Due Monday 02/22/21 (at the beginning of the class)

1. For each of the following systems a Hopf bifurcation occurs at the origin when $\alpha = 0$. Use numerical simulation to determine whether the bifurcation is subcritical or supercritical:

   (a) $\left\{ \begin{aligned} \dot{x} &= \alpha x + y \\ \dot{y} &= -x + \alpha y - x^2 y \end{aligned} \right.$

   (b) $\left\{ \begin{aligned} \dot{x} &= \alpha x + y - x^3 \\ \dot{y} &= -x + \alpha y + 2 y^3 \end{aligned} \right.$

   (c) $\left\{ \begin{aligned} \dot{x} &= \alpha x + y - x^2 \\ \dot{y} &= -x + \alpha y + 2 x^2 \end{aligned} \right.$

2. (a) Linearize the nonlinear system

$$\begin{aligned} \dot{x_1} &= x_2 + x_1 x_2^2 \\ \dot{x_2} &= -x_1 + x_1^2 x_2 \end{aligned}$$

   around the origin and classify the qualitative behavior of the linearized model. Show that the nonlinear system has no closed orbit.

   (b) Use Bendixon's criterion to show that the nonlinear system below has no closed orbits.

$$\begin{aligned} \dot{x_1} &= x_1 x_2^3 \\ \dot{x_2} &= x_1 \end{aligned}$$

3. Khalil, problem 2.20; see attached.

4. Khalil, problem 2.23; see attached.

(e) Using part (d), show that the period of oscillation of a closed trajectory through $(A, 0)$ is

$$T(A) = 2\sqrt{2}\int_0^A \frac{dy}{[G(A) - G(y)]^{1/2}}$$

(f) Discuss how the trajectory equation in part (d) can be used to construct the phase portrait of the system.

2.19 Use the previous exercise to construct the phase portrait and study periodic solutions for each of the following systems:

$$\textbf{(1)}\ \ g(x_1) = \sin x_1, \qquad \textbf{(2)}\ \ g(x_1) = x_1 + x_1^3, \qquad \textbf{(3)}\ \ g(x_1) = x_1^3$$

In each case, give the period of oscillation of the periodic orbit through the point $(1, 0)$.

2.20 For each of the following systems, show that the system has no limit cycles:

$$\begin{array}{llllll}
(1) & \dot{x}_1 & = & -x_1 + x_2, & \dot{x}_2 = & g(x_1) + ax_2, \ \ a \neq 1 \\
(2) & \dot{x}_1 & = & -x_1 + x_1^3 + x_1 x_2^2, & \dot{x}_2 = & -x_2 + x_2^3 + x_1^2 x_2 \\
(3) & \dot{x}_1 & = & 1 - x_1 x_2^2, & \dot{x}_2 = & x_1 \\
(4) & \dot{x}_1 & = & x_1 x_2, & \dot{x}_2 = & x_2 \\
(5) & \dot{x}_1 & = & x_2 \cos(x_1), & \dot{x}_2 = & \sin x_1
\end{array}$$

2.21 Consider the system

$$\dot{x}_1 = -x_1 + x_2(x_1 + a) - b, \qquad \dot{x}_2 = -cx_1(x_1 + a)$$

where a, b, and c are positive constants with $b > a$. Let

$$D = \left\{x \in R^2 \mid x_1 < -a \text{ and } x_2 < \frac{x_1 + b}{x_1 + a}\right\}$$

(a) Show that every trajectory starting in D stays in D for all future time.

(b) Show that there can be no periodic orbits through any point $x \in D$.

2.22 Consider the system

$$\dot{x}_1 = ax_1 - x_1 x_2, \qquad \dot{x}_2 = bx_1^2 - cx_2$$

where a, b, and c are positive constants with $c > a$. Let $D = \{x \in R^2 \mid x_2 \geq 0\}$.

(a) Show that every trajectory starting in D stays in D for all future time.

(b) Show that there can be no periodic orbits through any point $x \in D$.

2.23 ([85]) Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -[2b - g(x_1)]ax_2 - a^2x_1$$

where a and b are positive constants, and

$$g(x_1) = \begin{cases} 0, & |x_1| > 1 \\ k, & |x_1| \le 1 \end{cases}$$

(a) Show, using Bendixson's criterion, that there are no periodic orbits if $k < 2b$.

(b) Show, using the Poincaré–Bendixson criterion, that there is a periodic orbit if $k > 2b$.

2.24 Consider a second-order system and suppose that the set $M = \{x_1^2 + x_2^2 \le a^2\}$ has the property that every trajectory starting in M stays in M for all future time. Show that M contains an equilibrium point.

2.25 Verify Lemma 2.3 by examining the vector fields.

2.26 ([70]) For each of the following systems, show that the origin is not hyperbolic, find the index of the origin, and verify that it is different from ± 1:

$$\begin{array}{llll} (1) & \dot{x}_1 = x_1^2, & \dot{x}_2 = -x_2 \\ (2) & \dot{x}_1 = x_1^2 - x_2^2, & \dot{x}_2 = 2x_1x_2 \end{array}$$

2.27 For each of the following systems, find and classify bifurcations that occur as μ varies:

$$\begin{array}{lll} (1) & \dot{x}_1 = x_2, & \dot{x}_2 = \mu(x_1 + x_2) - x_2 - x_1^3 - 3x_1^2x_2 \\ (2) & \dot{x}_1 = -x_1^3 + x_2, & \dot{x}_2 = -(1+\mu^2)x_1 + 2\mu x_2 - \mu x_1^3 + 2(x_2 - \mu x_1)^3 \\ (3) & \dot{x}_1 = x_2, & \dot{x}_2 = \mu - x_2 - x_1^2 - 2x_1x_2 \\ (4) & \dot{x}_1 = x_2, & \dot{x}_2 = -(1+\mu^2)x_1 + 2\mu x_2 + \mu x_1^3 - x_1^2x_2 \\ (5) & \dot{x}_1 = x_2, & \dot{x}_2 = \mu(x_1 + x_2) - x_2 - x_1^3 + 3x_1^2x_2 \\ (6) & \dot{x}_1 = x_2, & \dot{x}_2 = \mu(x_1 + x_2) - x_2 - x_1^2 - 2x_1x_2 \end{array}$$