Lecture - 6
LINEAR PROGRAMMING

Reference: Book Chapter 11

INTRODUCTION

- **Linear programming (LP)** is a technique for optimization of a linear objective function, subject to linear equality and linear inequality constraints.
- Many practical problems in engineering and operations research can be expressed as linear programming problems.
- During World War II, George Dantzig of the U.S. Air Force used LP techniques for planning problems. He invented the *Simplex method*.

BASICS OF LINEAR PROGRAMMING

- A ***generic linear programming problem*** consisting of linear equality and linear inequality constraints

$$
\begin{aligned}
&\min_{x} \quad z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
&\text{such that} \\
&\quad a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\
&\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
&\quad a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \\
&\quad a_{eq_{11}}x_1 + a_{eq_{12}}x_2 + \ldots + a_{eq_{1n}}x_n = b_{eq_1} \\
&\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
&\quad a_{eq_{p1}}x_1 + a_{eq_{p2}}x_2 + \ldots + a_{eq_{pn}}x_n = b_{eq_p} \\
&\qquad x_{1-lb} \le x_1 \le x_{1-ub} \\
&\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
&\qquad x_{n-lb} \le x_n \le x_{n-ub}
\end{aligned}
$$

- **A matrix notation**

$$\min_{x} \; z = c^T x$$

linear objective function to be minimized

coefficients for each of the ***n*** design variables

such that

$$Ax \leq b$$

the left hand side matrix and the right hand side vector of coefficients of ***m*** **inequality constraints**

$$A_{eq} x = B_{eq}$$

The left hand side matrix and the right hand side vector of coefficients of *p* **equality constraints**

$$x_{lb} \leq x \leq x_{ub}$$

lower and upper bounds

GRAPHICAL SOLUTION APPROACH: Types of LP

- Graphical approach is a simple and easy technique to solve LP problems.
- The procedure involves plotting the contours of the objective function and the constraints.
- The feasible region of the constraints and the optimal solution is then identified graphically.
- There are four possible types of solutions in a generic LP problem:

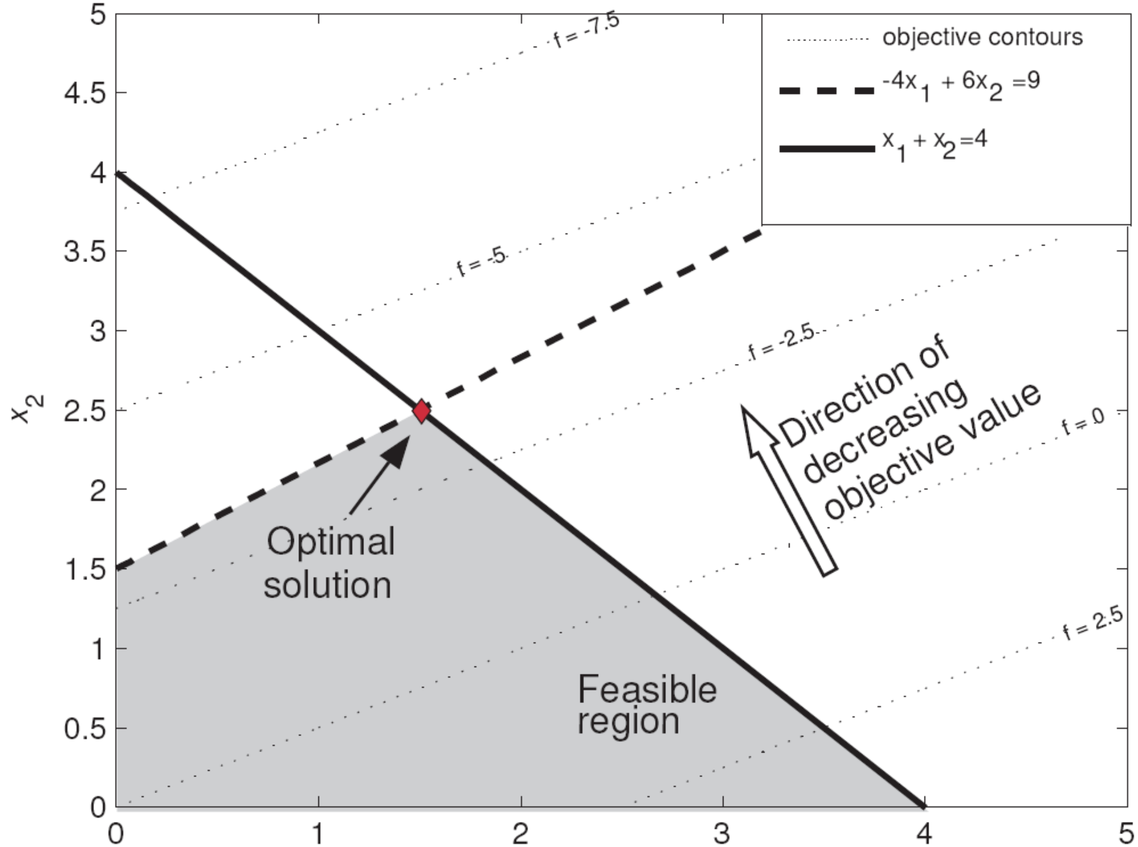
1. Unique solution
2. Segment solution
3. No solution
4. Solution at infinity

UNIQUE SOLUTION

$$\min_{x} \; x_1 - 2x_2$$

such that

$$-4x_1 + 6x_2 \le 9$$
$$x_1 + x_2 \le 4$$
$$x_1, x_2 \ge 0$$



- There is **a unique solution** for this problem, where the objective function contour has the least value while remaining in the feasible region.

SEGMENT SOLUTION

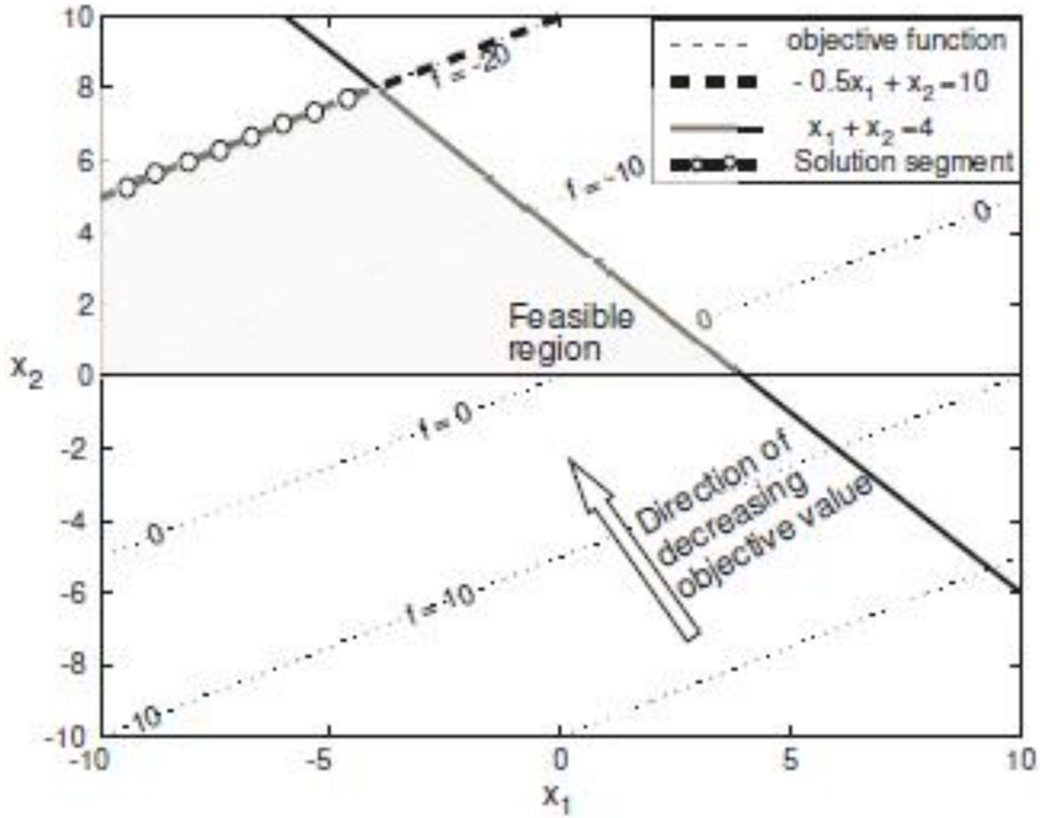
$$\min_{x} \; x_1 - 2x_2$$

such that

$$-0.5x_1 + x_2 \leq 10$$

$$x_1 + x_2 \leq 4$$

$$x_2 \geq 0$$



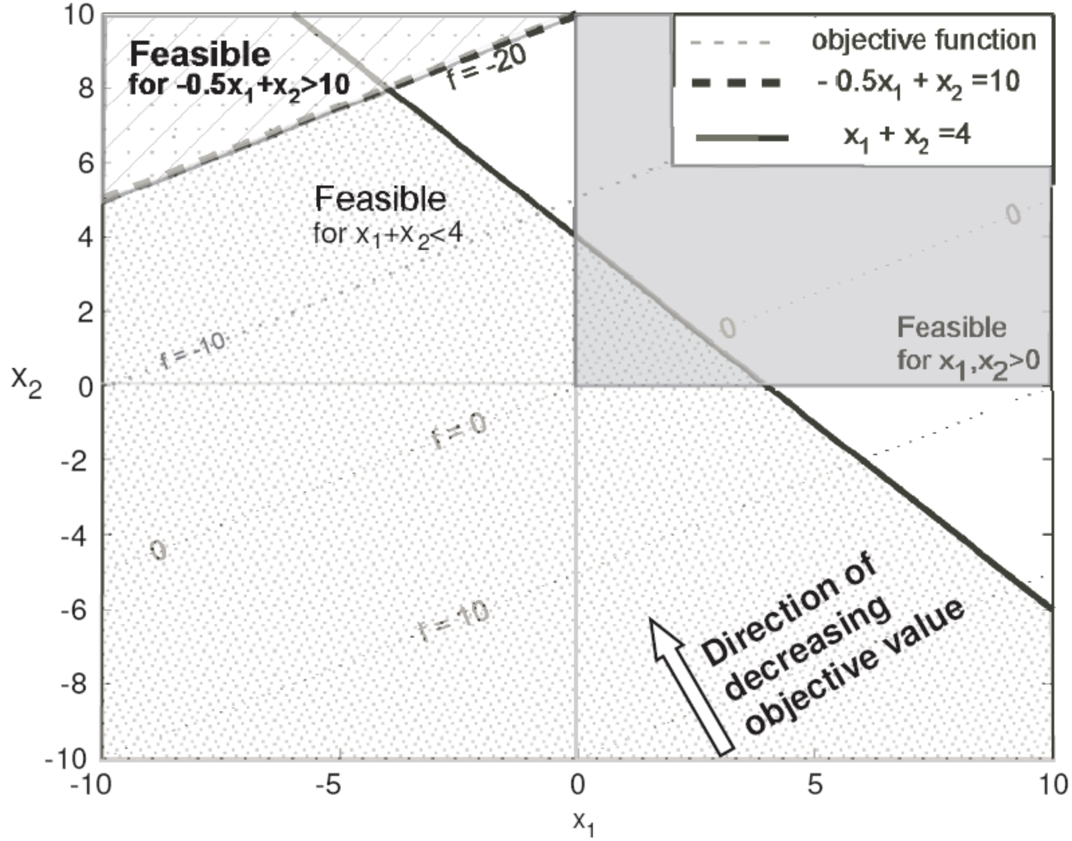
- The slope of the objective function and the constraint function are same.
- The objective function contour therefore coincides with the constraint function, and there are **infinitely many solutions along the segment**.

NO SOLUTION

$$\min_{x} \quad x_1 - 2x_2$$

such that

$$-0.5x_1 + x_2 \geq 10$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$



- The feasible regions of the inequality constraints do not intersect.
- Therefore, there is no solution that satisfies all constraints; there is **no solution** to this problem.

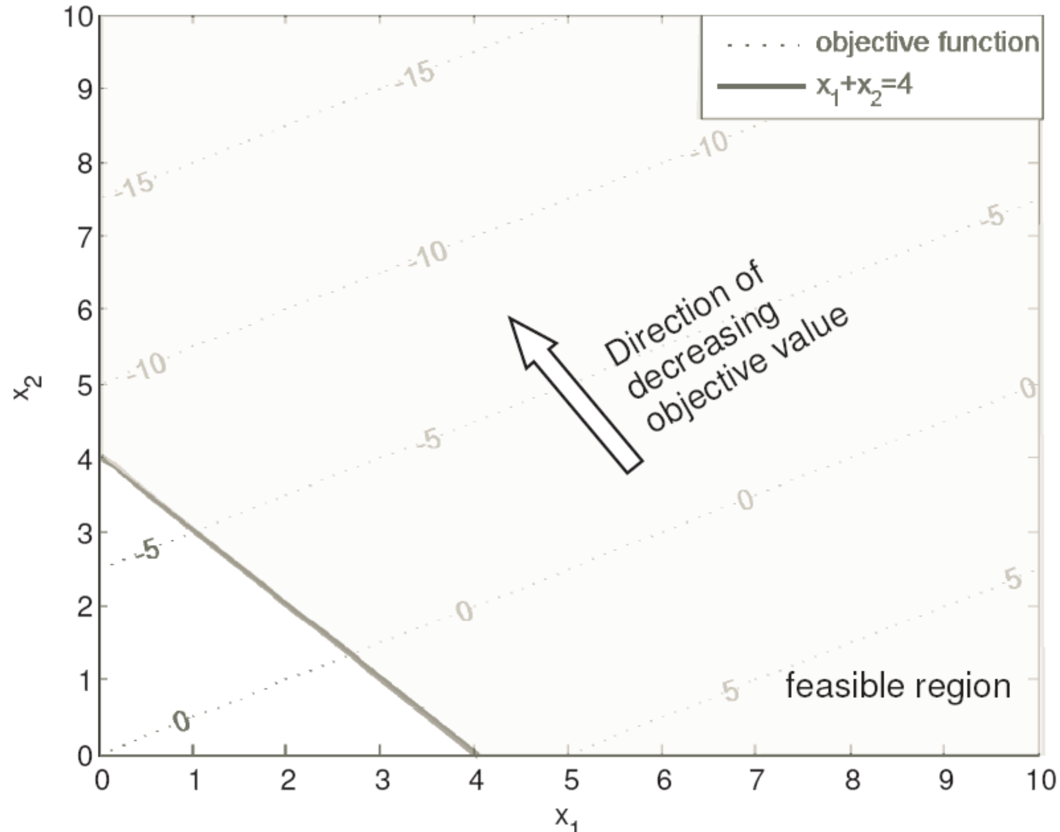
SOLUTION AT INFINITY

$$\min_x \; x_1 - 2x_2$$

such that

$$x_1 + x_2 \geq 4$$

$$x_1, x_2 \geq 0$$



- The feasible region is not bounded to yield a finite optimum value. The solution for this problem therefore lies at infinity.
- An unconstrained linear programming problem usually has solution at infinity. Such problems are rarely encountered in practice.

Linprog:

- The command **linprog** employs different strategies such as simplex method and interior point methods based on the size of the problem.
- The command allows for linear equality and linear inequality constraints and bounds on the design variables.
- The default problem formulation for **linprog** is given below.

$$\min_{x} \; f^T x$$

such that

$$Ax \leq b$$

$$A_{eq} x = b_{eq}$$

$$x_{lb} \leq x \leq x_{ub}$$

- ***f*** vector of coefficients of the objective function
- ***A*** and $\boldsymbol{A_{eq}}$ represent the matrices of the left hand side coefficients of the linear inequality and linear equality constraints, respectively
- ***b*** and $\boldsymbol{b_{eq}}$ represent the vectors of the right hand side values of the linear inequality and equality constraints, respectively

Example:

$$\min_{x} \; x_1 - 2x_2$$

such that

$$-9 - 4x_1 + 6x_2 \leq 0$$

$$x_1 + x_2 - 4 \leq 0$$

$$x_1, x_2 \geq 0$$

- The constraints can be rewritten as $-4x_1+6x_2 \leq 9$ and $x_1+x_2 \leq 4$ as per **Matlab's standard formulation**.
- The Matlab code to solve the above problem:

```
f = [1;-2]   % Defining Objective
A= [-4 6; 1  1];    % LHS inequalities
B = [ 9 ; 4]    % RHS inequalities
Aeq = [];      % No equalities
beq = [];      % No equalities
lb = [0;0]     % Lower bounds
ub = []        % No upper bounds
x0 = [1;1]     % Initial guess
x = linprog(f,A,B,Aeq,beq,lb,ub,x0)
```

```
x =

    1.5000
    2.5000
```

Example:

$$\min_{x} \; x_1 - 2x_2$$

such that

$$-9 - 4x_1 + 6x_2 \leq 0$$
$$x_1 + x_2 - 4 \leq 0$$
$$x_1, x_2 \geq 0$$

- **Note that** the warning displayed (in the Matlab's result) informs the reader that the interior point algorithm used by Matlab's solver does not need a starting point.
- The starting point provided by us has been ignored by the solver. Such messages are warnings that help the user understand the solver details, and should not be mistaken for error messages.

SIMPLEX METHOD BASICS

Standard Form

- to apply simplex method, we need to pose the problem in the so-called standard form.
- The standard form of an LP problem for simplex method is given by

$$\min_{x} \; z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

such that

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}$$

$$x_1, x_2, \ldots, x_n \geq 0$$

- **Note** that the standard formulation does not contain inequality constraints.

SIMPLEX METHOD BASICS

In a matrix notation, the standard formulation can be written as follows.

$$\min_{x} \quad z = c^T x$$

such that

$$Ax = b$$

$$x \geq 0$$

- *c = [c_1, ..., c_n]*: the vector of cost coefficients for the objective function
- ***A***: an m × n matrix for the linear equality constraints
- ***B***: the m × 1 vector of right hand side values

SIMPLEX METHOD BASICS

- The feasible region of the standard LP problem is a **convex polygon**.
- The optimal solution of the LP problem lies at one of the vertices of the polygon.
- **In the simplex method**, the solution process moves from one vertex of the polygon to the next along the boundary of the feasible region.

SIMPLEX METHOD BASICS

Transforming into standard form

- The standard definition of LP problems, there are no inequality constraints.
- Design variables have non-negativity constraints.
- If a given problem is not in this standard form, we will perform certain operations to reformulate the given problem into the standard form.

TRANSFORMING INTO STANDARD FORM

Inequality constraints

- If the given problem formulation contains inequality constraints of the form ***$g(x) \leq 0$***, we transform them into equality constraints by using the so-called **slack variables**.
- For constraints of the form **$g(x) \leq 0$**, we add to the left hand side of the constraint a **non-negative slack variable s_1** such that **$g(x)+s_1 =$** 0.
- The variable s_1 is called a slack variable because it represents the slack between the left hand side and the right hand side of the inequality.

TRANSFORMING INTO STANDARD FORM

Inequality constraints

- For inequalities of the form **$g(x) \geq 0$**, we transform them into equality constraints by using the so-called **surplus variables**.
- We subtract from the left hand side of the constraint **a non-negative surplus variable s_2** such that **$g(x) - s_2 = 0$**.
- The variable s_2 represents the surplus between the left hand side and right hand side of the inequality.
- The slack/surplus variables are unknowns, and will be determined as part of the LP solution process.

TRANSFORMING INTO STANDARD FORM

Example:

$$\min_{x} \; x_1 - 2x_2$$

such that

$$-4x_1 + 6x_2 \le 9$$

$$x_1 + x_2 \le 4$$

$$x_1, x_2 \ge 0$$

The standard form

$$\min_{x} \; x_1 - 2x_2$$

such that

$$-4x_1 + 6x_2 + s_1 = 9$$

$$x_1 + x_2 + s_2 = 4$$

$$x_1, x_2, s_1, s_2 \ge 0$$

TRANSFORMING INTO STANDARD FORM

Unbounded design variables

- In the standard form,
  - the design variables **should be non-negative**, $x \geq 0$.
  - the design variables **should not be indefinite** (i.e., no bounds may be specified).
- To put these design variables in a standard form:
  - If a design variable, x_i, does not have bounds imposed in the problem, we use $x_i = s_1 - s_2$, where $s_1, s_2 \geq 0$.
  - In the standard form, the variable x_i is then replaced by $s_1 - s_2$, and the additional constraints $s_1, s_2 \geq 0$.

TRANSFORMING INTO STANDARD FORM

Example:

$$\min_{x} \; x_1 - 2x_2 + 3x_3$$

such that

$$-4x_1 + 6x_2 + x_3 \leq 9$$
$$x_1 + x_2 - 2x_3 \leq 4$$
$$x_1, x_2 \geq 0$$

The standard form

$$\min_{x} \; x_1 - 2x_2 + 3(s_1 - s_2)$$

such that

$$-4x_1 + 6x_2 + (s_1 - s_2) + s_3 = 9$$
$$x_1 + x_2 - 2(s_1 - s_2) + s_4 = 4$$
$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

- The variable x_3 is unbounded in the above formulation. Assume that $\boldsymbol{x_3 = s_1 - s_2}$ and $\boldsymbol{s_1, s_2 \geq 0}$.

GAUSS JORDAN ELIMINATION

- The number of variables is not necessarily equal to the number of equations.
- If the number of variables is equal to the number of equality constraints, then the solution is **uniquely defined**.
- In most LP problems, there exist more variables than equations.
- This results in a so-called **under-determined** system of equations, resulting in **infinitely many feasible solutions** for the equality constraint set.

GAUSS JORDAN ELIMINATION

Example:

$$x_1 + x_2 - x_3 + 3x_4 = 2$$
$$-x_1 + 3x_2 - 5x_3 - 2x_4 = 5$$
$$x_1 + 2x_2 - x_4 = 6$$

- The above set of equations have more variables than equations.
- Therefore we have **infinitely many solutions for this case**.

GAUSS JORDAN ELIMINATION

- In order to efficiently deal with the constraint set of the LP problem, we will reduce the constraint set into a special form.
- The set of equations in the special form are said to be in a **canonical form**.
- <u>Note in the original constraint set and the canonical form are equivalent .</u>
- By transforming the LP constraint set into a canonical form (easier to solve), <u>we can find solutions more efficiently</u>.
- For solving LP problems, we use a canonical form known as the **reduced row echelon form**.
- This approach of using a reduced row echelon form to solve a set of linear equations is known as the **Gauss Jordan elimination**.

GAUSS JORDAN ELIMINATION

- A canonical form is usually defined with respect to a set of so-called **dependent or basic variables**, which are defined in terms of a set of **independent or non-basic variables**.
- A system of ***m*** equations and ***n*** variables is said to be in a **reduced row echelon form** with respect to a set of basic variables, x_1, ..., x_m, **if** all the basic variables have a coefficient of one in only one equation, have a zero coefficient in all other equations.

GAUSS JORDAN ELIMINATION

- A generic matrix based representation of a set of equations in a reduced row echelon form with ***m*** basic variables and ***p*** non-basic variables is

$$\begin{bmatrix} 1 & 0 & \dots & 0 & \vdots & d_{11} & \dots & d_{1p} \\ 0 & 1 & \dots & 0 & \vdots & d_{21} & \dots & d_{2p} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \vdots & d_{m1} & \dots & d_{mp} \end{bmatrix} \begin{bmatrix} x_{b1} \\ x_{b2} \\ \vdots \\ x_{bm} \\ \dots \\ x_{nb1} \\ \vdots \\ x_{nbp} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

d is the matrix of coefficients for the non-basic variables;
x_b is the set of basic variables;
x_{nb} is the set of non-basic variables.

- Note that the number of basic variables is equal to the number of equations.

Example:

The following equations are in a reduced row echelon form with respect of variables x_1, x_2, x_3, and x_4

$$\begin{gathered} x_1 + x_6 = 5 \\ x_2 - 3x_5 + 4x_6 = 10 \\ x_3 + 3x_5 = 2 \\ x_4 + 2x_5 - 5x_6 = 7 \end{gathered}$$

Writing the above equation set in a matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 & 4 \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 2 \\ 7 \end{bmatrix}$$

REDUCING TO A ROW ECHELON FORM

- A pivot operation consists of a series of elementary row operations to make a particular variable a basic variable
- Note that a particular basic variable, x_{bi}, could exist in some or all of the ***m*** equations.
- Each equation can have only one basic variable with unit coefficient.
- The choice of which equation corresponds to which basic variable is usually arbitrary or is based on algebraic convenience.

REDUCING TO A ROW ECHELON FORM

- There are two types of elementary row operations that can be performed **to reduce a set of equations into a reduced row echelon** form:

  (1) Multiply both sides of an equation with the same non-zero number.

  (2) Replace one equation by a linear combination of another equation.

REDUCING TO A ROW ECHELON FORM

Example:

- Let us reduce the following set of equations into a row echelon form.

$$
\begin{aligned}
R1 &\equiv x_1 + x_2 - x_3 + 3x_4 = 2 \\
R2 &\equiv -x_1 + 3x_2 - 7x_3 + x_4 = 6 \\
R3 &\equiv x_1 + 2x_2 \qquad\quad - 2x_4 = 7
\end{aligned}
$$

- Let us choose **x_1, x_2, and x_3** as basic variables.
- In order to obtain a row echelon form, <u>we need to perform operations such that the variables x_1, x_2, and x_3 appear only in one equation with a unit coefficient</u>, and do not appear in other equations.
- Let us use the following representation, where the first four columns represent the coefficients of each variable, and the last column represents the right hand side of the equation.

REDUCING TO A ROW ECHELON FORM

Example:

$$
\begin{array}{c|cccc|c}
 & x_1 & x_2 & x_3 & x_4 & b \\
R_1 & 1 & 1 & -1 & 3 & 2 \\
R_2 & -1 & 3 & -7 & 1 & 6 \\
R_3 & 1 & 2 & 0 & -2 & 7
\end{array}
$$

- Let us **first make x_1 a basic variable**. We choose x_1 to <u>have unit coefficient in</u> $\mathbf{R_1}$, and <u>zero coefficients in</u> $\mathbf{R_2}$ and $\mathbf{R_3}$.
- Replace R_2 by $R_2 + R_1$, and R_3 by $R_3 - R_1$, we obtain

$$
\begin{array}{c|cccc|c}
 & x_1 & x_2 & x_3 & x_4 & b \\
R_1 & 1 & 1 & -1 & 3 & 2 \\
R_2 & 0 & 4 & -8 & 4 & 8 \\
R_3 & 0 & 1 & 1 & -5 & 5
\end{array}
$$

REDUCING TO A ROW ECHELON FORM

Example:

- let us make x_2 a basic variable; we choose to make its coefficient one in R_2, and zeroes in other rows.
- We first divide R_2 by **4** to obtain a unit coefficient in R_2.

$$
\left[\begin{array}{c|cccc|c}
 & x_1 & x_2 & x_3 & x_4 & b \\
R1 & 1 & 1 & -1 & 3 & 2 \\
R2 & 0 & 1 & -2 & 1 & 2 \\
R3 & 0 & 1 & 1 & -5 & 5
\end{array}\right]
$$

- Replace R1 by $R_1 - R_2$ and R_3 by $R_3 - R_2$ to make the coefficients of x_2 zeroes in other equations.

$$
\left[\begin{array}{c|cccc|c}
 & x_1 & x_2 & x_3 & x_4 & b \\
R1 & 1 & 0 & 1 & 2 & 0 \\
R2 & 0 & 1 & -2 & 1 & 2 \\
R3 & 0 & 0 & 3 & -6 & 3
\end{array}\right]
$$

REDUCING TO A ROW ECHELON FORM

Example:

- make **x_3 a basic variable** by making its coefficient unity in R_3, and zeroes in other equations.
- We divide R_3 by 3 to obtain the following.

$$\begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & b \\ R1 & 1 & 0 & 1 & 2 & 0 \\ R2 & 0 & 1 & -2 & 1 & 2 \\ R3 & 0 & 0 & 1 & -2 & 1 \end{array}$$

- Replace R_2 by $R_2 + 2R_3$ and R_1 by $R_1 - R_3$ to make coefficients of x_3 zeroes in other equations.

$$\begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & b \\ R1 & 1 & 0 & 0 & 4 & -1 \\ R2 & 0 & 1 & 0 & -3 & 4 \\ R3 & 0 & 0 & 1 & -2 & 1 \end{array}$$

- The above set of equations are in **the reduced row echelon form with respect to basic variables x_1, x_2, and x_3**

BASIC SOLUTION

- A basic solution is obtained from the **canonical form** by setting the non-basic or independent variables to zero.
- A **basic feasible solution** is a basic solution in which the values of basic variables are nonnegative.

BASIC SOLUTION

Example:

- Consider the following canonical form

$$\left[\begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & b \\ R_1 & 1 & 0 & 0 & 4 & -1 \\ R_2 & 0 & 1 & 0 & -3 & 4 \\ R_3 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

- Set the non-basic variable, x_4, to zero.
- The basic solution can then be written as $x_1 = -1$, $x_2 = 4$, $x_3 = 1$, $x_4 = 0$.
- <u>Note that this basic solution is not a basic feasible solution for the standard LP formulation, since $x_1 < 0$.</u>

BASIC SOLUTION

- The choice of the set of basic variables is arbitrary, and can be decided as per computational convenience.
- For a generic problem, any set of m variables (recall we have m equations) from the possible n variables can be chosen as basic variables.
- This implies that the number of basic solutions for a generic standard LP problem with ***m*** constraints and ***n*** variables is given as

$$C_m^n = \frac{n!}{m!(n-m)!}$$

BASIC SOLUTION

Example:

$$
\begin{aligned}
R_1 &\equiv x_1 + x_2 - x_3 + 3x_4 = 2 \\
R_2 &\equiv -x_1 + 3x_2 - 7x_3 + x_4 = 6 \\
R_3 &\equiv x_1 + 2x_2 \qquad\qquad - 2x_4 = 7
\end{aligned}
$$

- Here, m = 3, and n = 4. Therefore we have

$$C_3^4 = \frac{4!}{3!(4-3)!} = 4$$

SIMPLEX ALGORITHM

- The optimal solution of the LP problem lies at one of the vertices of the feasible convex polygon.
- In the **Simplex method** the **solution process efficiently moves from one basic feasible solution to the next ensuring objective function reduction at each iteration**.
- We posed the standard LP problem as a minimization problem; the rules of the following algorithm apply to minimization problems only.

1- Transform into Standard LP Problem:

Transform the given problem into the standard LP formulation by adding slack/surplus variables.

$$\min_{x} \quad z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

$$\text{such that}$$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + d_1 s_1 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + d_2 s_2 &= b_2 \\
\vdots \qquad\qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + d_m s_m &= b_m \\
x_1, \ldots, x_n, s_1, \ldots s_m &\geq 0
\end{aligned}$$

Example:

$$\min_{x} \quad z = x_1 - 2x_2$$

such that

$$x_1 + x_2 \leq 4$$
$$-x_1 + x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

$$\min_{x} \quad z = x_1 - 2x_2$$

such that

$$x_1 + x_2 + s_1 = 4$$
$$-x_1 + x_2 + s_2 = 3$$
$$x_1, x_2, s_1, s_2 \geq 0$$

SIMPLEX ALGORITHM

2- Form Initial Simplex Tableau:

List the constraints and the objective function coefficients in the form of a table, known as the simplex tableau.

	x_1	$\ldots$	x_n	s_1	$\ldots$	s_p	b
Constraint 1	a_{11}	$\ldots$	a_{1n}	1	$\ldots$	0	b_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
Constraint m	a_{m1}	$\ldots$	a_{mn}	0	$\ldots$	1	b_m
Objective	c_1	$\ldots$	c_n	0	0	0	f

Example:

$$\min_{x} \quad z = x_1 - 2x_2$$

such that

$$x_1 + x_2 + s_1 = 4$$
$$-x_1 + x_2 + s_2 = 3$$
$$x_1, x_2, s_1, s_2 \geq 0$$

	x_1	x_2	s_1	s_2	b
R_1	1	1	1	0	4
R_2	-1	1	0	1	3
R_3	1	-2	0	0	f

SIMPLEX ALGORITHM

3- Choose Variable that Enters Basis – Identify Pivotal Column

- The simplex algorithm begins with the initial basic feasible solution.
- By observing the coefficients of the objective function row in the initial simplex tableau, the algorithm moves to the adjacent basic feasible solution that reduces the objective function.
- The other adjacent basic solution(s) with objective function value higher than the current solution are disregarded.

3- Choose Variable that Enters Basis – Identify Pivotal Column

- Choose the variable with the highest negative coefficient in the objective function row to <u>become the basic variable</u>.
- The variable <u>with the highest negative coefficient </u>has the potential to reduce the objective to the maximum extent.
- If make the corresponding variable a basic variable, the variable will be non-negative. The variable then multiplied with the highest negative coefficient in the objective function yields the most minimization.

Example:

	x_1	x_2	s_1	s_2	b
R_1	1	**1**	1	0	4
R_2	-1	**1**	0	1	3
R_3	1	**-2**	0	0	f

- Observe the entries of the objective function row (R_3).
- The coefficient of x_2 is negative.
- The current basic variables, s_1 and s_2, are not part of the objective function.
- If we make x_2 a basic variable, the objective function value can reduce from the current value.

4- Minimum Ratio rule – Identify Pivotal Row

- After add one variable to the basic variable, an existing basic variable must be made non-basic.
- **Minimum ratio rule:** Determine which basic variable is eliminated.
- Compute the following ratio for the selected column in the previous step corresponding to the variable that enters the basis, say x_j .

$$\min_{\text{for all } a_{ij}>0} \quad \frac{b_i}{a_{ij}}$$

- The row that satisfies the above minimum ratio rule is then selected as the pivotal row.

4- Minimum Ratio rule – Identify Pivotal Row

- **Special cases:**

  (1) If all the coefficients of the column corresponding to the chosen basic variable, are negative, we cannot compute the minimum ratio. In such cases, the LP problem has an unbounded solution.

  (2) If two or more rows have the same minimum ratio, choose any pivotal row of choice.

  (3) When one or more of the basic variables have zero values, the solution is said to be degenerate. This can happen when the right hand side value is zero, and consequently, the minimum ratio is zero. This usually implies that adding a new variable to the basic variable set may not reduce the objective function value.

5- Reduce to Canonical Form:

- Once we have chosen the pivotal row and the pivotal column based on the above rules, we can then **identify the pivotal element**.
- The constraint set is then transformed into a reduced row echelon form with respect to the newly identified incoming basic variable.

Example:

- Reduce the initial simplex tableau into the canonical form with respect to the newly added basic variable, x_2.

	x_1	x_2	s_1	s_2	b
R_1	2	**0**	1	-1	1
R_2	-1	**1**	0	1	3
R_3	-1	**0**	0	2	$f+6$

- The basic solution is $x_1 = 0$, $x_2 = 3$, $s_1 = 1$, $s_2 = 0$, and the function value is $f = -6$.
- <u>Note that x_2 has entered the basis, and s2 has left the basis.</u>
- The value of x_2 has increased from zero in the initial simplex tableau to three in the current iteration, and the function value has reduced from $f = 0$ to $f = -6$.

6- Check for optimality:

If the coefficients of the objective function are all nonnegative, we reached the optimum. **If not**, we repeat the simplex algorithm until the above termination criterion is met.

Example:

- The coefficient of x_1 in the objective function row is negative. Choose x_1 as the variable entering the basis

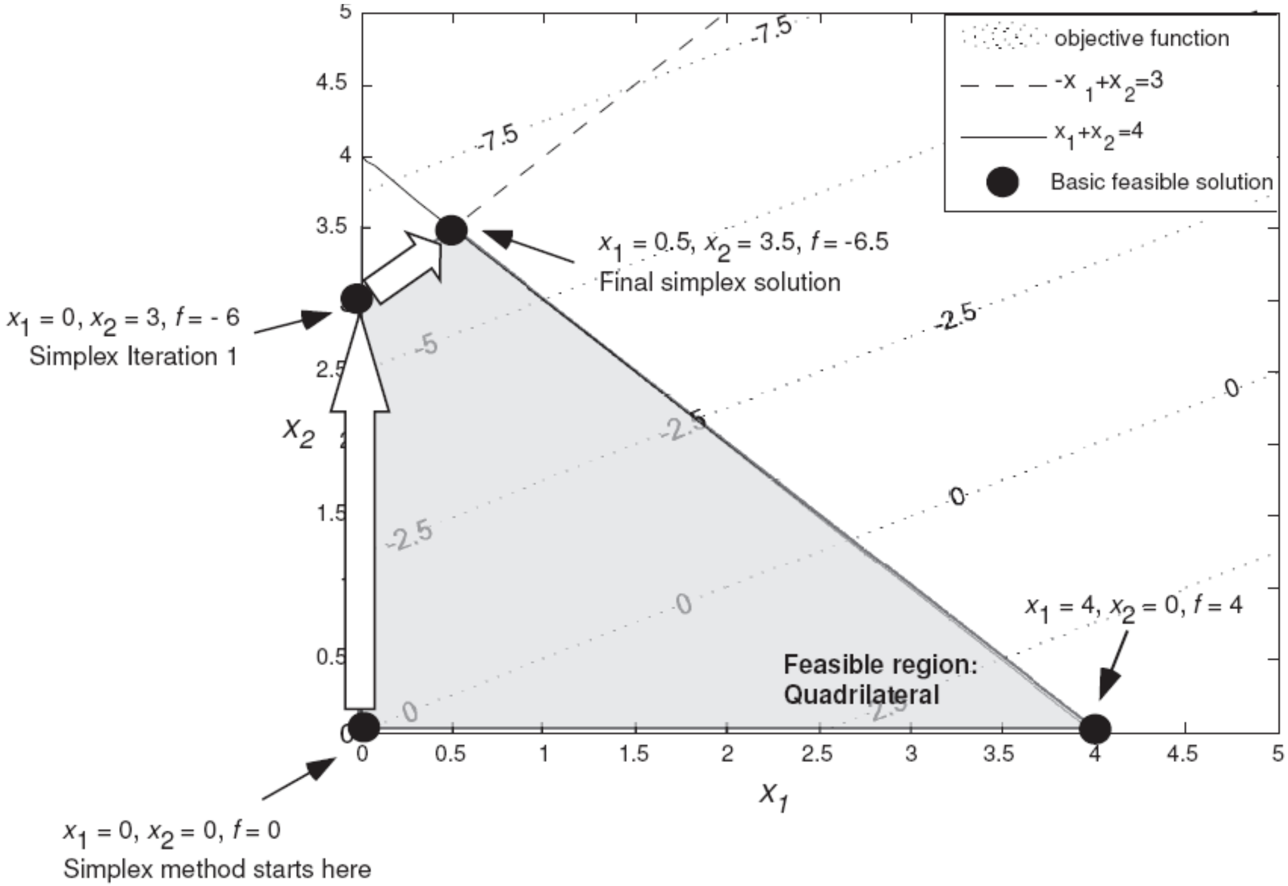
	x_1	x_2	s_1	s_2	b	$\frac{b_i}{a_{ij}}$
R_1	**2**	**0**	**1**	**-1**	**1**	1/2
R_2	**-1**	1	0	1	3	
R_3	**-1**	0	0	2	$f+6$	

Final simplex tableau

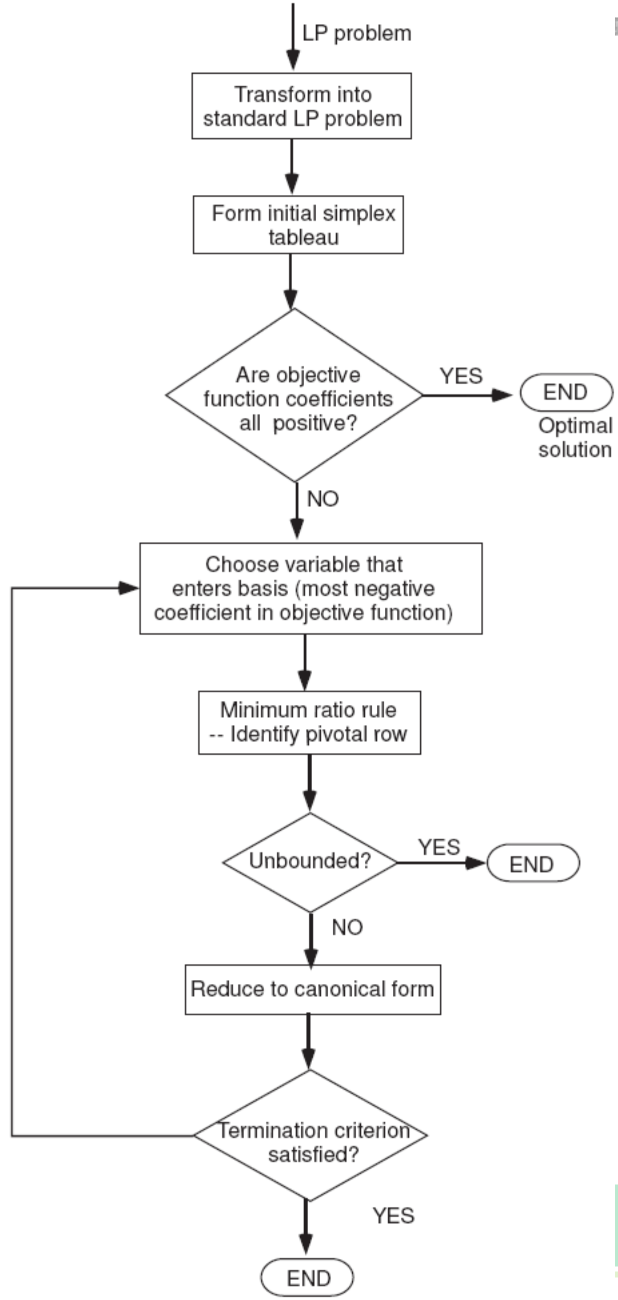
	x_1	x_2	s_1	s_2	b
R_1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
R_2	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{7}{2}$
R_3	0	0	$\frac{1}{2}$	$\frac{3}{2}$	$f+\frac{13}{2}$

SIMPLEX ALGORITHM

- The progression of the iterations of the simplex method.



SIMPLEX ALGORITHM



Other Special Cases of Simplex Method

- The discussion of simplex method in the previous subsection assumes that the LP problem at hand can be reduced into the standard formulation.
- In some cases, it may not be directly possible to do so.
- One requirement of the standard LP problem is that the variables must be non-negative, and the right hand side constants for constraints must be non-negative.
- In some problems, obtaining a standard LP problem is not straightforward.
- In such cases, use of so-called artificial variables is employed. Variations of simplex method known as two-phase simplex method and dual simplex method are used.

ENGINEERING EXAMPLES

Example 3.2 A manufacturing firm produces two machine parts using lathes, milling machines, and grinding machines. The different machining times required for each part, the machining times available on different machines, and the profit on each machine part are given in the following table.

Type of machine	Machining time required (min)		Maximum time available per week (min)
	Machine part I	Machine part II	
Lathes	10	5	2500
Milling machines	4	10	2000
Grinding machines	1	1.5	450
Profit per unit	\$50	\$100	

Determine the number of parts I and II to be manufactured per week to maximize the profit.

SOLUTION Let the number of machine parts I and II manufactured per week be denoted by x and y, respectively. The constraints due to the maximum time limitations on the various machines are given by

$$10x + 5y \leq 2500 \tag{E$_1$}$$

$$4x + 10y \leq 2000 \tag{E$_2$}$$

$$x + 1.5y \leq 450 \tag{E$_3$}$$

Since the variables x and y cannot take negative values, we have

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \end{aligned} \tag{E$_4$}$$

The total profit is given by

$$f(x, y) = 50x + 100y \tag{E$_5$}$$

ENGINEERING EXAMPLES

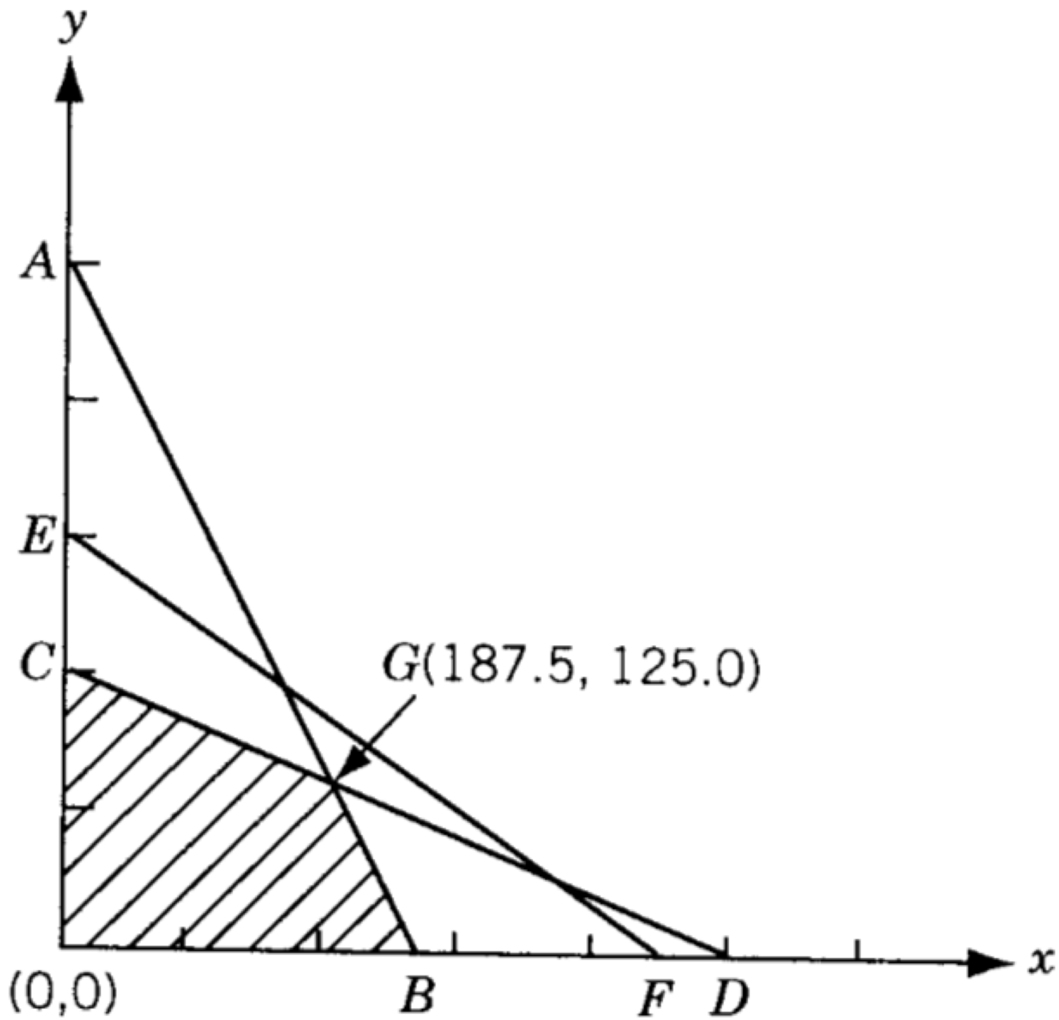


Figure 3.3 Feasible region given by Eqs. (E_1) to (E_4).

BASIC SOLUTIONS

Example 3.3 Find all the basic solutions corresponding to the system of equations

$$2x_1 + 3x_2 - 2x_3 - 7x_4 = 1 \quad (\mathrm{I}_0)$$

$$x_1 + x_2 + x_3 + 3x_4 = 6 \quad (\mathrm{II}_0)$$

$$x_1 - x_2 + x_3 + 5x_4 = 4 \quad (\mathrm{III}_0)$$

BASIC SOLUTIONS

SOLUTION First we reduce the system of equations into a canonical form with x_1, x_2, and x_3 as basic variables. For this, first we pivot on the element $a_{11} = 2$ to obtain

$$\begin{aligned}
x_1 + \tfrac{3}{2}x_2 - x_3 - \tfrac{7}{2}x_4 &= \tfrac{1}{2} & \mathrm{I}_1 &= \tfrac{1}{2}\mathrm{I}_0 \\
0 - \tfrac{1}{2}x_2 + 2x_3 + \tfrac{13}{2}x_4 &= \tfrac{11}{2} & \mathrm{II}_1 &= \mathrm{II}_0 - \mathrm{I}_1 \\
0 - \tfrac{5}{2}x_2 + 2x_3 + \tfrac{17}{2}x_4 &= \tfrac{7}{2} & \mathrm{III}_1 &= \mathrm{III}_0 - \mathrm{I}_1
\end{aligned}$$

Then we pivot on $a'_{22} = -\frac{1}{2}$, to obtain

$$\begin{aligned}
x_1 + 0 + 5x_3 + 16x_4 &= 17 & \mathrm{I}_2 &= \mathrm{I}_1 - \tfrac{3}{2}\mathrm{II}_2 \\
0 + x_2 - 4x_3 - 13x_4 &= -11 & \mathrm{II}_2 &= -2\mathrm{II}_1 \\
0 + 0 - 8x_3 - 24x_4 &= -24 & \mathrm{III}_2 &= \mathrm{III}_1 + \tfrac{5}{2}\mathrm{II}_2
\end{aligned}$$

BASIC SOLUTIONS

Finally we pivot on a'_{33} to obtain the required canonical form as

$$\begin{aligned} x_1 \quad\quad + x_4 &= 2 \qquad \mathrm{I}_3 = \mathrm{I}_2 - 5\mathrm{III}_3 \\ x_2 \quad - x_4 &= 1 \qquad \mathrm{II}_3 = \mathrm{II}_2 + 4\mathrm{III}_3 \\ x_3 + 3x_4 &= 3 \qquad \mathrm{III}_3 = -\tfrac{1}{8}\,\mathrm{III}_2 \end{aligned}$$

From this canonical form, we can readily write the solution of x_1, x_2, and x_3 in terms of the other variable x_4 as

$$x_1 = 2 - x_4$$

$$x_2 = 1 + x_4$$

$$x_3 = 3 - 3x_4$$

If Eqs. (I_0), (II_0), and (III_0) are the constraints of a linear programming problem, the solution obtained by setting the independent variable equal to zero is called a basic solution. In the present case, the basic solution is given by

$$x_1 = 2, \quad x_2 = 1, \quad x_3 = 3 \quad \text{(basic variables)}$$

and $x_4 = 0$ (nonbasic or independent variable). Since this basic solution has all $x_j \geq 0$ ($j = 1, 2, 3, 4$), it is a basic feasible solution.

BASIC SOLUTIONS

$$
\begin{aligned}
x_1 \qquad - \tfrac{1}{3}x_3 &= 1 \qquad \mathrm{I}_4 = \mathrm{I}_3 - \mathrm{III}_4 \\
x_2 + \tfrac{1}{3}x_3 &= 2 \qquad \mathrm{II}_4 = \mathrm{II}_3 + \mathrm{III}_4 \\
x_4 + \tfrac{1}{3}x_3 &= 1 \qquad \mathrm{III}_4 = \tfrac{1}{3}\mathrm{III}_3
\end{aligned}
$$

This canonical system gives the solution of x_1, x_2, and x_4 in terms of x_3 as

$$x_1 = 1 + \tfrac{1}{3}x_3$$

$$x_2 = 2 - \tfrac{1}{3}x_3$$

$$x_4 = 1 - \tfrac{1}{3}x_3$$

and the corresponding basic solution is given by

$$x_1 = 1, \quad x_2 = 2, \quad x_4 = 1 \quad \text{(basic variables)}$$

$$x_3 = 0 \quad \text{(nonbasic variable)}$$

BASIC SOLUTIONS

$$
\begin{aligned}
x_1 \qquad + x_2 &= 3 \qquad & \mathrm{I}_5 &= \mathrm{I}_4 + \tfrac{1}{3}\mathrm{II}_5 \\
x_3 \quad + 3x_2 &= 6 \qquad & \mathrm{II}_5 &= 3\mathrm{II}_4 \\
x_4 - x_2 &= -1 \qquad & \mathrm{III}_5 &= \mathrm{III}_4 - \tfrac{1}{3}\mathrm{II}_5
\end{aligned}
$$

The solution for x_1, x_3, and x_4 is given by

$$x_1 = 3 - x_2$$

$$x_3 = 6 - 3x_2$$

$$x_4 = -1 + x_2$$

from which the basic solution can be obtained as

$$x_1 = 3, \quad x_3 = 6, \quad x_4 = -1 \quad \text{(basic variables)}$$

$$x_2 = 0 \quad \text{(nonbasic variable)}$$

Since all the x_j are not nonnegative, this basic solution is not feasible.

BASIC SOLUTIONS

Finally, to obtain the canonical form in terms of the basic variables x_2, x_3, and x_4, we pivot on a''_{12} in Eq. (I_5), thereby bringing x_2 into the current basis in place of x_1. This gives

$$\begin{aligned} x_2 \quad + x_1 &= 3 & \mathrm{I}_6 &= \mathrm{I}_5 \\ x_3 \quad - 3x_1 &= -3 & \mathrm{II}_6 &= \mathrm{II}_5 - 3\mathrm{I}_6 \\ x_4 + x_1 &= 2 & \mathrm{III}_6 &= \mathrm{III}_5 + \mathrm{I}_6 \end{aligned}$$

This canonical form gives the solution for x_2, x_3, and x_4 in terms of x_1 as

$$x_2 = 3 - x_1$$

$$x_3 = -3 + 3x_1$$

$$x_4 = 2 - x_1$$

and the corresponding basic solution is

$$x_2 = 3, \quad x_3 = -3, \quad x_4 = 2 \quad \text{(basic variables)}$$

$$x_1 = 0 \quad \text{(nonbasic variable)}$$

This basic solution can also be seen to be infeasible due to the negative value for x_3.

IDENTIFYING AN OPTIMAL POINT

Theorem 3.7 A basic feasible solution is an optimal solution with a minimum objective function value of f_0'' if all the cost coefficients c_j'', $j = m+1, m+2, \ldots, n$, in Eqs. (3.21) are nonnegative.

Proof: From the last row of Eqs. (3.21), we can write that

$$f_0'' + \sum_{i=m+1}^{n} c_i'' x_i = f \tag{3.24}$$

IDENTIFYING AN OPTIMAL POINT

Since the variables $x_{m+1}, x_{m+2}, \ldots, x_n$ are presently zero and are constrained to be nonnegative, the only way any one of them can change is to become positive. But if $c_i'' > 0$ for $i = m + 1, m + 2, \ldots, n$, then increasing any x_i cannot decrease the value of the objective function f. Since no change in the nonbasic variables can cause f to decrease, the present solution must be optimal with the optimal value of f equal to f_0''.

A glance over c_i'' can also tell us if there are multiple optima. Let all $c_i'' > 0$, $i = m + 1, m + 2, \ldots, k - 1, k + 1, \ldots, n$, and let $c_k'' = 0$ for some nonbasic variable x_k. Then if the constraints allow that variable to be made positive (from its present value of zero), no change in f results, and there are multiple optima. It is possible, however, that the variable may not be allowed by the constraints to become positive; this may occur in the case of degenerate solutions. Thus as a corollary to the discussion above, we can state that a basic feasible solution is the unique optimal feasible solution if $c_j'' > 0$ for all nonbasic variables x_j, $j = m + 1, m + 2, \ldots, n$. If, after testing for optimality, the current basic feasible solution is found to be nonoptimal, an improved basic solution is obtained from the present canonical form as follows.

ADVANCED CONCEPTS

- **Two Phases of the Simplex Method**
- **Duality**
- **Transportation Problem**
- **Interior point methods.**

TWO-PHASE SIMPLEX METHOD

The problem is to find nonnegative values for the variables $x_1, x_2, \ldots, x_n$ that satisfy the equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{3.32}$$

and minimize the objective function given by

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = f \tag{3.33}$$

The general problems encountered in solving this problem are

1. An initial feasible canonical form may not be readily available. This is the case when the linear programming problem does not have slack variables for some of the equations or when the slack variables have negative coefficients.

2. The problem may have redundancies and/or inconsistencies, and may not be solvable in nonnegative numbers.

TWO-PHASE SIMPLEX METHOD

- Phase I of the simplex method uses the simplex algorithm itself to find whether the linear programming problem has a feasible solution. If a feasible solution exists, it provides a basic feasible solution in canonical form ready to initiate phase II of the method.
- Phase II, in turn, uses the simplex algorithm to find whether the problem has a bounded optimum. If a bounded optimum exists, it finds the basic feasible solution that is optimal.

TWO-PHASE SIMPLEX METHOD

1. Arrange the original system of Eqs. (3.32) so that all constant terms b_i are positive or zero by changing, where necessary, the signs on both sides of any of the equations.
2. Introduce to this system a set of artificial variables $y_1, y_2, \ldots, y_m$ (which serve as basic variables in phase I), where each $y_i \geq 0$, so that it becomes

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + y_1 \qquad\qquad &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \qquad + y_2 \qquad &= b_2 \\
&\ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \qquad\qquad + y_m &= b_m \\
b_i \geq 0 &
\end{aligned} \tag{3.34}$$

Note that in Eqs. (3.34), for a particular i, the a_{ij}'s and the b_i may be the negative of what they were in Eq. (3.32) because of step 1.

The objective function of Eq. (3.33) can be written as

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n + (-f) = 0 \tag{3.35}$$

TWO-PHASE SIMPLEX METHOD

3. *Phase I of the method*. Define a quantity w as the sum of the artificial variables

$$w = y_1 + y_2 + \cdots + y_m \tag{3.36}$$

and use the simplex algorithm to find $x_i \geq 0$ $(i = 1, 2, \ldots, n)$ and $y_i \geq 0$ $(i = 1, 2, \ldots, m)$ which minimize w and satisfy Eqs. (3.34) and (3.35). Consequently, consider the array

$$\begin{array}{lllll}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + y_1 & & & = & b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & + y_2 & & = & b_2 \\
\vdots & & & & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & + y_m & & = & b_m \\
c_1x_1 + c_2x_2 + \cdots + c_nx_n & & + (-f) & = & 0 \\
\qquad y_1 + y_2 + \cdots + y_m & & + (-w) & = & 0
\end{array} \tag{3.37}$$

This array is not in canonical form; however, it can be rewritten as a canonical system with basic variables $y_1, y_2, \ldots, y_m, -f$, and $-w$ by subtracting the sum of the first m equations from the last to obtain the new system

$$\begin{array}{lllll}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + y_1 & & & = & b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & + y_2 & & = & b_2 \\
\vdots & & & & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & + y_m & & = & b_m \\
c_1x_1 + c_2x_2 + \cdots + c_nx_n & & + (-f) & = & 0 \\
d_1x_1 + d_2x_2 + \cdots + d_nx_n & & + (-w) & = & -w_0
\end{array} \tag{3.38}$$

where

$$d_i = -(a_{1i} + a_{2i} + \cdots + a_{mi}), \quad i = 1, 2, \ldots, n \tag{3.39}$$

$$-w_0 = -(b_1 + b_2 + \cdots + b_m) \tag{3.40}$$

Equations (3.38) provide the initial basic feasible solution that is necessary for starting phase I.

4. In Eq. (3.37), the expression of w, in terms of the artificial variables $y_1, y_2, \ldots, y_m$ is known as the *infeasibility form*. w has the property that if as a result of phase I, with a minimum of $w > 0$, no feasible solution exists for the original linear programming problem stated in Eqs. (3.32) and (3.33), and thus the procedure is terminated. On the other hand, if the minimum of $w = 0$, the resulting array will be in canonical form and hence initiate phase II by eliminating the w equation as well as the columns corresponding to each of the artificial variables $y_1, y_2, \ldots, y_m$ from the array.

5. *Phase II of the method*. Apply the simplex algorithm to the adjusted canonical system at the end of phase I to obtain a solution, if a finite one exists, which optimizes the value of f.

TWO-PHASE SIMPLEX METHOD

Example 3.7

$$\text{Minimize } f = 2x_1 + 3x_2 + 2x_3 - x_4 + x_5$$

subject to the constraints

$$\begin{aligned} 3x_1 - 3x_2 + 4x_3 + 2x_4 - x_5 &= 0 \\ x_1 + x_2 + x_3 + 3x_4 + x_5 &= 2 \\ x_i \geq 0, \quad i = 1 \text{ to } 5 & \end{aligned}$$

SOLUTION

Step 1 As the constants on the right-hand side of the constraints are already nonnegative, the application of step 1 is unnecessary.

Step 2 Introducing the artificial variables $y_1 \geq 0$ and $y_2 \geq 0$, the equations can be written as follows:

$$\begin{aligned} 3x_1 - 3x_2 + 4x_3 + 2x_4 - x_5 + y_1 \qquad\qquad &= 0 \\ x_1 + x_2 + x_3 + 3x_4 + x_5 \qquad + y_2 \qquad &= 2 \\ 2x_1 + 3x_2 + 2x_3 - x_4 + x_5 \qquad\qquad - f &= 0 \end{aligned} \tag{E$_1$}$$

Step 3 By defining the infeasibility form w as

$$w = y_1 + y_2$$

TWO-PHASE SIMPLEX METHOD

This array can be rewritten as a canonical system with basic variables as y_1, y_2, $-f$, and $-w$ by subtracting the sum of the first two equations of (E_2) from the last equation of (E_2). Thus the last equation of (E_2) becomes

$$-4x_1 + 2x_2 - 5x_3 - 5x_4 + 0x_5 - w = -2 \qquad (E_3)$$

Since this canonical system [first three equations of (E_2), and (E_3)] provides an initial basic feasible solution, phase I of the simplex method can be started. The phase I computations are shown below in tableau form.

Basic variables	Admissible variables x_1	x_2	x_3	x_4	x_5	Artificial variables y_1	y_2	b_i''	Value of b_i''/a_{is}'' for $a_{is}'' > 0$	
y_1	3	−3	4	2 Pivot element	−1	1	0	0	0	← Smaller value (y_1 drops from next basis)
y_2	1	1	1	3	1	0	1	2	$\frac{2}{3}$	
$-f$	2	3	2	−1	1	0	0	0		
$-w$	−4	2	−5	−5	0	0	0	−2		
			↑	↑						

Most negative

Since there is a tie between d_3'' and d_4'', d_4'' is selected arbitrarily as the most negative d_i'' for pivoting (x_4 enters the next basis).
Result of pivoting:

TWO-PHASE SIMPLEX METHOD

x_4	$\frac{3}{2}$	$-\frac{3}{2}$	2	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0		
y_2	$-\frac{7}{2}$	$\boxed{\frac{11}{2}}$ Pivot element	-5	0	$\frac{5}{2}$	$-\frac{3}{2}$	1	2	$\frac{1}{11}$	$\leftarrow$ y_2 drops from next basis
$-f$	$\frac{7}{2}$	$\frac{3}{2}$	4	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0		
$-w$	$\frac{7}{2}$	$-\frac{11}{2}$	5	0	$-\frac{5}{2}$	$\frac{5}{2}$	0	-2		

$\uparrow$

Most negative d_i'' (x_2 enters next basis)

Result of pivoting (since y_1 and y_2 are dropped from basis, the columns corresponding to them need not be filled):

x_4	$\frac{6}{11}$	0	$\frac{7}{11}$	1	$\frac{2}{11}$	Dropped		$\frac{6}{11}$	$\frac{6}{2}$
x_2	$-\frac{7}{11}$	1	$-\frac{10}{11}$	0	$\frac{5}{11}$			$\frac{4}{11}$	$\frac{4}{5}$
$-f$	$\frac{98}{22}$	0	$\frac{118}{22}$	0	$-\frac{4}{22}$			$-\frac{6}{11}$	
$-w$	0	0	0	0	0			0	

TWO-PHASE SIMPLEX METHOD

Step 4 At this stage we notice that the present basic feasible solution does not contain any of the artificial variables y_1 and y_2, and also the value of w is reduced to 0. This indicates that phase I is completed.

Step 5 Now we start phase II computations by dropping the w row from further consideration. The results of phase II are again shown in tableau form:

Basic variables	Original variables					Constant	Value of b_i''/a_{is}'' for
	x_1	x_2	x_3	x_4	x_5	b_i''	$a_{is}'' > 0$
x_4	$\frac{6}{11}$	0	$\frac{7}{11}$	1	$\frac{2}{11}$	$\frac{6}{11}$	$\frac{6}{2}$
x_2	$-\frac{7}{11}$	1	$-\frac{10}{11}$	0	$\boxed{\frac{5}{11}}$ Pivot element	$\frac{4}{11}$	$\frac{4}{5}$ ← Smaller value (x_2 drops from next basis)
$-f$	$\frac{98}{22}$	0	$\frac{118}{22}$	0	$-\frac{4}{22}$	$-\frac{6}{11}$	

↑ Most negative c_i'' (x_5 enters next basis)

Result of pivoting:

x_4	$\frac{4}{5}$	$-\frac{2}{5}$	1	1	0	$\frac{2}{5}$
x_5	$-\frac{7}{5}$	$\frac{11}{5}$	-2	0	1	$\frac{4}{5}$
$-f$	$\frac{21}{5}$	$\frac{2}{5}$	5	0	0	$-\frac{2}{5}$

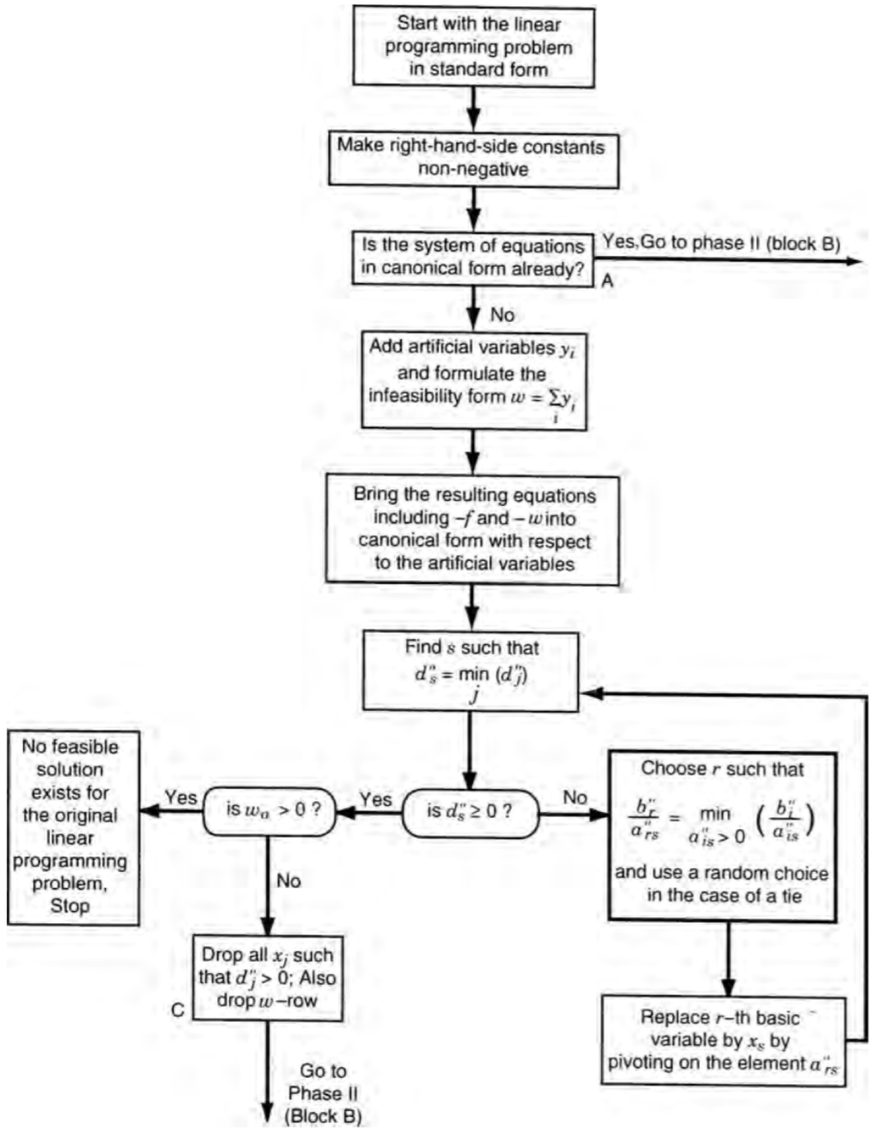
Now, since all c_i'' are nonnegative, phase II is completed. The (unique) optimal solution is given by

$$x_1 = x_2 = x_3 = 0 \quad \text{(nonbasic variables)}$$

$$x_4 = \frac{2}{5}, \quad x_5 = \frac{4}{5} \quad \text{(basic variables)}$$

$$f_{\min} = \frac{2}{5}$$

TWO-PHASE SIMPLEX METHOD



TWO-PHASE SIMPLEX METHOD

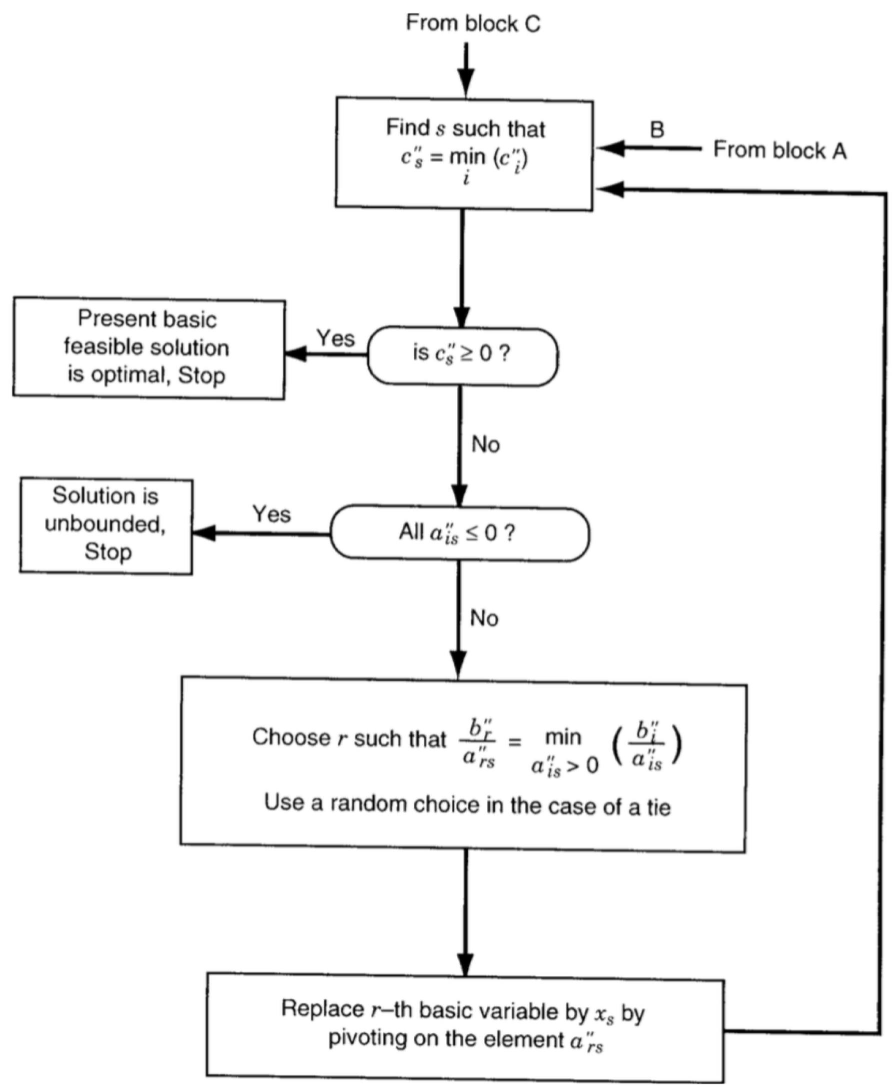


Figure 3.15 *(continued)*

DUALITY

- **Duality** is an important concept in linear programming problems.
- Each LP problem, known as the primal, had another corresponding LP problem, known as the dual, associated with it.
- The solutions of the primal and dual problems have interesting relationships.

Primal Problem.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \\
c_1x_1 + c_2x_2 + \cdots + c_nx_n &= f \\
(x_i \geq 0,\ i = 1 \text{ to } n, \text{ and } f &\text{ is to be minimized})
\end{aligned}
\tag{4.17}
$$

Dual Problem. As a definition, the dual problem can be formulated by transposing the rows and columns of Eq. (4.17) including the right-hand side and the objective function, reversing the inequalities and maximizing instead of minimizing. Thus by denoting the dual variables as $y_1, y_2, \ldots, y_m$, the dual problem becomes

$$
\begin{aligned}
a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m &\leq c_1 \\
a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}x_m &\leq c_2 \\
&\vdots \\
a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m &\leq c_n \\
b_1y_1 + b_2y_2 + \cdots + b_my_m &= v \\
(y_i \geq 0,\ i = 1 \text{ to } m, \text{ and } v &\text{ is to be maximized})
\end{aligned}
\tag{4.18}
$$

Equations (4.17) and (4.18) are called *symmetric primal–dual pairs* and it is easy to see from these relations that the dual of the dual is the primal.

DUALITY

Primal problem

$$\min_{x} \quad z = c^T x$$

such that

$$Ax \leq b$$

$$x \geq 0$$

The <u>corresponding dual problem</u>

$$\max_{x} \quad z_d = b^T y$$

such that

$$A^T y \geq c$$

$$y \geq 0$$

If the primal is a minimization problem, the dual will be a maximization problem, and vice versa.

Example:

The primal problem

$$\min_{x} \quad z = x_1 - x_2 - 2x_3 + 4x_4$$

such that

$$\begin{aligned} x_1 + 5x_2 - 2x_3 + 3x_4 &\leq 1 \\ 5x_1 + x_2 + 3x_3 + 8x_4 &\leq 5 \\ x_1 + 2x_2 + 3x_3 + 5x_4 &\leq 3 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The dual problem

$$\max_{y} \quad z_d = y_1 + 5y_2 + 3y_3$$

such that

$$\begin{aligned} y_1 + 5y_2 + y_3 &\geq 1 \\ 5y_1 + y_2 + 2y_3 &\geq -1 \\ -2y_1 + 3y_2 + 3y_3 &\geq -2 \\ 3y_1 + 8y_2 + 5y_3 &\geq 4 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

- The primal problem had three inequality constraints, whereas the dual problem has three design variables.
- The primal problem had four design variables, and the dual problem has four inequality constraints.

DUALITY

Example:

The primal problem

$$\min_x \quad z = x_1 - x_2 - 2x_3 + 4x_4$$

such that

$$x_1 + 5x_2 - 2x_3 + 3x_4 \leq 1$$
$$5x_1 + x_2 + 3x_3 + 8x_4 \leq 5$$
$$x_1 + 2x_2 + 3x_3 + 5x_4 \leq 3$$
$$x_1, x_2, x_3, x_4 \geq 0$$

The dual problem

$$\max_y \quad z_d = y_1 + 5y_2 + 3y_3$$

such that

$$y_1 + 5y_2 + y_3 \geq 1$$
$$5y_1 + y_2 + 2y_3 \geq -1$$
$$-2y_1 + 3y_2 + 3y_3 \geq -2$$
$$3y_1 + 8y_2 + 5y_3 \geq 4$$
$$y_1, y_2, y_3 \geq 0$$

- The primal is a minimization problem, and the dual is a maximization problem, and
- The primal inequalities are of the form ≤ 0, whereas the dual constraints are of the form ≥ 0.

DUALITY

Table 4.11 Correspondence Rules for Primal–Dual Relations

Primal quantity	Corresponding dual quantity
Objective function: Minimize $\mathbf{c}^{\mathrm{T}}\mathbf{X}$	Maximize $\mathbf{Y}^{\mathrm{T}}\mathbf{b}$
Variable $x_i \geq 0$	ith constraint $\mathbf{Y}^{\mathrm{T}}\mathbf{A}_i \leq c_i$ (inequality)
Variable x_i unrestricted in sign	ith constraint $\mathbf{Y}^{\mathrm{T}}\mathbf{A}_i = c_i$ (equality)
jth constraint, $\mathbf{A}_j\mathbf{X} = b_j$ (equality)	jth variable y_j unrestricted in sign
jth constraint, $\mathbf{A}_j\mathbf{X} \geq b_j$ (inequality)	jth variable $y_j \geq 0$
Coefficient matrix $\mathbf{A} \equiv [\mathbf{A}_1 \ldots \mathbf{A}_m]$	Coefficient matrix $\mathbf{A}^{\mathrm{T}} \equiv [\mathbf{A}_1, \ldots, \mathbf{A}_m]^{\mathrm{T}}$
Right-hand-side vector $\mathbf{b}$	Right-hand-side vector $\mathbf{c}$
Cost coefficients $\mathbf{c}$	Cost coefficients $\mathbf{b}$

DUALITY

Table 4.12 Primal–Dual Relations

Primal problem	Corresponding dual problem
Minimize $f = \sum_{i=1}^{n} c_i x_i$ subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i,\ i = 1, 2, \ldots, m^*$ $\sum_{j=1}^{n} a_{ij} x_j \geq b_i,\ i = m^* + 1, m^* + 2, \ldots, m$ where $x_i \geq 0, i = 1, 2, \ldots, n^*$; and x_i unrestricted in sign, $i = n^* + 1, n^* + 2, \ldots, n$	Maximize $v = \sum_{i=1}^{m} y_i b_i$ subject to $\sum_{i=1}^{m} y_i a_{ij} = c_j,\ j = n^* + 1, n^* + 2, \ldots, n$ $\sum_{i=1}^{m} y_i a_{ij} \leq c_j, j = 1, 2, \ldots, n^*$ where $y_i \geq 0, i = m^* + 1, m^* + 2, \ldots, m$; and y_i unrestricted in sign, $i = 1, 2, \ldots, m^*$

PRIMAL-DUAL RELATIONSHIPS

1. The dual of a dual problem is the primal problem.
2. Every feasible solution for the dual problem provides a lower bound on every feasible solution to the primal.
3. If the primal solution has a feasible solution, and the dual problem has a feasible solution, then there exist optimal feasible solutions such that the objective function values of the primal and the dual are the same. This is known as the strong duality theorem.
4. The objective function value of the dual problem evaluated at any feasible solution provides a lower bound on the objective function value of the primal problem evaluated at any feasible solution. **This is known as the weak duality theorem**.
5. If the primal problem is unbounded, the dual problem is infeasible.

DUAL SIMPLEX EXAMPLE

- Computationally, the dual simplex algorithm also involves a sequence of pivot operations, but with different rules (compared to the regular simplex method) for choosing the pivot element.

Let the problem to be solved be initially in canonical form with some of the $\bar{b}_i < 0$, the relative cost coefficients corresponding to the basic variables $\bar{c}_j = 0$, and all other $\bar{c}_j \geq 0$. Since some of the $\bar{b}_i$ are negative, the primal solution will be infeasible, and since all $\bar{c}_j \geq 0$, the corresponding dual solution will be feasible. Then the simplex method works according to the following iterative steps.

DUAL SIMPLEX EXAMPLE

1. Select row r as the pivot row such that

$$\bar{b}_r = \min \bar{b}_i < 0 \tag{4.22}$$

2. Select column s as the pivot column such that

$$\frac{\bar{c}_s}{-\bar{a}_{rs}} = \min_{\bar{a}_{rj}<0} \left(\frac{\bar{c}_j}{-\bar{a}_{rj}} \right) \tag{4.23}$$

   If all $\bar{a}_{rj} \geq 0$, the primal will not have any feasible (optimal) solution.

3. Carry out a pivot operation on $\bar{a}_{rs}$

4. Test for optimality: If all $\bar{b}_i \geq 0$, the current solution is optimal and hence stop the iterative procedure. Otherwise, go to step 1.

DUAL SIMPLEX EXAMPLE

$$\text{Minimize } f = 20x_1 + 16x_2$$

subject to

$$x_1 \geq 2.5$$
$$x_2 \geq 6$$
$$2x_1 + x_2 \geq 17$$
$$x_1 + x_2 \geq 12$$
$$x_1 \geq 0, \; x_2 \geq 0$$

SOLUTION By introducing the surplus variables x_3, x_4, x_5, and x_6, the problem can be stated in canonical form as
Minimize f
with

$$\begin{aligned}
-x_1 \qquad\quad + x_3 \qquad\qquad\qquad\qquad &= -2.5 \\
-x_2 \qquad\quad + x_4 \qquad\qquad\quad &= -6 \\
-2x_1 - x_2 \qquad\qquad + x_5 \qquad\quad &= -17 \\
-x_1 - x_2 \qquad\qquad\qquad + x_6 &= -12 \\
20x_1 + 16x_2 \qquad\qquad\qquad\qquad - f &= 0
\end{aligned} \tag{E$_1$}$$

$$x_i \geq 0, \quad i = 1 \text{ to } 6$$

DUAL SIMPLEX EXAMPLE

The basic solution corresponding to (E_1) is infeasible since $x_3 = -2.5$, $x_4 = -6$, $x_5 = -17$, and $x_6 = -12$. However, the objective equation shows optimality since the cost coefficients corresponding to the nonbasic variables are nonnegative ($\bar{c}_1 = 20$, $\bar{c}_2 = 16$). This shows that the solution is infeasible to the primal but feasible to the dual. Hence the dual simplex method can be applied to solve this problem as follows.

DUAL SIMPLEX EXAMPLE

Step 1 Write the system of equations (E_1) in tableau form:

Basic variables	Variables							
	x_1	x_2	x_3	x_4	x_5	x_6	$-f$	$\bar{b}_i$
x_3	-1	0	1	0	0	0	0	-2.5
x_4	0	-1	0	1	0	0	0	-6
x_5	$\boxed{-2}$ Pivot element	-1	0	0	1	0	0	$-17 \leftarrow$ Minimum, pivot row
x_6	-1	-1	0	0	0	1	0	-12
$-f$	20	16	0	0	0	0	1	0

Select the pivotal row r such that

$$\bar{b}_r = \min(\bar{b}_i < 0) = \bar{b}_3 = -17$$

in this case. Hence $r = 3$.

DUAL SIMPLEX EXAMPLE

Step 2 Select the pivotal column s as

$$\frac{\bar{c}_s}{-\bar{a}_{rs}} = \min_{\bar{a}_{rj}<0} \left(\frac{\bar{c}_j}{-\bar{a}_{rj}} \right)$$

Since

$$\frac{\bar{c}_1}{-\bar{a}_{31}} = \frac{20}{2} = 10, \quad \frac{\bar{c}_2}{-\bar{a}_{32}} = \frac{16}{1} = 16, \quad \text{and} \quad s = 1$$

DUAL SIMPLEX EXAMPLE

Step 3 The pivot operation is carried on $\bar{a}_{31}$ in the preceding table, and the result is as follows:

Basic variables	x_1	x_2	x_3	x_4	x_5	x_6	$-f$	$\bar{b}_i$	
x_3	0	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	0	6	
x_4	0	$\boxed{-1}$ Pivot element	0	1	0	0	0	-6	← Minimum, pivot row
x_1	1	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	$\frac{17}{2}$	
x_6	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	1	0	$-\frac{7}{2}$	
$-f$	0	6	0	0	10	0	1	-170	

Step 4 Since some of the $\bar{b}_i$ are < 0, the present solution is not optimum. Hence we proceed to the next iteration.

DUAL SIMPLEX EXAMPLE

Step 1 The pivot row corresponding to minimum ($\bar{b}_i < 0$) can be seen to be 2 in the preceding table.

Step 2 Since $\bar{a}_{22}$ is the only negative coefficient, it is taken as the pivot element.
Step 3 The result of pivot operation on $\bar{a}_{22}$ in the preceding table is as follows:

Basic variables	Variables							
	x_1	x_2	x_3	x_4	x_5	x_6	$-f$	$\bar{b}_i$
x_3	0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	3
x_2	0	1	0	-1	0	0	0	6
x_1	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{11}{2}$
x_6	0	0	0	$\boxed{-\frac{1}{2}}$ Pivot element	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$ ← Minimum, pivot row
$-f$	0	0	0	6	10	0	1	-206

Step 4 Since all $\bar{b}_i$ are not ≥ 0, the present solution is not optimum. Hence we go to the next iteration.

DUAL SIMPLEX EXAMPLE

Step 1 The pivot row (corresponding to minimum $\bar{b}_i \leq 0$) can be seen to be the fourth row.

Step 2 Since

$$\frac{\bar{c}_4}{-\bar{a}_{44}} = 12 \quad \text{and} \quad \frac{\bar{c}_5}{-\bar{a}_{45}} = 20$$

the pivot column is selected as $s = 4$.

Step 3 The pivot operation is carried on $\bar{a}_{44}$ in the preceding table, and the result is as follows:

Basic variables	Variables							
	x_1	x_2	x_3	x_4	x_5	x_6	$-f$	$\bar{b}_i$
x_3	0	0	1	0	-1	1	0	$\frac{5}{2}$
x_2	0	1	0	0	1	-2	0	7
x_1	1	0	0	0	-1	1	0	5
x_4	0	0	0	1	1	-2	0	1
$-f$	0	0	0	0	4	12	1	-212

DUAL SIMPLEX EXAMPLE

Step 4 Since all $\bar{b}_i$ are ≥ 0, the present solution is dual optimal and primal feasible. The solution is

$$x_1 = 5, \quad x_2 = 7, \quad x_3 = \tfrac{5}{2}, \quad x_4 = 1 \quad \text{(dual basic variables)}$$

$$x_5 = x_6 = 0 \quad \text{(dual nonbasic variables)}$$

$$f_{\min} = 212$$

$$\text{Minimize } f = 20x_1 + 16x_2$$

subject to

$$x_1 \geq 2.5$$

$$x_2 \geq 6$$

$$2x_1 + x_2 \geq 17$$

$$x_1 + x_2 \geq 12$$

$$x_1 \geq 0, \; x_2 \geq 0$$

TRANSPORTATION PROBLEM

Suppose that there are m origins $R_1, R_2, \cdots, R_m$ (e.g., warehouses) and n destinations, $D_1, D_2, \cdots, D_n$ (e.g., factories). Let a_i be the amount of a commodity available at origin i $(i = 1, 2, \ldots, m)$ and b_j be the amount required at destination j $(j = 1, 2, \ldots, n)$. Let c_{ij} be the cost per unit of transporting the commodity from origin i to destination j. The objective is to determine the amount of commodity (x_{ij}) transported from origin i to destination j such that the total transportation costs are minimized. This problem can be formulated mathematically as

$$\text{Minimize } f = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \tag{4.52}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, 2, \ldots, m \tag{4.53}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, 2, \ldots, n \tag{4.54}$$

$$x_{ij} \geq 0, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n \tag{4.55}$$

Clearly, this is a LP problem in mn variables and $m + n$ equality constraints.

TRANSPORTATION PROBLEM

Equations (4.53) state that the total amount of the commodity transported from the origin i to the various destinations must be equal to the amount available at origin i $(i = 1, 2, \ldots, m)$, while Eqs. (4.54) state that the total amount of the commodity received by destination j from all the sources must be equal to the amount required at the destination j $(j = 1, 2, \ldots, n)$. The nonnegativity conditions Eqs. (4.55) are added since negative values for any x_{ij} have no physical meaning. It is assumed that the total demand equals the total supply, that is,

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \tag{4.56}$$

Equation (4.56), called the *consistency condition*, must be satisfied if a solution is to exist. This can be seen easily since

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right) = \sum_{j=1}^{n} b_j \tag{4.57}$$

TRANSPORTATION PROBLEM

The special structure of the transportation matrix can be seen by writing the equations in standard form:

$$
\begin{aligned}
x_{11} + x_{12} + \cdots + x_{1n} \qquad\qquad\qquad\qquad\qquad\qquad &= a_1 \\
x_{21} + x_{22} + \cdots + x_{2n} \qquad\qquad\qquad &= a_2 \\
\vdots \qquad\qquad\qquad &\;\;\vdots \\
x_{m1} + x_{m2} + \cdots + x_{mn} &= a_m
\end{aligned}
\tag{4.58a}
$$

$$
\begin{aligned}
x_{11} \qquad + x_{21} \qquad + x_{m1} \quad &= b_1 \\
x_{12} \qquad + x_{22} \qquad + x_{m2} \quad &= b_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad & \\
x_{1n} \qquad + x_{2n} \qquad + x_{mn} \quad &= b_n
\end{aligned}
\tag{4.58b}
$$

$$
\begin{aligned}
c_{11}x_{11} + c_{12}x_{12} + \cdots + c_{1n}x_{1n} + c_{21}x_{21} + \cdots + c_{2n}x_{2n} + \cdots \\
+ c_{m1}x_{m1} + \cdots + c_{mn}x_{mn} = f
\end{aligned}
\tag{4.58c}
$$

- All the nonzero coefficients of the constraints are equal to 1.
- The constraint coefficients appear in a triangular form.
- Any variable appears only once in the first *m* equations and once in the next *n* equations.

TRANSPORTATION PROBLEM

To facilitate the identification of a starting solution, the system of equations is represented in the form of an array, called the **transportation array**.

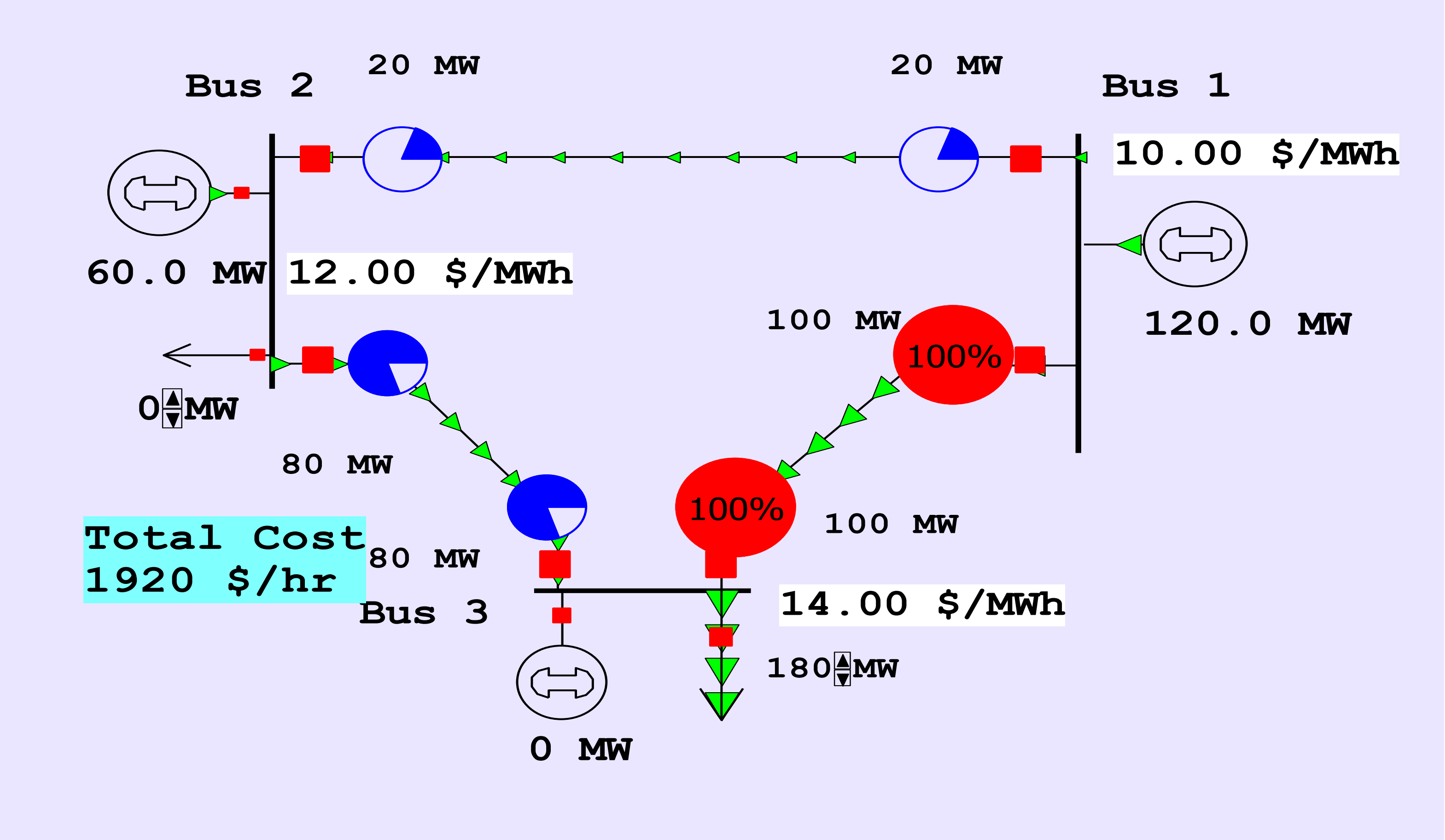
To / From		Destination j: 1	2	3	$\cdots$	n	Amount available a_i
Origin i	1	x_{11} c_{11}	x_{12} c_{12}	x_{13} c_{13}	$\cdots$	x_{1n} c_{1n}	a_1
	2	x_{21} c_{21}	x_{22} c_{22}	x_{23} c_{23}	$\cdots$	x_{2n} c_{2n}	a_2
	3	x_{31} c_{31}	x_{32} c_{32}	x_{33} c_{33}	$\cdots$	x_{3n} c_{3n}	a_3
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	m	x_{m1} c_{m1}	x_{m2} c_{m2}	x_{m3} c_{m3}	$\cdots$	x_{mn} c_{mn}	a_m
Amount required b_j		b_1	b_2	b_3	$\cdots$	b_n	

TRANSPORTATION PROBLEM

- ***Computational Procedure***
  - Determine a starting basic feasible solution.
  - Test the current basic feasible solution for optimality. If the current solution is optimal, stop the iterative process; otherwise, go to step 3.
  - Select a variable to enter the basis from among the current nonbasic variables.
  - Select a variable to leave from the basis from among the current basic variables (using the feasibility condition).
  - Find a new basic feasible solution and return to step 2.

OPTIMAL POWER FLOW

- The Optimal Power Flow (OPF) model represents the problem of determining the best operating levels for electric power plants in order to meet demands given throughout a transmission network, usually with the objective of minimizing operating cost.



OPTIMAL POWER FLOW

Three generator controls P_1, P_2, P_3

Incremental costs of 10, 12, 14 \$/MWh, respectively

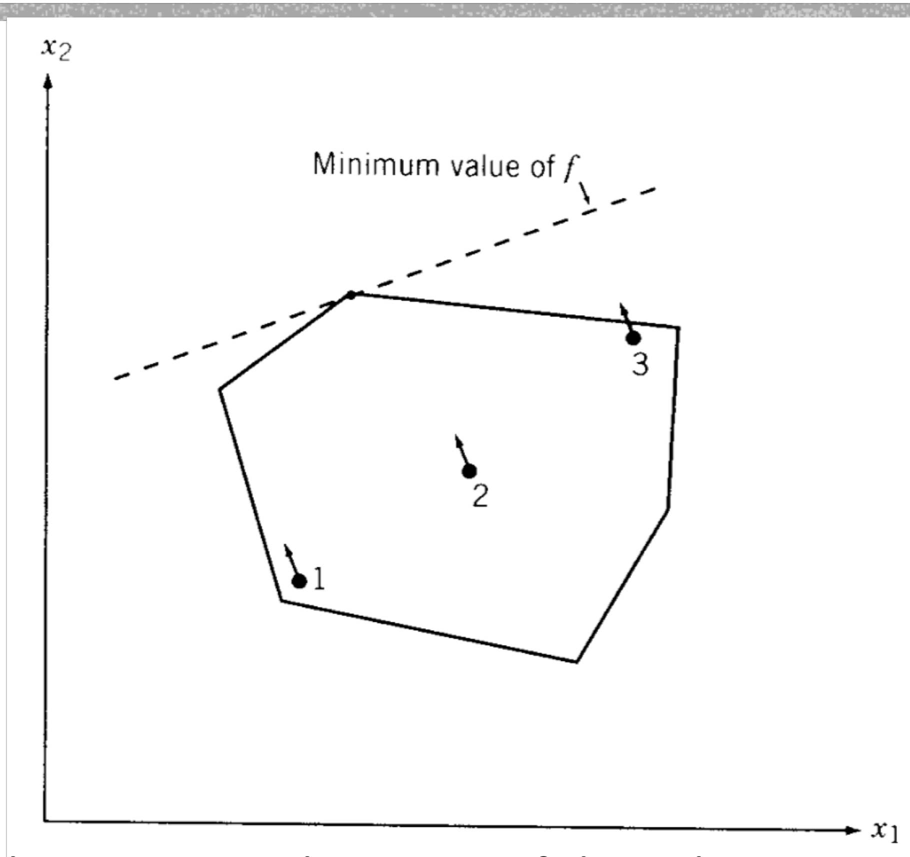
$$\text{min:} \quad 10P_1 + 12P_2 + 14P_3$$

$$\begin{aligned} \text{st:} \quad & P_1 + P_2 + P_3 && = 180 && \text{Power Balance} \\ & 0.66P_1 + 0.33P_2 && \leq 100 && \text{Line 1-3 Constraint} \\ & P_1, P_2, P_3 && \geq 0 \end{aligned}$$

INTERIOR POINT METHODS

- For large scale problems, simplex method can become cumbersome and computationally expensive.
- There exist another class of solution approaches for LP problems known as the interior point methods.
- Interior point methods move across the feasible region, based on pre-defined criteria, such as feasibility of constraints or objective function value.
- The interior point method solved problems involving 150,000 design variables and 12,000 constraints in 1 hour, while the simplex method required 4 hours for solving a smaller problem involving only 36,000 design variables and 10,000 constraints.
- It was found that the interior method is as much as 50 times faster than the simplex method for large problems.

INTERIOR POINT METHODS



- If the current solution is near the center of the polytope, we can move along the steepest descent direction to reduce the value of f by a maximum amount.
- The current solution can be improved substantially by moving along the steepest descent direction if it is near the center (point 2) but not near the boundary point (points 1 and 3).

INTERIOR POINT METHODS

Karmarkar's method requires the LP problem in the following form:

$$\text{Minimize } f = \mathbf{c}^{\mathrm{T}}\mathbf{X}$$

subject to

$$\begin{aligned} [a]\mathbf{X} &= \mathbf{0} \\ x_1 + x_2 + \cdots + x_n &= 1 \\ \mathbf{X} &\geq \mathbf{0} \end{aligned} \tag{4.59}$$

where $\mathbf{X} = \{x_1, x_2, \ldots, x_n\}^{\mathrm{T}}$, $\mathbf{c} = \{c_1, c_2, \ldots, c_n\}^{\mathrm{T}}$, and $[a]$ is an $m \times n$ matrix. In addition, an interior feasible starting solution to Eqs. (4.59) must be known. Usually,

$$\mathbf{X} = \left\{\frac{1}{n}, \frac{1}{n}, \cdots \frac{1}{n}\right\}^{\mathrm{T}}$$

is chosen as the starting point. In addition, the optimum value of f must be zero for the problem. Thus

$$\begin{aligned} \mathbf{X}^{(1)} &= \left\{\frac{1}{n} \quad \frac{1}{n} \cdots \frac{1}{n}\right\}^{\mathrm{T}} = \text{interior feasible} \\ f_{\min} &= 0 \end{aligned} \tag{4.60}$$

- Conversion of an LP Problem into the Required Form

Let the given LP problem be of the form

$$\text{Minimize } \mathbf{d}^{\mathrm{T}}\mathbf{X}$$

subject to

$$\begin{aligned} [\alpha]\mathbf{X} &= \mathbf{b} \\ \mathbf{X} &\geq \mathbf{0} \end{aligned} \tag{4.61}$$

To convert this problem into the form of Eq. (4.59), we use the procedure suggested in Ref. [4.20] and define integers m and n such that $\mathbf{X}$ will be an $(n-3)$-component vector and $[\alpha]$ will be a matrix of order $m-1 \times n-3$. We now define the vector $\bar{\mathbf{z}} = \{z_1, z_2, \cdots, z_{n-3}\}^{\mathrm{T}}$ as

$$\bar{\mathbf{z}} = \frac{\mathbf{X}}{\beta} \tag{4.62}$$

where β is a constant chosen to have a sufficiently large value such that

$$\beta > \sum_{i=1}^{n-3} x_i \tag{4.63}$$

for any feasible solution $\mathbf{X}$ (assuming that the solution is bounded). By using Eq. (4.62), the problem of Eq. (4.61) can be stated as follows:

$$\text{Minimize } \beta \mathbf{d}^{\mathrm{T}} \bar{\mathbf{z}}$$

subject to

$$[\alpha]\bar{\mathbf{z}} = \frac{1}{\beta}\mathbf{b} \qquad (4.64)$$
$$\bar{\mathbf{z}} \geq \mathbf{0}$$

We now define a new vector $\mathbf{z}$ as

$$\mathbf{z} = \begin{Bmatrix} \bar{\mathbf{z}} \\ z_{n-2} \\ z_{n-1} \\ z_n \end{Bmatrix}$$

and solve the following related problem instead of the problem in Eqs. (4.64):

$$\text{Minimize } \{\beta \mathbf{d}^{\mathrm{T}} \quad 0 \quad 0 \quad M\}\,\mathbf{z}$$

subject to

$$\begin{bmatrix} [\alpha] & \mathbf{0} & -\dfrac{n}{\beta}\mathbf{b} & \left(\dfrac{n}{\beta}\mathbf{b} - [\alpha]\mathbf{e}\right) \\ 0 & 0 & n & 0 \end{bmatrix} \mathbf{z} = \begin{Bmatrix} \mathbf{0} \\ 1 \end{Bmatrix}$$
$$\mathbf{e}^T \bar{\mathbf{z}} + z_{n-2} + z_{n-1} + z_n = 1 \qquad (4.65)$$
$$\mathbf{z} \geq \mathbf{0}$$

where $\mathbf{e}$ is an $(m-1)$-component vector whose elements are all equal to 1, z_{n-2} is a slack variable that absorbs the difference between 1 and the sum of other variables, z_{n-1} is constrained to have a value of $1/n$, and M is given a large value (corresponding to the artificial variable z_n) to force z_n to zero when the problem stated in Eqs. (4.61) has a feasible solution. Equations (4.65) are developed such that if $\mathbf{z}$ is a solution to these equations, $\mathbf{X} = \beta\bar{\mathbf{z}}$ will be a solution to Eqs. (4.61) if Eqs. (4.61) have a feasible solution. Also, it can be verified that the interior point $\mathbf{z} = (1/n)\mathbf{e}$ is a feasible solution to Eqs. (4.65). Equations (4.65) can be seen to be the desired form of Eqs. (4.61) except for a 1 on the right-hand side. This can be eliminated by subtracting the last constraint from the next-to-last constraint, to obtain the required form:

$$\text{Minimize } \{\beta\mathbf{d}^{\mathrm{T}} \quad 0 \quad 0 \quad M\}\mathbf{z}$$

subject to

$$\begin{bmatrix} [\alpha] & \mathbf{0} & -\dfrac{n}{\beta}\mathbf{b} & \left(\dfrac{n}{\beta}\mathbf{b} - [\alpha]\mathbf{e}\right) \\ -\mathbf{e}^T & -1 & (n-1) & -1 \end{bmatrix} \mathbf{z} = \begin{Bmatrix} \mathbf{0} \\ 0 \end{Bmatrix}$$

$$\mathbf{e}^{\mathrm{T}}\bar{\mathbf{z}} + z_{n-2} + z_{n-1} + z_n = 1 \tag{4.66}$$

$$\mathbf{z} \geq \mathbf{0}$$

- Note: When Eqs. (4.66) are solved, if the value of the artificial variable $z_n > 0$, the original problem in Eqs. (4.61) is infeasible. On the other hand, if the value of the slack variable $z_{n-2} = 0$, the solution of the problem given by Eqs. (4.61) is unbounded.

Example 4.12 Transform the following LP problem into a form required by Karmarkar's method:

$$\text{Minimize } 2x_1 + 3x_2$$

subject to

$$3x_1 + x_2 - 2x_3 = 3$$

$$5x_1 - 2x_2 = 2$$

$$x_i \geq 0, \quad i = 1, 2, 3$$

INTERIOR POINT METHODS

SOLUTION It can be seen that

$$\mathbf{d} = \{2 \quad 3 \quad 0\}^{\mathrm{T}},\ [\alpha] = \begin{bmatrix} 3 & 1 & -2 \\ 5 & -2 & 0 \end{bmatrix},\ \mathbf{b} = \begin{Bmatrix} 3 \\ 2 \end{Bmatrix},\ \text{and } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

We define the integers m and n as $n = 6$ and $m = 3$ and choose $\beta = 10$ so that

$$\bar{\mathbf{z}} = \frac{1}{10} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix}$$

Noting that $\mathbf{e} = \{1, 1, 1\}^{\mathrm{T}}$, Eqs. (4.66) can be expressed as

$$\text{Minimize } \{20 \quad 30 \quad 0 \quad 0 \quad 0 \quad M\}\,\mathbf{z}$$

subject to

$$\begin{bmatrix} [\alpha] & \mathbf{0} & -\dfrac{n}{\beta}\mathbf{b} & \left(\dfrac{n}{\beta}\mathbf{b} - [\alpha]\mathbf{e}\right) \\ -\mathbf{e}^T & -1 & (n-1) & -1 \end{bmatrix} \mathbf{z} = \begin{Bmatrix} \mathbf{0} \\ 0 \end{Bmatrix}$$

$$\mathbf{e}^{\mathrm{T}}\bar{\mathbf{z}} + z_{n-2} + z_{n-1} + z_n = 1$$

$$\left[\begin{bmatrix} 3 & 1 & -2 \\ 5 & -2 & 0 \end{bmatrix} \quad \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \frac{6}{10}\begin{Bmatrix} 3 \\ 2 \end{Bmatrix} \right.$$
$$\left. \times \left(\frac{6}{10}\begin{Bmatrix} 3 \\ 2 \end{Bmatrix} - \begin{bmatrix} 3 & 1 & -2 \\ 5 & -2 & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \right) \right] \mathbf{z} = \mathbf{0}$$

$$\begin{aligned} \{-\{1 \quad 1 \quad 1\} \quad -1 \quad 5 \quad -1\}\,\mathbf{z} &= 0 \\ z_1 + z_2 + z_3 + z_4 + z_5 + z_6 &= 1 \\ \mathbf{z} = \{z_1 \quad z_2 \quad z_3 \quad z_4 \quad z_5 \quad z_6\}^{\mathrm{T}} &\geq \mathbf{0} \end{aligned}$$

where M is a very large number. These equations can be seen to be in the desired form.

- **Algorithm**

Starting from an interior feasible point $\mathbf{X}^{(1)}$, Karmarkar's method finds a sequence of points $\mathbf{X}^{(2)}, \mathbf{X}^{(3)}, \cdots$ using the following iterative procedure:

1. Initialize the iterative process. Begin with the center point of the simplex as the initial feasible point

$$\mathbf{X}^{(1)} = \left\{ \frac{1}{n} \ \frac{1}{n} \cdots \frac{1}{n} \right\}^{\mathrm{T}}.$$

Set the iteration number as $k = 1$.

2. Test for optimality. Since $f = 0$ at the optimum point, we stop the procedure if the following convergence criterion is satisfied:

$$||\mathbf{c}^T \mathbf{X}^{(k)}|| \le \varepsilon \tag{4.67}$$

where ε is a small number. If Eq. (4.67) is not satisfied, go to step 3.

3. Compute the next point, $\mathbf{X}^{(k+1)}$. For this, we first find a point $\mathbf{Y}^{(k+1)}$ in the transformed unit simplex as

$$\mathbf{Y}^{(k+1)} = \left\{ \frac{1}{n} \quad \frac{1}{n} \cdots \frac{1}{n} \right\}^{\mathrm{T}} - \frac{\alpha([I] - [P]^{\mathrm{T}}([P][P]^{\mathrm{T}})^{-1}[P])[D(\mathbf{X}^{(k)})]\mathbf{c}}{||\mathbf{c}||\sqrt{n(n-1)}} \tag{4.68}$$

where $||\mathbf{c}||$ is the length of the vector $\mathbf{c}$, $[I]$ the identity matrix of order n, $[D(\mathbf{X}^{(k)})]$ an $n \times n$ matrix with all off-diagonal entries equal to 0, and diagonal entries equal to the components of the vector $\mathbf{X}^{(k)}$ as

$$[D(\mathbf{X}^{(k)})]_{ii} = x_i^{(k)}, \qquad i = 1, 2, \ldots, n \tag{4.69}$$

$[P]$ is an $(m+1) \times n$ matrix whose first m rows are given by $[a]\ [D(\mathbf{X}^{(k)})]$ and the last row is composed of 1's:

$$[P] = \begin{bmatrix} & [a][D(\mathbf{X}^{(k)})] & & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \tag{4.70}$$

and the value of the parameter α is usually chosen as $\alpha = \frac{1}{4}$ to ensure convergence. Once $\mathbf{Y}^{(k+1)}$ is found, the components of the new point $\mathbf{X}^{(k+1)}$ are determined as

$$x_i^{(k+1)} = \frac{x_i^{(k)} y_i^{(k+1)}}{\sum_{r=1}^{n} x_r^{(k)} y_r^{(k+1)}}, \qquad i = 1, 2, \ldots, n \tag{4.71}$$

Set the new iteration number as $k = k + 1$ and go to step 2.