

# MECH 6313 - Homework 6

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# 1 Problem 1

**Problem:** Show that the parallel connection of two passive dynamical systems is passive. Can you claim the same for the series connection of two passive systems?

**Solution:** Let two passive systems be defined as a system taking an input  $u$  and generating an output  $y$  as

$$H_1 : y_1 = h_1(u), \text{ s.t. } \langle y_1 | u \rangle \geq 0$$

and

$$H_2 : y_2 = h_2(u), \text{ s.t. } \langle y_2 | u \rangle \geq 0$$

with  $\langle y | u \rangle = \int_0^T y^T(t)u(t) dt$

## 1.1 Parallel Connection of Passive System

The parallel system  $H_p$  can then be defined by

$$H_p : h_p(u) = y_p = y_1 + y_2 = h_1(u) + h_2(u)$$

whose passivity can be proven directly by testing  $\langle y_p | u \rangle$  which is calculated as

$$\langle y_p | u \rangle = \int_0^T y_p^T u dt \tag{1}$$

$$= \int_0^T (y_1 + y_2)^T u dt \tag{2}$$

$$= \int_0^T y_1^T u + y_2^T u dt \tag{3}$$

$$= \int_0^T y_1^T u dt + \int_0^T y_2^T u dt \tag{4}$$

$$= \langle y_1 | u \rangle + \langle y_2 | u \rangle \tag{5}$$

Since  $\langle y_1 | u \rangle \geq 0$  and  $\langle y_2 | u \rangle \geq 0$ ,

$$\langle y_p | u \rangle \geq 0 \tag{6}$$

which proves, by definition, that  $H_p$  is passive.

## 1.2 Series Connection of Passive System

The series system  $H_s$  can be defined by

$$H_s : h_s(u) = y_s = h_1(u) \otimes h_2(u) = h_2(h_1(u))$$

whose passivity can be tested using  $\langle y_s | u \rangle$  which is calculated as:

$$\langle y_s | u \rangle = \int_0^T y_s^T u \, dt \quad (7)$$

$$= \int_0^T (h_1(u) \otimes h_2(u))^T u \, dt \quad (8)$$

$$= \int_0^T \left( \int_0^T h_1(t - \tau) h_2(\tau) \, d\tau \right) dt \quad (9)$$

$$= \int_0^T h_2(\tau) \left( \int_0^T h_1(t - \tau) \, dt \right) d\tau \quad (10)$$

which is not explicitly  $\geq 0$  so this method cannot prove passivity.

A different method of analysis can be done to prove that this is not passive in general, but a counter example from MATLAB (Appendix A) can be shown to not be passive due to a loss of positive realness of the transfer functions when placed in series:

$$G_1(s) = \frac{5s^2 + 3s + 1}{s^2 + 2s + 1}, \quad G_2(s) = \frac{s^3 + s^2 + 5s + 0.1}{s^3 + 2s^2 + 3s + 4}$$

and when combined in series the system is no longer passive due to a loss of positive realness.

$$\frac{5s^5 + 8s^4 + 29s^3 + 16.5s^2 + 5.3s + 0.1}{s^5 + 4s^4 + 8s^3 + 12s^2 + 11s + 4}$$

## 2 Problem 2

Let

$$H(s) = \frac{s + \lambda}{s^2 + as + b}$$

with  $a > 0$  and  $b > 0$ .

### 2.1 Part a

**Problem:** For which values of  $\lambda$  is  $H(s)$  Positive Real (PR)?

**Solution:** By definition, a transfer function must satisfy two conditions to be considered Postive Real:

1.  $\Re\{\lambda(H(s))\} \leq 0$ , any  $j\omega$  roots are simple, and any residuals are non negative.
2.  $\Re\{H(j\omega)\} \geq 0 \forall \omega \in \mathbb{R}$

The transfer function for this problem will always satisfy the first condition, however, the second condition is violated under the following conditions:

Setting

$$s = j\omega$$

$$H(j\omega) = \frac{j\omega + \lambda}{(j\omega)^2 + a(j\omega) + b} = \frac{j\omega + \lambda}{-\omega^2 + ja\omega + b} \quad (11)$$

$$= \frac{-\omega^2 - ja\omega + b}{-\omega^2 - ja\omega + b} \cdot \frac{j\omega + \lambda}{-\omega^2 + ja\omega + b} \quad (12)$$

$$= \frac{(a\omega^2 + \lambda(\omega^2 + b)) + j(\omega(\omega^2 + b) - a\lambda\omega)}{a^2\omega^2 + (\omega^2 + b)^2} \quad (13)$$

$$= \frac{a\omega^2 + \lambda(\omega^2 + b)}{a^2\omega^2 + (\omega^2 + b)^2} + j \frac{\omega(\omega^2 + b) - a\lambda\omega}{a^2\omega^2 + (\omega^2 + b)^2} \quad (14)$$

The real component being nonnegative can then be seen to occur when

$$a\omega^2 + \lambda(\omega^2 + b) \geq 0 \quad (15)$$

$$\lambda(\omega^2 + b) \geq -a\omega^2 \quad (16)$$

$$\lambda \geq \frac{-a\omega^2}{\omega^2 + b} \quad (17)$$

Since this mus apply  $\forall \omega \in \mathbb{R}$ , the following must be true

$$\lambda \geq 0 \quad (18)$$

## 2.2 Part b

**Problem:** Using the results from above, select  $\lambda_1, \lambda_2$  such that

$$H_1(s) = \frac{s + \lambda_1}{s^2 + s + 1} \text{ is PR} \quad (19)$$

$$H_2(s) = \frac{s + \lambda_2}{s^2 + s + 1} \text{ is not PR} \quad (20)$$

Then verify the PR properties for each using the Nyquist plots of  $H_1(s)$  and  $H_2(s)$ .

**Solution:** From the requirements set above, the zeros can be selected with  $\lambda_1 = 1$  and  $\lambda_2 = -1$  resulting in  $H_1(s)$  and  $H_2(s)$  being defined as

$$H_1(s) = \frac{s + 1}{s^2 + s + 1} \quad (21)$$

$$H_2(s) = \frac{s - 1}{s^2 + s + 1} \quad (22)$$

The nyquist plots, generated with the MATLAB code seen in Appendix A, can then be used to verify the PR properties. As can be seen in Figure 1, the nyquist diagram for  $H_1(s)$  never crosses into the LHP and therefore is Positive Real. Conversely, in Figure 2, the nyquist diagram for  $H_2(s)$  crosses into the LHP and therefore is not Positive Real.

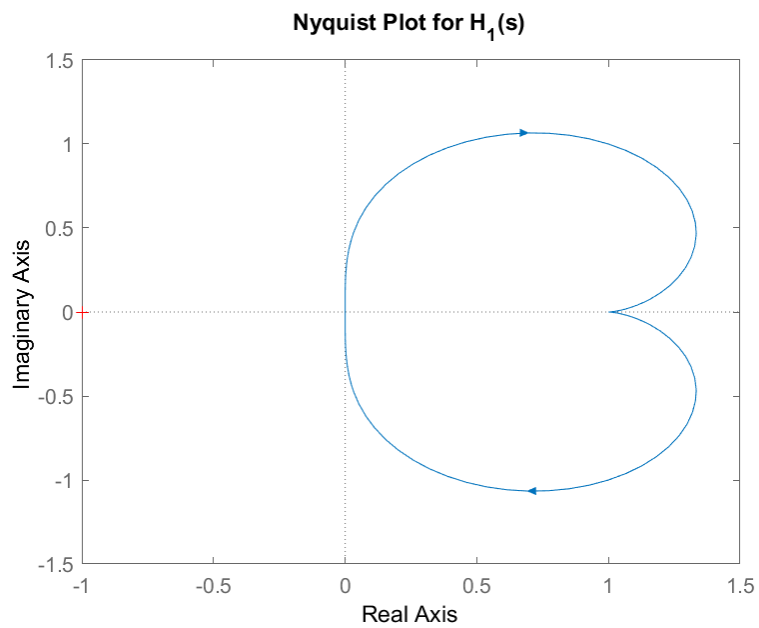


Figure 1: Nyquist Plot for the  $H_1(s)$  transfer function.

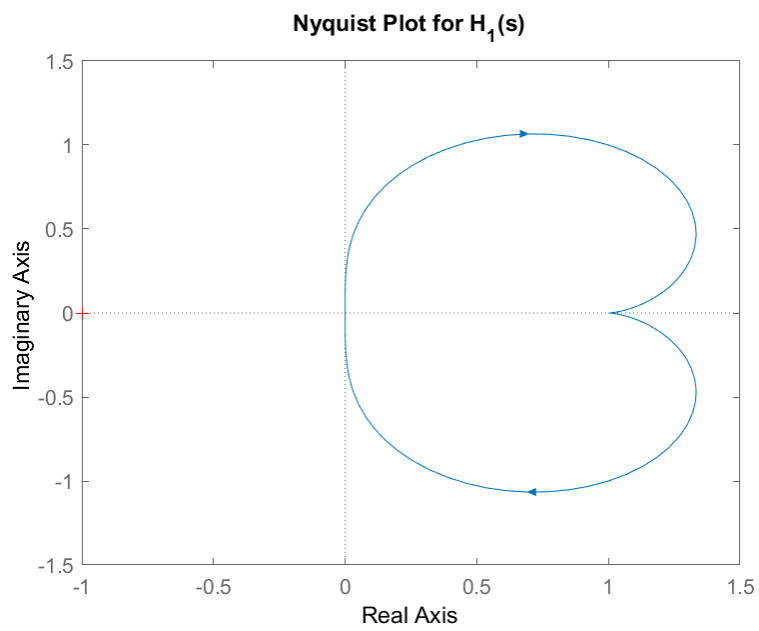


Figure 2: Nyquist Plot for the  $H_2(s)$  transfer function.

## 2.3 Part c

**Problem:** For  $H_1(s)$  and  $H_2(s)$ , write state-space realizations and solve for  $P = P^T > 0$  in the PR lemma and explain why it fails for  $H_2(s)$ .

**Solution:** Given a second order transfer function

$$\frac{s + b_0}{s^2 + a_1 s + a_0}$$

a state space system can be defined by:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} & B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} b_0 & 1 \end{bmatrix} & D &= 0 \end{aligned} \tag{23}$$

This can be applied to the systems and results in the state space representations of  $H_1$  as:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} & B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 1 \end{bmatrix} & D &= 0 \end{aligned} \tag{24}$$

and  $H_2$  as

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} & B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} -1 & 1 \end{bmatrix} & D &= 0 \end{aligned} \tag{25}$$

The

### 3 Problem 3

Consider the following 3-stage ring oscillator discussed in class:

$$\tau_1 \dot{x}_1 = -x_1 - \alpha_1 \tanh(\beta_1 x_3)$$

$$\tau_2 \dot{x}_2 = -x_2 - \alpha_2 \tanh(\beta_2 x_1)$$

$$\tau_3 \dot{x}_3 = -x_3 - \alpha_3 \tanh(\beta_3 x_2)$$

with  $\tau_i, \alpha_i, \beta_i > 0$  and  $x_i$  represents a voltage for  $i = 1, 2, 3$ .

#### 3.1 Part a

**Problem:** Suppose  $\alpha_1 \beta_1 = \alpha_2 \beta_2 = \alpha_3 \beta_3 = \mu$ , prove the origin is GAS when  $\mu < 2$ .

**Solution:**

idk what system this is referring to..

#### 3.2 Part b

**Problem:** Show that if  $\tau_1 = \tau_2 = \tau_3 = \tau$ , then  $\mu < 2$  is necessary for asymptotic stability. What type of bifurcation occurs at  $\mu = 2$ ?

**Solution:**

#### 3.3 Part c

**Problem:** Investigate the dynamic behavior of this system for  $\mu > 2$  with numerical simulations.

**Solution:**



## 4 Problem 4

## A MATLAB Code:

All code I write in this course can be found on my GitHub repository:

<https://github.com/jonaswagner2826/MECH6313>

Script 1: MECH6313\_HW6

```
1 % MECH 6313 - HW6
2 % Jonas Wagner
3 % 2021-04-27
4 %
5
6 clear
7 close all
8
9 pblm1 = false;
10 pblm2 = true;
11
12
13 if pblm1
14 %% Problem 1
15
16 G1 = tf([5 3 1], [1 2 1])
17 isPassive(G1)
18
19 G2 = tf([1 1 5 0.1], [1 2 3 4])
20 isPassive(G2)
21
22 Gp = G1 + G2
23 isPassive(Gp)
24
25 Gs = G1 * G2
26 isPassive(Gs)
27
28 end
29
30
31 if pblm2
32 %% Problem 2
33 pblm2a = false;
34 pblm2b = true;
35 pblm2c = true;
36
37 if pblm2a
38 % Part a
```

```

39 syms omega a b lambda
40 assume(a,'real'); assume(a > 0)
41 assume(b,'real'); assume(b > 0)
42 assume(omega,'real')
43 assume(lambda, 'real')
44
45 num = j*omega + lambda;
46 den = omega^2 + j*a*omega + b;
47
48 H_sym = num/den;
49 disp('H(s) = ')
50 pretty(H_sym)
51
52 H_real = real(H_sym);
53 disp('H_real = ')
54 pretty(H_real)
55
56 H_imag = imag(H_sym);
57 disp('H_imag = ')
58 pretty(imag(H_sym))
59 end
60
61 if pblm2b
62 % Part b
63 lambda1 = 1;
64 lambda2 = -1;
65
66 H1 = tf([1 lambda1], [1 1 1])
67 isPassive(H1)
68 figure
69 nyquist(H1)
70 title('Nyquist Plot for H_1(s)')
71 saveas(gcf, [pwd, '\Homework\HW6\fig\pblm2_H1.png'])
72
73 H2 = tf([1 lambda2], [1 1 1])
74 isPassive(H2)
75 figure
76 nyquist(H2)
77 title('Nyquist Plot for H_2(s)')
78 saveas(gcf, [pwd, '\Homework\HW6\fig\pblm2_H2.png'])
79 end
80
81 if pblm2c

```

```
82 H2_sys = ss([0, 1; -1, -1], [0; 1], [-1 1], 0)
83 tf (H2_sys)
84 end
85
86
87 end
```