Lecture - 6

LINEAR PROGRAMMING

Reference: Book Chapter 11

INTRODUCTION

- **Linear programming (LP)** is a technique for optimization of a linear objective function, subject to linear equality and linear inequality constraints.
- Many practical problems in engineering and operations research can be expressed as linear programming problems.
- During World War II, George Dantzig of the U.S. Air Force used LP techniques for planning problems. He invented the *Simplex method*.

BASICS OF LINEAR PROGRAMMING

- A ***generic linear programming problem*** consisting of linear equality and linear inequality constraints

$$
\begin{aligned}
&\min_{x} \quad z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
&\text{such that} \\
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\
&\qquad \vdots \qquad\qquad\qquad \vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \\
&a_{eq_{11}}x_1 + a_{eq_{12}}x_2 + \ldots + a_{eq_{1n}}x_n = b_{eq_1} \\
&\qquad \vdots \qquad\qquad\qquad \vdots \\
&a_{eq_{p1}}x_1 + a_{eq_{p2}}x_2 + \ldots + a_{eq_{pn}}x_n = b_{eq_p} \\
&x_{1-lb} \leq x_1 \leq x_{1-ub} \\
&\qquad \vdots \qquad\qquad\qquad \vdots \\
&x_{n-lb} \leq x_n \leq x_{n-ub}
\end{aligned}
$$

BASICS OF LINEAR PROGRAMMING

- **A matrix notation**

$$\min_{x} \; z = c^T x$$

linear objective function to be minimized

coefficients for each of the ***n*** design variables

such that

$$Ax \le b$$

the left hand side matrix and the right hand side vector of coefficients of ***m*** **inequality constraints**

$$A_{eq} x = B_{eq}$$

The left hand side matrix and the right hand side vector of coefficients of *p* **equality constraints**

$$x_{lb} \le x \le x_{ub}$$

lower and upper bounds

GRAPHICAL SOLUTION APPROACH: Types of LP

- Graphical approach is a simple and easy technique to solve LP problems.
- The procedure involves plotting the contours of the objective function and the constraints.
- The feasible region of the constraints and the optimal solution is then identified graphically.
- There are four possible types of solutions in a generic LP problem:

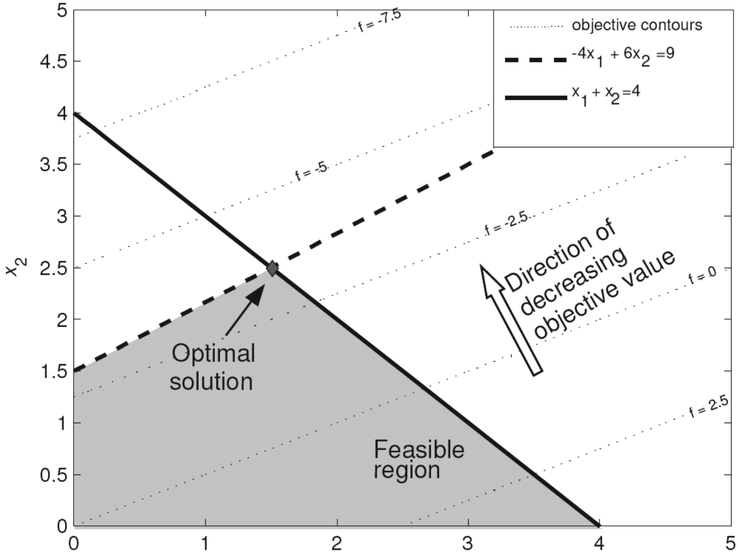
1. Unique solution
2. Segment solution
3. No solution
4. Solution at infinity

UNIQUE SOLUTION

$$\min_{x} \; x_1 - 2x_2$$

such that

$$-4x_1 + 6x_2 \leq 9$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$



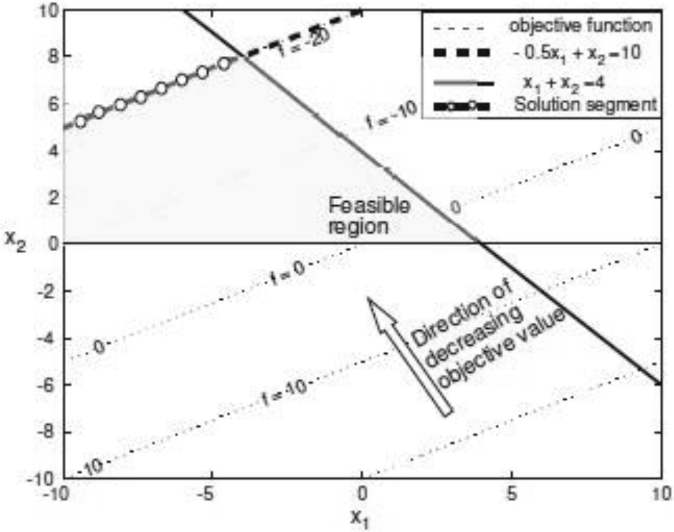
- There is **a unique solution** for this problem, where the objective function contour has the least value while remaining in the feasible region.

SEGMENT SOLUTION

$$\min_x \; x_1 - 2x_2$$

such that

$$-0.5x_1 + x_2 \leq 10$$
$$x_1 + x_2 \leq 4$$
$$x_2 \geq 0$$



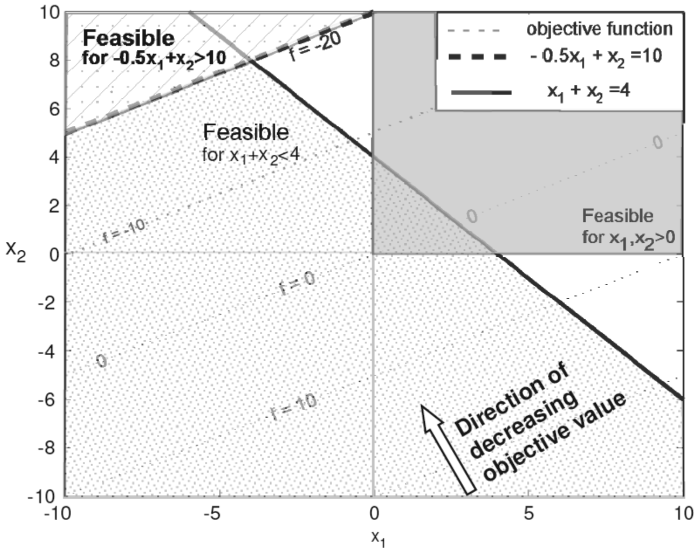
- The slope of the objective function and the constraint function are same.
- The objective function contour therefore coincides with the constraint function, and there are **infinitely many solutions along the segment**.

NO SOLUTION

$$\min_{x} \; x_1 - 2x_2$$

such that

$$-0.5x_1 + x_2 \geq 10$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$



- The feasible regions of the inequality constraints do not intersect.
- Therefore, there is no solution that satisfies all constraints; there is **no solution** to this problem.

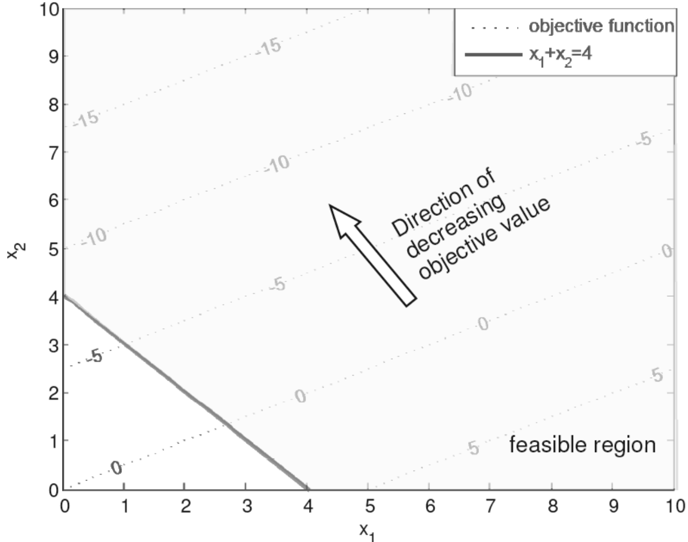
SOLUTION AT INFINITY

$$\min_{x} \; x_1 - 2x_2$$

such that

$$x_1 + x_2 \geq 4$$

$$x_1, x_2 \geq 0$$



- The feasible region is not bounded to yield a finite optimum value. The solution for this problem therefore lies at infinity.
- An unconstrained linear programming problem usually has solution at infinity. Such problems are rarely encountered in practice.

SOLVING LP PROBLEMS USING MATLAB

Linprog:

- The command **linprog** employs different strategies such as simplex method and interior point methods based on the size of the problem.
- The command allows for linear equality and linear inequality constraints and bounds on the design variables.
- The default problem formulation for **linprog** is given below.

$$\min_{x} \; f^T x$$

such that

$$Ax \leq b$$

$$A_{eq}x = b_{eq}$$

$$x_{lb} \leq x \leq x_{ub}$$

- ***f*** vector of coefficients of the objective function
- ***A*** and $\boldsymbol{A_{eq}}$ represent the matrices of the left hand side coefficients of the linear inequality and linear equality constraints, respectively
- ***b*** and $\boldsymbol{b_{eq}}$ represent the vectors of the right hand side values of the linear inequality and equality constraints, respectively

SOLVING LP PROBLEMS USING MATLAB

Example:

$$\min_{x} \; x_1 - 2x_2$$

such that

$$-9 - 4x_1 + 6x_2 \leq 0$$
$$x_1 + x_2 - 4 \leq 0$$
$$x_1, x_2 \geq 0$$

- The constraints can be rewritten as $-4x_1+6x_2 \leq 9$ and $x_1+x_2 \leq 4$ as per **Matlab's standard formulation**.
- The Matlab code to solve the above problem:

```
f = [1;-2]   % Defining Objective
A= [-4 6; 1  1];   % LHS inequalities
B = [ 9 ; 4]   % RHS inequalities
Aeq = [];     % No equalities
beq = [];     % No equalities
lb = [0;0]    % Lower bounds
ub = []       % No upper bounds
x0 = [1;1]    % Initial guess
x = linprog(f,A,B,Aeq,beq,lb,ub,x0)
```

```
x =

    1.5000
    2.5000
```

SOLVING LP PROBLEMS USING MATLAB

Example:

$$\min_{x} \; x_1 - 2x_2$$

such that

$$-9 - 4x_1 + 6x_2 \leq 0$$

$$x_1 + x_2 - 4 \leq 0$$

$$x_1, x_2 \geq 0$$

- **Note that** the warning displayed (in the Matlab's result) informs the reader that the interior point algorithm used by Matlab's solver does not need a starting point.
- The starting point provided by us has been ignored by the solver. Such messages are warnings that help the user understand the solver details, and should not be mistaken for error messages.

SIMPLEX METHOD BASICS

- ***Standard Form***
  - to apply simplex method, we need to pose the problem in the so-called standard form.
  - The standard form of an LP problem for simplex method is given by

$$\min_{x} \quad z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

such that

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}$$

$$x_1, x_2, \ldots, x_n \geq 0$$

  - **Note** that the standard formulation does not contain inequality constraints.

SIMPLEX METHOD BASICS

In a matrix notation, the standard formulation can be written as follows.

$$\min_{x} \quad z = c^T x$$

$$\text{such that}$$

$$Ax = b$$

$$x \geq 0$$

- $c = [c_1, ..., c_n]$: the vector of cost coefficients for the objective function
- ***A***: an m × n matrix for the linear equality constraints
- ***B***: the m × 1 vector of right hand side values

SIMPLEX METHOD BASICS

- The feasible region of the standard LP problem is a **convex polygon**.
- The optimal solution of the LP problem lies at one of the vertices of the polygon.
- **In the simplex method**, the solution process moves from one vertex of the polygon to the next along the boundary of the feasible region.

SIMPLEX METHOD BASICS

Transforming into standard form

- The standard definition of LP problems, there are no inequality constraints.
- Design variables have non-negativity constraints.
- If a given problem is not in this standard form, we will perform <u>certain operations to reformulate the given problem into the standard form</u>.

TRANSFORMING INTO STANDARD FORM

Inequality constraints

- If the given problem formulation contains inequality constraints of the form $g(x) \le 0$, we transform them into equality constraints by using the so-called **slack variables**.
- For constraints of the form $\mathbf{g(x) \le 0}$, we add to the left hand side of the constraint a **non-negative slack variable** $\mathbf{s_1}$ such that $\mathbf{g(x)+s_1} = 0$.
- The variable s_1 is called a slack variable because it represents the slack between the left hand side and the right hand side of the inequality.

TRANSFORMING INTO STANDARD FORM

Inequality constraints

- For inequalities of the form **$g(x) \geq 0$**, we transform them into equality constraints by using the so-called **surplus variables**.
- We subtract from the left hand side of the **constraint a non-negative surplus variable s_2** such that **$g(x) - s_2 = 0$**.
- The variable s_2 represents the surplus between the left hand side and right hand side of the inequality.
- The slack/surplus variables are unknowns, and will be determined as part of the LP solution process.

TRANSFORMING INTO STANDARD FORM

Example:

$$\min_{x} \ x_1 - 2x_2$$

such that

$$-4x_1 + 6x_2 \leq 9$$

$$x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

The standard form

$$\min_{x} \ x_1 - 2x_2$$

such that

$$-4x_1 + 6x_2 + s_1 = 9$$

$$x_1 + x_2 + s_2 = 4$$

$$x_1, x_2, s_1, s_2 \geq 0$$

TRANSFORMING INTO STANDARD FORM

Unbounded design variables

- In the standard form,
  - the design variables **should be non-negative**, $x \geq 0$.
  - the design variables **should not be indefinite** (i.e., no bounds may be specified).
- To put these design variables in a standard form:
  - If a design variable, x_i, does not have bounds imposed in the problem, we use $x_i = s_1 - s_2$, where $s_1, s_2 \geq 0$.
  - In the standard form, the variable x_i is then replaced by $s_1 - s_2$, and the additional constraints $s_1, s_2 \geq 0$.

TRANSFORMING INTO STANDARD FORM

Example:

$$\min_x \; x_1 - 2x_2 + 3x_3$$

such that

$$-4x_1 + 6x_2 + x_3 \leq 9$$
$$x_1 + x_2 - 2x_3 \leq 4$$
$$x_1, x_2 \geq 0$$

The standard form

$$\min_x \; x_1 - 2x_2 + 3(s_1 - s_2)$$

such that

$$-4x_1 + 6x_2 + (s_1 - s_2) + s_3 = 9$$
$$x_1 + x_2 - 2(s_1 - s_2) + s_4 = 4$$
$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

- The variable x_3 is unbounded in the above formulation. Assume that $\boldsymbol{x_3 = s_1 - s_2}$ and $\boldsymbol{s_1, s_2 \geq 0}$.

GAUSS JORDAN ELIMINATION

- The number of variables is not necessarily equal to the number of equations.
- If the number of variables is equal to the number of equality constraints, then the solution is **uniquely defined**.
- In most LP problems, there exist more variables than equations.
- This results in a so-called **under-determined** system of equations, resulting in **infinitely many feasible solutions** for the equality constraint set.

GAUSS JORDAN ELIMINATION

Example:

$$x_1 + x_2 - x_3 + 3x_4 = 2$$
$$-x_1 + 3x_2 - 5x_3 - 2x_4 = 5$$
$$x_1 + 2x_2 - x_4 = 6$$

- The above set of equations have more variables than equations.
- Therefore we have **infinitely many solutions for this case**.

GAUSS JORDAN ELIMINATION

- In order to efficiently deal with the constraint set of the LP problem, we will reduce the constraint set into a special form.
- The set of equations in the special form are said to be in a **canonical form**.
- Note in the original constraint set and the canonical form are equivalent .
- By transforming the LP constraint set into a canonical form (easier to solve), we can find solutions more efficiently.
- For solving LP problems, we use a canonical form known as the **reduced row echelon form**.
- This approach of using a reduced row echelon form to solve a set of linear equations is known as the **Gauss Jordan elimination**.

GAUSS JORDAN ELIMINATION

- A canonical form is usually defined with respect to a set of so-called **dependent or basic variables**, which are defined in terms of a set of **independent or non-basic variables**.
- A system of ***m*** equations and ***n*** variables is said to be in a **reduced row echelon form** with respect to a set of basic variables, x_1, ..., x_m, **if** all the basic variables have a coefficient of one in only one equation, have a zero coefficient in all other equations.

GAUSS JORDAN ELIMINATION

- A generic matrix based representation of a set of equations in a reduced row echelon form with ***m*** basic variables and ***p*** non-basic variables is

$$\begin{bmatrix} 1 & 0 & \dots & 0 & \vdots & d_{11} & \dots & d_{1p} \\ 0 & 1 & \dots & 0 & \vdots & d_{21} & \dots & d_{2p} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \vdots & d_{m1} & \dots & d_{mp} \end{bmatrix} \begin{bmatrix} x_{b1} \\ x_{b2} \\ \vdots \\ x_{bm} \\ \dots \\ x_{nb1} \\ \vdots \\ x_{nbp} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

d is the matrix of coefficients for the non-basic variables;
x_b is the set of basic variables;
x_{nb} is the set of non-basic variables.

- Note that the number of basic variables is equal to the number of equations.

GAUSS JORDAN ELIMINATION

Example:

The following equations are in a reduced row echelon form with respect of variables x_1, x_2, x_3, and x_4

$$x_1 + x_6 = 5$$
$$x_2 - 3x_5 + 4x_6 = 10$$
$$x_3 + 3x_5 = 2$$
$$x_4 + 2x_5 - 5x_6 = 7$$

Writing the above equation set in a matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 & 4 \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 2 \\ 7 \end{bmatrix}$$

REDUCING TO A ROW ECHELON FORM

- A pivot operation consists of a series of elementary row operations <u>to make a particular variable a basic variable</u>
- Note that a particular basic variable, x_{bi}, could exist in some or all of the ***m*** equations.
- Each equation can have only one basic variable with unit coefficient.
- The choice of which equation corresponds to which basic variable is usually arbitrary or is based on algebraic convenience.

REDUCING TO A ROW ECHELON FORM

- There are two types of elementary row operations that can be performed **to reduce a set of equations into a reduced row echelon** form:

  (1) Multiply both sides of an equation with the same non-zero number.

  (2) Replace one equation by a linear combination of another equation.

REDUCING TO A ROW ECHELON FORM

Example:

- Let us reduce the following set of equations into a row echelon form.

$$R1 \equiv x_1 + x_2 - x_3 + 3x_4 = 2$$
$$R2 \equiv -x_1 + 3x_2 - 7x_3 + x_4 = 6$$
$$R3 \equiv x_1 + 2x_2 \qquad\qquad - 2x_4 = 7$$

- Let us choose $\mathbf{x_1}$**,** $\mathbf{x_2}$**, and** $\mathbf{x_3}$ as basic variables.
- In order to obtain a row echelon form, <u>we need to perform operations such that the variables x_1, x_2, and x_3 appear only in one equation with a unit coefficient</u>, and do not appear in other equations.
- Let us use the following representation, where the first four columns represent the coefficients of each variable, and the last column represents the right hand side of the equation.

REDUCING TO A ROW ECHELON FORM

Example:

$$\begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & b \\ R_1 & 1 & 1 & -1 & 3 & 2 \\ R_2 & -1 & 3 & -7 & 1 & 6 \\ R_3 & 1 & 2 & 0 & -2 & 7 \end{array}$$

- Let us **first make x_1 a basic variable**. We choose x_1 to have unit coefficient in **R_1**, and zero coefficients in **R_2** and **R_3**.
- Replace R_2 by $R_2 + R_1$, and R_3 by $R_3 - R_1$, we obtain

$$\begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & b \\ R_1 & 1 & 1 & -1 & 3 & 2 \\ R_2 & 0 & 4 & -8 & 4 & 8 \\ R_3 & 0 & 1 & 1 & -5 & 5 \end{array}$$

REDUCING TO A ROW ECHELON FORM

Example:

- let us make x_2 a basic variable; we choose to make its coefficient one in R_2, and zeroes in other rows.
- We first divide R_2 by **4** to obtain a unit coefficient in R_2.

$$\begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & b \\ R1 & 1 & 1 & -1 & 3 & 2 \\ R2 & 0 & 1 & -2 & 1 & 2 \\ R3 & 0 & 1 & 1 & -5 & 5 \end{array}$$

- Replace R1 by $R_1 - R_2$ and R_3 by $R_3 - R_2$ to make the coefficients of x_2 zeroes in other equations.

$$\begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & b \\ R1 & 1 & 0 & 1 & 2 & 0 \\ R2 & 0 & 1 & -2 & 1 & 2 \\ R3 & 0 & 0 & 3 & -6 & 3 \end{array}$$

REDUCING TO A ROW ECHELON FORM

Example:

- make **x_3 a basic variable** by making its coefficient unity in R_3, and zeroes in other equations.
- We divide R_3 by 3 to obtain the following.

$$\begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & b \\ R1 & 1 & 0 & 1 & 2 & 0 \\ R2 & 0 & 1 & -2 & 1 & 2 \\ R3 & 0 & 0 & 1 & -2 & 1 \end{array}$$

- Replace R_2 by $R_2 + 2R_3$ and R_1 by $R_1 - R_3$ to make coefficients of x_3 zeroes in other equations.

$$\begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & b \\ R1 & 1 & 0 & 0 & 4 & -1 \\ R2 & 0 & 1 & 0 & -3 & 4 \\ R3 & 0 & 0 & 1 & -2 & 1 \end{array}$$

- The above set of equations are in **the reduced row echelon form with respect to basic variables x_1, x_2, and x_3**

BASIC SOLUTION

- A basic solution is obtained from the **canonical form** by setting the non-basic or independent variables to zero.
- A **basic feasible solution** is a basic solution in which the values of basic variables are nonnegative.

BASIC SOLUTION

Example:

- Consider the following canonical form

$$\left[\begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & b \\ R_1 & 1 & 0 & 0 & 4 & -1 \\ R_2 & 0 & 1 & 0 & -3 & 4 \\ R_3 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

- Set the non-basic variable, x_4, to zero.
- The basic solution can then be written as $x_1 = -1$, $x_2 = 4$, $x_3 = 1$, $x_4 = 0$.
- <u>Note that this basic solution is not a basic feasible solution for the standard LP formulation, since $x_1 < 0$.</u>

BASIC SOLUTION

- The choice of the set of basic variables is arbitrary, and can be decided as per computational convenience.
- For a generic problem, any set of m variables (recall we have m equations) from the possible n variables can be chosen as basic variables.
- This implies that the number of basic solutions for a generic standard LP problem with ***m*** constraints and ***n*** variables is given as

$$C_m^n = \frac{n!}{m!(n-m)!}$$

BASIC SOLUTION

Example:

$$R_1 \equiv x_1 + x_2 - x_3 + 3x_4 = 2$$
$$R_2 \equiv -x_1 + 3x_2 - 7x_3 + x_4 = 6$$
$$R_3 \equiv x_1 + 2x_2 \qquad - 2x_4 = 7$$

- Here, m = 3, and n = 4. Therefore we have

$$C_3^4 = \frac{4!}{3!(4-3)!} = 4$$

SIMPLEX ALGORITHM

- The optimal solution of the LP problem lies at one of the vertices of the feasible convex polygon.
- In the **Simplex method** the **solution process efficiently moves from one basic feasible solution to the next ensuring objective function reduction at each iteration**.
- We posed the standard LP problem as a minimization problem; the rules of the following algorithm apply to minimization problems only.

SIMPLEX ALGORITHM

1- Transform into Standard LP Problem:

Transform the given problem into the standard LP formulation by adding slack/surplus variables.

$$\min_{x} \; z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

such that

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + d_1s_1 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + d_2s_2 &= b_2 \\
\vdots \qquad\qquad\qquad & \;\; \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + d_ms_m &= b_m \\
x_1, \ldots, x_n, s_1, \ldots s_m &\geq 0
\end{aligned}$$

SIMPLEX ALGORITHM

Example:

$$\min_{x} \quad z = x_1 - 2x_2$$

such that

$$x_1 + x_2 \leq 4$$
$$-x_1 + x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

$\Rightarrow$

$$\min_{x} \quad z = x_1 - 2x_2$$

such that

$$x_1 + x_2 + s_1 = 4$$
$$-x_1 + x_2 + s_2 = 3$$
$$x_1, x_2, s_1, s_2 \geq 0$$

SIMPLEX ALGORITHM

2- Form Initial Simplex Tableau:

List the constraints and the objective function coefficients in the form of a table, known as the simplex tableau.

$$\left[\begin{array}{c|ccccccc}
 & x_1 & \dots & x_n & s_1 & \dots & s_p & b \\
\hline
\text{Constraint 1} & a_{11} & \dots & a_{1n} & 1 & \dots & 0 & b_1 \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\hline
\text{Constraint m} & a_{m1} & \dots & a_{mn} & 0 & \dots & 1 & b_m \\
\hline
\text{Objective} & c_1 & \dots & c_n & 0 & 0 & 0 & f
\end{array}\right]$$

SIMPLEX ALGORITHM

Example:

$$\min_{x} \quad z = x_1 - 2x_2$$

such that

$$x_1 + x_2 + s_1 = 4$$

$$-x_1 + x_2 + s_2 = 3$$

$$x_1, x_2, s_1, s_2 \geq 0$$

	x_1	x_2	s_1	s_2	b
R_1	1	1	1	0	4
R_2	-1	1	0	1	3
R_3	1	-2	0	0	f

SIMPLEX ALGORITHM

3- Choose Variable that Enters Basis – Identify Pivotal Column

- The simplex algorithm begins with the initial basic feasible solution.
- By observing the coefficients of the objective function row in the initial simplex tableau, the algorithm moves to the adjacent basic feasible solution that reduces the objective function.
- The other adjacent basic solution(s) with objective function value higher than the current solution are disregarded.

3- Choose Variable that Enters Basis – Identify Pivotal Column

- Choose the variable with the highest negative coefficient in the objective function row to become the basic variable.
- The variable with the highest negative coefficient has the potential to reduce the objective to the maximum extent.
- If make the corresponding variable a basic variable, the variable will be non-negative. The variable then multiplied with the highest negative coefficient in the objective function yields the most minimization.

SIMPLEX ALGORITHM

Example:

	x_1	x_2	s_1	s_2	b
R_1	1	**1**	1	0	4
R_2	-1	**1**	0	1	3
R_3	1	**-2**	0	0	f

- Observe the entries of the objective function row (R_3).
- The coefficient of x_2 is negative.
- The current basic variables, s_1 and s_2, are not part of the objective function.
- If we make x_2 a basic variable, the objective function value can reduce from the current value.

SIMPLEX ALGORITHM

4- Minimum Ratio rule – Identify Pivotal Row

- After add one variable to the basic variable, an existing basic variable must be made non-basic.
- **Minimum ratio rule:** Determine which basic variable is eliminated.
- Compute the following ratio for the selected column in the previous step corresponding to the variable that enters the basis, say x_j .

$$\min_{\text{for all } a_{ij}>0} \quad \frac{b_i}{a_{ij}}$$

- The row that satisfies the above minimum ratio rule is then selected as the pivotal row.

SIMPLEX ALGORITHM

4- Minimum Ratio rule – Identify Pivotal Row

- **Special cases:**

  (1) If all the coefficients of the column corresponding to the chosen basic variable, are negative, we cannot compute the minimum ratio. In such cases, <u>the LP problem has an unbounded solution</u>.

  (2) If two or more rows have the same minimum ratio, choose any pivotal row of choice.

  (3) When one or more of the basic variables have zero values, the solution is said to be degenerate. This can happen when the right hand side value is zero, and consequently, the minimum ratio is zero. This usually implies that adding a new variable to the basic variable set may not reduce the objective function value.

SIMPLEX ALGORITHM

5- Reduce to Canonical Form:

- Once we have chosen the pivotal row and the pivotal column based on the above rules, we can then **identify the pivotal element**.
- The constraint set is then transformed into a reduced row echelon form with respect to the newly identified incoming basic variable.

SIMPLEX ALGORITHM

Example:

- Reduce the initial simplex tableau into the canonical form with respect to the newly added basic variable, x_2.

	x_1	x_2	s_1	s_2	b
R_1	2	**0**	1	-1	1
R_2	-1	**1**	0	1	3
R_3	-1	**0**	0	2	$f+6$

- The basic solution is $x_1 = 0$, $x_2 = 3$, $s_1 = 1$, $s_2 = 0$, and the function value is $f = -6$.
- Note that x_2 has entered the basis, and s2 has left the basis.
- The value of x_2 has increased from zero in the initial simplex tableau to three in the current iteration, and the function value has reduced from $f = 0$ to $f = -6$.

6- Check for optimality:

If the coefficients of the objective function are all nonnegative, we reached the optimum. **If not**, we repeat the simplex algorithm until the above termination criterion is met.