

MECH6318 HW09

Friday, November 5, 2021 2:47 AM

13.1 Consider the following optimization problem:

$$\max_x f(x) = x_1^2 - x_2 \quad (13.139)$$

subject to

$$x_1^2 - x_2^2 = 1 \quad (13.140)$$

1. Find all the points that satisfy the KKT condition using Lagrange multipliers.
2. Use the variable elimination method to solve the problem (eliminate x_1). Compare the solutions obtained using the variable elimination method with those obtained in Question 1.

$$\begin{array}{l} \max_x f(x) = x_1^2 - x_2 \\ \text{s.t. } x_1^2 - x_2^2 - 1 = 0 \end{array} \left\{ \begin{array}{l} \text{max } -x_1^2 + x_2 = -f(x) \\ \text{s.t. } x_1^2 - x_2^2 - 1 = 0 \quad h_j(x) = 0 \end{array} \right.$$

$$\begin{aligned} 1) \quad L(x, \gamma) &= f(x) + \sum_{j=1}^n \gamma_j h_j(x) \\ &= -x_1^2 + x_2 + \gamma_1 (x_1^2 - x_2^2 - 1) \end{aligned}$$

$$h(x) = 0 = x_1^2 - x_2^2 - 1$$

$$(x_1, x_2) \in \{x_1, x_2 \in \mathbb{R} \mid x_1^2 - x_2^2 - 1 = 0\}$$

$$2) \quad \boxed{x_1^2 = \sqrt{x_2^2 + 1}} \quad \boxed{x_1^2 = x_2^2 + 1}$$

$$\boxed{x_1 = \sqrt{1 + x_2^2}}$$

$$\begin{aligned} \max_x -f(x) &= -x_1^2 + x_2 = -(1 + x_2^2) + x_2 \\ &= -x_2^2 + x_2 - 1 \end{aligned}$$

$$\nabla_x f(x) = -2x + 1 = 0$$

$$\nabla_{x_2} f(x) = -2x_2 + 1 = 0$$

$$x_2 = \frac{1}{2}$$

$$x_2 = \frac{1}{2}$$

$$x_1 = \sqrt{1 + x_2^2}$$

$$= \sqrt{1 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{5}{4}}$$

$$x_1 = \frac{\sqrt{5}}{2}$$



$$x_1 = \frac{\sqrt{5}}{2}$$

$$x_2 = \frac{1}{2}$$

13.2 Use the Lagrange multiplier method to solve the following problem:

$$\min_x f(x) = x_1^2 + x_2^2 + x_3^2 \quad (13.141)$$

subject to

$$x_1 + 2x_2 + 3x_3 = 7 \quad (13.142)$$

$$2x_1 + 2x_2 + x_3 = \frac{9}{2} \quad (13.143)$$

$$\min_x f(x) = x_1^2 + x_2^2 + x_3^2$$

$$\text{s.t. } h_1(x) = x_1 + 2x_2 + 3x_3 - 7 = 0$$

$$h_2(x) = 2x_1 + 2x_2 + x_3 - \frac{9}{2} = 0$$

$$L(x, \nu) = f(x) + \sum_{i=1}^n \nu_i h_i(x)$$

$$= x_1^2 + x_2^2 + x_3^2 + \nu_1(x_1 + 2x_2 + 3x_3 - 7) + \nu_2(2x_1 + 2x_2 + x_3 - \frac{9}{2})$$

$$\begin{aligned}\nabla_{x_1} L(x, v) &= 2x_1 + v_1 + 2v_2 = 0 \\ \nabla_{x_2} L(x, v) &= 2x_2 + 2v_1 + 2v_3 = 0 \\ \nabla_{x_3} L(x, v) &= 2x_3 + 3v_1 + v_3 = 0 \\ h_1(x) &= x_1 + 2x_2 + 3x_3 - 7 = 0 \\ h_2(x) &= 2x_1 + 2v_1 + x_3 - \frac{9}{2} = 0\end{aligned}$$

$$\left[\begin{array}{ccccc} x_1 & x_2 & x_3 & v_1 & v_2 \\ 2 & 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 3 & 3 & 1 \\ 1 & 2 & 3 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 7 \\ \frac{9}{2} \end{array} \right]$$

Problem 13.2

```
A = [2, 0, 0, 1, 2;
     0, 2, 0, 2, 2;
     0, 0, 3, 3, 1;
     1, 2, 3, 0, 0;
     2, 2, 1, 0, 0;
]
b = [0, 0, 0, 7, 9/2]'

syms x1 x2 x3 v1 v2
x_sym = [x1, x2, x3, v1, v2]'

results = solve(A * x_sym == b, x_sym');

x1 = double(results.x1)
x2 = double(results.x2)
x3 = double(results.x3)
v1 = double(results.v1)
v2 = double(results.v2)
```

$$A = 5 \times 5$$

$$\begin{matrix} 2 & 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 3 & 3 & 1 \\ 1 & 2 & 3 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \end{matrix}$$

$$b = 5 \times 1$$

$$\begin{matrix} 0 \\ 0 \\ 0 \\ 7.0000 \\ 4.5000 \end{matrix}$$

$$x_sym =$$

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}$$

$$\begin{matrix} x1 = 0.3723 \\ x2 = 1.1596 \\ x3 = 1.4362 \\ v1 = -1.5745 \\ v2 = 0.4149 \end{matrix}$$

13.3 Consider the following optimization problem:

$$\min_{H,D} f(H, D) = \pi DH + \frac{\pi D^2}{2} \quad (13.144)$$

subject to

$$g_1 \equiv 1,000\pi - \pi D^2 H \leq 0 \quad (13.145)$$

$$g_2 \equiv 4.5 - D \leq 0 \quad (13.146)$$

$$g_3 \equiv D - 12 \leq 0 \quad (13.147)$$

$$g_4 \equiv 10 - H \leq 0 \quad (13.148)$$

$$g_5 \equiv H - 18 \leq 0 \quad (13.149)$$

$$g_6 \equiv H - 18 \leq 0 \quad (13.150)$$

1. Write down the Karush-Kuhn-Tucker (KKT) necessary conditions for this problem
2. You are given that constraints g_1 and g_4 are active, and the other constraints are inactive. Given this information, simplify the KKT conditions found in No. 1.
3. Find all the possible KKT points using the simplified KKT conditions from No. 2.

$$\min_{H,D} f(H, D) = \pi DH + \frac{\pi D^2}{2}$$

$$\text{s.t. } g_1(H, D) = \pi(1 - D^2 H) \leq 0$$

$$g_2(H, D) = 4.5 - D \leq 0$$

$$g_3(H, D) = D - 12 \leq 0$$

$$g_4(H, D) = 10 - H \leq 0$$

$$g_5(H, D) = H - 18 \leq 0$$

$$L(H, D, \lambda) = f(H, D) + \sum_{i=1}^n \lambda_i g_i(H, D)$$

$$= \pi DH + \frac{\pi D^2}{2}$$

$$+ \lambda_1 (\pi(1 - D^2 H))$$

$$+ \lambda_2 (4.5 - D)$$

$$+ \lambda_3 (D - 12)$$

$$\begin{aligned} & + \lambda_2 (4.5 - D) \\ & + \lambda_3 (D - 12) \\ & + \lambda_4 (10 - H) \\ & + \lambda_5 (H - 18) \end{aligned}$$

$$\nabla_D L(H, D, \lambda) = \pi H + \pi D - 2\lambda_1, \pi D H - \lambda_2 + \lambda_3 = 0$$

$$\nabla_H L(H, D, \lambda) = \pi D - \lambda_1, \pi D^2 - \lambda_4 + \lambda_5 = 0$$

$$\lambda_1 g_1(H, D) = \lambda_1 (\pi(1 - D^2 H)) = 0$$

$$\lambda_2 g_2(H, D) = \lambda_2 (4.5 - D) = 0$$

$$\lambda_3 g_3(H, D) = \lambda_3 (D - 12) = 0$$

$$\lambda_4 g_4(H, D) = \lambda_4 (10 - H) = 0$$

$$\lambda_5 g_5(H, D) = \lambda_5 (H - 18) = 0$$

$$g_1(H, D) = \pi(1 - D^2 H) \leq 0 \quad \lambda_1 \geq 0$$

$$g_2(H, D) = 4.5 - D \leq 0 \quad \lambda_2 \geq 0$$

$$g_3(H, D) = D - 12 \leq 0 \quad \lambda_3 \geq 0$$

$$g_4(H, D) = 10 - H \leq 0 \quad \lambda_4 \geq 0$$

$$g_5(H, D) = H - 18 \leq 0 \quad \lambda_5 \geq 0$$

Problem 13.3

```
f = @(H,D) pi * D * H + (pi * D^2)/2
g1 = @(H,D) pi * (1 - D^2 * H)
g2 = @(H,D) 4.5 - D
g3 = @(H,D) D - 12
g4 = @(H,D) 10 - H
g5 = @(H,D) H - 18

L = @(H,D,lambda) (
    f(H,D) ...
    + lambda(1) * g1(H,D) ...
    + lambda(2) * g2(H,D) ...
    + lambda(3) * g3(H,D) ...
    + lambda(4) * g4(H,D) ...
    + lambda(5) * g5(H,D) ...
)

lambda = sym('lambda',[1,5]);
assume(lambda >= 0)

D_H_L = @(D,H) diff(L(H,D,lambda),H);
D_D_L = @(D,H) diff(L(H,D,lambda),D);
```

```
f = function_handle with value:
@(H,D)pi*D*H+(pi*D^2)/2
g1 = function_handle with value:
@(H,D)pi*(1-D^2*H)
g2 = function_handle with value:
@(H,D)4.5-D
g3 = function_handle with value:
@(H,D)D-12
g4 = function_handle with value:
@(H,D)10-H
g5 = function_handle with value:
@(H,D)H-18
L = function_handle with value:
@(H,D,lambda)(f(H,D)+lambda(1)*g1(H,D)+lambda(2)*g2(H,D)+lambda(3)*g3(H,D)+lambda(4)*g4(H,D)+lambda(5)*g5(H,D))
L_H = -lambda_1*pi*D^2 + pi*D - lambda_4 + lambda_5
L_D = lambda_3 - lambda_2 + pi*D + pi*H - 2*pi*D*H*lambda_1
```

```

results = solve([
    D_H_L(D,H) == 0, ...
    D_D_L(D,H) == 0, ...
    lambda(1) * g1(H,D) == 0, ...
    lambda(2) * g2(H,D) == 0, ...
    lambda(3) * g3(H,D) == 0, ...
    lambda(4) * g4(H,D) == 0, ...
    lambda(5) * g5(H,D) == 0, ...
    g1(H,D) <= 0, ...
    g2(H,D) <= 0, ...
    g3(H,D) <= 0, ...
    g4(H,D) <= 0, ...
    g5(H,D) <= 0
], ...
[D, H, lambda]...
);

D = double(results.D)
H = double(results.H)
lambda_1 = double(results.lambda1)
lambda_2 = double(results.lambda2)
lambda_3 = double(results.lambda3)
lambda_4 = double(results.lambda4)
lambda_5 = double(results.lambda5)

D = 4.5000
H = 10
lambda_1 = 0
lambda_2 = 45.5531
lambda_3 = 0
lambda_4 = 14.1372
lambda_5 = 0

```

13.4 Solve the constrained optimization problem below using the inverse penalty method

$$\min_x f(x) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2 \quad (13.151)$$

subject to

$$x_1 + x_2 - 2 \leq 0 \quad (13.152)$$

1. Perform four unconstrained optimizations using the following values for the penalty parameter: $R = 1, 0.1, 0.01$, and 0.001 .
2. For each R , prepare a table that provides the value of the penalty parameter, the values of the design variables, objective function and the constraint.
3. Can you guess where the constrained minimum might be?

$$\text{min}_x f(x) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2$$

$$\text{s.t. } g_i(x) = x_1 + x_2 - 2 \leq 0$$

Inverse Penalty Method:

$$\Omega = R \frac{1}{g_i(x)}$$

Penalty:

$$P(x, R) \approx f(x) + \Omega(R, g(x))$$

$$= (x_1 - 1.5)^2 + (x_2 - 1.5)^2 + R \left(\frac{1}{x_1 + x_2 - 2} \right)$$

$$= x_1^2 - 3x_1 + 2.25 + x_2^2 - 3x_2 + 2.25 + \left(\frac{R}{x_1 + x_2 - 2} \right)$$

$$= \left(x_1^3 - 3x_1^2 + 2.25x_1 + x_1x_2^2 + x_1^2x_2 + 3x_1x_2 + 2.25x_1 \right. \\ \left. + x_1x_2^2 - 3x_1x_2 + 2.25x_2^2 + x_2^3 - 3x_2^2 + 2.25x_2 \right. \\ \left. - 2x_1^2 + 6x_1 - 4.5 - 2x_2^2 + 6x_2 - 4.5 + R \right) / (x_1 + x_2 - 2)$$

$$= \frac{(x_1^3 - 5x_1^2 + 10.5x_1 + x_2^3 - 5x_2^2 + 10.5x_2 - 6x_1x_2 + x_1x_2^2 + x_1^2x_2)}{x_1 + x_2 - 2} - 9 + R$$

Problem 13.4

```

clear
f = @(x) (x(1) - 1.5)^2 + (x(2) - 1.5)^2;
g1 = @(x) x(1) + x(2) - 2;

x = sym('x',[2,1], 'real')
fx = f(x)
g1x = g1(x)

P = @(x,R) f(x) + R/(g1(x));
P_xR = P(x,sym('R'))

D_x_P = @(x,R) gradient(P(x,R),x);
D_x_P_R = D_x_P(x,sym('R'))

R = [1; 0.1; 0.01; 0.001];
X1 = arrayfun(@(R) double(solve(D_x_P(x,R),x).x1),R);
X2 = arrayfun(@(R) double(solve(D_x_P(x,R),x).x2),R);
G = arrayfun(@(x1,x2) g1([x1,x2]), X1, X2);
F = arrayfun(@(x1,x2) f([x1,x2]), X1, X2);
Omega = R./G;

Results = table(R,X1,X2,G,F,Omega)

```

$f_x =$

$$\left(x_1 - \frac{3}{2}\right)^2 + \left(x_2 - \frac{3}{2}\right)^2$$

$$g_{1x} = x_1 + x_2 - 2$$

$P_{xR} =$

$$\left(x_1 - \frac{3}{2}\right)^2 + \left(x_2 - \frac{3}{2}\right)^2 + \frac{R}{x_1 + x_2 - 2}$$

$D_{xP_R} =$

$$\begin{pmatrix} 2x_1 - \frac{R}{(x_1 + x_2 - 2)^2} - 3 \\ 2x_2 - \frac{R}{(x_1 + x_2 - 2)^2} - 3 \end{pmatrix}$$

Results = 4x6 table

	R	X1	X2	G	F	Omega
1	1.0000	1.7328	1.7328	1.4656	0.1084	0.6823
2	0.1000	1.5425	1.5425	1.0850	0.0036	0.0922
3	0.0100	1.5049	1.5049	1.0098	0.0000	0.0099
4	0.0010	1.5005	1.5005	1.0010	0.0000	0.0010

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```
clear
close all
```

Problem 13.1

```
% f = @(x) x(1)^2 - x(2)
% h1 = @(x) x(1)^2 - x(2) - 1
```

Part 1

```
% syms x1 x2
% eq1 = -x2^2 + x2 - 1 == 0;
% eq2 = x1^2 - x2^2 - 1 == 0;
% eq = [eq1; eq2]
%
% [X1, X2] = solve(eq, [x1, x2])
```

Problem 13.2

```
A = [2, 0, 0, 1, 2;
      0, 2, 0, 2, 2;
      0, 0, 3, 3, 1;
      1, 2, 3, 0, 0;
      2, 2, 1, 0, 0;
    ]
```

```
A = 5x5
 2     0     0     1     2
 0     2     0     2     2
 0     0     3     3     1
 1     2     3     0     0
 2     2     1     0     0
```

```
b = [0, 0, 0, 7, 9/2]'
```

```
b = 5x1
 0
 0
 0
 7.0000
 4.5000
```

```
syms x1 x2 x3 v1 v2
x_sym = [x1, x2, x3, v1, v2]'
```

```
x_sym =
```

$$\begin{pmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \\ \overline{v_1} \\ \overline{v_2} \end{pmatrix}$$

```
results = solve(A * x_sym == b, x_sym');
```

```
x1 = double(results.x1)
```

```
x1 = 0.3723
```

```
x2 = double(results.x2)
```

```
x2 = 1.1596
```

```
x3 = double(results.x3)
```

```
x3 = 1.4362
```

```
v1 = double(results.v1)
```

```
v1 = -1.5745
```

```
v2 = double(results.v2)
```

```
v2 = 0.4149
```

Problem 13.3

```
f = @(H,D) pi * D * H + (pi * D^2)/2
```

```
f = function_handle with value:  
@(H,D)pi*D*H+(pi*D^2)/2
```

```
g1 = @(H,D) pi * (1 - D^2 * H)
```

```
g1 = function_handle with value:  
@(H,D)pi*(1-D^2*H)
```

```
g2 = @(H,D) 4.5 - D
```

```
g2 = function_handle with value:  
@(H,D)4.5-D
```

```
g3 = @(H,D) D - 12
```

```
g3 = function_handle with value:  
@(H,D)D-12
```

```
g4 = @(H,D) 10 - H
```

```
g4 = function_handle with value:
```

```
@(H,D)10-H
```

```
g5 = @(H,D) H - 18
```

```
g5 = function_handle with value:  
@(H,D)H-18
```

```
L = @(H,D,lambda) (...  
f(H,D) ...  
+ lambda(1) * g1(H,D) ...  
+ lambda(2) * g2(H,D) ...  
+ lambda(3) * g3(H,D) ...  
+ lambda(4) * g4(H,D) ...  
+ lambda(5) * g5(H,D) ...  
)
```

```
L = function_handle with value:  
@(H,D,lambda)(f(H,D)+lambda(1)*g1(H,D)+lambda(2)*g2(H,D)+lambda(3)*g3(H,D)+lambda(4)*g4(H,D)+lambda(5)*g5(H,D))
```

```
lambda = sym('lambda',[1,5]);  
assume(lambda >= 0)  
  
D_H_L = @(D,H) diff(L(H,D,lambda),H);  
D_D_L = @(D,H) diff(L(H,D,lambda),D);  
  
syms D H  
L_H = D_H_L(D,H)
```

```
L_H = -λ₁ π D² + π D - λ₄ + λ₅
```

```
L_D = D_D_L(D,H)
```

```
L_D = λ₃ - λ₂ + π D + π H - 2 π D H λ₁
```

```
results = solve([ ...  
D_H_L(D,H) == 0, ...  
D_D_L(D,H) == 0, ...  
lambda(1) * g1(H,D) == 0, ...  
lambda(2) * g2(H,D) == 0, ...  
lambda(3) * g3(H,D) == 0, ...  
lambda(4) * g4(H,D) == 0, ...  
lambda(5) * g5(H,D) == 0, ...  
g1(H,D) <= 0, ...  
g2(H,D) <= 0, ...  
g3(H,D) <= 0, ...  
g4(H,D) <= 0, ...  
g5(H,D) <= 0  
, ...  
[D, H, lambda] ...  
)  
  
D = double(results.D)
```

```

D = 4.5000
H = double(results.H)
H = 10
lambda_1 = double(results.lambda1)
lambda_1 = 0
lambda_2 = double(results.lambda2)
lambda_2 = 45.5531
lambda_3 = double(results.lambda3)
lambda_3 = 0
lambda_4 = double(results.lambda4)
lambda_4 = 14.1372
lambda_5 = double(results.lambda5)
lambda_5 = 0

```

Problem 13.4

```

clear
f = @(x) (x(1) - 1.5)^2 + (x(2) - 1.5)^2;
g1 = @(x) x(1) + x(2) - 2;

x = sym('x',[2,1], 'real')

x =

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$


fx = f(x)

fx =

$$\left(x_1 - \frac{3}{2}\right)^2 + \left(x_2 - \frac{3}{2}\right)^2$$


g1x = g1(x)

g1x = x_1 + x_2 - 2

```

```

P = @(x,R) f(x) + R/(g1(x));
P_xR = P(x,sym('R'))
P_xR =

```

$$\left(x_1 - \frac{3}{2}\right)^2 + \left(x_2 - \frac{3}{2}\right)^2 + \frac{R}{x_1 + x_2 - 2}$$

```
D_x_P = @(x,R) gradient(P(x,R),x);
D_x_P_R = D_x_P(x,sym('R'))
```

$$D_x_P_R = \begin{pmatrix} 2x_1 - \frac{R}{(x_1 + x_2 - 2)^2} - 3 \\ 2x_2 - \frac{R}{(x_1 + x_2 - 2)^2} - 3 \end{pmatrix}$$

```
R = [1; 0.1; 0.01; 0.001];
X1 = arrayfun(@(R) double(solve(D_x_P(x,R),x).x1),R);
X2 = arrayfun(@(R) double(solve(D_x_P(x,R),x).x2),R);
G = arrayfun(@(x1,x2) g1([x1,x2]), X1, X2);
F = arrayfun(@(x1,x2) f([x1,x2]), X1, X2);
Omega = R./G;

Results = table(R,X1,X2,G,F,Omega)
```

Results = 4x6 table

	R	X1	X2	G	F	Omega
1	1.0000	1.7328	1.7328	1.4656	0.1084	0.6823
2	0.1000	1.5425	1.5425	1.0850	0.0036	0.0922
3	0.0100	1.5049	1.5049	1.0098	0.0000	0.0099
4	0.0010	1.5005	1.5005	1.0010	0.0000	0.0010