Lecture 13

03/10/2021

Last time : LDM

Today : Comparison functions

Stability of time varying systems

Comparison functions:

Moral: background material for stability of TV systems.

Also, useful in broader context, e.g., determining convergence rate via Lyapunov function.

3 type: K, K_∞, KL

class K functions:

cts functions $\alpha : [0,\infty) \longrightarrow [0,\infty)$

that have 2 properties: 1) $\alpha(0) = 0$

2) α strictly increasing

* α is a class K_∞ function if

in addition to conditions 1) & 2) above

3) $\lim_{r \to \infty} \alpha(r) = +\infty$

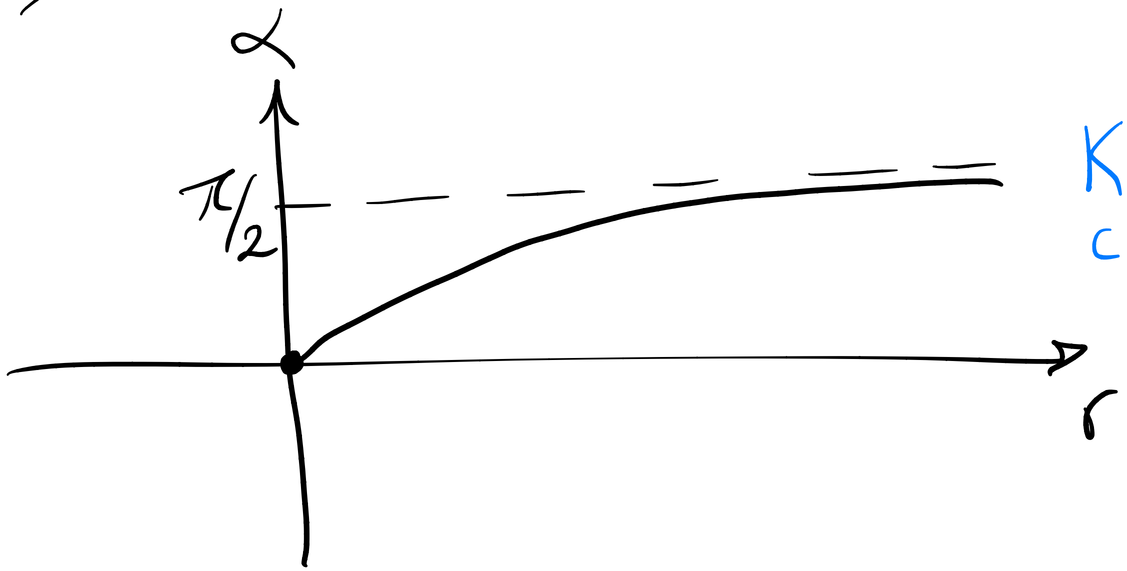
Note: For stability analysis we should think of the functions as being function of an initial condition of the system $\alpha(\|x_0\|)$

Ex

$$\left.\begin{aligned} &\alpha(r) = r^c \; ; \; c > 0 \\ &\alpha(0) = 0 \\ &\frac{d\alpha}{dr} = c r^{c-1} > 0 \\ &\alpha \text{ is increasing function} \end{aligned}\right\} \Rightarrow \text{class } K$$

Also, since $r^c \xrightarrow{r \to \infty} +\infty$, $\alpha(r)$ is class K_∞.

Ex $\alpha(r) = \tan^{-1}(r) = \operatorname{atan}(r)$



$$\text{K class}\left\{\begin{array}{l} 1)\ \operatorname{atan}(0) = 0 \\ 2)\ \dfrac{d\alpha}{dr} = \dfrac{1}{1+r^2} > 0 \end{array}\right.$$

$$3)\ \lim_{r \to \infty} \alpha(r) = \pi/2$$

(K class function but not K_∞)

* KL functions:

A <u>cts</u> function $\beta : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$

is class KL if:

a) $\beta(\cdot, s)$ is a class K (for any fixed s β is class K)

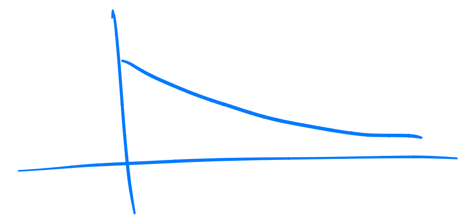
b) $\beta(r, \cdot)$ is decreasing and

$$\underbrace{\beta(r,s) \xrightarrow{s \to \infty} 0}_{\text{(for any fixed } r)}$$

Ex $\beta(r,s) = K r^c e^{-as}$ $\quad K, a > 0$, $\; c > 0$

if I fix $s = s_0$ I have $\beta(r, s_0) = \underbrace{K e^{-a s_0}}_{\text{constant}} r^c = m r^c$ K class

if I fix $r = r_0$ I have $\beta(r_0, s) = \underbrace{K r_0^c}_{\text{constant } m} e^{-as} = m e^{-as}$



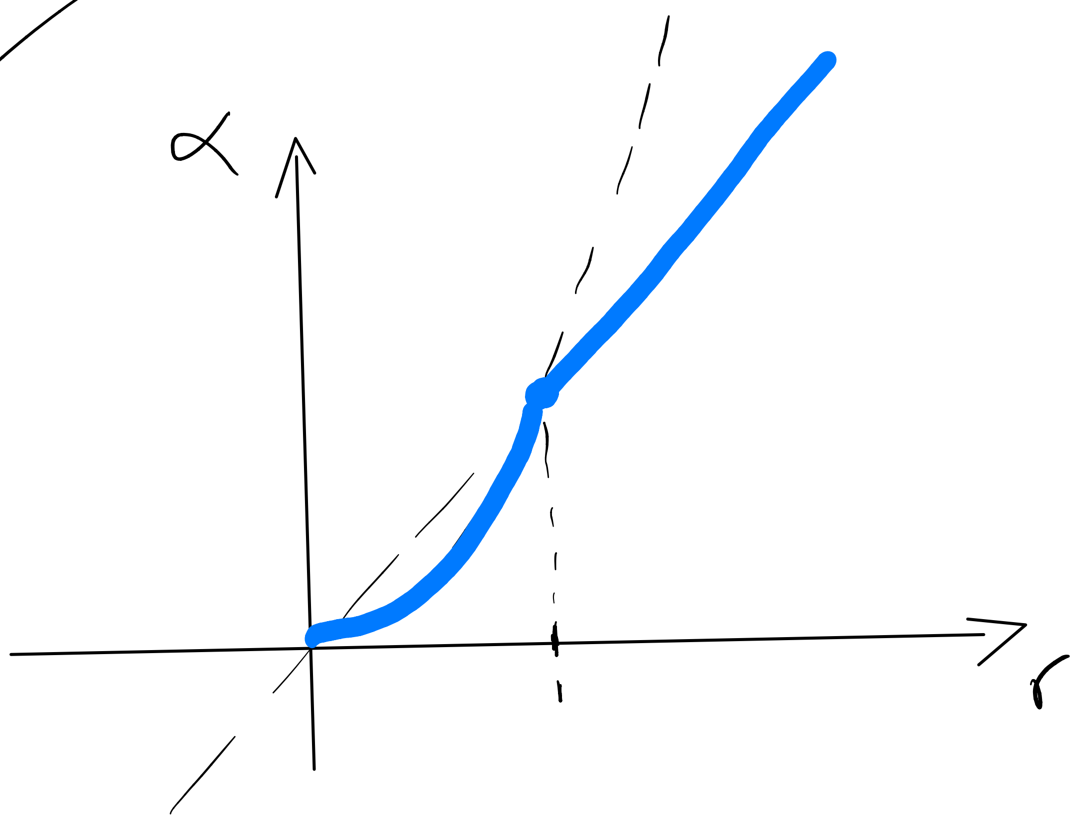
β is KL ✓

Ex $\beta(r,s) = \dfrac{r}{Krs+1}$; $K>0$

fixed $s=s_0$: $\left.\begin{array}{l} \beta(0,s_0)=0 \\ \dfrac{d\beta}{dr} = \dfrac{1}{(Krs+1)^2} > 0 \end{array}\right\}$ $\beta(\cdot,s)$ class 1

fixed $r=r_0$: $\left.\begin{array}{l} \beta(s,r_0) = \dfrac{r_0}{Kr_0 s+1} \\ \dfrac{d\beta}{ds} = \dfrac{-r_0(Kr_0)}{(Kr_0 s+1)^2} = \dfrac{-Kr_0^2}{(Kr_0 s+1)^2} < 0 \\ \lim_{s\to\infty} \beta(s,\cdot) = 0 \end{array}\right\}$ KL class ✓

Ex $\alpha(r) = \min(r, r^2)$



class K function

* Note we don't need function to be C^1 in order to be K or K_∞.

We want to study stability of $\begin{array}{l} \dot{x} = f(x,t) \\ f(0,t) = 0 \end{array}$ $(*)$

Time invariant system

$$\dot{z} = g(z) \qquad (1)$$

Consider trajectories $\phi(x_0, t_0; t)$

$$\dot{\bar{z}} = g(\bar{z}(t)) \qquad (2)$$

$$x(t) = z(t) - \bar{z}(t) \quad (3)$$

$$\frac{dx(t)}{dt} = \dot{x}(t) = \dot{z}(t) - \dot{\bar{z}}(t) = g(z(t)) - g(\bar{z}(t))$$

$$= g(x + \bar{z}(t)) - g(\bar{z}(t))$$

$$=: f(t, x)$$

$$z(t) = \bar{z}(t) + x(t)$$

original

time varying trajectory

fluctuations

The origin of $(*)$ $(\bar{x}=0)$ is stable if for every $\varepsilon>0$, there is $\delta(\varepsilon,t)>0$ st.

$$\|\underbrace{x(t_0)}_{x_0}\| < \delta(\varepsilon,t_0) \implies \|x(t)\| < \varepsilon$$

for all $t > t_0$

$x(t;t_0,x_0)$

trajectory that starts @ $x(t_0)$ in other notation

$\varphi(t, x(t_0))$

Note: Choice of initial time t_0 matters, e.g., we now have $\delta(\varepsilon,t_0)$

If in the above definition $\delta = \delta(\varepsilon)$ (independent of t_0)

then the origin is <u>uniformly</u> stable

aside

remember in time-invariant case we had

$\phi(x_0, t-t_0)$

$x(x_0, t-t_0)$

Ex $\dot{x} = \dfrac{-x}{1+t} = a(t)x$

$$\frac{dx}{dt} = \frac{-1}{1+t}x \Rightarrow \frac{dx}{x} = -\frac{1}{1+t}dt$$

$$\int_{x(t_0)}^{x(t)} \frac{dx}{x} = -\int_{t_0}^{t} \frac{1}{1+t}dt$$

$$\ln(x) - \ln x_0 = \ln\left(\frac{1+t_0}{1+t}\right)$$

$$\ln(x) = \ln x_0 + \ln\left(\frac{1+t_0}{1+t}\right)$$

apply $e^{(\cdot)}$ to both sides

$$x(t) = e^{\ln(x_0) + \ln\left(\frac{t_0+1}{t+1}\right)}$$

$$\boxed{x(t) = x(t_0)\,\frac{1+t_0}{1+t}}$$

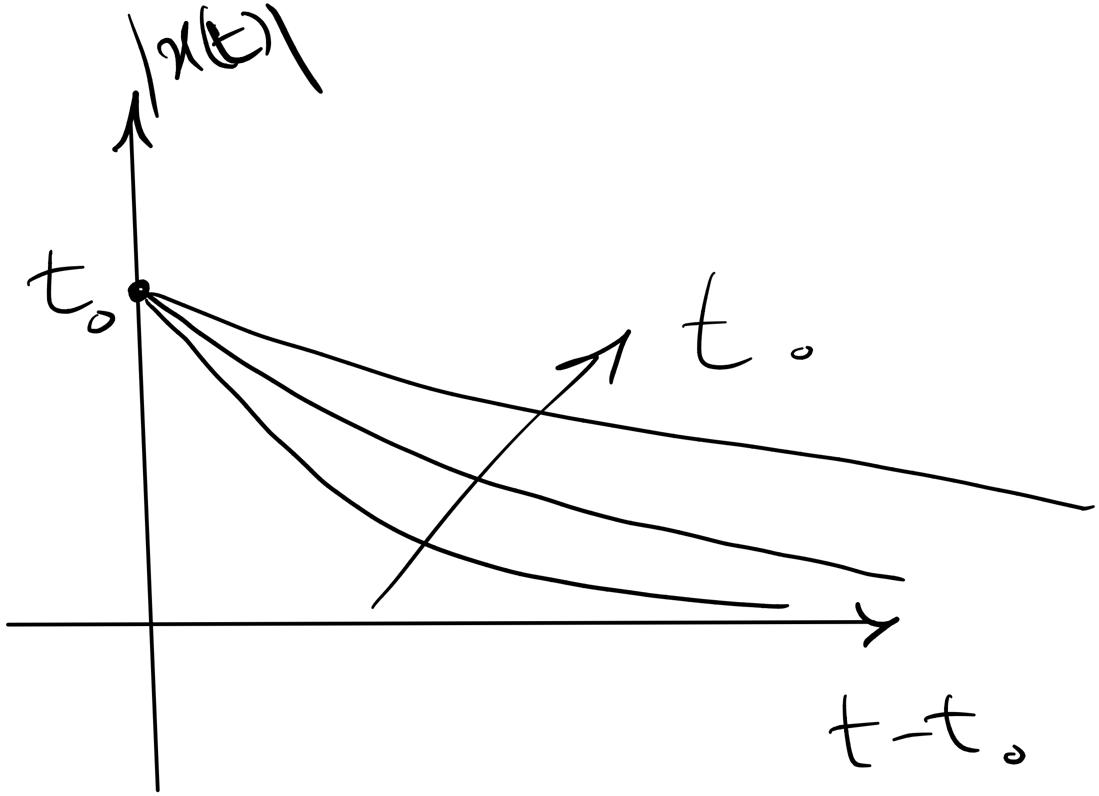
Since $t \geq t_0 \Rightarrow \frac{1+t_0}{1+t}$ decreasing

$\|x(t)\| \leq \|x_0\| \;\; \forall t \quad \bar{x} = 0$ clearly stable

$\delta = \varepsilon$ works here

and $\lim_{t\to\infty} \|x(t)\| = 0$ GAS

uniform stability



$$x(t) = x(t_0)\,\frac{1}{1+\frac{t-t_0}{1+t_0}}$$

Absence of a uniform rate of convergence.

Not uniformly asymptotically stable.

Note For uniform asymptotic stability the rate of convergence to zero needs to be independent of t_0 as well.

In general, it is more convenient to define stability of time-varying systems using comparison functions.

The origin of system (*) is:

a) uniformly stable if there is a class K function $\alpha(\cdot)$ and a constant $c>0$ st.

$$\|x(t)\| \leq \alpha(\|x_0\|) \text{ for all } t>t_0 \text{ and all}$$

$x(t_0)$ st. $\|x(t_0)\| < c$

b) uniformly asymptotically stable if there is a class KL function $\beta(\cdot,\cdot)$ and a constant $c>0$ st.

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0) \text{ for all } t>t_0 \text{ and for all}$$

$x(t_0)$ st. $\|x(t_0)\| < c$.

Note for global result β has to be K_∞ and c can be any constant.

c) uniformly exponentially stable if (b) holds

for $\beta(\|x_0\|, t-t_0) = K\|x_0\| e^{-a(t-t_0)}$

$a, K > 0$

Note! UES (unif. exponential stability) is the strongest notion of stability we can have.

* Lyapunov functions for time-varying systems $\dot{x} = f(x,t)$ are of the form $V(x,t)$.

$$\dot{V}(x,t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t)$$

at any given time

<u>Fact</u>, If $V(x)$ is positive definite, then we can find class K functions α_1 and α_2 st.

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

In addition, if $V(x)$ is radially unbounded then we can choose $\alpha_1(\cdot)$ to be K_∞.

Ex. $V(x) = x^T P x$ if $P = P^T > 0$

$$\underbrace{\lambda_{min}(P)\|x\|^2}_{\alpha_1} \leq V(x) \leq \underbrace{\lambda_{max}(P)\|x\|^2}_{\alpha_1}$$

$$\alpha_1(r) = \lambda_{min}(P)\, r^2$$

$$\alpha_2(r) = \lambda_{max}(P)\, r^2$$

Note! In the linear case: $\dot{x}(t) = A(t)\,x(t)$ LTV system

Q. How can we check stability of LTV system?
Are e-values of $A(t)$ enough?

~~stability $\iff$ $\text{Re}\{\lambda_i(A(t))\} < 0$ for all i and all $t \geq t_0$~~

$$\dot{x} = A(t)x \implies x(t) = \underbrace{\Phi(t,t_0)}_{\text{state transition matrix}} x(t_0)$$

aside

$$\frac{\partial \Phi(t,t_0)}{dt} = A(t)\,\Phi(t,t_0)$$

$$\Phi(t_0,t_0) = I \quad \text{initial condition}$$

for LTI

$$\Phi(t,t_0) = e^{A(t-t_0)}$$

$$\|x(t)\| \leqslant \|\Phi(t,t_0)\| \, \|x(t_0)\| \leqslant \underbrace{K \|x_0\| e^{-a(t-t_0)}}_{\text{if such a bound is found}}$$

we can conclude

$$\|\Phi(t,t_0)\| \leqslant K e^{-a(t-t_0)}$$

$$\forall t \geqslant t_0$$

$\Rightarrow$ U.E.S.

Q. How can we guarantee ex.p. bound on the norm of $\Phi(t,t_0)$ for all t???