MECH 6323 - HW 2

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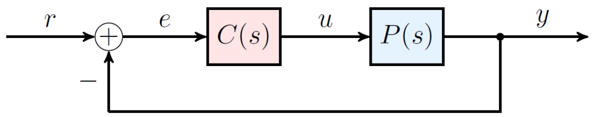
Contents

1 Problem 1

Consider the negative feedback interconnection:



1.1 (a)

Problem: If possible, give an example of P and C transfer functions such that $\frac{1}{1+PC}$ and $\frac{P}{1+PC}$ are stable, but $\frac{C}{1+PC}$ is not.

Solution:

$$P(s) = (s+1)(s-1)$$

$$C(s) = \frac{1}{(s-1)}$$

These then produces the two following stable transfer functions:

$$\frac{1}{1+PC} = \frac{1}{s+2}$$

$$\frac{P}{1+PC} = \frac{(s+1)(s-1)}{(s+2)}$$

However, then this transfer function is unstable:

$$\frac{C}{1+PC} = \frac{1}{(s-1)(s+2)}$$

1.2 (b)

Problem: If possible, give an example of P and C transfer functions such that $\frac{P}{1+PC}$ and $\frac{C}{1+PC}$ are stable, but $\frac{1}{1+PC}$ is not.

Solution: This is imposable to do for just a Transfer Function if assuming that pole-zero cancellations on the right-half plane are legitimate. If this is not the case, there are a myriad of cases in which this would be possible to construct a case where $1 + PC$ is unstable and this is due to cancellations that exist in the right-half plane, but this simple solution is not true under the previous assumption.

1.3 (c)

Problem: If possible, give an example of P and C transfer functions such that $\frac{1}{1+PC}$ and $\frac{C}{1+PC}$ are stable, but $\frac{P}{1+PC}$ is not.

Solution:

$$P(s) = \frac{1}{(s-1)}$$

$$C(s) = (s+1)(s-1)$$

These then produces the two following stable transfer functions:

$$\frac{1}{1+PC} = \frac{1}{s+2}$$

$$\frac{C}{1+PC} = \frac{(s+1)(s-1)}{(s+2)}$$

However, then this transfer function is unstable:

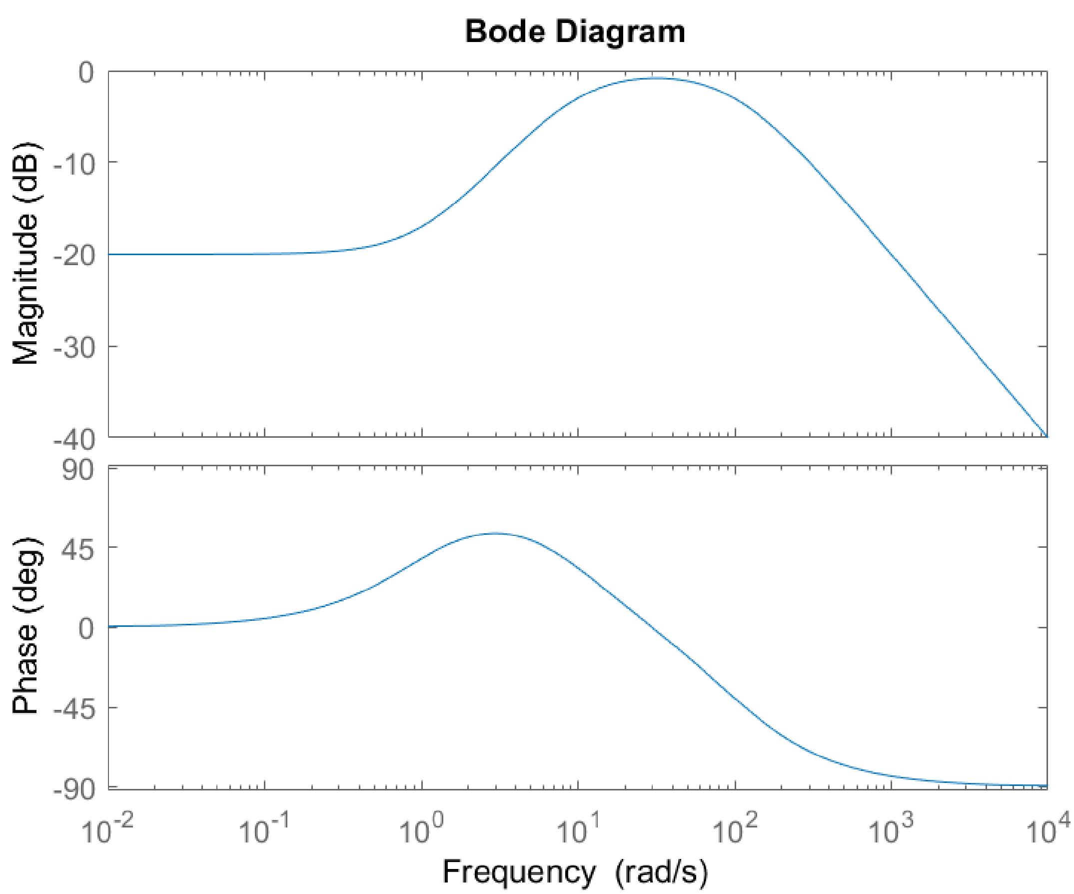
$$\frac{P}{1+PC} = \frac{1}{(s-1)(s+2)}$$

2 Problem 2

2.1 (a)

$$G = \frac{100s + 100}{s^2 + 110s + 1000} = \frac{100(s+1)}{(s+100)(s+10)}$$

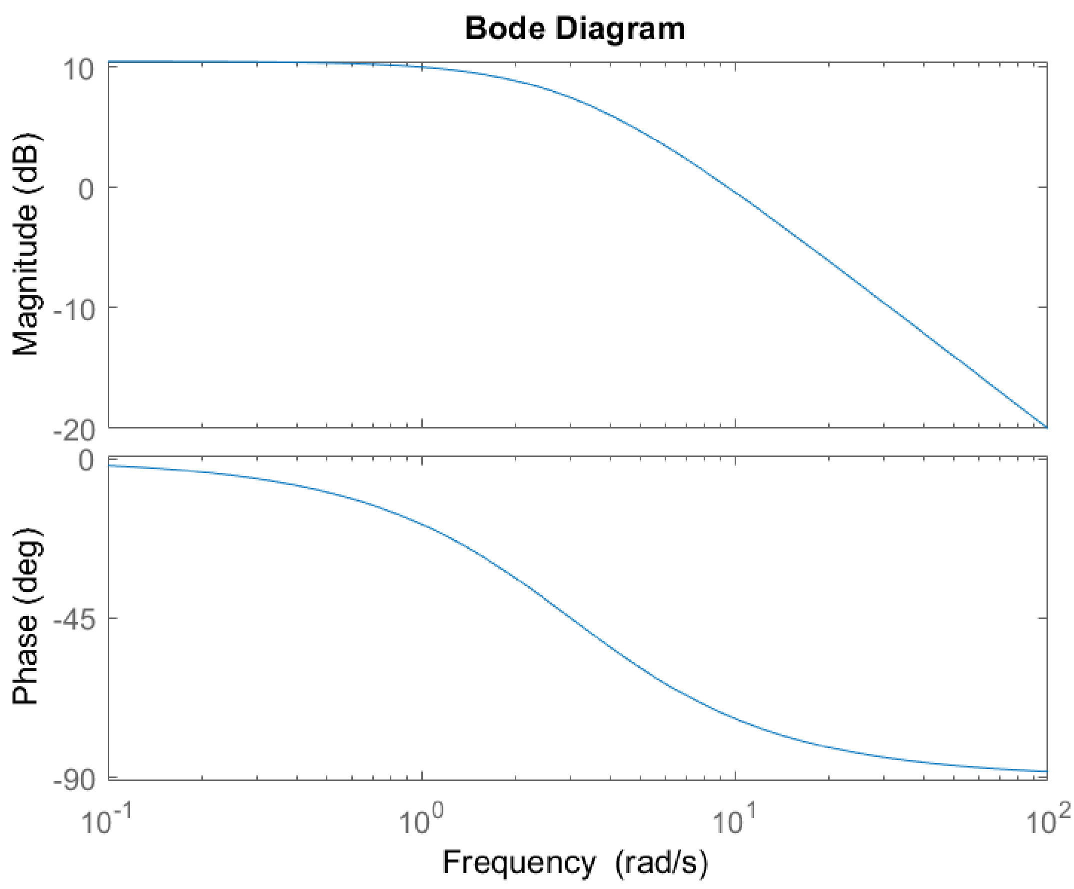
1. Zeros:
   (a) $z_1 = -1$
2. Poles:
   (a) $p_1 = -100$
   (b) $p_2 = -10$
3. Gain:
   (a) $K = 100$



2.2 (b)

$$G = \frac{10s}{s^2 + 3s} = \frac{10s}{s(s+3)}$$

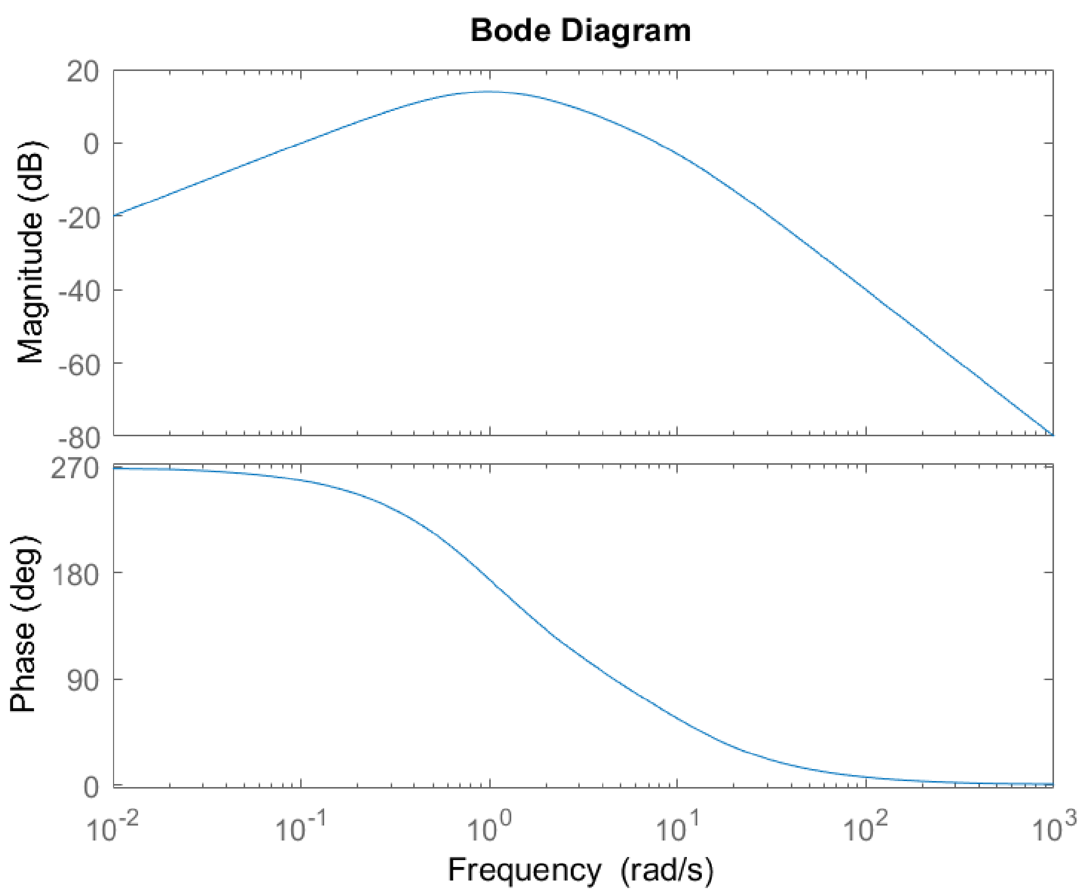
1. Zeros:
   (a) $z_1 = 0$
2. Poles:
   (a) $p_1 = 0$
   (b) $p_2 = -3$
3. Gain:
   (a) $K = 10$



2.3 (c)

$$G = \frac{-100s}{(s+1)^2(s+10)}$$

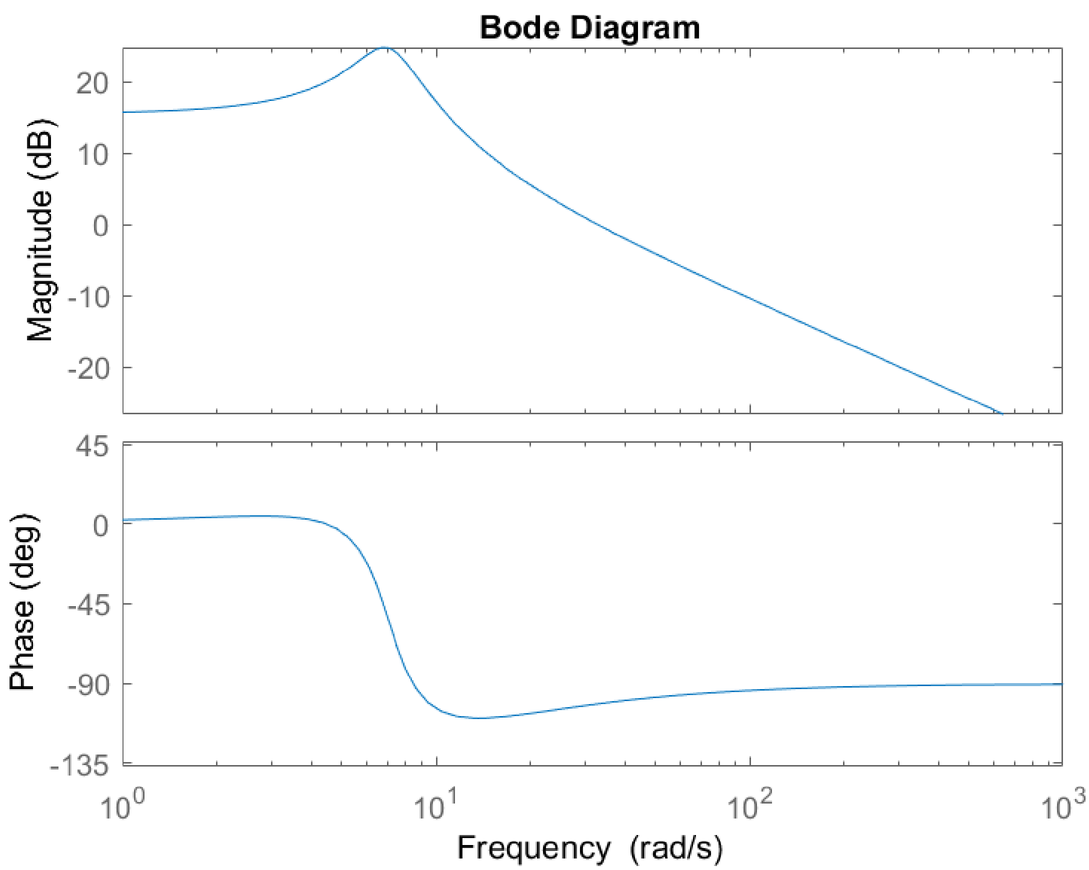
1. Zeros:
   (a) $z_1 = 0$
2. Poles:
   (a) $p_{1,2} = -1$
   (b) $p_3 = -10$
3. Gain:
   (a) $K = -100$



2.4 (d)

$$G = \frac{30(s+10)}{s^2+3s+50} = \frac{30(s+10)}{(s+1.5-j6.91)(s+1.5+j6.91)}$$

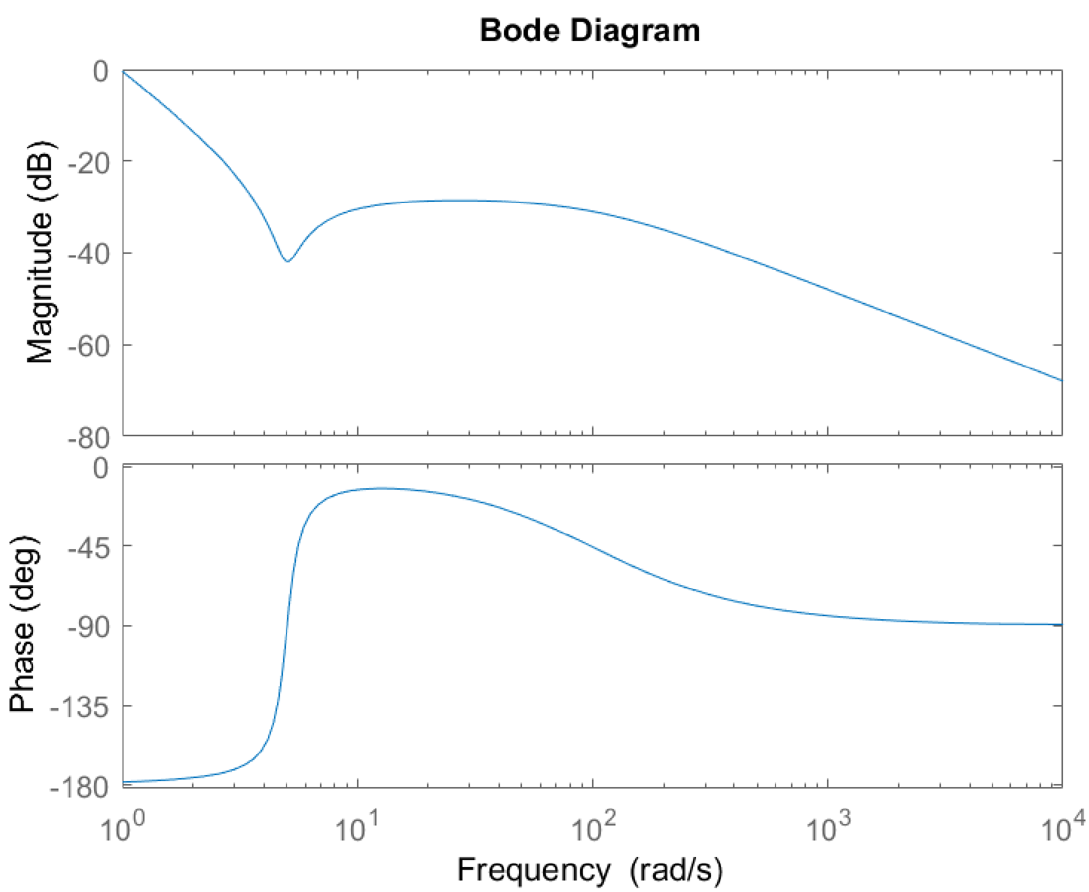
1. Zeros:
   (a) $z_1 = -10$
2. Poles:
   (a) $p_{1,2} = -1.5 \pm j6.91$
3. Gain:
   (a) $K = 30$



2.5 (e)

$$G = \frac{4(s^2+s+25)}{s^3+100s^2} = \frac{4(s+0.5+j4.9749)(s+0.5-j4.9749)}{s^2(s+100)}$$

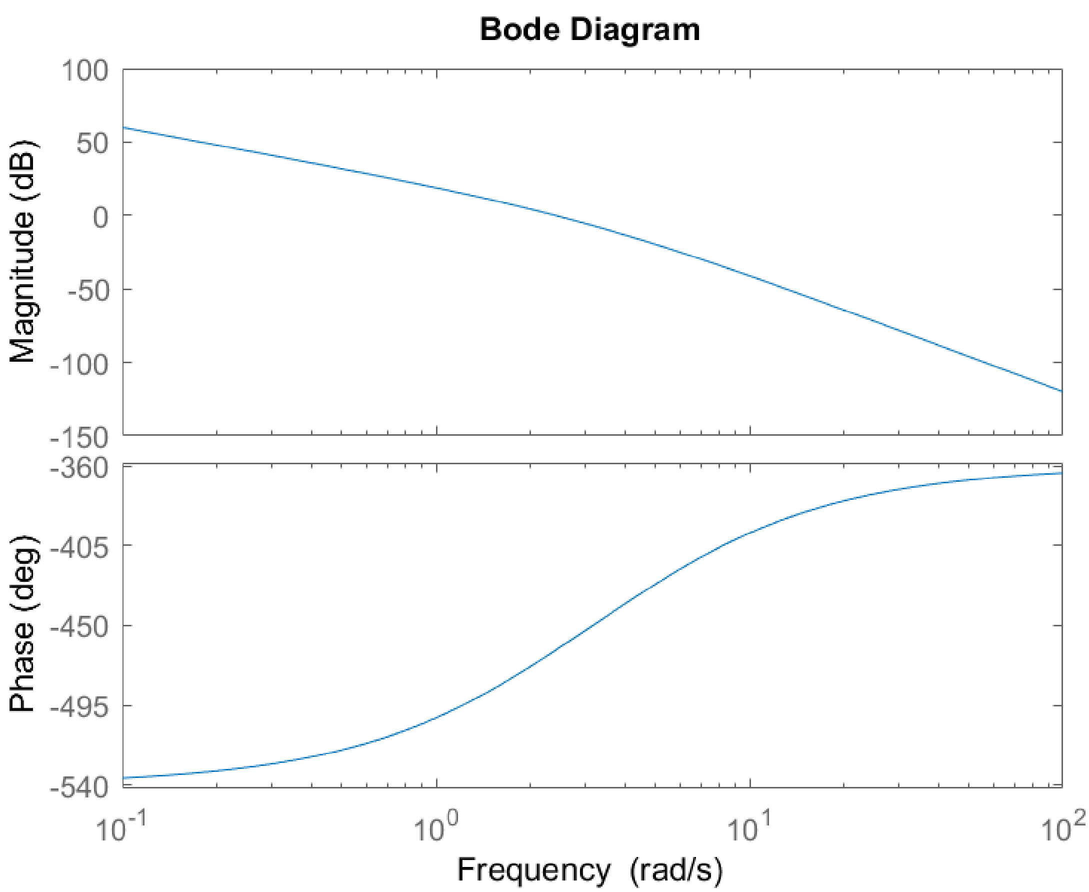
1. Zeros:
   (a) $z_{1,2} = -0.5 \pm j4.9749$
2. Poles:
   (a) $p_{1,2} = 0$
   (b) $p_3 = -100$
3. Gain:
   (a) $K = 4$



2.6 (f)

$$G = \frac{10}{s^2(1+0.2s)(1+0.5s)} = \frac{100}{s^2(s+5)(s+2)}$$

1. Zeros: NA
2. Poles:
   (a) $p_{1,2} = 0$
   (b) $p_3 = -5$
   (c) $p_4 = -2$
3. Gain:
   (a) $K = 100$



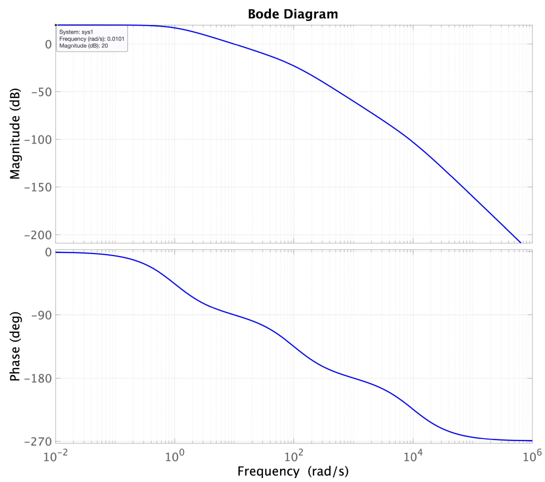
3 Problem 3

Problem: For each of the bode plots:

1. Determine the breakpoints and the transfer function.

2. Determine the gain cross-over frequency ω_c and the phase cross-over frequency ω_{180}.

3.1 Bode Plot 1:



3.1.1 Gain, Poles, and Zeros:

1. **Gain:** 20 db = 10

2. **Poles:**

   (a) $10^0 = 1$ rad/s

   (b) $10^2 = 100$ rad/s

   (c) $10^4 = 10,000$ rad/s

3. Zeros: (NA)

Transfer Function:

$$H(s) = \frac{10}{\left(1+\frac{s}{1}\right)\left(1+\frac{s}{100}\right)\left(1+\frac{s}{10000}\right)}$$

3.1.2 Cross-over Frequency:

1. $\omega_c = 10^1 = 10$ rad/s

2. $\omega_{180} = 10^3 = 100$ rad/s

3.2 Bode Plot 2:

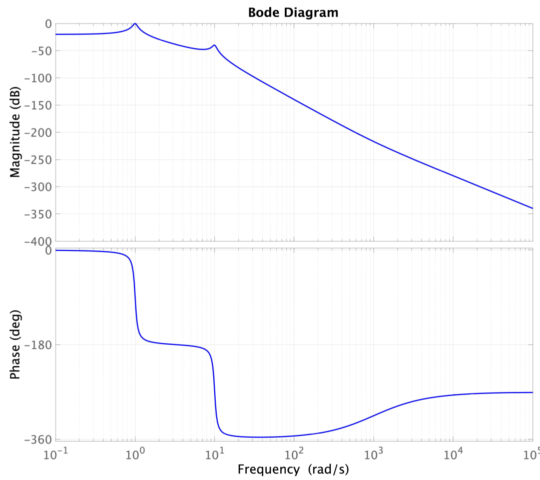


Figure 1: Bode Plot 2

3.2.1 Gain, Poles, and Zeros:

1. **Gain:** -20 db $= \frac{1}{10}$
2. **Poles:**
   (a) $10^0 = 1$ rad/s (complex)
   (b) $10^1 = 10$ rad/s (complex)
3. Zeros:
   (a) $10^3 = 1,000$ rad/s

3.2.2 Transfer Function:

$$H(s) = \frac{(1 + \frac{s}{1000})}{10\left(\frac{1}{1}\left(s^2 + 2(\frac{1}{10})(1)s + (1)^2\right)\right)\left(\frac{1}{10}\left(s^2 + 2(\frac{1}{10})(10)s + (10)^2\right)\right)} = \frac{(s + 1000)}{(s^2 + 0.2s + 1)(s^2 + 2s + 100)}$$

Assuming a Q-factor of around 10 to get the complex response.

3.2.3 Cross-over Frequency:

1. $\omega_c = 10^0 = 1$ rad/s
2. $\omega_{180} = 10^3 = 100$ rad/s

4 Problem 4

Consider the interconection of Problem 1 with the PI controller

$$C(s) = \frac{10(s+3)}{s}$$

and plant

$$P(s) = \frac{-0.5(s^2-2000)}{(s-3)(s^2+50s+1000)}$$

4.1 Is the feedback system stable? Why?

$$\frac{C(s)P(s)}{1+C(s)P(s)} = \frac{\frac{10(s+3)}{s}\frac{-0.5(s^2-2000)}{(s-3)(s^2+50s+1000)}}{1+\frac{10(s+3)}{s}\frac{-0.5(s^2-2000)}{(s-3)(s^2+50s+1000)}}$$
$$\approx \frac{-5s(s+3)(s-3)(s^2-2000)(s^2+50s+1000)}{s(s-3)(s^2+11.73s+73.9)(s^2+30.27s+406)(s^2+50s+1000)}$$

Yes and No. Internally it is not fully stable since it has a pole/zero pair at $s=3$; however, if we only care about TF after cancellations, then it is stable.

4.2 Find phase and cross-over frequencies.

Problem: Use the Bode plot of the open loop transfer function $L(s) = C(s)P(s)$ to find the phase cross-over frequencies ω_0 such that $L(j\omega_0) = 180\,\text{deg}$. Use this information to compute the gain margin(s) of the feedback system. Check your answers using the *allmargin* command in MATLAB.

Solution: As marked in the Bode Plot seen in Figure 4.2, the gain cross-over frequency is $\omega_c = 10$ resulting in a phase margin around 25 deg. Similarly, the phase cross-over occurs around $\omega_{180} = 4$ or $\omega_{180} = 25$, resulting in gain margins of around ± 10 dB or around $g_0 = 0.3$ and $g_0 = 3$ respectively.

Verification with *allmargin* resulted in similar and likely more precise and accurate results:

1. Gain Margin(s)
   (a) $g_0 = 0.3585$ at $\omega_{180} = 3.5966$
   (b) $g_0 = 2.6490$ at $\omega_{180} = 26.3797$
2. Phase Margin
   (a) $PM = 27.5718$ at $\omega_c = 10.2049$

4.3 Gain Margin Closed-loop poles

Problem: For each gain margin g_0 obtained in the previous part, construct the closed-loop using the perturbed loop transfer function $g_0L(s)$ and verify that the closed-loop has poles at $\pm\omega_0$.

Solution:

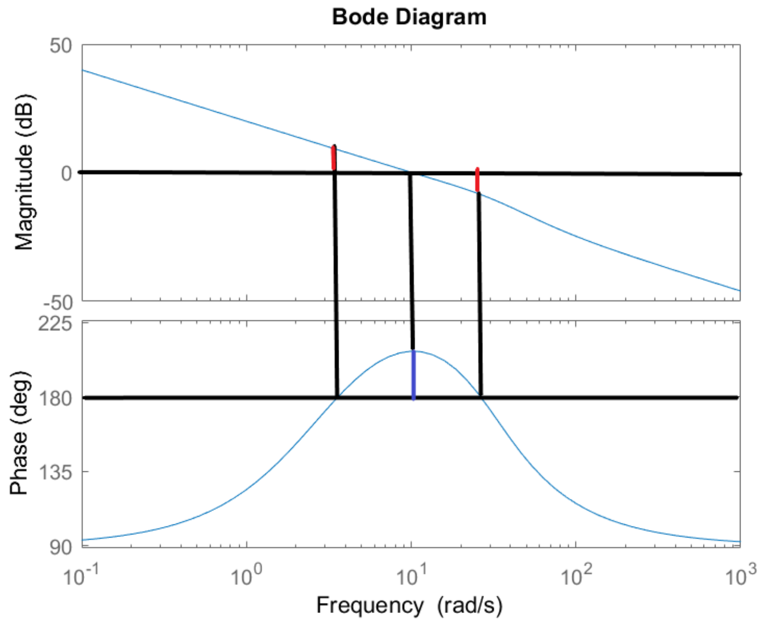


Figure 2: Open Loop Bode Plot of $L(S) = C(s)P(s)$

4.3.1 $g_0 = 0.3585$

Let $g_0 = 0.3585$,

$$g_0 L(s) = \frac{-1.7927(s+3)(s+44.72)(s-44.72)}{s(s-3)(s^2+50s+1000)}$$

and the closed-loop becomes

$$H = \frac{-1.7927(s-44.72)(s+44.72)(s+3)}{(s^2+0.0008164s+12.93)(s^2+45.21s+831.7)}$$

This has complex poles located at $-0.0004 \pm j3.596$, which is essentially roots at $\pm j\omega_{180}$.

4.3.2 $g_0 = 2.6490$

Let $g_0 = 2.6490$,

$$g_0 L(s) = \frac{-13.245(s+3)(s+44.72)(s-44.72)}{s(s-3)(s^2+50s+1000)}$$

and the closed-loop becomes

$$H = \frac{-13.245(s+3)(s+44.72)(s-44.72)}{(s+29.94)(s+3.814)(s^2+0.003402s+696)}$$

This has complex poles located at $-0.0017 \pm j26.3818$, which is essentially roots at $\pm j\omega_{180}$.

4.4 $\|S-T\|_\infty$

Problem: Compute $\|S-T\|_\infty$ and the corresponding frequency ω_p where the peak gain of $S-T$ is achieved.

Solution:

Sensitivity TF:

$$S(s) = \frac{1}{1+PC} = \frac{s(s-3)(s^2+50s+1000)}{(s^2+11.73s+73.9)(s^2+30.27s+406)}$$

Complementary Sensitivity TF:

$$T(s) = \frac{PC}{1+PC} = \frac{-5s(s-44.72)(s+44.72)(s+3)(s-3)(s^2+50s+1000)}{s(s-3)(s^2+11.73s+73.9)(s^2+30.27s+406)(s^2+50s+1000)}$$

$S(s) - T(s)$:

$$S(s)-T(s) = \frac{s(s-10.45)(s-3)(s+2.059)(s^2+11.73s+73.9)(s^2+30.27s+406)(s^2+50s+1000)(s^2+60.4s+1394)}{s(s-3)(s^2+11.73s+73.9)^2(s^2+30.27s+406)^2(s^2+50s+1000)}$$

$$= \frac{(s-10.45)(s+2.059)(s^2+60.4s+1394)}{(s^2+11.73s+73.9)(s^2+30.27s+406)}$$

Results:

$$\|S-T\|_\infty = 4.0763$$

$$\omega_p = 10.0798$$

4.5 Symmetric Disk Margin

Problem: What is the symmetric disk margin m for this plant and controller? Verify your answer using *dm = diskmargin(P*C)*. Note that the *diskmargin* command uses the convention $m = dm.DiskMargin\ /\ 2$.

Solution: By the Symmetric Disk Margin theorem, the disk margin defined for

$$\alpha \in \text{Disk}\left(\frac{1-m}{1+m}, \frac{1+m}{1-m}\right)$$

is the region that $C(s)$ stabilizes $\alpha P(s)$ for $m < 1$ satisfying

$$\|S-T\|_\infty \leq \frac{1}{m}$$

Therefore,

$$\overline{m}_{st} = \frac{1}{\|S-T\|_\infty} = \frac{1}{4.0763} = 0.453$$

4.6 α on Disk boundary

Problem: Construct an α on the boundary of Disk

$$\text{Disk}\left(\frac{1-m}{1+m}, \frac{1+m}{1-m}\right)$$

such that the perturbed closed-loop

$$S_\alpha = \frac{1}{1+\alpha L(s)}$$

has a pole at $j\omega_p$. Verify your construction by forming S_α and demonstrating that it has a pole at $j\omega_p$. Hint: Assume $\|S-T\|_\infty = \frac{1}{m}$ at frequency ω_p. Then there exists a complex number $S(j\omega_p) - T(j\omega_p) = \frac{1}{z}$ where $|z| = m$. Algebraically show that

$$\alpha = \frac{1+z}{1-z}$$

satisfies $1 + \alpha L(j\omega_p) = 0$ and this α is in the symmetric disk defined by m.

Solution: α can be constructed by first finding $z_0 = \frac{1}{S(j\omega_p) - T(j\omega_p)}$, calculating $z = m * \frac{z_0}{|z_0|}$, and then finding $\alpha = \frac{1+z}{1-z}$.

As demonstrated in MATLAB, this results in $\alpha = 0.8760 - j0.4572$. This is then verified as

$$\begin{aligned} 1 + \alpha L(j\omega_p) &\approx (0.8760 - j0.4572) \frac{-5(j10.0798 + 3)(j10.0798 + 44.72)(j10.0798 - 44.72)}{(j10.0798)(j10.0798 - 3)((j10.0798^2 + 50(j10.0798) + 1000)} \\ &= -2.7529e - 08 - j1.2029e - 06 \\ &\approx 0 \end{aligned}$$

A MATLAB Code:

See attached. Additionally, all the code I write in this course can be found on my GitHub repository:
https://github.com/jonaswagner2826/MECH6323

MECH 6323 - Homework 2

Author: Jonas Wanger

Date: 2022-02-13

Problem 2

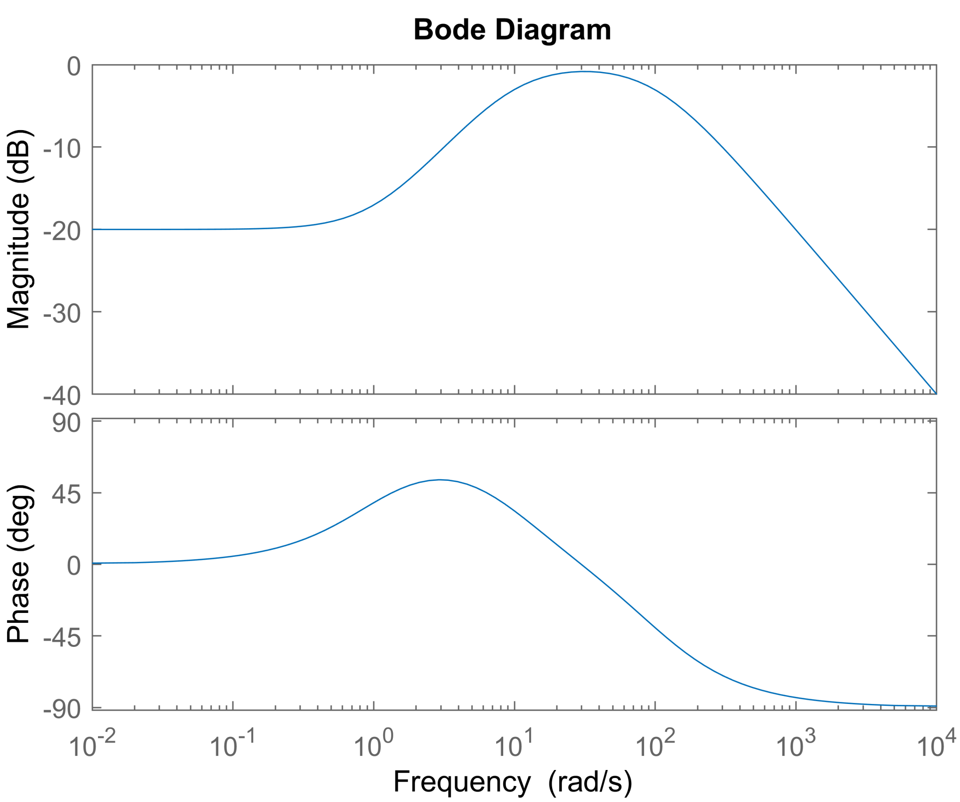
```
% a
G = zpk(tf([100 100],[1 110 1000]))
```

```
G =

    100 (s+1)
  --------------
  (s+100) (s+10)

Continuous-time zero/pole/gain model.
```

```
bode(G)
```



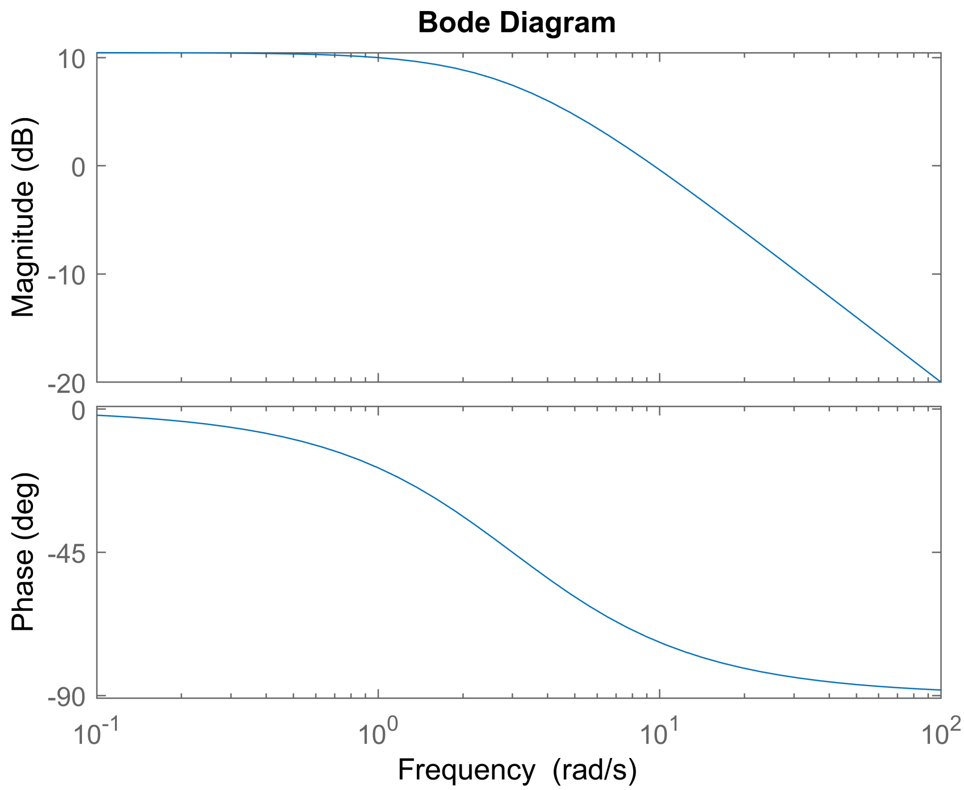

```
% b
G = zpk(tf([10 0],[1 3 0]))
```

```
G =

   10 s
  -------
  s (s+3)
```

```
Continuous-time zero/pole/gain model.
```

```
bode(G)
```



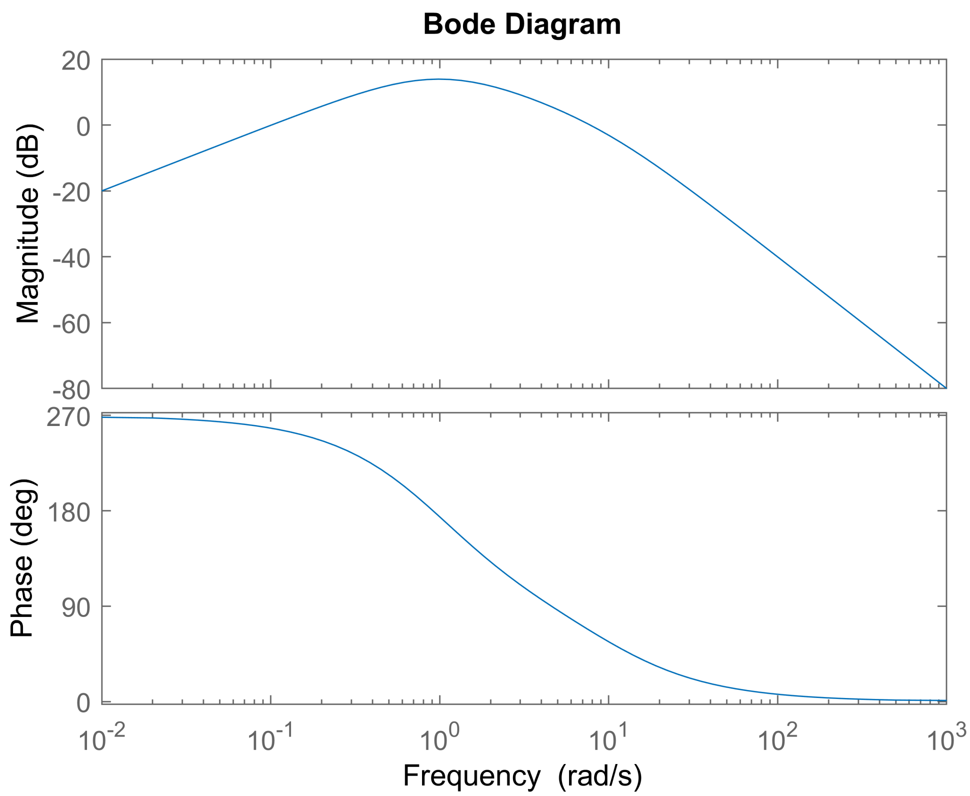

```
% c
G = zpk([0],[-1 -1 -10],-100)
```

```
G =

      -100 s
  --------------
  (s+1)^2 (s+10)

Continuous-time zero/pole/gain model.
```

```
bode(G)
```



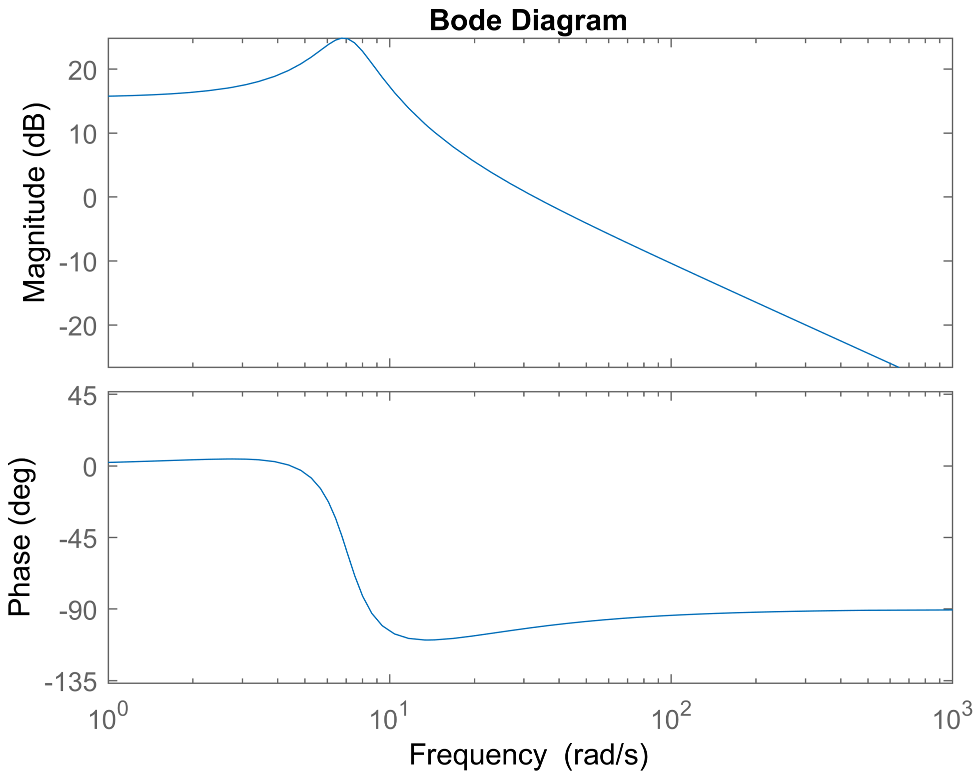

```
% d
G = zpk(tf(30*[1 10],[1 3 50]))
```

```
G =

      30 (s+10)
   ---------------
   (s^2 + 3s + 50)

Continuous-time zero/pole/gain model.
```

```
bode(G)
```



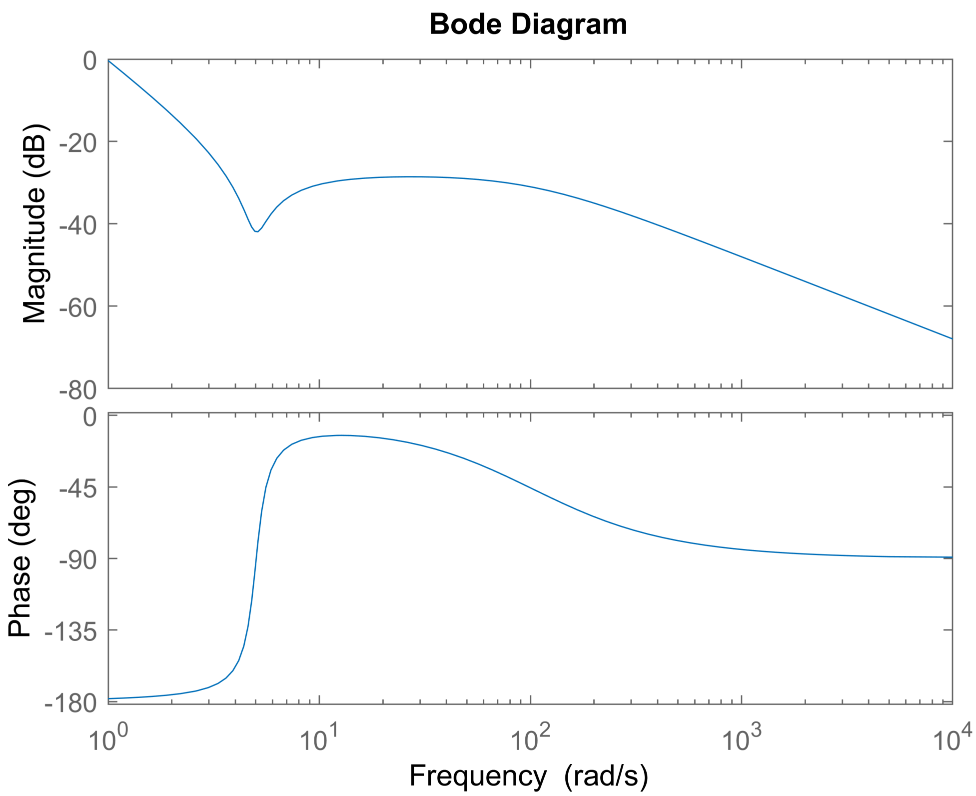

```
% e
G = zpk(tf(4*[1 1 25],[1 100 0 0]))
```

```
G =

  4 (s^2 + s + 25)
  ----------------
    s^2 (s+100)

Continuous-time zero/pole/gain model.
```

```
bode(G)
```



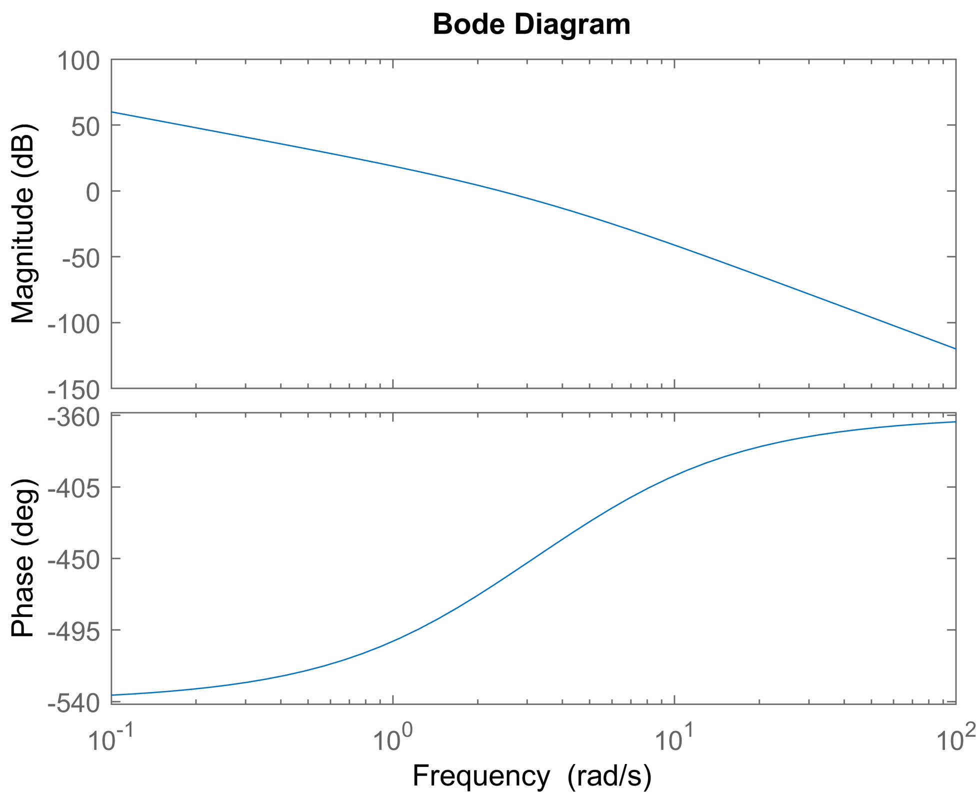

```
% f
G = zpk([],[0 0 -5 -2],100)
```

```
G =

        100
  ---------------
  s^2 (s-5) (s-2)

Continuous-time zero/pole/gain model.
```

```
bode(G)
```



Problem 3

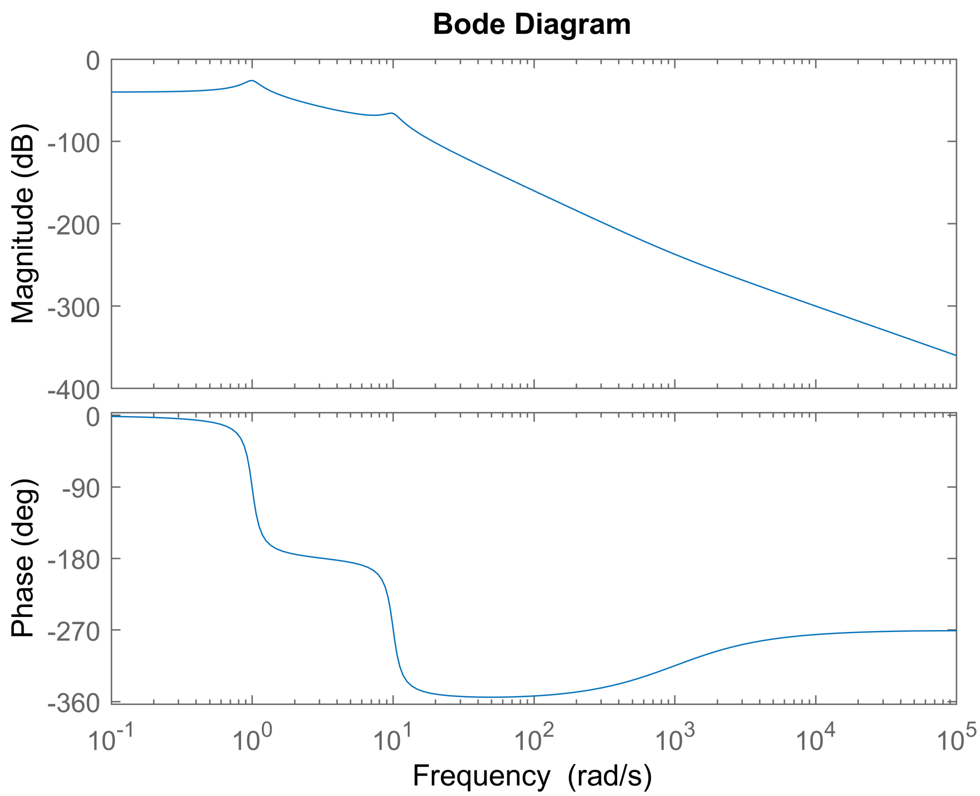
```
% 2nd TF
H1 = zpk(-1000,[],1/10);
H2 = tf([1/1], [1 2*(1/10)*1 1^2]);
H3 = tf([1/(10^2)], [1 2*(1/10)*10 (10)^2]);
H = H1 * H2 * H3
```

```
H =

            0.001 (s+1000)
  --------------------------------
  (s^2 + 0.2s + 1) (s^2 + 2s + 100)

Continuous-time zero/pole/gain model.
```

```
bode(H)
```



Problem 4

```
C = zpk(-3,0,10)
```

```
C =
 
  10 (s+3)
  --------
     s
 
Continuous-time zero/pole/gain model.
```

```
P1 = zpk([],[3],-0.5);
P2 = tf([1 0 -2000],[1 50 1000]);
P = P1 * P2
```

```
P =
 
  -0.5 (s-44.72) (s+44.72)
  ------------------------
  (s-3) (s^2 + 50s + 1000)
 
Continuous-time zero/pole/gain model.
```

```
% Part a
H = feedback(C*P,1)
```

```
H =
```

```
        -5 (s+3) (s+44.72) (s-44.72)
  -------------------------------------------
  (s^2 + 11.73s + 73.9) (s^2 + 30.27s + 406)

Continuous-time zero/pole/gain model.
```

```
isstable(H)
```

```
ans = logical
   1
```

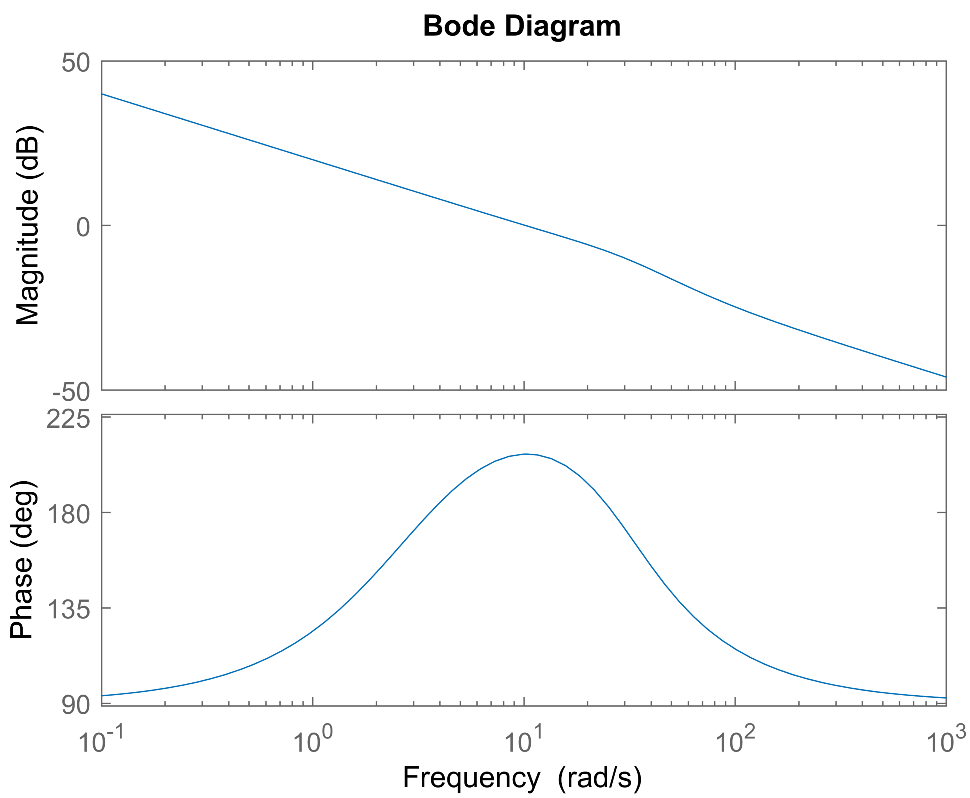
```
% Part b
L = C*P
```

```
L =

  -5 (s+3) (s+44.72) (s-44.72)
  ----------------------------
   s (s-3) (s^2 + 50s + 1000)

Continuous-time zero/pole/gain model.
```

```
bode(L)
```




```
margins = allmargin(L)
```

```
margins = struct with fields:
     GainMargin: [0.3585 2.6490]
    GMFrequency: [3.5966 26.3797]
    PhaseMargin: 27.5718
    PMFrequency: 10.2049
    DelayMargin: 0.0472
```

```
    DMFrequency: 10.2049
         Stable: 1
```

```
% Part c
g1 = margins.GainMargin(1)
```

```
g1 = 0.3585
```

```
L1 = g1 * L
```

```
L1 =
 
  -1.7927 (s+3) (s+44.72) (s-44.72)
  ---------------------------------
     s (s-3) (s^2 + 50s + 1000)
 
Continuous-time zero/pole/gain model.
```

```
H1 = feedback(L1,1)
```

```
H1 =
 
          -1.7927 (s-44.72) (s+44.72) (s+3)
  -------------------------------------------------
  (s^2 + 0.0008164s + 12.93) (s^2 + 45.21s + 831.7)
 
Continuous-time zero/pole/gain model.
```

```
poles1 = roots([1 0.0008164 12.93])
```

```
poles1 = 2×1 complex
  -0.0004 + 3.5958i
  -0.0004 - 3.5958i
```

```
g2 = margins.GainMargin(2)
```

```
g2 = 2.6490
```

```
L2 = g2 * L
```

```
L2 =
 
  -13.245 (s+3) (s+44.72) (s-44.72)
  ---------------------------------
     s (s-3) (s^2 + 50s + 1000)
 
Continuous-time zero/pole/gain model.
```

```
H2 = feedback(L2,1)
```

```
H2 =
 
       -13.245 (s+3) (s+44.72) (s-44.72)
  -------------------------------------------
  (s+29.94) (s+3.814) (s^2 + 0.003402s + 696)
 
Continuous-time zero/pole/gain model.
```

```
poles2 = roots([1 0.003402 696])
```

```
poles2 = 2×1 complex
  -0.0017 +26.3818i
  -0.0017 -26.3818i
```

```
% Part d
S = 1/(1+P*C)
```

```
S =

         s (s-3) (s^2 + 50s + 1000)
  ------------------------------------------
  (s^2 + 11.73s + 73.9) (s^2 + 30.27s + 406)

Continuous-time zero/pole/gain model.
```

```
T = (P*C)/(1+P*C)
```

```
T =

        -5 s (s-44.72) (s+44.72) (s+3) (s-3) (s^2 + 50s + 1000)
  --------------------------------------------------------------------
  s (s-3) (s^2 + 11.73s + 73.9) (s^2 + 30.27s + 406) (s^2 + 50s + 1000)

Continuous-time zero/pole/gain model.
```

```
S_minus_T = minreal(S-T)
```

```
S_minus_T =

   (s-10.45) (s+2.059) (s^2 + 60.4s + 1394)
  ------------------------------------------
  (s^2 + 11.73s + 73.9) (s^2 + 30.27s + 406)

Continuous-time zero/pole/gain model.
```

```
[NINF,w_p] = hinfnorm(S_minus_T)
```

```
NINF = 4.0763
w_p = 10.0798
```

```
% Part e
m = 1/NINF
```

```
m = 0.2453
```

```
dm = diskmargin(P*C)
```

```
dm = struct with fields:
           GainMargin: [0.6060 1.6501]
          PhaseMargin: [-27.5672 27.5672]
           DiskMargin: 0.4906
           LowerBound: 0.4906
           UpperBound: 0.4906
            Frequency: 10.0623
    WorstPerturbation: [1×1 ss]
```

```
m = dm.DiskMargin/2
```

```
m = 0.2453
```

```
% Part f
z = 1/evalfr(S-T,j*w_p);
z = (z / abs(z)) * abs(m)
```

```
z = -0.0063 - 0.2452i
```

```
alpha = (1+z)/(1-z)
```

```
alpha = 0.8760 - 0.4572i
```

```
alpha_L_plus_1 = alpha*evalfr(L,j*w_p) + 1
```

```
alpha_L_plus_1 = -2.7529e-08 - 1.2029e-06i
```