MECH 6323 - Homework 6

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```
clear
close all
```

Problem 1

Part a

(a) $\mu_\Delta(M) = 0 \implies M = 0.$

$$\mu_\Delta(M) := \left[\min \bar{\sigma}(\Delta) : |I - M\Delta| = 0\right]^{-1}$$

$\Rightarrow \nexists \Delta \Rightarrow |I - M\Delta| = 0$

However doen't say anything about $\underline{M}$

$\therefore$ False

Part b

(b) $\mu_\Delta(M_1 + M_2) \leq \mu_\Delta(M_1) + \mu_\Delta(M_2).$

$$\mu_\Delta(M_1 + M_2) \leq \min_D \|D(M_1 + M_2)D^{-1}\|$$

$$= \|DM_1D^{-1} + DM_2D^{-1}\|$$

$$\leq \underbrace{\|DM_1D^{-1}\|}_{\mu_\Delta(M_1)} + \underbrace{\|DM_2D^{-1}\|}_{\mu_\Delta(M_2)}$$

$$\mu_\Delta(M_1 + M_2) \leq \mu_\Delta(M_1) + \mu_\Delta(M_2)$$

Part c

(c) $\mu_\Delta(\alpha M) = |\alpha|\,\mu_\Delta(M)$.

$$\mu_\Delta(M) := \left[\min \bar{\sigma}(\Delta) : \; |I - \alpha M \Delta| = 0\right]^{-1}$$

$$\mu_\Delta(\alpha M) := \left[\min \bar{\sigma}(\alpha\Delta) : \; |I - M(\alpha\Delta)| = 0\right]^{-1}$$

$$\bar{\sigma}(\alpha\Delta) = \alpha\,\bar{\sigma}(\Delta) = |\alpha|\,\bar{\sigma}(\Delta)$$

σ are > 0

$$\Updownarrow$$

$$\mu_\Delta(\alpha M) = |\alpha|\,\mu_\Delta(M)$$

Part d

(d) $\mu_\Delta(I) = 1$.

$$\mu_\Delta(I) := \left[\min \bar{\sigma}(\Delta) : \; |I - I\Delta| = 0\right]^{-1}$$

$\Delta = I$

$$\bar{\sigma}(I) = 1$$

$$\boxed{\mu_\Delta(I) = 1}$$

Part e

```
N = sym('n',2,'real');
M = sym('m',2,'real');
MN = M*N;
sigma_delta_1 = max(svd(M));
mu_delta_2 = 1/max(svd(N\eye(2)));
mu_delta = 1/max(svd(MN\eye(2)));
solve(mu_delta > sigma_delta_1 + mu_delta_2, {M,N})
```

Part f

```
N = sym('n',2,'real');
M = sym('m',2,'real');
MN = M*N;
sigma_delta_1 = max(svd(N));
mu_delta_2 = 1/max(svd(M\eye(2)));
mu_delta = 1/max(svd(MN\eye(2)));
solve(mu_delta <= sigma_delta_1 + mu_delta_2,{M,N})
```

Problem 2

2. Let $\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$, where Δ_i are structured uncertainties. Show that

$$\mu_\Delta\left(\begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}\right) = \max\{\mu_{\Delta_1}(M_{11}), \mu_{\Delta_2}(M_{22})\}.$$

$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$ $\quad M \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}$

$$\mu_\Delta(M) := \left[\min \bar{\sigma}(\Delta) : |I - M\Delta| = 0\right]^{-1}$$

$$I - M\Delta = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} M_{11}\Delta_1 & M_{12}\Delta_2 \\ 0 & M_{22}\Delta_2 \end{bmatrix}$$

$$= \begin{bmatrix} I - M_{11}\Delta_1 & -M_{12}\Delta_2 \\ 0 & I - M_{22}\Delta_2 \end{bmatrix}$$

$$|I - M\Delta| = (I - M_{11}\Delta_1)(I - M_{22}\Delta_2) - \underbrace{(-M_{12}\Delta_2)(0)}_{0}$$

$$|I - M\Delta| = 0 \iff M_{11}\Delta_1 = I \wedge M_{22}\Delta_2 = I$$

$$\bar{\sigma}(\Delta) = \max\{\bar{\sigma}(\Delta_1), \bar{\sigma}(\Delta_2)\}$$

$$\min \bar{\sigma}(\Delta) \text{ will be } \min_{\Delta}\left(\max\left\{\begin{array}{l}\bar{\sigma}(\Delta_1) : I - M_{11}\Delta_1 = 0 \\ \bar{\sigma}(\Delta_2) : I - M_{22}\Delta_2 = 0\end{array}\right\}\right)$$

$$\Updownarrow$$

$$\boxed{\mu_\Delta(M) = \max\left(\mu_\Delta(M_{11}), \mu_\Delta(M_{22})\right)}$$

Problem 3

```
tol_0 = 0.001;
%from simplemu.m
% Example data
i=sqrt(-1);
mat=[0.10+0.07*i -0.1538+0.1615*i 0-0.56*i  0+42.0*i  4.-1.4*i
 0-0.2730*i -0.30-0.28*i 2.86+0.546*i -26.0+72.8*i 13.+5.4600*i
 0.10+0.1750*i 0.0769-0.1077*i -0.40+0.2100*i 5.0+3.5000*i 4.5-0.70*i
  0+0.0021*i -0.0038-0.0021*i 0.0060+0.0112*i 0.2000+0.4200*i  0.0600+0.0140*i
 0.0240+0.0280*i 0+0.0269*i -0.0660+0.0420*i 0+0.70*i -0.4210+0.49*i
];

% Sample uncertainty structures
blka = [5 0];
blkb = [3 0;2 0];
blkc = [1 1;1 1;1 1;2 0];
blkd = [1 1;1 1;1 1;1 1;1 1];
blke = [2 2;2 2;1 1];
blkf = [2 2;3 3];
blkg = [5 5];
blkh = [2 3;3 2];
blki = [1 4;4 1];

% Run easymu
[upp,low,pert,dleft,dright] = easymu(mat,blka);
[low,upp];
```

blke

```
[upp,low,pert,dleft,dright] = easymu(mat,blke);
pert
```

```
pert = 5×5 complex
  -0.0164 - 0.0447i  -0.0552 - 0.0645i   0.0000 + 0.0000i   0.0000 + 0.0000i ⋯
   0.0275 - 0.0026i   0.0444 - 0.0215i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0048 - 0.0024i  -0.0003 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0915 - 0.0650i   0.0028 - 0.0053i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
```

```
low
```

```
low = 8.8836
```

```
norm(pert)
```

```
ans = 0.1126
```

```
1/norm(pert)
```

```
ans = 8.8836
```

```
low - 1/norm(pert) < tol_0
```

```
ans = logical
   1
```

they be inversly related

```
det(eye(size(mat)) - mat*pert)
```

```
ans = 3.7583e-16 - 6.3936e-18i
```

```
det(eye(size(mat)) - mat*pert) < tol_0
```

```
ans = logical
   1
```

Which is essentially 0

```
upp
```

```
upp = 8.9017
```

```
norm(dleft*mat/dright)
```

```
ans = 8.9016
```

```
upp - norm(dleft*mat/dright) < tol_0
```

```
ans = logical
   1
```

blkf

```
[upp,low,pert,dleft,dright] = easymu(mat,blkg);
pert
```

```
pert = 5×5 complex
  -0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 + 0.0000i  -0.0000 + 0.0000i ⋯
  -0.0000 + 0.0000i  -0.0000 + 0.0000i  -0.0000 - 0.0000i  -0.0000 - 0.0000i
```

```
   0.0001 + 0.0000i   0.0003 - 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 - 0.0052i  -0.0033 - 0.0091i   0.0006 - 0.0005i   0.0000 - 0.0001i
   0.0009 - 0.0000i   0.0015 - 0.0006i   0.0001 + 0.0001i   0.0000 + 0.0000i
```

```
low
```

```
low = 89.4127
```

```
norm(pert)
```

```
ans = 0.0112
```

```
1/norm(pert)
```

```
ans = 89.4127
```

```
low - 1/norm(pert) < tol_0
```

```
ans = logical
   1
```

they be inversly related

```
det(eye(size(mat)) - mat*pert)
```

```
ans = -4.0660e-16 - 7.9568e-18i
```

```
det(eye(size(mat)) - mat*pert) < tol_0
```

```
ans = logical
   1
```

Which is essentially 0

```
upp
```

```
upp = 89.4127
```

```
norm(dleft*mat/dright)
```

```
ans = 89.4127
```

```
upp - norm(dleft*mat/dright) < tol_0
```

```
ans = logical
   1
```

Problem 4

Naive methods (for speed of writing)

```
% Parameters
m = 2150; %kg
b = 20; %N s/m
c = 150; %N/deg
tau = 0.1; %s
% System Definition
```

```
U_cmd = tf([1],[tau 1])
```

```
U_cmd =

      1
  ---------
  0.1 s + 1

Continuous-time transfer function.
```

```
P = U_cmd * tf([c], [m b])
```

```
P =

           150
  ---------------------
  215 s^2 + 2152 s + 20

Continuous-time transfer function.
```

Part a

```
B = zpk(tf([1 0.02], [0.5 1]))
```

```
B =

  2 (s+0.02)
  ----------
    (s+2)

Continuous-time zero/pole/gain model.
```

The magnitude of the frequency response to input must not excede a frequency dependent response with a maximum of 2 but only for freqencies above around 2 rad/s.

Part b

```
% arbritarily select large k_p
k_p = 50;
% controller: u = k_p * (r - y)
C = k_p;
% L(s) = C(s) * P(s)
L = zpk(P*C)
```

```
L =

        34.884
  -------------------
  (s+10) (s+0.009302)

Continuous-time zero/pole/gain model.
```

```
% T(s) = Y(s) / R(s) = L(s) / (1 + L(s))
T = zpk((P*C)/(1 + P*C))
```

```
T =

          34.884 (s+10) (s+0.009302)
```

```
  -----------------------------------------
  (s+10) (s+0.009302) (s^2 + 10.01s + 34.98)

Continuous-time zero/pole/gain model.
```

```
% or w/ minreal = feedback
T = feedback(L,1)
```

```
T =

          34.884
  ----------------------
  (s^2 + 10.01s + 34.98)

Continuous-time zero/pole/gain model.
```

The time constant $\tau_{cls} = \dfrac{1}{\omega_0}$ which for $T(s) = \dfrac{1}{s^2 + 2 * \omega_0 s + \omega_0^2}$ is simple to calculate... but also just using this:

```
damp(T)
```

```
         Pole              Damping       Frequency      Time Constant
                                       (rad/seconds)      (seconds)

 -5.00e+00 + 3.15e+00i     8.46e-01      5.91e+00         2.00e-01
 -5.00e+00 - 3.15e+00i     8.46e-01      5.91e+00         2.00e-01
```

```
[w, ~, ~] = damp(T);
tau_cls = 1./w
```

```
tau_cls = 2×1
    0.1691
    0.1691
```

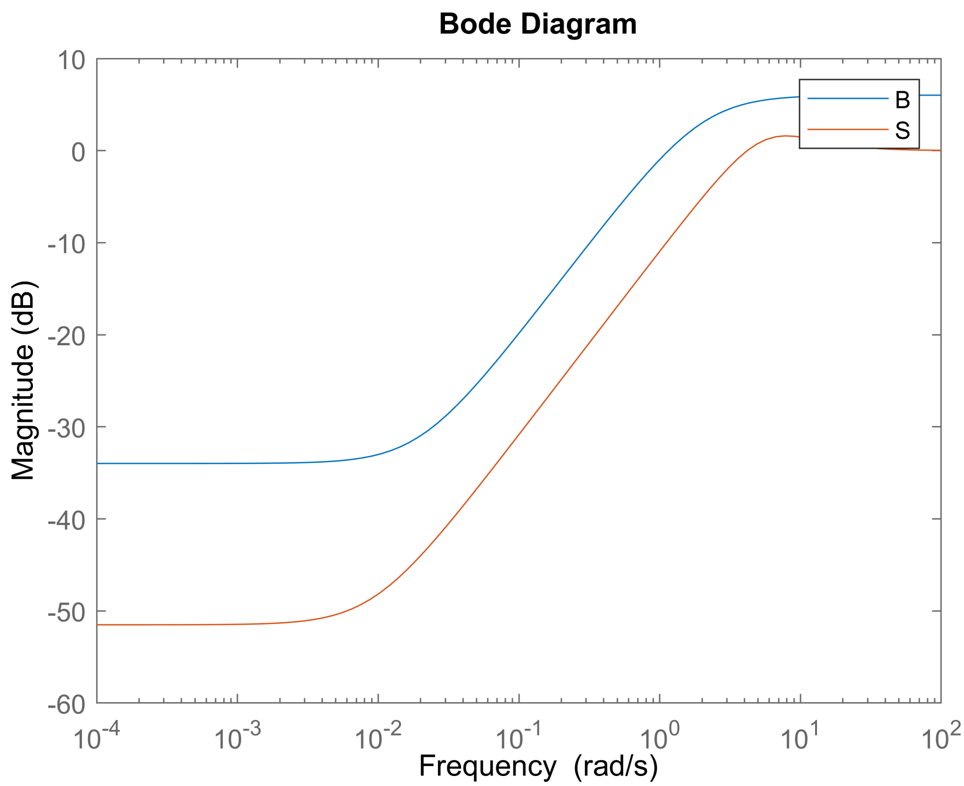
```
% S(s) = E(s) / R(s) = 1/(1+L(s))
S = zpk((1)/(1 + P*C))
```

```
S =

   (s+10) (s+0.009302)
  ----------------------
  (s^2 + 10.01s + 34.98)

Continuous-time zero/pole/gain model.
```

```
figure()
bodemag(B)
hold on
bodemag(S)
legend()
```



Part c

```
% m_u = 2150 + 150 * ureal('delta_m',0);
% b_u = b * (1 + 0.2*ureal('delta_b',0));
% c_u = c * (1 + 0.1*ureal('delta_c',0));
% tau_u = ureal('tau_u',0.1, 'Range', [0.05, 0.2]);
% P_u = tf([1], [tau_u 1]) * tf([c_u], [m_u b_u]);
% Ploting Relative Uncertainties
w = logspace(-2,4,500);
P_mag = bode(P,w);
figure
hold on
for i = 1:20
    % using braket indexing bc otherwise I get cell errors... cells are
    % wonderful
%     P_hat{i} = usample(P_u);

    % Just realized we were not supposed to use ureal... so:
    % --------------
    m_u{i} = 2150 + 150*(2*rand()-1);
    b_u{i} = b * (1 + 0.2*(2*rand()-1));
    c_u{i} = c * (1 + 0.1*(2*rand()-1));
    tau_u{i} = 0.05 + 0.15*rand();
    P_hat{i} = tf([1], [tau_u{i} 1]) * tf([c_u{i}], [m_u{i} b_u{i}]);
    % -------------------------------
```

```
    [P_hat_mag{i}, ~, ~] = bode(P_hat{i},w);
    R{i} = reshape(abs(P_hat_mag{i} - P_mag)./ abs(P_mag),[],1);
    plot(w,R{i})
end
% set(gca, 'YScale', 'log')
set(gca, 'XScale', 'log')
```

Arbitrarily selected values to make it fit... W(s) could probbably be derived as well based on the extremes of the uncertainty (by that I mean it is a simple caluculation with only 5 variables with defined ranges)

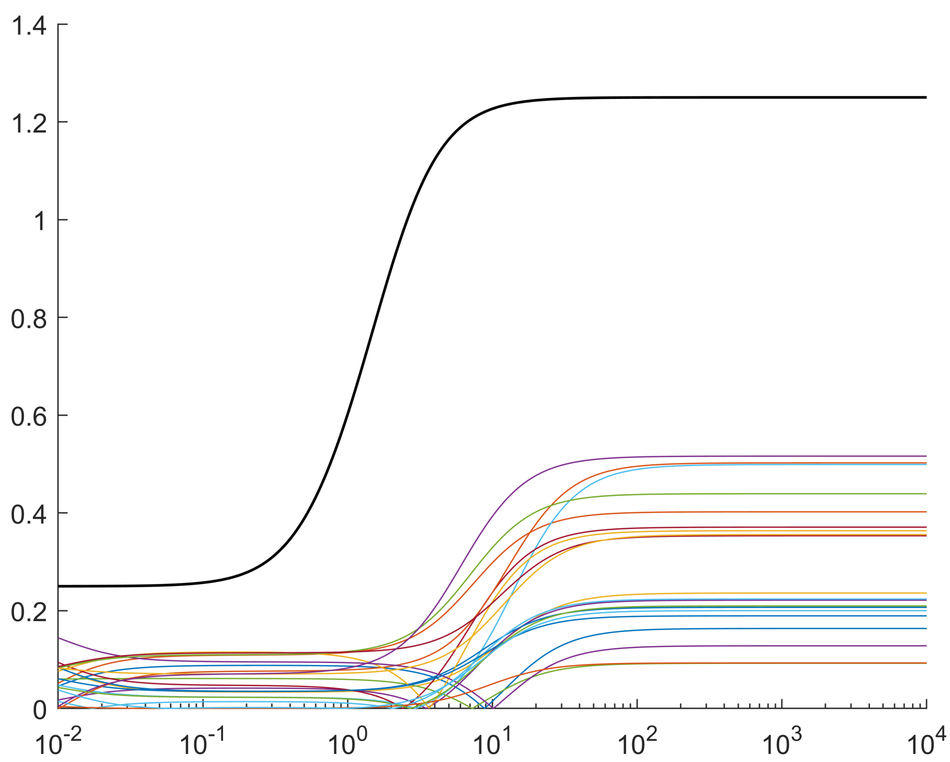
```
a1 = 1.25;
a2 = 0.5;
a3 = 2;
W_u = tf([a1 a2], [1 a3])
```

```
W_u =

  1.25 s + 0.5
  ------------
     s + 2

Continuous-time transfer function.
```

```
[W_u_mag, ~, ~] = bode(W_u,w);
plot(w,reshape(W_u_mag,[],1),'k',LineWidth=1)
```



Part d

```
for i = 1:20
    P_cls_stable(i) = isstable(1 + C * P_hat{i});
end
P_cls_stable
```

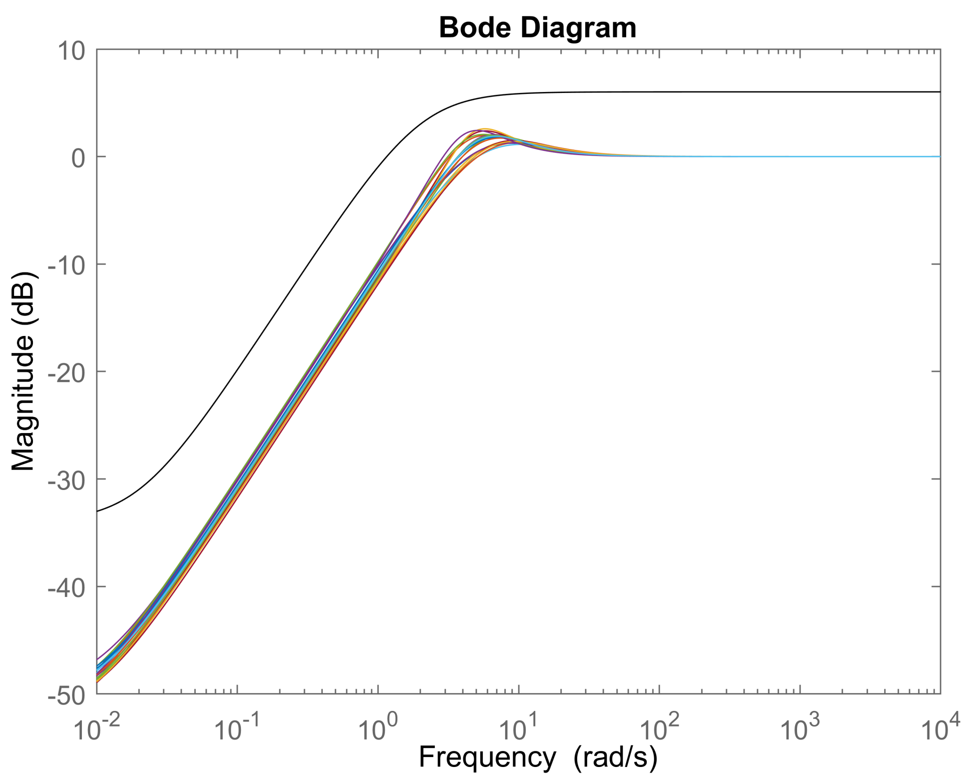
```
P_cls_stable = 1×20 logical array
   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
```

According to the simplistic test of stability for each of the sampled systems, the answer is clearly yes.

A better method of confirming this would be to actully prove it for the entire region, which I may or may not make time for...

Part e

```
figure
hold on
for i = 1:20
    S_hat{i} = (1) / (1 + C*P_hat{i});
    bodemag(S_hat{i},w)
end
bodemag(B,w, 'k')
```



Acording to bode diagram, the controller does meet performance specifications.