SIMPLEX ALGORITHM

Example:

- The coefficient of x_1 in the objective function row is negative. Choose x_1 as the variable entering the basis

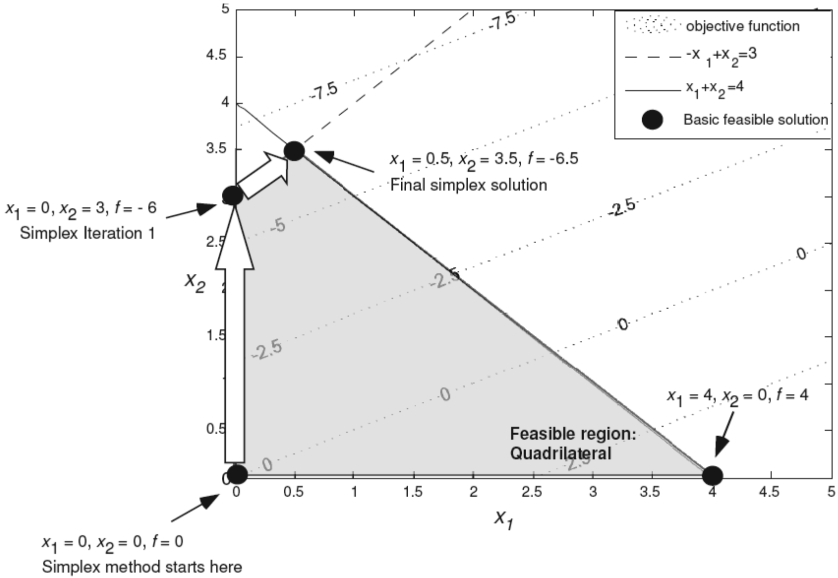
	x_1	x_2	s_1	s_2	b	$\frac{b_i}{a_{ij}}$
R_1	**2**	**0**	**1**	**-1**	**1**	1/2
R_2	**-1**	1	0	1	3	
R_3	**-1**	0	0	2	$f+6$	

Final simplex tableau

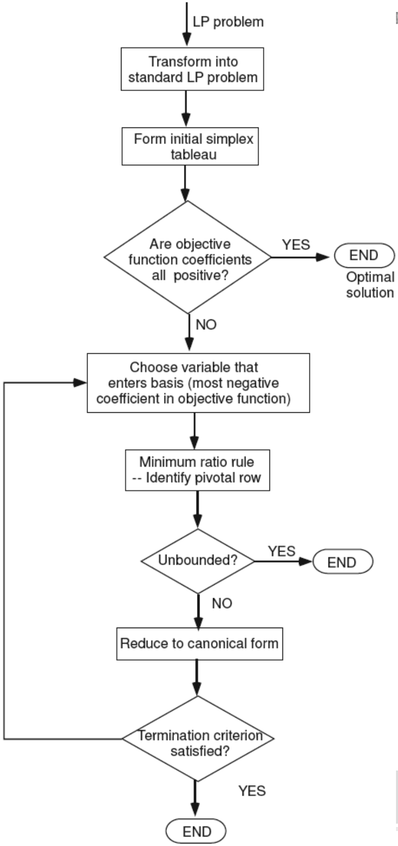
	x_1	x_2	s_1	s_2	b
R_1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
R_2	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{7}{2}$
R_3	0	0	$\frac{1}{2}$	$\frac{3}{2}$	$f+\frac{13}{2}$

SIMPLEX ALGORITHM

- The progression of the iterations of the simplex method.



SIMPLEX ALGORITHM



Other Special Cases of Simplex Method

- The discussion of simplex method in the previous subsection assumes that the LP problem at hand can be reduced into the standard formulation.
- In some cases, it may not be directly possible to do so.
- One requirement of the standard LP problem is that the variables must be non-negative, and the right hand side constants for constraints must be non-negative.
- In some problems, obtaining a standard LP problem is not straightforward.
- In such cases, use of so-called artificial variables is employed. Variations of simplex method known as two-phase simplex method and dual simplex method are used.

ENGINEERING EXAMPLES

Example 3.2 A manufacturing firm produces two machine parts using lathes, milling machines, and grinding machines. The different machining times required for each part, the machining times available on different machines, and the profit on each machine part are given in the following table.

	Machining time required (min)		
Type of machine	Machine part I	Machine part II	Maximum time available per week (min)
Lathes	10	5	2500
Milling machines	4	10	2000
Grinding machines	1	1.5	450
Profit per unit	\$50	\$100	

Determine the number of parts I and II to be manufactured per week to maximize the profit.

ENGINEERING EXAMPLES

SOLUTION Let the number of machine parts I and II manufactured per week be denoted by x and y, respectively. The constraints due to the maximum time limitations on the various machines are given by

$$10x + 5y \leq 2500 \tag{E$_1$}$$

$$4x + 10y \leq 2000 \tag{E$_2$}$$

$$x + 1.5y \leq 450 \tag{E$_3$}$$

Since the variables x and y cannot take negative values, we have

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \end{aligned} \tag{E$_4$}$$

The total profit is given by

$$f(x, y) = 50x + 100y \tag{E$_5$}$$

ENGINEERING EXAMPLES

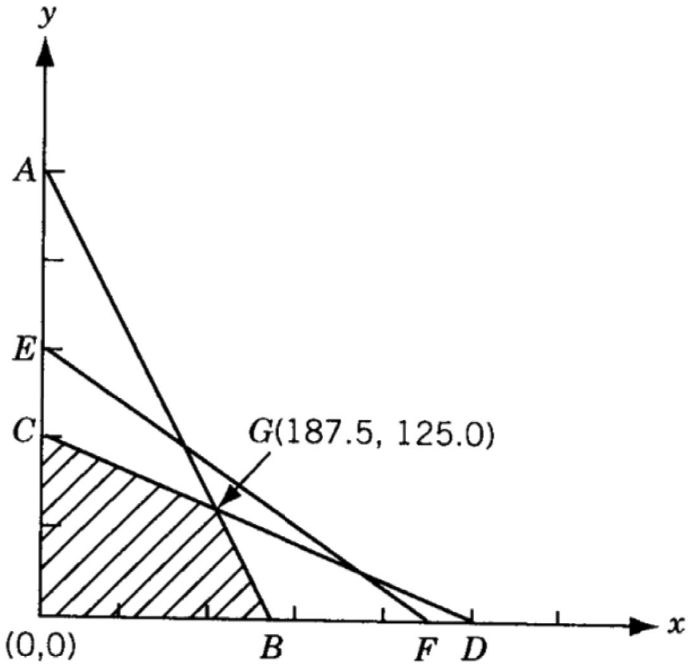


Figure 3.3 Feasible region given by Eqs. (E_1) to (E_4).

BASIC SOLUTIONS

Example 3.3 Find all the basic solutions corresponding to the system of equations

$$2x_1 + 3x_2 - 2x_3 - 7x_4 = 1 \qquad (\mathrm{I}_0)$$

$$x_1 + x_2 + x_3 + 3x_4 = 6 \qquad (\mathrm{II}_0)$$

$$x_1 - x_2 + x_3 + 5x_4 = 4 \qquad (\mathrm{III}_0)$$

BASIC SOLUTIONS

SOLUTION First we reduce the system of equations into a canonical form with x_1, x_2, and x_3 as basic variables. For this, first we pivot on the element $a_{11} = 2$ to obtain

$$
\begin{aligned}
x_1 + \tfrac{3}{2}x_2 - x_3 - \tfrac{7}{2}x_4 &= \tfrac{1}{2} & \mathrm{I}_1 &= \tfrac{1}{2}\mathrm{I}_0 \\
0 - \tfrac{1}{2}x_2 + 2x_3 + \tfrac{13}{2}x_4 &= \tfrac{11}{2} & \mathrm{II}_1 &= \mathrm{II}_0 - \mathrm{I}_1 \\
0 - \tfrac{5}{2}x_2 + 2x_3 + \tfrac{17}{2}x_4 &= \tfrac{7}{2} & \mathrm{III}_1 &= \mathrm{III}_0 - \mathrm{I}_1
\end{aligned}
$$

Then we pivot on $a'_{22} = -\frac{1}{2}$, to obtain

$$
\begin{aligned}
x_1 + 0 + 5x_3 + 16x_4 &= 17 & \mathrm{I}_2 &= \mathrm{I}_1 - \tfrac{3}{2}\mathrm{II}_2 \\
0 + x_2 - 4x_3 - 13x_4 &= -11 & \mathrm{II}_2 &= -2\mathrm{II}_1 \\
0 + 0 - 8x_3 - 24x_4 &= -24 & \mathrm{III}_2 &= \mathrm{III}_1 + \tfrac{5}{2}\mathrm{II}_2
\end{aligned}
$$

BASIC SOLUTIONS

Finally we pivot on a'_{33} to obtain the required canonical form as

$$\begin{array}{llll} x_1 \qquad + x_4 = 2 & \mathrm{I}_3 = \mathrm{I}_2 - 5\mathrm{III}_3 \\ \quad x_2 \quad - x_4 = 1 & \mathrm{II}_3 = \mathrm{II}_2 + 4\mathrm{III}_3 \\ \qquad x_3 + 3x_4 = 3 & \mathrm{III}_3 = -\frac{1}{8}\,\mathrm{III}_2 \end{array}$$

From this canonical form, we can readily write the solution of x_1, x_2, and x_3 in terms of the other variable x_4 as

$$x_1 = 2 - x_4$$

$$x_2 = 1 + x_4$$

$$x_3 = 3 - 3x_4$$

If Eqs. (I_0), (II_0), and (III_0) are the constraints of a linear programming problem, the solution obtained by setting the independent variable equal to zero is called a basic solution. In the present case, the basic solution is given by

$$x_1 = 2, \quad x_2 = 1, \quad x_3 = 3 \quad \text{(basic variables)}$$

and $x_4 = 0$ (nonbasic or independent variable). Since this basic solution has all $x_j \geq 0$ ($j = 1, 2, 3, 4$), it is a basic feasible solution.

BASIC SOLUTIONS

$$
\begin{aligned}
x_1 \qquad - \tfrac{1}{3}x_3 &= 1 \qquad \mathrm{I}_4 = \mathrm{I}_3 - \mathrm{III}_4 \\
x_2 \quad + \tfrac{1}{3}x_3 &= 2 \qquad \mathrm{II}_4 = \mathrm{II}_3 + \mathrm{III}_4 \\
x_4 + \tfrac{1}{3}x_3 &= 1 \qquad \mathrm{III}_4 = \tfrac{1}{3}\mathrm{III}_3
\end{aligned}
$$

This canonical system gives the solution of x_1, x_2, and x_4 in terms of x_3 as

$$
\begin{aligned}
x_1 &= 1 + \tfrac{1}{3}x_3 \\
x_2 &= 2 - \tfrac{1}{3}x_3 \\
x_4 &= 1 - \tfrac{1}{3}x_3
\end{aligned}
$$

and the corresponding basic solution is given by

$$x_1 = 1, \quad x_2 = 2, \quad x_4 = 1 \quad \text{(basic variables)}$$

$$x_3 = 0 \quad \text{(nonbasic variable)}$$

BASIC SOLUTIONS

$$
\begin{aligned}
x_1 \quad\quad + x_2 &= 3 \qquad \mathrm{I}_5 = \mathrm{I}_4 + \tfrac{1}{3}\mathrm{II}_5 \\
x_3 \quad + 3x_2 &= 6 \qquad \mathrm{II}_5 = 3\mathrm{II}_4 \\
x_4 - x_2 &= -1 \quad \mathrm{III}_5 = \mathrm{III}_4 - \tfrac{1}{3}\mathrm{II}_5
\end{aligned}
$$

The solution for x_1, x_3, and x_4 is given by

$$
\begin{aligned}
x_1 &= 3 - x_2 \\
x_3 &= 6 - 3x_2 \\
x_4 &= -1 + x_2
\end{aligned}
$$

from which the basic solution can be obtained as

$$
\begin{aligned}
&x_1 = 3, \quad x_3 = 6, \quad x_4 = -1 \quad \text{(basic variables)} \\
&x_2 = 0 \quad \text{(nonbasic variable)}
\end{aligned}
$$

Since all the x_j are not nonnegative, this basic solution is not feasible.

BASIC SOLUTIONS

Finally, to obtain the canonical form in terms of the basic variables x_2, x_3, and x_4, we pivot on a''_{12} in Eq. (I_5), thereby bringing x_2 into the current basis in place of x_1. This gives

$$\begin{aligned} x_2 \qquad + x_1 &= 3 & \mathrm{I}_6 &= \mathrm{I}_5 \\ x_3 \quad - 3x_1 &= -3 & \mathrm{II}_6 &= \mathrm{II}_5 - 3\mathrm{I}_6 \\ x_4 + x_1 &= 2 & \mathrm{III}_6 &= \mathrm{III}_5 + \mathrm{I}_6 \end{aligned}$$

This canonical form gives the solution for x_2, x_3, and x_4 in terms of x_1 as

$$x_2 = 3 - x_1$$

$$x_3 = -3 + 3x_1$$

$$x_4 = 2 - x_1$$

and the corresponding basic solution is

$$x_2 = 3, \quad x_3 = -3, \quad x_4 = 2 \quad \text{(basic variables)}$$

$$x_1 = 0 \quad \text{(nonbasic variable)}$$

This basic solution can also be seen to be infeasible due to the negative value for x_3.

IDENTIFYING AN OPTIMAL POINT

Theorem 3.7 A basic feasible solution is an optimal solution with a minimum objective function value of f_0'' if all the cost coefficients c_j'', $j = m+1, m+2, \ldots, n$, in Eqs. (3.21) are nonnegative.

Proof: From the last row of Eqs. (3.21), we can write that

$$f_0'' + \sum_{i=m+1}^{n} c_i'' x_i = f \tag{3.24}$$

IDENTIFYING AN OPTIMAL POINT

Since the variables $x_{m+1}, x_{m+2}, \ldots, x_n$ are presently zero and are constrained to be nonnegative, the only way any one of them can change is to become positive. But if $c_i'' > 0$ for $i = m+1, m+2, \ldots, n$, then increasing any x_i cannot decrease the value of the objective function f. Since no change in the nonbasic variables can cause f to decrease, the present solution must be optimal with the optimal value of f equal to f_0''.

A glance over c_i'' can also tell us if there are multiple optima. Let all $c_i'' > 0$, $i = m+1, m+2, \ldots, k-1, k+1, \ldots, n$, and let $c_k'' = 0$ for some nonbasic variable x_k. Then if the constraints allow that variable to be made positive (from its present value of zero), no change in f results, and there are multiple optima. It is possible, however, that the variable may not be allowed by the constraints to become positive; this may occur in the case of degenerate solutions. Thus as a corollary to the discussion above, we can state that a basic feasible solution is the unique optimal feasible solution if $c_j'' > 0$ for all nonbasic variables x_j, $j = m+1, m+2, \ldots, n$. If, after testing for optimality, the current basic feasible solution is found to be nonoptimal, an improved basic solution is obtained from the present canonical form as follows.

ADVANCED CONCEPTS

- **Two Phases of the Simplex Method**
- **Duality**
- **Transportation Problem**
- **Interior point methods.**

TWO-PHASE SIMPLEX METHOD

The problem is to find nonnegative values for the variables $x_1, x_2, \ldots, x_n$ that satisfy the equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{3.32}$$

and minimize the objective function given by

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = f \tag{3.33}$$

The general problems encountered in solving this problem are

1. An initial feasible canonical form may not be readily available. This is the case when the linear programming problem does not have slack variables for some of the equations or when the slack variables have negative coefficients.
2. The problem may have redundancies and/or inconsistencies, and may not be solvable in nonnegative numbers.

TWO-PHASE SIMPLEX METHOD

- Phase I of the simplex method uses the simplex algorithm itself to find whether the linear programming problem has a feasible solution. If a feasible solution exists, it provides a basic feasible solution in canonical form ready to initiate phase II of the method.
- Phase II, in turn, uses the simplex algorithm to find whether the problem has a bounded optimum. If a bounded optimum exists, it finds the basic feasible solution that is optimal.

TWO-PHASE SIMPLEX METHOD

1. Arrange the original system of Eqs. (3.32) so that all constant terms b_i are positive or zero by changing, where necessary, the signs on both sides of any of the equations.
2. Introduce to this system a set of artificial variables $y_1, y_2, \ldots, y_m$ (which serve as basic variables in phase I), where each $y_i \geq 0$, so that it becomes

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + y_1 \qquad\qquad &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \qquad + y_2 \qquad &= b_2 \\
&\ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \qquad\qquad + y_m &= b_m \\
b_i \geq 0 &
\end{aligned} \tag{3.34}$$

Note that in Eqs. (3.34), for a particular i, the a_{ij}'s and the b_i may be the negative of what they were in Eq. (3.32) because of step 1.

The objective function of Eq. (3.33) can be written as

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n + (-f) = 0 \tag{3.35}$$

3. *Phase I of the method*. Define a quantity w as the sum of the artificial variables

$$w = y_1 + y_2 + \cdots + y_m \tag{3.36}$$

and use the simplex algorithm to find $x_i \geq 0$ $(i = 1, 2, \ldots, n)$ and $y_i \geq 0$ $(i = 1, 2, \ldots, m)$ which minimize w and satisfy Eqs. (3.34) and (3.35). Consequently, consider the array

$$\begin{array}{llll}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + y_1 & & = & b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \quad + y_2 & & = & b_2 \\
\vdots & & \vdots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \qquad + y_m & & = & b_m \\
c_1x_1 + c_2x_2 + \cdots + c_nx_n & +(-f) & = & 0 \\
\qquad y_1 + y_2 + \cdots + y_m & +(-w) & = & 0
\end{array} \tag{3.37}$$

This array is not in canonical form; however, it can be rewritten as a canonical system with basic variables $y_1, y_2, \ldots, y_m, -f$, and $-w$ by subtracting the sum of the first m equations from the last to obtain the new system

$$\begin{array}{llll}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + y_1 & & = & b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \quad + y_2 & & = & b_2 \\
\vdots & & \vdots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \qquad + y_m & & = & b_m \\
c_1x_1 + c_2x_2 + \cdots + c_nx_n & +(-f) & = & 0 \\
d_1x_1 + d_2x_2 + \cdots + d_nx_n & +(-w) & = & -w_0
\end{array} \tag{3.38}$$

TWO-PHASE SIMPLEX METHOD

where

$$d_i = -(a_{1i} + a_{2i} + \cdots + a_{mi}), \quad i = 1, 2, \ldots, n \tag{3.39}$$

$$-w_0 = -(b_1 + b_2 + \cdots + b_m) \tag{3.40}$$

Equations (3.38) provide the initial basic feasible solution that is necessary for starting phase I.

4. In Eq. (3.37), the expression of w, in terms of the artificial variables $y_1, y_2, \ldots, y_m$ is known as the *infeasibility form*. w has the property that if as a result of phase I, with a minimum of $w > 0$, no feasible solution exists for the original linear programming problem stated in Eqs. (3.32) and (3.33), and thus the procedure is terminated. On the other hand, if the minimum of $w = 0$, the resulting array will be in canonical form and hence initiate phase II by eliminating the w equation as well as the columns corresponding to each of the artificial variables $y_1, y_2, \ldots, y_m$ from the array.

5. *Phase II of the method*. Apply the simplex algorithm to the adjusted canonical system at the end of phase I to obtain a solution, if a finite one exists, which optimizes the value of f.

TWO-PHASE SIMPLEX METHOD

Example 3.7

$$\text{Minimize } f = 2x_1 + 3x_2 + 2x_3 - x_4 + x_5$$

subject to the constraints

$$3x_1 - 3x_2 + 4x_3 + 2x_4 - x_5 = 0$$
$$x_1 + x_2 + x_3 + 3x_4 + x_5 = 2$$
$$x_i \geq 0, \quad i = 1 \text{ to } 5$$

SOLUTION

Step 1 As the constants on the right-hand side of the constraints are already nonnegative, the application of step 1 is unnecessary.

Step 2 Introducing the artificial variables $y_1 \geq 0$ and $y_2 \geq 0$, the equations can be written as follows:

$$\begin{aligned} 3x_1 - 3x_2 + 4x_3 + 2x_4 - x_5 + y_1 \qquad\qquad &= 0 \\ x_1 + x_2 + x_3 + 3x_4 + x_5 \qquad + y_2 \qquad &= 2 \qquad (E_1) \\ 2x_1 + 3x_2 + 2x_3 - x_4 + x_5 \qquad\qquad - f &= 0 \end{aligned}$$

Step 3 By defining the infeasibility form w as

$$w = y_1 + y_2$$

TWO-PHASE SIMPLEX METHOD

This array can be rewritten as a canonical system with basic variables as y_1, y_2, $-f$, and $-w$ by subtracting the sum of the first two equations of (E_2) from the last equation of (E_2). Thus the last equation of (E_2) becomes

$$-4x_1 + 2x_2 - 5x_3 - 5x_4 + 0x_5 - w = -2 \qquad (E_3)$$

Since this canonical system [first three equations of (E_2), and (E_3)] provides an initial basic feasible solution, phase I of the simplex method can be started. The phase I computations are shown below in tableau form.

Basic variables	Admissible variables x_1	x_2	x_3	x_4	x_5	Artificial variables y_1	y_2	b''_i	Value of b''_i/a''_{is} for $a''_{is} > 0$	
y_1	3	−3	4	2 Pivot element	−1	1	0	0	0	← Smaller value (y_1 drops from next basis)
y_2	1	1	1	3	1	0	1	2	$\frac{2}{3}$	
$-f$	2	3	2	−1	1	0	0	0		
$-w$	−4	2	−5	−5	0	0	0	−2		
			↑	↑						
			Most negative							

Since there is a tie between d''_3 and d''_4, d''_4 is selected arbitrarily as the most negative d''_i for pivoting (x_4 enters the next basis).
Result of pivoting:

TWO-PHASE SIMPLEX METHOD

x_4	$\frac{3}{2}$	$-\frac{3}{2}$	2	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0		
y_2	$-\frac{7}{2}$	$\boxed{\frac{11}{2}}$ Pivot element	-5	0	$\frac{5}{2}$	$-\frac{3}{2}$	1	2	$\frac{1}{11}$	$\leftarrow$ y_2 drops from next basis
$-f$	$\frac{7}{2}$	$\frac{3}{2}$	4	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0		
$-w$	$\frac{7}{2}$	$-\frac{11}{2}$	5	0	$-\frac{5}{2}$	$\frac{5}{2}$	0	-2		

$\uparrow$

Most negative d_i'' (x_2 enters next basis)

Result of pivoting (since y_1 and y_2 are dropped from basis, the columns corresponding to them need not be filled):

x_4	$\frac{6}{11}$	0	$\frac{7}{11}$	1	$\frac{2}{11}$	Dropped	$\frac{6}{11}$	$\frac{6}{2}$
x_2	$-\frac{7}{11}$	1	$-\frac{10}{11}$	0	$\frac{5}{11}$		$\frac{4}{11}$	$\frac{4}{5}$
$-f$	$\frac{98}{22}$	0	$\frac{118}{22}$	0	$-\frac{4}{22}$		$-\frac{6}{11}$	
$-w$	0	0	0	0	0		0	

TWO-PHASE SIMPLEX METHOD

Step 4 At this stage we notice that the present basic feasible solution does not contain any of the artificial variables y_1 and y_2, and also the value of w is reduced to 0. This indicates that phase I is completed.

Step 5 Now we start phase II computations by dropping the w row from further consideration. The results of phase II are again shown in tableau form:

Basic variables	Original variables					Constant	Value of b_i''/a_{is}'' for
	x_1	x_2	x_3	x_4	x_5	b_i''	$a_{is}''>0$
x_4	$\frac{6}{11}$	0	$\frac{7}{11}$	1	$\frac{2}{11}$	$\frac{6}{11}$	$\frac{6}{2}$
x_2	$-\frac{7}{11}$	1	$-\frac{10}{11}$	0	$\boxed{\frac{5}{11}}$ Pivot element	$\frac{4}{11}$	$\frac{4}{5}$ ← Smaller value (x_2 drops from next basis)
$-f$	$\frac{98}{22}$	0	$\frac{118}{22}$	0	$-\frac{4}{22}$	$-\frac{6}{11}$	

↑ Most negative c_i'' (x_5 enters next basis)

Result of pivoting:

x_4	$\frac{4}{5}$	$-\frac{2}{5}$	1	1	0	$\frac{2}{5}$
x_5	$-\frac{7}{5}$	$\frac{11}{5}$	-2	0	1	$\frac{4}{5}$
$-f$	$\frac{21}{5}$	$\frac{2}{5}$	5	0	0	$-\frac{2}{5}$

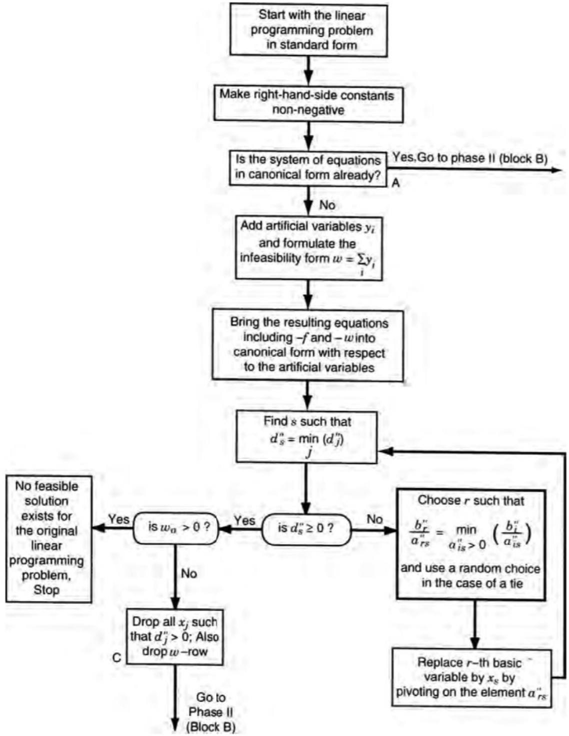
Now, since all c_i'' are nonnegative, phase II is completed. The (unique) optimal solution is given by

$$x_1 = x_2 = x_3 = 0 \quad \text{(nonbasic variables)}$$

$$x_4 = \tfrac{2}{5}, \quad x_5 = \tfrac{4}{5} \quad \text{(basic variables)}$$

$$f_{\min} = \tfrac{2}{5}$$

TWO-PHASE SIMPLEX METHOD



TWO-PHASE SIMPLEX METHOD

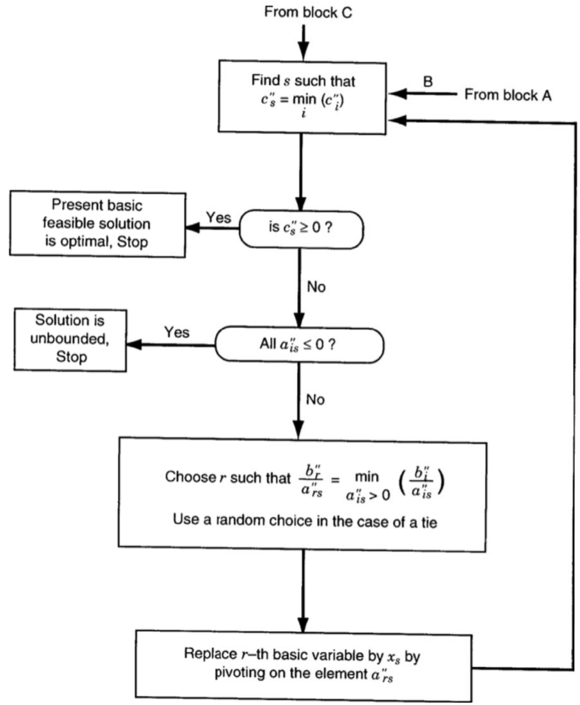


Figure 3.15 (*continued*)

DUALITY

- **Duality** is an important concept in linear programming problems.
- Each LP problem, known as the primal, had another corresponding LP problem, known as the dual, associated with it.
- The solutions of the primal and dual problems have interesting relationships.

DUALITY

Primal Problem.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \\
c_1x_1 + c_2x_2 + \cdots + c_nx_n &= f
\end{aligned}
\tag{4.17}
$$

$(x_i \geq 0, i = 1 \text{ to } n, \text{ and } f \text{ is to be minimized})$

Dual Problem. As a definition, the dual problem can be formulated by transposing the rows and columns of Eq. (4.17) including the right-hand side and the objective function, reversing the inequalities and maximizing instead of minimizing. Thus by denoting the dual variables as $y_1, y_2, \ldots, y_m$, the dual problem becomes

$$
\begin{aligned}
a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m &\leq c_1 \\
a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}x_m &\leq c_2 \\
&\vdots \\
a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m &\leq c_n \\
b_1y_1 + b_2y_2 + \cdots + b_my_m &= v
\end{aligned}
\tag{4.18}
$$

$(y_i \geq 0, i = 1 \text{ to } m, \text{ and } v \text{ is to be maximized})$

Equations (4.17) and (4.18) are called *symmetric primal–dual pairs* and it is easy to see from these relations that the dual of the dual is the primal.

DUALITY

Primal problem

$$\min_{x} \quad z = c^T x$$

$$\text{such that}$$

$$Ax \leq b$$

$$x \geq 0$$

The <u>corresponding dual problem</u>

$$\max_{x} \quad z_d = b^T y$$

$$\text{such that}$$

$$A^T y \geq c$$

$$y \geq 0$$

If the primal is a minimization problem, the dual will be a maximization problem, and vice versa.

DUALITY

Example:

The primal problem

$$\min_{x} \quad z = x_1 - x_2 - 2x_3 + 4x_4$$

such that

$$x_1 + 5x_2 - 2x_3 + 3x_4 \leq 1$$
$$5x_1 + x_2 + 3x_3 + 8x_4 \leq 5$$
$$x_1 + 2x_2 + 3x_3 + 5x_4 \leq 3$$
$$x_1, x_2, x_3, x_4 \geq 0$$

The dual problem

$$\max_{y} \quad z_d = y_1 + 5y_2 + 3y_3$$

such that

$$y_1 + 5y_2 + y_3 \geq 1$$
$$5y_1 + y_2 + 2y_3 \geq -1$$
$$-2y_1 + 3y_2 + 3y_3 \geq -2$$
$$3y_1 + 8y_2 + 5y_3 \geq 4$$
$$y_1, y_2, y_3 \geq 0$$

- The primal problem had three inequality constraints, whereas the dual problem has three design variables.
- The primal problem had four design variables, and the dual problem has four inequality constraints.

DUALITY

Example:

The primal problem

$$\min_{x} \quad z = x_1 - x_2 - 2x_3 + 4x_4$$

such that

$$x_1 + 5x_2 - 2x_3 + 3x_4 \leq 1$$
$$5x_1 + x_2 + 3x_3 + 8x_4 \leq 5$$
$$x_1 + 2x_2 + 3x_3 + 5x_4 \leq 3$$
$$x_1, x_2, x_3, x_4 \geq 0$$

The dual problem

$$\max_{y} \quad z_d = y_1 + 5y_2 + 3y_3$$

such that

$$y_1 + 5y_2 + y_3 \geq 1$$
$$5y_1 + y_2 + 2y_3 \geq -1$$
$$-2y_1 + 3y_2 + 3y_3 \geq -2$$
$$3y_1 + 8y_2 + 5y_3 \geq 4$$
$$y_1, y_2, y_3 \geq 0$$

- The primal is a minimization problem, and the dual is a maximization problem, and
- The primal inequalities are of the form ≤ 0, whereas the dual constraints are of the form ≥ 0.

DUALITY

Table 4.11 Correspondence Rules for Primal–Dual Relations

Primal quantity	Corresponding dual quantity
Objective function: Minimize $\mathbf{c}^{\mathrm{T}}\mathbf{X}$	Maximize $\mathbf{Y}^{\mathrm{T}}\mathbf{b}$
Variable $x_i \geq 0$	ith constraint $\mathbf{Y}^{\mathrm{T}}\mathbf{A}_i \leq c_i$ (inequality)
Variable x_i unrestricted in sign	ith constraint $\mathbf{Y}^{\mathrm{T}}\mathbf{A}_i = c_i$ (equality)
jth constraint, $\mathbf{A}_j\mathbf{X} = b_j$ (equality)	jth variable y_j unrestricted in sign
jth constraint, $\mathbf{A}_j\mathbf{X} \geq b_j$ (inequality)	jth variable $y_j \geq 0$
Coefficient matrix $\mathbf{A} \equiv [\mathbf{A}_1 \ldots \mathbf{A}_m]$	Coefficient matrix $\mathbf{A}^{\mathrm{T}} \equiv [\mathbf{A}_1, \ldots, \mathbf{A}_m]^{\mathrm{T}}$
Right-hand-side vector $\mathbf{b}$	Right-hand-side vector $\mathbf{c}$
Cost coefficients $\mathbf{c}$	Cost coefficients $\mathbf{b}$

DUALITY

Table 4.12 Primal–Dual Relations

Primal problem	Corresponding dual problem
Minimize $f = \sum_{i=1}^{n} c_i x_i$ subject to	Maximize $v = \sum_{i=1}^{m} y_i b_i$ subject to
$\sum_{j=1}^{n} a_{ij} x_j = b_i, \ i = 1, 2, \ldots, \ m^*$	$\sum_{i=1}^{m} y_i a_{ij} = c_j, \ j = n^* + 1, \ n^* + 2, \ldots, n$
$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \ i = m^* + 1, \ m^* + 2, \ldots, m$	$\sum_{i=1}^{m} y_i a_{ij} \leq c_j, j = 1, 2, \ldots, n^*$
where	where
$x_i \geq 0, i = 1, 2, \ldots, n^*;$	$y_i \geq 0, i = m^* + 1, m^* + 2, \ldots, m;$
and	and
x_i unrestricted in sign, $i = n^* + 1, n^* + 2, \ldots, n$	y_i unrestricted in sign, $i = 1, 2, \ldots, m^*$

PRIMAL-DUAL RELATIONSHIPS

1. The dual of a dual problem is the primal problem.
2. Every feasible solution for the dual problem provides a lower bound on every feasible solution to the primal.
3. If the primal solution has a feasible solution, and the dual problem has a feasible solution, then there exist optimal feasible solutions such that the objective function values of the primal and the dual are the same. This is known as the strong duality theorem.
4. The objective function value of the dual problem evaluated at any feasible solution provides a lower bound on the objective function value of the primal problem evaluated at any feasible solution. **This is known as the weak duality theorem**.
5. If the primal problem is unbounded, the dual problem is infeasible.

DUAL SIMPLEX EXAMPLE

- Computationally, the dual simplex algorithm also involves a sequence of pivot operations, but with different rules (compared to the regular simplex method) for choosing the pivot element.

Let the problem to be solved be initially in canonical form with some of the $\overline{b}_i < 0$, the relative cost coefficients corresponding to the basic variables $\overline{c}_j = 0$, and all other $\overline{c}_j \geq 0$. Since some of the $\overline{b}_i$ are negative, the primal solution will be infeasible, and since all $\overline{c}_j \geq 0$, the corresponding dual solution will be feasible. Then the simplex method works according to the following iterative steps.

DUAL SIMPLEX EXAMPLE

1. Select row r as the pivot row such that

$$\bar{b}_r = \min \bar{b}_i < 0 \qquad (4.22)$$

2. Select column s as the pivot column such that

$$\frac{\bar{c}_s}{-\bar{a}_{rs}} = \min_{\bar{a}_{rj}<0} \left(\frac{\bar{c}_j}{-\bar{a}_{rj}} \right) \qquad (4.23)$$

   If all $\bar{a}_{rj} \geq 0$, the primal will not have any feasible (optimal) solution.

3. Carry out a pivot operation on $\bar{a}_{rs}$
4. Test for optimality: If all $\bar{b}_i \geq 0$, the current solution is optimal and hence stop the iterative procedure. Otherwise, go to step 1.

DUAL SIMPLEX EXAMPLE

$$\text{Minimize } f = 20x_1 + 16x_2$$

subject to

$$x_1 \geq 2.5$$
$$x_2 \geq 6$$
$$2x_1 + x_2 \geq 17$$
$$x_1 + x_2 \geq 12$$
$$x_1 \geq 0, \ x_2 \geq 0$$

SOLUTION By introducing the surplus variables x_3, x_4, x_5, and x_6, the problem can be stated in canonical form as
Minimize f
with

$$\begin{array}{llllllll}
-x_1 & & +x_3 & & & & & = -2.5 \\
 & -x_2 & & +x_4 & & & & = -6 \\
-2x_1 & -x_2 & & & +x_5 & & & = -17 \\
-x_1 & -x_2 & & & & +x_6 & & = -12 \\
20x_1 & +16x_2 & & & & & -f & = 0
\end{array} \qquad (\text{E}_1)$$

$$x_i \geq 0, \quad i = 1 \text{ to } 6$$

DUAL SIMPLEX EXAMPLE

The basic solution corresponding to (E_1) is infeasible since $x_3 = -2.5$, $x_4 = -6$, $x_5 = -17$, and $x_6 = -12$. However, the objective equation shows optimality since the cost coefficients corresponding to the nonbasic variables are nonnegative ($\bar{c}_1 = 20$, $\bar{c}_2 = 16$). This shows that the solution is infeasible to the primal but feasible to the dual. Hence the dual simplex method can be applied to solve this problem as follows.

DUAL SIMPLEX EXAMPLE

Step 1 Write the system of equations (E_1) in tableau form:

Basic variables	Variables							
	x_1	x_2	x_3	x_4	x_5	x_6	$-f$	$\overline{b}_i$
x_3	−1	0	1	0	0	0	0	−2.5
x_4	0	−1	0	1	0	0	0	−6
x_5	$\boxed{-2}$ Pivot element	−1	0	0	1	0	0	−17 ← Minimum, pivot row
x_6	−1	−1	0	0	0	1	0	−12
$-f$	20	16	0	0	0	0	1	0

Select the pivotal row r such that

$$\overline{b}_r = \min(\overline{b}_i < 0) = \overline{b}_3 = -17$$

in this case. Hence $r = 3$.

DUAL SIMPLEX EXAMPLE

Step 2 Select the pivotal column s as

$$\frac{\bar{c}_s}{-\bar{a}_{rs}} = \min_{\bar{a}_{rj}<0} \left(\frac{\bar{c}_j}{-\bar{a}_{rj}} \right)$$

Since

$$\frac{\bar{c}_1}{-\bar{a}_{31}} = \frac{20}{2} = 10, \quad \frac{\bar{c}_2}{-\bar{a}_{32}} = \frac{16}{1} = 16, \quad \text{and} \quad s = 1$$

DUAL SIMPLEX EXAMPLE

Step 3 The pivot operation is carried on $\bar{a}_{31}$ in the preceding table, and the result is as follows:

Basic variables	Variables x_1	x_2	x_3	x_4	x_5	x_6	$-f$	$\bar{b}_i$
x_3	0	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	0	6
x_4	0	$\boxed{-1}$ Pivot element	0	1	0	0	0	$-6 \leftarrow$ Minimum, pivot row
x_1	1	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	$\frac{17}{2}$
x_6	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	1	0	$-\frac{7}{2}$
$-f$	0	6	0	0	10	0	1	-170

Step 4 Since some of the $\bar{b}_i$ are < 0, the present solution is not optimum. Hence we proceed to the next iteration.

DUAL SIMPLEX EXAMPLE

Step 1 The pivot row corresponding to minimum ($\bar{b}_i < 0$) can be seen to be 2 in the preceding table.

Step 2 Since $\bar{a}_{22}$ is the only negative coefficient, it is taken as the pivot element.

Step 3 The result of pivot operation on $\bar{a}_{22}$ in the preceding table is as follows:

Basic variables	Variables							
	x_1	x_2	x_3	x_4	x_5	x_6	$-f$	$\bar{b}_i$
x_3	0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	3
x_2	0	1	0	-1	0	0	0	6
x_1	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{11}{2}$
x_6	0	0	0	$\boxed{-\frac{1}{2}}$ Pivot element	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$ ← Minimum, pivot row
$-f$	0	0	0	6	10	0	1	-206

Step 4 Since all $\bar{b}_i$ are not ≥ 0, the present solution is not optimum. Hence we go to the next iteration.

DUAL SIMPLEX EXAMPLE

Step 1 The pivot row (corresponding to minimum $\bar{b}_i \leq 0$) can be seen to be the fourth row.

Step 2 Since

$$\frac{\bar{c}_4}{-\bar{a}_{44}} = 12 \quad \text{and} \quad \frac{\bar{c}_5}{-\bar{a}_{45}} = 20$$

the pivot column is selected as $s = 4$.

Step 3 The pivot operation is carried on $\bar{a}_{44}$ in the preceding table, and the result is as follows:

Basic variables	Variables							
	x_1	x_2	x_3	x_4	x_5	x_6	$-f$	$\bar{b}_i$
x_3	0	0	1	0	−1	1	0	$\frac{5}{2}$
x_2	0	1	0	0	1	−2	0	7
x_1	1	0	0	0	−1	1	0	5
x_4	0	0	0	1	1	−2	0	1
$-f$	0	0	0	0	4	12	1	−212

DUAL SIMPLEX EXAMPLE

Step 4 Since all $\overline{b}_i$ are ≥ 0, the present solution is dual optimal and primal feasible. The solution is

$$x_1 = 5, \quad x_2 = 7, \quad x_3 = \tfrac{5}{2}, \quad x_4 = 1 \quad \text{(dual basic variables)}$$

$$x_5 = x_6 = 0 \quad \text{(dual nonbasic variables)}$$

$$f_{\min} = 212$$

$$\text{Minimize } f = 20x_1 + 16x_2$$

subject to

$$x_1 \geq 2.5$$

$$x_2 \geq 6$$

$$2x_1 + x_2 \geq 17$$

$$x_1 + x_2 \geq 12$$

$$x_1 \geq 0, \; x_2 \geq 0$$

TRANSPORTATION PROBLEM

Suppose that there are m origins $R_1, R_2, \cdots, R_m$ (e.g., warehouses) and n destinations, $D_1, D_2, \cdots, D_n$ (e.g., factories). Let a_i be the amount of a commodity available at origin i ($i = 1, 2, \ldots, m$) and b_j be the amount required at destination j ($j = 1, 2, \ldots, n$). Let c_{ij} be the cost per unit of transporting the commodity from origin i to destination j. The objective is to determine the amount of commodity (x_{ij}) transported from origin i to destination j such that the total transportation costs are minimized. This problem can be formulated mathematically as

$$\text{Minimize } f = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \tag{4.52}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, 2, \ldots, m \tag{4.53}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, 2, \ldots, n \tag{4.54}$$

$$x_{ij} \geq 0, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n \tag{4.55}$$

Clearly, this is a LP problem in mn variables and $m + n$ equality constraints.

TRANSPORTATION PROBLEM

Equations (4.53) state that the total amount of the commodity transported from the origin i to the various destinations must be equal to the amount available at origin i $(i = 1, 2, \ldots, m)$, while Eqs. (4.54) state that the total amount of the commodity received by destination j from all the sources must be equal to the amount required at the destination j $(j = 1, 2, \ldots, n)$. The nonnegativity conditions Eqs. (4.55) are added since negative values for any x_{ij} have no physical meaning. It is assumed that the total demand equals the total supply, that is,

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \tag{4.56}$$

Equation (4.56), called the *consistency condition*, must be satisfied if a solution is to exist. This can be seen easily since

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right) = \sum_{j=1}^{n} b_j \tag{4.57}$$

TRANSPORTATION PROBLEM

The special structure of the transportation matrix can be seen by writing the equations in standard form:

$$
\begin{aligned}
&x_{11} + x_{12} + \cdots + x_{1n} &&= a_1 \\
&\qquad x_{21} + x_{22} + \cdots + x_{2n} &&= a_2 \\
&\qquad\qquad \vdots &&\;\;\vdots \\
&\qquad\qquad\qquad x_{m1} + x_{m2} + \cdots + x_{mn} &&= a_m
\end{aligned}
\qquad (4.58a)
$$

$$
\begin{aligned}
&x_{11} \quad + x_{21} \quad + x_{m1} &&= b_1 \\
&\;x_{12} \quad + x_{22} \quad + x_{m2} &&= b_2 \\
&\;\;\vdots \qquad\quad \vdots \qquad\quad \vdots && \\
&\;\;x_{1n} \quad + x_{2n} \quad + x_{mn} &&= b_n
\end{aligned}
\qquad (4.58b)
$$

$$
\begin{aligned}
&c_{11}x_{11} + c_{12}x_{12} + \cdots + c_{1n}x_{1n} + c_{21}x_{21} + \cdots + c_{2n}x_{2n} + \cdots \\
&\qquad + c_{m1}x_{m1} + \cdots + c_{mn}x_{mn} = f
\end{aligned}
\qquad (4.58c)
$$

- All the nonzero coefficients of the constraints are equal to 1.
- The constraint coefficients appear in a triangular form.
- Any variable appears only once in the first *m* equations and once in the next *n* equations.

TRANSPORTATION PROBLEM

To facilitate the identification of a starting solution, the system of equations is represented in the form of an array, called the **transportation array**.

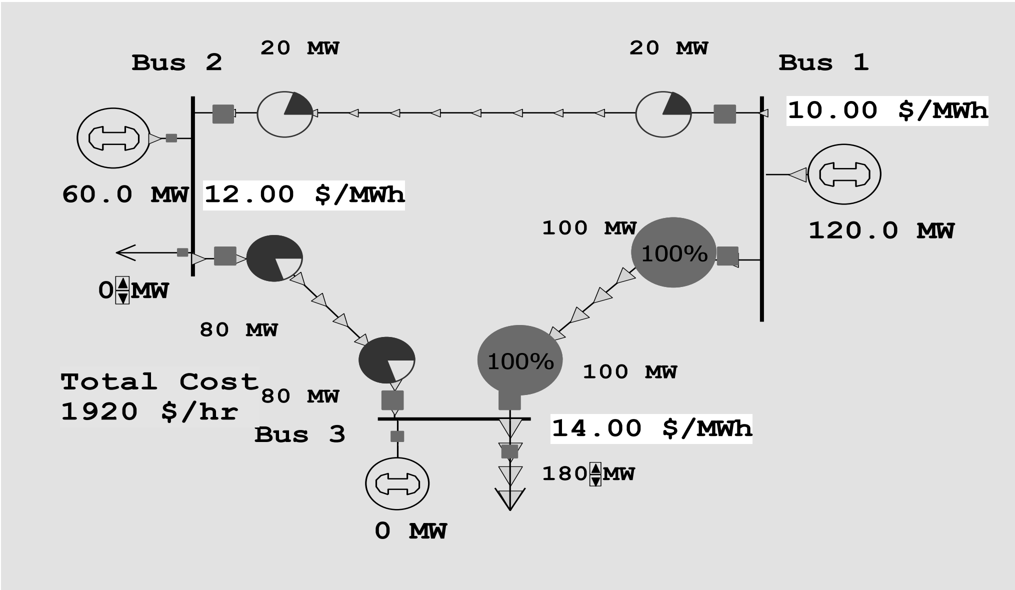
From \ To		Destination j: 1	2	3	$\cdots$	n	Amount available a_i
Origin i	1	x_{11} c_{11}	x_{12} c_{12}	x_{13} c_{13}	$\cdots$	x_{1n} c_{1n}	a_1
	2	x_{21} c_{21}	x_{22} c_{22}	x_{23} c_{23}	$\cdots$	x_{2n} c_{2n}	a_2
	3	x_{31} c_{31}	x_{32} c_{32}	x_{33} c_{33}	$\cdots$	x_{3n} c_{3n}	a_3
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	m	x_{m1} c_{m1}	x_{m2} c_{m2}	x_{m3} c_{m3}	$\cdots$	x_{mn} c_{mn}	a_m
Amount required b_j		b_1	b_2	b_3	$\cdots$	b_n	

TRANSPORTATION PROBLEM

- ***Computational Procedure***
  - Determine a starting basic feasible solution.
  - Test the current basic feasible solution for optimality. If the current solution is optimal, stop the iterative process; otherwise, go to step 3.
  - Select a variable to enter the basis from among the current nonbasic variables.
  - Select a variable to leave from the basis from among the current basic variables (using the feasibility condition).
  - Find a new basic feasible solution and return to step 2.

OPTIMAL POWER FLOW

- The Optimal Power Flow (OPF) model represents the problem of determining the best operating levels for electric power plants in order to meet demands given throughout a transmission network, usually with the objective of minimizing operating cost.



OPTIMAL POWER FLOW

Three generator controls P_1, P_2, P_3

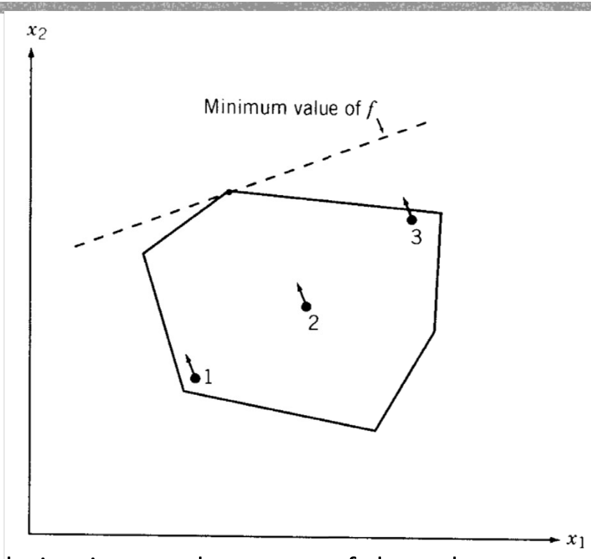
Incremental costs of 10, 12, 14 $/MWh, respectively

$$
\begin{aligned}
\text{min:} \quad & 10P_1 + 12P_2 + 14P_3 & & \\
\text{st:} \quad & P_1 + P_2 + P_3 & = 180 & \quad \text{Power Balance} \\
& 0.66P_1 + 0.33P_2 & \leq 100 & \quad \text{Line 1-3 Constraint} \\
& P_1, P_2, P_3 & \geq 0 &
\end{aligned}
$$

INTERIOR POINT METHODS

- For large scale problems, simplex method can become cumbersome and computationally expensive.
- There exist another class of solution approaches for LP problems known as the interior point methods.
- Interior point methods move across the feasible region, based on pre-defined criteria, such as feasibility of constraints or objective function value.
- The interior point method solved problems involving 150,000 design variables and 12,000 constraints in 1 hour, while the simplex method required 4 hours for solving a smaller problem involving only 36,000 design variables and 10,000 constraints.
- It was found that the interior method is as much as 50 times faster than the simplex method for large problems.

INTERIOR POINT METHODS



- If the current solution is near the center of the polytope, we can move along the steepest descent direction to reduce the value of f by a maximum amount.
- The current solution can be improved substantially by moving along the steepest descent direction if it is near the center (point 2) but not near the boundary point (points 1 and 3).

INTERIOR POINT METHODS

Karmarkar's method requires the LP problem in the following form:

$$\text{Minimize } f = \mathbf{c}^{\mathrm{T}}\mathbf{X}$$

subject to

$$\begin{aligned} [a]\mathbf{X} &= \mathbf{0} \\ x_1 + x_2 + \cdots + x_n &= 1 \qquad (4.59) \\ \mathbf{X} &\geq \mathbf{0} \end{aligned}$$

where $\mathbf{X} = \{x_1, x_2, \ldots, x_n\}^{\mathrm{T}}$, $\mathbf{c} = \{c_1, c_2, \ldots, c_n\}^{\mathrm{T}}$, and $[a]$ is an $m \times n$ matrix. In addition, an interior feasible starting solution to Eqs. (4.59) must be known. Usually,

$$\mathbf{X} = \left\{\frac{1}{n}, \frac{1}{n}, \cdots \frac{1}{n}\right\}^{\mathrm{T}}$$

is chosen as the starting point. In addition, the optimum value of f must be zero for the problem. Thus

$$\begin{aligned} \mathbf{X}^{(1)} &= \left\{\frac{1}{n} \quad \frac{1}{n} \cdots \frac{1}{n}\right\}^{\mathrm{T}} = \text{interior feasible} \\ f_{\min} &= 0 \end{aligned} \qquad (4.60)$$

- Conversion of an LP Problem into the Required Form

Let the given LP problem be of the form

$$\text{Minimize } \mathbf{d}^{\mathrm{T}}\mathbf{X}$$

subject to

$$[\alpha]\mathbf{X} = \mathbf{b}$$
$$\mathbf{X} \geq \mathbf{0} \tag{4.61}$$

To convert this problem into the form of Eq. (4.59), we use the procedure suggested in Ref. [4.20] and define integers m and n such that $\mathbf{X}$ will be an $(n-3)$-component vector and $[\alpha]$ will be a matrix of order $m-1 \times n-3$. We now define the vector $\bar{\mathbf{z}} = \{z_1, z_2, \cdots, z_{n-3}\}^{\mathrm{T}}$ as

$$\bar{\mathbf{z}} = \frac{\mathbf{X}}{\beta} \tag{4.62}$$

where β is a constant chosen to have a sufficiently large value such that

$$\beta > \sum_{i=1}^{n-3} x_i \tag{4.63}$$

for any feasible solution $\mathbf{X}$ (assuming that the solution is bounded). By using Eq. (4.62), the problem of Eq. (4.61) can be stated as follows:

$$\text{Minimize } \beta \mathbf{d}^{\mathrm{T}} \overline{\mathbf{z}}$$

subject to

$$\begin{aligned} [\alpha]\overline{\mathbf{z}} &= \frac{1}{\beta}\mathbf{b} \\ \overline{\mathbf{z}} &\geq \mathbf{0} \end{aligned} \tag{4.64}$$

INTERIOR POINT METHODS

We now define a new vector $\mathbf{z}$ as

$$\mathbf{z} = \begin{Bmatrix} \overline{\mathbf{z}} \\ z_{n-2} \\ z_{n-1} \\ z_n \end{Bmatrix}$$

and solve the following related problem instead of the problem in Eqs. (4.64):

$$\text{Minimize } \{\beta \mathbf{d}^{\mathrm{T}} \quad 0 \quad 0 \quad M\}\,\mathbf{z}$$

subject to

$$\begin{bmatrix} [\alpha] & \mathbf{0} & -\dfrac{n}{\beta}\mathbf{b} & \left(\dfrac{n}{\beta}\mathbf{b} - [\alpha]\mathbf{e}\right) \\ 0 & 0 & n & 0 \end{bmatrix} \mathbf{z} = \begin{Bmatrix} \mathbf{0} \\ 1 \end{Bmatrix}$$

$$\mathbf{e}^T \overline{\mathbf{z}} + z_{n-2} + z_{n-1} + z_n = 1 \qquad (4.65)$$

$$\mathbf{z} \geq \mathbf{0}$$

where $\mathbf{e}$ is an $(m-1)$-component vector whose elements are all equal to 1, z_{n-2} is a slack variable that absorbs the difference between 1 and the sum of other variables, z_{n-1} is constrained to have a value of $1/n$, and M is given a large value (corresponding to the artificial variable z_n) to force z_n to zero when the problem stated in Eqs. (4.61) has a feasible solution. Equations (4.65) are developed such that if $\mathbf{z}$ is a solution to these equations, $\mathbf{X} = \beta\overline{\mathbf{z}}$ will be a solution to Eqs. (4.61) if Eqs. (4.61) have a feasible solution. Also, it can be verified that the interior point $\mathbf{z} = (1/n)\mathbf{e}$ is a feasible solution to Eqs. (4.65). Equations (4.65) can be seen to be the desired form of Eqs. (4.61) except for a 1 on the right-hand side. This can be eliminated by subtracting the last constraint from the next-to-last constraint, to obtain the required form:

$$\text{Minimize } \{\beta\mathbf{d}^{\mathrm{T}} \quad 0 \quad 0 \quad M\}\mathbf{z}$$

subject to

$$\begin{bmatrix} [\alpha] & \mathbf{0} & -\dfrac{n}{\beta}\mathbf{b} & \left(\dfrac{n}{\beta}\mathbf{b} - [\alpha]\mathbf{e}\right) \\ -\mathbf{e}^T & -1 & (n-1) & -1 \end{bmatrix} \mathbf{z} = \begin{Bmatrix} \mathbf{0} \\ 0 \end{Bmatrix}$$

$$\mathbf{e}^{\mathrm{T}}\overline{\mathbf{z}} + z_{n-2} + z_{n-1} + z_n = 1 \qquad (4.66)$$

$$\mathbf{z} \geq \mathbf{0}$$

INTERIOR POINT METHODS

- Note: When Eqs. (4.66) are solved, if the value of the artificial variable $z_n > 0$, the original problem in Eqs. (4.61) is infeasible. On the other hand, if the value of the slack variable $z_{n-2} = 0$, the solution of the problem given by Eqs. (4.61) is unbounded.

Example 4.12 Transform the following LP problem into a form required by Karmarkar's method:

$$\text{Minimize } 2x_1 + 3x_2$$

subject to

$$3x_1 + x_2 - 2x_3 = 3$$

$$5x_1 - 2x_2 = 2$$

$$x_i \geq 0, \quad i = 1, 2, 3$$

INTERIOR POINT METHODS

SOLUTION It can be seen that

$$\mathbf{d} = \{2 \quad 3 \quad 0\}^{\mathrm{T}},\ [\alpha] = \begin{bmatrix} 3 & 1 & -2 \\ 5 & -2 & 0 \end{bmatrix},\ \mathbf{b} = \begin{Bmatrix} 3 \\ 2 \end{Bmatrix},\ \text{and } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

We define the integers m and n as $n = 6$ and $m = 3$ and choose $\beta = 10$ so that

$$\bar{\mathbf{z}} = \frac{1}{10}\begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix}$$

Noting that $\mathbf{e} = \{1, 1, 1\}^{\mathrm{T}}$, Eqs. (4.66) can be expressed as

$$\text{Minimize } \{20 \quad 30 \quad 0 \quad 0 \quad 0 \quad M\}\,\mathbf{z}$$

subject to

$$\left[\begin{bmatrix} 3 & 1 & -2 \\ 5 & -2 & 0 \end{bmatrix} \quad \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \frac{6}{10}\begin{Bmatrix} 3 \\ 2 \end{Bmatrix} \right.$$

$$\left. \times \left(\frac{6}{10}\begin{Bmatrix} 3 \\ 2 \end{Bmatrix} - \begin{bmatrix} 3 & 1 & -2 \\ 5 & -2 & 0 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}\right)\right]\mathbf{z} = \mathbf{0}$$

$$\begin{aligned} \{-\{1 \quad 1 \quad 1\} \quad -1 \quad 5 \quad -1\}\,\mathbf{z} &= 0 \\ z_1 + z_2 + z_3 + z_4 + z_5 + z_6 &= 1 \\ \mathbf{z} = \{z_1 \quad z_2 \quad z_3 \quad z_4 \quad z_5 \quad z_6\}^{\mathrm{T}} &\geq \mathbf{0} \end{aligned}$$

$$\begin{bmatrix} [\alpha] & \mathbf{0} & -\frac{n}{\beta}\mathbf{b} & \left(\frac{n}{\beta}\mathbf{b} - [\alpha]\mathbf{e}\right) \\ -\mathbf{e}^T & -1 & (n-1) & -1 \end{bmatrix}\mathbf{z} = \begin{Bmatrix} \mathbf{0} \\ 0 \end{Bmatrix}$$

$$\mathbf{e}^{\mathrm{T}}\bar{\mathbf{z}} + z_{n-2} + z_{n-1} + z_n = 1$$

where M is a very large number. These equations can be seen to be in the desired form.

Algorithm

Starting from an interior feasible point $\mathbf{X}^{(1)}$, Karmarkar's method finds a sequence of points $\mathbf{X}^{(2)}, \mathbf{X}^{(3)}, \cdots$ using the following iterative procedure:

1. Initialize the iterative process. Begin with the center point of the simplex as the initial feasible point

$$\mathbf{X}^{(1)} = \left\{ \frac{1}{n} \ \frac{1}{n} \cdots \frac{1}{n} \right\}^{\mathrm{T}}.$$

Set the iteration number as $k = 1$.

2. Test for optimality. Since $f = 0$ at the optimum point, we stop the procedure if the following convergence criterion is satisfied:

$$||\mathbf{c}^T \mathbf{X}^{(k)}|| \le \varepsilon \tag{4.67}$$

where ε is a small number. If Eq. (4.67) is not satisfied, go to step 3.

3. Compute the next point, $\mathbf{X}^{(k+1)}$. For this, we first find a point $\mathbf{Y}^{(k+1)}$ in the transformed unit simplex as

$$\mathbf{Y}^{(k+1)} = \left\{\frac{1}{n} \quad \frac{1}{n} \cdots \frac{1}{n}\right\}^{\mathrm{T}} - \frac{\alpha([I] - [P]^{\mathrm{T}}([P][P]^{\mathrm{T}})^{-1}[P])[D(\mathbf{X}^{(k)})]\mathbf{c}}{\|\mathbf{c}\| \sqrt{n(n-1)}} \tag{4.68}$$

where $||\mathbf{c}||$ is the length of the vector $\mathbf{c}$, $[I]$ the identity matrix of order n, $[D(\mathbf{X}^{(k)})]$ an $n \times n$ matrix with all off-diagonal entries equal to 0, and diagonal entries equal to the components of the vector $\mathbf{X}^{(k)}$ as

$$[D(\mathbf{X}^{(k)})]_{ii} = x_i^{(k)}, \qquad i = 1, 2, \ldots, n \tag{4.69}$$

$[P]$ is an $(m + 1) \times n$ matrix whose first m rows are given by $[a]$ $[D(\mathbf{X}^{(k)})]$ and the last row is composed of 1's:

$$[P] = \begin{bmatrix} & [a][D(\mathbf{X}^{(k)})] & & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \tag{4.70}$$

and the value of the parameter α is usually chosen as $\alpha = \frac{1}{4}$ to ensure convergence. Once $\mathbf{Y}^{(k+1)}$ is found, the components of the new point $\mathbf{X}^{(k+1)}$ are determined as

$$x_i^{(k+1)} = \frac{x_i^{(k)} y_i^{(k+1)}}{\sum_{r=1}^{n} x_r^{(k)} y_r^{(k+1)}}, \qquad i = 1, 2, \ldots, n \tag{4.71}$$

Set the new iteration number as $k = k + 1$ and go to step 2.