MECH 6313 - Homework 6

Jonas Wagner

2021, April 28

Contents

1 Problem 1

Problem: Show that the parallel connection of two passive dynamical systems is passive. Can you claim the same for the series connection of two passive systems?

Solution: Let two passive systems be defined as a system taking an input u and generating an output y as

$$H_1 : y_1 = h_1(u), \text{ s.t. } \langle y_1|u\rangle \geq 0$$

and

$$H_2 : y_2 = h_2(u), \text{ s.t. } \langle y_2|u\rangle \geq 0$$

with $\langle y|u\rangle = \int_0^T y^T(t)u(t)\,\mathrm{d}t$

1.1 Parallel Connection of Passive System

The parallel system H_p can then be defined by

$$H_p : h_p(u) = y_p = y_1 + y_2 = h_1(u) + h_2(u)$$

whose passivity can be proven directly by testing $\langle y_p|u\rangle$ which is calculated as

$$\begin{aligned}
\langle y_p|u\rangle &= \int_0^T y_p^T u\,\mathrm{d}t &(1)\\
&= \int_0^T (y_1 + y_2)^T u\,\mathrm{d}t &(2)\\
&= \int_0^T y_1^T u + y_2^T u\,\mathrm{d}t &(3)\\
&= \int_0^T y_1^T u\,\mathrm{d}t + \int_0^T y_2^T u\,\mathrm{d}t &(4)\\
&= \langle y_1|u\rangle + \langle y_2|u\rangle &(5)
\end{aligned}$$

Since $\langle y_1|u\rangle \geq 0$ and $\langle y_2|u\rangle \geq 0$,

$$\langle y_p|u\rangle \geq 0 \qquad (6)$$

which proves, by definition, that H_p is passive.

1.2 Series Connection of Passive System

The series system H_s can be defined by

$$H_s : h_s(u) = y_s = h_1(u) \circledast h_2(u) = h_2(h_1(u))$$

whose passivity can be tested using $\langle y_s|u\rangle$ which is calculated as:

$$\begin{aligned}
\langle y_s|u\rangle &= \int_0^T y_s^T u \,\mathrm{d}t && (7)\\
&= \int_0^T (h_1(u) \circledast h_2(u))^T u \,\mathrm{d}t && (8)\\
&= \int_0^T \left(\int_0^T h_1(t-\tau)h_2(\tau)\,\mathrm{d}\tau\right)\mathrm{d}t && (9)\\
&= \int_0^T h_2(\tau)\left(\int_0^T h_1(t-\tau)\,\mathrm{d}t\right)\mathrm{d}\tau && (10)
\end{aligned}$$

which is not explicitly ≥ 0 so this method cannot prove passivity.

A different method of analysis can be done to prove that this is not passive in general, but a counter example from MATLAB can be shown to not be passive due to a loss of positive realness of the transfer functions when placed in series:

$$G_1(s) = \frac{5s^2 + 3s + 1}{s^2 + 2s + 1},\ G_2(s) = \frac{s^3 + s^2 + 5s + 0.1}{s^3 + 2s^2 + 3s + 4}$$

and when combined in series the system is no longer passive due to a loss of positive realness.

2 Problem 2

Let

$$H(s) = \frac{s+\lambda}{s^2+as+b}$$

with $a > 0$ and $b > 0$.

2.1 Part a

Problem: For which values of λ is $H(s)$ Positive Real (PR)?

Solution:

2.2 Part b

Problem: Using the results from above, select λ_1, λ_2 such that

$$H_1(s) = \frac{s+\lambda_1}{s^2+s+1} \text{ is PR}$$

$$H_2(s) = \frac{s+\lambda_2}{s^2+s+1} \text{ is not PR}$$

Then verify the PR properties for each using the Nyquist plots of $H_1(s)$ and $H_2(s)$.

Solution:

2.3 Part c

Problem: For $H_1(s)$ and $H_2(s)$, write state-space realizations and solve for $P = P^T > 0$ in the PR lemma and explain why it fails for $H_2(s)$.

Solution:

3 Problem 3

Consider the following 3-stage ring oscillator discussed in class:

$$\tau_1 \dot{x}_1 = -x_1 - \alpha_1 \tanh(\beta_1 x_3)$$
$$\tau_2 \dot{x}_2 = -x_2 - \alpha_2 \tanh(\beta_2 x_1)$$
$$\tau_3 \dot{x}_3 = -x_3 - \alpha_3 \tanh(\beta_3 x_2)$$

with $\tau_i, \alpha_i, \beta_i > 0$ and x_i represents a voltage for $i = 1, 2, 3$.

3.1 Part a

Problem: Suppose $\alpha_1\beta_1 = \alpha_2\beta_2 = \alpha_3\beta_3 = \mu$, prove the origin is GAS when $\mu < 2$.

Solution:

idk what system this is refering to..

3.2 Part b

Problem: Show that if $\tau_1 = \tau_2 = \tau_3 = \tau$, then $\mu < 2$ is necessary for asymptotic stability. What type of bifurcation occurs at $\mu = 2$?

Solution:

3.3 Part c

Problem: Investigate the dynamic behavior of this system for $\mu > 2$ with numerical simulations.

Solution:

4 Problem 4

A MATLAB Code:

All code I write in this course can be found on my GitHub repository:
https://github.com/jonaswagner2826/MECH6313