

# MECH 6323 - HW 2

Jonas Wagner

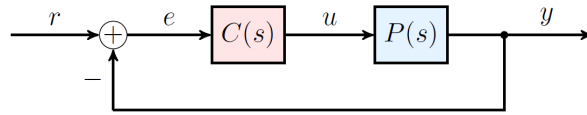
2022, February 14

## Contents

<b>1 Problem 1</b>	<b>2</b>
1.1 . . . . .	2
<b>2 Problem 2</b>	<b>3</b>
2.1 (a) . . . . .	3
2.2 (b) . . . . .	4
2.3 (c) . . . . .	5
2.4 (d) . . . . .	6
2.5 (e) . . . . .	7
2.6 (f) . . . . .	8
<b>3 Problem 3</b>	<b>9</b>
3.1 Bode Plot 1: . . . . .	9
3.1.1 Gain, Poles, and Zeros: . . . . .	9
3.1.2 Cross-over Frequency: . . . . .	9
3.2 Bode Plot 2: . . . . .	10
3.2.1 Gain, Poles, and Zeros: . . . . .	10
3.2.2 Transfer Function: . . . . .	10
3.2.3 Cross-over Frequency: . . . . .	10
<b>4 Problem 4</b>	<b>11</b>
4.1 Is the feedback system stable? Why? . . . . .	11
4.2 Find phase and cross-over frequencies. . . . .	11
4.3 Gain Margin Closed-loop poles . . . . .	11
4.3.1 $g_0 = 0.3585$ . . . . .	12
4.3.2 $g_0 = 2.6490$ . . . . .	12
4.4 $\ S - T\ _\infty$ . . . . .	12
4.5 Symmetric Disk Margin . . . . .	13
4.6 $\alpha$ on Disk boundary . . . . .	13
<b>A MATLAB Code:</b>	<b>15</b>

# 1 Problem 1

Consider the negative feedback interconnection:



## 1.1

**Problem:** If possible, give an example of  $P$  and  $C$  transfer functions such that  $\frac{1}{1+PC}$  and  $\frac{P}{1+PC}$  are stable, but  $\frac{C}{1+PC}$  is not.

**Solution:**

$$P(s) =$$

$$C(s) =$$

These then produces the following stable transfer functions:

$$\frac{1}{1+PC} =$$

$$\frac{P}{1+PC} =$$

However, then this transfer function is unstable:

$$\frac{C}{1+PC} =$$

## 2 Problem 2

### 2.1 (a)

$$G = \frac{100s + 100}{s^2 + 110s + 1000} = \frac{100(s + 1)}{(s + 100)(s + 10)}$$

1. Zeros:

(a)  $z_1 = -1$

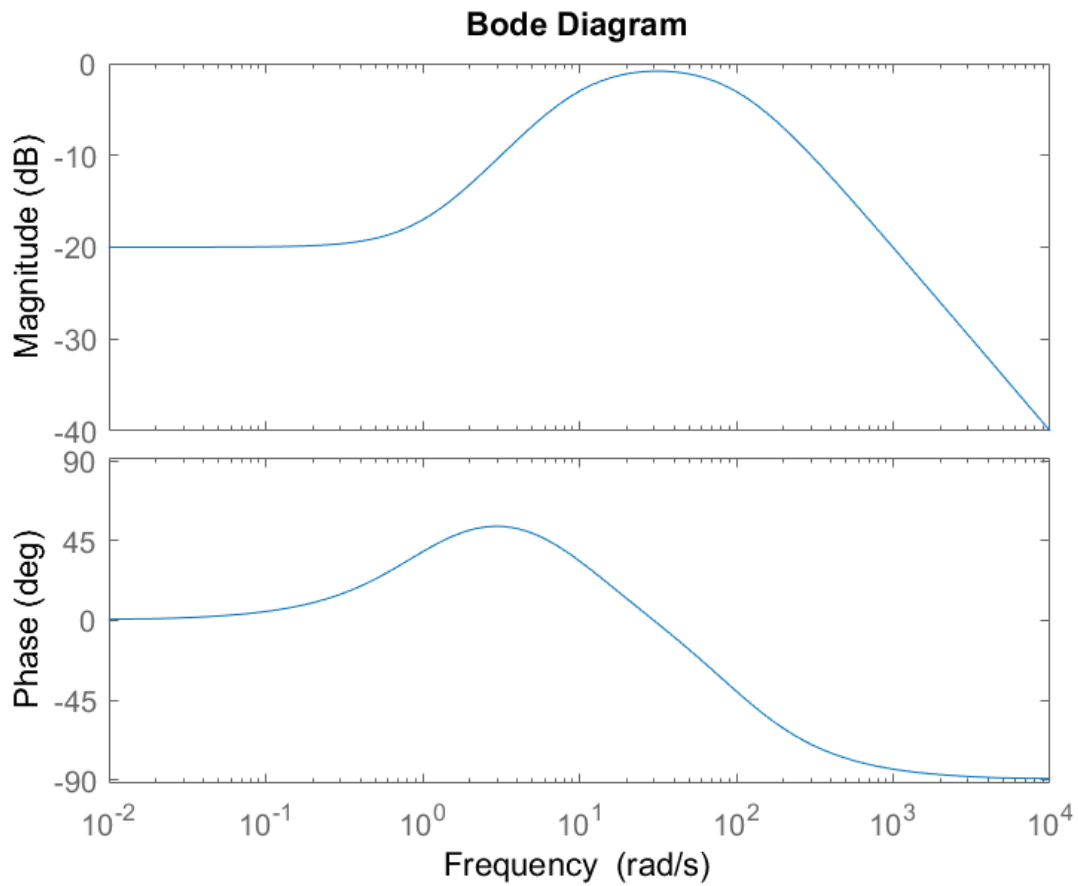
2. Poles:

(a)  $p_1 = -100$

(b)  $p_2 = -10$

3. Gain:

(a)  $K = 100$



## 2.2 (b)

$$G = \frac{10s}{s^2 + 3s} = \frac{10s}{s(s+3)}$$

1. Zeros:

(a)  $z_1 = 0$

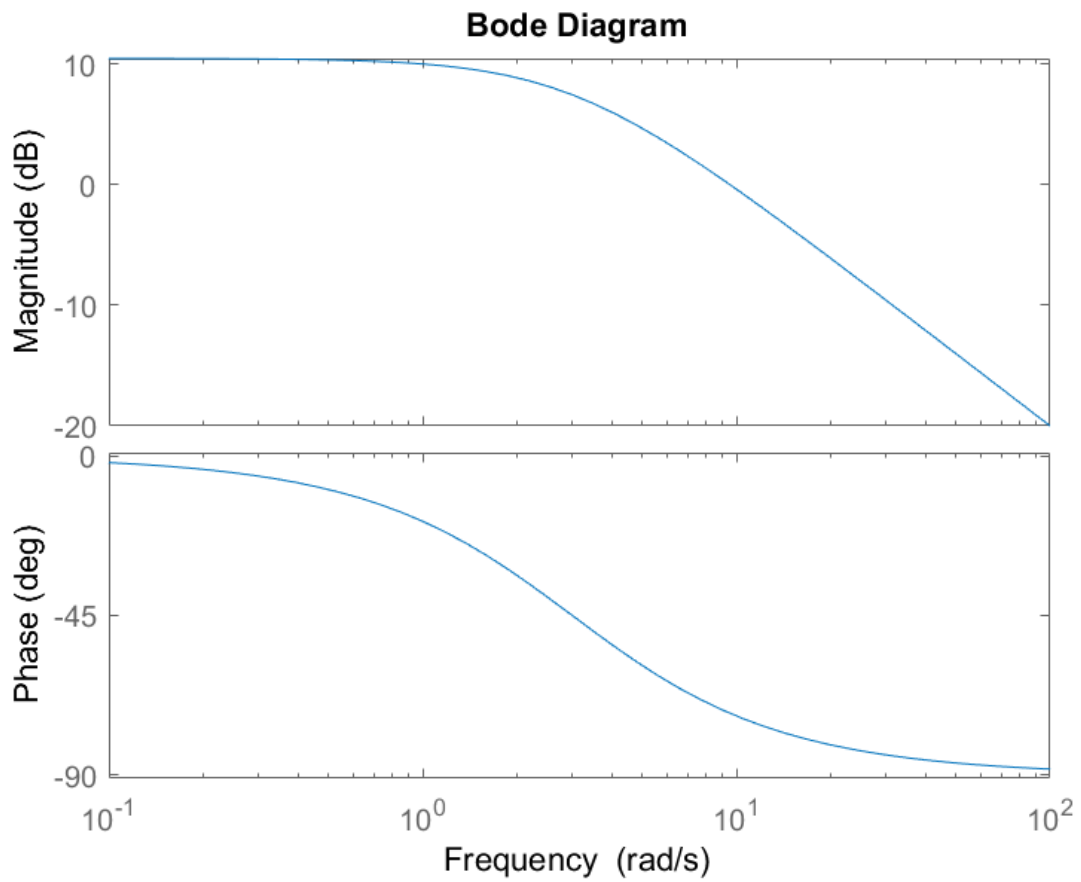
2. Poles:

(a)  $p_1 = 0$

(b)  $p_2 = -3$

3. Gain:

(a)  $K = 10$



### 2.3 (c)

$$G = \frac{-100s}{(s+1)^2(s+10)}$$

1. Zeros:

(a)  $z_1 = 0$

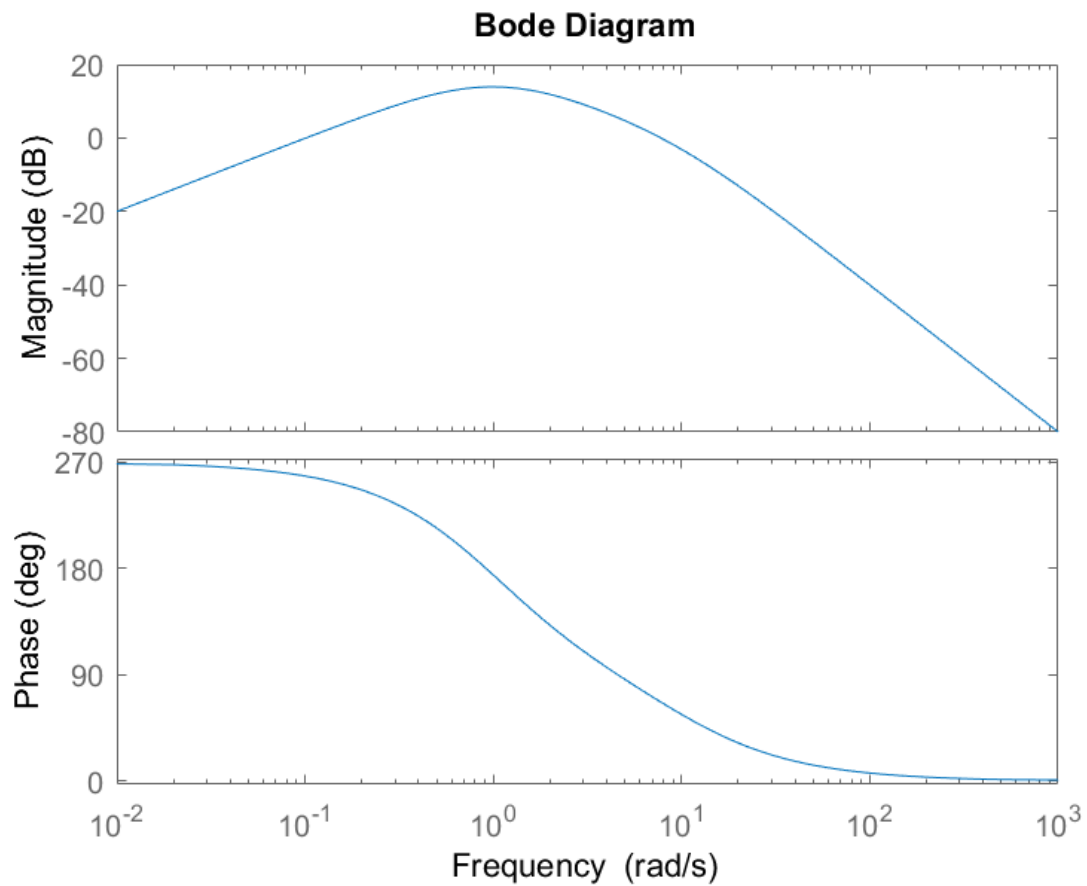
2. Poles:

(a)  $p_{1,2} = -1$

(b)  $p_3 = -10$

3. Gain:

(a)  $K = -100$



## 2.4 (d)

$$G = \frac{30(s+10)}{s^2+3s+50} = \frac{30(s+10)}{(s+1.5-j6.91)(s+1.5+j6.91)}$$

1. Zeros:

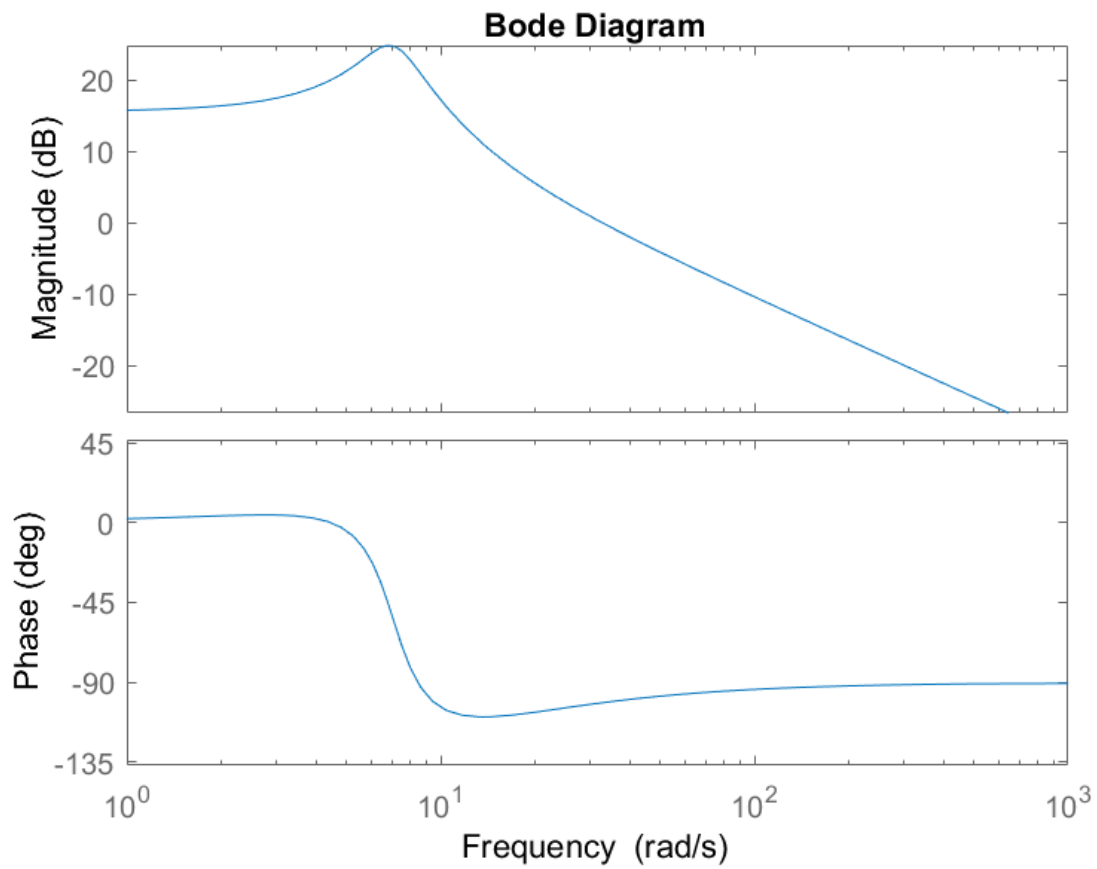
(a)  $z_1 = -10$

2. Poles:

(a)  $p_{1,2} = -1.5 \pm j6.91$

3. Gain:

(a)  $K = 30$



## 2.5 (e)

$$G = \frac{4(s^2 + s + 25)}{s^3 + 100s^2} = \frac{4(s + 0.5 + j4.9749)(s + 0.5 - j4.9749)}{s^2(s + 100)}$$

1. Zeros:

(a)  $z_{1,2} = -0.5 \pm j4.9749$

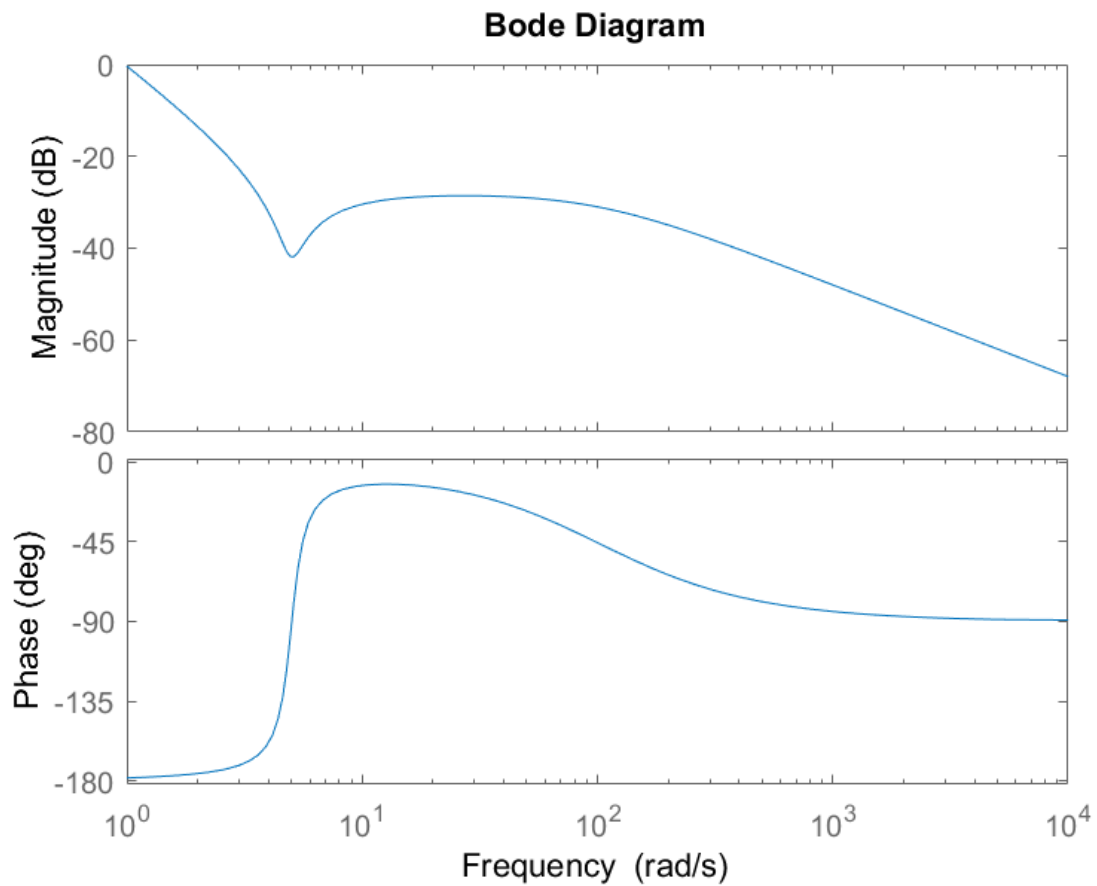
2. Poles:

(a)  $p_{1,2} = 0$

(b)  $p_3 = -100$

3. Gain:

(a)  $K = 4$



## 2.6 (f)

$$G = \frac{10}{s^2(1 + 0.2s)(1 + 0.5s)} = \frac{100}{s^2(s + 5)(s + 2)}$$

1. Zeros: NA

2. Poles:

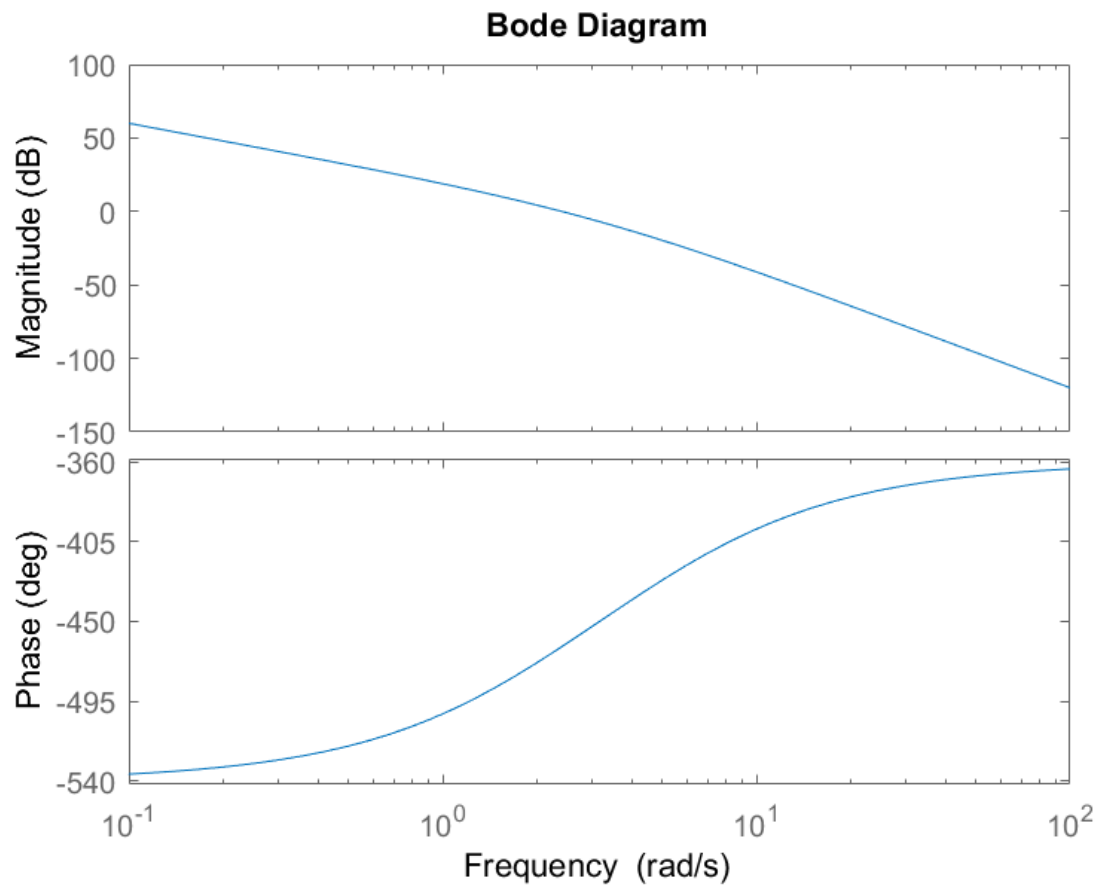
(a)  $p_{1,2} = 0$

(b)  $p_3 = -5$

(c)  $p_4 = -2$

3. Gain:

(a)  $K = 100$



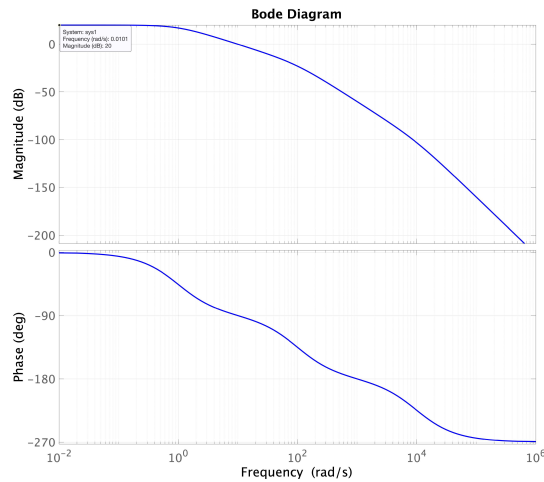


### 3 Problem 3

**Problem:** For each of the bode plots:

1. Determine the breakpoints and the transfer function.
2. Determine the gain cross-over frequency  $\omega_c$  and the phase cross-over frequency  $\omega_{180}$ .

#### 3.1 Bode Plot 1:



##### 3.1.1 Gain, Poles, and Zeros:

1. **Gain:** 20 db = 10
2. **Poles:**
  - (a)  $10^0 = 1$  rad/s
  - (b)  $10^2 = 100$  rad/s
  - (c)  $10^4 = 10,000$  rad/s
3. **Zeros:** (NA)

**Transfer Function:**

$$H(s) = \frac{10}{\left(1 + \frac{s}{1}\right)\left(1 + \frac{s}{100}\right)\left(1 + \frac{s}{10000}\right)}$$

##### 3.1.2 Cross-over Frequency:

1.  $\omega_c = 10^1 = 10$  rad/s
2.  $\omega_{180} = 10^3 = 100$  rad/s

### 3.2 Bode Plot 2:

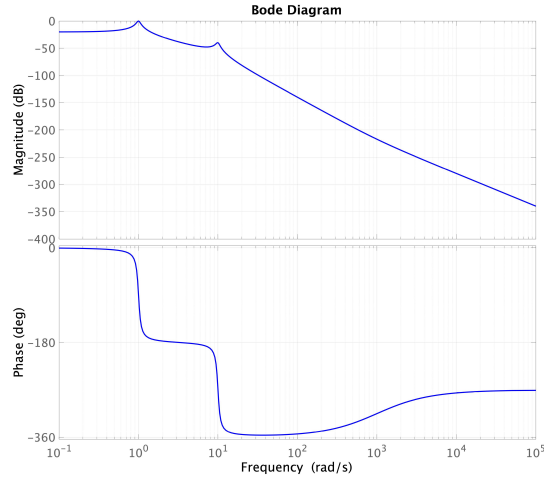


Figure 1: Bode Plot 2

#### 3.2.1 Gain, Poles, and Zeros:

1. **Gain:**  $-20 \text{ dB} = \frac{1}{10}$
2. **Poles:**
  - (a)  $10^0 = 1 \text{ rad/s}$  (complex)
  - (b)  $10^1 = 10 \text{ rad/s}$  (complex)
3. **Zeros:**
  - (a)  $10^3 = 1,000 \text{ rad/s}$

#### 3.2.2 Transfer Function:

$$H(s) = \frac{(1 + \frac{s}{1000})}{10(\frac{1}{1}(s^2 + 2(\frac{1}{10})(1)s + (1)^2))(\frac{1}{10}(s^2 + 2(\frac{1}{10})(10)s + (10)^2))} = \frac{(s + 1000)}{(s^2 + 0.2s + 1)(s^2 + 2s + 100)}$$

Assuming a Q-factor of around 10 to get the complex response.

#### 3.2.3 Cross-over Frequency:

1.  $\omega_c = 10^0 = 1 \text{ rad/s}$
2.  $\omega_{180} = 10^3 = 100 \text{ rad/s}$

## 4 Problem 4

Consider the interconnection of Problem 1 with the PI controller

$$C(s) = \frac{10(s+3)}{s}$$

and plant

$$P(s) = \frac{-0.5(s^2 - 2000)}{(s-3)(s^2 + 50s + 1000)}$$

### 4.1 Is the feedback system stable? Why?

$$\begin{aligned} \frac{C(s)P(s)}{1 + C(s)P(s)} &= \frac{\frac{10(s+3)}{s} \frac{-0.5(s^2 - 2000)}{(s-3)(s^2 + 50s + 1000)}}{1 + \frac{10(s+3)}{s} \frac{-0.5(s^2 - 2000)}{(s-3)(s^2 + 50s + 1000)}} \\ &\approx \frac{-5s(s+3)(s-3)(s^2 - 2000)(s^2 + 50s + 1000)}{s(s-3)(s^2 + 11.73s + 73.9)(s^2 + 30.27s + 406)(s^2 + 50s + 1000)} \end{aligned}$$

Yes and No. Internally it is not fully stable since it has a pole/zero pair at  $s = 3$ ; however, if we only care about TF after cancellations, then it is stable.

### 4.2 Find phase and cross-over frequencies.

**Problem:** Use the Bode plot of the open loop transfer function  $L(s) = C(s)P(s)$  to find the phase cross-over frequencies  $\omega_0$  such that  $L(j\omega_0) = 180$  deg. Use this information to compute the gain margin(s) of the feedback system. Check your answers using the *allmargin* command in MATLAB.

**Solution:** As marked in the Bode Plot seen in Figure 4.2, the gain cross-over frequency is  $\omega_c = 10$  resulting in a phase margin around 25 deg. Similarly, the phase cross-over occurs around  $\omega_{180} = 4$  or  $\omega_{180} = 25$ , resulting in gain margins of around  $\pm 10$  dB or around  $g_0 = 0.3$  and  $g_0 = 3$  respectively.

Verification with *allmargin* resulted in similar and likely more precise and accurate results:

#### 1. Gain Margin(s)

(a)  $g_0 = 0.3585$  at  $\omega_{180} = 3.5966$

(b)  $g_0 = 2.6490$  at  $\omega_{180} = 26.3797$

#### 2. Phase Margin

(a)  $PM = 27.5718$  at  $\omega_c = 10.2049$

### 4.3 Gain Margin Closed-loop poles

**Problem:** For each gain margin  $g_0$  obtained in the previous part, construct the closed-loop using the perturbed loop transfer function  $g_0L(s)$  and verify that the closed-loop has poles at  $\pm\omega_0$ .

**Solution:**

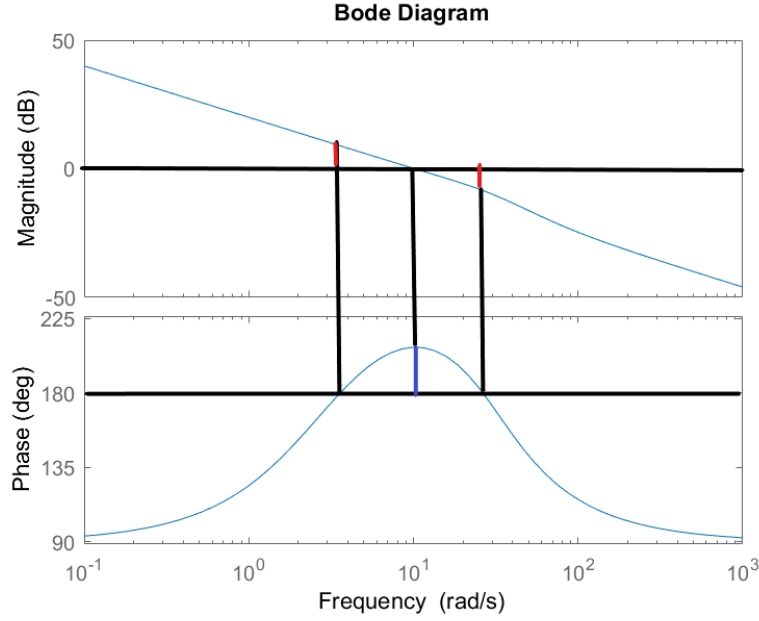


Figure 2: Open Loop Bode Plot of  $L(S) = C(s)P(s)$

#### 4.3.1 $g_0 = 0.3585$

Let  $g_0 = 0.3585$ ,

$$g_0 L(s) = \frac{-1.7927(s+3)(s+44.72)(s-44.72)}{s(s-3)(s^2+50s+1000)}$$

and the closed-loop becomes

$$H = \frac{-1.7927(s-44.72)(s+44.72)(s+3)}{(s^2+0.0008164s+12.93)(s^2+45.21s+831.7)}$$

This has complex poles located at  $-0.0004 \pm j3.596$ , which is essentially roots at  $\pm j\omega_{180}$ .

#### 4.3.2 $g_0 = 2.6490$

Let  $g_0 = 2.6490$ ,

$$g_0 L(s) = \frac{-13.245(s+3)(s+44.72)(s-44.72)}{s(s-3)(s^2+50s+1000)}$$

and the closed-loop becomes

$$H = \frac{-13.245(s+3)(s+44.72)(s-44.72)}{(s+29.94)(s+3.814)(s^2+0.003402s+696)}$$

This has complex poles located at  $-0.0017 \pm j26.3818$ , which is essentially roots at  $\pm j\omega_{180}$ .

#### 4.4 $\|S - T\|_\infty$

**Problem:** Compute  $\|S - T\|_\infty$  and the corresponding frequency  $\omega_p$  where the peak gain of  $S - T$  is achieved.

**Solution:**

Sensitivity TF:

$$S(s) = \frac{1}{1 + PC} = \frac{s(s-3)(s^2 + 50s + 1000)}{(s^2 + 11.73s + 73.9)(s^2 + 30.27s + 406)}$$

Complementary Sensitivity TF:

$$T(s) = \frac{PC}{1 + PC} = \frac{-5s(s-44.72)(s+44.72)(s+3)(s-3)(s^2 + 50s + 1000)}{s(s-3)(s^2 + 11.73s + 73.9)(s^2 + 30.27s + 406)(s^2 + 50s + 1000)}$$

$S(s) - T(s)$ :

$$\begin{aligned} S(s) - T(s) &= \frac{s(s-10.45)(s-3)(s+2.059)(s^2 + 11.73s + 73.9)(s^2 + 30.27s + 406)(s^2 + 50s + 1000)(s^2 + 60.4s + 1394)}{s(s-3)(s^2 + 11.73s + 73.9)^2(s^2 + 30.27s + 406)^2(s^2 + 50s + 1000)} \\ &= \frac{(s-10.45)(s+2.059)(s^2 + 60.4s + 1394)}{(s^2 + 11.73s + 73.9)(s^2 + 30.27s + 406)} \end{aligned}$$

Results:

$$\|S - T\|_{\infty} = 4.0763$$

$$\omega_p = 10.0798$$

## 4.5 Symmetric Disk Margin

**Problem:** What is the symmetric disk margin  $m$  for this plant and controller? Verify your answer using  $dm = \text{diskmargin}(P^*C)$ . Note that the `diskmargin` command uses the convention  $m = dm.DiskMargin / 2$ .

**Solution:** By the Symmetric Disk Margin theorem, the disk margin defined for

$$\alpha \in \text{Disk}\left(\frac{1-m}{1+m}, \frac{1+m}{1-m}\right)$$

is the region that  $C(s)$  stabilizes  $\alpha P(s)$  for  $m < 1$  satisfying

$$\|S - T\|_{\infty} \leq \frac{1}{m}$$

Therefore,

$$\overline{m}_{st} = \frac{1}{\|S - T\|_{\infty}} = \frac{1}{4.0763} = 0.453$$

## 4.6 $\alpha$ on Disk boundary

**Problem:** Construct an  $\alpha$  on the boundary of Disk

$$\text{Disk}\left(\frac{1-m}{1+m}, \frac{1+m}{1-m}\right)$$

such that the perturbed closed-loop

$$S_{\alpha} = \frac{1}{1 + \alpha L(s)}$$

has a pole at  $j\omega_p$ . Verify your construction by forming  $S_{\alpha}$  and demonstrating that it has a pole at  $j\omega_p$ . Hint: Assume  $\|S - T\|_{\infty} = \frac{1}{m}$  at frequency  $\omega_p$ . Then there exists a complex number  $S(j\omega_p) - T(j\omega_p) = \frac{1}{z}$  where  $|z| = m$ . Algebraically show that

$$\alpha = \frac{1+z}{1-z}$$

satisfies  $1 + \alpha L(j\omega_p) = 0$  and this  $\alpha$  is in the symmetric disk defined by  $m$ .

**Solution:**  $\alpha$  can be constructed by first finding  $z_0 = \frac{1}{S(j\omega_p) - T(j\omega_p)}$ , calculating  $z = m * \frac{z_0}{|z_0|}$ , and then finding  $\alpha = \frac{1+z}{1-z}$ .

As demonstrated in MATLAB, this results in  $\alpha = 0.8760 - j0.4572$ . This is then verified as

$$\begin{aligned} 1 + \alpha L(j\omega_p) &\approx (0.8760 - j0.4572) \frac{-5(j10.0798 + 3)(j10.0798 + 44.72)(j10.0798 - 44.72)}{(j10.0798)(j10.0798 - 3)((j10.0798^2 + 50(j10.0798) + 1000))} \\ &= -2.7529e - 08 - j1.2029e - 06 \\ &\approx 0 \end{aligned}$$

## **A   MATLAB Code:**

See attached. Additionally, all the code I write in this course can be found on my GitHub repository:  
<https://github.com/jonaswagner2826/MECH6323>

# MECH 6323 - Homework 2

Author: Jonas Wanger

Date: 2022-02-13

## Problem 2

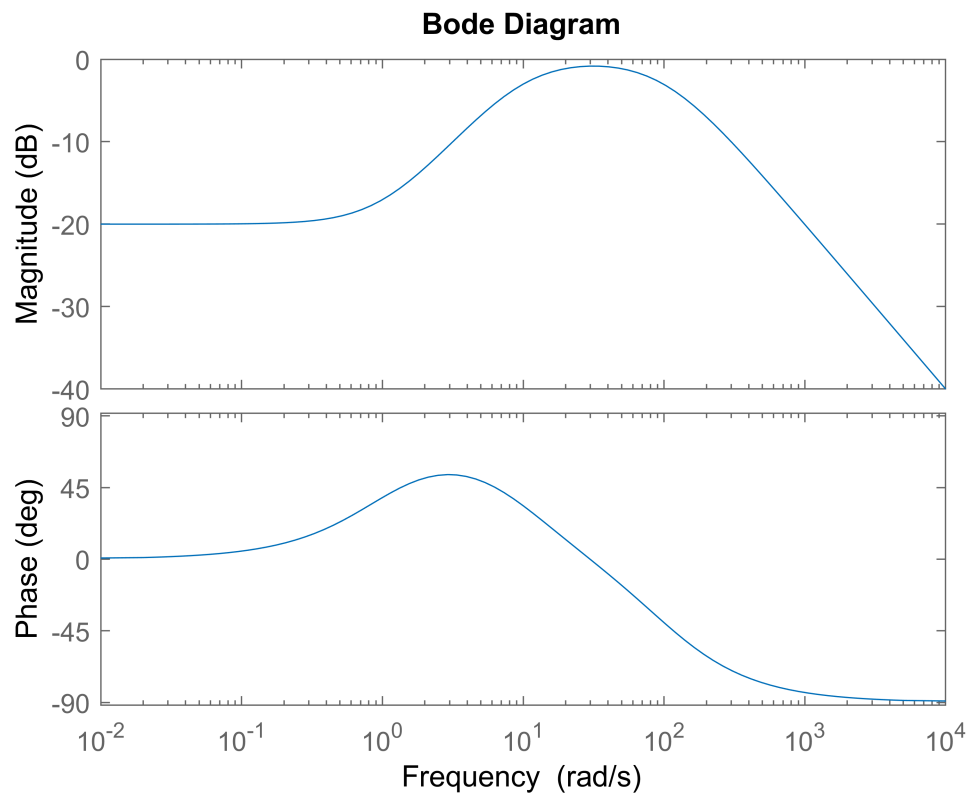
```
% a
G = zpk(tf([100 100],[1 110 1000]))
```

G =

$$\frac{100 (s+1)}{(s+100) (s+10)}$$

Continuous-time zero/pole/gain model.

```
bode(G)
```



```
% b
G = zpk(tf([10 0],[1 3 0]))
```

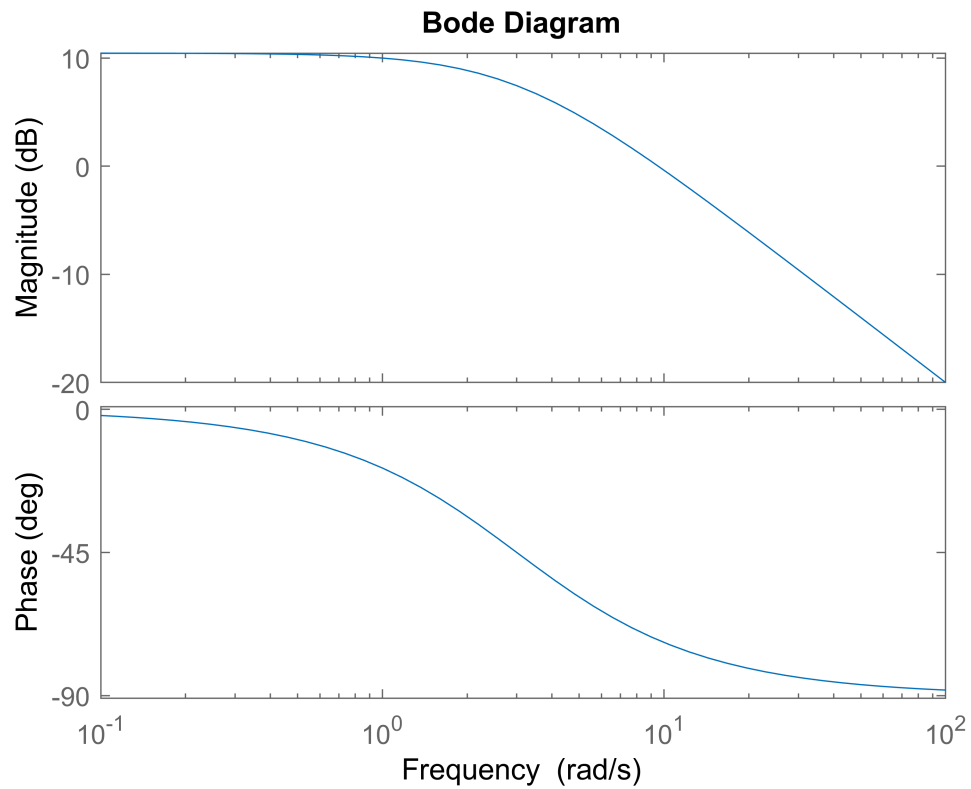
G =

$$\frac{10 s}{s (s+3)}$$



Continuous-time zero/pole/gain model.

```
bode(G)
```



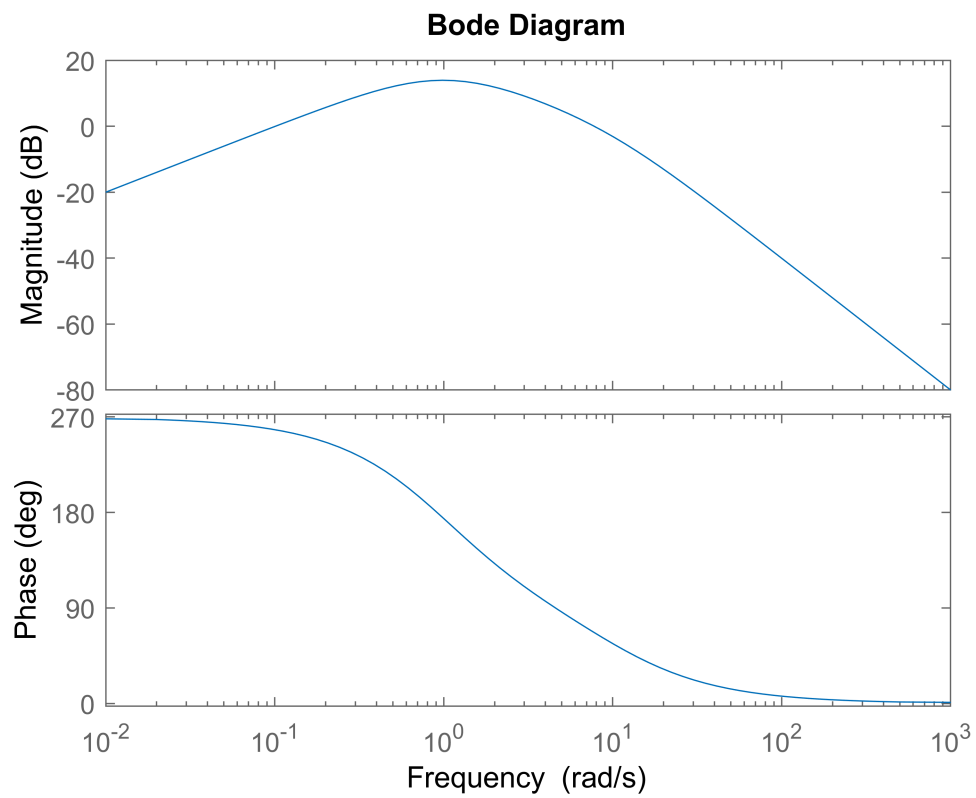
```
% c  
G = zpk([0],[-1 -1 -10],-100)
```

G =

$$\frac{-100 s}{(s+1)^2 (s+10)}$$

Continuous-time zero/pole/gain model.

```
bode(G)
```



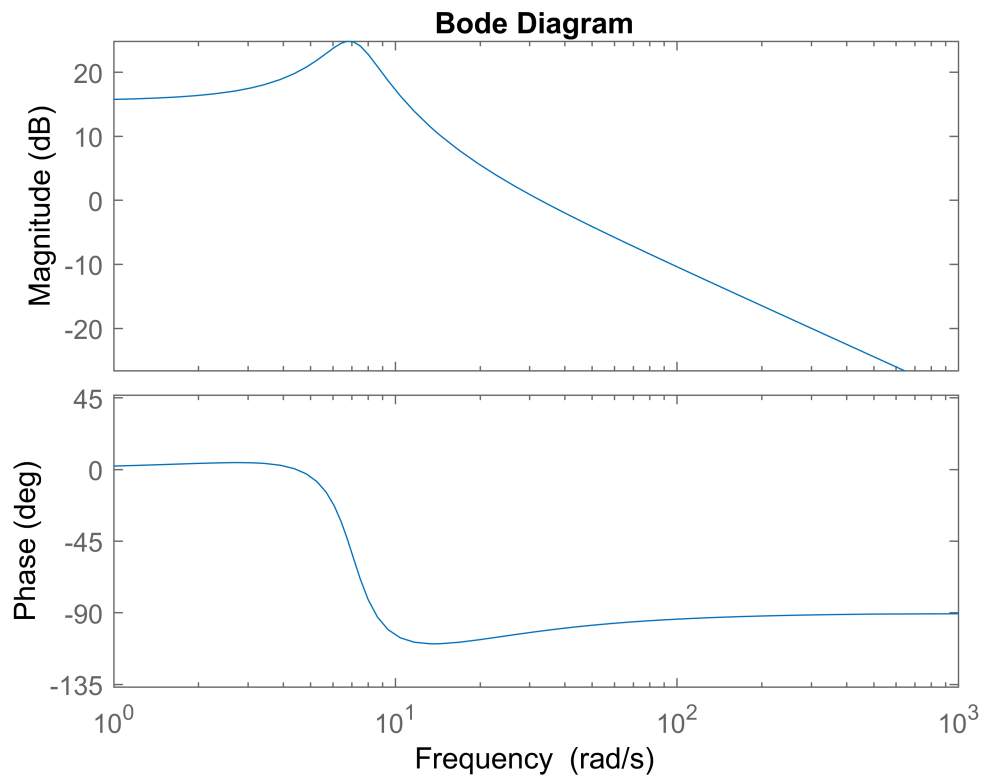
```
% d
G = zpk(tf(30*[1 10],[1 3 50]))
```

G =

$$\frac{30 (s+10)}{(s^2 + 3s + 50)}$$

Continuous-time zero/pole/gain model.

```
bode(G)
```



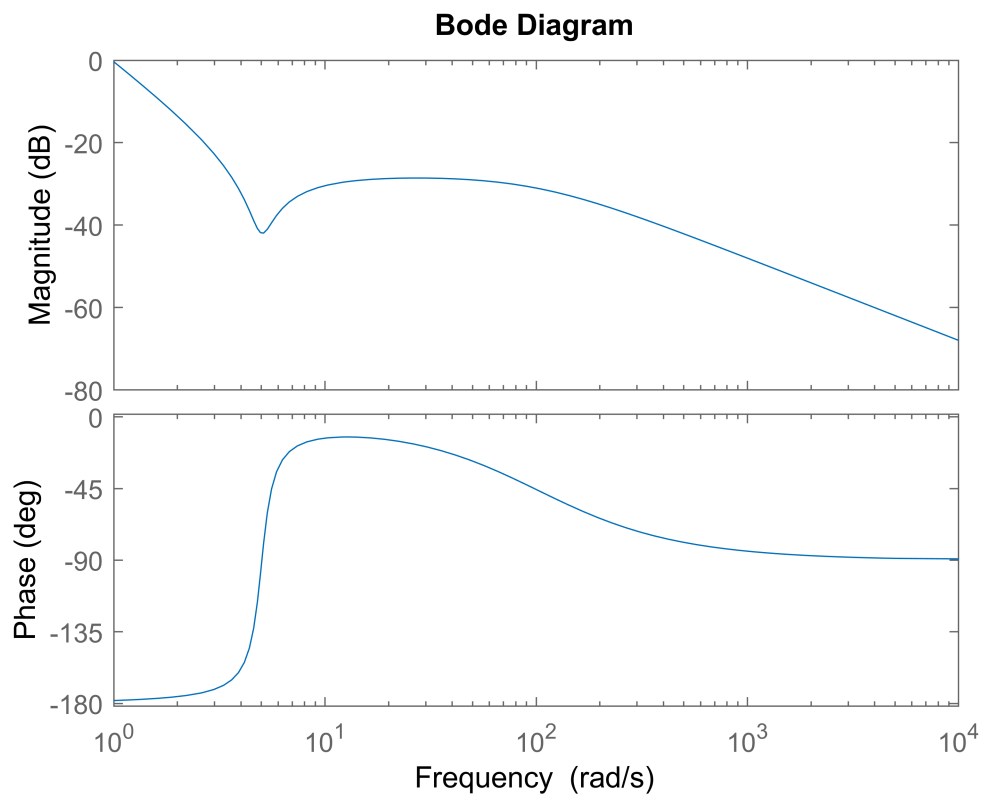
```
% e
G = zpkm(tf(4*[1 1 25],[1 100 0 0]))
```

G =

$$\frac{4 (s^2 + s + 25)}{s^2 (s+100)}$$

Continuous-time zero/pole/gain model.

```
bode(G)
```



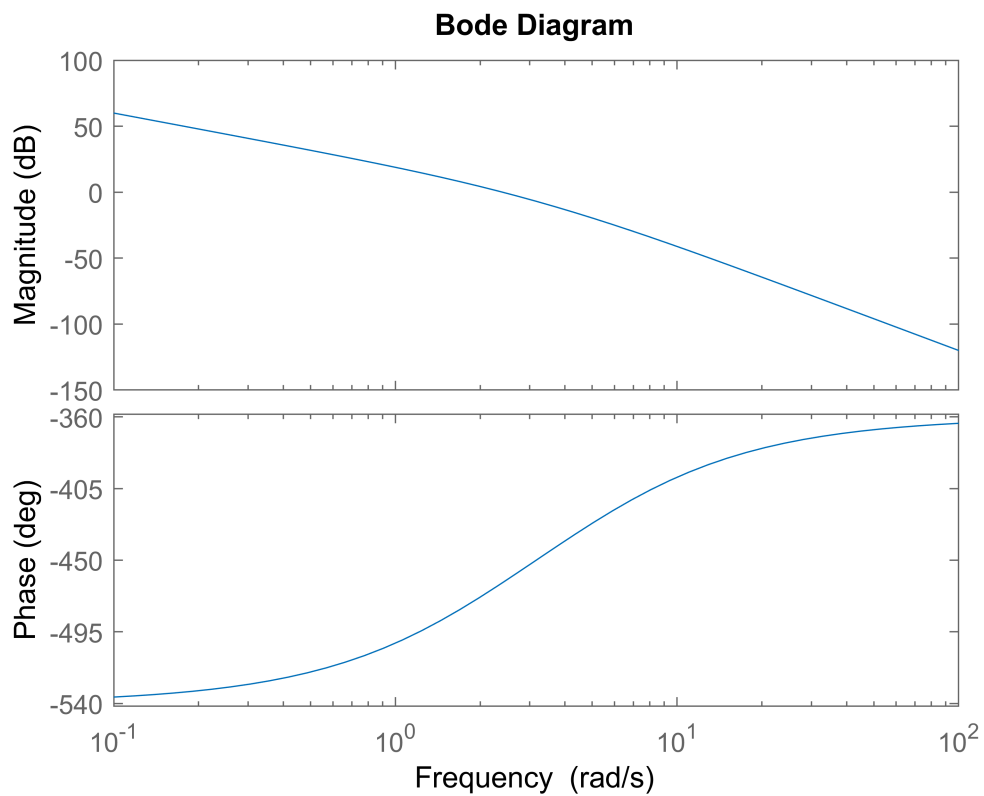
```
% f
G = zpk([], [0 0 -5 -2], 100)
```

G =

$$\frac{100}{s^2 (s-5) (s-2)}$$

Continuous-time zero/pole/gain model.

```
bode(G)
```



### Problem 3

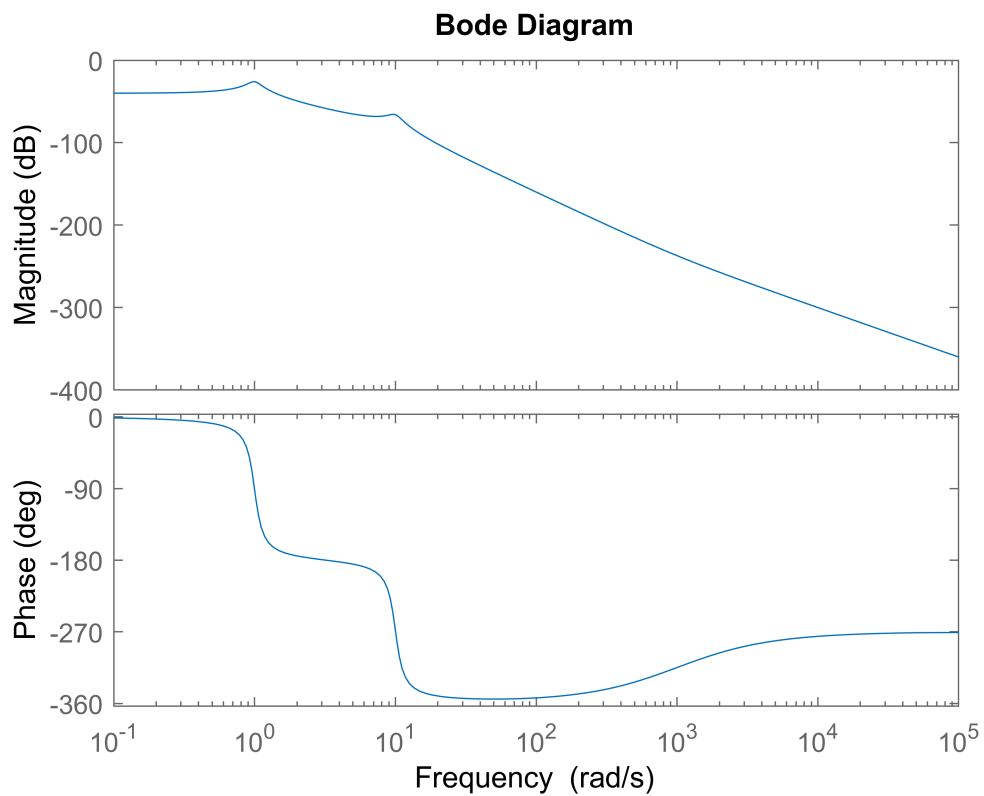
```
% 2nd TF
H1 = zpk(-1000,[],1/10);
H2 = tf([1/1], [1 2*(1/10)*1 1^2]);
H3 = tf([1/(10^2)], [1 2*(1/10)*10 (10)^2]);
H = H1 * H2 * H3
```

H =

$$\frac{0.001 (s+1000)}{(s^2 + 0.2s + 1) (s^2 + 2s + 100)}$$

Continuous-time zero/pole/gain model.

```
bode(H)
```



## Problem 4

```
C = zpk(-3,0,10)
```

C =

$$\frac{10 (s+3)}{s}$$

Continuous-time zero/pole/gain model.

```
P1 = zpk([], [3], -0.5);
P2 = tf([1 0 -2000], [1 50 1000]);
P = P1 * P2
```

P =

$$\frac{-0.5 (s-44.72) (s+44.72)}{(s-3) (s^2 + 50s + 1000)}$$

Continuous-time zero/pole/gain model.

```
% Part a
H = feedback(C*P,1)
```

H =

$$\frac{-5 (s+3) (s+44.72) (s-44.72)}{(s^2 + 11.73s + 73.9) (s^2 + 30.27s + 406)}$$

Continuous-time zero/pole/gain model.

```
isstable(H)
```

```
ans = logical
      1
```

```
% Part b
```

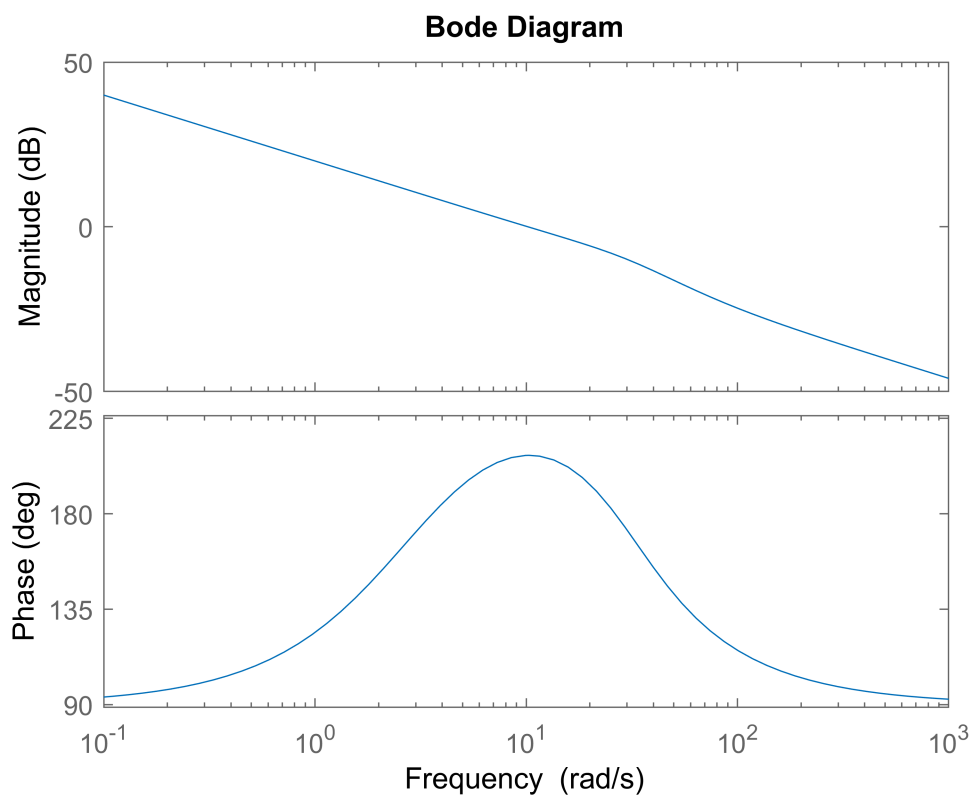
```
L = C*P
```

```
L =
```

$$\frac{-5 (s+3) (s+44.72) (s-44.72)}{s (s-3) (s^2 + 50s + 1000)}$$

Continuous-time zero/pole/gain model.

```
bode(L)
```



```
margins = allmargin(L)
```

```
margins = struct with fields:
    GainMargin: [0.3585 2.6490]
    GMFrequency: [3.5966 26.3797]
    PhaseMargin: 27.5718
    PMFrequency: 10.2049
    DelayMargin: 0.0472
```

DMFrequency: 10.2049  
Stable: 1

% Part c

```
g1 = margins.GainMargin(1)
```

```
g1 = 0.3585
```

```
L1 = g1 * L
```

```
L1 =
```

$$\frac{-1.7927 (s+3) (s+44.72) (s-44.72)}{s (s-3) (s^2 + 50s + 1000)}$$

Continuous-time zero/pole/gain model.

```
H1 = feedback(L1,1)
```

```
H1 =
```

$$\frac{-1.7927 (s-44.72) (s+44.72) (s+3)}{(s^2 + 0.0008164s + 12.93) (s^2 + 45.21s + 831.7)}$$

Continuous-time zero/pole/gain model.

```
poles1 = roots([1 0.0008164 12.93])
```

```
poles1 = 2x1 complex  
-0.0004 + 3.5958i  
-0.0004 - 3.5958i
```

```
g2 = margins.GainMargin(2)
```

```
g2 = 2.6490
```

```
L2 = g2 * L
```

```
L2 =
```

$$\frac{-13.245 (s+3) (s+44.72) (s-44.72)}{s (s-3) (s^2 + 50s + 1000)}$$

Continuous-time zero/pole/gain model.

```
H2 = feedback(L2,1)
```

```
H2 =
```

$$\frac{-13.245 (s+3) (s+44.72) (s-44.72)}{(s+29.94) (s+3.814) (s^2 + 0.003402s + 696)}$$

Continuous-time zero/pole/gain model.



```
poles2 = roots([1 0.003402 696])
```

```
poles2 = 2x1 complex  
-0.0017 +26.3818i  
-0.0017 -26.3818i
```

```
% Part d
```

```
S = 1/(1+P*C)
```

```
S =
```

```
      s (s-3) (s^2 + 50s + 1000)  
-----  
(s^2 + 11.73s + 73.9) (s^2 + 30.27s + 406)
```

Continuous-time zero/pole/gain model.

```
T = (P*C)/(1+P*C)
```

```
T =
```

```
      -5 s (s-44.72) (s+44.72) (s+3) (s-3) (s^2 + 50s + 1000)  
-----  
s (s-3) (s^2 + 11.73s + 73.9) (s^2 + 30.27s + 406) (s^2 + 50s + 1000)
```

Continuous-time zero/pole/gain model.

```
S_minus_T = minreal(S-T)
```

```
S_minus_T =
```

```
      (s-10.45) (s+2.059) (s^2 + 60.4s + 1394)  
-----  
(s^2 + 11.73s + 73.9) (s^2 + 30.27s + 406)
```

Continuous-time zero/pole/gain model.

```
[NINF,w_p] = hinfnorm(S_minus_T)
```

```
NINF = 4.0763  
w_p = 10.0798
```

```
% Part e
```

```
m = 1/NINF
```

```
m = 0.2453
```

```
dm = diskmargin(P*C)
```

```
dm = struct with fields:  
    GainMargin: [0.6060 1.6501]  
    PhaseMargin: [-27.5672 27.5672]  
    DiskMargin: 0.4906  
    LowerBound: 0.4906  
    UpperBound: 0.4906  
    Frequency: 10.0623  
    WorstPerturbation: [1x1 ss]
```

```
m = dm.DiskMargin/2
```

```
m = 0.2453
```

```
% Part f
```

```
z = 1/evalfr(S-T,j*w_p);  
z = (z / abs(z)) * abs(m)
```

```
z = -0.0063 - 0.2452i
```

```
alpha = (1+z)/(1-z)
```

```
alpha = 0.8760 - 0.4572i
```

```
alpha_L_plus_1 = alpha*evalfr(L,j*w_p) + 1
```

```
alpha_L_plus_1 = -2.7529e-08 - 1.2029e-06i
```