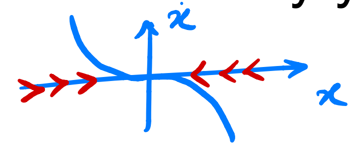
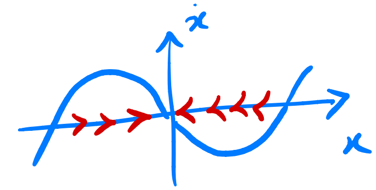
Nonlinear Sys. Quiz 4 Mar 24, 2021

Problem 1: Comment on the stability of the systems below. Justify your statement.

a) $\dot{x} = -x^3$ GAS



b) $\dot{x} = -\sin(x)$ LAS



There are multiple methods that can be used, the simplest being the phase-plane quiver plots. Obviously, (b) cannot have global stability features due to multiple e.p.'s.

Problem 2: What can we conclude regarding the stability of the origin using the the candidate Lyapunov function $V(x) = x^T P x$ when $P = P^T > 0$ and $A^T P + PA \leq 0$?

Stability in the sense of Lyaponov, unless observability guarantees allows us to envoke LaSalle to get GAS.

Lecture 15

03/24/2021

Last time: stability of time varying systems

Today: A brief recap.

-> special case of LTV
Differential Lyapunov eq.n.

- Extensions of LaSalle's principle to TV systems stability of $\dot{x} = f(x,t)$

Lyapunov-based analysis of TV systems:

↳ uniform stability

1° $\alpha_1(\|x\|) \leq W_1(x) \leq V(x,t) \leq W_2(x) \leq \alpha_2(\|x_2\|)$

2° $\dot{V}(x,t) = \partial V/\partial t + [\nabla_x V(x,t)]^T f(x,t) \leq -W_3(x)$

W_1, W_2 positive definite functions of x

1° If W_3 is locally positive semi-definite

$\Rightarrow \bar{x} = 0$ is U.S.

2° If W_3 is locally positive definite

$\Rightarrow \bar{x} = 0$ is U.A.S.

3. If W_3 is globally positive definite &
W_1 is radially unbounded

$\Rightarrow \bar{x} = 0$ is GUAS

4° If $W_i = K_i \|x\|^a \Rightarrow \bar{x} = 0$ GUES

Main challenge in going through proof e.g. when we had for UAS

$$\dot{V}(x,t) \leq -\alpha_3(\|x\|)$$

was to see how we could get rid of $\|x\|$ on the rhs and rewrite this condition as a differential equation in V.

scalar diff e.q.

tools: comparison & composition rules.

Ex. $\dot{x} = -g(t)\, x^3$ nonlinear TV system

$$g(t) \geq 1 \quad \forall\, t \geq t_0$$

Lyapunov function candidate:

1° $V(x) = \frac{1}{2} x^2$ no explicit time dependence

2° $\dot{V}(x) = x\dot{x} = -g(t)\, x^4 \leq -x^4 < 0$

$\forall\, x \in \mathbb{R} \setminus \{0\}$

$$\left.\begin{aligned} W_1(x) = W_2(x) = \frac{1}{2} x^2 \\ W_3(x) = x^4 \end{aligned}\right\} \Rightarrow \text{GUAS}$$

$$\dot{V} \leq -x^4 \Rightarrow \frac{dV}{dt} \leq -x^4$$

here we have $V = \frac{1}{2}x^2 \Rightarrow \boxed{x^2 = 2V}$

$$\frac{dV}{dt} \leq -(2V)^2 \Rightarrow \frac{dV}{dt} \leq -4V^2 \quad (*)$$

solution to $(*)$ satisfies:

$$V(x(t)) \leq \frac{V(x(t_0))}{4V(x(t_0))(t-t_0)+1}$$

No exponential stability

Fact: No expential stability w/o stable linearization

→ If linearization is unstable @ the origin we can conclude that we do not have exponential decay. In other words, nonlinealities cannot affect stability (locally) and cause stability to happen.

back to the example:

$$\left.\begin{aligned}\dot{x} &= -g(t)x^3 \\ \bar{x} &= 0\end{aligned}\right\} \quad \left.\frac{\partial f(x,t)}{\partial x}\right|_{\bar{x}=0} = -3g(t)\left.\bar{x}^2\right|_{\bar{x}=0}$$

$$= 0$$

linearization $\quad \dot{\tilde{x}} = 0\,\tilde{x}$

$\rightarrow$ Marginally stable $\longrightarrow$ for sure we have lack of exp. stability of $\dot{x} = -g(t)x^3$

Linear time varying (LTV) system

$$\dot{x} = A(t)x$$

source of time variation

Ex. Mass-spring system:

$$A(t) = \begin{bmatrix} 0 & 1 \\ -\frac{K(t)}{m} & 0 \end{bmatrix}$$

If $K = \text{const} \longrightarrow$ LTI system

$V(x) = x^T P x$ Lyap. function

constant p.d. matrix

For LTV systems, we can still use quadratic Lyapunov functions BUT P may depend on time:

$$V(x,t) = x^T \cdot P(t) \cdot x$$

where $P(t) = P^T(t) > 0 \quad \forall t \geq t_0$

Here, condition 1° & 2° simplify to

1° $$k_1 I \leqslant P(t) \leqslant k_2 I \qquad k_i > 0$$
$i = 1, 2$

2° $$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) = -Q(t)$$

want: $Q(t) \geqslant k_3 I \;;\; k_3 > 0$

proof

2°

$$\dot V = \dot x^T P x + \underbrace{x^T \dot P x}_{= 0 \text{ in LTI systems}} + x^T P \dot x \to A(t)x$$

$$\dot V = x^T \underbrace{\left[A^T(t)P(t) + \dot P(t) + P(t)A(t)\right]}_{-Q(t)} x$$

$$= -x^T Q(t) x \leq -k_3\|x\|^2$$

1°

$$k_1\|x\|^2 \leq x^T P(t) x \leq k_2 \|x\|^2$$

$\Rightarrow$ GUES!

Q. Does the converse hold?

A. Yes!

Thm Let $\bar{x}=0$ of $\dot{x}=A(t)x$ be UES

and

1° $A(t)$ continuous + bounded

2° $Q(t)=Q(t)^T$ is continuous and

$$k_3 I \leq Q(t) \leq k_4 I \qquad k_i > 0$$

Then, there is $P(t)=P^T(t)$ such that

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + Q(t) = 0$$

and

$$0 < k_1 I \leq P(t) \leq k_2 I .$$

Recall the LTI case: $\dot{x} = Ax$; A constant matrix

Lyapunov eq: $A^T P + PA + Q = 0$

$$P = \int_0^\infty e^{A^T t} Q e^{At} \, dt \; ; \; Q = Q^T > 0$$

for LTV: sol'n to $\underbrace{\text{D}}_{\text{differential}}\underbrace{\text{LE}}_{}$ → Lyap. eq.

$$P(t) = \int_{t_0}^{t} \Phi^T(\tau, t_0)\, Q(\tau)\, \underbrace{\Phi(\tau, t_0)}_{\text{for LTI: } e^{At}}\, d\tau$$

state transition Matrix

If therefore we obtain $\Phi(t, t_0)$ for a given LTV system, we can find $P(t)$ and use it to construct a candidate Lyap. function.

☆

1° $$\frac{\partial \Phi(t,t_0)}{\partial t} = A(t)\,\Phi(t,t_0)$$

2° $$\Phi(t_0,t_0) = I$$

Summary

Fact: for Linear systems

Asymptotic stability $\iff$ Exponential stability

BUT

$$\mathrm{Re}\{\lambda_i(A(t))\} < 0 \quad \forall\, t \in [t_0, +\infty) \;\not\Longrightarrow\; \text{Exp. stability}$$

useless test.

Ex. $\dot{x} = A(t)x$

$$A(t) = \begin{bmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t\cos t \\ 1 - 1.5\sin t\cos t & -1 + 1.5\sin^2 t \end{bmatrix}$$

E-values of $A(t)$:

$$\lambda_{1,2}(A(t)) = \underbrace{-1/4}_{\text{negative}} \pm j\frac{\sqrt{7}}{4} \qquad \forall t$$

negative, YET

state transition

$$\Phi(t,0) = \begin{bmatrix} e^{\frac{1}{2}t}\cos t & e^{-t}\sin t \\ e^{-\frac{1}{2}t}\sin t & e^{-t}\cos t \end{bmatrix}$$

There is no $k, \lambda > 0$ s.t. $\|\Phi(t,t_0)\| \leq k e^{-\lambda(t-t_0)}$

The (1,1) entry prevents us from bounding the norm of the state-transition matrix by exp. decaying function

In fact!

$$\lim_{t \to \infty} \| \Phi(t, 0) \| = +\infty$$

This system is UNSTABLE