(u)$

$$\int_0^T y^T(t)\,u(t)\,dt \geq 0$$

has to hold for all times T and all input trajectories u

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accounts for IC's

$$\underbrace{\int_0^T y^T(t)\,u(t)\,dt}_{\langle y_T,\, u_T \rangle} \geq \begin{cases} -\beta & \text{passive (P)} \\[2ex] \delta \langle u_T, u_T \rangle - \beta & \text{input strictly passive (ISP)} \\[2ex] \varepsilon \langle y_T, y_T \rangle - \beta & \text{output strictly passive (OSP)} \end{cases}$$

State-space conditions for passivity:

$$\dot{x} = f(x) + g(x)u$$

$$y = h(x)$$

If there is a continuous differentiable positive definite storage function $V(x)$ st.

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x) \leq \begin{cases} y^T u & \text{passive} \\ y^T u - \delta u^T u & \text{input strictly passive} \\ y^T u - \varepsilon y^T y & \text{output strictly passive} \end{cases}$$

Proof. integrate this inequality

$$-V(x(0)) \leq V(x(T)) - V(x(0)) \leq \langle y_T, u_T \rangle - \begin{cases} 0 \\ \delta \langle u_T, u_T \rangle \\ \varepsilon \langle y_T, y_T \rangle \end{cases}$$

. . . $\blacksquare$

Ex1 $\dot{x} = u$ $\quad x \in \mathbb{R}$

$y = x$

$$V(x) = \frac{1}{2}x^2 \longrightarrow \dot{V}(x) = x\dot{x} = xu = yu$$

thus passive

Note if y is velocity and u is forcing

$\Rightarrow y \cdot u$ = velocity × force = power supplied to system

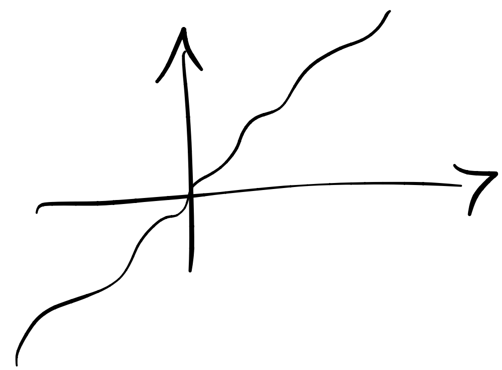
no storage mechanism

$\Rightarrow V = \frac{1}{2}x^2$ is interpreted as a kinetic energy of the system and the <u>rate of change</u> in kinetic energy is equal to the energy supplied to this system.

Ex2

$$\dot{x} = u$$
$$y = h(x)$$

and $x \cdot h(x) > 0$ for all $x \neq 0$



$$V(x) = \int_0^x h(\xi)\, d\xi$$ → can be interpreted as potential energy

$$\dot{V}(x) = h(x)\dot{x} = h(x)u = yu$$

$\Rightarrow$ passive

Ex 3

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{dP(x_1)}{dx_1} - Kx_2 + u$$

potential energy

velocity

force

$$y = x_2$$

$$V(x) = P(x_1) + \frac{1}{2}x_2^2$$

(potential) + (kinetic)

$$\dot{V}(x) = \frac{dP}{dx_1}\dot{x}_1 + x_2\dot{x}_2 = \cancel{\frac{dP}{dx_1}x_2} + x_2\left(\cancel{-\frac{dP}{dx_1}} - Kx_2 + u\right)$$

$$= -K x_2^2 + x_2 u$$

$$= \underbrace{-K y^2}_{\text{output strictly passive}} + y u \leq y \cdot u$$

Consequences (implications) of passivity:

$$\dot{V} = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x) u \leq y^T u \qquad (*)$$

1) $\frac{\partial V}{\partial x} f(x) \leq 0$ (set $u = 0$ in $(*)$)

$\downarrow$

stability in the sense of Lyapunov

2) $\frac{\partial V}{\partial x} g(x) = y^T$

$$\dot{V} = \frac{\partial V}{\partial x} f(x) + \left(\frac{\partial V}{\partial x} g(x) - y^T \right) u > 0$$ if I choose u appropriately

if $\frac{\partial V}{\partial x} g(x) = y^T$

then we cannot make $\dot{V} > 0$ based on a choice of input.

Note if $*$ holds $\Longleftrightarrow$ (1) and (2) hold.

linear case:

$$V(x) = \frac{1}{2} x^T P x$$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

implications of passivity:

(1) $\frac{\partial V}{\partial x} f(x) \leqslant 0 \iff \frac{1}{2} x^T (A^T P + PA) x \leqslant 0$

$$A^T P + PA \leqslant 0$$ standard stability

(2) $\frac{\partial V}{\partial x} g(x) = y^T$

$= x^T PB = x^T C^T \Rightarrow PB = C^T$

restriction needed to get passivity in addition to stability

Passivity : stability $\longrightarrow$ $A^TP + PA \leq 0$

$+$

input-output constraint $\longrightarrow$ $PB = C^T$

structural constraint

input matrix

output matrix

For LTI systems, passivity <=> positive realness

Def: A proper rational transfer function $H(s)$ with real coefficients is positive real (PR) if:

$H = \frac{p(s)}{q(s)}$ $\qquad \deg\{q(s)\} \geq \deg\{p(s)\}$

a) poles of $H(s)$ satisfy $Re(\lambda_i) \leq 0$, poles on $j\omega$-axis are simple (not repeated) and the associated residues (coefficients of partial fractional expansion) are non-negative.

b) $\operatorname{Re}\{H(j\omega)\} \geq 0 \quad \forall \omega \in \mathbb{R}$

* In addition, $H(s)$ is strictictly positive real (SPR) if $H(s-\varepsilon)$ is PR for some $\varepsilon > 0$.

Ex. a) $\frac{1}{s} \xrightarrow{s=j\omega} -j/\omega$ $\quad Re\{H(j\omega)\}=0$

PR but not SPR

$\frac{1}{s-\varepsilon}$ $\varepsilon>0$ not PR

b) $-\frac{1}{s}$ not PR because of negative residue

c) $\frac{1}{s+a}$, $a>0$

$$H(j\omega) = \frac{1}{a+j\omega} = \frac{a-j\omega}{a-j\omega}\,\frac{1}{a+j\omega} = \frac{a}{a^2+\omega^2} - \frac{j\omega}{a^2+\omega^2}$$

≥ 0

a shift can also be tolerated $\Rightarrow$ SPR