MECH6318 HW09

Friday, November 5, 2021 2:47 AM

13.1 Consider the following optimization problem:

$$\max_x \; f(x) = x_1^2 - x_2 \qquad (13.139)$$

subject to

$$x_1^2 - x_2^2 = 1 \qquad (13.140)$$

1. Find all the points that satisfy the KKT condition using Lagrange multipliers.
2. Use the variable elimination method to solve the problem (eliminate x_1). Compare the solutions obtained using the variable elimination method with those obtained in Question 1.

$$\max_x \; f(x) = x_1^2 - x_2$$

$$\text{s.t.} \quad x_1^2 - x_2^2 - 1 = 0$$

$$\min_x \; -x_1^2 + x_2 \qquad -f(x)$$

$$\text{s.t.} \quad x_1^2 - x_2^2 - 1 = 0 \qquad h_j(x) = 0$$

1) $$L(x, v) = f(x) + \sum_{j=1}^{n} v_j h_j(x)$$

$$= -x_1^2 + x_2 + v_1(x_1^2 - x_2^2 - 1)$$

$$h(x) = 0 = x_1^2 - x_2^2 - 1$$

$$(x_1, x_2) \in \{ x_1, x_2 \in \mathbb{R} \mid x_1^2 - x_2^2 - 1 = 0 \}$$

2) $$\sqrt{x_1^2} = \sqrt{x_2^2 + 1} \qquad x_1^2 = \underbrace{x_2^2 + 1}$$

$$x_1 = \sqrt{1 + x_2^2}$$

$$\min_x -f(x) = -x_1^2 + x_2 = -(1 + x_2^2) + x_2$$

$$= -x_2^2 + x_2 - 1$$

$$\nabla_? f(x) = -2x_? + 1 = 0$$

$$\nabla_{x_2} f(x) = -2x_2 + 1 = 0$$

$$\frac{\cancel{2} x_2}{\cancel{2}} = \frac{1}{2}$$

$$\boxed{x_2 = \frac{1}{2}}$$

$$x_1 = \sqrt{1 + x_2^2}$$

$$= \sqrt{1 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{5}{4}}$$

$$\boxed{x_1 = \frac{\sqrt{5}}{2}}$$

$$\boxed{\begin{aligned} x_1 &= \frac{\sqrt{5}}{2} \\ x_2 &= \frac{1}{2} \end{aligned}}$$

13.2 Use the Lagrange multiplier method to solve the following problem:

$$\min_x f(x) = x_1^2 + x_2^2 + x_3^2 \quad (13.141)$$

subject to

$$x_1 + 2x_2 + 3x_3 = 7 \quad (13.142)$$

$$2x_1 + 2x_2 + x_3 = \frac{9}{2} \quad (13.143)$$

$$\min_x f(x) = x_1^2 + x_2^2 + x_3^2$$

$$\text{s.t.} \quad h_1(x) = x_1 + 2x_2 + 3x_3 - 7 = 0$$

$$h_2(x) = 2x_1 + 2x_2 + x_3 - \frac{9}{2} = 0$$

$$L(x, v) = f(x) + \sum_{i=1}^{n} v_i h_i(x)$$

$$= x_1^2 + x_2^2 + x_3^2 + v_1(x_1 + 2x_2 + 3x_3 - 7) + v_2\left(2x_1 + 2x_2 + x_3 - \frac{9}{2}\right)$$

$$\nabla_{x_1} L(x,v) = 2x_1 + v_1 + 2v_2 = 0$$

$$\nabla_{x_2} L(x,v) = 2x_2 + 2v_1 + 2v_2 = 0$$

$$\nabla_{x_3} L(x,v) = 2x_3 + 3v_1 + v_2 = 0$$

$$h_1(x) = x_1 + 2x_2 + 3x_3 - 7 = 0$$

$$h_2(x) = 2x_1 + 2x_2 + x_3 - \frac{9}{2} = 0$$

$$\Rightarrow \quad \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & v_1 & v_2 \end{array} \\ \begin{bmatrix} 2 & 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 3 & 3 & 1 \\ 1 & 2 & 3 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7 \\ \frac{9}{2} \end{bmatrix}$$

Problem 13.2

```
A = [2, 0, 0, 1, 2;
     0, 2, 0, 2, 2;
     0, 0, 3, 3, 1;
     1, 2, 3, 0, 0;
     2, 2, 1, 0, 0;
    ]
b = [0, 0, 0, 7, 9/2]'

syms x1 x2 x3 v1 v2
x_sym = [x1, x2, x3, v1, v2]'

results = solve(A * x_sym == b, x_sym');

x1 = double(results.x1)
x2 = double(results.x2)
x3 = double(results.x3)
v1 = double(results.v1)
v2 = double(results.v2)
```

```
A = 5×5
     2     0     0     1     2
     0     2     0     2     2
     0     0     3     3     1
     1     2     3     0     0
     2     2     1     0     0

b = 5×1
          0
          0
          0
     7.0000
     4.5000

x_sym =
```

$$\begin{pmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \\ \overline{v_1} \\ \overline{v_2} \end{pmatrix}$$

```
x1 = 0.3723
x2 = 1.1596
x3 = 1.4362
v1 = -1.5745
v2 = 0.4149
```

13.3 Consider the following optimization problem:

$$\min_{H,D} f(H,D) = \pi DH + \frac{\pi D^2}{2} \tag{13.144}$$

subject to

$$g_1 \equiv 1{,}000\pi - \pi D^2 H \leq 0 \tag{13.145}$$
$$g_2 \equiv 4.5 - D \leq 0 \tag{13.146}$$
$$g_3 \equiv D - 12 \leq 0 \tag{13.147}$$
$$g_4 \equiv 10 - H \leq 0 \tag{13.148}$$
$$g_5 \equiv H - 18 \leq 0 \tag{13.149}$$
$$(13.150)$$

1. Write down the Karush-Kuhn Tucker (KKT) necessary conditions for this problem
2. You are given that constraints g_1 and g_4 are active, and the other constraints are inactive. Given this information, simplify the KKT conditions found in No. 1.
3. Find all the possible KKT points using the simplified KKT conditions from No. 2.

$$\min_{H,D} f(H,D) = \pi DH + \frac{\pi D^2}{2}$$

s.t.
$$g_1(H,D) = \pi(1 - D^2 H) \leq 0$$
$$g_2(H,D) = 4.5 - D \leq 0$$
$$g_3(H,D) = D - 12 \leq 0$$
$$g_4(H,D) = 10 - H \leq 0$$
$$g_5(H,D) = H - 18 \leq 0$$

$$L(H,D,\lambda) = f(H,D) + \sum_{i=1}^{n} \lambda_i g_i(H,D)$$
$$= \pi DH + \frac{\pi D^2}{2}$$
$$+ \lambda_1 (\pi(1 - D^2 H))$$
$$+ \lambda_2 (4.5 - D)$$
$$+ \lambda_3 (D - 12)$$

$$
\begin{aligned}
&+\lambda_2 (4.5 - D) \\
&+\lambda_3 (D - 12) \\
&+\lambda_4 (10 - H) \\
&+\lambda_5 (H - 18)
\end{aligned}
$$

$$\nabla_D L(H, D, \lambda) = \pi H + \pi D - 2\lambda_1 \pi D H - \lambda_2 + \lambda_3 = 0$$

$$\nabla_H L(H, D, \lambda) = \pi D - \lambda_1 \pi D^2 - \lambda_4 + \lambda_5 = 0$$

$$
\begin{aligned}
\lambda_1 g_1(H, D) &= \lambda_1 (\pi (1 - D^2 H)) = 0 \\
\lambda_2 g_2(H, D) &= \lambda_2 (4.5 - D) = 0 \\
\lambda_3 g_3(H, D) &= \lambda_3 (D - 12) = 0 \\
\lambda_4 g_4(H, D) &= \lambda_4 (10 - H) = 0 \\
\lambda_5 g_5(H, D) &= \lambda_5 (H - 18) = 0
\end{aligned}
$$

$$
\begin{aligned}
g_1(H, D) &= \pi(1 - D^2 H) \leq 0 & \lambda_1 &\geq 0 \\
g_2(H, D) &= 4.5 - D \leq 0 & \lambda_2 &\geq 0 \\
g_3(H, D) &= D - 12 \leq 0 & \lambda_3 &\geq 0 \\
g_4(H, D) &= 10 - H \leq 0 & \lambda_4 &\geq 0 \\
g_5(H, D) &= H - 18 \leq 0 & \lambda_5 &\geq 0
\end{aligned}
$$

Problem 13.3

```
f = @(H,D) pi * D * H + (pi * D^2)/2
g1 = @(H,D) pi * (1 - D^2 * H)
g2 = @(H,D) 4.5 - D
g3 = @(H,D) D - 12
g4 = @(H,D) 10 - H
g5 = @(H,D) H - 18

L = @(H,D,lambda) (...
    f(H,D) ...
    + lambda(1) * g1(H,D) ...
    + lambda(2) * g2(H,D) ...
    + lambda(3) * g3(H,D) ...
    + lambda(4) * g4(H,D) ...
    + lambda(5) * g5(H,D)...
)

lambda = sym('lambda',[1,5]);
assume(lambda >= 0)

D_H_L = @(D,H) diff(L(H,D,lambda),H);
D_D_L = @(D,H) diff(L(H,D,lambda),D);
```

```
f = function_handle with value:
    @(H,D)pi*D*H+(pi*D^2)/2
g1 = function_handle with value:
    @(H,D)pi*(1-D^2*H)
g2 = function_handle with value:
    @(H,D)4.5-D
g3 = function_handle with value:
    @(H,D)D-12
g4 = function_handle with value:
    @(H,D)10-H
g5 = function_handle with value:
    @(H,D)H-18
L = function_handle with value:
    @(H,D,lambda)(f(H,D)+lambda(1)*g1(H,D)+lambda(2)*g2(H,D)+lambda(3)*g3(H,D)+lambda(4)*g4(H,D)+lambda(5)*g5(H,D))
```

L_H = $-\lambda_1\,\pi\,\mathrm{D}^2 + \pi\,\mathrm{D} - \lambda_4 + \lambda_5$

L_D = $\lambda_3 - \lambda_2 + \pi\,\mathrm{D} + \pi\,H - 2\,\pi\,\mathrm{D}\,H\,\lambda_1$

```
results = solve([...
    D_H_L(D,H) == 0,...
    D_D_L(D,H) == 0,...
    lambda(1) * g1(H,D) == 0,...
    lambda(2) * g2(H,D) == 0,...
    lambda(3) * g3(H,D) == 0,...
    lambda(4) * g4(H,D) == 0,...
    lambda(5) * g5(H,D) == 0,...
    g1(H,D) <= 0,...
    g2(H,D) <= 0,...
    g3(H,D) <= 0,...
    g4(H,D) <= 0,...
    g5(H,D) <= 0
    ], ...
    [D, H, lambda]...
);

D = double(results.D)
H = double(results.H)
lambda_1 = double(results.lambda1)
lambda_2 = double(results.lambda2)
lambda_3 = double(results.lambda3)
lambda_4 = double(results.lambda4)
lambda_5 = double(results.lambda5)
```

```
D = 4.5000
H = 10
lambda_1 = 0
lambda_2 = 45.5531
lambda_3 = 0
lambda_4 = 14.1372
lambda_5 = 0
```

13.4 Solve the constrained optimization problem below using the inverse penalty metho

$$\min_x f(x) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2 \quad (13.151)$$

subject to

$$x_1 + x_2 - 2 \leq 0 \quad (13.152)$$

1. Perform four unconstrained optimizations using the following values for the penalty parameter: $R = 1, 0.1, 0.01$, and 0.001.
2. For each R, prepare a table that provides the value of the penalty parameter, the values of the design variables, objective function and the constraint.
3. Can you guess where the constrained minimum might be?

$$\min_x f(x) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2$$

$$\text{s.t.} \quad g_1(x) = x_1 + x_2 - 2 \leq 0$$

Inverse Penalty Method:

$$\boxed{\Omega = R\frac{1}{g_i(x)}}$$

Penalty:

$$P(x,R) = f(x) + \Omega(R, g(x))$$

$$= (x_1 - 1.5)^2 + (x_2 - 1.5)^2 + R\left(\frac{1}{x_1 + x_2 - 2}\right)$$

$$= x_1^2 - 3x_1 + 2.25 + x_2^2 - 3x_2 + 2.25 + \left(\frac{R}{x_1 + x_2 - 2}\right)$$

$$= \Big(x_1^3 - 3x_1^2 + 2.25x_1 + x_1x_2^2 - 3x_1x_2 + 2.25x_1$$
$$+ x_1^2x_2 - 3x_1x_2 + 2.25x_2 + x_2^3 - 3x_2^2 + 2.25x_2$$
$$- 2x_1^2 + 6x_1 - 4.5 - 2x_2^2 + 6x_2 - 4.5 + R\Big) / (x_1 + x_2 - 2)$$

$$= \frac{x_1^3 - 5x_1^2 + 10.5x_1 + x_2^3 - 5x_2^2 + 10.5x_2 - 6x_1x_2 + x_1x_2^2 + x_1^2x_2 - 9 + R}{x_1 + x_2 - 2}$$

Problem 13.4

```
clear
f = @(x) (x(1) - 1.5)^2 + (x(2) - 1.5)^2;
g1 = @(x) x(1) + x(2) - 2;

x = sym('x',[2,1],'real')
fx = f(x)
g1x = g1(x)

P = @(x,R) f(x) + R/(g1(x));
P_xR = P(x,sym('R'))

D_x_P = @(x,R) gradient(P(x,R),x);
D_x_P_R = D_x_P(x,sym('R'))

R = [1; 0.1; 0.01; 0.001];
X1 = arrayfun(@(R) double(solve(D_x_P(x,R),x).x1),R);
X2 = arrayfun(@(R) double(solve(D_x_P(x,R),x).x2),R);
G = arrayfun(@(x1,x2) g1([x1,x2]), X1, X2);
F = arrayfun(@(x1,x2) f([x1,x2]), X1, X2);
Omega = R./G;

Results = table(R,X1,X2,G,F,Omega)
```

fx =

$$\left(x_1 - \frac{3}{2}\right)^2 + \left(x_2 - \frac{3}{2}\right)^2$$

g1x = $x_1 + x_2 - 2$

P_xR =

$$\left(x_1 - \frac{3}{2}\right)^2 + \left(x_2 - \frac{3}{2}\right)^2 + \frac{R}{x_1 + x_2 - 2}$$

D_x_P_R =

$$\begin{pmatrix} 2\,x_1 - \dfrac{R}{(x_1 + x_2 - 2)^2} - 3 \\ 2\,x_2 - \dfrac{R}{(x_1 + x_2 - 2)^2} - 3 \end{pmatrix}$$

Results = 4×6 table

	R	X1	X2	G	F	Omega
1	1.0000	1.7328	1.7328	1.4656	0.1084	0.6823
2	0.1000	1.5425	1.5425	1.0850	0.0036	0.0922
3	0.0100	1.5049	1.5049	1.0098	0.0000	0.0099
4	0.0010	1.5005	1.5005	1.0010	0.0000	0.0010