MECH 6313 - Term Exam

Name: Jonas Wagner
UTD ID: 2021531784

2021, April 30

Contents

1 Problem 1

Consider the system:

$$\begin{aligned}\tau\dot{x} &= x - \frac{1}{3}x^3 - y \\ \dot{y} &= x + \mu\end{aligned} \tag{1.1}$$

where $\tau > 0$ and $\mu \geq 0$ are constants.

1.1 Part a

Problem: Determine the equilibrium points and classify their stability properties depending on the values of parameter μ.

Solution:

1.1.1 Equilibrium Point Identification

The equilibrium points exist whenever $\dot{x} = \dot{y} = 0$ and can be identified as follows:

$$\begin{aligned}\tau(0) &= x - \frac{1}{3}x^3 - y \\ (0) &= x + \mu\end{aligned} \tag{1.2}$$

which becomes:

$$\begin{aligned}&y = x - \frac{1}{3}x^3 \\ &x = -\mu\end{aligned} \tag{1.3}$$

and can then substituted in as:

$$\begin{aligned}&x_{eq} = -\mu \\ &y_{eq} = -\mu - \frac{1}{3}(-\mu)^3\end{aligned} \tag{1.4}$$

This results in the equilibrium points being defined in terms of μ as:

$$\boxed{\begin{aligned}&x_{eq} = -\mu \\ &y_{eq} = \frac{1}{3}\mu^3 - \mu\end{aligned}} \tag{1.5}$$

1.1.2 System Linearization

The stability around an equilibrium point can be evaluated by looking at the linearized model, which can be found as follows:

Let the state-variables be defined as:

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

The nonlinear state equation would then be defined as:

$$\dot{X} = f(x) = \begin{bmatrix} \dfrac{x_1 - \frac{1}{3}x_1^3 - x_2}{\tau} \\ x_1 + \mu \end{bmatrix} \tag{1.6}$$

Then the equilibrium point is described as

$$X_{eq} = \begin{bmatrix} -\mu \\ \frac{1}{3}\mu^3 - \mu \end{bmatrix}$$

and the jacobian can be computed as:

$$J_x = \frac{\mathrm{d}f}{\mathrm{d}X} = \begin{bmatrix} \frac{\mathrm{d}f_1}{\mathrm{d}x_1} & \frac{\mathrm{d}f_1}{\mathrm{d}x_2} \\ \frac{\mathrm{d}f_2}{\mathrm{d}x_1} & \frac{\mathrm{d}f_2}{\mathrm{d}x_2} \end{bmatrix} \tag{1.7}$$

$$= \begin{bmatrix} \dfrac{1 - x_1^2}{\tau} & -1 \\ 1 & 0 \end{bmatrix} \tag{1.8}$$

The state dynamics around the equilibrium are described by this Jacobian evaluated at $X = X_{eq}$:

$$A = J_x\Big|_{X=X_{eq}} = \begin{bmatrix} \dfrac{1 - x_1^2}{\tau} & -1 \\ 1 & 0 \end{bmatrix}\Bigg|_{x_1=-\mu, x_2=\frac{1}{3}\mu^3-\mu} \tag{1.9}$$

$$= \begin{bmatrix} \dfrac{1 - (-\mu)^2}{\tau} & -1 \\ 1 & 0 \end{bmatrix} \tag{1.10}$$

$$= \begin{bmatrix} \frac{1}{\tau} - \frac{\mu^2}{\tau} & -1 \\ 1 & 0 \end{bmatrix} \tag{1.11}$$

The dynamics of this linearized system are described by the characteristic polynomial calculated as:

$$\Delta(s) = \det(sI - A) = \det \begin{bmatrix} s - \left(\frac{1}{\tau} - \frac{\mu^2}{\tau}\right) & 1 \\ -1 & s \end{bmatrix} \tag{1.12}$$

$$= s\left(s - \left(\frac{1}{\tau} - \frac{\mu^2}{\tau}\right)\right) - (1)(-1) \tag{1.13}$$

$$\boxed{\Delta(s) = s^2 - \left(\frac{1}{\tau} - \frac{\mu^2}{\tau}\right)s + 1} \tag{1.14}$$

1.1.3 Linearized Model Stability

The roots of $\Delta(s)$ are the eigenvalues of the linearization and are dependent on μ and τ calculated as:

$$\Lambda(A) = \lambda_{1,2} = \frac{\left(\frac{1}{\tau} - \frac{\mu^2}{\tau}\right) \pm \sqrt{\left(\frac{1}{\tau} - \frac{\mu^2}{\tau}\right)^2 - 4(1)(1)}}{2(1)} \tag{1.15}$$

$$= \frac{1}{2}\left(\frac{1}{\tau} - \frac{\mu^2}{\tau}\right) \pm \frac{1}{2}\sqrt{\left(\frac{1}{\tau} - \frac{\mu^2}{\tau}\right)^2 - 4} \tag{1.16}$$

or in a factored form:

$$= \frac{1}{2\tau}\left(1 - \mu^2\right) \pm \frac{1}{2\tau}\sqrt{\left(1 - \mu^2\right)^2 - 4\tau^2} \tag{1.17}$$

$$= \frac{1}{2\tau}\left(\left(1 - \mu^2\right) \pm \sqrt{\mu^4 - 2\mu^2 + 1 - 4\tau^2}\right) \tag{1.18}$$

or in a fully factored form:

$$= \frac{1 - \mu^2}{2\tau}\left(1 \pm \sqrt{1 - \frac{4\tau^2}{\left(1 - \mu^2\right)^2}}\right) \tag{1.19}$$

or in condenced form:

$$= \left(\frac{1}{2\tau} - \frac{\mu^2}{2\tau}\right) \pm \sqrt{\left(\frac{1}{2\tau} - \frac{\mu^2}{2\tau}\right)^2 - 1} \tag{1.20}$$

The roots are entirely **real** when:

$$\left(1 - \mu^2\right)^2 - 4\tau^2 > 0 \tag{1.21}$$

$$\left(1 - \mu^2\right)^2 > 4\tau^2 \tag{1.22}$$

$$1 - \mu^2 > 2\tau \tag{1.23}$$

$$\mu^2 + 2\tau > 1 \tag{1.24}$$

in which case, the linearized system is **stable** only when:

$$0 > \Re(\lambda_1) = \frac{1}{2\tau}\left(\left(1 - \mu^2\right) + \sqrt{\left(1 - \mu^2\right)^2 - 4\tau^2}\right) \tag{1.25}$$

$$\boxed{\Re(\lambda_1) = 1 - \mu^2 + \sqrt{\left(1 - \mu^2\right)^2 - 4\tau^2} < 0} \tag{1.26}$$

The system has **complex roots** when :

$$\left(1 - \mu^2\right)^2 - 4\tau^2 < 0 \tag{1.27}$$

$$\left(1 - \mu^2\right)^2 < 4\tau^2 \tag{1.28}$$

$$1 - \mu^2 < 2\tau \tag{1.29}$$

$$\mu^2 + 2\tau < 1 \tag{1.30}$$

in which case, the linearized system is only **stable** when

$$0 > \Re(\lambda_{1,2}) = \frac{1}{2\tau}\left(1 - \mu^2\right) \tag{1.31}$$

$$= 1 - \mu^2 \tag{1.32}$$

$$\boxed{\mu^2 > 1} \tag{1.33}$$

1.2 Part b

Problem: At which value of μ does a bifurcation occur and what type of bifurcation is it?

Solution: The stability properties of the linearized system indicate that a hopf bifurcation occur when

$$\mu = 1$$

and the system condenses the system into a single equalibrium point at the (parameter dependent) equalibrium point of

$$X_{eq} = \begin{bmatrix} -\mu \\ \frac{1}{3}\mu^3 - \mu \end{bmatrix}$$

.

1.3 Part c

Problem: Assume $\tau << 1$ and sketch the phase portrait for two values of μ, one just below and one just above the bifurcation value.

Solution: As can be seen in the two phase portraits, Figure 1 and Figure 2, the stable focus becomes unstable when bifurcation occurs.

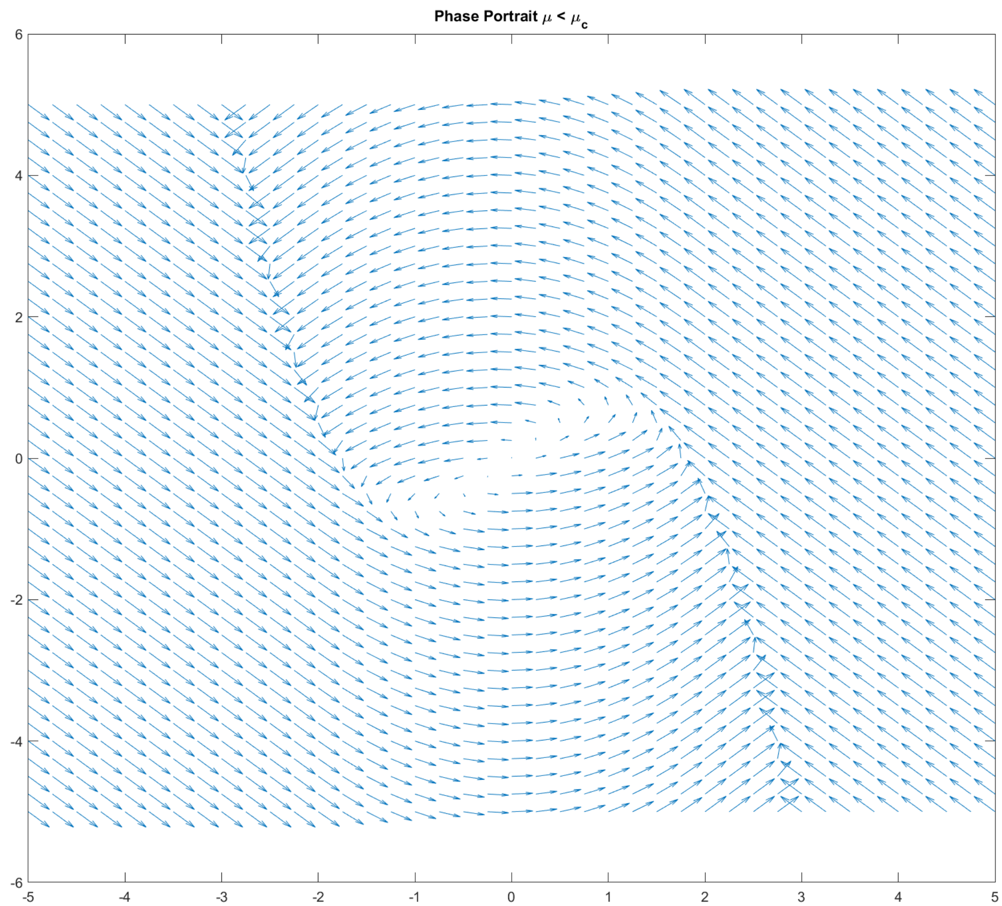

Figure 1: Phase Portrait for Problem 1 with the parameter below the critical value.

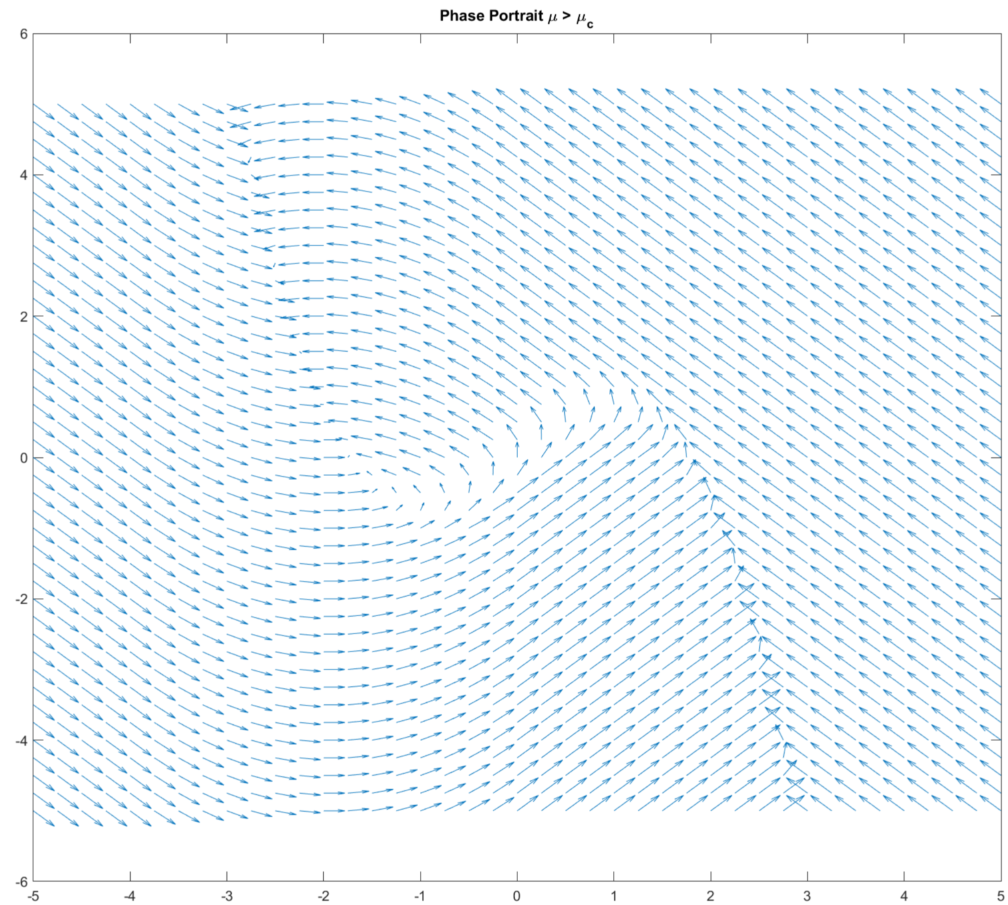

Figure 2: Phase Portrait for Problem 1 with the parameter above the critical value.

2 Problem 2:

Consider the system:

$$\begin{aligned}\dot{x}_1 &= -\frac{1}{2}\tan\left(\frac{\pi x_1}{2}\right) + x_2 \\ \dot{x}_2 &= x_1 - \frac{1}{2}\tan\left(\frac{\pi x_2}{2}\right)\end{aligned} \tag{2.1}$$

2.1 Part a

Problem: Find all equilibrium points of this system.

Solution: The equilibrium points exist whenever $\dot{X} = \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = 0$ and can be identified as follows:

$$\begin{aligned}(0) &= -\frac{1}{2}\tan\left(\frac{\pi x_1}{2}\right) + x_2 \\ (0) &= x_1 - \frac{1}{2}\tan\left(\frac{\pi x_2}{2}\right)\end{aligned} \tag{2.2}$$

which becomes:

$$\begin{aligned}x_2 &= \frac{1}{2}\tan\left(\frac{\pi x_1}{2}\right) \\ x_1 &= \frac{1}{2}\tan\left(\frac{\pi x_2}{2}\right)\end{aligned} \tag{2.3}$$

There are an infinite number of solutions to this set of equations, each of which are equilibrium points.

At each asymptote, there are unstable equilibrium points

$$\boxed{X_{eq} = \begin{bmatrix}1 \\ 1\end{bmatrix} + i\begin{bmatrix}2 \\ 0\end{bmatrix} + j\begin{bmatrix}0 \\ 2\end{bmatrix}} \tag{2.4}$$

with $i = \dots, -1, 0, 1, \dots$ and $j = \dots, -1, 0, 1, \dots$

In addition, any time the each subsequent tangent function intersect with each other another another equilibrium point exists, including, but not exhausted:

$$\boxed{x = \frac{1}{2}\tan\left(\frac{\pi x}{2}\right) \text{ or } x = -\frac{1}{2}\tan\left(\frac{\pi x}{2}\right)} \tag{2.5}$$

Additionally, within the region around the origin $x \in [-1, 1]$ and $y \in [-1, 1]$, 3 distinct equilibrium points exist:

$$\boxed{\begin{aligned}X_{eq} &= \begin{bmatrix}0 \\ 0\end{bmatrix} \\ X_{eq} &= \begin{bmatrix}0.5 \\ 0.5\end{bmatrix} \\ X_{eq} &= \begin{bmatrix}-0.5 \\ -0.5\end{bmatrix}\end{aligned}} \tag{2.6}$$

2.2 Part b

Problem: Use linearization to study the stability of each equilibrium point.

Solution: The linearization around individual equilibrium points is given by evaluating the Jacobian matrix which is calculated as:

$$A = J_x\Big|_{X=X_{eq}} = \begin{bmatrix} \frac{\mathrm{d}f_1}{\mathrm{d}x_1} & \frac{\mathrm{d}f_1}{\mathrm{d}x_2} \\ \frac{\mathrm{d}f_2}{\mathrm{d}x_1} & \frac{\mathrm{d}f_2}{\mathrm{d}x_2} \end{bmatrix}\Bigg|_{X=X_{eq}} \tag{2.7}$$

$$= \begin{bmatrix} -\frac{\pi}{4}\left(\tan^2\left(\frac{\pi x_1}{2}\right)+1\right) & 1 \\ 1 & -\frac{\pi}{4}\left(\tan^2\left(\frac{\pi x_2}{2}\right)+1\right) \end{bmatrix}\Bigg|_{X=X_{eq}} \tag{2.8}$$

Using the nlsys class I developed in MATLAB, see Appendix A, multiple Equilibrium points were linearized and stability was analyzed.

It was determined based on the eigenvalues of the linear systems that all the "equilibrium points" occurring at asymtopes (2.4) were all unstable, the asymptopes (that were checked) satisfying (2.5) were asymptotically stable, and the analysis of the 3 important equilibrium points (2.6) are addressed below:

The origin itself was determined to be unstable given the dynamics matrix and eigenvalues:

$$\boxed{X_{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad A = \begin{bmatrix} -0.7854 & 1 \\ 1 & -0.7854 \end{bmatrix} \quad \begin{matrix} \lambda_1 = -1.7854 \\ \lambda_2 = 0.2146 \end{matrix}} \tag{2.9}$$

The quadrant 1 and 3 systems were determined to be asymptotically stable given the dynamics matrix and eigenvalues:

$$\boxed{X_{eq} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \quad A = \begin{bmatrix} -1.5708 & 1 \\ 1 & -1.5708 \end{bmatrix} \quad \begin{matrix} \lambda_1 = -2.5708 \\ \lambda_2 = -0.5708 \end{matrix}} \tag{2.10}$$

$$\boxed{X_{eq} = \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix} \quad A = \begin{bmatrix} -1.5708 & 1 \\ 1 & -1.5708 \end{bmatrix} \quad \begin{matrix} \lambda_1 = -2.5708 \\ \lambda_2 = -0.5708 \end{matrix}} \tag{2.11}$$

2.3 Part c

Problem: Using quadratic Lyapunov functions, estimate the region of attraction of each asymptotically stable equilibrium point.

Solution:

For the asymptotically stable equalibrium point in quadrant 1, a relative system can be defined with

$$\tilde{x} = x - x_{eq}$$

resulting in

$$f(\tilde{x}) = \begin{bmatrix} -\frac{1}{2}\tan\left(\frac{\pi(\tilde{x}_1 + x_{1eq})}{2}\right) + \tilde{x}_2 + x_{2eq} \\ -\frac{1}{2}\tan\left(\frac{\pi(\tilde{x}_2 + x_{2eq})}{2}\right) + \tilde{x}_1 + x_{1eq} \end{bmatrix} \tag{2.12}$$

$$= \begin{bmatrix} -\frac{1}{2}\tan\left(\frac{\pi\tilde{x}_1}{2} + \frac{\pi}{4}\right) + \tilde{x}_2 + \frac{1}{2} \\ -\frac{1}{2}\tan\left(\frac{\pi\tilde{x}_2}{2} + \frac{\pi}{4}\right) + \tilde{x}_1 + \frac{1}{2} \end{bmatrix} \tag{2.13}$$

Additionally, an invarient set boundary can be defined by

$$\tilde{x}_1^2 + \tilde{x}_2^2 = r^2$$

which represents a simple radial boundary.
The maximum positive invariant set boundary can be calculated by solving for r s.t.

$$f^T(x) \cdot \boldsymbol{\nabla} V(x) \leq 0$$

and is shown as follows:

$$V(x) = \tilde{x}_1^2 + \tilde{x}_2^2 = r^2 \tag{2.14}$$

$$\boldsymbol{\nabla} V(x) = \begin{bmatrix} 2\tilde{x}_1 \\ 2\tilde{x}_2 \end{bmatrix} \tag{2.15}$$

$$f^T(x) \cdot \boldsymbol{\nabla} V(x) = \begin{bmatrix} -\frac{1}{2}\tan\left(\frac{\pi\tilde{x}_1}{2} + \frac{\pi}{4}\right) + \tilde{x}_2 + \frac{1}{2} \\ -\frac{1}{2}\tan\left(\frac{\pi\tilde{x}_2}{2} + \frac{\pi}{4}\right) + \tilde{x}_1 + \frac{1}{2} \end{bmatrix}^T \begin{bmatrix} 2\tilde{x}_1 \\ 2\tilde{x}_2 \end{bmatrix} \tag{2.16}$$

$$\begin{aligned} &= \left(-\frac{1}{2}\tan\left(\frac{\pi\tilde{x}_1}{2} + \frac{\pi}{4}\right) + \tilde{x}_2 + \frac{1}{2}\right)(2\tilde{x}_1) \\ &\quad + \left(-\frac{1}{2}\tan\left(\frac{\pi\tilde{x}_2}{2} + \frac{\pi}{4}\right) + \tilde{x}_1 + \frac{1}{2}\right)(2\tilde{x}_2) \end{aligned} \tag{2.17}$$

$$\begin{aligned} &= \tilde{x}_1\tan\left(\frac{\pi\tilde{x}_1}{2} + \frac{\pi}{4}\right) + \tilde{x}_2\tan\left(\frac{\pi\tilde{x}_2}{2} + \frac{\pi}{4}\right) \\ &\quad + 4\tilde{x}_1\tilde{x}_2 + \tilde{x}_1 + \tilde{x}_2 \end{aligned} \tag{2.18}$$

Which when solved results in a conservative region of attraction of radius $r \leq 0.5$. (or maybe strictly less-than... calculator acted funny when solving it)

Additionally, when the process was repeated for the 3rd quadrant equilibrium point, the results were the same and the conservative region of attraction had a radius of $r = 0.5$.

A few other equilibrium points were tested and it was clear that the asymptotic eq-points were unstable, but of the other stable eq-points (done numerically) they were all very very conservative results using this method. (SOS is probably a better option... or perhaps a more complicated elliptical bound instead)

2.3.1 Part d

Problem: Plot the phase portrait of the system and show on it the exact regions of attraction as well as your estimates.

Solution: The phase portrait of the system was plotted along with the important indicators for equilibrium points and boundary regions as seen in Figure 3, Figure 4, and Figure 5.

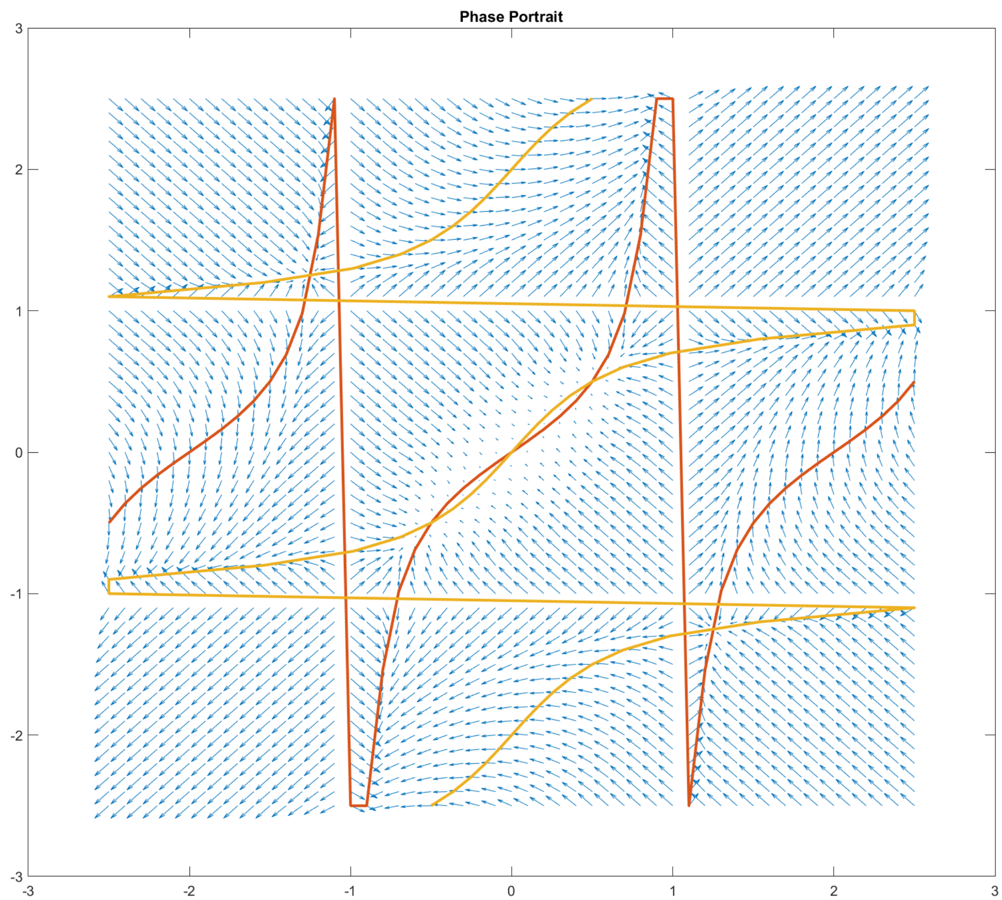

Figure 3: Phase Portrait for Problem 2

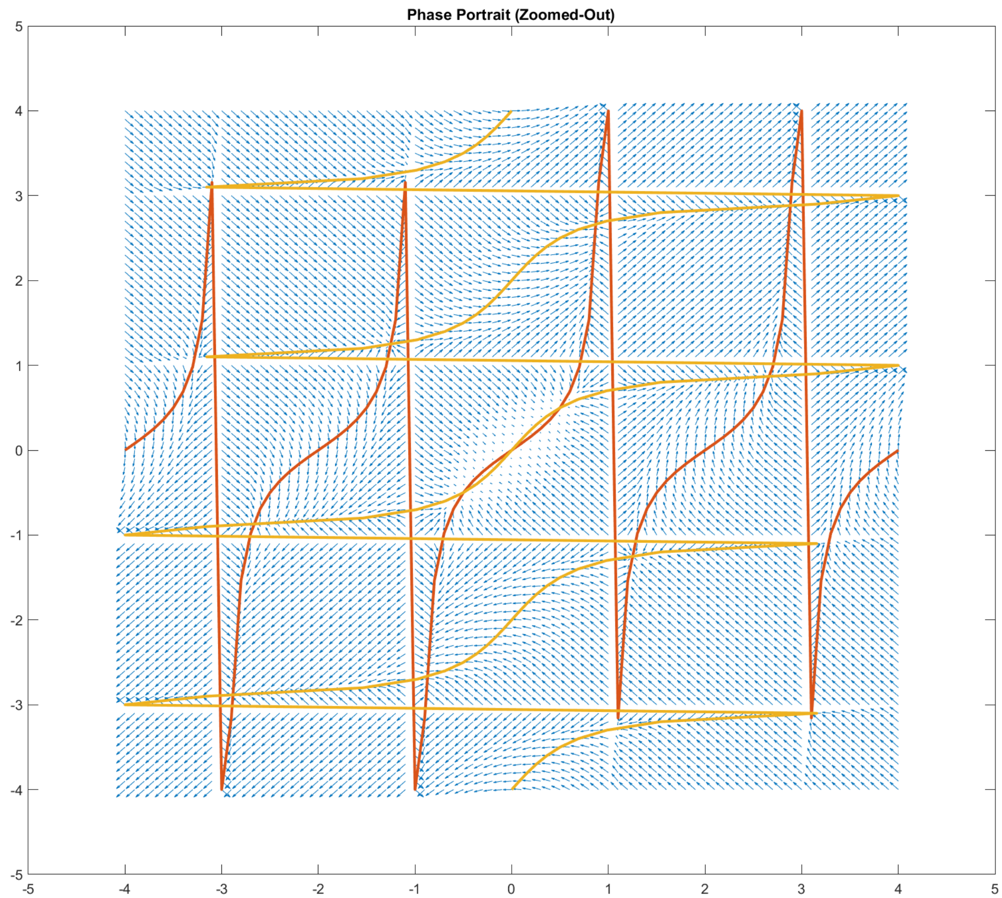

Figure 4: Phase Portrait for Problem 2 that is zoomed out to demonstrate the infinite nature of equilibrium points.

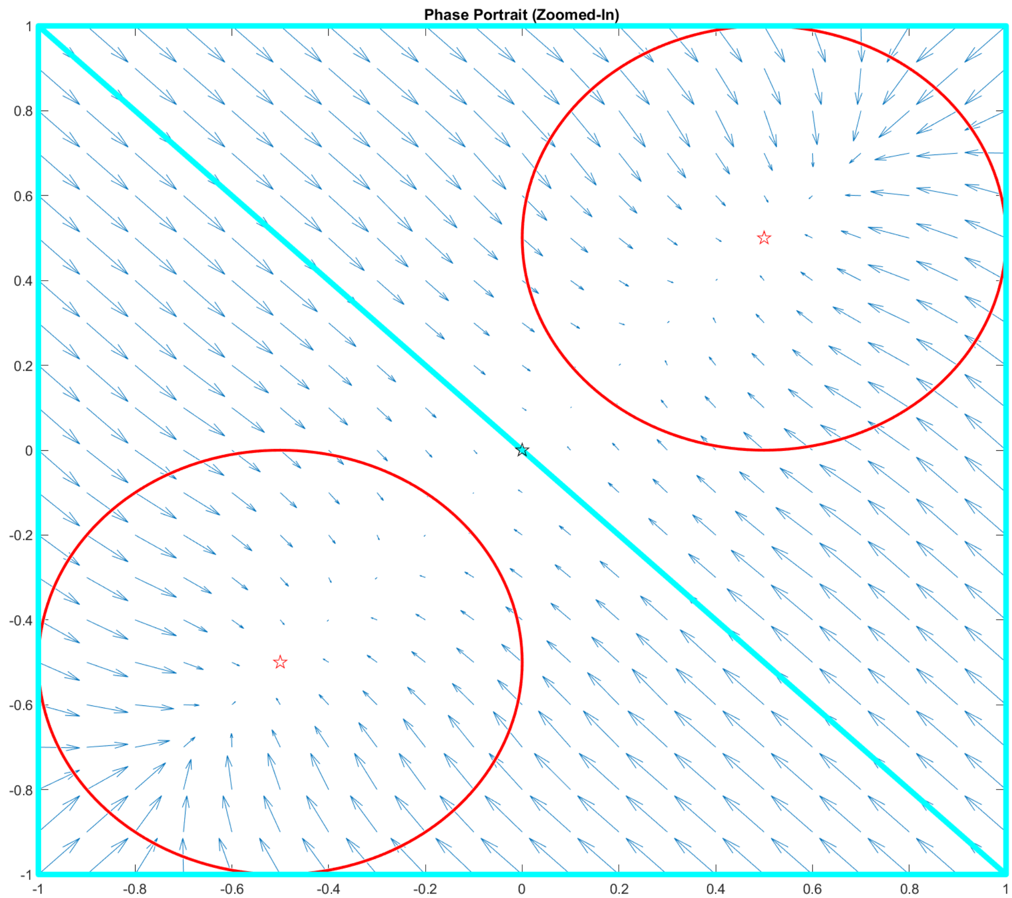

Figure 5: Phase Portrait for Problem 2 focusing on the origin with regions of convergences and equilibrium points.

3 Problem 3

Problem: Prove that the origin is the globally asymptotically stable equilibrium point of the system

$$
\begin{aligned}
\dot{x}_1 &= -x_1 - \mathrm{sat}(x_3) \\
\dot{x}_1 &= -x_2 - \mathrm{sat}(x_1) \\
\dot{x}_1 &= -x_3 - \mathrm{sat}(x_2)
\end{aligned} \tag{3.1}
$$

where

$$
\mathrm{sat}(x) := \mathrm{sign}(x)\min\{1, |x|\} \tag{3.2}
$$

Solution:

3.1 System and Storage Function Definition

This system can be rewritten in a coupling feedback system H and K defined as:

$$
H = \begin{bmatrix} H_1 & & \\ & H_2 & \\ & & H_3 \end{bmatrix}, \qquad H_i = \begin{cases} \dot{x}_i = f(x_i) + g(x_i)u_i \\ y_i = h(x_i) \end{cases} \tag{3.3}
$$

$$
K = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \tag{3.4}
$$

where K is the coupling matrix of the individual nonlinear subsystems with the following specific definitions:

$$
H_i = \begin{cases} \dot{x}_i = -x_i + u_i \\ y_i = h_i(x_i) \end{cases} \tag{3.5}
$$

where $h_i(x_i) = \mathrm{sat}(x_i)$.

A storage function for each of the individual subsystems can be defined as:

$$
V_i(x_i) = \int_0^{x_i} h_i(\eta)\mathrm{d}\eta \tag{3.6}
$$

Taking a look at the change of the storage function over time, the output strict passivity can be proven by:

$$
\begin{aligned}
\dot{V}_i(x_i) &= \frac{\mathrm{d}V_i}{\mathrm{d}x_i}\dot{x}_i && (3.7) \\
&= \frac{\mathrm{d}}{\mathrm{d}x_i}\int_0^{x_i} h_i(\eta)\,\mathrm{d}\eta\,\dot{x}_i && (3.8) \\
&= h_i(x_i)\dot{x}_i && (3.9)
\end{aligned}
$$

taking the definition for $\dot{x}_i$ and relating $h_i(x_i) = y_i$,

$$
\begin{aligned}
&= h_i(x_i)(-x_i + u_i) && (3.10) \\
&= -x_i h_i(x_i) + u_i y_i && (3.11)
\end{aligned}
$$

3.2 Probing Input/Output Passivity

In order to guarantee Input passivity, the system must satisfy the following inequality:

$$xh(x) \leq \delta_i x^2 \tag{3.12}$$

$$x(\text{sat}(x)) \leq \delta x^2 \tag{3.13}$$

by definition, $\text{sat}(x) := \text{sign}(x)\min\{1, |x|\}$, thus the following inequalities apply:

$$\begin{cases} \text{sat}(x) > 0, & x > 0 \\ \text{sat}(x) < 0, & x < 0 \end{cases} \tag{3.14}$$

therefore, the input passivity equality holds.

Since the input passivity holds, a δ_i will exist s.t.,

$$x_i h_i(x_i) \leq \delta_i x^2 \tag{3.15}$$

$$x_i(h_i(x_i) - \delta_i x_i) \leq 0 \tag{3.16}$$

clearly, $x_i h_i(x_i)$ can then be bounded from below by:

$$x_i h_i(x) \geq \frac{1}{\delta_i} h_i^2(x_i) \tag{3.17}$$

$$-x_i h_i(x) \geq -\frac{1}{\delta_i} h_i^2(x_i) \tag{3.18}$$

since $y_i = h_i(x_i)$,

$$-x_i h_i(x) \geq -\frac{1}{\delta_i} y_i^2(x_i) \tag{3.19}$$

Therefore, this demonstrates Output Strict Passivity:

$$\dot{V}_i \leq -\frac{1}{\delta_i} y_i^2 + y_i u_i \tag{3.20}$$

or with $d_i = \frac{1}{\delta_i}$ and

$$\dot{V}_i \leq d_i y_i^2 + y_i u_i \tag{3.21}$$

and the passivity theorem can then be applied.

3.3 Applying Passivity Theorem

Let

$$\epsilon_i = \frac{1}{\delta_i}$$

and then define

$$A = -\text{diag}\{\epsilon_i\} + K \tag{3.22}$$

$$P = \text{diag}\{d_i\} \tag{3.23}$$

which for this 3^{rd}-order system can be written as

$$A = -\begin{bmatrix} \epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -\epsilon_1 & 0 & -1 \\ -1 & -\epsilon_2 & 0 \\ 0 & -1 & -\epsilon_3 \end{bmatrix} \tag{3.24}$$

$$P = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \tag{3.25}$$

Appropriate values for A and P can be found to prove stability of the full feedback interconnection using the following inequality:

$$A^T P + PA \leq 0 \tag{3.26}$$

This can then be further developed to prove Global Asymptotic Stability by making it a strict inequality like so:

$$A^T P + PA < 0 \tag{3.27}$$

This can be written with the actual matrices as

$$\begin{bmatrix} -\epsilon_1 & 0 & -1 \\ -1 & -\epsilon_2 & 0 \\ 0 & -1 & -\epsilon_3 \end{bmatrix}^T \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} + \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \begin{bmatrix} -\epsilon_1 & 0 & -1 \\ -1 & -\epsilon_2 & 0 \\ 0 & -1 & -\epsilon_3 \end{bmatrix} < 0 \tag{3.28}$$

$$\begin{bmatrix} -d_1\epsilon_1 & -d_1 & 0 \\ 0 & -d_2\epsilon_2 & -d_2 \\ -d_3 & 0 & -d_3\epsilon_3 \end{bmatrix} + \begin{bmatrix} -d_2\epsilon_1 & 0 & -d_1 \\ -d_2 & -d_2\epsilon_2 & 0 \\ 0 & -d_3 & -d_3\epsilon_3 \end{bmatrix} < 0 \tag{3.29}$$

$$\begin{bmatrix} -2d_1\epsilon_1 & -d_1 & -d_1 \\ -d_2 & -2d_2\epsilon_2 & -d_2 \\ -d_3 & -d_3 & -2d_3\epsilon_3 \end{bmatrix} < 0 \tag{3.30}$$

or equivalently

$$\begin{bmatrix} 2d_1\epsilon_1 & d_1 & d_1 \\ d_2 & 2d_2\epsilon_2 & d_2 \\ d_3 & d_3 & 2d_3\epsilon_3 \end{bmatrix} > 0 \tag{3.31}$$

The positive definiteness can be determined by

$$2d_1\epsilon_1 > 0 \tag{3.32}$$

$$\begin{vmatrix} 2d_1\epsilon_1 & d_1 \\ d_2 & 2d_2\epsilon_2 \end{vmatrix} = d_1d_2(4\epsilon_1\epsilon_2 - 1) > 0 \tag{3.33}$$

$$\begin{aligned}\begin{vmatrix} 2d_1\epsilon_1 & d_1 & d_1 \\ d_2 & 2d_2\epsilon_2 & d_2 \\ d_3 & d_3 & 2d_3\epsilon_3 \end{vmatrix} &= d_1d_2d_3(8\epsilon_1\epsilon_2\epsilon_3 - 2\epsilon_1) \\ &\quad -d_1d_2d_3(2\epsilon_3 - 1) + d_1d_2d_3(1 - 2\epsilon_2) \\ &= d_1d_2d_3(8\epsilon_1\epsilon_2\epsilon_3 - 2\epsilon_1 - 2\epsilon_2 - 2\epsilon_3) > 0 \end{aligned} \tag{3.34}$$

From this and the definition of $d_i > 0$, these inequalities can be equated to

$$\epsilon_1 > 0 \tag{3.35}$$

$$4\epsilon_1\epsilon_2 - 1 > 0 \tag{3.36}$$

$$4\epsilon_1\epsilon_2\epsilon_3 - \epsilon_1 - \epsilon_2 - \epsilon_3 > 0 \tag{3.37}$$

Returning to the original definition of $\epsilon_i = \frac{1}{\delta_i}$ and the limitation of $x\text{sat}(x) \leq \delta_i x^2$, it can be seen that a selection of $\delta_i = 1\ \forall i = 1, 2, 3$ is valid and thus

$$\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$$

which can be used to satisfy the inequalities:

$$(1) = 1 > 0 \tag{3.38}$$

$$4(1)(1) - 1 = 3 > 0 \tag{3.39}$$

$$4(1)(1)(1) - (1) - (1) - (1) = 1 > 0 \tag{3.40}$$

Therefore, it can be seen said that the origin for the coupled feedback system is Globally Asymptotically Stable.

4 Problem 4

Problem: Comment on the existence/uniqueness of solutions for the systems below. Provide your reasons.

4.1 Part a

$$\dot{x} = x^2 \tag{4.1}$$

Solution: Assuming that the system is defined for $x \in \Re$, existence and uniqueness can be guaranteed by ensuring certain continuity conditions exist.
Since the function $f(x) = x^2$ is a continuous function $\forall x \in \Re$, it can be said that a solution does exist.
Additionally, since $f(x)$ is locally Lipschitz continuous, i.e. $\frac{\mathrm{d}f}{\mathrm{d}x} = 2x$ is continuous, it can be said that a unique solution exists for $t \in [0, t_f)$.
However, $f(x)$ is not globally Lipschitz continous, since $\left\|\frac{\mathrm{d}f}{\mathrm{d}x}\right\| = \|2x\| \nleq L \forall x \in \Re^n$, (which can be more rigorously proven as this was only a sufficient condition) the uniqueness of a solution cannot be garunteed for $t \in [0, \infty)$.

4.2 Part b

$$\dot{x} = \sqrt{x} \tag{4.2}$$

Solution: Assuming that the system is defined for $x \geq 0$, existence and uniqueness can be guaranteed by ensuring certain continuity conditions exist.
Since the function $f(x) = x^2$ is a continuous function $\forall x \geq 0$, it can be said that a solution does exist.
However, a unique solution cannot be guaranteed as the function is not Liptchitz continuous directly around $x = 0$ as the slope becomes infinite and cannot be bounded by a Liptchitz constant.

4.3 Part c

$$\dot{x} = 1 + \frac{1 + x^3}{1 + x^4} \tag{4.3}$$

Solution: Assuming that the system is defined for $x \in \Re$, existence and uniqueness can be guaranteed by ensuring certain continuity conditions exist.
Since the function $f(x)$ is a continuous function $\forall x \in \Re$, it can be said that a solution does exist.
Additionally, the system is also continuously differentiable:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(1 + \frac{1 + x^3}{1 + x^4}\right) = \frac{-x^2(x^4 - 2x - 3)}{(1 + x^4)^2}$$

and its derivative is bounded

$$\left\|\frac{-x^2(x^4 - 2x - 3)}{(1 + x^4)^2}\right\| \leq L$$

by the positive constant $L < \infty$. This implys that the system is globally Lipshitz continuous and therefore a unique solution is guaranteed to exists for $t \in [0, \infty)$.

5 Problem 5

Problem: Show that the following system contains no closed orbits.

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^3 + 1 \\ \dot{x}_2 &= -4x_1^2 + 3x_2 \end{aligned} \tag{5.1}$$

Solution: Sufficient conditions to proving that no closed orbits exist are that If $\boldsymbol{\nabla} \cdot f \neq 0 \forall x \in D$ and does not change sign within a simply connected region D. Let $D = x \in \Re^2$. The divergence is given as:

$$\boldsymbol{\nabla} \cdot f = \frac{\mathrm{d}f_1}{\mathrm{d}x_1} + \frac{\mathrm{d}f_2}{\mathrm{d}x_2} \tag{5.2}$$

$$= -1 + 3 \tag{5.3}$$

$$= 4 \tag{5.4}$$

Since $\boldsymbol{\nabla} \cdot f$ is constant (and not identically zero) within the entire region D, there is sufficient evidence to say that no periodic orbits exist and therefore the system has no closed orbits.

6 Problem 6

Problem: Prove that the origin is the globally asymptotically stable equilibrium of the following system.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(\sin(x_1) + 2)(x_1 + x_2) \end{aligned} \tag{6.1}$$

Solution:

6.1 Initial Linearized System Stability

The linearization around individual equilibrium points is given by evaluating the Jacobian matrix which is calculated as:

$$A = J_x\Big|_{X=X_{eq}} = \begin{bmatrix} \frac{\mathrm{d}f_1}{\mathrm{d}x_1} & \frac{\mathrm{d}f_1}{\mathrm{d}x_2} \\ \frac{\mathrm{d}f_2}{\mathrm{d}x_1} & \frac{\mathrm{d}f_2}{\mathrm{d}x_2} \end{bmatrix}\Bigg|_{X=X_{eq}} \tag{6.2}$$

$$= \begin{bmatrix} 0 & 1 \\ -(\sin(x_1) + x_1\cos(x_1) + 2 + x_2\cos(x_1)) & -(\sin(x_1)+2) \end{bmatrix}\Bigg|_{x_1=x_2=0} \tag{6.3}$$

$$= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \tag{6.4}$$

The dynamics of this linearized system are described by the characteristic polynomial calculated as:

$$\Delta(s) = \det(sI - A) = \det\begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix} \tag{6.5}$$

$$= s(s+2) - (-1)(2) \tag{6.6}$$

$$\boxed{\Delta(s) = s^2 + 2s + 2} \tag{6.7}$$

The roots of this polynomial are then calculated as the eigenvalues:

$$\boxed{\lambda_{1,2} = -1 \pm j1} \tag{6.8}$$

From this it is apparent that, locally, there exists a stable focus around the origin.

6.2 Lyapnov Method

The maximum positive invariant set boundary can be calculated by solving for r s.t.

$$f^T(x) \cdot \boldsymbol{\nabla} V(x) \leq 0$$

and is shown as follows:

$$V(x) = x_1^2 + x_2^2 = r^2 \tag{6.9}$$

$$\boldsymbol{\nabla} V(x) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \tag{6.10}$$

$$f^T(x) \cdot \boldsymbol{\nabla} V(x) = \begin{bmatrix} x_2 & -(\sin(x_1) + 2)(x_1 + x_2) \end{bmatrix} \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \tag{6.11}$$

$$= 2x_1 x_2 - 2x_2(x_1 \sin(x_1) + 2x_1 + x_2 \sin(x_1) + 2x_2) \tag{6.12}$$

$$= 2x_1 x_2 - 2x_1 x_2 \sin(x_1) - 4x_1 x_2 - 2x_2^2 \sin(x_1) - 4x_2^2 \tag{6.13}$$

$$= -2x_1 x_2(1 + \sin(x_1)) - 2x_2^2(2 + \sin(x_1)) \tag{6.14}$$

The second term is clearly not problimatic:

$$-2x_2^2(2 + \sin(x_1)) \leq 0, \ \forall x_1, x_2 \in \Re$$

However, the first term is not as straight forward. It is true that

$$1 + \sin(x_1) \geq 0$$

and that in quadrants 1 and 3, the term satisfies the requirements. In quadrants 2 and 4 it becomes problematic as the term does not always remain negative. However, it is possible to prove that the boundary region itself is unbounded as

$$|2x_1 x_2(1 + sin(x_1))| \leq \left|2x_2^2(2 + sin(x_1))\right|, \forall \ x_1, x_2 \in \Re$$

This leads to the conclusion that the system is Globally Asymptotically Stable since the region of convergence is $x_1, x_2 \in \Re$.

A MATLAB Code:

All code I write in this course can be found on my GitHub repository:
https://github.com/jonaswagner2826/MECH6313

Script 1: MECH6313_Exam

```
% MECH 6313 - Exam

clear
close all

pblm1 = true;
pblm2 = true;
pblm3 = false;
pblm4 = false;
pblm5 = false;
pblm6 = false;

if pblm1
%% Problem 1

satVal = 2;

tau = 0.25;
mu = 0.1;
% Phase Plot 1
figure('position',[0,0,1200,1000])
xmax = 5;
ymax = 5;
xmin = -5;
ymin = -5;
xstep = 0.25;
ystep = xstep;

[X,Y] = meshgrid(xmin:xstep:xmax,ymin:ystep:ymax);
DX = max(min((X - (X.^3)/3 - Y)/tau, satVal), -satVal);
DY = max(min(X + mu, satVal), -satVal);

quiver(X,Y,DX,DY)
title('Phase Portrait \mu < \mu_c')
hold on
% x = [xmin:xstep:xmax];
% y = max(min(1/2*tan(pi/2 * x), xmax), xmin);
% plot(x,y, 'LineWidth', 2)
```

```
% plot(y,x, 'LineWidth', 2)
saveas(gcf,[pwd,'\Exam\fig\pblm1_phaseplot_mu01.png'])



% tau = 0.1;
mu = 2;
% Phase Plot 2
figure('position',[0,0,1200,1000])
% xmax = 1;
% ymax = xmax;
% xmin = -1.5*xmax;
% ymin = xmin;
% xstep = 0.5;
% ystep = xstep;

[X,Y] = meshgrid(xmin:xstep:xmax,ymin:ystep:ymax);
DX = max(min((X - (X.^3)/3 - Y)/tau, satVal), -satVal);
DY = max(min(X + mu, satVal), -satVal);

quiver(X,Y,DX,DY)
title('Phase Portrait \mu > \mu_c')
hold on
% x = [xmin:xstep:xmax];
% y = max(min(1/2*tan(pi/2 * x), xmax), xmin);
% plot(x,y, 'LineWidth', 2)
% plot(y,x, 'LineWidth', 2)
saveas(gcf,[pwd,'\Exam\fig\pblm1_phaseplot_mu2.png'])


end

if pblm2
%% Problem 2
solveEqPnt = false;
phasePlt = true;
linSysCalc = false;


if solveEqPnt
% --------------------------------------------
% Equalibrium Points
syms x1 x2
```

```
eq1 = 0 == -1/2 * tan(pi*x1/2) + x2;
eq2 = 0 == x1 -1/2 * tan(pi*x1/2);


[x1_eq, x2_eq] = vpasolve([eq1, eq2], [x1,x2]);

eq3 = x1 == 1/2 * tan(pi*x1/2);

x3_eq = solve(eq3, x1);
end

if phasePlt
% ---------------------------------------------
% Phase Plot 1
figure('position',[0,0,1200,1000])
xmax = 2.5;
ymax = xmax;
xmin = -xmax;
ymin = -ymax;
xstep = 0.1;
ystep = xstep;

[X,Y] = meshgrid(xmin:xstep:xmax,ymin:ystep:ymax);
DX = max(min(-1/2 * tan(pi*X/2) + Y, 1), -1);
DY = max(min(-1/2 * tan(pi*Y/2) + X, 1), -1);

quiver(X,Y,DX,DY)
title('Phase Portrait')
hold on
x = [xmin:xstep:xmax];
y = max(min(1/2*tan(pi/2 * x), xmax), xmin);
plot(x,y, 'LineWidth', 2)
plot(y,x, 'LineWidth', 2)
saveas(gcf,[pwd,'\Exam\fig\pblm2_phaseplot.png'])


% Phase Plot 2
figure('position',[0,0,1200,1000])
xmax = 1;
ymax = xmax;
xmin = -xmax;
ymin = -ymax;
xstep = 0.1;
```

```
ystep = xstep;

[X,Y] = meshgrid(xmin:xstep:xmax,ymin:ystep:ymax);
DX = max(min(-1/2 * tan(pi*X/2) + Y, 1), -1);
DY = max(min(-1/2 * tan(pi*Y/2) + X, 1), -1);

quiver(X,Y,DX,DY)
title('Phase Portrait (Zoomed-In)')
hold on
% Region of Attraction
viscircles([[0.5,0.5;-0.5,-0.5]],[0.5,0.5])
x=[-1,-1,1,1,-1,1];
y=[1,-1,-1,1,1,-1];
plot(x,y,'c','LineWidth',4)
% Eq-Points
scatter([-0.5,0.5],[-0.5,0.5],100,'rp')
scatter([0],[0],100,'kp')
saveas(gcf,[pwd,'\Exam\fig\pblm2_phaseplot_origin.png'])


% Phase Plot 3
figure('position',[0,0,1200,1000])
xmax = 4;
ymax = xmax;
xmin = -xmax;
ymin = -ymax;
xstep = 0.1;
ystep = xstep;

[X,Y] = meshgrid(xmin:xstep:xmax,ymin:ystep:ymax);
DX = max(min(-1/2 * tan(pi*X/2) + Y, 1), -1);
DY = max(min(-1/2 * tan(pi*Y/2) + X, 1), -1);

quiver(X,Y,DX,DY)
title('Phase Portrait (Zoomed-Out)')
hold on
x = [xmin:xstep:xmax];
y = max(min(1/2*tan(pi/2 * x), xmax), xmin);
plot(x,y, 'LineWidth', 2)
plot(y,x, 'LineWidth', 2)
saveas(gcf,[pwd,'\Exam\fig\pblm2_phaseplot_zoomOut.png'])
end

```

```
if linSysCalc
% -------------------------------------------------
% Linearized System Calc
sys2a = nlsys(@pblm2a)
syms x1 x2
linsys2a_sym = sys2a.linearize([x1;x2])
linsys2_0 = sys2a.linearize([0;0])
eig(linsys2_0)
linsys2_p05 = sys2a.linearize([0.5;0.5])
eig(linsys2_p05)
linsys2_n05 = sys2a.linearize([-0.5;-0.5])
eig(linsys2_n05)
linsys2_p1p1 = sys2a.linearize([1;1])
linsys2_n1n1 = sys2a.linearize([-1;-1])
linsys2_p1n1 = sys2a.linearize([1;-1])
linsys2_p125n125 = sys2a.linearize([1.25;-1.25])
eig(linsys2_p125n125)
linsys2_n1p1 = sys2a.linearize([-1;1])
linsys2_n125p125 = sys2a.linearize([-1.25;1.25])
eig(linsys2_n125p125)
end


end

if pblm3
%% Problem 3
end

if pblm4
%% Problem 4
end

if pblm5
%% Problem 5

syms x1 x2
eq1 = 0 == -x1 + x2^3 + 1;
eq2 = 0 == -4*x1^2 + 3*x2;
solve([eq1,eq2],[x1,x2])

```

```
sys5 = nlsys(@pblm5a)

% Phase Plot 2
figure()
xmax = 5;
ymax = xmax;
xmin = -xmax;
ymin = -ymax;
xstep = 0.1;
ystep = xstep;

[X,Y] = meshgrid(xmin:xstep:xmax,ymin:ystep:ymax);
DX = -X + Y^3 + 1%max(min(, 1), -1);
DY = -4*X^2 + 3*Y%max(min(, 1), -1);

quiver(X,Y,DX,DY)


end

if pblm6
%% Problem 6
sys6 = nlsys(@pblm6a)

linsys6 = sys6.linearize([0;0])


end


%% Local Functions
function y = pblm1a(x,u)
    % pblm1c function
    arguments
        x (2,:) = [0; 0];
        u (1,:) = 0;
    end

    % Array Sizes
    n = 2;
    p = 1;


```

```
    % Parameters
    tau = 0.1;
    mu = 0.9;


    % State Upadate Eqs
    y(1,1) = (x(1) - (x(1)^3)/3 - x(2))/tau;
    y(2,1) = x(1) + mu;

    if nargin == 0
        y = [n;p];
    end
end

function y = pblm1b(x,u)
    % pblm1c function
    arguments
        x (2,:) = [0; 0];
        u (1,:) = 0;
    end

    % Array Sizes
    n = 2;
    p = 1;

    % Parameters
    tau = 0.1;
    mu = 1.1;


    % State Upadate Eqs
    y(1,1) = (x(1) - (x(1)^3)/3 - x(2))/tau;
    y(2,1) = x(1) + mu;

    if nargin == 0
        y = [n;p];
    end
end


function y = pblm2b(x,u)
    % pblm1c function
    arguments
```

```
        x (2,:) = [0; 0];
        u (1,:) = 0;
    end

    % Array Sizes
    n = 2;
    p = 1;


    % State Upadate Eqs
    y(1,1) = -1/2 * tan(pi*x(1)/2) + x(2);
    y(2,1) = x(1) -1/2 * tan(pi*x(2)/2);

    if nargin == 0
        y = [n;p];
    end
end

function y = pblm5a(x,u)
    % pblm1c function
    arguments
        x (2,:) = [0; 0];
        u (1,:) = 0;
    end

    % Array Sizes
    n = 2;
    p = 1;


    % State Upadate Eqs
    y(1,1) = -x(1) + x(2)^3 + 1;
    y(2,1) = -4*x(1)^2 + 3*x(2);

    if nargin == 0
        y = [n;p];
    end
end


function y = pblm6a(x,u)
    % pblm1c function
    arguments
```

```
        x (2,:) = [0; 0];
        u (1,:) = 0;
    end

    % Array Sizes
    n = 2;
    p = 1;


    % State Upadate Eqs
    y(1,1) = x(2);
    y(2,1) = -(sin(x(1)) + 2) * (x(1) + x(2));

    if nargin == 0
        y = [n;p];
    end
end
```