MECH6318 HW01.pdf

Tuesday, August 31, 2021 2:37 PM

Fall 2021 | MECH 6318
Engineering Optimization – Prof. Jie Zhang
HOMEWORK #1
August 31, 2021

DUE: Tuesday, Sep 07, 2021 **5pm (central time)**
Submit the HW to eLearning

Points Distribution
30 points maximum
-5 to 0 points reserved for **Neatness and Professional Presentation**
(legible, stapled, show key Matlab commands, properly labeled plots, etc.)

Book Problems:

Problems 2.10, 2.14, 2.16, 2.18, 2.22, 2.26, 2.27

Additional problem (required solving by hand):

Consider the following function and do all the following by hand:

$$f(x) = 3x_1^2 + 3x_2^2 + 4x_1x_2 + (4x_3 + 2)^2 + 4x_1$$

Part (a)
What are the gradient and hessian of *f(x)*?

Part (b)
What are the stationary point(s) of *f(x)*?

Part (c)
Using the eigenvalues of the hessian, can the stationary point(s) be classified into a minimum, maximum, or saddle point?

2.10 Let `A=[1 0 2;0 3 4;2 1 3]`and `B=[0 2 4;2 3 4;5 1 3]`. Do the following problems by hand and verify using MATLAB. Turn in your hand written results and a print out of the results at the MATLAB Command Window. If you think any of the operations below cannot be performed, explain why. (a) A^T, (b) $(A^T)^T$, (c) $(A + B)^T$, and (d) $A^T + B^T$ (e) What matrix properties do you observe in these results?

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 3 & 4 \\ 5 & 1 & 3 \end{bmatrix}$$

```
% Problem 2.10 ------
A =[1, 0, 2;
    0, 3, 4;
    2, 1, 3]
```

```
A = 3×3
     1     0     2
     0     3     4
     2     1     3
```

a) $$A^T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 2 & 4 & 3 \end{bmatrix}$$

```
B =[0, 2, 4;
    2, 3, 4;
    5, 1, 3]
```

```
B = 3×3
     0     2     4
```

b) $$(A^T)^T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \end{bmatrix}$$

b) $(A^T)^T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix}$

c) $(A+B)^T = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 6 & 8 \\ 7 & 2 & 6 \end{bmatrix}^T$

$= \begin{bmatrix} 1 & 2 & 7 \\ 2 & 6 & 2 \\ 6 & 8 & 6 \end{bmatrix}$

d) $A^T + B^T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 2 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 5 \\ 2 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 7 \\ 2 & 6 & 2 \\ 6 & 8 & 6 \end{bmatrix}$

e) $(A^T)^T = A$

$(A+B)^T = A^T + B^T$

```
B = 3×3
     0     2     4
     2     3     4
     5     1     3
```

```
% Part a
A'
```

```
ans = 3×3
     1     0     2
     0     3     1
     2     4     3
```

```
% Part b
(A')'
```

```
ans = 3×3
     1     0     2
     0     3     4
     2     1     3
```

```
% Part c
(A + B)'
```

```
ans = 3×3
     1     2     7
     2     6     2
     6     8     6
```

```
% Part d
A' + B'
```

```
ans = 3×3
     1     2     7
     2     6     2
     6     8     6
```

2.14 Find the inverse of a 3 × 3 Identity matrix by hand. What do you observe?

$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad I^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

```
% Problem 2. 14 ---
I = eye(3)
```

```
I = 3×3
     1     0     0
     0     1     0
     0     0     1
```

```
I'
```

```
ans = 3×3
     1     0     0
     0     1     0
     0     0     1
```

2.16 Find the eigenvalues of `A=[3 2 1;2 2 1;1 1 1]` by hand and using MATLAB. What can you say about the definiteness of the matrix? Give reasons. (You can use MATLAB to solve for the cubic equation.) Use MATLAB to compute the eigenvectors of A. Turn in your hand written calculations and the printouts of the Command Window.

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} (s-3) & -2 & -1 \\ -2 & (s-2) & -1 \\ -1 & -1 & (s-1) \end{bmatrix}$$

$$\begin{aligned}
\Delta(s) = |sI - A| &= (s-3)\big((s-2)(s-1) - (-1)(-1)\big) \\
&\quad - (-2)\big(-2(s-1) - (-1)(-1)\big) \\
&\quad + (-1)\big((-2)(-1) - (-1)(s-2)\big) \\
&= (s-3)(s^2 - 3s + 2 - 1) \\
&\quad + 2(-2s - 1) \\
&\quad - 1(\cancel{2} + s - \cancel{2}) \\
&= s^3 - 3s^2 + s - 3s^2 + 9s - 1 \\
&\quad - 4s \\
&\quad - s
\end{aligned}$$

$$\boxed{\Delta(s) = s^3 - 6s^2 + 5s - 1}$$

$$\lambda_1 = 5.0489$$
$$\lambda_2 = 0.6431$$
$$\lambda_3 = 0.3080$$

```
% Problem 2.16 --------------------
% by hand calc
Delta = [1, -6, 5, -1]
```

```
Delta = 1×4
     1    -6     5    -1
```

```
roots(Delta)
```

```
ans = 3×1
    5.0489
    0.6431
    0.3080
```

```
% with MATLAB
A =[3, 2, 1;
    2, 2, 1;
    1, 1, 1]
```

```
A = 3×3
     3     2     1
     2     2     1
     1     1     1
```

```
poly(A)
```

```
ans = 1×4
    1.0000   -6.0000    5.0000   -1.0000
```

```
eig(A)
```

```
ans = 3×1
    0.3080
    0.6431
    5.0489
```

2.18 Determine if the function is convex or concave.

(a) $f(x) = e^{-x}$

(b) $f(x) = xlog(x)$

(c) $f(x) = 1/x^2$

a) $f(x) = e^{-x}$

$f'(x) = -e^{-x}$

$f''(x) = e^{-x}$

$f''(x) > 0 \quad \forall x \longrightarrow$ Concave $\forall x$

b) $f(x) = x \log(x)$

$f'(x) = \log(x) + 1$

$\underbrace{f''(x) = \frac{1}{x}}$

$f''(x) > 0 \quad \forall x > 0 \longrightarrow$ Concave $\forall x > 0$

$f''(x) < 0 \quad \forall x < 0 \longrightarrow$ Convex $\forall x < 0$

c) $f(x) = x^{-2}$

$f'(x) = -2x^{-3}$

$\underbrace{f''(x) = 6x^{-4}}$

$f''(x) \geq 0 \quad \forall x \longrightarrow$ Concave $\forall x$

2.22 X is a design variable vector. Find the Hessian of the following functions.

(a) $f(X) = \sum_{j=1}^{4} \sum_{i=1}^{4} c_{ij} x_i x_j$

(b) $f(X) = (\sin x_1 + \cos x_2)^N$, N is an integer

(c) $f(X) = x_1 \ln x_2 + x_2 \ln x_1$, at $(x_1, x_2) = (1, 1)$

a) $H_f(x) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

b) Using Wolfram Alpha

$$\begin{pmatrix} (N-1)\,N\cos^2(x1)\,(\sin(x1)+\cos(x2))^{N-2} - N\sin(x1)\,(\sin(x1)+\cos(x2))^{N-1} & -(N-1)\,N\cos(x1)\sin(x2)\,(\sin(x1)+\cos(x2))^{N-2} \\ -(N-1)\,N\cos(x1)\sin(x2)\,(\sin(x1)+\cos(x2))^{N-2} & (N-1)\,N\sin^2(x2)\,(\sin(x1)+\cos(x2))^{N-2} - N\cos(x2)\,(\sin(x1)+\cos(x2))^{N-1} \end{pmatrix}$$

c) $H_f(x) = x_1 \ln x_2 + x_2 \ln x_1 \quad @\ x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\left.\begin{bmatrix} 0 & \frac{1}{x_2} \\ \frac{1}{x_2} & 0 \end{bmatrix}\right|_{(1,1)}$$

$$H_f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Intermediate Problems

2.26 Let $f(x) = \sin(x)$ be a function that you are interested in optimizing. Answer the following questions.

(a) What are the necessary conditions for a solution to be an optimum of $f(x)$?

(b) Using the necessary conditions obtained in (a), and considering the interval $0 \le x \le 2\pi$, obtain the stationary point(s).

(c) Confirm whether the above point(s) are inflection points, maxima, or minima. If they are maximum (or minimum) points, are they global maximum (or minimum) in the given interval?

(d) Plot the function $\sin(x)$ over the interval $0 \le x \le 2\pi$. Show all the stationary points on it, and label them appropriately (maximum, minimum, or inflection).

$f(x) = \sin(x)$

a) $f'(x) = \cos(x) = 0$

b) $x = \frac{\pi}{2}, \frac{3\pi}{2}$

c) $f''(x) = -\sin(x)$

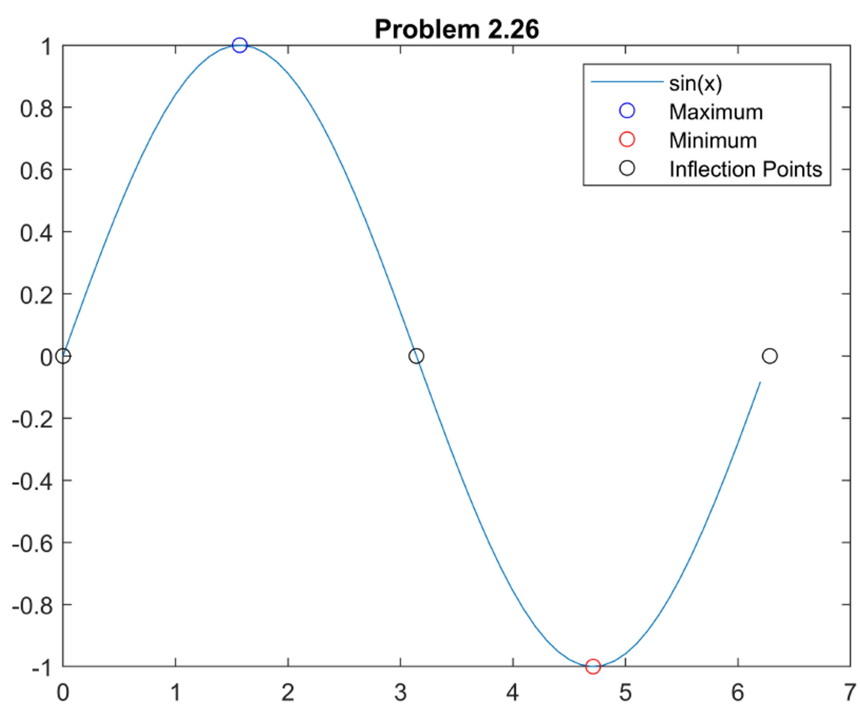
$-\sin\left(\frac{\pi}{2}\right) = -1 < 0 \rightarrow$ Maximum

$-\sin\left(\frac{\pi}{2}\right) = -1 < 0 \rightarrow$ Maximum

$-\sin\left(\frac{3\pi}{2}\right) = 1 > 0 \rightarrow$ Minimum

```
% Problem 2.26 ---------------------------------------------
x = 0:0.1:2*pi;
sinx = sin(x);

% Part d
figure()
plot(x,sinx,'DisplayName','sin(x)')
hold on
scatter([pi/2], [1], 'b', 'DisplayName', 'Maximum')
scatter([3*pi/2],[-1],'r','DisplayName','Minimum')
scatter([0, pi,2*pi], [0,0,0], 'k', 'DisplayName',...
    'Inflection Points')
legend
title('Problem 2.26')
```



2.27 Consider the single variable function $f(x) = e^{-ax^2}$, where a is a constant. This function is often used as a "radial basis function" for function approximation.

(a) Is the point $x = 0$ a stationary point for (i) $a > 0$, and (ii) $a < 0$. What happens if $a = 0$? Is $x = 0$ still a stationary point?

(b) If $x = 0$ is a stationary point, classify it as a minimum, maximum, or an inflection point for (i) $a > 0$, (ii) $a < 0$, and (iii) $a = 0$.

(c) Prepare a plot of $f(x)$ for $a = 1$, $a = 2$, and $a = 3$. Plot all three curves on the same figure. By observing the plot, do you think $f(x) = e^{-ax^2}, a > 0$ has a global minimum? If so, what is the value of x and $f(x)$ at the minimum?

$f(x) = e^{-ax^2}$

a) $f'(x) = -2axe^{-ax^2}$

i) $a > 0 \rightarrow f'(0) = 0 \rightarrow$ stationary point

ii) $a < 0 \rightarrow f'(0) = 0 \rightarrow$ stationary point

iii) $a = 0 \rightarrow f'(0) = 0 \rightarrow$ stationary point

b) $f''(x) = 2ae^{-ax^2}(2ax^2 - 1)$

i) $a > 0 \rightarrow f''(0) = -2|a| < 0 \rightarrow$ Maximum

ii) $a < 0 \rightarrow f''(0) = +2|a| > 0 \rightarrow$ Minimum

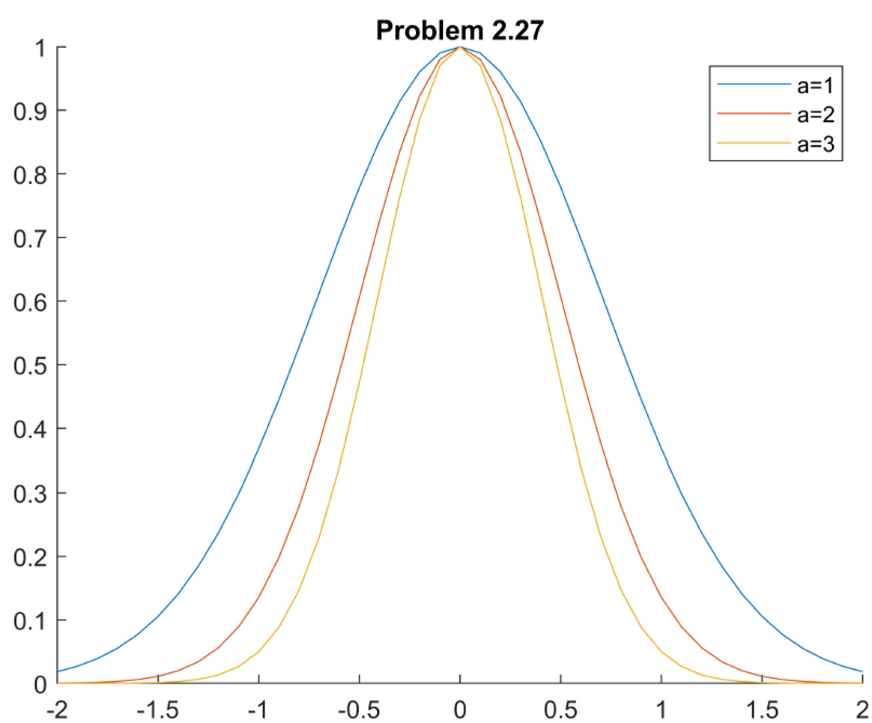
iii) $a = 0 \rightarrow f''(0) = 0 \rightarrow$ Inflection Point

```
% Problem 2.27 -------------------------------------------
x = -2:0.1:2;
fa1 = exp(-1*x.^2);
fa2 = exp(-2*x.^2);
fa3 = exp(-3*x.^2);

figure()
hold on
plot(x,fa1,'DisplayName','a=1')
plot(x,fa2,'DisplayName','a=2')
plot(x,fa3,'DisplayName','a=3')
```



```
legend
title('Problem 2.27')
```



Consider the following function and do all the following by hand:

$$f(x) = 3x_1^2 + 3x_2^2 + 4x_1x_2 + (4x_3 + 2)^2 + 4x_1$$

Part (a)
What are the gradient and hessian of *f(x)*?

Part (b)
What are the stationary point(s) of *f(x)*?

Part (c)
Using the eigenvalues of the hessian, can the stationary point(s) be classified into a minimum, maximum, or saddle point?

$$16x_3^2 + 16x_3 + 4$$

$$f(x) = 3x_1^2 + 3x_2^2 + 4x_1x_2 + (4x_3+2)^2 + 4x_1$$

$$\nabla f(x) = \begin{bmatrix} 6x_1 + 4x_2 + 4 \\ 6x_2 + 4x_1 \\ 32x_3 + 16 \end{bmatrix}$$

$$H_f = \begin{bmatrix} 6 & 4 & 0 \\ 4 & 6 & 0 \\ 0 & 0 & 32 \end{bmatrix}$$

b) stationary points:

$$x_3 = -\frac{1}{2}$$

$$x_1 = \frac{6}{5}$$

$$x_2 = -\frac{4}{5}$$

$$x = \begin{bmatrix} \frac{6}{5} \\ \frac{-4}{5} \\ \frac{-1}{2} \end{bmatrix}$$

c) Eigen values:

$$\lambda_1 = 32$$

$$\lambda_2 = 10$$

$\lambda_1 = 54$

$\lambda_2 = 10$

$\lambda_3 = 2$

$\mathrm{eig}_i(H_f(\tilde{X})) > 0 \ \forall i \longrightarrow$ Minimum