Lecture 24

04/27/2021

Last time: Positive realness + implications

KYP Lemma & Passivity theory

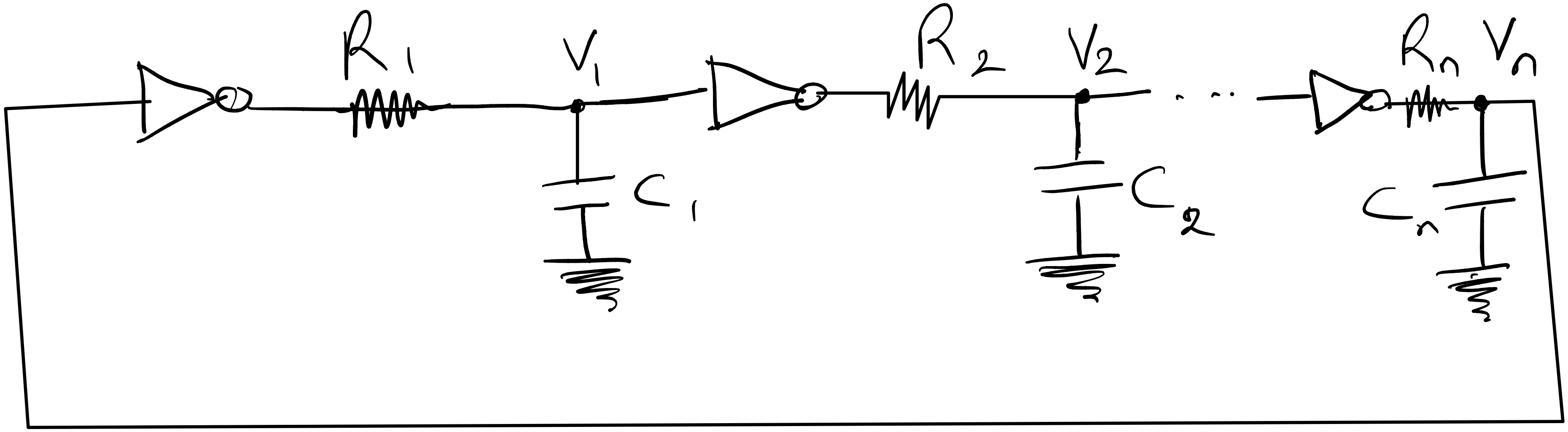
Today: An example

Input to state stability

(if time permits) feedback linearization

Ex. n-stage ring oscillator

aside
$i_c = C \frac{dV}{dt}$



nonlinear

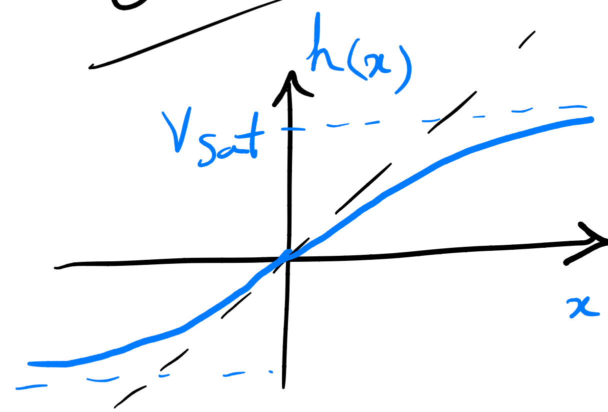
$$R_1 C_1 \dot{V}_1 = -V_1 - h_n(V_n)$$

$$R_2 C_2 \dot{V}_2 = -V_2 - h_1(V_1)$$

$$\vdots$$

$$R_n C_n \dot{V}_n = -V_n - h_{n-1}(V_{n-1})$$

Typically

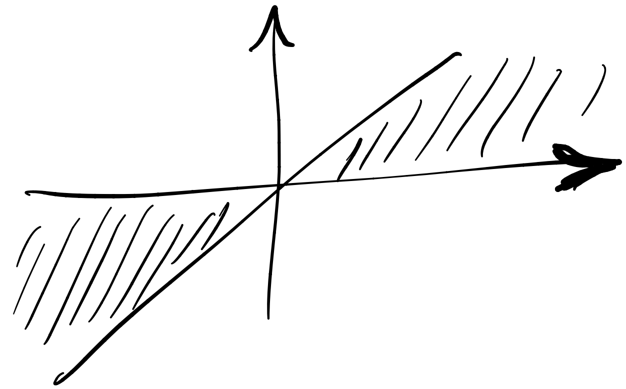


$$h_i(x) = V_{sat} \tanh\left(\frac{x}{V_s}\right)$$

1) $x h_i(x) > 0$

2) $x h_i(x) \leqslant \delta_i x^2$

input passivity



choose storage function as : $V_i(x_i) = R_i C_i \int_0^{x_i} h_i(\xi) d\xi$

$$H_i : \quad R_i C_i \dot{x}_i = -x_i + u_i$$

$$y_i = h_i(x_i)$$

where $u_i = -y_{i-1} \big|_{\text{mod } n}$ gives the coupling in the model

$$K = \begin{bmatrix} 0 & 0 & \cdots & -1 \\ -1 & 0 & \cdots & 0 \\ & -1 & \ddots & \vdots \\ & & \ddots & 0 \\ \mathbf{0} & & -1 & 0 \end{bmatrix}$$

$$\dot{V}_i = R_i C_i h_i(x_i)\dot{x}_i = -x_i h_i(x_i) + \overbrace{h_i(x_i)}^{y_i} u_i$$

$$= -x_i h_i(x_i) + y_i u_i$$

$$x h_i(x) \leq \delta_i x^2$$

$$x(h_i(x) - \delta_i x) \leq 0 \implies \begin{array}{ll} x > 0 & h_i(x) \leq \delta_i x \\ x < 0 & h_i(x) \geq \delta_i x \end{array}$$

$$\hookrightarrow \times\ h_i(x) \implies \forall x \quad h_i^2(x) \leq \delta_i x h_i(x)$$

So we can always lower bound $\delta_i x h_i(x)$

$$\Rightarrow \quad x h_i(x) \geq \frac{1}{\delta_i} h_i^2(x)$$

$$-x h_i(x) \geq -\frac{1}{\delta_i} \underbrace{h_i^2(x)}_{y_i^2}$$

output strict passive!

Thus,

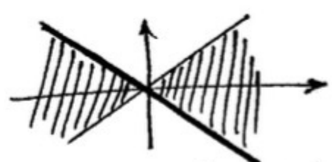
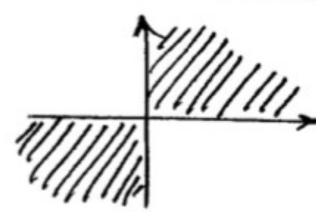
$$\boxed{\dot{V}_i \leq -\frac{1}{\delta_i} y_i^2 + y_i u_i} \quad (*)$$

So stability of this fbk interconnection is satisfied if we can find a matrix P (feasibility problem)

that satisfies the Lyapunov inequality with structured matrx K provided that (*) is satisfied.

V_i's came from $\displaystyle -\underbrace{R_i C_i}_{\text{time constant of each individual subsystem}} \int_0^{x_i} h_i(\xi)\, d\xi$

Summary

	L_2 gain	Passivity
Definition	$\exists \gamma, \beta$ st. $\|y_T\|_2 \leq \gamma \|u_T\|_2 + \beta$	$\langle u_T, y_T \rangle \geq -\beta$ forall T
state space verification with storage fcn $V(x)$	$\dot{V} \leq -\frac{1}{2} y^T y + \frac{\gamma}{2} u^T u$	$\dot{V} \leq u^T y$ (or $y^T u$)
Equivalent condition for $\dot{x} = f(x) + g(x)u$, $y = h(x)$	$\frac{\partial V}{\partial x} f(x) + \frac{1}{2} h^T(x) h(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} g(x) g^T(x) \frac{\partial V}{\partial x} \leq 0$ (HJ)	$\frac{\partial V}{\partial x} f(x) \leq 0$, $\frac{\partial V}{\partial x} g(x) = h^T(x)$
linear case $\dot{x} = Ax + Bu$, $y = Cx$	$A^T P + PA + C^T C + \frac{1}{\gamma^2} PBB^T P \leq 0$ (Bounded real lemma)	$A^T P + PA \leq 0$, $PB = C^T$ (KYP lemma)
Freq. domain condition $H(s) = C(sI-A)^{-1}B$	$\|H(j\omega)\| < \gamma$ forall ω	$\mathrm{Re}\{H(j\omega)\} \geq 0$ (Nyquist) ... (Bode) → phase properties
Memoryless	 Bow tie!	
Stability thm for intercon.	Small gain thm.	Passivity thm.

Input to state stability

Linear systems: $\dot{x} = Ax + Bu$

stability of $\dot{x} = Ax$ guarantees boundedness of the state for bounded inputs

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

vector norm

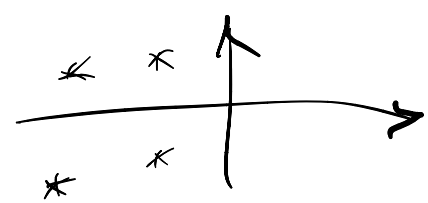
$$\|x(t)\| \leq \underbrace{\|e^{At}\|}_{\substack{\text{induced norm} \\ \text{(largest singular value of matrix exp.)}}} \|x_0\| + \int_0^t \|e^{A(t-\tau)}\| \, \|B\| \, \|u(\tau)\| \, d\tau$$

$$\leqslant K e^{-\alpha t} \|x_0\| + \|B\| \sup_{0 \leqslant \tau \leqslant t} \|u(\tau)\| \int_0^t K e^{-\alpha t} \, d\tau$$

$\Rightarrow$

$$\boxed{\|x(t)\| \leqslant \underbrace{K e^{-\alpha t} \|x_0\|}_{\text{effect of I.C.}} + \underbrace{\frac{K}{\alpha} \|B\| \sup_{0 \leqslant \tau \leqslant t} \|u(\tau)\|}_{\text{effect of input}}}$$

we derive this bound using assumption that $\dot{x} = Ax$ is stable
(e-values of A in the LHP)



Unfortunately, for nonlinear systems this property doesn't hold.

Ex (counter examples)

$$\dot{x} = -x + xu$$

Note! $\dot{x} = -x$ is stable but any u with $|u(t)| > 1 \ \forall t$ is going to generate unbounded response.

e.g. $u(t) = 2 \quad \forall t$

$$\dot{x} = -x + 2x = x \implies x(t) = e^{t} x_0$$

Def. A system $\dot{x} = f(x,u)$ is input-to-state stable (ISS) if:

$$\|x(t)\| \leq \underbrace{\beta(\|x_0\|, t)}_{\text{class KL function}} + \underbrace{\gamma\left(\sup_{0 \leq \tau \leq t} \|u(\tau)\|\right)}_{\text{class K function}}$$

For linear system:

$$\beta(r,s) = K e^{-\alpha s} r$$

$$\gamma(s) = \frac{K}{\alpha} \|B\| s$$

Implication of ISS:

1) $\dot{x} = f(x,u)$ is ISS $\Rightarrow$ $\dot{x} = f(x,0)$ is GAS

2) if $u(t) \xrightarrow{t\to\infty} 0$ $\Rightarrow$ $x(t) \xrightarrow{t\to\infty} 0$

A dissipation like inequality for ISS:

If there are class K_∞ functions $\alpha_i(\cdot)$; $i = 1,2,3,4$ and a cts differentiable function $V(x)$ st.

* $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$

* $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq -\alpha_3(\|x\|) + \alpha_4(\|u\|)$

we have ISS.

~~Proof.~~ $\longrightarrow$ (Khalil)

Ex. $\dot{x} = -x^p + x^q u$

p is an odd integer

ISS if $p > q$

$$V(x) = \frac{1}{2} x^2 \Rightarrow \dot{V} = x\dot{x} = -x^{p+1} + x^{q+1} u$$

Young's inequality : $a.b \leqslant \frac{\alpha^r}{r} |a|^r + \frac{1}{s\alpha^s} |b|^s$

$r > 1$
$s > 1$ & $(r-1)(s-1) = 1$; $\alpha > 0$

$$\Rightarrow \quad \dot{V} = -x^{p+1} + x^{q+1} u$$

$$x^{q+1} u \leq \frac{\alpha^r}{r} |x|^{(q+1)r} + \frac{1}{s\alpha^r} |u|^s$$

choose: $\quad r = \dfrac{p+1}{q+1} > 1 \quad ; \quad s = 1 + \dfrac{1}{r-1}$

and α s.t. $\alpha^r / r = \dfrac{1}{2}$

$$\dot{V} \leq \underbrace{-\frac{1}{2} |x|^{p+1}}_{\alpha_3(|x|)} + \underbrace{\frac{1}{s\alpha^4} |u|^s}_{\alpha_4(\|u\|)} \quad \Rightarrow \text{ISS} \checkmark$$

Note! ISS if $p > q$

Compare with $\dot{x} = -x + xu$