Lecture 26

05/03/2021

Last time: input-output linearization

zero-dynamic

Today: Normal form

Recap :

$$\dot{x} = f(x) + g(x)u$$

$$y = h(x)$$

$$y = h(x) \longrightarrow \dot{y} = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}\left(f(x) + g(x)u\right)$$

$$= \underbrace{\frac{\partial h}{\partial x} f(x)}_{L_f h(x)} + \underbrace{\frac{\partial h}{\partial x} g(x)}_{L_g h(x)} u$$

$L_g h(x) \neq 0 \Rightarrow r.d. = 1$

if not continue to differentiate

→ example of a system that doesn't have globally defined relative degree:

$$\dot{x}_1 = x_2 + x_3^3$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = u$$

$$y = x_1 \longrightarrow \dot{y} = \dot{x}_1 = x_2 + x_3^3$$

$$\ddot{y} = \dot{x}_2 + 2x_3^2\dot{x}_3 = x_3 + \underbrace{3x_3^2}_{}u$$

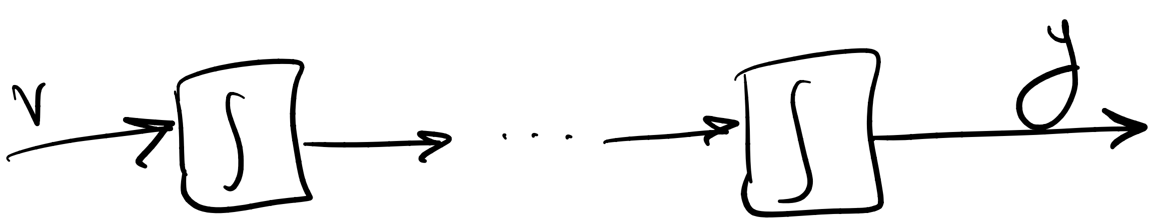
Vanishes when $x_3 = 0$

this system doesn't have a well-defined relative degree when $x_3 = 0$ we would not be able to determine the influence of u on $\breve{y}$ or the derivative of x_1.

recall:

$$y^{(r)} = L_g^{r} h(x) + \underbrace{L_g L_f^{r-1} h(x)}_{\neq 0} \cdot u$$

$$u = \frac{1}{L_g L_f^{r-1} h(x)} \left\{ -L_f^{r} h(x) + \underline{\underline{v}} \right\}$$

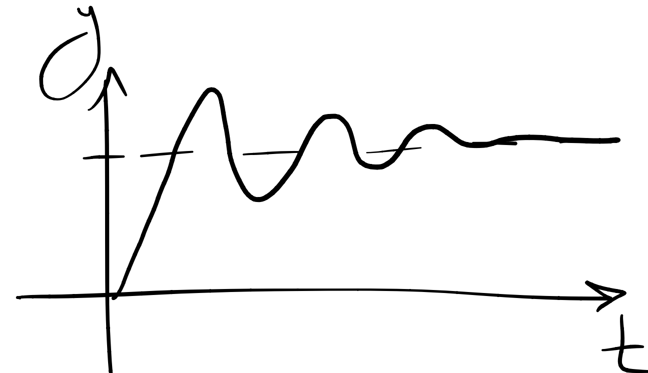


One of the applications for this type of nonlinear control is <u>output tracking</u>, but it does show limitations in ideal conditions and there may be robustness issues when nonlinearities are not well characterized.

reminder

$$H_1(s) = s+1 \quad \text{and} \quad H_2(s) = s-1$$

the system with RHP zeros is non-minimum phase.

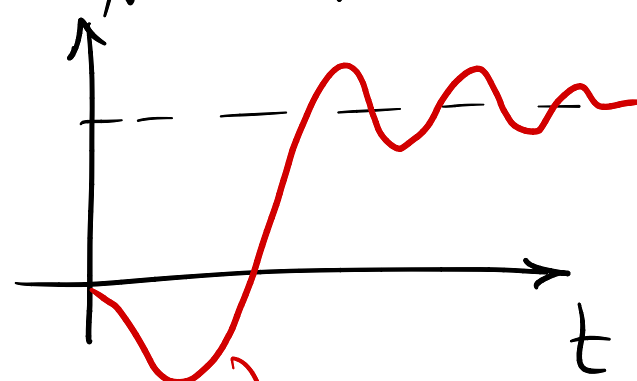


Ex.

$$G(s) = \frac{s-1}{s^2+3s+6}$$

then $\ddot{y} + 3\dot{y} + 6y = \dot{u} - u$

Non min-phase



transient dip away from desired output

Typically, a part of a non-minimum phase system will not be controlled with reference tracking schemes and would therefore require a form of feedback.

BIG PICTURE

zero-dynamics generalizes RHP zeros to nonlinear dynamics

Normal Form : a set of coordinates that clearly displays the zero dynamics.

consider the nonlinear system given by

$$\begin{cases} \dot{x} = f(x) + g(x)\, u \\ y = h(x) \end{cases}$$

if this system has relative degree r $(r<n)$ then there are variables

$$\begin{cases} Z \in \mathbb{R}^{n-1} \\ \xi \in \mathbb{R}^{r} \end{cases}$$ such that there is mapping

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ s.t. $T(x) = \begin{bmatrix} Z \\ \xi \end{bmatrix}$ where $T(0) = 0$.

The mapping T is called a diffeomorphism (i.e., its inverse T^{-1} exists where T and T^{-1} are conitinuous and differentiable)

The dynamics in the new cordinates are given by:

$$\begin{cases} \dot{z} = f_0(z, \xi) \\ \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_r \end{pmatrix} = \begin{pmatrix} \xi_2 \\ \xi_3 \\ \\ b(z,\xi) + a(z,\xi)u \end{pmatrix} \end{cases}$$

Note: for ξ-dynamics, there is a cascade connection therefore, we can find u st. we follow a a desired ξ trajectory

Note: ξs are the variables that represent the output and its derivatives.

$$b(z,\xi) = L_f^r h(x)$$

$$a(z,\xi) = L_g L_f^{r-1} h(x)$$

$$\xi = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \end{bmatrix}$$

Two requirements:

(1) ξ and z must be linearly independent

(2) Control u must not be in the equation for z.

As $r < n$, we could find an input/output linearizing controller (if $a(z,\xi)$ is well-defined & invertible!)

$$u = \frac{1}{a(z,\xi)} \left(-b(z,\xi) + v \right)$$

where $v = -K_1 \xi_1 - \cdots - K_r \xi_r$. This will allow use to achieve the form:

$$\dot{\xi} = A_\xi \xi \quad \text{with} \quad A_\xi = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \\ & & & 0 & 1 \\ -K_1 & -K_2 & \cdots & & -K_r \end{bmatrix} \in \mathbb{R}^{r \times r}$$

stability properties of I/O feedback controller are determined completely be the choice of K_i's.

Finally, we call the form

$$\dot{\xi} = A_\xi \xi$$
$$\dot{z} = f_0(z, \xi)$$

the normal form.

The zero-dynamics are recovered by plugging $\xi = 0$ into the normal form.

Why $\xi = 0$? linearization gives local stability properties of

$$\boxed{\begin{bmatrix} \dot{\xi} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_\xi & 0 \\ \left.\frac{\partial f_0}{\partial \xi}\right|_{\xi=0} & \left.\frac{\partial f_0}{\partial z}\right|_{\xi=0} \end{bmatrix} \begin{bmatrix} \xi \\ z \end{bmatrix}}$$

Cascade connection:

$\xi = 0$ means that stability properties are captured by Jacobian $\left.\frac{\partial f_0}{\partial z}\right|_{\xi=0}$.

Ex. Consider the system: $\dot{x}_1 = x_2$ with $y = x_1$
$$\dot{x}_2 = \alpha x_3 + u$$
$$\dot{x}_3 = \beta x_3 - u$$

① Differentiate output to find relative degree.

$$y = x_1 \longrightarrow \dot{y} = \dot{x}_1 = x_2 \longrightarrow \ddot{y} = \alpha x_3 + u$$

relative degree r.d. $= r = 2 < 3 = n$

part of the system will remain after I/O linearization.

(2) Therefore, $\xi_1 = x_1 \quad \Rightarrow \quad \dot{\xi}_1 = \xi_2$

$\xi_2 = x_2 \qquad\qquad \dot{\xi}_2 = \alpha x_3 + u$

Aside Note! Cannot choose $x_3 = z$ otherwise we would have input in our equation for z!

Also, $T(x)$ would be trivially $x = I\left(\xi_z\right)$.

(3) Instead, define $z = x_2 + x_3$ (to cancel input in $\dot{z}$)

$$\dot{z} = \dot{x}_2 + \dot{x}_3 = (\alpha + \beta)x_3 = (\alpha + \beta)z - (\alpha + \beta)\xi_2$$

Therefore,

Normal form

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \alpha(z - \xi_2) + u \\ \dot{z} = (\alpha+\beta)z - (\alpha+\beta)\xi_2 \end{cases}$$

and the zero dynamics are given by $(\xi_1, \xi_2) = (0, 0)$

s.t. $\Rightarrow$ $\boxed{\dot{z} = (\alpha+\beta)z}$

stability of z will be completely determined by $\mathrm{sgn}(\alpha+\beta)$.

Transfer function from u to y:

$$\frac{Y(s)}{U(s)} = \frac{s-(\alpha+\beta)}{s^2(s-\beta)}$$

numerator determines zeros of the T.F.

Therefore, for the purpose of reference tracking if we have zeros of the T.F. in the RHP then we will encounter problems of stability in aerodynamics and we cannot guarantee stability of nonlinear dynamics.

Summary:

Nonliner system $\longrightarrow$ r.d. $\longrightarrow$ Normal form $\longrightarrow$ linearizing controller u

$$\begin{bmatrix} \dot{\xi} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_\xi & 0 \\ \partial f/\partial \xi & \partial f/\partial z \big|_{\xi=0} \end{bmatrix} \begin{bmatrix} \xi \\ z \end{bmatrix}$$

the stability (LAS) of the normal form and original nonlinear dynamics is equavalent to $\frac{\partial f_0}{\partial z}$ and A_ξ being Hurwitz.

Obviously, A_ξ will be Hurwitz b/c of our choice of linearizing controller u (and param. v) so the LAS of nonlinear system essentially amounts to the stability of the zero-dynamics

$$\left.\frac{\partial f_0}{\partial z}\right|_0$$

Note: If you have input to state stability you will have GAS as well.