Nonlinear Sys. Quiz 3 Mar 3, 2021

Problem 1: What does it take for the solution of a nonlinear system $\dot{x} = f(x)$ to exist?

piecewise continuity in t + Continuity in x

Problem 2: Comment on the continuous dependence of the solution to system $\dot{x} = -ax^3$ on parameter a.

$f(x) = -ax^3$ $\frac{\partial f}{\partial x} = -3ax^2$

'a' is a parameter. cts dependence on 'a' follows from cts dependence on IC's $\Leftarrow$ cts dependence on initial conditions $\Leftarrow$ Locally Lipschitz $\Leftarrow$ cts differentiable

Problem 3: True/False?

a) A Lipschitz function is differentiable? F

b) A Lipschitz function is continuous? T

c) Existence and uniqueness of solutions is guaranteed for all linear systems. T

Note, technically for time varying linear systems you still require piecewise continuity w.r.t. time

Lecture 11

03/03/2021

Last time : Lyapunov's indirect method

Today : How to construct Lyapunov functions?

Ex scalar case

$\dot{x} = -g(x)$ $x(t) \in \mathbb{R}$

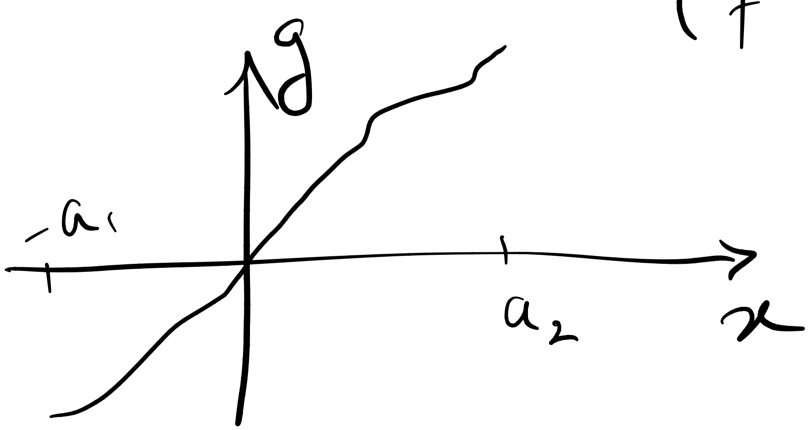
We restrict class of functions g to make some theoretical progress ...

1) $g(0) = 0$

$\bar{x} = 0$ equilibrium pt.

2) g : locally Lipschitz continuous

(for existence + uniquness)

3) 

$x g(x) \geq 0$ key feature

$x \in [-a_1, a_2]$

* Fact: for scalar systems, a quadratic Lyapunov function is a good candidate.

Lyapunov function candidate:

$$V(x) = \frac{1}{2} x^2$$

$$\dot{V} = x \dot{x} = -x g(x)$$

Based on the properties of g, we know that

$$-x g(x) < 0 \quad \forall x \in (-a_1, a_2) \setminus \{0\}$$

thus we have LAS of $\bar{x}=0$.

However, if $x g(x) > 0 \quad \forall x \in \mathbb{R} \setminus \{0\}$

then $\bar{x}=0$ is GAS!

The question that we should pose now is:

Is this Lyapunov function the only viable candidate?

Answer: NO

Another Lyapunov function candidate:

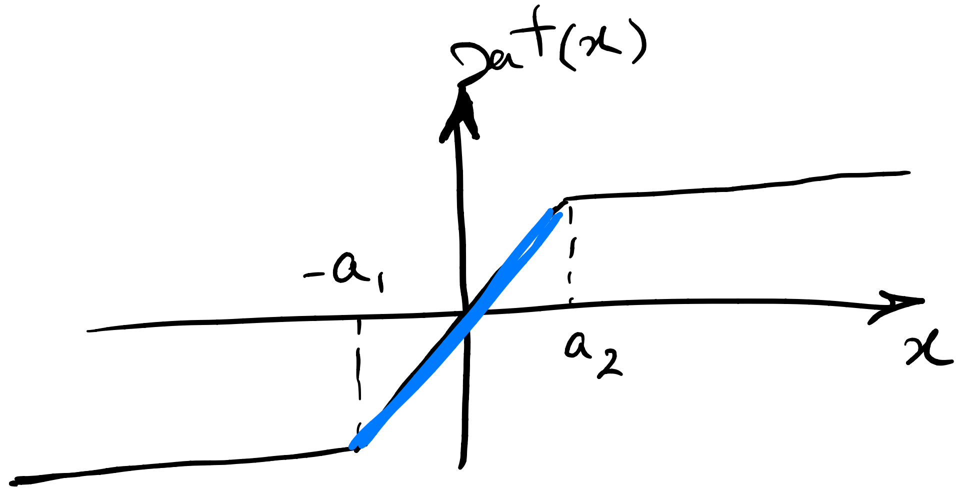
✓ $V(x) = x^{2n}$, $n = 1, 2, 3, \dots$

? $V(x) = \int_0^x g(\xi)\, d\xi$

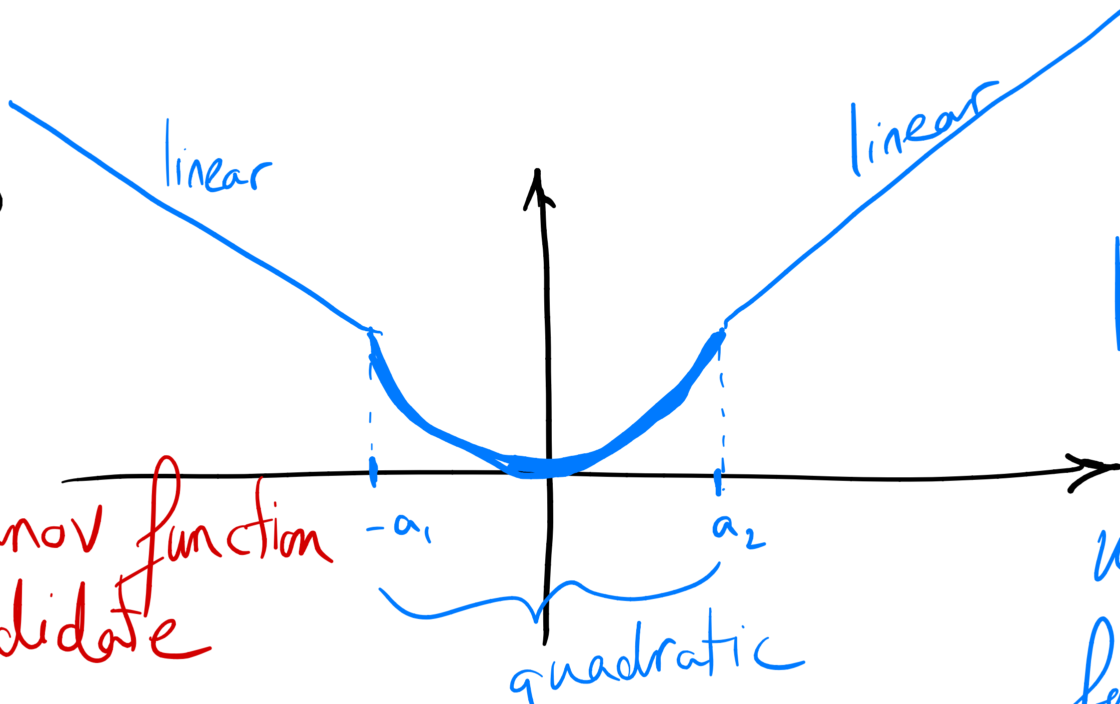
$\dot{V} = g(x)(-g(x)) = -g^2(x) < 0 \quad \forall x \in (a_1, a_2)$

Ex.

$$\dot{x} = -\mathrm{sat}(x)$$



$x\,\mathrm{sat}(x)$



Huber function

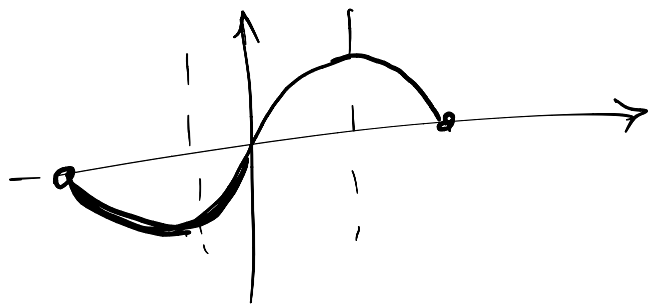
useful function for eliminating outliers.

quadratic Lyapunov function is a good candidate

$\hookrightarrow$ globally asymptotic stability.

Ex $\dot{x} = -g(x) = -x^3$ GAS

$\dot{x} = -\sin(x)$ LAS



Ex 2nd order system:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -g(x_1) - b x_2$$

pendulum fits into this class of models with $g(x_1) = \sin x_1$

Attempt: $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$

$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = \overbrace{x_1x_2 - x_2\,g(x_1)}^{?}\;\overbrace{-\,b x_2^2}^{\text{negativ term}}$$

lets identify "good" & "bad" term

$$x_1x_2 \leq \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

$$\downarrow$$

$$x_2\,g(x_1) \leq \frac{1}{2}\left(x_2^2 + \underbrace{g^2(x_1)}_{\text{positive term}}\right)$$

$\therefore$ We cannot conclude anything about stability properties using this Lyapunov function.

* Unfortunately, this Lyapunov function candidate is NOT a Lyapunov function for our system.

What can we do here? Change form of Lyapunov function.

How about using $V(x) = \int_0^{x_1} g(\xi)\, d\xi + \frac{1}{2} x_1^2$

$$\dot{V}(x) = g(x_1)\,\dot{x}_1 + x_2 \dot{x}_2 = \cancel{g(x_1) x_2} + x_2(\cancel{-g(x_1)} - b x_2)$$

$$= 0 x_1^2 - b x_2^2 \leq 0 \quad \forall (x_1, x_2) \in \Omega$$

stable (in the sense of Lyapunov)

What if we need to prove LAS?

choose a different Lyapunov function

Invoke La Salle's invariance principle

LaSalle's principle: a tool for assessing asymptotic stability properties $\bar{x}=0$ for time-invariant $\dot{x}=f(x)$ when $\dot{V}(x)$ is only negative semi-definite (i.e. $\dot{V}(x) \leq 0$)

Theorem Let $\Omega_c := \{x \,/\, V(x) \leq c\}$ and

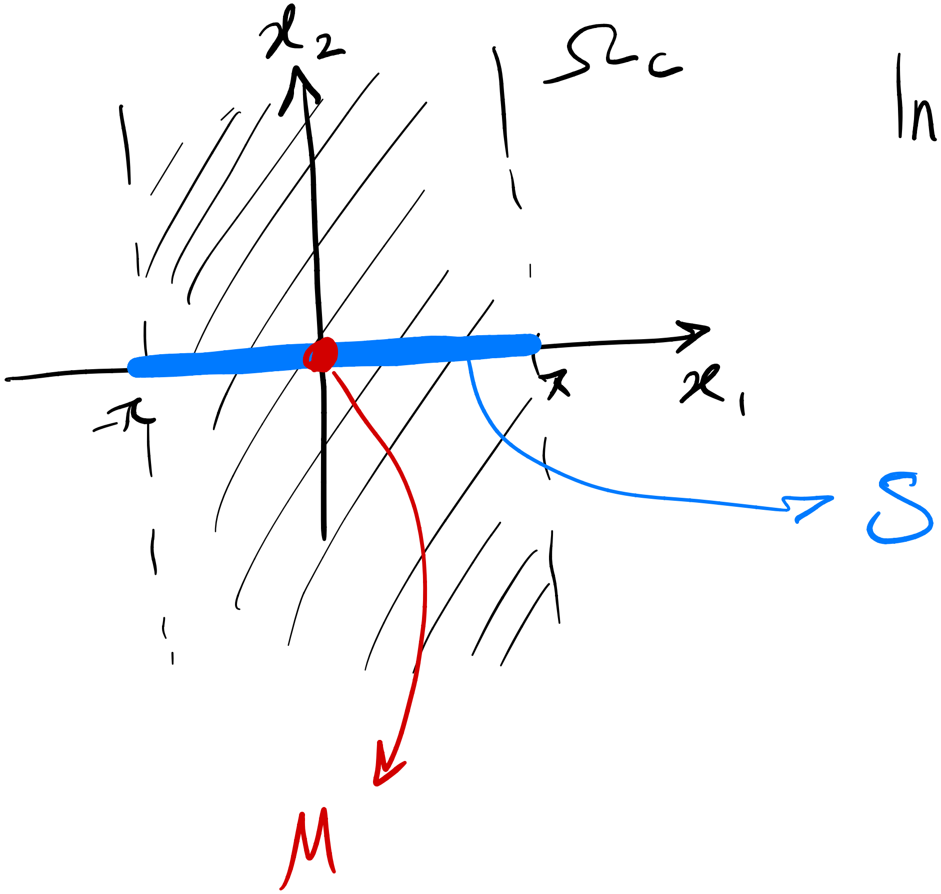
$\dot{V}(x) \leq 0, \ \forall x \in \Omega_c$

Define: $S = \{x \in \Omega_c \,/\, \dot{V}(x) = 0\}$ and let

M be the largest invariant set in S, then

for every $x(0) \in \Omega_c \Rightarrow \lim_{t \to \infty} x(t) \in M$.

Ex $\dot{V} = -bx_2^2 = 0 \implies x_2 = 0$



Invariant set : $x_2 \equiv 0$
$(x_2(t)=0, \forall t)$

then $\dot{x}_2 = 0 = -g(x_1) - b\cancel{x_2}$
$\quad\quad\quad\quad\quad\quad 0$
$= -g(x_1)$
then $x_1 = 0$
(Recall $g(x_1) = a\sin(x_1)$)

therefore $M = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \lim_{t\to\infty} x(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Interpretation for LTI systems...

But before that, Lyapunov theory for LTI systems.

$\dot{x} = Ax$ $\quad\quad A \in \mathbb{R}^{n \times n}$: constant matrix

Quadratic Lyapunov function condition

$$V(x) = x^T P x$$

where P is symmetric positive definite

$P = P^T > 0$ (eigenvalues are positive) why does P have to be symmetric ... ?

If $P \neq P^T$, then we can write

$$P = \frac{1}{2}\underbrace{(P+P^T)}_{P_s} + \frac{1}{2}\underbrace{(P-P^T)}_{P_a}$$

Fact $\quad x^T P x = x^T P_s x$

why?

$$x^T \overbrace{(P-P^T)}^{P_a} x = x^T P x - (Px)^T x = \underbrace{x^T y}_{\text{scalars}} - \underbrace{y^T x}_{\text{scalar}} = 0$$

Fact Any symmetric matrix $P = P^T$ is unitarily diagonalizable

$$P = U \Lambda U^T$$

$$U U^T = U^T U = I$$

Now we can write:

$$V(x) = x^T P x = x^T U \Lambda U^T x = z^T \underset{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}{\Lambda} z = \sum_{i=1}^{n} \lambda_i z_i^2 > 0$$

$$\forall x \Rightarrow \lambda_i > 0 \quad \forall i$$

Q. Can we check for GAS?

i.e., is $V(x) = x^T P x$ radially unbounded?

A. Yes

Fact $\quad P = P^T > 0$

$$\underbrace{\lambda_{min}(P)}_{>0} \|x\|_2^2 \leq x^T P x \leq \lambda_{max}(P) \|x\|_2^2$$

$$\Rightarrow \lim_{\|x\| \to \infty} x^T P x = +\infty$$

radial unbounded!

Thus, if P is positive definite, the radial unboundness comes for free.

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} \Big|_{\dot{x} = Ax} = (Ax)^T P x + x^T P (Ax)$$

$$= x^T A^T P x + x^T P A x = x^T \underbrace{(A^T P + PA)}_{-Q} x < 0$$

$$= -x^T Q x < 0$$

$$\iff \quad x^T Q x > 0$$

If Q is positive definite $Q = Q^T > 0$ then

$\bar{x} = 0$ is GAS.