Lecture 14

03/22/2021

Last time: Comparison function

stability analysis of TV systems

Today: Recap. + finish discussion

Lyapunov-based analysis

$$\dot{x} = f(x,t) \qquad (*)$$

time-varying system

$f(0,t) = 0$, for all times

$\bar{x} = 0$ equilibrium point of system $(*)$

- Stability definitions

- Uniform stability (US)

$$\|x(t)\| \leqslant \underbrace{\alpha}_{\text{Class-}\mathcal{K}\text{ function}}(\|\underbrace{x_0}_{x(t_0)}\|)$$

Class-$\mathcal{K}$ function $\begin{cases} \text{cts func.} \\ \alpha(0)=0 \\ \text{strictly increasing} \end{cases}$

- Uniform Asymptotic Stability (UAS)

$$\|x(t)\| \leqslant \underbrace{\beta}_{\mathcal{KL}\text{ function}}(\|x(t_0)\|, t-t_0)$$

- Uniform Exponential Stability (UES)

$$\underbrace{\|x(t)\| \leq K\|x(t_0)\|}_{\text{class } K} \underbrace{e^{-a(t-t_0)}}_{\text{monotonically decreasing portion}}$$

$$\lim_{t \to \infty} e^{-a(t-t_0)} = 0$$

Ex Time-invariant nonlinear systems

$$\dot{x} = f(x) \; ; \; f(0) = 0$$

Fact: $\bar{x} = 0$ locally exponentially stable

$\Updownarrow$

linearization around $\bar{x} = 0$ is stable

$$\boxed{\text{Local E.S.} \iff Re\{\lambda_i(A)\} < 0 \qquad A := \left.\frac{\partial f}{\partial x}\right|_{x = \bar{x} = 0}}$$

Ex. $\dot{x} = -x^3$ GAS $\left[V(x) = \frac{1}{2}x^2\right]$

linearization

$\dot{x} = 0\,x$ $\longrightarrow$ stability in the sense of Lyap.

(Marginal stability)

$\Rightarrow$ $\bar{x} = 0$ of $\dot{x} = -x^3$ is NOT exponentially stable.

Thus, for nonlinear systems

Asymptotic stability $\neq$ Exponential stability

Ex. $V(x) = x^T P x$

Then, $\begin{cases} \alpha_1(\|x\|) = \lambda_{min}(P) \cdot \|x\|_2^2 \\ \alpha_2(\|x\|) = \lambda_{max}(P) \cdot \|x\|_2^2 \end{cases}$

$$\alpha_1(\|x\|) \leq \|V(x)\| \leq \alpha_2(\|x\|)$$

Today: How can we build on properties of $V(x)$ to classify stability of nonlinear (TV) sys?

Q. How can we check stability of $\bar{x}=0$?

e-value enough?

$$\cancel{\text{stability} \iff \mathrm{Re}\{\lambda_i(A(t))\} < 0 \quad \forall t \geq t_0}$$

T.V. system

even in the case of LTV systems

$$\dot{x} = A(t)\,x \implies x(t) = \Phi(t,t_0)\,x(t_0)$$

state-transition

$$\Phi(t_0,t_0) = I$$

$$\frac{\partial \Phi(t,t_0)}{\partial t} = A(t)\,\Phi(t,t_0)$$

LTI

$$\Phi(t,t_0) = e^{A(t-t_0)}$$

Q. How to establish a bound on $\|x(t)\|$?

$$\|x(t)\| = \|\Phi(t,t_0)\,x(t_0)\| \leq \|\Phi(t,t_0)\|\,\|x(t_0)\|$$

If $\quad \|\Phi(t,t_0)\| \leq K e^{-a(t-t_0)}$

$\forall t \geq t_0$

Then $\Rightarrow$ UES ✓

Challenge:

- No explicit expression for $\Phi(t,t_0)$
- E-value test <u>inconclusive</u> (doesn't work)

In fact, there are problems in which

$$\mathrm{Re}\{\lambda_i(A)\} < 0 \quad , \forall i \ \& \ t \geq t_0$$

Yet
$$\lim_{t \to \infty} \|\Phi(t,t_0)\| = +\infty$$

Examples in Khalil

What do we do???

A. Lyapunov-based analysis

→ the most general tool for stability analysis.

Thm. ★ Given $\dot{x} = f(x,t)$ with $f(0,t) = 0$ (e.p.)

if:

1. $$W_1(x) \leq V(x,t) \leq W_2(x)$$

and

2. $$\frac{\partial V(x,t)}{\partial t} = \partial V/\partial t + \left(\nabla_x V(x,t)\right)^T f(x,t) \leq 0$$

for some positive definite functions W_1 and W_2 on a domain D that contains $\bar{x} = 0$, then $\bar{x} = 0$ is U.S.!

★ In addition to being U.S. if

3. $\frac{dV(x,t)}{\partial t} \leq -W_3(x)$ on $D(\ni 0)$

where $W_3(x)$ is positive definite function, then

$\bar{x}=0$ is U.A.S. (on D).

★ If $\bar{x}=0$ is U.A.S. on $\mathbb{R}^n$ and $W_1(x)$ is radially unbounded then $\bar{x}=0$ is Globally U.A.S.

* If $W_i(x) = K_i \|x\|^a$ for $K_i > 0$, $a > 0$

(the same 'a' for all W_i's)

then $\bar{x} = 0$ is Uniformly Exponentially Stable (UES).

Note

$$\cancel{UAS \iff UES}$$

$$UES \implies UAS$$ (but not vice versa)

Ex. $\dot{x} = -x^3$ time-invariant system

$W_1 \leq V(x) \leq W_2$

$$V(x) = \frac{1}{2}x^2$$

$$\dot{V} = x\dot{x} = -x^4$$

$\Rightarrow$ $W_1(x) = W_2(x) = \frac{1}{2}x^2$

$W_3(x) = x^4$

This system is UAS but we cannot conclud UES.

Proof. 1. let $W_1(x) \leqslant V(x,t) \leqslant W_2(x)$ and

$$\left.\frac{dV(x,t)}{dt}\right|_{\dot{x}=f(x,t)} \leqslant 0$$

(non increasing function of time)

Recall: For W_1 & W_2 positive definite there are class-K functions α_1 & α_2 st.

1. $$\boxed{\alpha_1(\|x\|) \leqslant W_1(x)} \leqslant V(x,t) \leqslant \boxed{W_2(x) \leqslant \alpha_2(\|x\|)}$$

also given $\partial V / \partial t \leq 0 \Rightarrow \boxed{V(x,t) \leq V(x(t_0),t_0)}$

From here we can conclude that

$$\alpha_1(\|x(t)\|) \leq V(x,t) \leq V(x(t_0),t_0) \leq \alpha_2(\|x(t_0)\|)$$

$$\alpha_1(\|x(t)\|) \leq \alpha_2(\|x(t_0)\|) \quad \text{forall } t \geq t_0$$

$$\|x(t)\| \leq [\alpha_1^{-1} \circ \alpha_2](\|x(t_0)\|)$$

Notation

$$(\alpha_1^{-1} \circ \alpha_2)(r) = \alpha_1^{-1}(\alpha_2(r))$$

aside

Fact: $\alpha := \alpha_1^{-1} \circ \alpha_2$ locally (around zero) class K-function

if α_1 & α_2 are class-K themselves

Lemma 4.2 (Khalil)

a) Inverse of a class K function is well-defind locally (globally if K_∞) and it itself is a class-K function.

b) The composition of class-K functions is also class K.

Ex. $\underbrace{\alpha_1(r) = r^2}_{\text{class } K} \Rightarrow \underbrace{\alpha_1^{-1}(y) = \sqrt{y} = y^{1/2}}_{\text{class } K}$

$\underbrace{\alpha_1(r_1) = r_1^{\,2} \; ; \; \alpha_2(r_2) = r_2^{\,1/3}}_{\text{class } K} \Rightarrow (\alpha_1 \circ \alpha_2)(r_2) = \alpha_1(r_2^{\,1/3})$

$= r_2^{\,2/3}$

class K

Thus, $\|x(t)\| \leq \alpha(\|x(t_0)\|)$

$$\alpha := \alpha_1^{-1} \circ \alpha_2 \quad \text{class } K$$

$\Rightarrow \bar{x} = 0$ of $\dot{x} = f(x,t)$ is U.S. $\blacksquare$

Let $\dfrac{\partial V(x,t)}{\partial t} \leq 0$ and $\left.\dfrac{\partial V(x,t)}{\partial t}\right|_{\dot{x}=f(x,t)} \leq -W_3(x)$

Since W_3 is positive definite, there is a class-K function α_3 s.t. $\alpha_3(\|x\|) \leq W_3(x)$

$$-W_3(x) \leq -\alpha_3(\|x\|)$$

$$\rightarrow \quad \frac{\partial V(x,t)}{\partial t} \leq -\alpha_3(\|x\|) \quad (i)$$

Also, note that we had

$$\alpha_1(\|x\|) \leq \boxed{V(x,t) \leq \alpha_2(\|x\|)}$$

$$\alpha_2^{-1}(V(x,t)) \leq \|x\|$$

$$\alpha_3(\alpha_2^{-1}(V(x,t))) \leq \alpha_3(\|x\|) \quad (ii)$$

$$(i) + (ii) \Rightarrow \quad -\alpha_3(\|x\|) \leq -(\alpha_3 \circ \alpha_2^{-1})(V(x,t))$$

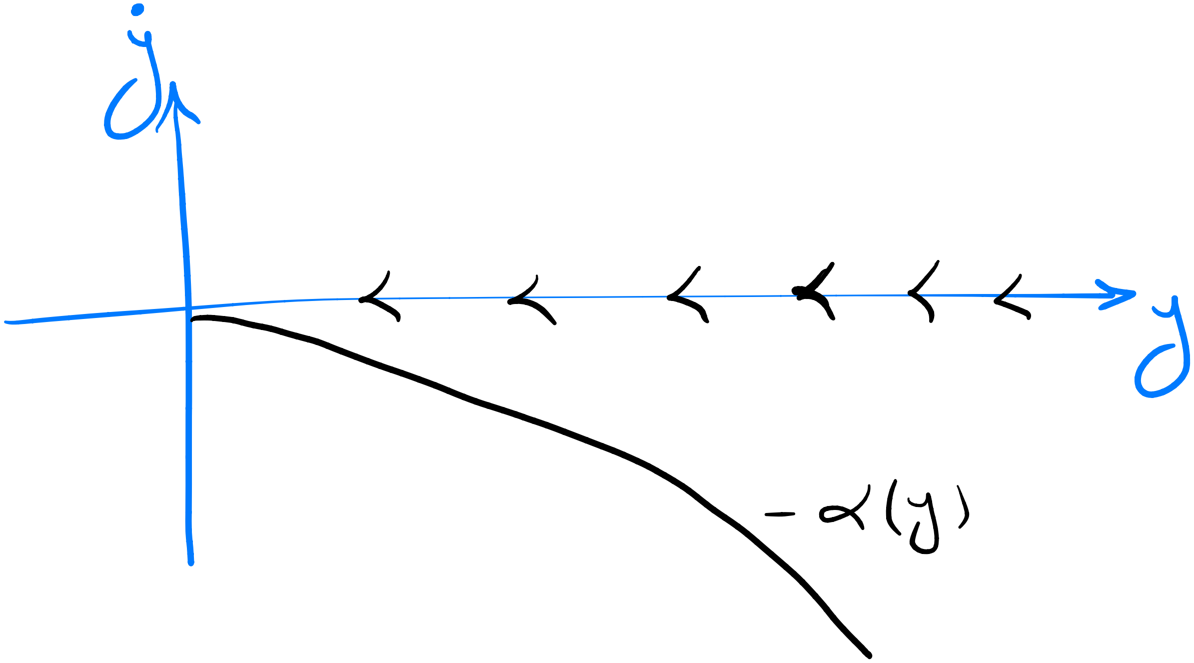
and

$$\boxed{\begin{gathered}\frac{dV(x,t)}{dt} \leq -\alpha\big(V(x,t)\big) \\ \text{where } \alpha := \alpha_3 \circ \alpha_2^{-1} \; : \text{class-}K\end{gathered}} \quad (iii)$$

Scalar differentially inequality

If ODE scalar $\dot{y} = -\alpha(y)$

$y \geq 0$ + α class K



Ex. $\alpha(y) = a \cdot y \qquad a > 0$

$$\dot{y} = -ay \Rightarrow y(t) = \underbrace{y_0}_{y(t_0)} \underbrace{e^{-a(t-t_0)}}_{\text{monotonically decreasing}}$$

$$V(t) \leq V(t_0)\, e^{-a(t-t_0)}$$

Ex. $\alpha(y) = K y^2 \quad K>0$

$$\dot{y} = -Ky^2 \Rightarrow \underbrace{y(t) = \frac{y(t_0)}{Ky(t_0)(t-t_0)+1}}_{KL}$$

$$y(t) = \beta(y(t_0), t-t_0)$$

$\beta \rightarrow$ class KL function

from (iii) we can conclude that

$$\frac{dV(x,t)}{dt} \leq -\alpha\big(V(x,t)\big)$$

$$\Downarrow$$

$$\alpha_1(\|x(t)\|) \leq \boxed{V(x,t) \leq \beta\big(V(x_0,t_0),\, t-t_0\big)}$$

$$V(x_0,t_0) \leq \alpha_2(\|x_0\|,t_0)$$

$$\alpha_1\big(\|x(t)\|\big) \leq \beta\big(V(x_0,t_0),\, t-t_0\big)$$

$$\leq \beta\big(\alpha_2(\|x_0\|,t_0),\, t-t_0\big)$$

aside (Khalil) $\alpha_1^{-1} \circ \beta = \tilde{\beta}$ class KL

if α_1 is class K

and β is KL

$$\Rightarrow \quad \|x(t)\| \leq \underbrace{\tilde{\beta}}_{\text{class KL}}(\|x(t_0)\|, t-t_0)$$

UAS!

* for UES :

i-independent

$$\alpha_i(\|x\|) = K_i \|x\|^a \quad ; i=1,2,3 \quad \text{(iv)}$$

Recall $\dot{V} \leq -\alpha_3(\alpha_2^{-1}(\|x\|))$

from the proof of UAS

$\alpha_i = K_i \|x\|^a$

For $\alpha_2^{-1}(\|x\|) \Longrightarrow \|x\| = \left\| \frac{y}{K_2} \right\|^{1/a}$

$$\alpha_3\left(\alpha_2^{-1}(\|x\|)\right) = K_3\left(\left(\frac{V}{K_2}\right)^{\cancel{1/\alpha}}\right)^{\cancel{\alpha}} = \frac{K_3}{K_2}V$$

from this inequality and (iv) we have

$$\dot{V}(t) \leq -\frac{K_3}{K_2}V(t)$$

$$V(t) \leq V(t_0)\, e^{-K_3/K_2\,(t-t_0)}$$

UAS