Lecture 2

01/25/2021

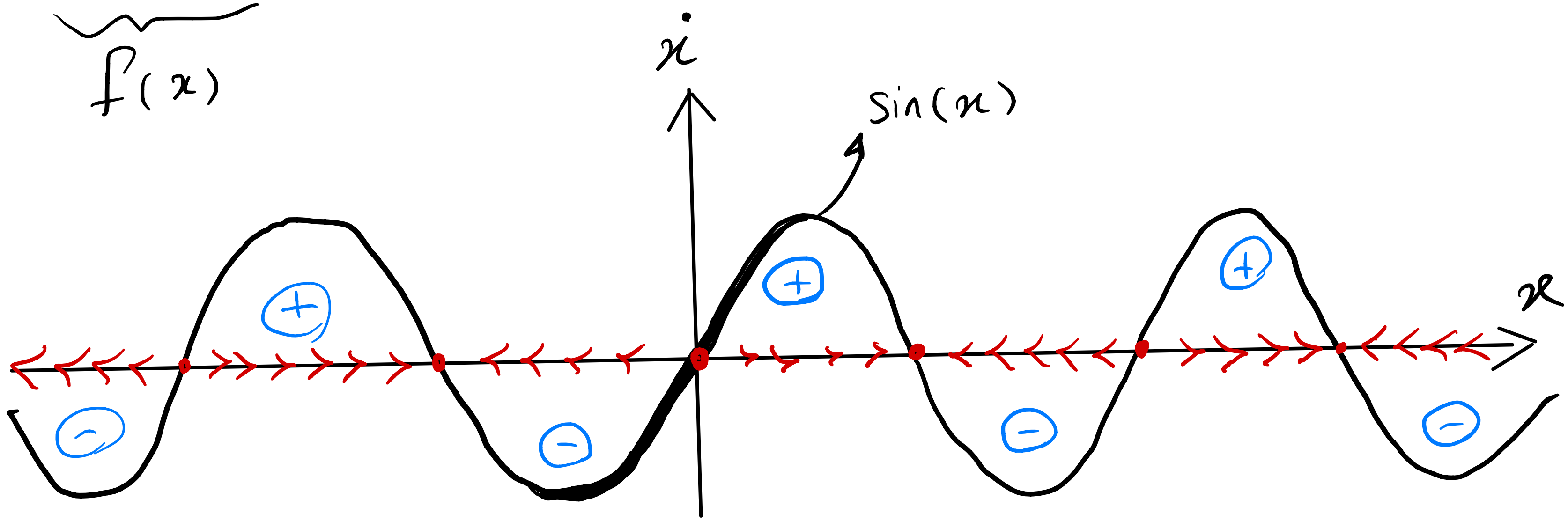
Ex

$$\dot{x} = f(x) = \sin(x)$$

Eq. pts. $\bar{x}$: $f(\bar{x}) = 0 \Rightarrow \bar{x} = K\pi = \begin{cases} x_{eq_1} & K \text{ even} \\ x_{eq_2} & K \text{ odd} \end{cases}$

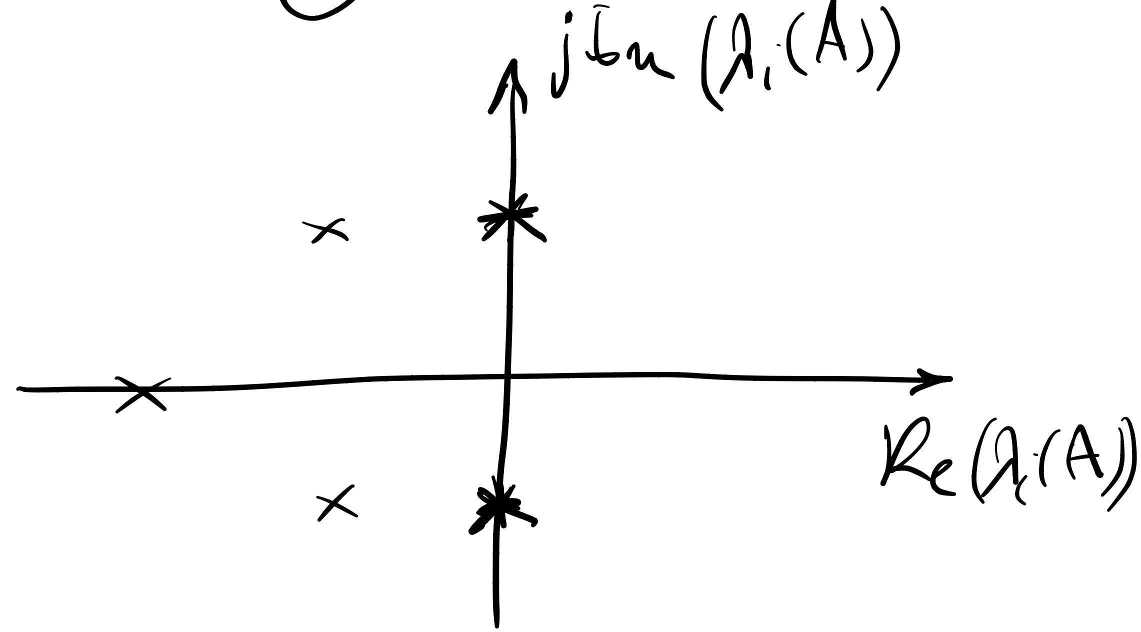
$$A = \left.\frac{\partial f}{\partial x}\right|_{x = x_{eq}} = \left.\cos(x)\right|_{x = x_{eq}} = \begin{cases} 1 & K \text{ even} \\ -1 & K \text{ odd} \end{cases}$$

$$\dot{x} = \underbrace{\sin(x)}_{f(x)}$$



Issues: 1) Only provides local stability results $\rightarrow$ can't say anything about global properties of the system

2) If $Re(\lambda_i(A)) \leq 0$ with some e-values having $Re(\lambda_i(A)) = 0$



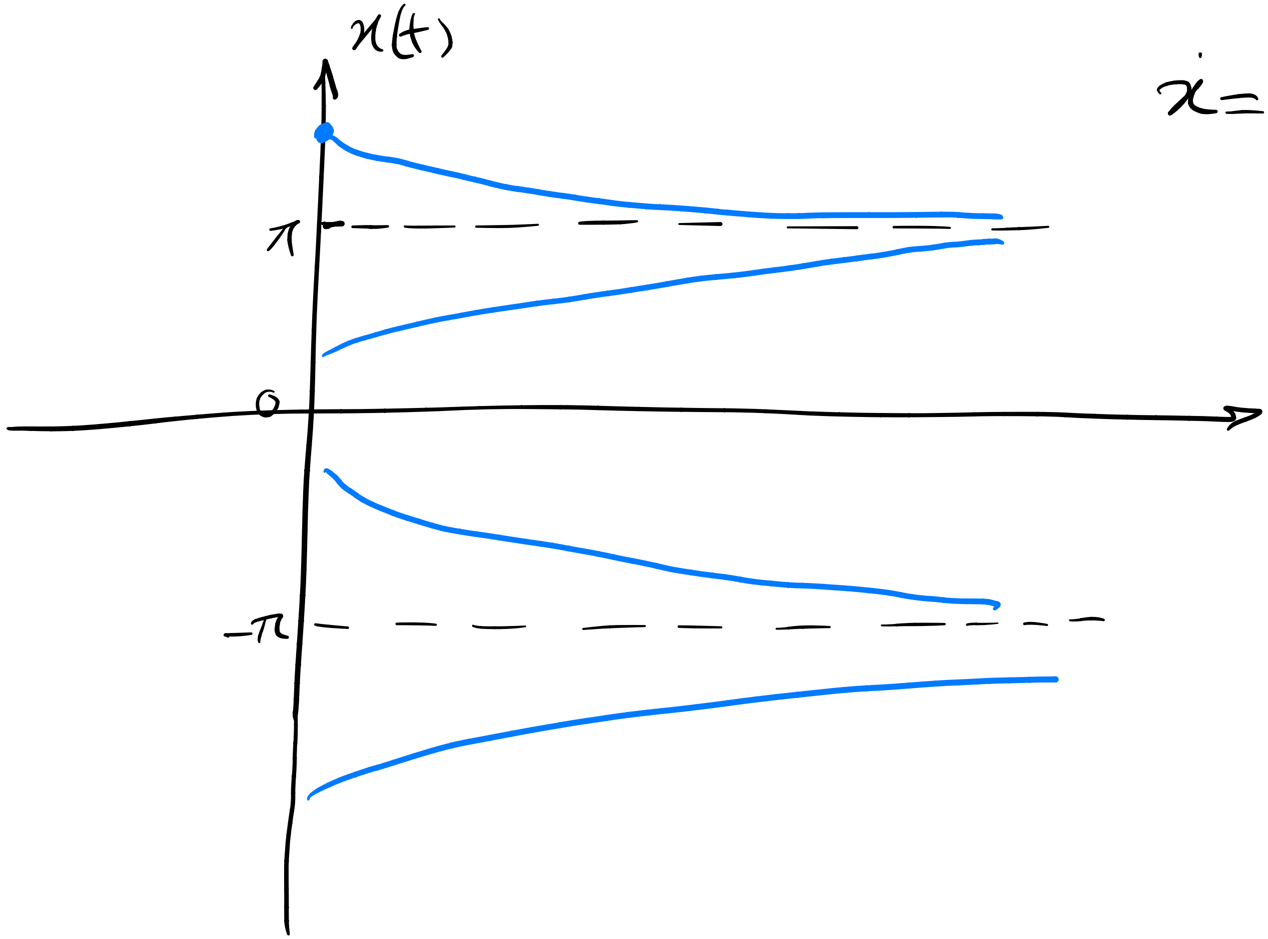
then linearization does NOT provide info about stability (and we need to examine <u>H.O.T.</u>)

$$f(x) = \underbrace{f(x_{eq}) + \frac{\partial f}{\partial x}\bigg|_{x=x_{eq}} x}_{} + \underbrace{H.O.T.}_{O(x^2)}$$

$$x(t) = x_0 e^{At}$$

$$\exists\, i\ ;\ \lambda_i(A) = \lambda_{i,imag}\, j$$

$$\Bigg\}\ x(t) = x_0 e^{\lambda_{i,imag} j t}$$



$\dot{x} = \sin(x)$

Ex

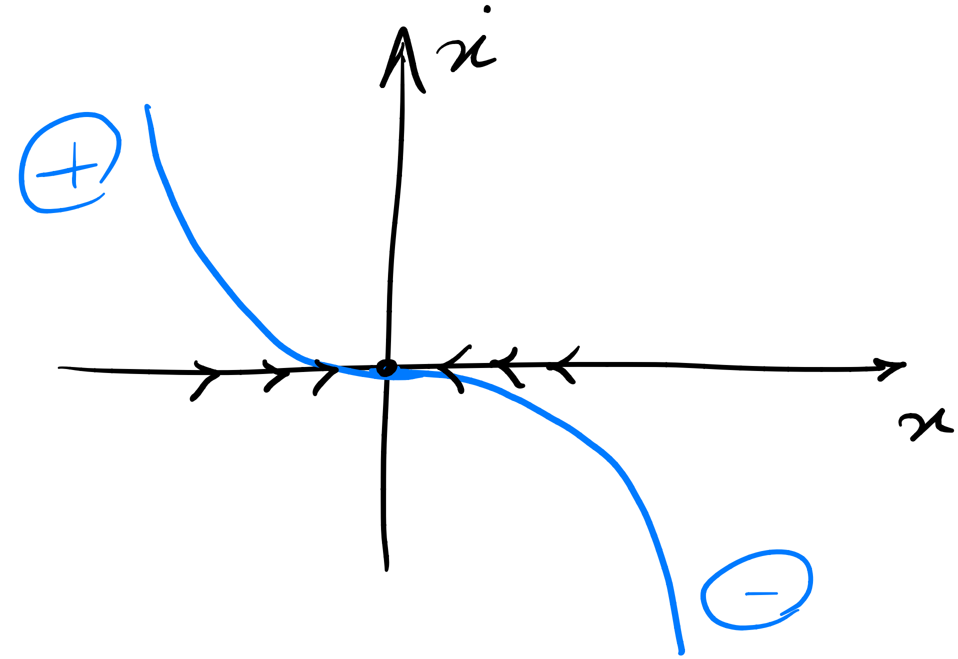
a) $\dot{x} = -x^3$

b) $\dot{x} = +x^3$

$$\left.\begin{array}{l} \dot{x} = -x^3 \\ \dot{x} = +x^3 \end{array}\right\} \quad x_{eq} = 0 \xRightarrow{\text{linearization}} A = 0$$

$$\dot{\tilde{x}} = 0\,\tilde{x}$$

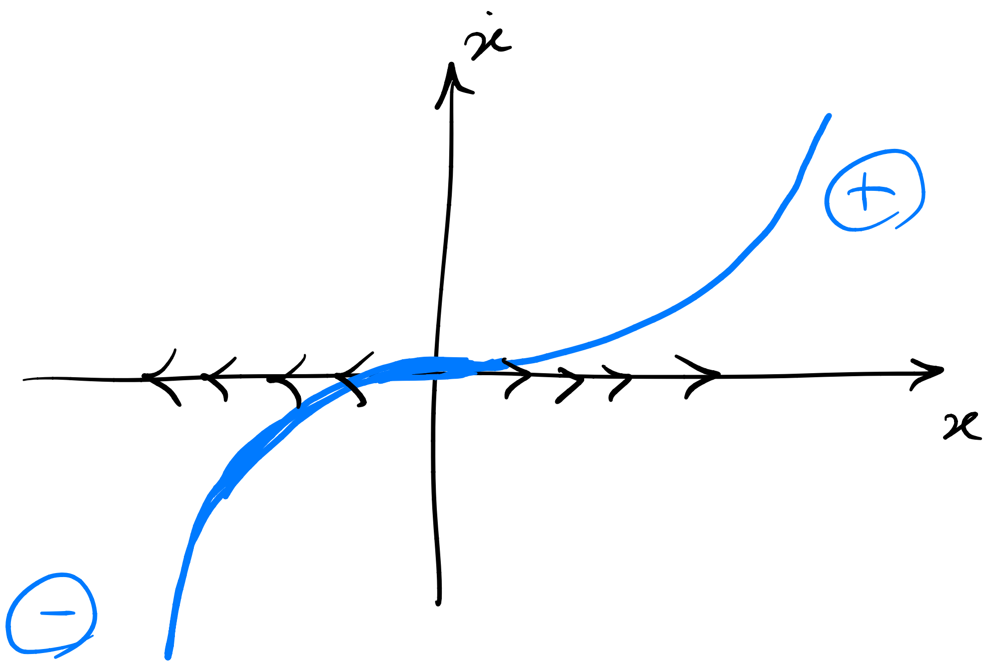
a)



$\bar{x} = 0$

globally asymptotically stable

b)



$$\dot{x} = x^3$$

$$\downarrow$$

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

"Essentially nonlinear phenomena"

1. Finite escape time

$\left(x(t) \to \infty \quad \text{even for } t < +\infty, \text{ finite } t \right)$

Can't happen in linear case: $\underbrace{x(t) = e^{At} x_0}_{t \to \infty \iff \|x(t)\| \to \infty}$

Ex. $\dot{x} = x^2 \qquad x(t) \in \mathbb{R}$

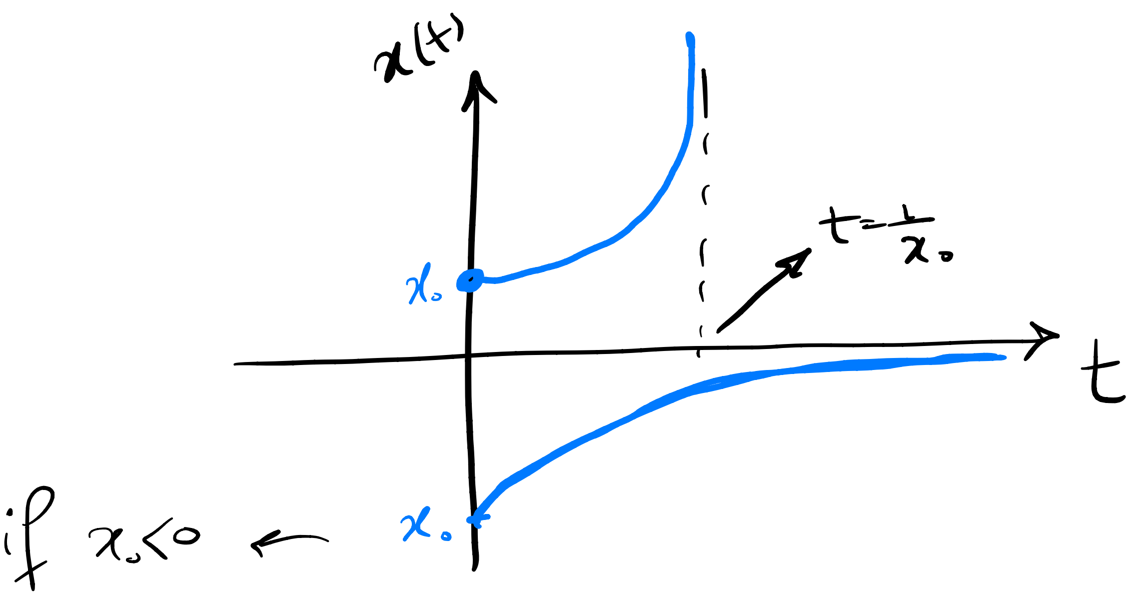
$\downarrow$

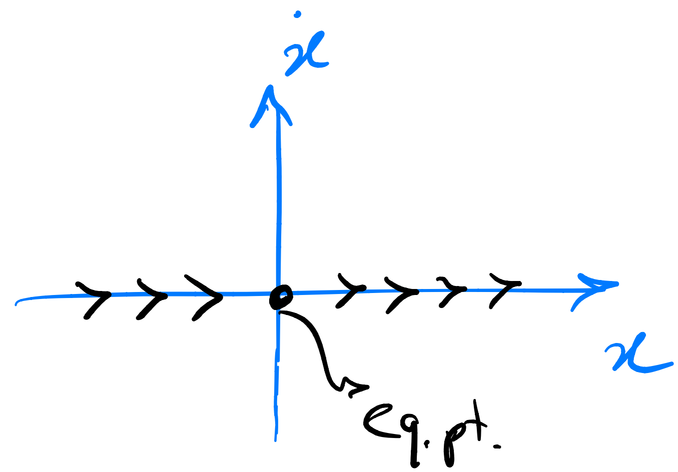
$$\frac{dx}{dt} = x^2 \implies \int_{x_0}^{x(t)} \frac{dx}{x^2} = \int_0^t dt$$

$$\Rightarrow \quad -\frac{1}{x(t)} + \frac{1}{x_0} = t - 0$$

$$\Rightarrow \quad x(t) = \frac{1}{\frac{1}{x_0} - t}$$

if $x_0 > 0 \Rightarrow t \to \frac{1}{x_0} \Rightarrow x(t) \longrightarrow \infty$



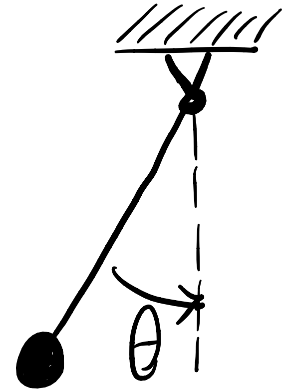


2. Multiple isolated equilibria

Ex.

1) $\dot{x} = \sin(x)$

2) pendulum



$$m l \ddot{\theta} + K l \theta + m g \sin(\theta) = 0$$

$$\ddot{\theta} = -\frac{g}{l} \sin(\theta) - \frac{K}{m} \dot{\theta}$$

$$\left.\begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} \end{aligned}\right\} \Rightarrow \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} x_2 \\ -\frac{g}{l} \sin(x_1) - \frac{K}{m} x_2 \end{bmatrix}}_{f(x)} \quad ; \; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

e.p.? $\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix} = \begin{bmatrix} k\pi \\ 0\end{bmatrix}$; $k = 0, \pm 1, \pm 2, \dots$

$$A = \frac{\partial f}{\partial x}\bigg|_{\bar{x}} = \begin{bmatrix} 0 & 1 \\ -g/l \cos(\bar{x}_1) & -K/m \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 & 1 \\ -g/l & -K/m \end{bmatrix} & \text{stable} \\ \begin{bmatrix} 0 & 1 \\ +g/l & -K/m \end{bmatrix} & \text{unstable} \end{cases}$$

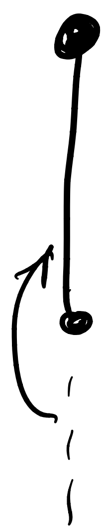
k even

down

K odd

up

$\theta = \pi$



Ex. logistics equations

$$\dot{x} = \alpha \left(1 - x/k\right) x \quad ; \quad x(t) \in \mathbb{R} \quad \left(\begin{array}{c}\text{1st-order} \\ \text{system}\end{array}\right)$$

Models population growth

$$\left(\underbrace{\alpha, k}_{\text{constants}} > 0\right)$$

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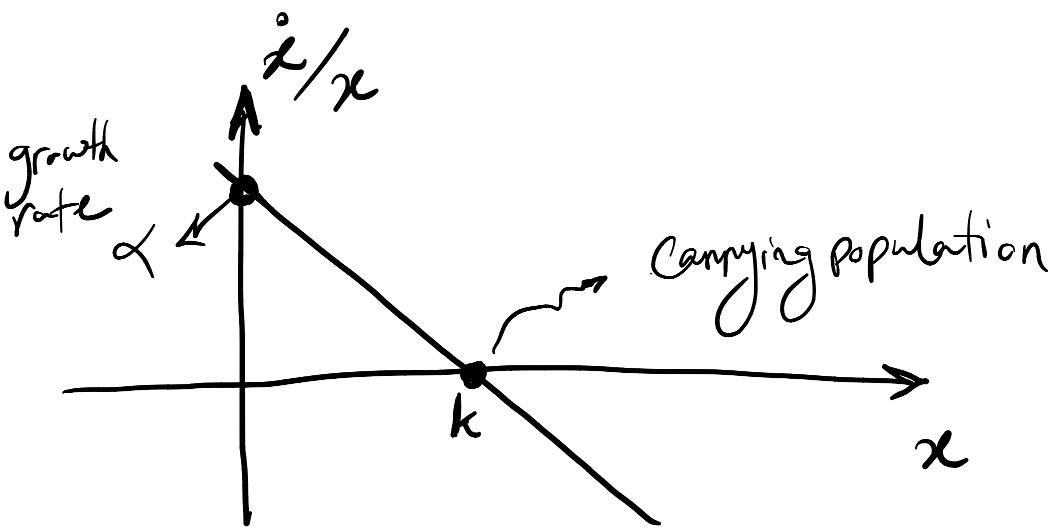
aside

simpler model: $\dot{x} = \alpha x$ $\quad$ α growth rate $(\alpha > 0)$

$$x(t) = e^{\alpha t} x(0) \qquad \left(\begin{array}{c}\text{rate of pop. growth} \\ \text{constant per capita}\end{array}\right)$$

Issue: population can grow unbounded!

* logistics eq'n provides a fix by assuming $\frac{\dot{x}}{x}$ decays linearly w.r.t. x (more sophisticated model)



e.p.: $\dot{\bar{x}} = 0$

$$f(\bar{x}) = \alpha\left(1 - \frac{\bar{x}}{k}\right)\bar{x} = 0$$

$\bar{x} = 0$ no population

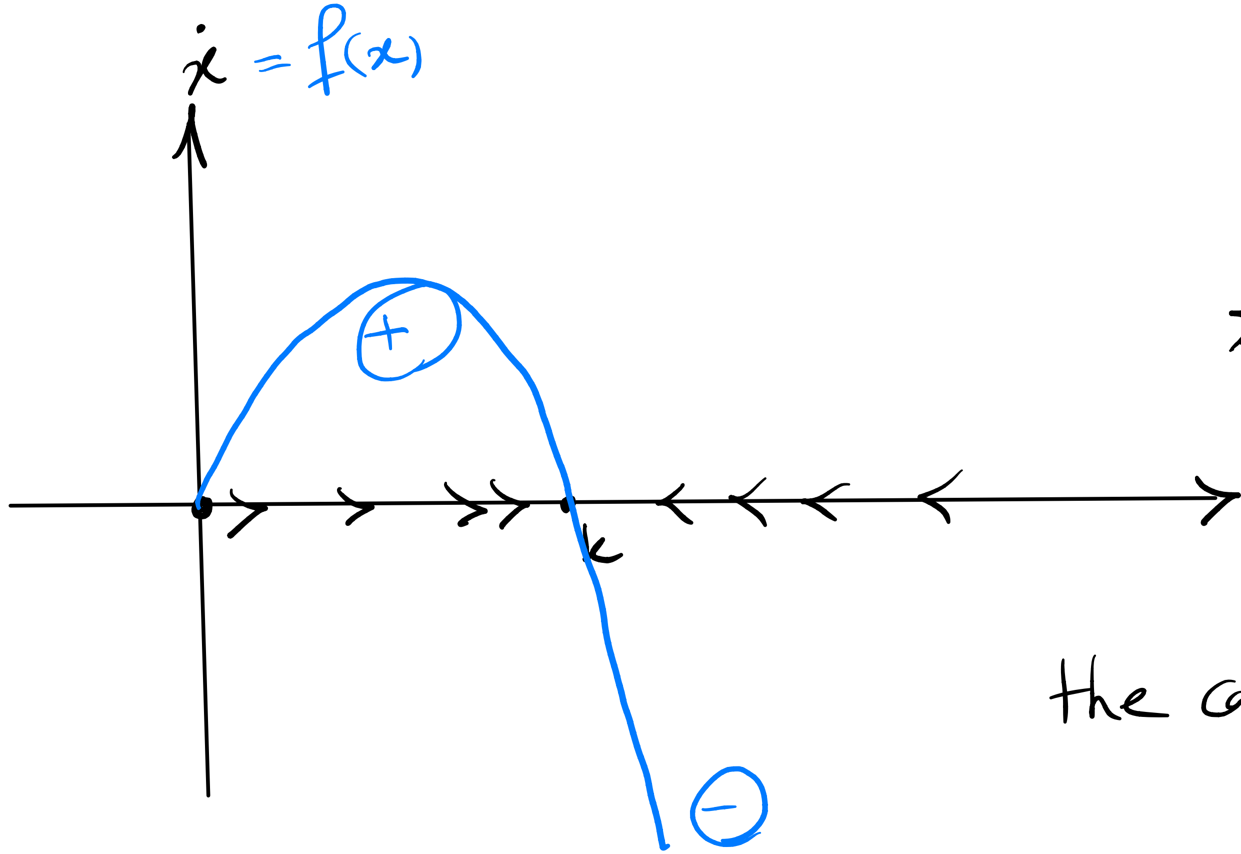
$\bar{x} = k > 0$ carrying capacity

linearization

$$\left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} = \left.\left(\alpha - \frac{2\alpha\bar{x}}{k}\right)\right|_{\bar{x}}$$

$\bar{x}=0$: $A=\alpha$ unstable

$\bar{x}=k$: $A=-\alpha$ locally asymp. stable

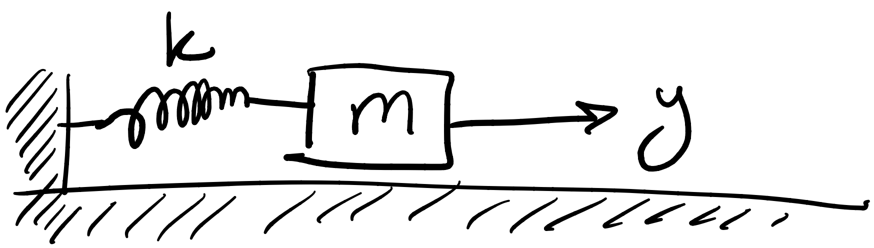


★ no matter how big the population is, it will settle at the carrying capacity.

However, it will stay at $\emptyset$ if it is at $\emptyset$ initially (not self evolving)

3) Limit Cycles

Ex.

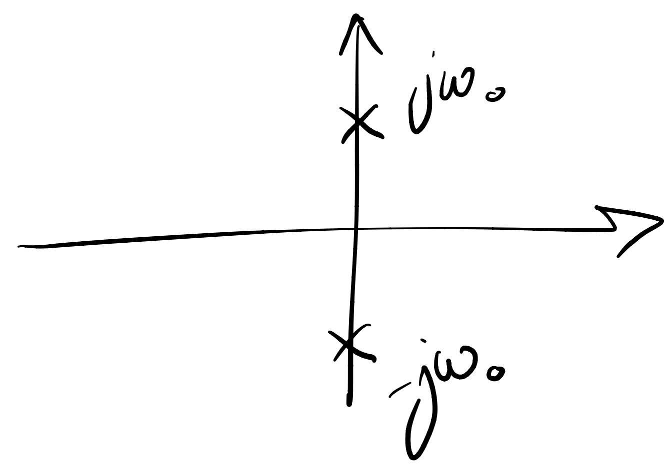


$$m\ddot{y} + ky = 0$$

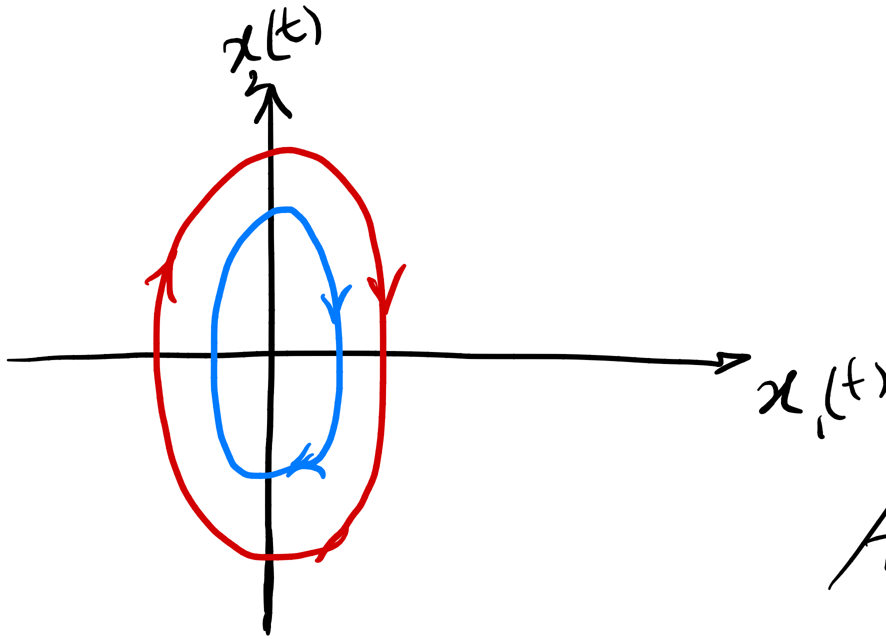
(Harmonic oscillator)

(LC circuit in EE)

$$x_1 = y, \quad x_2 = \dot{y} \;\rightarrow\; \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



$$\omega_0 = \sqrt{k/m}$$



$$x(t) = e^{At} x_0$$

Amplitude of oscillations depends on initial conditions.

★ not very robust: small amount of damping would decrease oscillation amplitude and bring them to zero as $t \to \infty$.

Moral (structurally) robust oscillations are impossible to achieve in unforced LTI systems (you need nonlinearity)