MECH 6313 - Term Exam

Name: Jonas Wagner
UTD ID: 2021531784

2021, April 30

Contents

1 Problem 1

Consider the system:

$$
\begin{aligned}
\tau \dot{x} &= x - \frac{1}{3}x^3 - y \\
\dot{y} &= x + \mu
\end{aligned}
\tag{1.1}
$$

where $\tau > 0$ and $\mu \geq 0$ are constants.

1.1 Part a

Problem: Determine the equilibrium points and classify their stability properties depending on the values of parameter μ.

Solution:

1.1.1 Equilibrium Point Identification

The equilibrium points exist whenever $\dot{x} = \dot{y} = 0$ and can be identified as follows:

$$
\begin{aligned}
\tau(0) &= x - \frac{1}{3}x^3 - y \\
(0) &= x + \mu
\end{aligned}
\tag{1.2}
$$

which becomes:

$$
\begin{aligned}
y &= x - \frac{1}{3}x^3 \\
x &= -\mu
\end{aligned}
\tag{1.3}
$$

and can then substituted in as:

$$
\begin{aligned}
x_{eq} &= -\mu \\
y_{eq} &= -\mu - \frac{1}{3}(-\mu)^3
\end{aligned}
\tag{1.4}
$$

This results in the equilibrium points being defined in terms of μ as:

$$
\boxed{
\begin{aligned}
x_{eq} &= -\mu \\
y_{eq} &= \frac{1}{3}\mu^3 - \mu
\end{aligned}
}
\tag{1.5}
$$

1.1.2 System Linearization

The stability around an equilibrium point can be evaluated by looking at the linearized model, which can be found as follows:

Let the state-variables be defined as:

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

The nonlinear state equation would then be defined as:

$$\dot{X} = f(x) = \begin{bmatrix} \dfrac{x_1 - \frac{1}{3}x_1^3 - x_2}{\tau} \\ x_1 + \mu \end{bmatrix} \tag{1.6}$$

Then the equilibrium point is described as

$$X_{eq} = \begin{bmatrix} -\mu \\ \frac{1}{3}\mu^3 - \mu \end{bmatrix}$$

and the jacobian can be computed as:

$$J_x = \frac{\mathrm{d}f}{\mathrm{d}X} = \begin{bmatrix} \frac{\mathrm{d}f_1}{\mathrm{d}x_1} & \frac{\mathrm{d}f_1}{\mathrm{d}x_2} \\ \frac{\mathrm{d}f_2}{\mathrm{d}x_1} & \frac{\mathrm{d}f_2}{\mathrm{d}x_2} \end{bmatrix} \tag{1.7}$$

$$= \begin{bmatrix} \dfrac{1 - x_1^2}{\tau} & -1 \\ 1 & 0 \end{bmatrix} \tag{1.8}$$

The state dynamics around the equilibrium are described by this Jacobian evaluated at $X = X_{eq}$:

$$A = J_x\Bigg|_{X=X_{eq}} = \begin{bmatrix} \dfrac{1 - x_1^2}{\tau} & -1 \\ 1 & 0 \end{bmatrix}\Bigg|_{x_1=-\mu, x_2=\frac{1}{3}\mu^3-\mu} \tag{1.9}$$

$$= \begin{bmatrix} \dfrac{1 - (-\mu)^2}{\tau} & -1 \\ 1 & 0 \end{bmatrix} \tag{1.10}$$

$$= \begin{bmatrix} \frac{1}{\tau} - \frac{\mu^2}{\tau} & -1 \\ 1 & 0 \end{bmatrix} \tag{1.11}$$

The dynamics of this linearized system are described by the characteristic polynomial calculated as:

$$\Delta(s) = \det(sI - A) = \det \begin{bmatrix} s - \left(\frac{1}{\tau} - \frac{\mu^2}{\tau}\right) & 1 \\ -1 & s \end{bmatrix} \tag{1.12}$$

$$= s\left(s - \left(\frac{1}{\tau} - \frac{\mu^2}{\tau}\right)\right) - (1)(-1) \tag{1.13}$$

$$\boxed{\Delta(s) = s^2 - \left(\frac{1}{\tau} - \frac{\mu^2}{\tau}\right)s + 1} \tag{1.14}$$

1.1.3 Linearized Model Stability

The roots of $\Delta(s)$ are the eigenvalues of the linearization and are dependent on μ and τ calculated as:

$$\Lambda(A) = \lambda_{1,2} = \frac{\left(\frac{1}{\tau} - \frac{\mu^2}{\tau}\right) \pm \sqrt{\left(\frac{1}{\tau} - \frac{\mu^2}{\tau}\right)^2 - 4(1)(1)}}{2(1)} \tag{1.15}$$

$$= \frac{1}{2}\left(\frac{1}{\tau} - \frac{\mu^2}{\tau}\right) \pm \frac{1}{2}\sqrt{\left(\frac{1}{\tau} - \frac{\mu^2}{\tau}\right)^2 - 4} \tag{1.16}$$

or in a factored form:

$$= \frac{1}{2\tau}\left(1-\mu^2\right) \pm \frac{1}{2\tau}\sqrt{\left(1-\mu^2\right)^2 - 4\tau^2} \tag{1.17}$$

$$= \frac{1}{2\tau}\left(\left(1-\mu^2\right) \pm \sqrt{\mu^4 - 2\mu^2 + 1 - 4\tau^2}\right) \tag{1.18}$$

or in a fully factored form:

$$= \frac{1-\mu^2}{2\tau}\left(1 \pm \sqrt{1 - \frac{4\tau^2}{\left(1-\mu^2\right)^2}}\right) \tag{1.19}$$

or in condenced form:

$$= \left(\frac{1}{2\tau} - \frac{\mu^2}{2\tau}\right) \pm \sqrt{\left(\frac{1}{2\tau} - \frac{\mu^2}{2\tau}\right)^2 - 1} \tag{1.20}$$

The roots are entirely **real** when:

$$\left(1-\mu^2\right)^2 - 4\tau^2 > 0 \tag{1.21}$$

$$\left(1-\mu^2\right)^2 > 4\tau^2 \tag{1.22}$$

$$1-\mu^2 > 2\tau \tag{1.23}$$

$$\mu^2 + 2\tau > 1 \tag{1.24}$$

in which case, the linearized system is **stable** only when:

$$0 > \Re(\lambda_1) = \frac{1}{2\tau}\left(\left(1-\mu^2\right) + \sqrt{\left(1-\mu^2\right)^2 - 4\tau^2}\right) \tag{1.25}$$

$$\boxed{\Re(\lambda_1) = 1 - \mu^2 + \sqrt{\left(1-\mu^2\right)^2 - 4\tau^2} < 0} \tag{1.26}$$

The system has **complex roots** when :

$$\left(1-\mu^2\right)^2 - 4\tau^2 < 0 \tag{1.27}$$

$$\left(1-\mu^2\right)^2 < 4\tau^2 \tag{1.28}$$

$$1-\mu^2 < 2\tau \tag{1.29}$$

$$\mu^2 + 2\tau < 1 \tag{1.30}$$

in which case, the linearized system is only **stable** when

$$0 > \Re(\lambda_{1,2}) = \frac{1}{2\tau}\left(1-\mu^2\right) \tag{1.31}$$

$$= 1 - \mu^2 \tag{1.32}$$

$$\boxed{\mu^2 > 1} \tag{1.33}$$

1.2 Part b

Problem: At which value of μ does a bifurcation occur and what type of bifurcation is it?

Solution: The stability properties of the linearized system indicate that a hopf bifurcation occur when

$$\mu = 1$$

and the system condenses the system into a single equalibrium point at the (parameter dependent) equalibrium point of

$$X_{eq} = \begin{bmatrix} -\mu \\ \frac{1}{3}\mu^3 - \mu \end{bmatrix}$$

.

1.3 Part c

Problem: Assume $\tau << 1$ and sketch the phase portrait for two values of μ, one just below and one just above the bifurcation value.

Solution:

2 Problem 2:

Consider the system:

$$\begin{aligned}
\dot{x}_1 &= -\frac{1}{2}\tan\!\left(\frac{\pi x_1}{2}\right) + x_2 \\
\dot{x}_2 &= x_1 - \frac{1}{2}\tan\!\left(\frac{\pi x_2}{2}\right)
\end{aligned} \tag{2.1}$$

A MATLAB Code:

All code I write in this course can be found on my GitHub repository:
https://github.com/jonaswagner2826/MECH6313