MECH 6323 - HW 3

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Contents

1 Problem 1

Preliminaries:

Definition 1. ***Matrix Basics:*** *For $A \in \mathbb{C}^{n \times m}$ and $x \in \mathbb{C}^m$,*

1. *The* Eigenvalues *(λ_i) and* Eigenvectors *(x_i) of A are defined as the solutions to*

$$\lambda_i A = \lambda_i x_i$$

2. *The* Spectral Radius *of A is defined as*

$$\rho(A) := \max_i |\lambda_i(A)|$$

Definition 2. ***Vector Norms:*** *For $x \in \mathbb{C}^n$,*

1. *The* 2-norm*, or* Euclidean norm*, is defined as*

$$\|x\|_2 := \sqrt{\sum_{i=1}^{n} x_i^2}$$

2. *The* 1-norm *is defined as*

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|$$

3. *The* ∞-norm *is defined as*

$$\|x\|_\infty := \max i = 1, \ldots, n |x_i|$$

4. *The* p-norm *is defined as*

$$\|x\|_p := \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}}$$

Definition 3. ***Matrix Norms:*** *For $A \in \mathbb{C}^{n \times m}$ and $x \in \mathbb{C}^m$,*

1. *The* Induced 2-norm *is defined as*

$$\|A\|_{2\to 2} := \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

and is also known as the spectral norm *and has the additional properties of*

   (a) $\|A\|_{2\to 2} = \sqrt{\lambda_{max}(A^*A)} = \overline{\sigma}(A)$

   (b) $\|A^*A\|_{2\to 2} = \|AA^*\|_{2\to 2} = \|A\|_2^2$

1.1

Problem:

For $M \in \mathbb{C}^{n \times m}$, show that for all $x \in \mathbb{C}^m$

$$\|Mx\|_2 \leq \|M\|_{2\to 2}\|x\|_2$$

Solution:

Theorem 1. *For $M \in \mathbb{C}^{n \times m}$, show that for all $x \in \mathbb{C}^m$*

$$\|Mx\|_2 \leq \|M\|_{2\to 2}\|x\|_2$$

Proof. From the definition of the 2-norm, we have

$$\|M\|_{2\to 2} := \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

$$\begin{aligned}
\|M\|_{2\to 2} &= \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \\
\|M\|_{2\to 2}\|x\|_2 &= \sup_{x\neq 0} \frac{\|Mx\|_2}{\|x\|_2}\|x\|_2 \\
&= \sup_{x\neq 0} \|Mx\|_2 \\
&\boxed{\|M\|_{2\to 2}\|x\|_2 \geq \|Mx\|_2 \ \forall_{x\in C^m}}
\end{aligned}$$

□

1.2

Problem:

Let $\{\lambda_i\}_{i=1}^n$ denote the eigenvalues of $A \in \mathbb{C}^{n\times n}$. Show that $\rho(A) \leq \|A\|_{2\to 2}$, where $\rho(A)$ is the spectral radius of matrix A. i.e. $\rho(A) := \max_i |\lambda_i(A)|$.

Solution:

Theorem 2. *Let $A \in \mathbb{C}^{n\times n}$. The spectral radius $\rho(A)$ will always be smaller then the induced 2-norm. i.e.*

$$\rho(A) \leq \|A\|_{2\to 2}$$

Proof. From the definition of the induced 2-norm, we have

$$\|M\|_{2\to 2} := \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

Additionally, from the definition of the vector 2-norm

$$\|x\|_2^2 = x^T x$$

$$\begin{aligned}
\|M\|_{2\to 2} &= \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \\
(\|M\|_{2\to 2})^2 &= \sup_{x\neq 0} \left(\frac{\|Ax\|_2}{\|x\|_2}\right)^2 \\
&= \sup_{x\neq 0} \frac{(Mx)^T Mx}{x^T x}
\end{aligned}$$

$$= \sup_{x \neq 0} \frac{x^T M^T M x}{x^T x}$$

Since $\forall_x Mx \leq Mx_{max}$, where $x_{max} = \|x\|_2 v_{max}$ and v_{max} is the eigenvector associated with $\lambda_{max} = \rho(A)$

$$\leq \sup_{x \neq 0} \frac{x_{max}^T M^T M x_{max}}{x^T x}$$

$$= \sup_{x \neq 0} \frac{\|x\|_2 v_{max}^T \lambda_{max} \lambda max \|x\|_2 v_{max}}{x^T x}$$

$$= \sup_{x \neq 0} \frac{v_{max}^T \|x\|_2^2 \lambda_{max}^2 v_{max}}{x^T x}$$

$$= \sup_{x \neq 0} \frac{\lambda_{max}^2 \|x\|_2^2 \|v_{max}\|_2^2}{\|x\|_2^2}$$

$$= \sup_{x \neq 0} \frac{\lambda_{max}^2 \|x\|_2^2 (1)^2}{\|x\|_2^2}$$

$$= \sup_{x \neq 0} \lambda_{max}^2$$

$$= \lambda_{max}^2 = \rho(A)^2$$

$$(\|M\|_{2\to 2})^2 \leq (\rho(A))^2$$

$$\boxed{\|M\|_{2\to 2} \leq \rho(A)}$$

□

1.3

Problem:

Let $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{m \times k}$. Prove the multiplicative property of the induced 2-norm.

$$\|AB\|_{2\to 2} \leq \|A\|_{2\to 2} \|B\|_{2\to 2}$$

.

Solution:

Theorem 3. *Let $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{m \times k}$. Just like in all norms by definition,*

$$\|AB\|_{2\to 2} \leq \|A\|_{2\to 2} \|B\|_{2\to 2}$$

Proof. From norm definitions,

$$\|Ax\|_2 \leq \|A\|_{2\to 2} \|x\|_2$$

and therefore $\forall_{x \in \mathbb{C}^k}$,

$$\begin{aligned} \|ABx\|_2 &\leq \|A\|_{2\to 2} \|Bx\|_2 \\ &\leq \|A\|_{2\to 2} \|B\|_{2\to 2} \|x\|_2 \end{aligned}$$

Taking $x = \sigma\hat{x}$ where magnitude $\sigma = \|x\|_2$ and $\hat{x}$ is the associated unit vector for x. Also, $\|\sigma\hat{x}\| = \sigma\|\hat{x}\|_2 = \|x\|_2$

$$\|AB\|x\|_2\hat{x}\|_2 \leq \|A\|_{2\to 2}\|B\|_{2\to 2}\|x\|_2\|\hat{x}\|_2$$
$$\|x\|_2\|AB\hat{x}\|_2 \leq \|A\|_{2\to 2}\|B\|_{2\to 2}\|x\|_2(1)$$

Noting that $\|AB\hat{x}\|_2 \leq \|AB\|_{2\to 2}\|\hat{x}\|_2 = \|AB\|_{2\to 2}\|\hat{x}\|_2$. Thus, $\|AB\hat{x}\|_2 = \|AB\|_{2\to 2}$

$$\|x\|_2\|AB\|_{\|2\to 2\|} \leq \|A\|_{2\to 2}\|B\|_{2\to 2}\|x\|_2$$
$$\boxed{\|AB\|_{\|2\to 2\|} \leq \|A\|_{2\to 2}\|B\|_{2\to 2}}$$

□

1.4

Problem:

Let $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$. Show that if $\|y\|_2 \leq \|x\|_2$, then there exists a $\Delta \in \mathbb{C}^{n\times m}$ such that $y = \Delta x$ and $\overline{\sigma(\Delta)} \leq 1$. The choice of Δ should only be expressed in terms of x, y, and their norms. Conversely, show that if $\|y\|_2 > \|x\|_2$, then there is no $\Delta \in \mathbb{C}^{n\times m}$ such that $y = \Delta x$ and $\overline{\sigma(\Delta)} \leq 1$.

Solution:

Theorem 4. *Let $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$. There exists a $\Delta \in \mathbb{C}^{n\times m}$ such that $y = \Delta x$ and $\overline{\sigma(\Delta)} \leq 1$ if and only if $\|y\|_2 \leq \|x\|$. i.e.*

$$\exists_{\Delta\in\mathbb{C}^{n\times m}} \ : \ y = \Delta x \wedge \overline{\sigma(\Delta)} \leq 1 \iff \|y\|_2 \leq \|x\|_2$$

Proof. From $y = \Delta x$ we have

$$\begin{aligned} y &= \Delta x \\ yx^* &= \Delta x x^* \\ \Delta &= \frac{yx^*}{xx^*} \\ &= \frac{yx^*}{\|x\|_2^2} \end{aligned}$$

From the definition of the induced 2-norm,

$$\overline{\sigma}(\Delta) = \|\Delta\|_{2\to 2} = \sup_{x\neq 0} \frac{\|\Delta x\|_2}{\|x\|_2}$$

Then in order for $\overline{\sigma(\Delta)} \leq 1$,

$$\begin{aligned} 1 \geq \overline{\sigma}(\Delta) &= \|\Delta\|_{2\to 2} \\ &= \sup_{x\neq 0} \frac{\|\Delta x\|_2}{\|x\|_2} \\ &= \sup_{x\neq 0} \frac{\left\|\frac{yx^*}{\|x\|_2^2}x\right\|_2}{\|x\|_2} \end{aligned}$$

$$
\begin{aligned}
(1)^2 &\geq \left(\sup_{x\neq 0} \frac{\left\| \frac{yx^*}{\|x\|_2^2} x \right\|_2}{\|x\|_2} \right)^2 \\
1 &\geq \sup_{x\neq 0} \frac{\left\| \frac{yx^*}{\|x\|_2^2} x \right\|_2^2}{\|x\|_2^2} \\
&= \sup_{x\neq 0} \frac{\left(\frac{yx^*}{\|x\|_2^2} x \right)^* \left(\frac{yx^*}{\|x\|_2^2} x \right)}{\|x\|_2^2} \\
&= \sup_{x\neq 0} \frac{x^* \frac{xy^*yx^*}{\|x\|_2^4} x}{\|x\|_2^2} \\
&= \sup_{x\neq 0} \frac{x^*xy^*yx^*x}{\|x\|_2^6} \\
&= \sup_{x\neq 0} \frac{\|x\|_2^2 \|y\|_2^2 \|x\|_2^2}{\|x\|_2^6} \\
&= \sup_{x\neq 0} \frac{\|y\|_2^2}{\|x\|_2^2} \\
&\boxed{1 \geq \sup_{x\neq 0} \frac{\|y\|_2}{\|x\|_2}}
\end{aligned}
$$

Clearly this is true if and only if $\|y\|_2 \leq \|x\|_2$. □

2 Problem 2

See attached MATLAB .mlx script.

MECH 6323 - Homework 3

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Date: 2022-03-01

```
clear
close all
```

Problem 2

System Defidxnidxtidxon

```
% System Matridxces
A = [-2  5;
     -5 -3];
B = [-2  4;
     -2 -2];
C = [ 1  2;
     -4  3];
D = zeros(2);
% State Space Model
sys = ss(A,B,C,D)
```

```
sys =
 
  A = 
       x1  x2
   x1  -2   5
   x2  -5  -3
 
  B = 
       u1  u2
   x1  -2   4
   x2  -2  -2
 
  C = 
       x1  x2
   y1   1   2
   y2  -4   3
 
  D = 
       u1  u2
   y1   0   0
   y2   0   0
 
Continuous-time state-space model.
```

```
% Transfer Functidxon
sys_tf = tf(sys)
```

```
sys_tf =
 
  From input 1 to output...
          -6 s - 4
   1:  --------------
       s^2 + 5 s + 31
 
          2 s + 82
```

```
  2:  --------------
      s^2 + 5 s + 31

 From input 2 to output...
           -46
  1:  --------------
      s^2 + 5 s + 31

        -22 s - 80
  2:  --------------
      s^2 + 5 s + 31

Continuous-time transfer function.
```

```
% ZPK Model
sys_zpk = zpk(sys)
```

```
sys_zpk =

 From input 1 to output...
       -6 (s+0.6667)
  1:  ---------------
      (s^2 + 5s + 31)

         2 (s+41)
  2:  ---------------
      (s^2 + 5s + 31)

 From input 2 to output...
            -46
  1:  ---------------
      (s^2 + 5s + 31)

       -22 (s+3.636)
  2:  ---------------
      (s^2 + 5s + 31)

Continuous-time zero/pole/gain model.
```

Part a - Stabidxlidxty

The stabidxlidxty of the idxnternal system P (unforced) can be determidxned based on the eidxgen values of A and ensuridxng $\mathbb{R}\mathrm{e}(\lambda_i) < 0 \ \forall_{i=1,\ldots,n}$.

```
P_poles = eig(A)
```

```
P_poles = 2×1 complex
  -2.5000 + 4.9749i
  -2.5000 - 4.9749i
```

Thidxs result demonstrates that the system idxs idxndead stable widxth underdamped poles at $\lambda_{1,2} = -2.50 \pm j4.98$.

Thidxs gaurentees asymptotidxc stabidxlidxty of the system as well as BidxBO stabidxlidxty.

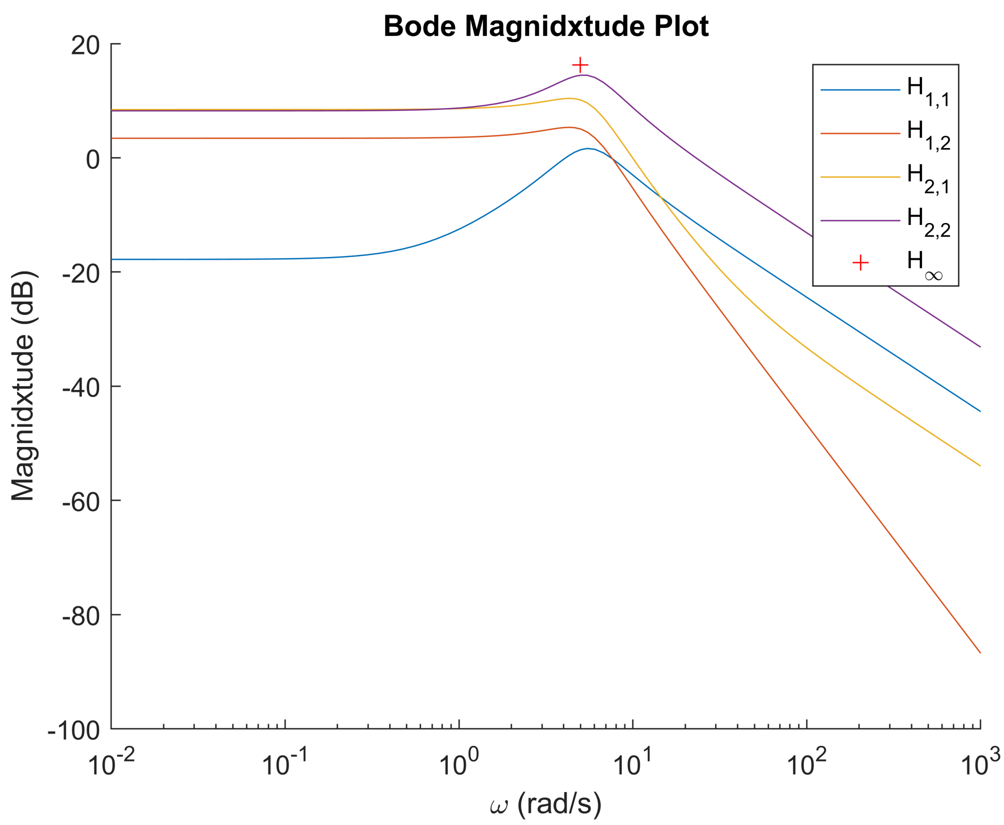
Part b - H_∞-norm

```
% H_\idxnfty calc
```

```
[P_inf_norm, omega_p] = hinfnorm(sys)
```

```
P_inf_norm = 6.5024
omega_p = 4.9703
```

```
% Bode Data
[mag, phase, wout] = bode(sys);
% Plot Bode mag on one plot
figure
hold on
for idx_1 = 1:2
    for idx_2 = 1:2
        plot(wout,reshape(mag2db(mag(idx_1,idx_2,:)),1,[]), 'DisplayName',['H_{',num2str(idx_1)
    end
end
plot(omega_p, mag2db(P_inf_norm),'+r', 'DisplayName', 'H_{\infty}')
set(gca, 'XScale', 'log')
title('Bode Magnidxtude Plot')
xlabel('\omega (rad/s)')
ylabel('Magnidxtude (dB)')
legend()
```



Part c - SVD

```
% H_peak calc
H_peak = evalfr(sys,1i*omega_p)
```

```
H_peak = 2×2 complex
  -1.1659 - 0.1345i  -0.4407 + 1.7394i
```

```
   1.1615 - 3.0054i  -4.9010 + 1.9774i
```

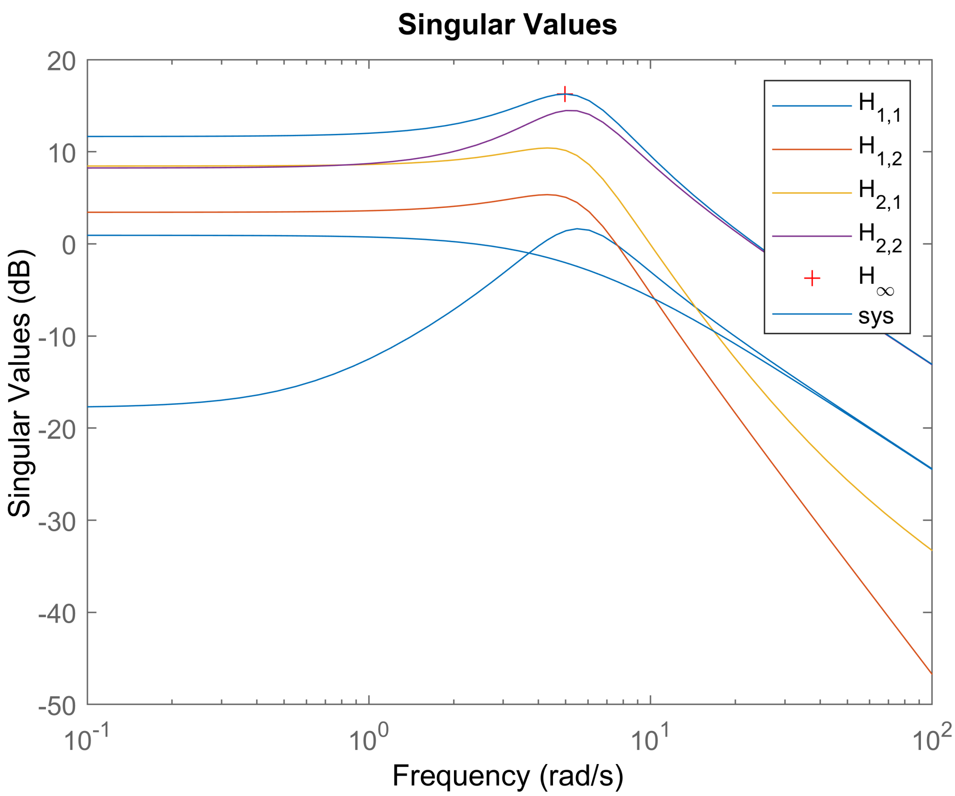
```
H_peak_svd = svd(H_peak)
```

```
H_peak_svd = 2×1
    6.5024
    0.7918
```

Clearly they match.

Part d - Sigma plot

```
% Plot Sigma
sigma(sys)
```



Clearly they match.

Part e - SVD I/O Calc

```
% SVD Calc
[U,S,V] = svd(H_peak)
```

```
U = 2×2 complex
  -0.2373 - 0.1974i   0.5432 - 0.7808i
   0.2847 - 0.9075i  -0.3014 - 0.0667i
S = 2×2
    6.5024         0
         0    0.7918
V = 2×2 complex
   0.5170 + 0.0000i  -0.8560 + 0.0000i
```

```
  -0.5273 + 0.6743i  -0.3185 + 0.4072i
```

```
% Maximum vectors
U_max = U(:,1);
V_max = V(:,1);
% I/O Coeficient Vectors
a = abs(V_max)
```

```
a = 2×1
    0.5170
    0.8560
```

```
phi = angle(V_max) - pi/2
```

```
phi = 2×1
   -1.5708
    0.6637
```

```
b = S(1,1) * abs(U_max)
```

```
b = 2×1
    2.0074
    6.1848
```

```
psi = angle(U_max) - pi/2
```

```
psi = 2×1
   -4.0185
   -2.8376
```

```
% norm gain check
IO_norm_gain = norm(b)/norm(a)
```

```
IO_norm_gain = 6.5024
```

Part f - Simulate System

```
%% Sim Setup
dt = 0.1;
tf = 5;
t = 0:dt:tf;

%% Input/Output Production
u = a .*  sin(omega_p * t + phi)
```

```
u = 2×51
   -0.5170   -0.4544   -0.2819   -0.0412    0.2095    0.4095    0.5104    0.4878 ···
    0.5273    0.7850    0.8528    0.7142    0.4027   -0.0062   -0.4136   -0.7209
```

```
y = b .* sin(omega_p * t + psi)
```

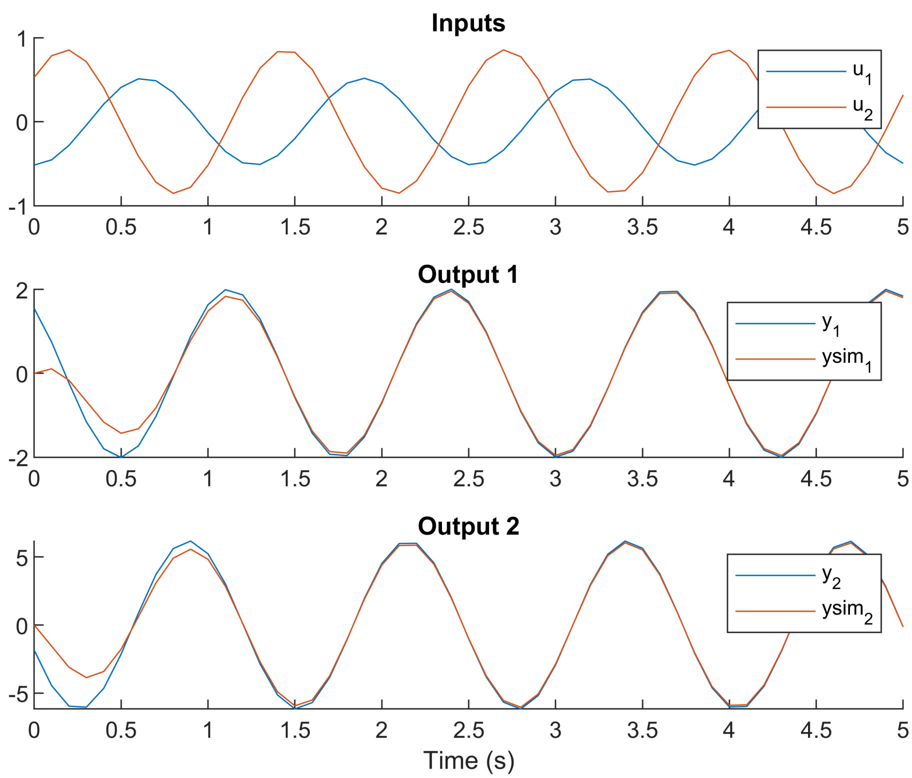
```
y = 2×51
    1.5432    0.7443   -0.2347   -1.1569   -1.7992   -2.0060   -1.7274   -1.0308 ···
   -1.8515   -4.4412   -5.9562   -6.0298   -4.6443   -2.1349    0.8912    3.7015
```

Simulat System

```
ysim = lsim(sys,u,t)';
```

Plot Sim

```
figure
% Plot Inputs
subplot(3,1,1)
hold on
plot(t,u(1,:),'DisplayName','u_1')
plot(t,u(2,:),'DisplayName','u_2')
legend()
title('Inputs')
% Plot Outputs
for idx = 1:2
    subplot(3,1,idx+1)
    hold on
    plot(t,y(idx,:),'DisplayName',['y_',num2str(idx)])
    plot(t,ysim(idx,:),'DisplayName',['ysim_',num2str(idx)])
    legend()
    title(['Output ',num2str(idx)])
end
xlabel('Time (s)')
```



It is evident that works.

3 Problem 3

Problem:

Let S and T denote the sensitivity and complementary sensitivity closed-loop transfer functions. Prove that

$$\|S\|_\infty \geq \|T\|_\infty - 1$$

Preliminaries

Definition 4. *Let P and C represent the plant and controller transfer functions respectively. Within a standard unity feedback system,*

1. *the* sensitivity closed-loop transfer function *is defined as:*

$$S = \frac{1}{1 + PC}$$

2. *the* complementary sensitivity closed-loop transfer function *is defined as:*

$$T = \frac{PC}{1 + PC}$$

Solution:

Theorem 5. *Let P and C represent the plant and controller transfer functions respectively. Within a standard unity feedback system,*

$$\|S\|_\infty \geq \|T\|_\infty - 1$$

Proof. From the definitions,

$$T - S = \frac{PC}{1 + PC} - \frac{1}{1 + PC} = \frac{-1 + PC}{1 + PC}$$

Applying the $\infty-$norm,

$$\|T - s\|_\infty = \left\| \frac{-1 + PC}{1 + PC} \right\|_i nfty$$

Since $-1 + PC < 1 + PC$,

$$\|T - S\|_\infty \leq 1$$

From the triangular inequality we have

$$\|T - S\|_\infty \geq \|T\|_\infty - \|S\|_\infty$$

And thus,

$$\boxed{1 \geq \|T - S\|_\infty \geq \|T\|_\infty - \|S\|_\infty \implies \|S\|_\infty \geq \|T\|_\infty - 1}$$

□

A MATLAB Code:

See attached. Additionally, all the code I write in this course can be found on my GitHub repository: https://github.com/jonaswagner2826/MECH6323