# MECH6318 HW05

Monday, October 4, 2021 1:28 PM



Fall 2021 | MECH 6318
Engineering Optimization – Prof. Jie Zhang
HOMEWORK #5
September 28, 2021

DUE: Tuesday, October 5, 2021 **5PM central time**
**Submit the HW to eLearning**

**Points Distribution**
30 points maximum
-5 to 0 points reserved for **Neatness and Professional Presentation**
(legible, stapled, show key Matlab commands, properly labeled plots, etc.)

## Book Problems:

Problems 12.4 parts 1a (Bisection method) and 1c (Golden section metod); Problem 12.5 part 1c (Quadratic approximation method).

Please show your complete work, and present the results in a tabular form as shown in Table 12.6.

**12.4** Consider the following function:

$$f(x) = -3x^3 + \frac{12}{x^2} + 2e^{x^2}, \quad \frac{1}{2} \le x \le 2$$

We wish to find the minimum of the above function in the given range of *x*.

1. Using the following listed methods, estimate the minimum of the above function using up to three iterations. Solve Parts (a) through (c) by hand, and please show your complete work.
   (a) Interval Halving Method (*i.e.*, Bisection method)
   (b) Newton-Raphson Method using $x_0 = 1$ as the starting point
   (c) Golden Section Method
   (d) Plot the function *f*(*x*) using MATLAB, and identify the optimum solution in the range of *x*. For Parts (a) through (c), calculate the percentage error between the optimum *x* value and the value of *x* after three iterations. Present your results in a tabular form, as shown below, for methods in Parts (a), (b), and (c).

| Iteration | (a) | | (b) | | (c) | |
|---|---|---|---|---|---|---|
| | *x* | *f*(*x*) | *x* | *f*(*x*) | *x* | *f*(*x*) |
| 1 | | | | | | |
| 2 | | | | | | |
| 3 | | | | | | |

**12.5** Consider the following function:

$$f(x) = -3x^3 + \frac{12}{x^2} + 2e^{x^2}, \quad \frac{1}{2} \le x \le 2$$

We want to find the minimum of the above function in the given range of *x*.

1. Using the following methods, estimate the minimum of the above function by implementing up to three iterations. Solve Parts (a) through (c) by hand, and please show your complete work.
   (a) Bisection Method
   (b) Secant Method
   (c) Successive quadratic estimation Present your results in a tabular form, as shown below, for methods in Parts (a), (b), and (c).

## 12.4-1a: (Bisection)

**BISECTION**

➢*Procedure*

1. Specify the convergence tolerance ε > 0. Specify the convergence tolerance of the gradient γ > 0. Set the number of iterations, k, as 0. The bounds are $l_0 = a$, and $r_0 = b$.
2. If $r_k - l_k < \varepsilon$, **stop**. The midpoint $x_k = (l_k+r_k)/2$ is taken as the minimum, x∗, and f(x∗) is the optimal solution.
3. Evaluate the gradient at the middle point, $f'(x_k)=f'((l_k+r_k)/2)$.
4. If $||f'(x_k)|| < \gamma$, **stop**. The $x_k$ is taken as the minimum, x∗, and the corresponding function value f(x∗) is the optimal solution.
5. Evaluate the product $f'(l_k)f'(x_k)$. If it is negative, $l_{k+1} = l_k$ and $r_{k+1} = x_k$. If it is positive, $l_{k+1} = x_k$ and $r_{k+1} = r_k$.
6. k = k + 1. Go to step 2.

$f(x) = -3x^3 + \frac{12}{x^2} + 2e^{x^2}, \quad \frac{1}{2} \le x \le 2$

$\min_x f(x) = -3x^3 + \frac{12}{x^2} + 2e^{x^2}$ $\qquad f'(x) = -9x^2 - \frac{24}{x^3} + 4xe^{x^2}$

$\frac{1}{2} \le x \le 2$ $\qquad \gamma = 0.05$

$\varepsilon = 1$

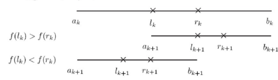
| N | a | b | $c = \frac{a+b}{2}$ | $f'(a) = -9x^2 - \frac{24}{x^3} + 4xe^{x^2}$ | $f'(c) = -9x^2 - \frac{24}{x^3} + 4xe^{x^2}$ | sign(f'(a)) | sign(f'(c)) | $= \to a = c$ <br> $\ne \to a = a$ | $= \to b = b$ <br> $\ne \to b = c$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.5 | 2 | $\frac{2.5}{2} = 1.25$ | −191.68 | −2.496 | − | − | a = c | b = b |
| 2 | 1.25 | 2 | 1.625 | −2.496 | 61.8 | − | + | a = a | b = c |
| 3 | 1.25 | 1.625 | 1.4375 | −2.496 | 18.72 | − | + | a = a | b = c |
| 4 | 1.25 | 1.4375 | 1.34375 | −2.496 | 6.559 | − | + | a = a | b = c |
| 5 | 1.25 | 1.34375 | 1.296875 | −2.496 | 1.746 | − | + | a = a | b = c |
| 6 | 1.25 | 1.296875 | 1.2734 | | | | | | |

(1.250, 1.297) $\boxed{x^* = 1.273 \quad f(x^*) = 11.328}$

## 12.4-1c: (Golden Section)

### GOLDEN SECTION

- Suppose f(x) is inside the interval [a, b], and f(x) is continuous.
  1. Choose ε > 0 as the convergence tolerance of interval. Set k = 0. The bounds are $a_0 = a$ and $b_0 = b$. Set $l_0 = b_0 - \tau(b_0 - a_0)$, and $r_0 = a_0 + \tau(b_0 - a_0)$.
  2. **If $b_k - a_k < \varepsilon$, stop**. The midpoint $x_k = (a_k+b_k)/2$ is taken as the optimal value, x*, and f(x*) is the optimal solution.
- Suppose f(x) is inside the interval [a, b], and f(x) is continuous.
  3. **If $f(l_k) > f(r_k)$**, the parameters are updated as follows. $a_{k+1} = l_k$; $b_{k+1} = b_k$; $l_{k+1} = r_k$; $r_{k+1} = a_{k+1} + \tau(b_{k+1} - a_{k+1})$.
  4. **If $f(l_k) < f(r_k)$**, the parameters are updated as follows. $a_{k+1} = a_k$; $b_{k+1} = r_k$; $r_{k+1} = l_k$; $l_{k+1} = b_{k+1} - \tau(b_{k+1} - a_{k+1})$
  5. k = k + 1, go to Step 2.




- The length of the interval is reduced by τ at each iteration.

$f(x) = -3x^3 + \frac{12}{x^2} + 2e^{x^2}, \quad \frac{1}{2} \le x \le 2$

$\min_x f(x) = -3x^3 + \frac{12}{x^2} + 2e^{x^2}$ $\qquad \cancel{f'(x) = -9x^2 - \frac{24}{x^3} + 4xe^{x^2}}$

$\frac{1}{2} \le x \le 2$

$\varepsilon = 0.1$ $\qquad \tau = 0.6$

| N | a | b | b − a | l | r | $f(l) = -3x^3 + \frac{12}{x^2} + 2e^{x^2}$ | $f(r) = -3x^3 + \frac{12}{x^2} + 2e^{x^2}$ | $f(l) > f(r)$ <br> a = l, b = b, l = r, r = a + b(1−τ) | $f(l) < f(r)$ <br> a = a, b = r, r = l, l = b − τ(b − a) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.5 | 2 | 1.5 | 1.1 | 1.4 | 12.6313 | 12.0891 | X | |
| 2 | 1.1 | 2 | 0.9 | 1.4 | 1.64 | 12.0891 | 20.6804 | | X |
| 3 | 1.1 | 1.64 | 0.54 | 1.316 | 1.4 | 11.3939 | 12.0891 | | X |
| 4 | 1.1 | 1.4 | 0.3 | 1.22 | 1.316 | 11.4748 | 11.3939 | X | |
| 5 | 1.22 | 1.4 | 0.18 | 1.316 | 1.328 | 11.3939 | 11.4448 | | X |
| 6 | 1.22 | 1.328 | 0.108 | 1.2632 | 1.316 | 11.3367 | 11.3939 | | X |

0.096

(1.22, 1.316) → $\boxed{x^* = 1.268 \quad f(x^*) = 11.3312}$

## 12.5-1c: (Quadratic Approximation)

### QUADRATIC APPROXIMATION

**➢Procedure**

1. Define the tolerance of function value difference, γ > 0, and the tolerance of the variable, ε > 0. Set Δx. Set k = 1.
2. Set the initial point, $x^1_1$. Compute $x^1_2 = x^1_1 + \Delta x$. Evaluate $f(x^1_1)$ and $f(x^1_2)$.

   If $f(x^1_1) > f(x^1_2)$, then $x^1_3 = x^1_2 + \Delta x$.
   If $f(x^1_1) < f(x^1_2)$, then $x^1_3 = x^1_1 - \Delta x$.
   Evaluate $f(x^1_3)$.
3. Compare the function values at the three points, $x^k_1$, $x^k_2$, and $x^k_3$. Find out the minimum $f^k_{min} = \min\{f(x^k_1), (x^k_2), f(x^k_3)\}$, and the corresponding point $x^k_{min}$.

**➢Procedure**

4. Using the three points, $x^k_1$, $x^k_2$, and $x^k_3$, construct a quadratic approximation. Compute the minimum point, $x^*_k$. Evaluate $f(x^*_k)$.
5. If $|| f(x^*_k) - f^k_{min} || < \gamma$ and $|| x^*_k - x^k_{min} || < \varepsilon$, take $x^*_k$ as the minimum. Terminate the iteration.
6. Take the current **best point $x^*_k$** and the two points bracketing it, as the three points for the next quadratic approximation. k = k + 1. Go to Step 3.

- Assume an objective function is **unimodal and continuous inside an interval**, then it can be approximated by a quadratic approximation.
- It consists of a sequence of interval reducing and iterative approximation in the reduced intervals.
- A quadratic approximating function is constructed, given three consecutive points and their corresponding function values.

$\tilde{f}(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2)$

$f(x_1) = \tilde{f}(x_1) = c_0$ $\qquad c_0 = f(x_1)$

$f(x_2) = \tilde{f}(x_2) = c_0 + c_1(x_2 - x_1)$ $\qquad c_1 = \frac{f(x_2) - c_0}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

$f(x_3) = \tilde{f}(x_3) = c_0 + c_1(x_3 - x_1) + c_2(x_3 - x_1)(x_3 - x_2)$

$c_2 = \frac{f(x_3) - c_0 - c_1(x_3 - x_1)}{(x_3 - x_1)(x_3 - x_2)} = \frac{1}{x_3 - x_2}\left(\frac{f(x_3) - f(x_1)}{x_3 - x_1} - \frac{f(x_2) - f(x_1)}{x_2 - x_1}\right)$

$f(x) = -3x^3 + \frac{12}{x^2} + 2e^{x^2}, \quad \frac{1}{2} \le x \le 2$

$\min_x f(x) = -3x^3 + \frac{12}{x^2} + 2e^{x^2}$

$f(x) = -3x^3 + \frac{12}{x^2} + 2e^{x^2}, \quad \frac{1}{2} \le x \le 2$

$$\min_x f(x) = -3x^3 + \frac{12}{x^2} + 2e^{x^2}$$

$\frac{1}{2} \le x \le 2$

$\gamma = 0.01 \qquad \Delta x = 0.1 \qquad x_0 = 1$

$\varepsilon = 0.05$

$x_2 = x_1 + \Delta x$

$f(x_1) > f(x_2): x_3 = x_2 + \Delta x$

$f(x_1) < f(x_2): x_3 = x_1 - \Delta x$

$C_0 = f(x_1)$ $\qquad C_1 = \frac{f(x_2) - C_0}{x_2 - x_1}$ $\qquad C_2 = \frac{1}{x_3 - x_2}\left(\frac{f(x_3) - f(x_1)}{x_3 - x_1} - \frac{f(x_2) - f(x_1)}{x_2 - x_1}\right)$

$$x^* = \frac{x_1 + x_2}{2} - \frac{C_1}{2C_2}$$

| N | $x_1$ | $x_2$ | $f(x_1)$ | $f(x_2)$ | $x_3$ | $f(x_3)$ | $f(x_{min})$ | $x_{min}$ | $C_0$ | $C_1$ | $C_2$ | $x^*$ | $f(x^*)$ | $f(x^*)-f(x_{min})$ | $x^*-x_{min}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1.1 | 14.44 | 12.63 | 1.2 | 11.59 | 11.59 | 1.2 | 14.44 | -18.05 | 38.23 | 1.286 | 11.32 | -0.26 | 0.086 |
| 2 | 1.286 | 1.2 | 11.33 | 11.59 | 1.3 | 11.35 | 11.33 | 1.286 | 11.33 | -3.04 | 44.19 | 1.277 | 11.33 | -0.003 | -0.0088 |
| 3 | | | | | | | | | | | | | | | |
| 4 | | | | | | | | | | | | | | | |
| 5 | | | | | | | | | | | | | | | |
| 6 | | | | | | | | | | | | | | | |

$< \gamma$

$< \varepsilon$

$$\boxed{\begin{aligned} x^* &= 1.277 \\ f(x^*) &= 11.32 \end{aligned}}$$