MECH 6323 - Homework 5

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```
close all
clear
% clc
```

Problem 1

$$M^{-1}M = I$$

$$\begin{bmatrix} I & -K \\ -G_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} I & -K \\ -G_{22} & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} A + (B)(-G_{22}) & (A)(-K) + (B)(I) \\ (C)(I) + (D)(-G_{22}) & (C)(-K) + (D)(I) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} A - BG_{22} & -AK + B \\ C - DG_{22} & -CK + D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$B = AK$$

$$C = DG_{22}$$

$$A - AKG_{22} = I$$

$$D - DG_{22}K = I$$

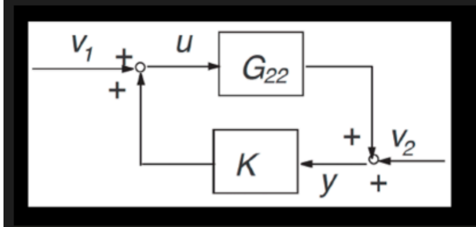
$$A(I - KG_{22}) = I$$

$$D(I - G_{22}K) = I$$

$$A = (I - KG_{22})^{-1}$$

$$D = (I - G_{22}K)^{-1}$$

$$B = (I - KG_{22})^{-1}K$$



$$u = V_1 + Ky$$
$$y = V_2 + G_{22}\, u$$

$$u = V_1 + K(V_2 + G_{22}\, u)$$
$$y = V_2 + G_{22}(V_1 + Ky)$$

$$u = KG_{22}\, u + V_1 + KV_2$$
$$y = G_{22}Ky + G_{22}V_1 + V_2$$

$$(I - KG_{22})\, u = V_1 + KV_2$$
$$(I - G_{22}K)\, y = G_{22}V_1 + V_2$$

$$u = (I - KG_{22})^{-1} \begin{bmatrix} 1 & K \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$
$$y = (I - G_{22}K)^{-1} \begin{bmatrix} G_{22} & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\begin{bmatrix} u \\ \cdots \end{bmatrix} = \begin{bmatrix} (I - KG_{22})^{-1} & (I - KG_{22})^{-1} K \\ \cdots & \cdots \end{bmatrix} \begin{bmatrix} V_1 \\ \cdots \end{bmatrix}$$

Problem 2

$H(G_{22}, K)$

(a) $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is stabilizable and detectable;

$\Updownarrow$

G_{22} $\qquad$ K

(b) (A, B_2, C_2, D_{22}) and (A_K, B_K, C_K, D_K) are stabilizable and detectable.

$G_{22} \} - \lambda_i^G$

$K \} - \lambda_i^K$

$H(G_{22}, K) \} - \lambda_i^H = \lambda_i^G \cup \lambda_i^K$

G_{22} and K stabilizable + detectable

$\Updownarrow$

$\operatorname{rank}\begin{bmatrix}(\lambda_i^G I - A) & B_2\end{bmatrix} = n \qquad \operatorname{rank}\begin{bmatrix}(\lambda_i^G I - A) \\ C_2\end{bmatrix} = n \qquad \forall_i : \lambda_i^G \geq 0$

$\operatorname{rank}\begin{bmatrix}(\lambda_i^K I - A_K) & B_K\end{bmatrix} = n \qquad \operatorname{rank}\begin{bmatrix}(\lambda_i^K I - A_K) \\ C_K\end{bmatrix} = n \qquad \forall_i : \lambda_i^K \geq 0$

$$\operatorname{rank}\left[(\lambda_i I - \tilde{A}) \quad \tilde{B}\right] = n$$

$$= \operatorname{rank}\left[(\lambda_i I - (A_1 + BD^{-1}C)) \quad BD^{-1}\right]$$

$$= \operatorname{rank}\left[\left(\lambda_i I - \left(\begin{bmatrix} A & B_2C_K \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2D_K \\ B_K \end{bmatrix}(I - D_{22}D_K)^{-1}\begin{bmatrix} C_2 & D_{22}C_K \end{bmatrix}\right)\right) \quad \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix}\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} D_K \\ I \end{bmatrix}(I - D_{22}D_K)^{-1}\begin{bmatrix} D_{22} & I \end{bmatrix}\right)\right]$$

$\forall_i: \lambda_i^G \geq 0$, this is of rank n since the G_{22} matrices (A, B_2) create n independent rows by themselves

$\forall_i: \lambda_i^K \geq 0$, this is of rank n since the K matrices (A_K, B_K) create n independent rows by themselves

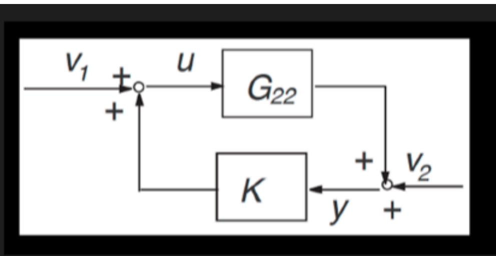
$$\operatorname{rank}\begin{bmatrix} (\lambda_i^K I - \tilde{A}) \\ \tilde{C} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \lambda_i I - (A_1 + BD^{-1}C) \\ D^{-1}C \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} \left(\lambda_i I - \left(\begin{bmatrix} A & B_2C_K \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2D_K \\ B_K \end{bmatrix}(I - D_{22}D_K)^{-1}\begin{bmatrix} C_2 & D_{22}C_K \end{bmatrix}\right)\right) \\ \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} D_K \\ I \end{bmatrix}(I - D_{22}D_K)^{-1}\begin{bmatrix} D_{22} & I \end{bmatrix}\right)\begin{bmatrix} 0 & -C_K \\ -C_2 & 0 \end{bmatrix} \end{bmatrix}$$

$\forall_i: \lambda_i^G \geq 0$, this is of rank n since the G_{22} matrices (A, C_2) create n independent rows by themselves

$\forall_i: \lambda_i^K \geq 0$, this is of rank n since the K matrices (A_K, C_K) create n independent rows by themselves

Problem 3



$$u = v_1 + Ky$$
$$y = v_2 + G_{22}u$$

$$\begin{bmatrix} I & -K \\ -G_{22} & I \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$H(G_{22}, K) = \begin{bmatrix} I & -K \\ -G_{22} & I \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (I - KG_{22})^{-1} & (I - KG_{22})^{-1}K \\ (I - G_{22}K)^{-1}G_{22} & (I - G_{22}K)^{-1} \end{bmatrix}$$

- Closed-loop stability means that all elements of $H(G_{22}, K)$ are stable.
- Matrix M is stable iff
  $$Re\{\lambda_i\} < 0 \quad \forall_{i=1,\dots,n}$$
- Matrix multiplicity maintains stability of eigenvalues if all eigenvalues are stable

(i) If K is stable then the closed loop interconnection is stable if and only if $G_{22}(I - KG_{22})^{-1}$ is stable.

i) K stable $\Rightarrow$

H stable $\Leftrightarrow G_{22}(I - KG_{22})^{-1}$

($\Rightarrow$)

K stable $\wedge$ $G_{22}(I - KG_{22})^{-1}$ stable

$\Downarrow$

$(I - KG_{22})^{-1}$ stable, G_{22} stable

$\Rightarrow (I - KG_{22})^{-1}K$ stable

$(I - KG_{22})^{-1}$ stable $\wedge$ K stable $\Rightarrow (I - G_{22}K)^{-1}$ stable

$\Rightarrow (I - G_{22}K)^{-1}G_{22}$ stable

$\therefore H$ stable

($\Leftarrow$)

H stable $\Rightarrow (I - G_{22}K)^{-1}G_{22}$ stable

K stable $\wedge$ $(I - G_{22}K)^{-1}G_{22}$ stable

$\Downarrow$

$G_{22}(I - KG_{22})^{-1}$ stable

$\blacksquare$

(ii) If G_{22} is stable then the closed loop interconnection is stable if and only if $K(I - G_{22}K)^{-1}$ is stable.

2) G_{22} stable $\Rightarrow$

$$H \text{ stable} \Leftrightarrow K(I - G_{22}K)^{-1}$$

$(\Rightarrow)$

G_{22} stable $\wedge$ $K(I - G_{22}K)^{-1}$ stable

$\Downarrow$

$(I - G_{22}K)^{-1}$ stable, K stable,

$(I - G_{22}K)^{-1}$ stable $\Rightarrow$ $(I - KG_{22})^{-1}$ stable

$\wedge\, G_{22} \wedge K$ stable

$\Rightarrow (I - KG_{22})^{-1}K$ stable

$\Rightarrow (I - G_{22}K)^{-1}G_{22}$ stable

$\therefore H$ stable

$(\Leftarrow)$

H stable $\Rightarrow$ $(I - KG_{22})^{-1}K$ stable

G_{22} stable $\wedge$ $(I - KG_{22})^{-1}G_{22}$ stable

$\Downarrow$

$G_{22}(I - G_{22}K)^{-1}$ stable

$\blacksquare$

Problem 4

Consider the standard negative feedback loop with the nominal plant dynamics $P(s) = \dfrac{2}{s+1}$ and controller $K(s) = 20$.

Assume the "true" dynamics lie within the following multiplicative uncertainty set:

$$\mathcal{M} := \{\hat{P} = P(1 + W_u\Delta) : ||\Delta||_\infty < 1 \text{ and } \Delta \text{ stable}\}$$

Assume the uncertainty weight is $W_u(s) = \dfrac{2s+1}{s+10}$.

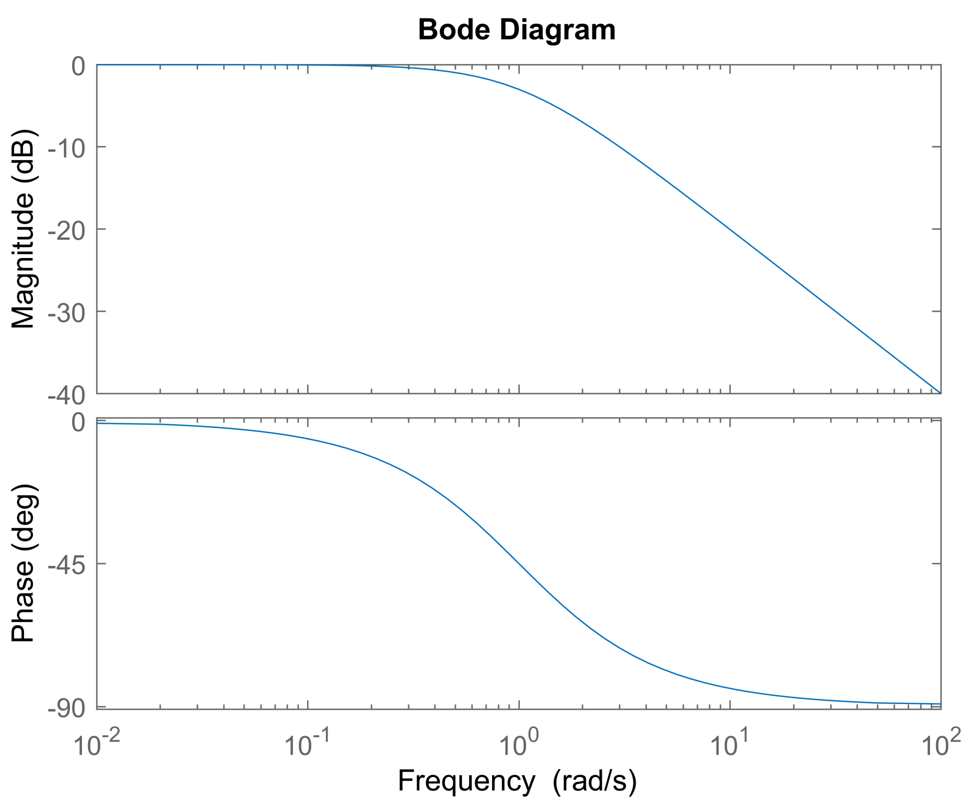
```
P = tf([1], [1 1])
```

```
P =

    1
  -----
  s + 1

Continuous-time transfer function.
```

```
bode(P)
```




```
K = tf([20], [1])
```

```
K =
```

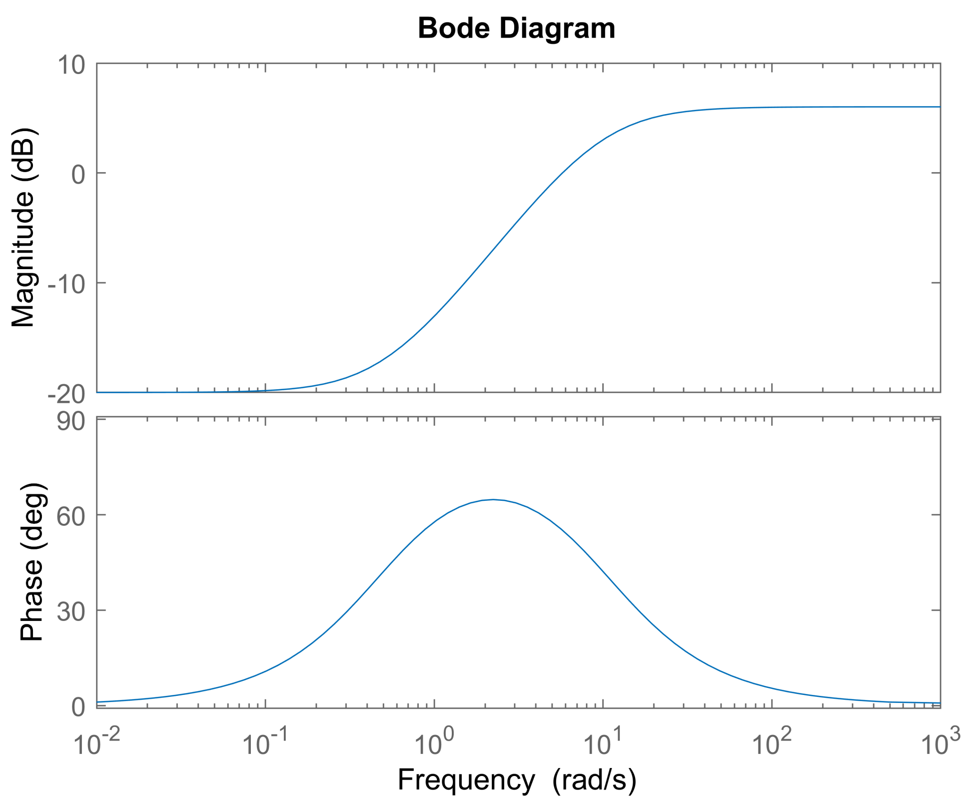
```
  20

Static gain.
```

```
Wu = tf([2 1], [1 10])
```

```
Wu =
 
  2 s + 1
  -------
  s + 10
 
Continuous-time transfer function.
```

```
bode(Wu)
```



(a) Provide an interpretation for the uncertainty described by the weight W_u.

A frequency dependent uncertainty that describes the modeling inacuracies of the system for frequencies increasing from

(b)

Is the nominal system stable?

```
closed_nom_stable = isstable(1+K*P)
```

```
closed_nom_stable = logical
   1
```

Yes

What are the gain and phase margins of the nominal loop $L = PK$?

```
L = P * K
```

```
L =
 
   20
  -----
  s + 1
 
Continuous-time transfer function.
```

```
[Gm_L, Pm_L, Wcg_L, Wcp_L] = margin(L);
disp(['Gain Marigin: ', num2str(Gm_L)])
```

```
Gain Marigin: Inf
```

```
disp(['Phas Marigin: ', num2str(Pm_L)])
```

```
Phas Marigin: 92.8663
```

(c)

The robust stability condition for this type of multiplicative uncertainty is stated as:

K stabilizes all $\hat{P} \in \mathscr{M}$ if and only if $||W_u T||_\infty \leq 1$.

```
T = feedback(L,1)
```

```
T =
 
    20
  ------
  s + 21
 
Continuous-time transfer function.
```

```
WuT = Wu * T
```

```
WuT =
 
     40 s + 20
  ----------------
  s^2 + 31 s + 210
 
Continuous-time transfer function.
```

```
WuT_norm = norm(WuT,'inf')
```

```
WuT_norm = 1.2911
```

Does robustly stabilize all models in based on this condition?

No. Since $||W_u T||_\infty = 1.2911 > 1$, the robust stability condition is not satisfied.

(d)

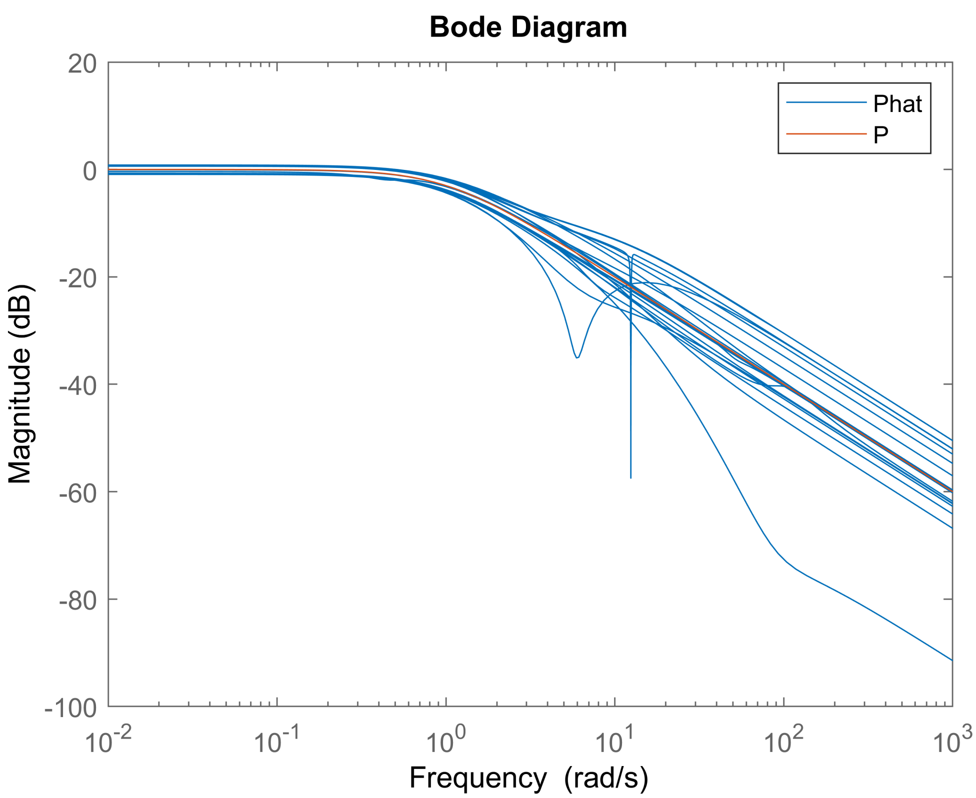
Construct the uncertain model Phat

```
Delta = ultidyn('Delta', [1 1]);
Phat = P * (1 + Wu * Delta);
```

Generate a Bode magnitude plot with 10 samples drawn from the uncertainty set and draw the nominal response P on

the same plot.

```
pblm4_d = figure();
hold on
bodemag(Phat)
bodemag(P)
legend
```



(e)

Perform the robustness test with robstab

```
Lhat = Phat * K;
That = feedback(Lhat,1);
opts = robOptions('Display','on','Sensitivity','on');
[stabmarg, destabunc, report] = robstab(That, opts);
```

```
Computing peak...  Percent completed: 100/100
System is not robustly stable for the modeled uncertainty.
```

```
  -- It can tolerate up to 77.3% of the modeled uncertainty.
  -- There is a destabilizing perturbation amounting to 77.5% of the modeled uncertainty.
  -- This perturbation causes an instability at the frequency 14.5 rad/seconds.
  -- Sensitivity with respect to each uncertain element is:
       100% for Delta. Increasing Delta by 25% decreases the margin by 25%.
```

Does the result obtained with robstab agree with your conclusions in part (c)?

Yes. The system is not robustly stable becouse a destabilizing pertubation exists with 77.5% model uncertainty at frequency 14.5 rad/s.