Due Saturday April 2nd, at 12:00pm CST.

Your name:

Your UTD ID:

Important points:

- This is a take-home, open lecture/notes exam. You are not allowed to look at your homework assignments.
- You are not allowed to talk to anybody about this exam until you submit it.
- There are 5 problems. The last one is MATLAB-based and will probably take a bit longer to complete. Use your time wisely!
- Think before you start solving the problems! If you are spending too much time on any of the problems, you're probably on the wrong track.
- You **cannot** use MATLAB except in the last quesiton.
- I will not be responding to emails during the exam time. Explain your solutions clearly and provide all necessary details including the listings of your MATLAB scripts.
- Your answers should be combined into a single PDF file and submitted via eLearning. To submit, click on the midterm assignment in the **Exams** folder and use the ASSIGNMENT SUBMISSION feature to browse your computer and upload your file. Note that I will not be accepting submissions via email after the deadline. **Don't wait until the last minute!**

Good luck!

1. [15 points] Consider the feedback system shown in Figure 1 where $G(s)$ is the nominal plant model and $\Delta(s)$ is stable. Assume that $G(s)$, $K(s)$, and $\Delta(s)$ are all SISO.

   (a) The dashed box represents an uncertain model $\hat{G}(s)$ that depends on both $G(s)$ and $\Delta(s)$. What is the set of models $\mathcal{A}$ corresponding to this block diagram?

   (b) What can you conclude about the classical gain margins if the feedback system is stable for all $\|\Delta\|_\infty < 0.5$?

   (c) Find a necessary and sufficient condition for $K(s)$ to stabilize all $\hat{G}(s) \in \mathcal{A}$. Briefly describe a proof that your condition is sufficient for $K(s)$ to achieve robust stability. You do not need to prove necessity.

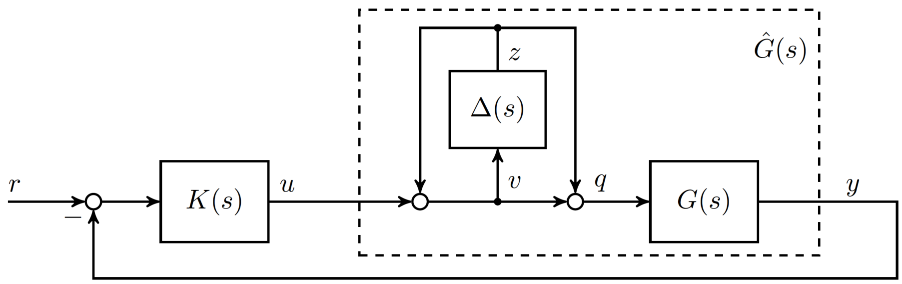

Fig. 1: Feedback system

2. [20 points] Consider a string of two vehicles as shown in Figure 2. The leading (first) vehicle is tracking a reference r_1 and the following (second) vehicle is tracking a reference $r_2 = x_1 - \delta$ to maintain a distance of δ with the first vehicle. The second vehicle uses a radar device to measure the distance to the first vehicle and compute the error $e_2 = r_2 - x_2$. Let each vehicle be modeled by the transfer function $G(s)$ and assume that both vehicles use the same control law $K(s)$. Figure 3 shows the feedback diagram for the two vehicle string.

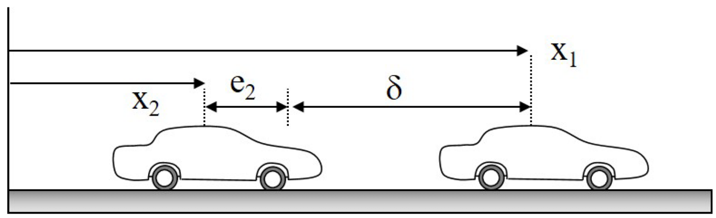

Fig. 2: String of two vehicles

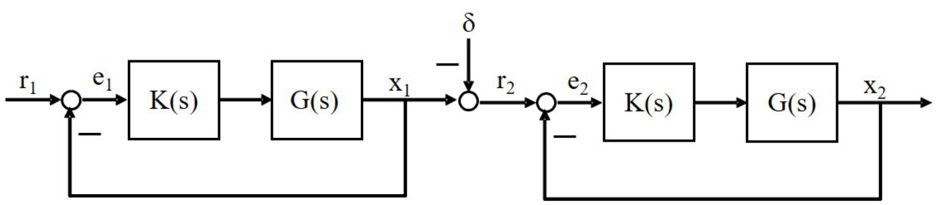

Fig. 3: Feedback diagram for two-vehicle system

(a) Compute the transfer function from: (i) r_1 to e_1, (ii) r_1 to x_1, (iii) r_2 to e_2, and (iv) r_2 to x_2. Your answers should be expressed in terms of the sensitivity $S(s) = \frac{1}{1+G(s)K(s)}$ and complementary sensitivity $T(s) = \frac{G(s)K(s)}{1+G(s)K(s)}$.

(b) What is the transfer function from the first vehicle's reference r_1 to the tracking error for the second vehicle e_2? Note: By linearity you may assume $\delta = 0$ in this calculation.

(c) The goal is for the first vehicle to track the reference command r_1. In addition, the second vehicle should achieve a much smaller tracking error than the first vehicle, i.e., we would like $|e_2|$ to be much smaller than $|e_1|$. Use your results from the previous parts to express these objectives in terms of $S(s)$ and $T(s)$. You may assume that the reference command r_1 mainly consists of low frequency content.

(d) Is it possible to achieve the two goals in part (c)? If yes, describe how you would design $K(s)$ to achieve these goals. If no, then describe a constraint that prevents you from achieving both goals.

3. [15 points] Let $A \in \mathbb{C}^{n \times m}$ where $n > m$. The singular value decomposition (SVD) of A is given by

$$A \;=\; U\Sigma V^* \;=\; \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \hat{\Sigma} \\ 0_{n-m,m} \end{bmatrix} V^*$$

where U and V are unitary matrices and $\hat{\Sigma} = \mathrm{diag}(\sigma_1, \ldots, \sigma_m)$. Assume A is full rank and hence $\sigma_i > 0$ for $i = 1, \ldots, m$.

(a) What is the SVD of $(A^*A)^{-1}A^*$?

(b) Show that for any vectors $x \in \mathbb{C}^m$ and $b \in \mathbb{C}^n$,

$$\|Ax \,-\, b\|_2^2 \;=\; \|\hat{\Sigma}\, V^*x \,-\, U_1^*b\|_2^2 \,+\, \|U_2^*b\|_2^2. \tag{1}$$

(c) Use Equation (1) to specify the vector x that solves

$$\min_x \; \|Ax \,-\, b\|_2$$

How is the solution related to the matrix in part (a)?

4. [20 points] Consider the following set of models:

$$\mathcal{I} := \left\{ \frac{G(s)}{1 + w(s)\Delta(s)} \;\middle|\; \Delta \text{ stable, } \|\Delta\|_\infty < 1 \right\}$$

$G(s)$ is SISO and $w(s)$ is a stable, proper rational transfer function.

(a) Draw a block diagram showing the structure of any $G_p \in \mathcal{I}$. Your diagram should only include blocks for G, w, and Δ.

(b) Consider the feedback diagram show in Figure 4. Assume the controller K stabilizes the nominal system G. Find a necessary and sufficient condition for K to stabilize all $G_p \in \mathcal{I}$. Provide a proof of sufficiency, i.e., prove that K achieves robust stability if your condition is satisfied. You do not need to prove necessity.

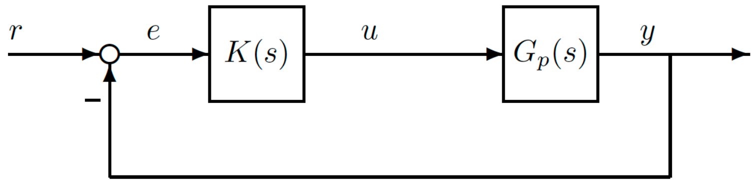

Fig. 4: Feedback loop

5. [30 points] Figure 5 shows the block diagram for a plane autopilot system with state-space representation

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A - BD_c C & BC_c \\ -B_c C & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} BD_c \\ B_c \end{bmatrix} r \;:=\; A_{\text{cl}} \begin{bmatrix} x \\ x_c \end{bmatrix} + B_{\text{cl}}\, r.$$

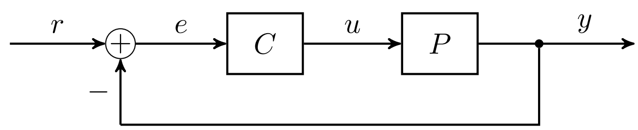

Fig. 5: Autopilot feedback system

The controller state matrices are given by

$$A_c := \begin{bmatrix} 0 & 0 \\ K_q\, a_q & 0 \end{bmatrix}, \; B_c := \begin{bmatrix} K_a\, a_z & 0 \\ K_a K_q\, a_q & K_q\, a_q \end{bmatrix}, \; C_c := \begin{bmatrix} K_q & 1 \end{bmatrix}, \; D_c := \begin{bmatrix} K_a K_q & K_q \end{bmatrix}$$

where $K_a = -0.0015$, $K_q = -0.32$, $a_z = 2$, and $a_q = 6$. The plant state matrices are given by

$$A := \begin{bmatrix} Z_\alpha & 1 & Z_\delta & 0 \\ M_\alpha & 0 & M_\delta & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega^2 & -2\zeta\omega \end{bmatrix}, \; B := \begin{bmatrix} 0 \\ 0 \\ 0 \\ \omega^2 \end{bmatrix}, \; C := \begin{bmatrix} VZ_\alpha & 0 & VZ_\delta & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; D := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $V = 886.78$, $\zeta = 0.6$, and $\omega = 113$. The aerodynamic coefficients have significant uncertainty which we model as:

$$\begin{aligned} Z_\alpha &= -1.3046\,(1 + k\,\delta_1)\,, \quad & Z_\delta &= -0.2142\,(1 + k\,\delta_2) \\ M_\alpha &= 47.7109\,(1 + k\,\delta_3)\,, \quad & M_\delta &= -104.8346\,(1 + k\,\delta_4)\,. \end{aligned}$$

Each δ_i is norm-bounded, $|\delta_i| \leq 1$, and k measures the size of the allowable uncertainty. We want to answer the following question: What is the largest value of k for which we can guarantee that the closed loop system is stable for all possible values of $|\delta_i| \leq 1$? This value, denoted $k_{\max}$ is called the stability margin of the system.

(a) Show that the nominal system is stable by computing the closed loop eigenvalues with all $\delta_i = 0$.

(b) Estimate the stability margin with a simple random search. Choose $k_{\text{try}} = 10$ and randomly set each δ_i to have a value in $-1 \leq \delta_i \leq 1$. Check the eigenvalues of the closed loop system for this perturbation. If the system is unstable, then store the perturbation and reduce k_{try}, e.g. $k_{\text{try}} = 0.99\, k_{\text{try}}$. Repeat this for 10000 random samples. Let $\bar{k}$ denote the smallest value of k_{try} for which you found a destabilizing perturbation. Record your $\bar{k}$ and the δ_i that caused instability. Explain why $k_{\max} \leq \bar{k}$.

(c) Construct the uncertain closed loop system in MATLAB and plot its frequency response using the ufrd command. Next, find the bounds on the stability margin using the robstab command and compare the resulting bounds with the $\bar{k}$ you computed in part (b). Provide an interpretation for these bounds based on the structured singular value.

(d) Verify that the uncertainty returned by the robstab command causes the closed-loop to become unstable.