Nonlinear Sys. Quiz 4 Apr 7, 2021

Problem 1: True or False? Explain why.

a) $\dot{x} = -g(t)x^5$; $g(t) \geq 1 \ \forall t \geq t_0$ is GUES. F

b) $\dot{x} = -g(t)x$; $g(t) \geq 1 \ \forall t \geq t_0$ is GUES. T

c) An LTI system with a dynamic generator A with $\mathrm{Re}\{\lambda_i(A)\} < 0 \ \forall i$ is Exponetially Stable T

Problem 2: Comment on the convergence properties of the gradient-based algorithm $\dot{\hat{\theta}}(t) = \psi(t)\psi^T(t)\hat{\theta}(t) - \psi(t)y(t)$ in obtaining the value of the true constant parameters $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ when we only have access to linear measurements:

$$y(t) = \theta_1 + 2\theta_2 .$$

$\psi = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $-\psi\psi^T \Rightarrow$ not Hurwitz (rank 1)

not PE $\rightarrow$ no UO. $\rightarrow$ no convergence

Problem 1)

a) $V = \frac{1}{2}x^2 > 0$, $\dot{V} = -g(t)x^6 < 0$ F

$W_1 = W_2 = \frac{1}{2}x^2$

$W_3 = x^6$

GUAS but not GUES

b) LTV $V = 1/2 x^2 > 0$, $\dot{V} = -g(t)x^2 < 0$ T

$\text{GUAS} \equiv \text{GUES}$

c) True for LTI systems T

Problem 2) Sol'n on first page

Lecture 19

04/07/2021

Last time: Model reference adaptive control (MRAC)

Goal: Design u so that unknown dynamics $\dot{y} = ay + u$ follows reference model $\dot{y}_m = -a_m y_m + r$

ref. signal

positive const.

Today: Integrator Backstepping

* A few words on Lyapunov-based control design

first-order nonlinear system:

$$\dot{x} = ax^2 + u \qquad x(t), u(t) \in \mathbb{R}$$

a: constant parameter

Goal: Design $u = \alpha(x)$ s.t. $\dot{x} = ax^2 + \alpha(x)$ is GAS.

Note: nonlinear version of $\dot{y} = ay + u$

If $a = \text{const.}$ is known, then

$$u(t) = -a x^2(t) - K x(t) \qquad K > 0$$

Closed-loop system : LTI

$$\dot{x} = a x^2 + u = -K x \implies x(t) = e^{-Kx} x(0)$$

Note: this is a special case of Lyapunov based control design framework.

$$\boxed{\dot{x} = ax^2 + u}$$

Lyapunov function: $\quad V(x) = \frac{1}{2}x^2$

$$\dot{V} = x\dot{x} = x(ax^2 + u)$$

$$= x\left[\cancel{ax^2} - \cancel{ax^2} - Kx^{2i+1}\right]$$

$$= -Kx^{2i} < 0$$

$$x \in \mathbb{R} \setminus \{0\}$$

$K > 0$

$i = 0, 1, 2, \ldots$

Q. How can we use a Lyapunov-based approach to design $u = \alpha(x)$ to stabilize system of the form:

$$\boxed{\begin{aligned} \dot{x}_1 &= a x_1^2 + x_2 \\ \dot{x}_2 &= u \end{aligned}}$$

One integrator seperates u from $a x_1^2$!!!

$\downarrow$

Cannot cancel $a x_1^2$ by control

Key tool: Integrator Backstepping

Backstepping

Consider $\dot{x} = \underbrace{f(x)}_{\substack{\text{vector} \\ \text{field}}} + \underbrace{g(x)}_{\substack{\text{vector} \\ \text{field}}} u$ Scalar control input

$$x(t) \in \mathbb{R}^n$$

assume $u = \alpha(x)$ such that we have

$V(x)$: pos. definite radially unbounded

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial x}\left[f(x) + g(x)\alpha(x)\right] \leq -W(x) < 0$$

$\Rightarrow$ GAS

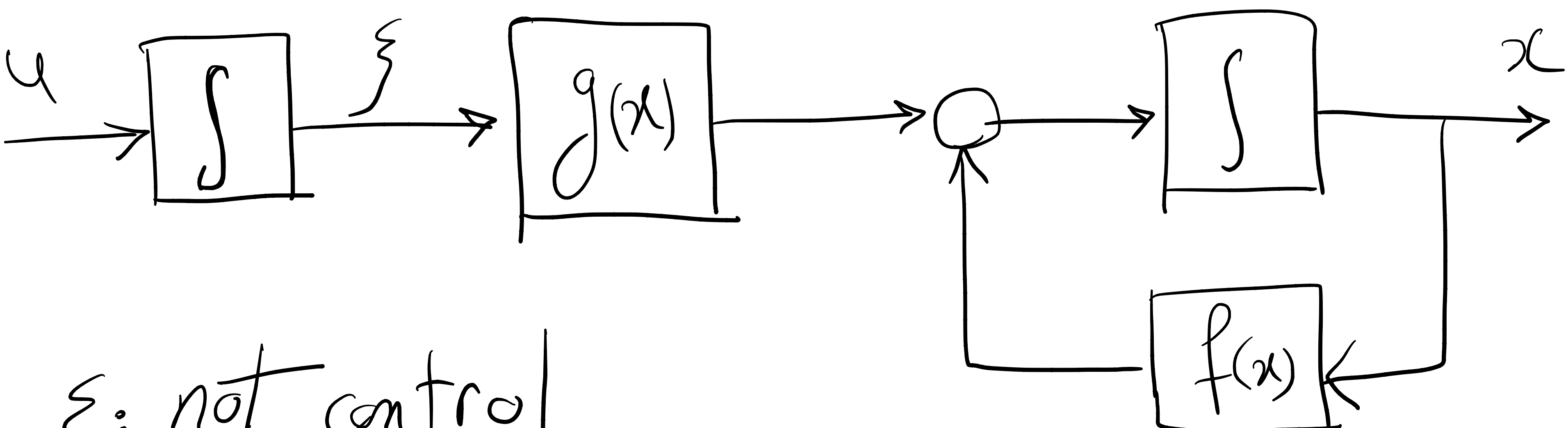
Now consider:

$$\begin{cases} \dot{x} = f(x) + g(x)\xi & (\mathrm{I}) \\ \dot{\xi} = u & (\mathrm{II}) \end{cases}$$

with $\xi = \alpha(x)$ satisfying above condition.

Q. Can we design $u = \beta(x, \xi)$ s.t. the origin of (I)+(II), i.e., $(\bar{x}, \bar{\xi}) = (0, 0)$, is GAS?

Answer: Yes. Approach → backstepping



ξ: not control

↳ think of it as a virtual control which results from an approach that aims to minimize difference between ξ and $\alpha(x)$.

In order for this penalization of the difference $\xi - \alpha(x)$ to happen we may augment $V(x)$ $\left[\frac{\partial V}{\partial x}\left(f(x)+g(x)\alpha(x)\right)=-W(x)<0\right]$ with $(\xi-\alpha(x))^2$:

$$V_a(x,\xi) = V(x) + \frac{1}{2}(\xi-\alpha(x))^2$$

$$z := \xi - \alpha(x)$$

$$\dot{z} = \dot{\xi} - \frac{\partial \alpha}{\partial x}(\dot{x}) = u - \frac{\partial \alpha}{\partial x}\left(f(x) + g(x)\xi\right)$$

$$V_a(x, z) = V(x) + \frac{1}{2} z^2$$

$$\dot{V}_a(x, z) = \dot{V} + z\dot{z} = \frac{\partial V}{\partial x}\Big[f(x) + g(x)\big(\overbrace{\alpha(x) + z}^{\xi}\big)\Big]$$

$$+ z\left[u - \frac{\partial \alpha}{\partial x}\left(f(x) + g(x)\xi\right)\right]$$

$$\dot{V}_a = \underbrace{\frac{\partial V}{\partial x}\left[f(x)+g(x)\alpha(x)\right]}_{-W(x)} + z\left[u - \frac{\partial \alpha}{\partial x}\left(f(x)+g(x)\xi\right) + \frac{\partial V}{\partial x} g(x)\right]$$

$-W(x)$ ✓ $+$

Since we can measure z there is no need to set this to zero!

design u so that we get $-kz^2$

Made possible by:

$$u = \frac{\partial \alpha}{\partial x}\left[f(x)+g(x)\xi\right] - \frac{\partial V}{\partial x} g(x) - kz \; ; \quad k > 0$$

Ex. $\dot{x}_1 = x_1^2 + x_2$

$\dot{x}_2 = u$

step1: consider $\dot{x}_1 = x_1^2 + x_2$ and think of x_2 as a control

propose: $V_1(x_1) = \frac{1}{2} x_1^2$

$$\dot{V}_1 = x_1 \dot{x}_1 = x_1 (x_1^2 + x_2)$$

e.g. $x_2 = -x_1^2 - K_1 x_1 \quad (K_1 > 0)$

or $x_2 = -x_1^2 - K_1 x_1^{937} \quad (K_1 > 0)$

abstractly : $x_2 = \alpha(x_1)$

if x_2 was the control $\Rightarrow$ done!

But it is not $\longrightarrow$ need to account for the fact that

x_2 is a state variable (not control)

step 2) $z_2 = x_2 - \alpha(x_1)$

$$\dot{z}_2 = \dot{x}_2 - \dot{\alpha}(x_1)$$

$$= u - \frac{\partial \alpha}{\partial x} \dot{x}_1 = u - \frac{\partial \alpha}{\partial x}\left[x_1^2 + x_2\right]$$

$$z_2 = x_2 - \alpha(x_1) \xrightarrow[\text{to eliminat } x_2]{} \dot{z}_2 = u - \frac{\partial \alpha}{\partial x_1}\left[x_1^2 + z_2 + \alpha(x_1)\right]$$

Augment Lyapunov function from step (1):

$$V_a(x_1, z_2) = V_1(x_1) + \frac{1}{2} z_2^2 = \frac{1}{2} x_1^2 + \frac{1}{2} z_2^2$$

$$\dot{V}_a(x_1, z_2) = \dot{V}_1 + z_2 \dot{z}_2 = x_1(x_1^2 + \alpha(x_1) + z_2) + z_2(u - \dot{\alpha})$$

$$= \underbrace{x_1(x_1^2 + \alpha(x_1))}_{-W_1(x_1)} + z_2(u - \dot{\alpha} + x_1)$$

Choose: $\boxed{u = -x_1 + \dot{\alpha} - K_2 z_2 \qquad (K_2 > 0)}$ (*)

will yield

$$\dot{V}_a = -W_1(x_1) - K_2 z_2^2 < 0$$

GAS of the origin of (I) + (II) ✓

Note! No reason to explicitly differentiate α in the expression for control (*). Use instead the analytical expression for $\dot{\alpha}$:

$$\partial\alpha/\partial t = \partial\alpha/\partial x_1 \, \dot{x}_1 = \partial\alpha/\partial x_1 \left(x_1^2 + x_2\right)$$

Thus, 'u' given by (*) is indeed a static nonlinear state feedback control law, which provides GAS.

(it is in the form $u = \beta(x_1, x_2)$)

Therefore, backstepping provides a family of control laws in a constructive manner to achieve GAS.

Note Backstepping yields:

→ A family of globally stabilizing control laws (parameterized by different choices of the stabilizing function)

Finally, Control Lypunov Function (CLF) can be used to obain additional control laws

↓

(later)

Additional comment:

$$\begin{cases} \dot{x} = f(x) + g(x)\,\xi \\ \dot{\xi} = f_a(x,\xi) + g_a(x,\xi)\,u \end{cases}$$

if $g_a(x,\xi) \neq 0$ for all x, ξ (same as before)

$$u = \frac{1}{g_a(x,\xi)} \left(\cdots \right)$$

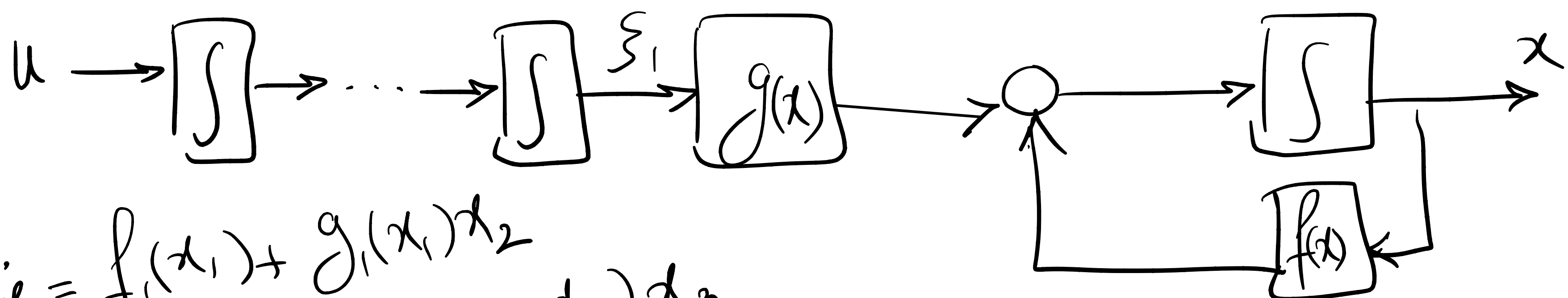
Ex

$$\dot{x}_1 = x_1^2 + x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = u$$

$z_3 = x_3 - \alpha_2(x_1, x_2)$ and ... (same as before)



$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2$$

$$\dot{x}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)x_3$$

$$\dot{x}_3 = f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)u$$

$$\vdots$$