Due Sunday 03/13/20 (10:00pm); Beware of daylight saving adjustments

1. Determine the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of the following systems:

   (a) $H(s) = \dfrac{1}{s+a}$, with $a > 0$. How do these norms compare to each other for different values of a? What happens for $a = 0$?

   (b) $$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u \\ y &= x_1. \end{aligned}$$

2. Consider the system parameterized by k and R

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -(1+k^2)/R & 0 \\ k & -(2+k^2)/R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = x_2.$$

   (a) For what values of k and R is this system stable?

   (b) Derive the formula for the $\mathcal{H}_2$ norm of this system as a function of k and R. Using this formula, plot the $\mathcal{H}_2$ norm as a function of k for $R = 1$ and $R = 1000$, and as a function of R for $k = 2$.

   (c) Find the solution of the unforced system (i.e., determine operator $G(t)$ that maps the initial conditions to the output $y(t)$, $y(t) = G(t)x_0$).

   (d) Plot the maximal singular value of $G(t)$ as a function of time (on the time interval $t \in (0, 1000)$) for two different cases: (i) $R = 1000$, $k = 0$; (ii) $R = 1000$, $k = 2$. How do these two cases compare to each other? Explain the obtained results.

3. (a) Prove that $\underline{\sigma} = \min_{x \neq 0} \dfrac{\|Ax\|_2}{\|x\|_2}$.

   (b) Prove that $\bar{\sigma}(A^{-1}) = \dfrac{1}{\underline{\sigma}(A)}$.

   (c) Give and example of a 2×2 matrix $A(\epsilon)$ that has stable eigenvalues that are constant and independent of ϵ, but $\bar{\sigma}(A(\epsilon)) \to \infty$ with $\epsilon \to \infty$.

   (d) Construct matrices $A(\epsilon)$, B, C, D (note that B, C, and D are constant matrices) such that if $G(s) = \left[\begin{array}{c|c} A(\epsilon) & B \\ \hline C & D \end{array}\right]$ then $\|G(s)\|_{\mathcal{H}_\infty} \to \infty$ with $\epsilon \to \infty$, but the poles of G are independent of ϵ. Interpret this result.

4. Prove that if G_1 and G_2 have state-space realizations $\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$ and $\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$, respectively, then their serial and parallel interconnection yield

$$G_1 G_2 = \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1C_2 & A_1 & B_1D_2 \\ \hline D_1C_2 & C_1 & D_1D_2 \end{array}\right]$$

and

$$G_1 + G_2 = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

respectively. Suppose $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is square and D is invertible then

$$G^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right].$$

5. Write $T = GK(I + GK)^{-1}$ as an LFT of K, i.e., find P such that $T = F_\ell(P, K)$.

6. Write K as an LFT of $T = GK(I + GK)^{-1}$, i.e., find J such that $K = F_\ell(J, T)$.

7. For the state-space description represented by (A, B, C, D), find H such that

$$F_\ell(H, 1/s) = C\,(sI - A)^{-1}\,B + D.$$