Lecture 21

04/14/2021

→ Next class quiz is Monday 19th

Last time: Input-output stability

Finit gain $\mathcal{L}_p$ stable

$$\|y_T\|_p \leq K\|u_T\|_p + \beta$$

K ↳ const.

Today: condition on $\mathcal{L}_2$ stability of nonlinear sys.

L_2 stability

$$\dot{x} = f(x) + g(x)u$$

$$y = h(x)$$

$*$

This nonlinear system is L_2 stable if there is a cts. differentiable positive definite $V(x)$

$$\frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} g(x) g^T(x) \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2} h^T(x) h(x) \leq 0$$

Hamilton-Jacobi inequality (HJ)

then $\circledast$ has an L_2 gain $\leq \gamma$.

proof. let (HJ) hold

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \underbrace{\frac{\partial V}{\partial x} f(x)}_{L_f V} + \underbrace{\frac{\partial V}{\partial x} g(x)}_{L_g V} u$$

$$\dot{V} \leq \frac{\partial V}{\partial x} f(x) + \frac{1}{2\alpha} \frac{\partial V}{\partial x} g(x) g(x)^T \left(\frac{\partial V}{\partial x}\right)^T + \frac{\alpha}{2} u^T u$$

aside: $a \cdot b \leq \frac{1}{2\alpha} a^2 + \frac{\alpha}{2} b^2 \iff 0 \leq \left(\frac{a}{\sqrt{\alpha}} - \sqrt{\alpha}\, b\right)^2$

Now from (HJ) with $\alpha = \gamma^2$

$$\Rightarrow \quad \dot{V} \leq -\frac{1}{2} h^T(x)\, h(x) + \frac{\gamma^2}{2} u^T u$$

$$= -\frac{1}{2} y^T(t)\, y(t) + \gamma^2/_2\, u^T(t)\, u(t)$$

Now if we integrate both sides from 0 to T:

$$V(x(T)) - V(x(0)) \leq -\frac{1}{2} \| y_T \|_2^2 + \frac{\gamma^2}{2} \| u_T \|_2^2$$

$$-V(x(0)) \leq V(x(T)) - V(x(0)) \Rightarrow$$

$$\|y_T\|_2^2 \leq \gamma^2 \|u_T\|_2^2 + 2V(x(0))$$

we know $\sqrt{a^2+b^2} \leq |a| + |b|$

$$\Rightarrow \|y_T\|_2 \leq \underset{\big\downarrow\, K}{\gamma} \|u_T\|_2 + \underbrace{\sqrt{2V(x(0))}}_{\beta}$$

$\blacksquare$

Note Lyapunov like functions, $V(x)$, that are used to establish input-output stability are known as "storage functions".

For linear systems:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

(HJ) holds with $V(x) = \frac{1}{2} x^T P x$ simplifies to

$$x^T\left(A^TP + PA + \frac{1}{\gamma^2} PBB^TP + C^TC\right)x \preceq 0$$

$$A^TP + PA + \frac{1}{\gamma^2} PBB^TP + C^TC \preceq 0$$

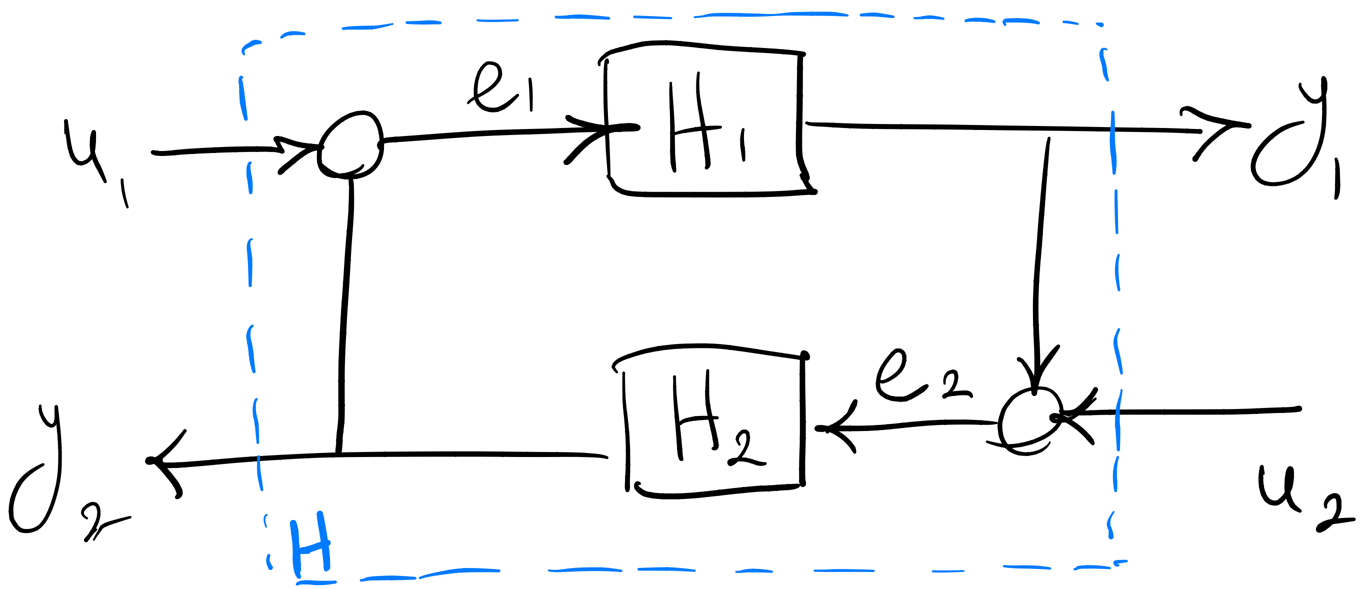
Bounded Real Lemma:

Suppose A is Hurwitz ($\lambda_i(A) < 0 \quad i = 1, \ldots, n$) and let γ^* denote the $\underbrace{L_2\text{-induced gain}}_{(H_\infty \text{ norm})}$ of $\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx\end{array}$

(H_∞ norm) → peak of Bode-mag plot then for every $\gamma > \gamma^*$, there is $P = P^T > 0$ st.

$$A^T P + PA + \frac{1}{\gamma^2} P B B^T P + CC < 0.$$

Small Gain Thm:



$$H : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \longrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Suppose H_i has L_p gain $\leq \gamma_i$ $\forall i = 1, 2, \dots$

If $\gamma_1 \gamma_2 < 1$ then the feedback interconnection

H is L_p stable.

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proof: H_1: $\|y_{1T}\|_p \leq \gamma_1 \|e_{1T}\|_p + \beta_1$

H_2: $\|y_{2T}\|_p \leq \gamma_2 \|e_{2T}\|_p + \beta_2$

$e_1 = u_1 + y_2$

$e_2 = u_2 + y_1$

$$\|y_{1T}\|_\rho \leq \gamma_1 \|u_{1T} + u_{2T}\|_\rho + \beta_1$$

$$\leq \gamma_1 \|u_{1T}\|_\rho + \gamma_1 \|y_{2T}\|_\rho + \beta_1$$

$$\leq \gamma_1 \|u_{1T}\|_\rho + \gamma_1\gamma_2 \|y_{2T}\|_\rho + \gamma_1\gamma_2 \|y_{1T}\|_\rho$$

$$+ \gamma_1 \beta_2 + \beta_1$$

$$\Rightarrow \|y_{1T}\|_\rho \leq \frac{\gamma_1}{1-\gamma_1\gamma_2} \|u_{1T}\|_\rho + \frac{\gamma_1\gamma_2}{1-\gamma_1\gamma_2} \|u_{2T}\|_\rho + \frac{\beta_1 + \gamma_1\beta_2}{1-\gamma_1\gamma_2}$$

and similarly for $\|y_{2T}\|_p$

if $\gamma_1\gamma_2 \neq 1 \longrightarrow$ if $\gamma_1\gamma_2 = 1$ well-posedness are violated

if $\gamma_1\gamma_2 > 1 \longrightarrow$ positiveness of Lp gains is violated

Note that there is a sufficient condition on the stability of such interconnected systems, but it is highly conservative.

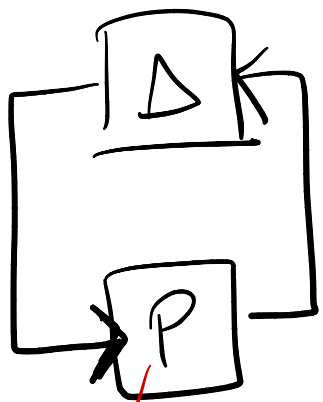
$\xrightarrow{\text{why?}}$ info. from phase is not taken into account.

Robust control

modeling uncertainty

Δ can be anything as long as it is norm bounded

L_2 gain $\leq \gamma_\Delta$



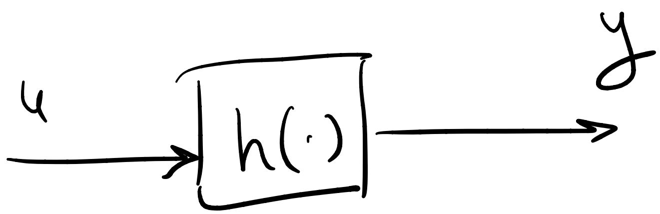
plant

In adaptive control, the structure of uncertainty Δ is taken into account.

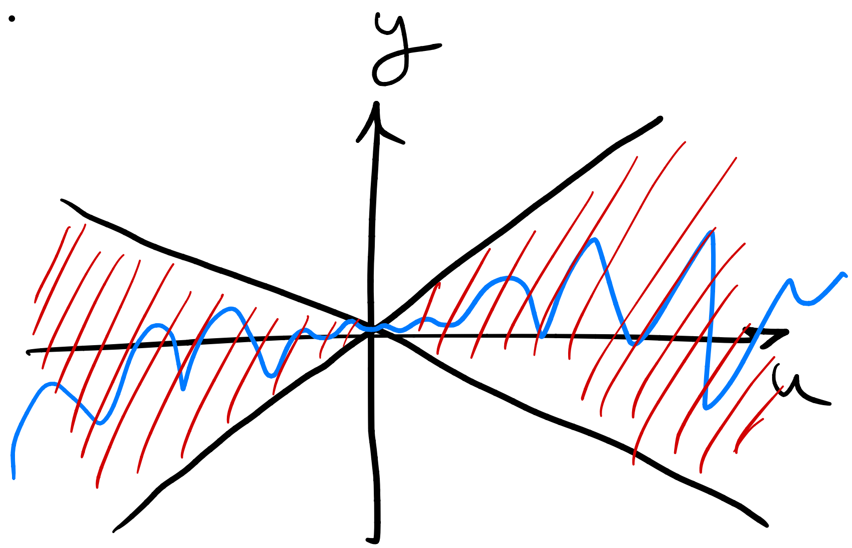
If L_2 gain $\gamma_P \leq \frac{1}{\gamma_\Delta} \Rightarrow$ robust stability

Passivity:

Q: What does it means for L_p gain $\leq \gamma$ in the case of memoryless functions?



$y = h(u)$



$$\boxed{|y| = |h(u)| \leq K|u|}$$

$K > 0$

bow tie!

$H : L_{2e} \to L_{2e}$ passive if for any

$u \in L_{2e}$ and any $T > 0$

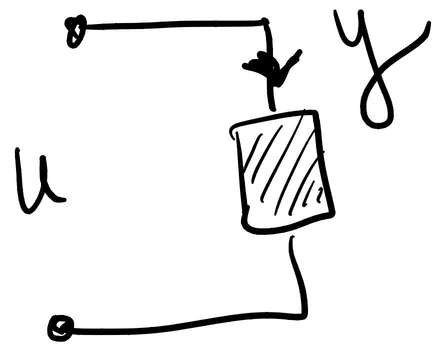
$$\langle y_T, u_T \rangle \geq -\beta$$

accounts for initial conditions

$$\left(x(0) = 0 \Rightarrow \langle y_T, u_T \rangle \geq 0 \right)$$

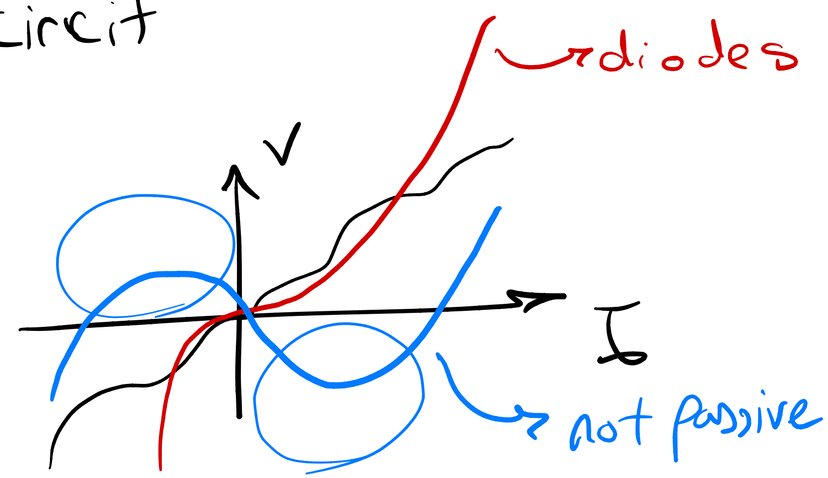
$$\langle y_T, u_T \rangle := \langle y, u \rangle_{L_2[0,T]} = \int_0^T y^T(t)\, u(t)\, dt$$

In the case of electronic circuits



element is passive if the power introduced to the element is nonnegative.

$$\langle y, u \rangle = y^T u \xrightarrow[\text{electric circit}]{\text{for}} P = VI$$



e.g. linear resistors

$$\langle y_T, u_T \rangle \geq \begin{cases} -\beta & \text{passive} \\ \delta \langle u_T, u_T \rangle - \beta & \text{input strictly passive} \\ \varepsilon \langle y_T, y_T \rangle - \beta & \text{output strictly passiv} \end{cases}$$

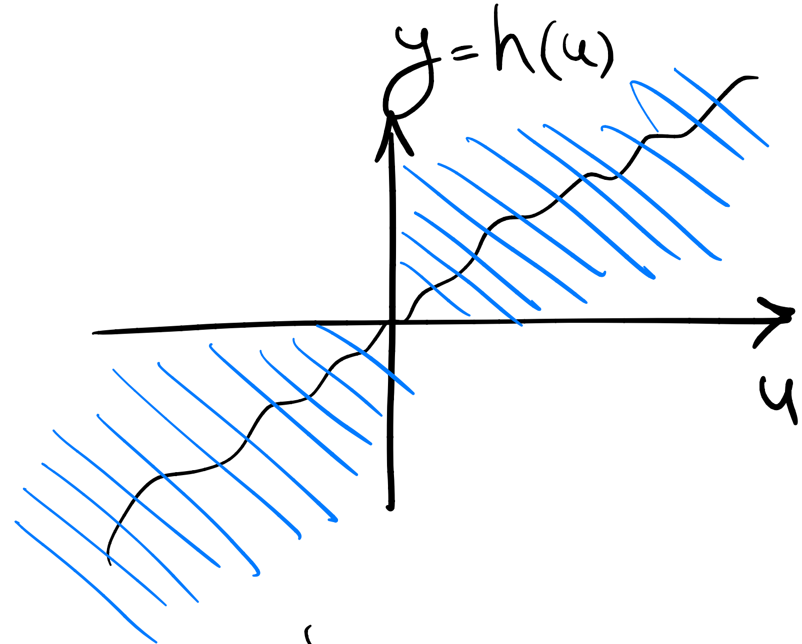
For memoryless (static) nonlinearity,

$$u \longrightarrow \boxed{h(\cdot)} \longrightarrow y = h(u)$$

Passive

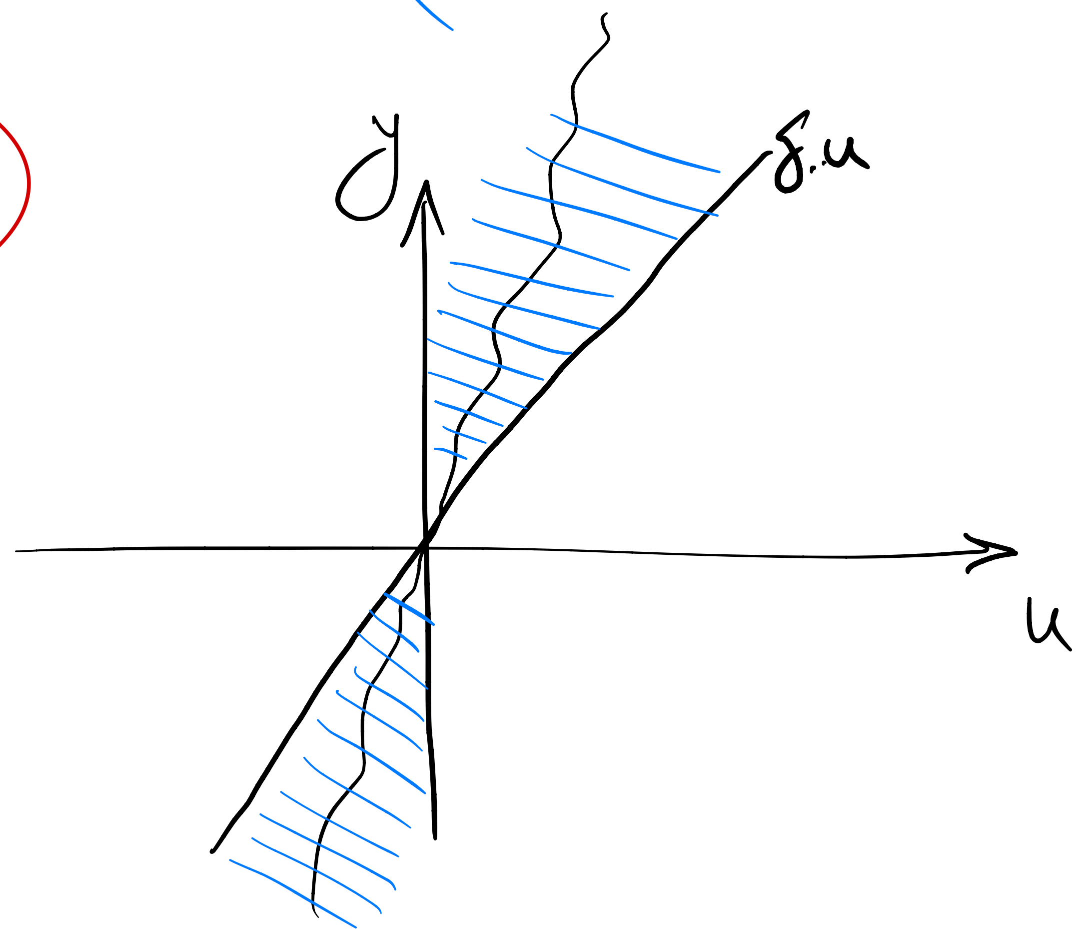
$y \cdot u \geq 0$

$h(u) \cdot u \geq 0$



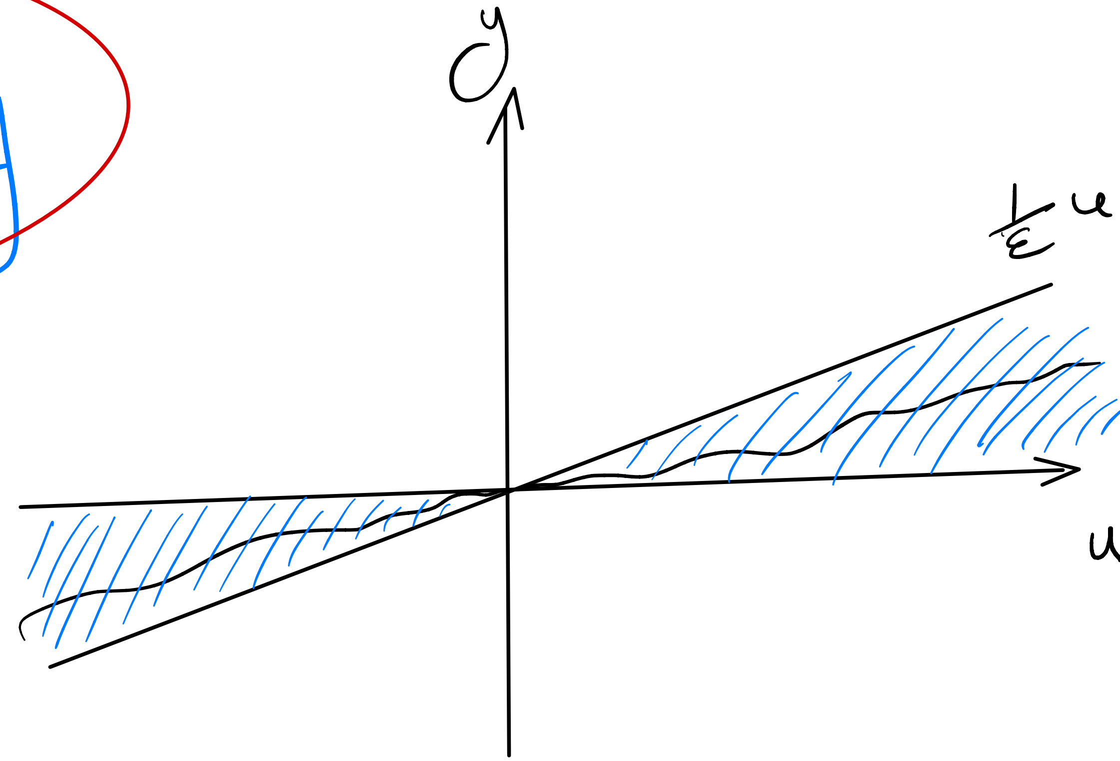
Input strict passivity

$y \cdot u \geq \delta u^2$



Output strict passivity

$y \cdot u \geq \varepsilon y^2$



Note!

Remember that Lp stability had a bow-tie region!