MECH 6323 - HW 3

Jonas Wagner

2022, March 1st

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1 Problem 1

Preliminaries:

Definition 1. ***Matrix Basics:*** *For $A \in \mathbb{C}^{n \times m}$ and $x \in \mathbb{C}^m$,*

1. *The* Eigenvalues *(λ_i) and* Eigenvectors *(x_i) of A are defined as the solutions to*

$$\lambda_i A = \lambda_i x_i$$

2. *The* Spectral Radius *of A is defined as*

$$\rho(A) := \max_i |\lambda_i(A)|$$

Definition 2. ***Vector Norms:*** *For $x \in \mathbb{C}^n$,*

1. *The* 2-norm, *or* Euclidean norm, *is defined as*

$$\|x\|_2 := \sqrt{\sum_{i=1}^{n} x_i^2}$$

2. *The* 1-norm *is defined as*

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|$$

3. *The* ∞-norm *is defined as*

$$\|x\|_\infty := \max i = 1, \ldots, n |x_i|$$

4. *The* p-norm *is defined as*

$$\|x\|_p := \left[\sum_{i=1}^{n} |x_i|^p \right]^{\frac{1}{p}}$$

Definition 3. ***Matrix Norms:*** *For $A \in \mathbb{C}^{n \times m}$ and $x \in \mathbb{C}^m$,*

1. *The* Induced 2-norm *is defined as*

$$\|A\|_{2\to2} := \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

and is also known as the spectral norm *and has the additional properties of*

   (a) $\|A\|_{2\to2} = \sqrt{\lambda_{max}(A^*A)} = \overline{\sigma}(A)$

   (b) $\|A^*A\|_{2\to2} = \|AA^*\|_{2\to2} = \|A\|_2^2$

1.1

Problem:

For $M \in \mathbb{C}^{n \times m}$, show that for all $x \in \mathbb{C}^m$

$$\|Mx\|_2 \leq \|M\|_{2\to2} \|x\|_2$$

Solution:

Theorem 1. *For $M \in \mathbb{C}^{n \times m}$, show that for all $x \in \mathbb{C}^m$*

$$\|Mx\|_2 \leq \|M\|_{2\to 2}\|x\|_2$$

Proof. From the definition of the 2-norm, we have

$$\|M\|_{2\to 2} := \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

$$\begin{aligned}
\|M\|_{2\to 2} &= \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \\
\|M\|_{2\to 2}\|x\|_2 &= \sup_{x\neq 0} \frac{\|Mx\|_2}{\|x\|_2}\|x\|_2 \\
&= \sup_{x\neq 0} \|Mx\|_2 \\
&\boxed{\|M\|_{2\to 2}\|x\|_2 \geq \|Mx\|_2 \ \forall_{x\in C^m}}
\end{aligned}$$

□

1.2

Problem:

Let $\{\lambda_i\}_{i=1}^n$ denote the eigenvalues of $A \in \mathbb{C}^{n\times n}$. Show that $\rho(A) \leq \|A\|_{2\to 2}$, where $\rho(A)$ is the spectral radius of matrix A. i.e. $\rho(A) := \max_i |\lambda_i(A)|$.

Solution:

Theorem 2. *Let $A \in \mathbb{C}^{n\times n}$. The spectral radius $\rho(A)$ will always be smaller then the induced 2-norm. i.e.*

$$\rho(A) \leq \|A\|_{2\to 2}$$

Proof. From the definition of the induced 2-norm, we have

$$\|M\|_{2\to 2} := \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

Additionally, from the definition of the vector 2-norm

$$\|x\|_2^2 = x^T x$$

$$\begin{aligned}
\|M\|_{2\to 2} &= \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \\
(\|M\|_{2\to 2})^2 &= \sup_{x\neq 0} \left(\frac{\|Ax\|_2}{\|x\|_2}\right)^2 \\
&= \sup_{x\neq 0} \frac{(Mx)^T Mx}{x^T x}
\end{aligned}$$

$$= \sup_{x \neq 0} \frac{x^T M^T M x}{x^T x}$$

Since $\forall_x Mx \leq Mx_{max}$, where $x_{max} = \|x\|_2 v_{max}$ and v_{max} is the eigenvector associated with $\lambda_{max} = \rho(A)$

$$\begin{aligned}
&\leq \sup_{x \neq 0} \frac{x_{max}^T M^T M x_{max}}{x^T x} \\
&= \sup_{x \neq 0} \frac{\|x\|_2 v_{max}^T \lambda_{max} \lambda max \|x\|_2 v_{max}}{x^T x} \\
&= \sup_{x \neq 0} \frac{v_{max}^T \|x\|_2^2 \lambda_{max}^2 v_{max}}{x^T x} \\
&= \sup_{x \neq 0} \frac{\lambda_{max}^2 \|x\|_2^2 \|v_{max}\|_2^2}{\|x\|_2^2} \\
&= \sup_{x \neq 0} \frac{\lambda_{max}^2 \|x\|_2^2 (1)^2}{\|x\|_2^2} \\
&= \sup_{x \neq 0} \lambda_{max}^2 \\
&= \lambda_{max}^2 = \rho(A)^2
\end{aligned}$$

$$(\|M\|_{2\to 2})^2 \leq (\rho(A))^2$$

$$\boxed{\|M\|_{2\to 2} \leq \rho(A)}$$

□

1.3

Problem:

Let $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{m \times k}$. Prove the multiplicative property of the induced 2-norm.

$$\|AB\|_{2\to 2} \leq \|A\|_{2\to 2} \|B\|_{2\to 2}$$

.

Solution:

Theorem 3. *Let $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{m \times k}$. Just like in all norms by definition,*

$$\|AB\|_{2\to 2} \leq \|A\|_{2\to 2} \|B\|_{2\to 2}$$

Proof. From norm definitions,

$$\|Ax\|_2 \leq \|A\|_{2\to 2} \|x\|_2$$

and therefore $\forall_{x \in \mathbb{C}^k}$,

$$\begin{aligned}
\|ABx\|_2 &\leq \|A\|_{2\to 2} \|Bx\|_2 \\
&\leq \|A\|_{2\to 2} \|B\|_{2\to 2} \|x\|_2
\end{aligned}$$

Taking $x = \sigma\hat{x}$ where magnitude $\sigma = \|x\|_2$ and $\hat{x}$ is the associated unit vector for x. Also, $\|\sigma\hat{x}\| = \sigma\|\hat{x}\|_2 = \|x\|_2$

$$\|AB\|x\|_2\hat{x}\|_2 \leq \|A\|_{2\to 2}\|B\|_{2\to 2}\|x\|_2\|\hat{x}\|_2$$
$$\|x\|_2\|AB\hat{x}\|_2 \leq \|A\|_{2\to 2}\|B\|_{2\to 2}\|x\|_2(1)$$

Noting that $\|AB\hat{x}\|_2 \leq \|AB\|_{2\to 2}\|\hat{x}\|_2 = \|AB\|_{2\to 2}\|\hat{x}\|_2$. Thus, $\|AB\hat{x}\|_2 = \|AB\|_{2\to 2}$

$$\|x\|_2\|AB\|_{\|2\to 2\|} \leq \|A\|_{2\to 2}\|B\|_{2\to 2}\|x\|_2$$
$$\boxed{\|AB\|_{\|2\to 2\|} \leq \|A\|_{2\to 2}\|B\|_{2\to 2}}$$

□

1.4

Problem:

Let $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$. Show that if $\|y\|_2 \leq \|x\|_2$, then there exists a $\Delta \in \mathbb{C}^{n\times m}$ such that $y = \Delta x$ and $\overline{\sigma(\Delta)} \leq 1$. The choice of Δ should only be expressed in terms of x, y, and their norms. Conversely, show that if $\|y\|_2 > \|x\|_2$, then there is no $\Delta \in \mathbb{C}^{n\times m}$ such that $y = \Delta x$ and $\overline{\sigma(\Delta)} \leq 1$.

Solution:

Theorem 4. *Let $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$. There exists a $\Delta \in \mathbb{C}^{n\times m}$ such that $y = \Delta x$ and $\overline{\sigma(\Delta)} \leq 1$ if and only if $\|y\|_2 \leq \|x\|$. i.e.*

$$\exists_{\Delta\in\mathbb{C}^{n\times m}} \ : \ y = \Delta x \wedge \overline{\sigma(\Delta)} \leq 1 \iff \|y\|_2 \leq \|x\|_2$$

Proof. From $y = \Delta x$ we have

$$\begin{aligned} y &= \Delta x \\ yx^* &= \Delta x x^* \\ \Delta &= \frac{yx^*}{xx^*} \\ &= \frac{yx^*}{\|x\|_2^2} \end{aligned}$$

From the definition of the induced 2-norm,

$$\overline{\sigma}(\Delta) = \|\Delta\|_{2\to 2} = \sup_{x\neq 0}\frac{\|\Delta x\|_2}{\|x\|_2}$$

Then in order for $\overline{\sigma(\Delta)} \leq 1$,

$$\begin{aligned} 1 \geq \overline{\sigma}(\Delta) = \|\Delta\|_{2\to 2}& \\ &= \sup_{x\neq 0}\frac{\|\Delta x\|_2}{\|x\|_2} \\ &= \sup_{x\neq 0}\frac{\left\|\frac{yx^*}{\|x\|_2^2}x\right\|_2}{x^*x} \end{aligned}$$

$\square$

2 Problem 2

See attached MATLAB .mlx script (and pdf version in appendix)

3 Problem 3

Problem:

Let S and T denote the sensitivity and complementary sensitivity closed-loop transfer functions. Prove that

$$\|S\|_\infty \geq \|T\|_\infty - 1$$

Preliminaries

Definition 4. *Let P and C represent the plant and controller transfer functions respectively. Within a standard unity feedback system,*

1. *the* <u>sensitivity closed-loop transfer function</u> *is defined as:*

$$S = \frac{1}{1+PC}$$

2. *the* <u>complementary sensitivity closed-loop transfer function</u> *is defined as:*

$$T = \frac{PC}{1+PC}$$

Solution:

Theorem 5. *Let P and C represent the plant and controller transfer functions respectively. Within a standard unity feedback system,*

$$\|S\|_\infty \geq \|T\|_\infty - 1$$

Proof. From the definitions,

$$T - S = \frac{PC}{1+PC} - \frac{1}{1+PC} = \frac{-1+PC}{1+PC}$$

Applying the $\infty-$norm,

$$\|T - s\|_\infty = \left\| \frac{-1+PC}{1+PC} \right\|_i nfty$$

Since $-1 + PC < 1 + PC$,

$$\|T - S\|_\infty \leq 1$$

From the triangular inequality we have

$$\|T - S\|_\infty \geq \|T\|_\infty - \|S\|_\infty$$

And thus,

$$\boxed{1 \geq \|T - S\|_\infty \geq \|T\|_\infty - \|S\|_\infty \implies \|S\|_\infty \geq \|T\|_\infty - 1}$$

□

A MATLAB Code:

See attached. Additionally, all the code I write in this course can be found on my GitHub repository:
https://github.com/jonaswagner2826/MECH6323