Due Wednesday 04/28/2021 (at the beginning of the class)

1. Show that the parallel connection of two passive dynamical systems is passive. Can you claim the same for the series connection of two passive systems? Either provide a proof or a counterexample.

2. Let
$$H(s) = \frac{s+\lambda}{s^2 + a\,s + b}$$
with $a > 0$, $b > 0$.

   (a) For which values of λ is $H(s)$ Positive Real (PR)?

   (b) Using the answer to part (a), select two values, λ_1 and λ_2, such that
$$\begin{aligned} H_1(s) &= \frac{s+\lambda_1}{s^2+s+1} \quad \text{is PR,} \\ H_2(s) &= \frac{s+\lambda_2}{s^2+s+1} \quad \text{is not.} \end{aligned}$$
   Verify the PR property or its absence from the Nyquist plots of $H_1(s)$ and $H_2(s)$. (You can use the MATLAB **nyquist** command.)

   (c) For $H_1(s)$ and $H_2(s)$, write a state-space realization and solve for $P = P^T > 0$ in the PR lemma. Explain why your attempt fails for $H_2(s)$.

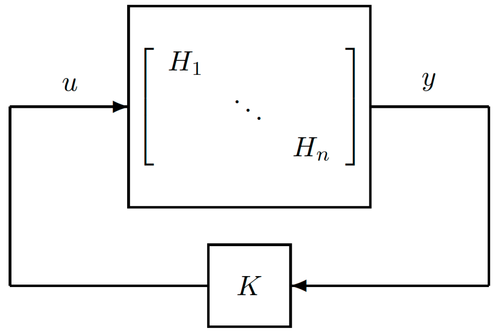
3. Consider the following model for a three-stage ring oscillator, discussed in class:
$$\begin{aligned} \tau_1\dot{x}_1 &= -x_1 - \alpha_1 \tanh(\beta_1 x_3) \\ \tau_2\dot{x}_2 &= -x_2 - \alpha_2 \tanh(\beta_2 x_1) \\ \tau_3\dot{x}_3 &= -x_3 - \alpha_3 \tanh(\beta_3 x_2) \end{aligned}$$
where τ_i, α_i, and β_i are positive constants and x_i represents voltages, $i = 1, 2, 3$.

   (a) Suppose $\alpha_1\beta_1 = \alpha_2\beta_2 = \alpha_3\beta_3 =: \mu$, and prove that the origin is globally asymptotically stable when $\mu < 2$.

   (b) Show that if $\tau_1 = \tau_2 = \tau_3 =: \tau$, then $\mu < 2$ is also necessary for asymptotic stability. What type of bifurcation occurs at $\mu = 2$?

   (c) Investigate the dynamical behavior of this system for $\mu > 2$ with numerical simulations. (You can take $\tau = 1$ for simplicity. Note that changing τ simply scales the time variable, i.e., if $x(t)$ is a solution for $\tau = 1$, then $x(t/\tau)$ is a solution for $\tau \neq 1$.)

4. Consider systems H_i ($i = 1, \ldots, n$), whose inputs u_i and outputs y_i are coupled according to
$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = K \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$
as in the figure shown below, where K is an $n \times n$ matrix.



(a) Suppose each H_i satisfies the dissipation inequality:

$$\dot{V}_i(x_i) \;\leq\; -y_i^2 \;+\; \gamma_i^2 u_i^2$$

with a positive definite storage function of its state vector x_i.

(b) Determine a matrix inequality that restricts the matrices:

$$D \;:=\; \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}, \quad \Gamma \;:=\; \begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_n \end{bmatrix}$$

and K, such that $V(x) \;=\; \sum_{i=1}^n d_i\, V_i(x_i)$ is a Lyapunov function for the interconnected system.

(c) Investigate when an appropriate matrix D satisfying this inequality exists for $K \in \mathbb{R}^{2\times 2}$ given by

$$K \;=\; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$