MECH 6313 - Homework 6

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Contents

1 Problem 1

Problem: Show that the parallel connection of two passive dynamical systems is passive. Can you claim the same for the series connection of two passive systems?

Solution: Let two passive systems be defined as a system taking an input u and generating an output y as

$$H_1 : y_1 = h_1(u), \text{ s.t. } \langle y_1 | u \rangle \geq 0$$

and

$$H_2 : y_2 = h_2(u), \text{ s.t. } \langle y_2 | u \rangle \geq 0$$

with $\langle y|u \rangle = \int_0^T y^T(t) u(t) \, \mathrm{d}t$

1.1 Parallel Connection of Passive System

The parallel system H_p can then be defined by

$$H_p : h_p(u) = y_p = y_1 + y_2 = h_1(u) + h_2(u)$$

whose passivity can be proven directly by testing $\langle y_p | u \rangle$ which is calculated as

$$\begin{aligned} \langle y_p | u \rangle &= \int_0^T y_p^T u \, \mathrm{d}t && (1) \\ &= \int_0^T (y_1 + y_2)^T u \, \mathrm{d}t && (2) \\ &= \int_0^T y_1^T u + y_2^T u \, \mathrm{d}t && (3) \\ &= \int_0^T y_1^T u \, \mathrm{d}t + \int_0^T y_2^T u \, \mathrm{d}t && (4) \\ &= \langle y_1 | u \rangle + \langle y_2 | u \rangle && (5) \end{aligned}$$

Since $\langle y_1 | u \rangle \geq 0$ and $\langle y_2 | u \rangle \geq 0$,

$$\langle y_p | u \rangle \geq 0 \qquad (6)$$

which proves, by definition, that H_p is passive.

1.2 Series Connection of Passive System

The series system H_s can be defined by

$$H_s : h_s(u) = y_s = h_1(u) \circledast h_2(u) = h_2(h_1(u))$$

whose passivity can be tested using $\langle y_s|u\rangle$ which is calculated as:

$$\begin{aligned}
\langle y_s|u\rangle &= \int_0^T y_s^T u \,\mathrm{d}t && (7)\\
&= \int_0^T (h_1(u) \circledast h_2(u))^T u \,\mathrm{d}t && (8)\\
&= \int_0^T \left(\int_0^T h_1(t-\tau) h_2(\tau) \,\mathrm{d}\tau \right) \mathrm{d}t && (9)\\
&= \int_0^T h_2(\tau) \left(\int_0^T h_1(t-\tau) \,\mathrm{d}t \right) \mathrm{d}\tau && (10)
\end{aligned}$$

which is not explicitly ≥ 0 so this method cannot prove passivity.

A different method of analysis can be done to prove that this is not passive in general, but a counter example from MATLAB (Appendix A) can be shown to not be passive due to a loss of positive realness of the transfer functions when placed in series:

$$G_1(s) = \frac{5s^2 + 3s + 1}{s^2 + 2s + 1},\ G_2(s) = \frac{s^3 + s^2 + 5s + 0.1}{s^3 + 2s^2 + 3s + 4}$$

and when combined in series the system is no longer passive due to a loss of positive realness.

$$\frac{5s^5 + 8s^4 + 29s^3 + 16.5s^2 + 5.3s + 0.1}{s^5 + 4s^4 + 8s^3 + 12s^2 + 11s + 4}$$

2 Problem 2

Let

$$H(s) = \frac{s+\lambda}{s^2+as+b}$$

with $a > 0$ and $b > 0$.

2.1 Part a

Problem: For which values of λ is $H(s)$ Positive Real (PR)?

Solution: By definition, a transfer function must satisfy two conditions to be considered Postive Real:

1. $\Re\{\lambda(H(s))\} \leq 0$, any $j\omega$ roots are simple, and any residuals are non negative.
2. $\Re\{H(j\omega)\} \geq 0 \ \forall \omega \in \Re$

The transfer function for this problem will always satisfy the first condition, however, the second condition is violated under the following conditions:

Setting

$$s = j\omega$$

$$\begin{aligned}
H(j\omega) &= \frac{j\omega+\lambda}{(j\omega)^2+a(j\omega)+b} = \frac{j\omega+\lambda}{-\omega^2+ja\omega+b} && (11)\\
&= \frac{-\omega^2-ja\omega+b}{-\omega^2-ja\omega+b}\cdot\frac{j\omega+\lambda}{-\omega^2+ja\omega+b} && (12)\\
&= \frac{\left(a\omega^2+\lambda(\omega^2+b)\right)+j\left(\omega(\omega^2+b)-a\lambda\omega\right)}{a^2\omega^2+(\omega^2+b)^2} && (13)\\
&= \frac{a\omega^2+\lambda(\omega^2+b)}{a^2\omega^2+(\omega^2+b)^2}+j\frac{\omega(\omega^2+b)-a\lambda\omega}{a^2\omega^2+(\omega^2+b)^2} && (14)
\end{aligned}$$

The real component being nonnegative can then be seen to occur when

$$\begin{aligned}
a\omega^2+\lambda(\omega^2+b) &\geq 0 && (15)\\
\lambda(\omega^2+b) &\geq -a\omega^2 && (16)\\
\lambda &\geq \frac{-a\omega^2}{\omega^2+b} && (17)
\end{aligned}$$

Since this mus apply $\forall\omega \in \Re$, the following must be true

$$\lambda \geq 0 \tag{18}$$

2.2 Part b

Problem: Using the results from above, select λ_1, λ_2 such that

$$H_1(s) = \frac{s+\lambda_1}{s^2+s+1} \text{ is PR} \tag{19}$$

$$H_2(s) = \frac{s+\lambda_2}{s^2+s+1} \text{ is not PR} \tag{20}$$

Then verify the PR properties for each using the Nyquist plots of $H_1(s)$ and $H_2(s)$.

Solution: From the requirements set above, the zeros can be selected with $\lambda_1 = 1$ and $\lambda_2 = -1$ resulting in $H_1(s)$ and $H_2(s)$ being defined as

$$H_1(s) = \frac{s+1}{s^2+s+1} \tag{21}$$

$$H_2(s) = \frac{s-1}{s^2+s+1} \tag{22}$$

The nyquist plots, generated with the MATLAB code seen in Appendix A, can then be used to verify the PR properties. As can be seen in Figure 1, the nyquist diagram for $H_1(s)$ never crosses into the LHP and therefore is Positive Real. Conversely, in Figure 2, the nyquist diagram for $H_2(s)$ crosses into the LHP and therefore is not Positive Real.

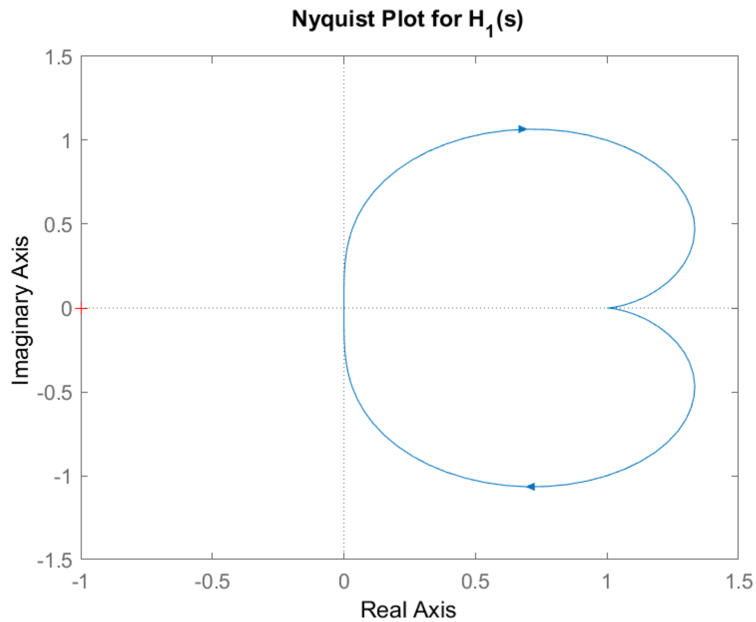


Figure 1: Nyquist Plot for the $H_1(s)$ transfer function.

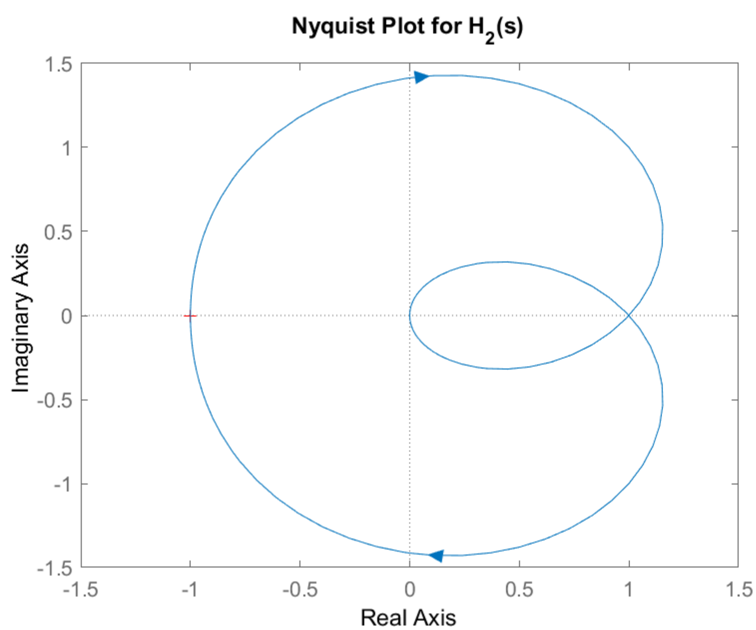


Figure 2: Nyquist Plot for the $H_2(s)$ transfer function.

2.3 Part c

Problem: For $H_1(s)$ and $H_2(s)$, write state-space realizations and solve for $P = P^T > 0$ in the PR lemma and explain why it fails for $H_2(s)$.

Solution: Given a second order transfer function

$$\frac{s + b_0}{s^2 + a_1 s + a_0}$$

a state space system can be defined by:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} & B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} b_0 & 1 \end{bmatrix} & D &= 0 \end{aligned} \tag{23}$$

This can be applied to the systems and results in the state space representations of H_1 as:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} & B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 1 \end{bmatrix} & D &= 0 \end{aligned} \tag{24}$$

and H_2 as

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} & B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C &= \begin{bmatrix} -1 & 1 \end{bmatrix} & D &= 0 \end{aligned} \tag{25}$$

Each of these systems can be tested for passivity by solving for a $P = P^T > 0$ s.t.

$$A^T P + PA < 0 \tag{26}$$

$$PB = C^T \tag{27}$$

Additionally, let

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$H_1(s)$ was found in (24)to be

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad D = 0$$

and can be proven to be passive by solving the following LMI:

$$A^T P + PA \leq 0 \tag{28}$$
$$PB - C^T = 0 \tag{29}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \leq 0 \tag{30}$$
$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \end{bmatrix}^T = 0 \tag{31}$$

$$\begin{bmatrix} -2p_{12} & p_{11} - p_{12} - p_{22} \\ p_{11} - p_{12} - p_{22} & 2p_{12} - 2p_{22} \end{bmatrix} \leq 0 \tag{32}$$
$$p_{12} - 1 = 0 \tag{33}$$
$$p_{22} - 1 = 0 \tag{34}$$

Two coefficients can then clearly be defined resulting in

$$p_{12} = 1 \tag{35}$$
$$p_{22} = 1 \tag{36}$$
$$\begin{bmatrix} -2(1) & p_{11} - (1) - (1) \\ p_{11} - (1) - (1) & 2(1) - 2(1) \end{bmatrix} \leq 0 \tag{37}$$

$$\begin{bmatrix} -2 & p_{11} - 2 \\ p_{11} - 2 & 0 \end{bmatrix} \leq 0 \tag{38}$$

Which is equivalent to the standard PSD equation:

$$\begin{bmatrix} 2 & 2 - p_{11} \\ 2 - p_{11} & 0 \end{bmatrix} \geq 0 \tag{39}$$

And to ensure this is PSD, the principle minors can be analyzed:

$$a = 2 > 0 \tag{40}$$
$$\det \begin{bmatrix} 2 & 2 - p_{11} \\ 2 - p_{11} & 0 \end{bmatrix} \geq 0 \tag{41}$$
$$-(2 - p_{11})(2 - p_{11}) \geq 0 \tag{42}$$
$$-4 + 2p_{11} - p_{11}^2 \geq 0 \tag{43}$$

it is clear that a solution of $p_{11} = 0$ results in a PSD matrix and thus a Positive Real and passive system.

$H_2(s)$ was found in (25) to be

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} -1 & 1 \end{bmatrix} \qquad D = 0$$

and can be proven to be passive by solving the following LMI

$$A^T P + PA \leq 0 \tag{44}$$
$$PB - C^T = 0 \tag{45}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \leq 0 \tag{46}$$
$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \end{bmatrix}^T = 0 \tag{47}$$

$$\begin{bmatrix} -2p_{12} & p_{11} - p_{12} - p_{22} \\ p_{11} - p_{12} - p_{22} & 2p_{12} - 2p_{22} \end{bmatrix} \leq 0 \tag{48}$$
$$p_{12} + 1 = 0 \tag{49}$$
$$p_{22} - 1 = 0 \tag{50}$$

Two coefficients can then clearly be defined resulting in

$$p_{12} = -1 \tag{51}$$
$$p_{22} = 1 \tag{52}$$
$$\begin{bmatrix} -2(-1) & p_{11} - (-1) - (1) \\ p_{11} - (-1) - (1) & 2(-1) - 2(1) \end{bmatrix} \leq 0 \tag{53}$$

$$\begin{bmatrix} 2 & p_{11} \\ p_{11} & -4 \end{bmatrix} \leq 0 \tag{54}$$

Which is equivalent to the standard PSD equation:

$$\begin{bmatrix} -2 & -p_{11} \\ -p_{11} & 4 \end{bmatrix} \geq 0 \tag{55}$$

And this is clearly not capable of being PSD due to the first principle minor being negative. Therefore, the system is not Positive Real and cannot be passive.

3 Problem 3

Consider the following 3-stage ring oscillator discussed in class:

$$\begin{aligned}
\tau_1 \dot{x}_1 &= -x_1 - \alpha_1 \tanh(\beta_1 x_3) \\
\tau_2 \dot{x}_2 &= -x_2 - \alpha_2 \tanh(\beta_2 x_1) \\
\tau_3 \dot{x}_3 &= -x_3 - \alpha_3 \tanh(\beta_3 x_2)
\end{aligned}$$

with $\tau_i, \alpha_i, \beta_i > 0$ and x_i represents a voltage for $i = 1, 2, 3$.

3.1 Part a

Problem: Suppose $\alpha_1\beta_1 = \alpha_2\beta_2 = \alpha_3\beta_3 = \mu$, prove the origin is GAS when $\mu < 2$.

Solution: The 3-stage ring oscillator can be rewritten in a coupling feedback system H and K defined as:

$$H = \begin{bmatrix} H_1 & & \\ & H_2 & \\ & & H_3 \end{bmatrix}, \qquad H_i = \begin{cases} \dot{x}_i = f(x_i) + g(x_i)u_i \\ y_i = h(x_i) \end{cases} \tag{56}$$

$$K = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \tag{57}$$

where K is the coupling matrix of the individual nonlinear subsystems with the following specific definitions:

$$H_i = \begin{cases} \dot{x}_i = \dfrac{-x_i + u_i}{\tau_i} \\ y_i = \alpha_j \tanh(\beta_j x_i) \end{cases} \tag{58}$$

with j being defined as either $i+1$ or 1 if $i = n$, resulting in

$$H_1 = \begin{cases} \dot{x}_1 = \dfrac{-x_1 + u_1}{\tau_1} \\ y_1 = \alpha_2 \tanh(\beta_2 x_1) \end{cases} \tag{59}$$

$$H_2 = \begin{cases} \dot{x}_2 = \dfrac{-x_2 + u_2}{\tau_2} \\ y_2 = \alpha_3 \tanh(\beta_3 x_2) \end{cases} \tag{60}$$

$$H_3 = \begin{cases} \dot{x}_3 = \dfrac{-x_3 + u_3}{\tau_3} \\ y_3 = \alpha_1 \tanh(\beta_1 x_3) \end{cases} \tag{61}$$

From this model a storage function can be defined for each of the coupled systems as

$$V_i(x_i) = \tau_i \int_0^{x_i} h_i(\eta)\,\mathrm{d}\eta \tag{62}$$

and each individual subsystem storage functions are given as:

$$V_1(x_1) = \tau_1 \int_0^{x_1} \alpha_2 \tanh(\beta_2 x_1) \tag{63}$$

$$V_2(x_2) = \tau_2 \int_0^{x_2} \alpha_3 \tanh(\beta_3 x_2) \tag{64}$$

$$V_3(x_3) = \tau_3 \int_0^{x_3} \alpha_1 \tanh(\beta_1 x_3) \tag{65}$$

Taking a look at the change of the storage function over time, the output strict passivity can be proven by:

$$\dot{V}_i(x_i) = \frac{\mathrm{d}V_i}{\mathrm{d}x_i}\dot{x}_i \tag{66}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x_i}\tau_i \int_0^{x_i} h_i(\eta)\,\mathrm{d}\eta\,\dot{x}_i \tag{67}$$

$$= \tau_i h_i(x_i)\dot{x}_i \tag{68}$$

taking the definition for $\dot{x}_i$ and relating $h_i(x_i) = y_i$,

$$= \tau_i h_i(x_i)\frac{-x_i + u_i}{\tau_i} \tag{69}$$

$$= -x_i h_i(x_i) + u_i y_i \tag{70}$$

In order to guarantee Input passivity, the system must satisfy the following inequality:

$$xh(x) \le \delta_i x^2 \tag{71}$$

$$x(\alpha \tanh(\beta x)) \le \delta x^2 \tag{72}$$

by definition, $\alpha, \beta > 0$ and $\delta \ge 0$, and thus the following inequalities apply:

$$\begin{cases} \alpha \tanh(\beta x) > 0, & x > 0 \\ \alpha \tanh(\beta x) < 0, & x < 0 \end{cases} \tag{73}$$

therefore, when

$$|\alpha \tanh(\beta x)| \le |\delta x|$$

the input passivity equality holds. This occurs whenever

$$\alpha\beta = \mu \le \delta$$

, so δ is the limiting factor that is restricted by the storage function calculation.

In the case that input passivity is satisfied by $h_i(x_i)$, a δ_i will exist s.t.,

$$x_i h_i(x_i) \le \delta_i x^2 \tag{74}$$

$$x_i(h_i(x_i) - \delta_i x_i) \le 0 \tag{75}$$

clearly, $x_i h_i(x_i)$ can then be bounded from below by:

$$x_i h_i(x) \ge \frac{1}{\delta_i} h_i^2(x_i) \tag{76}$$

$$-x_i h_i(x) \ge -\frac{1}{\delta_i} h_i^2(x_i) \tag{77}$$

since $y_i = h_i(x_i)$,

$$-x_i h_i(x) \ge -\frac{1}{\delta_i} y_i^2(x_i) \tag{78}$$

Therefore, this demonstrates Output Strict Passivity:

$$\dot{V}_i \le -\frac{1}{\delta_i} y_i^2 + y_i u_i \tag{79}$$

or with $d_i = \frac{1}{\delta_i}$ and

$$\dot{V}_i \le d_i y_i^2 + y_i u_i \tag{80}$$

and the passivity theorem can then be applied.

Let

$$\epsilon_i = \frac{1}{\delta_i} \le \frac{1}{\mu}$$

and then define

$$A = -\text{diag}\{\epsilon_i\} + K \tag{81}$$

$$P = \text{diag}\{d_i\} \tag{82}$$

which for this 3^{rd}-order system can be written as

$$A = -\begin{bmatrix} \epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -\epsilon_1 & 0 & -1 \\ -1 & -\epsilon_2 & 0 \\ 0 & -1 & -\epsilon_3 \end{bmatrix} \tag{83}$$

$$P = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \tag{84}$$

From this, appropriate values for A and P can be found using CVX to prove stability of the full feedback interconnection by solving the feasibility problem of

$$A^T P + PA \leq 0 \tag{85}$$

with C defined as the output matrix associated with the actual state-space representation.

This can then be further developed to prove Global Asymptotic Stability by making it a strict inequality like so:

$$A^T P + PA < 0 \tag{86}$$

Which can be expanded with the actual matrices as

$$\begin{bmatrix} -\epsilon_1 & 0 & -1 \\ -1 & -\epsilon_2 & 0 \\ 0 & -1 & -\epsilon_3 \end{bmatrix}^T \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} + \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \begin{bmatrix} -\epsilon_1 & 0 & -1 \\ -1 & -\epsilon_2 & 0 \\ 0 & -1 & -\epsilon_3 \end{bmatrix} < 0 \tag{87}$$

$$\begin{bmatrix} -d_1\epsilon_1 & -d_1 & 0 \\ 0 & -d_2\epsilon_2 & -d_2 \\ -d_3 & 0 & -d_3\epsilon_3 \end{bmatrix} + \begin{bmatrix} -d_2\epsilon_1 & 0 & -d_1 \\ -d_2 & -d_2\epsilon_2 & 0 \\ 0 & -d_3 & -d_3\epsilon_3 \end{bmatrix} < 0 \tag{88}$$

$$\begin{bmatrix} -2d_1\epsilon_1 & -d_1 & -d_1 \\ -d_2 & -2d_2\epsilon_2 & -d_2 \\ -d_3 & -d_3 & -2d_3\epsilon_3 \end{bmatrix} < 0 \tag{89}$$

or equivalently

$$\begin{bmatrix} 2d_1\epsilon_1 & d_1 & d_1 \\ d_2 & 2d_2\epsilon_2 & d_2 \\ d_3 & d_3 & 2d_3\epsilon_3 \end{bmatrix} > 0 \tag{90}$$

The positive definess can be determined by

$$2d_1\epsilon_1 > 0 \tag{91}$$

$$\begin{vmatrix} 2d_1\epsilon_1 & d_1 \\ d_2 & 2d_2\epsilon_2 \end{vmatrix} = d_1 d_2(4\epsilon_1\epsilon_2 - 1) > 0 \tag{92}$$

$$\begin{vmatrix} 2d_1\epsilon_1 & d_1 & d_1 \\ d_2 & 2d_2\epsilon_2 & d_2 \\ d_3 & d_3 & 2d_3\epsilon_3 \end{vmatrix} = d_1d_2d_3(8\epsilon_1\epsilon_2\epsilon_3 - 2\epsilon_1)$$

$$-d_1d_2d_3(2\epsilon_3 - 1) + d_1d_2d_3(1 - 2\epsilon_2)$$

$$= d_1d_2d_3(8\epsilon_1\epsilon_2\epsilon_3 - 2\epsilon_1 - 2\epsilon_2 - 2\epsilon_3) > 0 \tag{93}$$

From this and the definition of $d_i > 0$, these inequalities can be equated to

$$\epsilon_1 > 0 \tag{94}$$

$$4\epsilon_1\epsilon_2 - 1 > 0 \tag{95}$$

$$4\epsilon_1\epsilon_2\epsilon_3 - \epsilon_1 - \epsilon_2 - \epsilon_3 > 0 \tag{96}$$

Returning to the identity of

$$\epsilon_i = \frac{1}{\delta_i} \leq \frac{1}{\mu}$$

and the knowledge that $\mu < 2$, it is possible to demonstrate that $\exists \epsilon_i,$ s.t. the inequalities are satisfied.

3.2 Part b

Problem: Show that if $\tau_1 = \tau_2 = \tau_3 = \tau$, then $\mu < 2$ is necessary for asymptotic stability. What type of bifurcation occurs at $\mu = 2$?

Solution: In the case that $\tau_1 = \tau_2 = \tau_3 = \tau$, the subsystems can be directly related and the inequality $A^TP + PA < 0$ no longer is true when $\mu \geq 2$ as this results in the inequality to be only stable until eventually a limit cycle is generated. This limit cycle occurance at the biforcation point of $\mu = 2$ indicates that a super-critical hopf-bifurcation occurs

3.3 Part c

Problem: Investigate the dynamic behavior of this system for $\mu > 2$ with numerical simulations.
Solution: The system was modeled in Simulink using the models show in Figure 3 and Figure 4. This was tested for many values of of μ using the MATLAB scripts in Appendix A and B.

The results for various values of μ can be seen in Figure 5, Figure 6, Figure 7, Figure 9, and Figure 10. It is apparent that Asymptotic stability is lost and a limit cycle is gained, in other words, a super-critical hopf-bifurcation occurs. The limit cycle appears to maintain a similar frequency, but as μ grows (which really means β increases based on my code) the magnitude of the limit cycle seems to grow. When it reaches a point where the theoretical magnitude is above the initial conditions it is then truncated (i.e. clipping occurs).

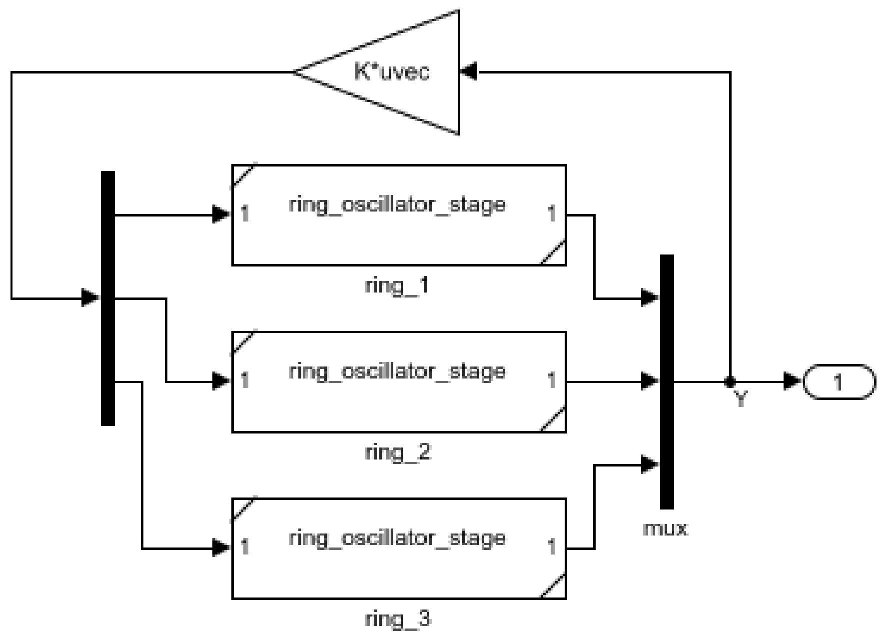


Figure 3: Simulink Model of the feedback system composed of individual nonlinear subsystems

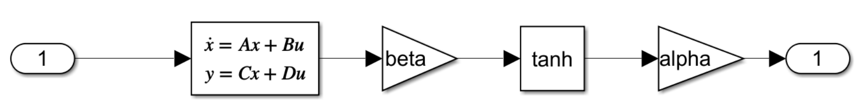


Figure 4: Simulink Model of an individual nonlinear subsystem

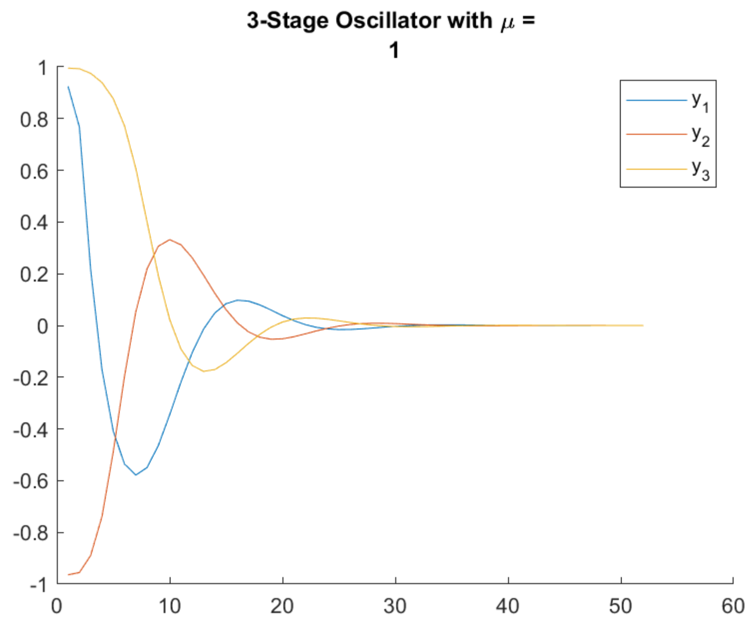


Figure 5: Results for the outputs of the 3-stage ring oscillator with mu = 1

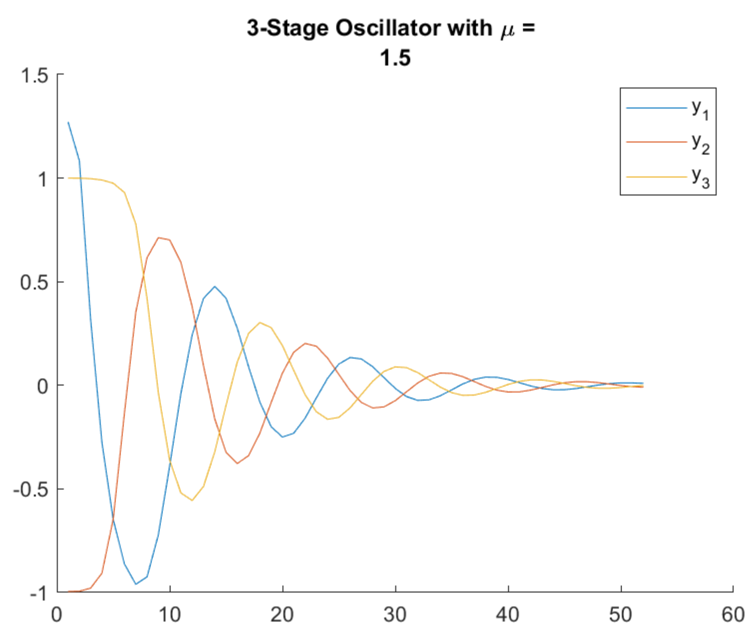


Figure 6: Results for the outputs of the 3-stage ring oscillator with mu = 1.5

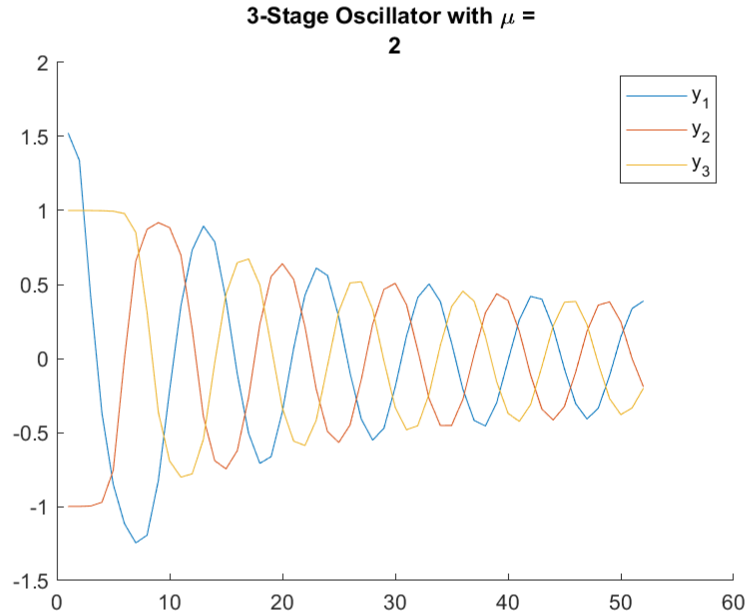


Figure 7: Results for the outputs of the 3-stage ring oscillator with mu = 2

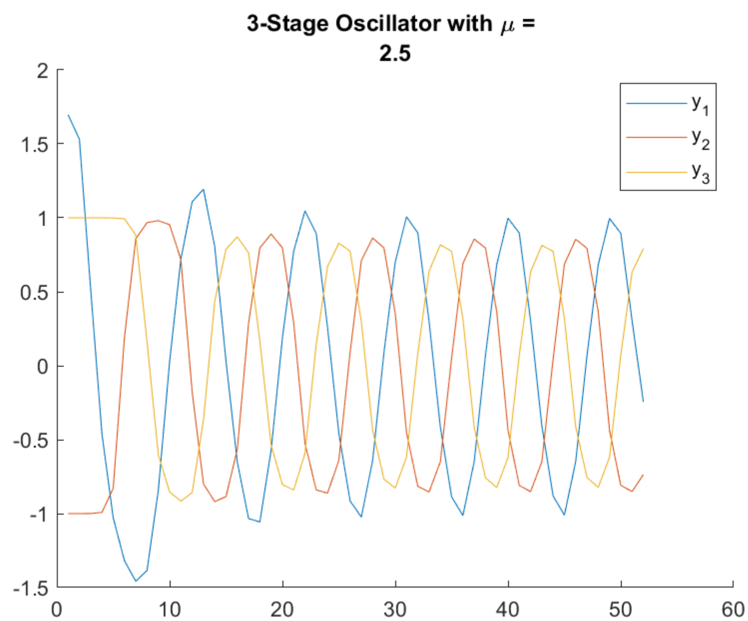


Figure 8: Results for the outputs of the 3-stage ring oscillator with mu = 2.5

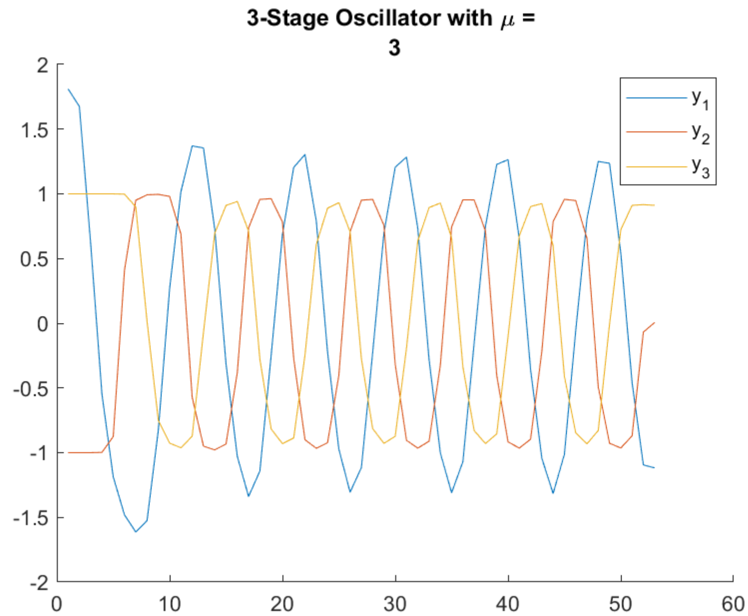


Figure 9: Results for the outputs of the 3-stage ring oscillator with mu = 3

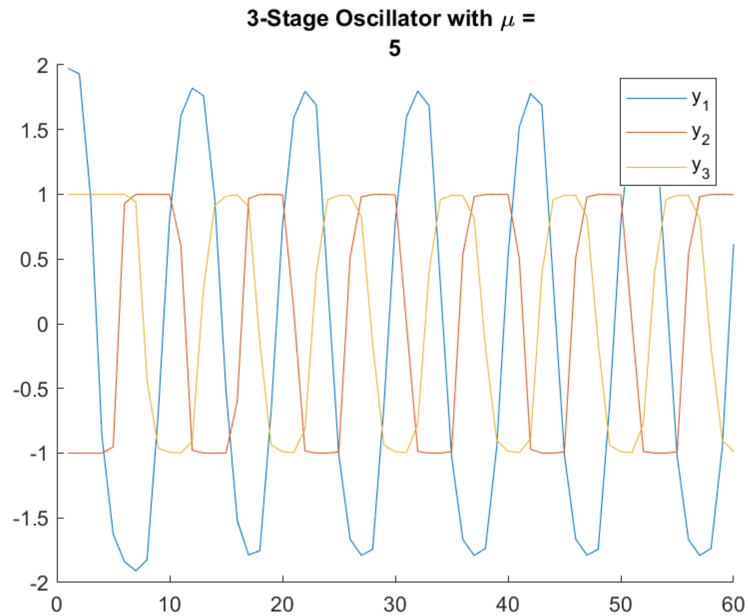


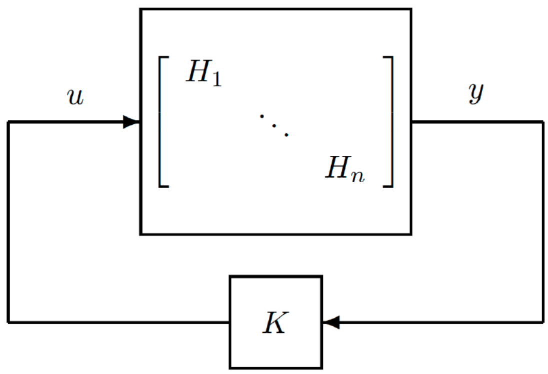
Figure 10: Results for the outputs of the 3-stage ring oscillator with mu = 5

4 Problem 4

Consider a set of systems, $H_i (i = 1, \dots, n)$, who are coupled together in a feedback given by

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = K \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

for inputs u_i and outputs y_i, with feedback matrix $K \in \Re^{n \times n}$.



4.1 Part a (Extra note)

Suppose each H_i satisfies the dissipation inequality:

$$\dot{V}_i(x_i) \leq -y_i^2 + \gamma_i^2 u_i^2$$

for a Positive Definite function,$V_i(x_i)$, of x_i.

4.2 Part b

Problem: Determine a matrix inequality that restrict the matrices

$$D := \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix},\ \Gamma := \begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_n \end{bmatrix}$$

and K, such that

$$V(x) = \sum_{i=1}^{n} d_i V_i(x_i)$$

is a Lyapunov function for the interconnected system.

Solution: The stability of a an interconnected system, by definition of the Lyapunov Function, can be proven by ensuring the that $\dot{V}(x)$ is negative definite.

First, Let

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{n} d_i V_i,\ d_i \geq 0 && (97)\\ &\leq \sum_{i=1}^{n} d_i\big(-y_i^2 + \gamma_i^2 u_i^2\big) && (98)\\ &\leq \sum_{i=1}^{n} -d_i y_i^2 + d_i\gamma_i^2 u_i^2 && (99) \end{aligned}$$

which can be put into a Matrix form as

$$\dot{V} \leq y^T(-D)y + u^T\Gamma^T D\Gamma u \tag{100}$$

and since $u = Ky$, this becomes

$$\begin{aligned} \dot{V} &\leq y^T(-D)y + y^T K^T\Gamma^T D\Gamma K y && (101)\\ &\leq y^T\big(-D + K^T\Gamma^T D\Gamma K\big)y && (102) \end{aligned}$$

Since this exists as an upper bound on $\dot{V}$, the following inequalities are sufficent to say $\dot{V} < 0$ (aka stable)

$$-D + K^T\Gamma^T D\Gamma K < 0 \tag{103}$$

since D is a diagonal matrix (and thus communtable with other matrices), this can be further reformulated

$$D\big(-I + K^T\Gamma^T\Gamma K\big) < 0 \tag{104}$$

which is equivalent to[1]

$$\frac{1}{2}\Big\{\big(-I + K^T\Gamma^T\Gamma K\big)^T D + D\big(-I + K^T\Gamma^T\Gamma K\big)\Big\} < 0 \tag{105}$$

or

$$\big(-I + K^T\Gamma^T\Gamma K\big)^T D + D\big(-I + K^T\Gamma^T\Gamma K\big) < 0 \tag{106}$$

which can be used to test the feasibility and find the elements of D to ensure stability.

[1] This probably wasn't necessary and lead to the more difficult solution on the next page...

4.3 Part c

Problem: Investigate when $\exists$ D s.t. the inequality is satisfied for

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution: The inequality from the previous problem can be used to asses what conditions must be met for D to exist. Let,

$$D = \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix}$$

and

$$\Gamma = \begin{bmatrix} \gamma_1 & \\ & \gamma_2 \end{bmatrix}$$

These $n = 2$ matrices can then be plugged into the inequality as follows

$$\left(-I + K^T\Gamma^T\Gamma K\right)^T D + D\left(-I + K^T\Gamma^T\Gamma K\right) < 0 \tag{107}$$

$$\begin{aligned}\left(-\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} \gamma_1 & \\ & \gamma_2 \end{bmatrix}^T \begin{bmatrix} \gamma_1 & \\ & \gamma_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)^T \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix} \\ + \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix} \left(-\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} \gamma_1 & \\ & \gamma_2 \end{bmatrix}^T \begin{bmatrix} \gamma_1 & \\ & \gamma_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) < 0\end{aligned} \tag{108}$$

$$\left(-\begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} \gamma_1^2 & \\ & \gamma_2^2 \end{bmatrix}\right)^T \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix} + \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix} \left(-\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} + \begin{bmatrix} \gamma_1^2 & \\ & \gamma_2^2 \end{bmatrix}\right) < 0 \tag{109}$$

$$\begin{bmatrix} \gamma_1^2 - 1 & \\ & \gamma_2^2 - 1 \end{bmatrix}^T \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix} + \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix} \begin{bmatrix} \gamma_1^2 - 1 & \\ & \gamma_2^2 - 1 \end{bmatrix} < 0 \tag{110}$$

which (due to communicability of diagonal matrices) can be rewritten as

$$2\begin{bmatrix} d_1(\gamma_1^2 - 1) & \\ & d_2(\gamma_2^2 - 1) \end{bmatrix} < 0 \tag{111}$$

or by analyzing the diagonal values themselves, we have

$$\begin{cases} d_1(\gamma_1^2 - 1) < 0 \\ d_2(\gamma_2^2 - 1) < 0 \end{cases} \tag{112}$$

in other word, d_1 and d_2 (and therefore D) is possible iff

$$\gamma_1^2 < 1 \text{ and } \gamma_2^2 < 1$$

A MATLAB Code:

All code I write in this course can be found on my GitHub repository:
https://github.com/jonaswagner2826/MECH6313

Script 1: MECH6313_HW6

```
% MECH 6313 - HW6
% Jonas Wagner
% 2021-04-27
%

clear
close all

pblm1 = false;
pblm2 = false;
pblm3 = true;


if pblm1
%% Problem 1

G1 = tf([5 3 1], [1 2 1])
isPassive(G1)

G2 = tf([1 1 5 0.1], [1 2 3 4])
isPassive(G2)

Gp = G1 + G2
isPassive(Gp)

Gs = G1 * G2
isPassive(Gs)

end


if pblm2
%% Problem 2
pblm2a = false;
pblm2b = false;
pblm2c = true;

if pblm2a
```

```
% Part a
syms omega a b lambda
assume(a,'real'); assume(a > 0)
assume(b,'real'); assume(b > 0)
assume(omega,'real')
assume(lambda, 'real')

num = j*omega + lambda;
den = omega^2 + j*a*omega + b;

H_sym = num/den;
disp('H(s) = ')
pretty(H_sym)

H_real = real(H_sym);
disp('H_real = ')
pretty(H_real)

H_imag = imag(H_sym);
disp('H_imag = ')
pretty(imag(H_sym))
end

if pblm2b
% Part b
lambda1 = 1;
lambda2 = -1;

H1 = tf([1 lambda1], [1 1 1])
isPassive(H1)
figure
nyquist(H1)
title('Nyquist Plot for H_1(s)')
saveas(gcf, [pwd, '\Homework\HW6\fig\pblm2_H1.png'])

H2 = tf([1 lambda2], [1 1 1])
isPassive(H2)
figure
nyquist(H2)
title('Nyquist Plot for H_2(s)')
saveas(gcf, [pwd, '\Homework\HW6\fig\pblm2_H2.png'])
end
```

```
if pblm2c
% Part c
H1_sys = ss([0, 1; -1, -1], [0; 1], [1 1], 0)
tf (H1_sys)
[A,B,C,D] = ssdata(H1_sys)
syms p11 p12 p22
P = [p11, p12; p12, p22]
A'*P + P * A
P * B - C'

H2_sys = ss([0, 1; -1, -1], [0; 1], [-1 1], 0)
tf(H2_sys)
[A,B,C,D] = ssdata(H2_sys)
syms p11 p12 p22
P = [p11, p12; p12, p22]
A'*P + P * A
P * B - C'
end

end


if pblm3
%% Problem 3

mu = 1;
MECH6313_HW6_pblm3

mu = 1.5;
MECH6313_HW6_pblm3

mu = 2;
MECH6313_HW6_pblm3

mu = 2.5;
MECH6313_HW6_pblm3

mu = 3;
MECH6313_HW6_pblm3

mu = 5;
MECH6313_HW6_pblm3

```

```

end
```

B MATLAB Code for Problem 3:

Script 2: MECH6313_HW6_pblm3

```
% MECH 6313 - HW5
% Jonas Wagner
% 2021-04-08


% mu = 15; %Change this... or make it a global variable for looping...

%% Settings
% generateModel = false;
openModel = true;
simulateModel = true;
plotResults = true;

% Name of the simulink model
[cfolder,~,~] = fileparts(mfilename('fullpath'));
subfolder = ''; %include '/' at end of subfolder name
fname = 'HW6_pblm3';
simTime = '20';


%% System Definitions

% Ring Oscillator Parameters
% Based on the model w/ \dot{x} = -x + u & y = alpha*tanh(beta*x)
tau = 1;
alpha = 1;
beta = mu/alpha;
x0 = 1;

tau1 = tau;
alpha1 = 2*alpha;
beta1 = beta/2;
x01 = x0;

tau2 = tau;
alpha2 = alpha;
beta2 = beta;
x02 = -2*x0;

tau3 = tau;
```

```
alpha3 = alpha;
beta3 = beta;
x03 = 3*x0;

% Coupling Matrix
K = toeplitz([0, -1, 0],[0, 0, -1]);



if openModel
    open(fname); % Don't need to open to run
end

% Auto Arrange
Simulink.BlockDiagram.arrangeSystem(gcs) %Auto Arrange
print(['-s', gcs], '-dpng',... % Print model to figure
    [cfolder, '\' subfolder, 'fig\', 'sim_model_', fname, '.png'])

if simulateModel
%% Simulate System
simConfig.SaveState = 'on';
simOut = sim(fname, 'SaveState', 'on', 'StartTime', '0', 'StopTime', simTime);

% Sim Data
Y_out = simOut.yout{1}.Values;
Xout = simOut.xout{1}.Values.Data; %Only works by grabbing states of first block (LTI_sys
    )
end

if plotResults
%% Plot Results

fig = figure;
hold on
plot(Y_out.Data(:,1))
plot(Y_out.Data(:,2))
plot(Y_out.Data(:,3))
title(['3-Stage Oscillator with \mu = ', string(mu)])
legend('y_1','y_2','y_3')

mu_str = string(1*(mu));
saveas(fig, string([cfolder, '\',subfolder, 'fig\', fname, '_results','_mu_',num2str(mu),
    '.png']))
```

```
% close all
end
```