Due Monday 03/14/21 (end of the day: 11:59 pm)

1. Consider the following system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2\\ \dot{x}_2 &= \frac{x_1^2}{1+x_1^2} - 0.5\,x_2\end{aligned}$$

   (a) Define the shifted state variables $\tilde{x}_1 = x_1 - 1$, $\tilde{x}_2 = x_2 - 1$ so that the equilibrium $x = (1,1)$ becomes $\tilde{x} = (0,0)$. Rewrite the state equations above in terms of $\tilde{x}_1$ and $\tilde{x}_2$, eliminating all x_1 and x_2 terms.

   (b) Linearize the system you found in the previous part at the origin and show that one of the eigenvalues is zero. Next, find variables y and z to bring the system to the form discussed in the class lecture on center manifold theory.

   (c) Determine the stability properties of $\tilde{x} = (0,0)$.

   (d) Use numerical simulation to plot the phase portrait in the original (x_1, x_2) coordinates and superimpose the lines $y = 0$ and $z = 0$ on the same plot. Discuss whether the phase portrait is consistent with the properties of the center manifold discussed in the class.

2. Strogatz, Problem 3.7.3; see attached.

3. Strogatz, Problem 3.7.4; see attached.

4. Khalil, Problem 3.8; see attached.

5. Khalil, Problem 3.13; see attached. For the initial condition $(x_1(0), x_2(0)) = (1, -1)$, simulate the sensitivity equations and plot the time dependence of the corresponding sensitivity functions.

At high temperature the spins point in random directions and so $m \approx 0$; the material is in the *paramagnetic* state. As the temperature is lowered, m remains near zero until a critical temperature T_c is reached. Then a ***phase transition*** occurs and the material spontaneously magnetizes. Now $m > 0$; we have a *ferromagnet.*

But the symmetry between up and down spins means that there are *two* possible ferromagnetic states. This symmetry can be broken by applying an external magnetic field h, which favors either the up or down direction. Then, in an approximation called *mean-field theory,* the equation governing the equilibrium value of m is

$$h = T\tanh^{-1} m - Jnm$$

where J and n are constants; $J > 0$ is the ferromagnetic coupling strength and n is the number of neighbors of each spin (Ma 1985, p. 459).

a) Analyze the solutions m^* of $h = T\tanh^{-1} m - Jnm$, using a graphical approach.
b) For the special case $h = 0$, find the critical temperature T_c at which a phase transition occurs.

3.7 Insect Outbreak

3.7.1 (Warm-up question about insect outbreak model) Show that the fixed point $x^* = 0$ is *always unstable* for Equation (3.7.3).

3.7.2 (Bifurcation curves for insect outbreak model)
a) Using Equations (3.7.8) and (3.7.9), sketch $r(x)$ and $k(x)$ vs. x. Determine the limiting behavior of $r(x)$ and $k(x)$ as $x \to 1$ and $x \to \infty$.
b) Find the exact values of r, k, and x at the cusp point shown in Figure 3.7.5.

3.7.3 (A model of a fishery) The equation $\dot{N} = rN(1 - \frac{N}{K}) - H$ provides an extremely simple model of a fishery. In the absence of fishing, the population is assumed to grow logistically. The effects of fishing are modeled by the term $-H$, which says that fish are caught or "harvested" at a constant rate $H > 0$, independent of their population N. (This assumes that the fishermen aren't worried about fishing the population dry—they simply catch the same number of fish every day.)
a) Show that the system can be rewritten in dimensionless form as

$$\frac{dx}{d\tau} = x(1 - x) - h,$$

for suitably defined dimensionless quantities x, τ, and h.
b) Plot the vector field for different values of h.
c) Show that a bifurcation occurs at a certain value h_c, and classify this bifurcation.
d) Discuss the long-term behavior of the fish population for $h < h_c$ and $h > h_c$, and give the biological interpretation in each case.

There's something silly about this model—the population can become negative! A better model would have a fixed point at zero population for all values of H. See the next exercise for such an improvement.

3.7.4 (Improved model of a fishery) A refinement of the model in the last exercise is

$$\dot{N} = rN\left(1-\frac{N}{K}\right) - H\frac{N}{A+N}$$

where $H > 0$ and $A > 0$. This model is more realistic in two respects: it has a fixed point at $N = 0$ for all values of the parameters, and the rate at which fish are caught decreases with N. This is plausible—when fewer fish are available, it is harder to find them and so the daily catch drops.

a) Give a biological interpretation of the parameter A; what does it measure?

b) Show that the system can be rewritten in dimensionless form as

$$\frac{dx}{d\tau} = x(1-x) - h\frac{x}{a+x},$$

for suitably defined dimensionless quantities x, τ, a, and h.

c) Show that the system can have one, two, or three fixed points, depending on the values of a and h. Classify the stability of the fixed points in each case.

d) Analyze the dynamics near $x = 0$ and show that a bifurcation occurs when $h = a$. What type of bifurcation is it?

e) Show that another bifurcation occurs when $h = \frac{1}{4}(a+1)^2$, for $a < a_c$, where a_c is to be determined. Classify this bifurcation.

f) Plot the stability diagram of the system in (a, h) parameter space. Can hysteresis occur in any of the stability regions?

3.7.5 (A biochemical switch) Zebra stripes and butterfly wing patterns are two of the most spectacular examples of biological pattern formation. Explaining the development of these patterns is one of the outstanding problems of biology; see Murray (1989) for an excellent review of our current knowledge.

As one ingredient in a model of pattern formation, Lewis et al. (1977) considered a simple example of a biochemical switch, in which a gene G is activated by a biochemical signal substance S. For example, the gene may normally be inactive but can be "switched on" to produce a pigment or other gene product when the concentration of S exceeds a certain threshold. Let $g(t)$ denote the concentration of the gene product, and assume that the concentration s_0 of S is fixed. The model is

$$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^{\,2} + g^2}$$

where the k's are positive constants. The production of g is stimulated by s_0 at a

3.6 Let $f(t,x)$ be piecewise continuous in t, locally Lipschitz in x, and

$$\|f(t,x)\| \le k_1 + k_2\|x\|, \quad \forall\ (t,x) \in [t_0,\infty) \times R^n$$

(a) Show that the solution of (3.1) satisfies

$$\|x(t)\| \le \|x_0\| \exp[k_2(t-t_0)] + \frac{k_1}{k_2}\{\exp[k_2(t-t_0)] - 1\}$$

for all $t \ge t_0$ for which the solution exists.

(b) Can the solution have a finite escape time?

3.7 Let $g : R^n \to R^n$ be continuously differentiable for all $x \in R^n$ and define $f(x)$ by

$$f(x) = \frac{1}{1+g^T(x)g(x)}g(x)$$

Show that $\dot{x} = f(x)$, with $x(0) = x_0$, has a unique solution defined for all $t \ge 0$.

3.8 Show that the state equation

$$\begin{aligned} \dot{x}_1 &= -x_1 + \frac{2x_2}{1+x_2^2}, \quad x_1(0) = a \\ \dot{x}_2 &= -x_2 + \frac{2x_1}{1+x_1^2}, \quad x_2(0) = b \end{aligned}$$

has a unique solution defined for all $t \ge 0$.

3.9 Suppose that the second-order system $\dot{x} = f(x)$, with a locally Lipschitz $f(x)$, has a limit cycle. Show that any solution that starts in the region enclosed by the limit cycle cannot have a finite escape time.

3.10 Derive the sensitivity equations for the tunnel-diode circuit of Example 2.1 as L and C vary from their nominal values.

3.11 Derive the sensitivity equations for the Van der Pol oscillator of Example 2.6 as ε varies from its nominal value. Use the state equation in the x-coordinates.

3.12 Repeat the previous exercise by using the state equation in the z-coordinates.

3.13 Derive the sensitivity equations for the system

$$\dot{x}_1 = \tan^{-1}(ax_1) - x_1x_2, \qquad \dot{x}_2 = bx_1^2 - cx_2$$

as the parameters a, b, c vary from their nominal values $a_0 = 1$, $b_0 = 0$, and $c_0 = 1$.