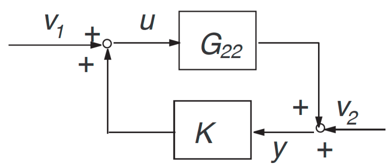
Due Sunday 03/27/2022 (10:00 pm)

Final Project in lieu of Final Exam Please start thinking about your final project. Your project should use the techniques and tools covered in this course on a particular physical system. This may or may not be related to your research. In the project, you can use analysis tools to assess the robustness of a given feedback system. Alternatively, you can use optimal control techniques (LQR, $\mathcal{H}_1$, $\mathcal{H}_2$) to synthesize a controller. For now, please specify the system that you will consider. Give a brief 1-page summary describing the physical system that you will study. This should include a description of the (linearized) dynamics as well as some initial simulation results (e.g., open-loop step response if the system is stable). You may work individually or in a group of two. Just make sure you mention the names of both group members on the summary. **Deadline: Please email me your 1-page project summaries before the beginning of class on April 6th.**

1. Consider the feedback configuration in the figure below.



Prove that

$$\begin{bmatrix} I & -K \\ -G_{22} & I \end{bmatrix}^{-1} = \underbrace{\begin{bmatrix} (I - KG_{22})^{-1} & (I - KG_{22})^{-1}K \\ (I - G_{22}K)^{-1}G_{22} & (I - G_{22}K)^{-1} \end{bmatrix}}_{H(G_{22},K)}.$$

2. Consider the block diagram in the previous exercise. Suppose G_{22} and K have minimal state space realizations $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ and $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$. Let

$$T = \left[\begin{array}{c|c} \overbrace{\begin{matrix} A & 0 \\ 0 & A_K \end{matrix}}^{A_1} & \overbrace{\begin{matrix} B_2 & 0 \\ 0 & B_K \end{matrix}}^{B} \\ \hline \underbrace{\begin{matrix} 0 & -C_K \\ -C_2 & 0 \end{matrix}}_{-C} & \underbrace{\begin{matrix} I & -D_K \\ -D_{22} & I \end{matrix}}_{D} \end{array}\right].$$

Thus,

$$T^{-1} = \left[\begin{array}{c|c} A_1 + BD^{-1}C & BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right] =: \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$$

where

$$\begin{aligned}\bar{D} \,=\, D^{-1} &= \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} I + (I - D_{22}D_K)^{-1}D_{22} & D_K(I - D_{22}D_K)^{-1} \\ (I - D_{22}D_K)^{-1}D_{22} & (I - D_{22}D_K)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} (I - D_{22}D_K)^{-1}D_{22} & D_K(I - D_{22}D_K)^{-1} \\ (I - D_{22}D_K)^{-1}D_{22} & (I - D_{22}D_K)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} D_K \\ I \end{bmatrix} (I \,-\, D_{22}D_K)^{-1} \begin{bmatrix} D_{22} & I \end{bmatrix}.\end{aligned}$$

Thus,

$$\bar{A} \,=\, A_1 \,+\, BD^{-1}C \,=\, \begin{bmatrix} A & B_2C_K \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2D_K \\ B_K \end{bmatrix} (I \,-\, D_{22}D_K)^{-1} \begin{bmatrix} C_2 & D_{22}C_K \end{bmatrix}.$$

Prove that the following are equivalent:

(a) $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is stabilizable and detectable;

(b) (A, B_2, C_2, D_{22}) and (A_K, B_K, C_K, D_K) are stabilizable and detectable.

3. For the feedback configuration in the first exercise, prove that:

   1. If K is stable then the closed loop interconnection is stable if and only if $G_{22}\,(I \,-\, KG_{22})^{-1}$ is stable.
   2. If G_{22} is stable then the closed loop interconnection is stable if and only if $K\,(I \,-\, G_{22}K)^{-1}$ is stable.

4. Consider the standard negative feedback loop with the nominal plant dynamics $P(s) \,=\, \frac{1}{s+1}$ and controller $K(s) = 20$. Assume the "true" dynamics lie within the following multiplicative uncertainty set:

$$\mathcal{M} \,:=\, \left\{ \hat{P} \,=\, P\,(1 \,+\, W_u\Delta) : \|\Delta\|_\infty \,<\, 1 \text{ and } \Delta \text{ stable} \right\}.$$

Assume the uncertainty weight is $W_u(s) = \frac{2s+1}{s+10}$.

(a) Provide an interpretation for the uncertainty described by the weight W_u.

(b) Is the nominal feedback system stable? What are the gain and phase margins of the nominal loop $L = PK$?

(c) The robust stability condition for this type of multiplicative uncertainty is stated as: K stabilizes all $\hat{P} \in \mathcal{M}$ if and only if $\|W_uT\|_\infty \leq 1$. Does K robustly stabilize all models in $\mathcal{M}$ based on this condition?

(d) We can construct the uncertainty set $\mathcal{M}$ in MATLAB using the following commands:

```
>> Delta = ultidyn('Delta', [1  1]);
>> Phat = P * (1 + Wu * Delta);
>> Lhat = Phat * K;
>> That = feedback(Lhat, 1);
```

The ultidyn command constructs an uncertain, LTI transfer function object that satisfies $\|\Delta\|_\infty < 1$. The function inputs are the name and input/output size of the object. Construct the uncertain model Phat using the commands above. Generate a Bode magnitude plot with 10 samples drawn from the uncertainty set and draw the nominal response P on the same plot. Note: The command bodemag(Phat) will automatically generate 10 samples and draw their plots.

(e) Finally, we can perform the robustness test using the following command:

$$>> [\text{stabmarg}, \text{destabunc}, \text{report}] = \text{robstab}(\text{That})$$

Refer to the function documentation for robstab for a short description of the input/output arguments. Does the result obtained with robstab agree with your conclusions in part (c)?