Lecture 18

04/05/2021

Last time: parameter estimation

$$y = \underbrace{\psi^T}_{\text{regressor}} \theta$$

$\theta \longrightarrow$ unknown constant param.

Convergence of gradient-based estimation algorithm

Today: Model reference adaptive control (MRAC)

Ex $\rightarrow$ $$\begin{cases} \dot{x}_1 = -ax + w^T(t)\, x_2 \\ \dot{x}_2 = -W(t)\, x_1 \end{cases}$$

$x_1(t) \in \mathbb{R}$; $a > 0$ $\quad$ $w(t), x_2(t) \in \mathbb{R}^{n-1}$

$$A(t) = \underbrace{\left.\begin{bmatrix} -a & w^T(t) \\ -w(t) & \underset{(n-1)\times(n-1)}{0} \end{bmatrix}\right\} n}_{n} \in \mathbb{R}^{n \times n}$$

$$V(x_1, x_2) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^T x_2 = x^T P x$$

$$\dot{V}(x_1, x_2) = x_1 \dot{x}_1 + x_2^T \dot{x}_2 = x_1(-a x_1 + \cancel{w^T(t) x_2})$$

$$\cancel{- x_2^T w(t) x_1}$$

$$\dot{V}(x_1, x_2) = -a x_1^2 + 0 \, \|x_2\|_2^2 \leqq 0$$

negative semi-definite

$$a > 0$$

$$x_1(t) \in \mathbb{R}$$

$$x_2(t) \in \mathbb{R}^{n-1}$$

$\bar{x} = 0$ is uniformly stable

LTV $\Rightarrow$ No La'Salle

We showed that if for $\dot{x} = A(t)x$

$$P = \frac{1}{2}I \Rightarrow \dot{V} = -x^T \underbrace{C^T C}_{Q} x$$

$$C = [\sqrt{a} \quad 0] \qquad C^T C = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$$

Last time

$$\left.\begin{aligned} \dot{x} &= A(t)x(t) \\ y &= C(t)x(t) \end{aligned}\right\} \text{UAS} \iff \begin{gathered}\text{U.O.} \\ (A(t), C(t))\end{gathered}$$

Challenge : state-transition matrix of $A(t)$?

<u>Key result</u>

$$\begin{gathered}\text{U.O.} \\ (A(t), C(t))\end{gathered} \iff \begin{gathered}\text{U.O.} \\ (A(t)+L(t)C(t), C(t))\end{gathered}$$

In this example:

$$\underbrace{A(t)+L(t)C(t)}_{2\times 2} = \begin{bmatrix} -a & w^T(t) \\ -w(t) & 0 \end{bmatrix} + \underbrace{\begin{bmatrix} l_1(t) \\ l_2(t) \end{bmatrix}}_{\text{column vector}} \begin{bmatrix} \sqrt{a} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -a + l_1(t)\sqrt{a} & w^T(t) \\ -w(t) + l_2(t)\sqrt{a} & 0 \end{bmatrix} =$$

choose

$$\begin{bmatrix} l_1(t) \\ l_2(t) \end{bmatrix} = \begin{bmatrix} \sqrt{a} \\ \frac{1}{\sqrt{a}} w(t) \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & w^T(t) \\ 0 & 0 \end{bmatrix}$$

U.O.

$$\dot{x}_1 = -a x_1 + w^T(t) x_2$$
$$\dot{x}_2 = -w(t) x_1$$
$$y = \begin{bmatrix} \sqrt{a} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

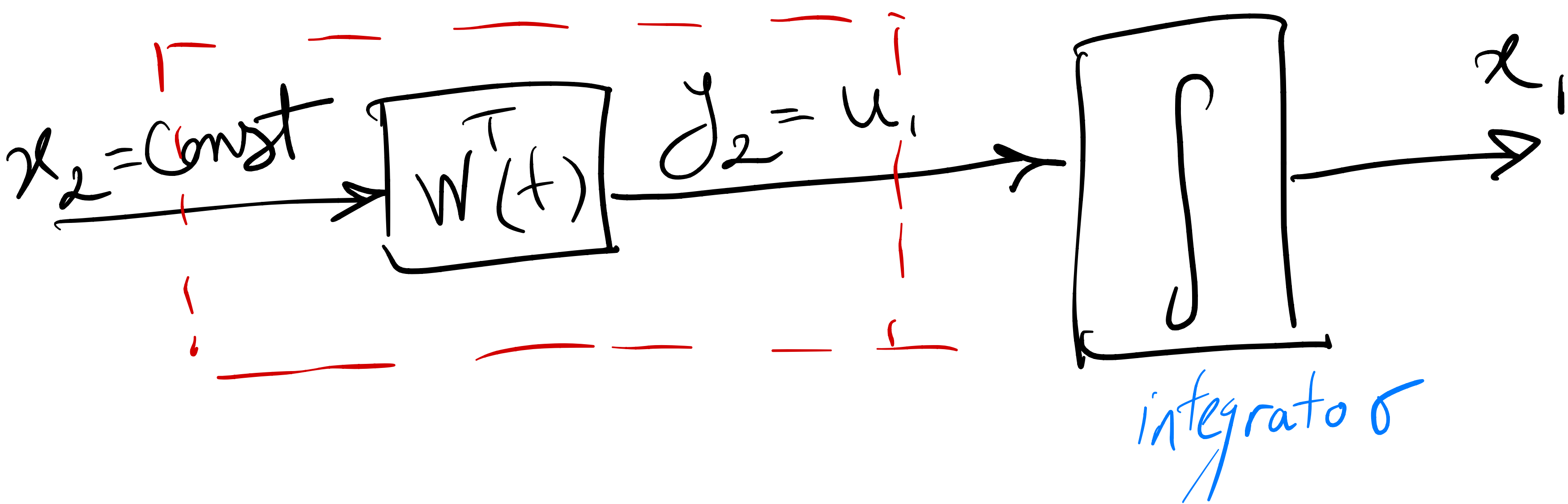
$\Longleftrightarrow$

U.O.

$$\dot{x}_1 = w^T(t)\, x_2$$
$$\dot{x}_2 = 0$$
$$y = \begin{bmatrix} \sqrt{a} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Block diagram

$\dot{x}_1 = u_1$



<u>Fact</u> passing output through an integrator doesn't change U.O.

U.O.

$$\dot{x}_1 = W^T(t)\,x_2$$

$$\dot{x}_2 = 0$$

$$y = \begin{bmatrix} \sqrt{a} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Leftrightarrow$ (*)

U.O.

$$\dot{x}_2 = 0$$

$$y_2 = W^T(t)\,x_2$$

$A_2 = 0 \quad C_2(t) = W^T(t)$

U.O. of $(*)$:

If $\exists\, \delta, \alpha > 0$ st

$$\int_{t_0}^{t_0+\delta} w(t)\, w^T(t)\, dt \geq \alpha I \quad \text{forall } t_0$$

P.E. of $w(t)$

$\Rightarrow$ U.A.S.

Takeaway

Uniform observability is preserved through intergtion.

Model Reference Adaptive Control (MRAC)

(1) $\dot{y} = a y + u$ $\quad a \in \mathbb{R}$

scalar

constant but unknown

Goal: design control input $u(t)$ s.t. our system behaves as a given reference model:

positive constant

(2) $\dot{y}_m = -a_m y_m + r$

given reference signal

model

If a was known

$$\dot{y} = ay + \boxed{u}$$

and $y(t)$ was measured, then

$$\boxed{u(t) = -(a + a_m)\, y(t) + r(t)} \qquad (3)$$

$$\Rightarrow \dot{y} = -a_m y + r \qquad (4)$$

Define error : $e(t) = y(t) - y_m(t)$

$$\dot{e}(t) = \dot{y}(t) - \dot{y}_m(t) \xRightarrow{(2)+(4)} \dot{e} = -a_m e$$

$$\Rightarrow e(t) = \exp(-a_m t)\, e(0)$$

Challenge: What can we do when $a = \text{const}$ but UNKNOWN ?

Tow options: 1° estimate a ; design u

2° estimate the feedback gain in $u(t)$ directly

We will pursue option 2°:

Note: when a was known $\Rightarrow K = a + a_m$

i.e., $u(t) = -K y(t) + r(t)$

with 'a' being unknown, let choose

$$u(t) = -\underbrace{\hat{K}(t)}_{\text{estimate of feedback gain } K \text{ at time } t} \underbrace{y(t)}_{\text{measurement output}} + \underbrace{r(t)}_{\text{known reference signal}}$$

let: $\tilde{K}(t) := K - \hat{K}(t)$ feedback gain estimation error

Note for $a = \text{const.}$ but unknown, we expect that there is constant feedback gain K that would work. $\longrightarrow$ $\left(a - K = -a_m\right)$

$$\dot{y} = ay + u$$

$$u = -\hat{K}y + r$$

$$\Rightarrow \dot{y} = ay - \hat{K}y + r$$

$$= ay - [K - \tilde{K}]y + r$$

$$\Rightarrow \dot{y} = \underbrace{(a-K)}_{-a_m}y + \tilde{K}y + r$$

$$= -a_m y + \tilde{K}y + r \quad (\mathrm{I})$$

additional term that comes from the fact that we don't know parameter a.

Reference model: $\dot{y}_m = -a_m y_m + r$ (II)

(I) − (II) ; $e := y - y_m$

$$\boxed{\dot{e}(t) = -a_m e(t) + \tilde{K}(t) \cdot y(t)}$$

Q. How to design parameter update law for $\hat{K}(t)$ s.t. $\tilde{K}y$ doesn't cause problem.

Clearly, the only term that we can influence in the problematic last term is $\tilde{K}(t)$.

[$y(t)$ measurement output $\qquad$ $e(t)$: output error]

Have to find equation for $\dot{\tilde{K}}$ that doesn't containt K or $\tilde{K}$.

$$\dot{e} = -a_m e + \tilde{K} y$$

Lapunov function candidate:

$$V(e, \tilde{K}) = \frac{1}{2} e^2(t) + \frac{1}{2} \tilde{K}^2(t)$$

aside Note that $\tilde{K}(t) = \overset{\text{const.}}{K} - \hat{K}$

$$\boxed{\dot{\tilde{K}}(t) = -\dot{\hat{K}}(t)} \qquad (\dot{K} = 0 \Leftarrow K = \text{const})$$

Objective: drive $e := y - y_m$ and

$\tilde{K} := K - \hat{K}$ to zero

$$\dot{V} = e \cdot \dot{e} + \tilde{K} \cdot \dot{\tilde{K}} = e(-a_m e + \tilde{K} y) + \tilde{K} \dot{\tilde{K}}$$

$$\Rightarrow \dot{V} = \underbrace{-a_m e^2}_{\leq 0} + \tilde{K}\left[\dot{\tilde{K}} + e y\right]$$

Can this term be ≤ 0 ?

Note: $\tilde{K} = -ey$ gives

$$\boxed{\dot{V} = -a_m e^2 - 0.\tilde{K}^2 \leq 0}$$

$$\dot{\tilde{K}} = -ey = -(y - y_m)y$$

output of reference model

measured outputs

We can indeed implement

$$\dot{\hat{K}}(t) = -\dot{\tilde{K}}(t) = +e(t).y(t)$$

since $e(t)$, $\tilde{K}(t)$ are bounded;

$$\lim_{t \to \infty} e(t) = 0$$

Q. Can we make the 2nd term in $\dot{V}$ NEGATIVE ???

How about: $\dot{\tilde{K}} = -ey - \underbrace{\alpha}_{>0} \tilde{K}$

No! because rhs depends on $\tilde{K}$!

$$\dot{\hat{K}}(t) = -\dot{\tilde{K}}(t) = e(t)y(t) + \alpha\tilde{K}(t)$$

Not implementable

$\tilde{K} = K - \hat{K}$

unknown

So, the best that we can do is to make the second term in $\dot{V}$ zero !!!

$$\boxed{\begin{aligned}\dot{e} &= -a_m e + y(t)\tilde{K}\\ \dot{\tilde{K}} &= -y(t)e\end{aligned}}$$

$$A(t) = \begin{bmatrix} -a_m & y(t) \\ -y(t) & 0 \end{bmatrix}$$

Q. Conditions for U.A.S and $\lim_{t\to\infty} \tilde{K}(t) \to 0$

$$V(e,\tilde{K}) = \frac{1}{2}e^2 + \frac{1}{2}\tilde{K}^2$$

$$\dot{V} = -\begin{bmatrix} e & \tilde{K} \end{bmatrix} \begin{bmatrix} \sqrt{a_m} \\ 0 \end{bmatrix} \begin{bmatrix} \sqrt{a_m} & 0 \end{bmatrix} \begin{bmatrix} e \\ \tilde{K} \end{bmatrix} \leq 0$$

Exact structure as the 1st example today with

$$x_1 = e$$
$$x_2 = \tilde{K}$$
$$w(t) = y(t)$$

$$\text{U.O.} \iff \int_{t_0}^{t_0+\delta} y^2(t)\, dt \geq \alpha I \quad \text{for all } t_0$$

In order to have U.O. and thus U.A.S it is of essence to carefully choose $r(t)$ in the reference model $\dot{y}_m = -a_m y_m + r$.

to recap

closed loop system

$$\dot{y} = ay + u \quad : \text{plant}$$

$$\dot{y}_m = -a_m y_m + r \quad : \text{reference model}$$

$$\dot{\hat{k}} = +y(y - y_m) \quad : \text{parameter update for } \hat{k}$$

$$u = -\hat{k}y + r \quad : \text{control signal}$$

adaptive controller: B/C. dynamic system contain e.g. for $\hat{k}$.

Compare this with the static controller

from when a is known

$$u = -(a + a_m)\, y + r.$$