Problem01

Saturday, April 2, 2022 05:58

1. [15 points] Consider the feedback system shown in Figure 1 where $G(s)$ is the nominal plant model and $\Delta(s)$ is stable. Assume that $G(s)$, $K(s)$, and $\Delta(s)$ are all SISO.

   (a) The dashed box represents an uncertain model $\hat{G}(s)$ that depends on both $G(s)$ and $\Delta(s)$. What is the set of models $\mathcal{A}$ corresponding to this block diagram?

   (b) What can you conclude about the classical gain margins if the feedback system is stable for all $\|\Delta\|_\infty < 0.5$?

   (c) Find a necessary and sufficient condition for $K(s)$ to stabilize all $\hat{G}(s) \in \mathcal{A}$. Briefly describe a proof that your condition is sufficient for $K(s)$ to achieve robust stability. You do not need to prove necessity.

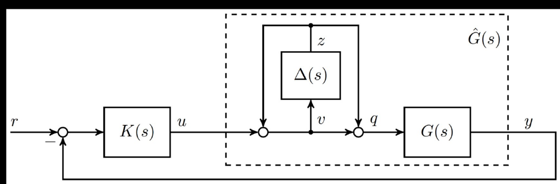


Fig. 1: Feedback system

a) $y = G(s)\, q$

$q = v + z$

$v = u + z$

$z = \Delta(s)\, v$

$y = G(s)(v + z)$

$z = \Delta(s)(u + z)$

$(1 - \Delta(s))\, z = u$

$z = (1 - \Delta(s))^{-1} u$

$(v + z) = u + z + z$

$(v + z) = u + 2z$

$(v + z) = u + 2(1 - \Delta(s))^{-1} u$

$(v + z) = \left(1 + 2(1 - \Delta(s))^{-1}\right) u$

$$y = G(s)\left(1 + 2(1 - \Delta(s))^{-1}\right) u$$

$$\hat{G}(s) = \left(1 + 2(1 - \Delta(s))^{-1}\right) G(s)$$

MIMO

$$\boxed{\mathcal{A} := \left\{ \hat{G}(s) = \left(1 + 2(1 - \Delta(s))^{-1}\right) G(s) \;:\; \Delta(s) \text{ stable} \right\}}$$

since we assume SISO,

$$\hat{G}(s) = \frac{(1 - \Delta(s)) + 2}{1 - \Delta(s)}\, G(s)$$

$$\hat{G}(s) = \frac{3 - \Delta(s)}{1 - \Delta(s)} G(s)$$

$$\mathcal{A} := \left\{ \hat{G}(s) = \frac{3 - \Delta(s)}{1 - \Delta(s)} G(s) \;:\; \forall s : s - \Delta(s) = 0 \Rightarrow \mathrm{Re}\{s\} < 0 \right\}$$

---

b) $\|\Delta(s)\| < 0.5$

$$y = \left(\frac{3 - \Delta(s)}{1 - \Delta(s)}\right) G(s)\, K(s)\, (y - r)$$

$$\frac{\left((3 - \Delta(s))\, G(s)\, K(s) - (1 - \Delta(s))\right)}{\cancel{1 - \Delta(s)}}\, y = \frac{(3 - \Delta(s))\, G(s)\, K(s)\, r(s)}{\cancel{1 - \Delta(s)}}$$

$$y = \frac{3GK - GK\Delta}{(3GK - 1) + (1 - GK)\Delta}\, r$$

$$\left(3GK - 1 - \frac{1 + GK}{2},\; 3GK - 1 + \frac{1 - GK}{2}\right)$$

$$\left(\frac{5GK - 3}{2},\; \frac{5GK + 3}{2}\right)$$

c)

Problem02

Saturday, April 2, 2022 06:00

2. [20 points] Consider a string of two vehicles as shown in Figure 2. The leading (first) vehicle is tracking a reference r_1 and the following (second) vehicle is tracking a reference $r_2 = x_1 - \delta$ to maintain a distance of δ with the first vehicle. The second vehicle uses a radar device to measure the distance to the first vehicle and compute the error $e_2 = r_2 - x_2$. Let each vehicle be modeled by the transfer function $G(s)$ and assume that both vehicles use the same control law $K(s)$. Figure 3 shows the feedback diagram for the two vehicle string.

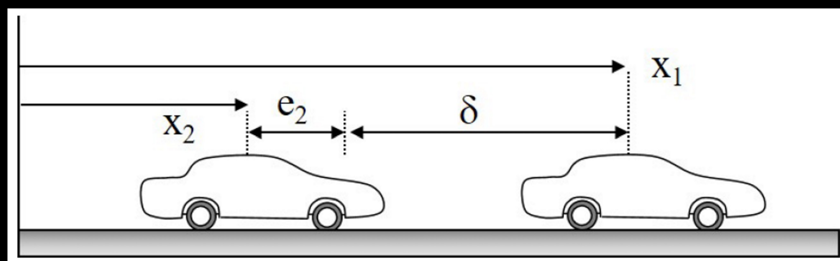


Fig. 2: String of two vehicles

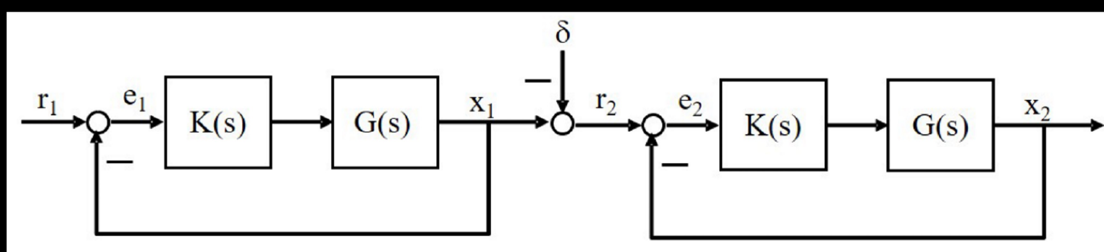


Fig. 3: Feedback diagram for two-vehicle system

(a) Compute the transfer function from: (i) r_1 to e_1, (ii) r_1 to x_1, (iii) r_2 to e_2, and (iv) r_2 to x_2. Your answers should be expressed in terms of the sensitivity $S(s) = \frac{1}{1+G(s)K(s)}$ and complementary sensitivity $T(s) = \frac{G(s)K(s)}{1+G(s)K(s)}$.

$e_1 = r_1 - x_1$

$x_1 = KG e_1$

$e_1 = r_1 - KGe_1$

i)

$r_1 = (1+KG)e_1 \Rightarrow \boxed{\frac{e_1}{r_1} = \frac{1}{1+KG} = S(s)}$

i)

$$r_1 = (1+KG)e_1 \Rightarrow \boxed{\frac{e_1}{r_1} = \frac{1}{1+KG} = S(s)}$$

ii)

$$x_1 = KG(r_1 - x_1)$$

$$(1+KG)x_1 = KG\,r_1 \Rightarrow \boxed{\frac{x_1}{r_1} = \frac{KG}{1+KG} = T(s)}$$

iii)

$$e_2 = r_2 - x_2$$

$$r_2 = (1+KG)e_1$$

$$\boxed{\frac{e_2}{r_2} = \frac{1}{1+KG} = S(s)}$$

$$x_2 = KG\,e_2$$

$$x_2 = KG\left(\frac{1}{1+KG}\right)r_2 \Rightarrow \boxed{\frac{x_2}{r_2} = \frac{KG}{1+KG} = T(s)}$$

(b) What is the transfer function from the first vehicle's reference r_1 to the tracking error for the second vehicle e_2? Note: By linearity you may assume $\delta = 0$ in this calculation.

$$r_1 \rightarrow e_2$$

$$\frac{x_1}{r_1} = T(s) = \frac{r_2}{r_1} \Rightarrow \boxed{\frac{e_2}{r_1} = S(s)\,T(s)}$$

r_1

$$\frac{e_2}{r_2} = S(s)$$

$r_2 = x_1$

r_1

(c) The goal is for the first vehicle to track the reference command r_1. In addition, the second vehicle should achieve a much smaller tracking error than the first vehicle, i.e., we would like $|e_2|$ to be much smaller than $|e_1|$. Use your results from the previous parts to express these objectives in terms of $S(s)$ and $T(s)$. You may assume that the reference command r_1 mainly consists of low frequency content.

(d) Is it possible to achieve the two goals in part (c)? If yes, describe how you would design $K(s)$ to achieve these goals. If no, then describe a constraint that prevents you from achieving both goals.

Problem03

Saturday, April 2, 2022 06:02

3. [15 points] Let $A \in \mathbb{C}^{n\times m}$ where $n > m$. The singular value decomposition (SVD) of A is given by

$$A = U\Sigma V^* = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \hat{\Sigma} \\ 0_{n-m,m} \end{bmatrix} V^*$$

where U and V are unitary matrices and $\hat{\Sigma} = \mathrm{diag}(\sigma_1, \ldots, \sigma_m)$. Assume A is full rank and hence $\sigma_i > 0$ for $i = 1, \ldots, m$.

(a) What is the SVD of $(A^*A)^{-1}A^*$?

$$A = U\Sigma V^*$$

$$A^* = V\Sigma^* U^*$$

$$A^*A = V\Sigma^* U^* U \Sigma V^*$$

$$U^*U = I \Downarrow$$

$$A^*A = V\Sigma^*\Sigma V^*$$

$$(A^*A)^{-1} = (V^*)^{-1}(\Sigma^*\Sigma)^{-1}V^{-1}$$

$$(A^*A)^{-1}A^* = (V^*)^{-1}\Sigma^{*-1}\Sigma^{-1}\underbrace{\cancel{(V)^{-1}V}}_{I}\Sigma^* U^*$$

$$= (V^*)^{-1}\cancel{\Sigma^{*-1}}\,\Sigma^{-1}\cancel{\Sigma^*}\,U^*$$

comutable bc diagonal

$$\boxed{(A^*A)^{-1}A^* = (V^*)^{-1}\Sigma^{-1}U^*}$$

(b) Show that for any vectors $x \in \mathbb{C}^m$ and $b \in \mathbb{C}^n$,

$$\|Ax - b\|_2^2 = \|\hat{\Sigma}V^*x - U_1^*b\|_2^2 + \|U_2^*b\|_2^2. \tag{1}$$

$$\|Ax-b\|_2^2 = (Ax-b)^*(Ax-b) = (x^*A^* - b^*)(Ax-b)$$

$$= x^*A^*Ax - x^*A^*b - b^*Ax + b^*b$$

$A^*A = V\Sigma^*\Sigma V^*$

$$= x^*V\Sigma^*\Sigma Vx - x^*V\Sigma^*U^*b - b^*U\Sigma V^*x + b^*b$$

$$= x^*V\Sigma^*\Sigma Vx - x^*V\Sigma^*U^*b - b^*U\Sigma V^*x + b^*(U_1U_1^* + U_2U_2^*)b \qquad 1$$

$$\|\hat{\Sigma}V^*x - U_1^*b\|_2^2 + \|U_2^*b\|_2^2. = (\hat{\Sigma}V^*x - U_1^*b)(x^*V\hat{\Sigma}^* - b^*U_1) + b^*U_2U_2^*b$$

$$= x^*V\Sigma^*\Sigma Vx - x^*V\Sigma^*U_1^*b - b^*U_1\Sigma V^*x + b^*U_1U_1^*b + \underbrace{b^*U_2U_2^*b}_{2}$$

$1 \leftrightarrow 2$

(c) Use Equation (1) to specify the vector x that solves

$$\min_x \|Ax - b\|_2$$

How is the solution related to the matrix in part (a)?

$$\frac{d}{dx} = 0 = 2\Sigma Vx - 2V\Sigma^*U_1b + 0 + 0$$

$$\cancel{2}V\Sigma^*U_1b = \cancel{2}\Sigma Vx$$

$$x = V^{-1}\Sigma^{-1}(V\Sigma^*U_1)b \longrightarrow (A^*A)^{-1}A$$

This least-squares problem has a psudo-inverse solution which is the equivelent to the one in (a)

Problem04

Saturday, April 2, 2022 06:03

4. [20 points] Consider the following set of models:

$$\mathcal{I} := \left\{ \frac{G(s)}{1 + w(s)\Delta(s)} \;\middle|\; \Delta \text{ stable}, \; \|\Delta\|_\infty < 1 \right\}$$

$G(s)$ is SISO and $w(s)$ is a stable, proper rational transfer function.

(a) Draw a block diagram showing the structure of any $G_p \in \mathcal{I}$. Your diagram should only include blocks for G, w, and Δ.

(b) Consider the feedback diagram show in Figure 4. Assume the controller K stabilizes the nominal system G. Find a necessary and sufficient condition for K to stabilize all $G_p \in \mathcal{I}$. Provide a proof of sufficiency, i.e., prove that K achieves robust stability if your condition is satisfied. You do not need to prove necessity.

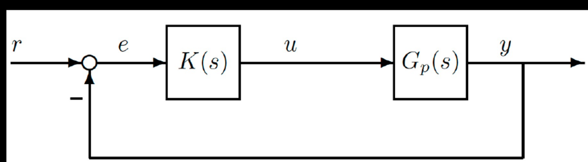
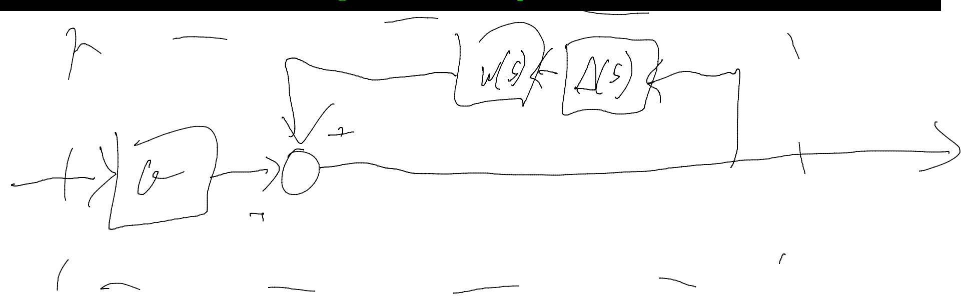


Fig. 4: Feedback loop

a)



b)

$$\frac{K G}{1 + w(G W)} \quad \text{stable} \; \forall w$$

Proof:

This can be proven w/ Nyquist critera...

(don't have time to write out specifics but essentially that it must just remain

MECH 6323 - Robust Control - Midterm Exam

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Date: 2022-04-02

```
clear
close all
```

Problem 4 - Plane Autopilot System

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Controller Definition

Parameters

```
K_a = -1.5e-3;
K_q = -0.32;
a_z = 2;
a_q = 6;
```

State Matrices

```
A_C = [ 0           0
        K_q*a_q   0];
B_C = [ K_a*a_z           0
        K_a*K_q*a_q     K_q*a_q];
C_C = [ K_q    1];
D_C = [ K_a*K_q       K_q];
```

System Definition

```
sys_C = ss(A_C, B_C, C_C, D_C)
```

```
sys_C =

  A =
          x1     x2
   x1      0      0
   x2  -1.92      0

  B =
             u1       u2
   x1    -0.003        0
   x2   0.00288    -1.92

  C =
          x1     x2
   y1  -0.32      1

  D =
             u1       u2
   y1   0.00048    -0.32

Continuous-time state-space model.
```

```
tf_C = tf(sys_C)
```

```
tf_C =

  From input 1 to output:
  0.00048 s^2 + 0.00384 s + 0.00576
  ---------------------------------
                s^2

  From input 2 to output:
  -0.32 s - 1.92
  --------------
        s

Continuous-time transfer function.
```

Plant Definition

Parameters

```
V = 886.78;
zeta = 0.6;
omega = 113;
```

Uncertain Coeficients

Nominal Values

```
Z_alpha_0 = -1.3046;
Z_delta_0 = -0.2142;
M_alpha_0 = 47.7109;
M_delta_0 = -104.83436;
```

Uncertain Dynamics

```
Z_alpha = @(k, delta) Z_alpha_0 * (1 + k * delta);
Z_delta = @(k, delta) Z_delta_0 * (1 + k * delta);
M_alpha = @(k, delta) M_alpha_0 * (1 + k * delta);
M_delta = @(k, delta) M_delta_0 * (1 + k * delta);
```

System Matrices

```
A_P = @(Z_alpha, Z_delta, M_alpha, M_delta) [
    Z_alpha 1 Z_delta     0
    M_alpha 0 M_delta     0
    0     0 0         1
    0     0 -omega^2 -2*zeta*omega
];
B_P = [
    0
    0
    0
    omega^2
];
```

```
C_P = @(Z_alpha, Z_delta) [
    V*Z_alpha   0   V*Z_delta   0
    0           1   0           0
];
D_P = [
    0
    0
];
```

Nominal System

Nominal System Matrices

```
A_P_0 = A_P(Z_alpha_0, Z_delta_0, M_alpha_0, M_delta_0)
```

```
A_P_0 = 4×4
10^4 ×
   -0.0001    0.0001   -0.0000         0
    0.0048         0   -0.0105         0
         0         0         0    0.0001
         0         0   -1.2769   -0.0136
```

```
% double(vpa(subs(A_P, ...
%     [Z_alpha, Z_delta, M_alpha, M_delta], ...
%     [Z_alpha_0, Z_delta_0, M_alpha_0, M_delta_0]), 4));
B_P_0 = B_P
```

```
B_P_0 = 4×1
           0
           0
           0
       12769
```

```
% double(vpa(subs(B_P, ...
%     [Z_alpha, Z_delta, M_alpha, M_delta], ...
%     [Z_alpha_0, Z_delta_0, M_alpha_0, M_delta_0]), 4));
C_P_0 = C_P(Z_alpha_0, Z_delta_0)
```

```
C_P_0 = 2×4
10^3 ×
   -1.1569         0   -0.1899         0
         0    0.0010         0         0
```

```
% double(vpa(subs(C_P, ...
%     [Z_alpha, Z_delta, M_alpha, M_delta], ...
%     [Z_alpha_0, Z_delta_0, M_alpha_0, M_delta_0]), 4));
D_P_0 = D_P
```

```
D_P_0 = 2×1
     0
     0
```

```
% double(vpa(subs(D_P, ...
%     [Z_alpha, Z_delta, M_alpha, M_delta], ...
%     [Z_alpha_0, Z_delta_0, M_alpha_0, M_delta_0]), 4));
```

Nominal State-space System

```
sys_P_0 = ss(A_P_0, B_P_0, C_P_0, D_P_0)
```

```
sys_P_0 =

  A =
               x1          x2          x3          x4
   x1      -1.305           1     -0.2142           0
   x2       47.71           0      -104.8           0
   x3           0           0           0           1
   x4           0           0  -1.277e+04      -135.6

  B =
              u1
   x1          0
   x2          0
   x3          0
   x4  1.277e+04

  C =
           x1      x2      x3      x4
   y1   -1157       0  -189.9       0
   y2       0       1       0       0

  D =
       u1
   y1   0
   y2   0

Continuous-time state-space model.
```

Nominal Transfer Function

```
tf_P_0 = tf(sys_P_0)
```

```
tf_P_0 =

  From input to output...
              -2.425e06 s^2 + 2.585e-08 s + 1.664e09
   1:  -----------------------------------------------------
       s^4 + 136.9 s^3 + 1.29e04 s^2 + 1.019e04 s - 6.092e05

                      -1.339e06 s - 1.877e06
   2:  -----------------------------------------------------
       s^4 + 136.9 s^3 + 1.29e04 s^2 + 1.019e04 s - 6.092e05

Continuous-time transfer function.
```

Uncertain System Dynamics

Uncertain Matrices

```
A_P = @(k, Delta) A_P(  Z_alpha(k, Delta(1)), ...
                        Z_delta(k, Delta(2)), ...
                        M_alpha(k, Delta(3)), ...
                        M_delta(k, Delta(4)));
C_P = @(k, Delta) C_P(  Z_alpha(k, Delta(1)), ...
                        Z_delta(k, Delta(2)));
```

Uncertain System

```
sys_P = @(k, Delta) ss(A_P(k, Delta), B_P, C_P(k, Delta), D_P);
```

Feedback System Definition

Open Loop System: $L(s) = C(s)P(s)$

Nominal

```
sys_L_0 = series(sys_C, sys_P_0)
```

```
sys_L_0 =

  A = 
                x1          x2          x3          x4          x5          x6
   x1       -1.305           1     -0.2142           0           0           0
   x2        47.71           0      -104.8           0           0           0
   x3            0           0           0           1           0           0
   x4            0           0  -1.277e+04      -135.6       -4086   1.277e+04
   x5            0           0           0           0           0           0
   x6            0           0           0           0       -1.92           0

  B = 
             u1       u2
   x1         0        0
   x2         0        0
   x3         0        0
   x4     6.129    -4086
   x5    -0.003        0
   x6   0.00288    -1.92

  C = 
            x1      x2      x3      x4      x5      x6
   y1    -1157       0  -189.9       0       0       0
   y2        0       1       0       0       0       0

  D = 
        u1  u2
   y1    0   0
   y2    0   0

Continuous-time state-space model.
```

Uncertain

```
sys_L = @(k, Delta) series(sys_C, sys_P(k, Delta));
```

Closed Loop System: $S(s) = \dfrac{C(s)P(s)}{1 + C(s)P(s)}$

Nominal

```
sys_S_0 = feedback(sys_L_0, eye(size(C_P_0,1)))
```

```
sys_S_0 =

  A = 
               x1          x2          x3          x4          x5          x6
   x1      -1.305           1     -0.2142           0           0           0
   x2       47.71           0      -104.8           0           0           0
```

```
  x3         0         0          0          1          0          0
  x4      7091      4086  -1.16e+04     -135.6      -4086  1.277e+04
  x5    -3.471         0    -0.5698          0          0          0
  x6     3.332      1.92     0.5471          0      -1.92          0

 B =
           u1       u2
  x1        0        0
  x2        0        0
  x3        0        0
  x4    6.129    -4086
  x5   -0.003        0
  x6  0.00288    -1.92

 C =
          x1      x2      x3      x4      x5      x6
  y1   -1157       0  -189.9       0       0       0
  y2       0       1       0       0       0       0

 D =
      u1  u2
  y1   0   0
  y2   0   0

Continuous-time state-space model.
```

Uncertain

```
sys_S = @(k, Delta) feedback(sys_L(k,Delta), eye(size(C_P_0,1)));
```

(a)

```
eig_S_0 = eig(sys_S_0)
```

```
eig_S_0 = 6×1 complex
 -39.2033 +71.5951i
 -39.2033 -71.5951i
 -49.2597 + 0.0000i
  -1.8309 + 0.0000i
  -3.7038 + 1.4953i
  -3.7038 - 1.4953i
```

Since $\mathscr{R}(\lambda_i) < 0 \ \forall_i$, the closed loop system $S(s)$ is stable.

```
sys_S_stable = isstable(sys_S_0)
```

```
sys_S_stable = logical
   1
```

(b)

Random Testing Code

```
i_worst = 1;
k_try = 10;
N_samples = 10000;
Delta_data = 2 * rand([4, N_samples]) - 1;
```

```
for i = 1:N_samples
    % Better Implimentation
    sys_S_test = @(k) sys_S(k, Delta_data(:,i));
    if isstable(sys_S_test(k_try))
        break
    else
        while ~isstable(sys_S_test(k_try))
            k_try = 0.99 * k_try;
            i_worst = i;
        end
    end
    % Alternative (as described in the problem itself)
%     Delta = Delta_data(:,i);
%     sys_S_unstable = true;
%     while sys_S_unstable
%         sys_S_test = sys_S(k_try, Delta);
%         if max(real(eig(sys_S_test))) >= 0
%             k_try = 0.99 * k_try;
%             i_worst = i;
%         else
%             sys_S_unstable = false;
%         end
%     end
end
```

Random Testing Results

```
k_bar = k_try
```

```
k_bar = 1.1639
```

```
Delta_worst = Delta_data(:,i)
```

```
Delta_worst = 4×1
    0.9599
    0.9308
   -0.6538
   -0.5855
```

Since this gets randomized everytime, $\bar{k}$ changes but often gets down below 1 to 0.7-ish, but also stays at 10 sometimes.

Why must $k_{max} \leq \bar{k}$?

Well, many reasons. We know that k_{max} is the largest possible k that maintains stability of the closed-loop system, and therefore all unstabilizing k due to some disturbance would be greater then the lowest-upper bound on k, i.e. $\bar{k} \geq \sup_k S(s)$ stable. Therefore, $k_{max} = \sup_k S(s) \leq \bar{k}$.

(c)

Uncertain System

Bounded Uncertainty

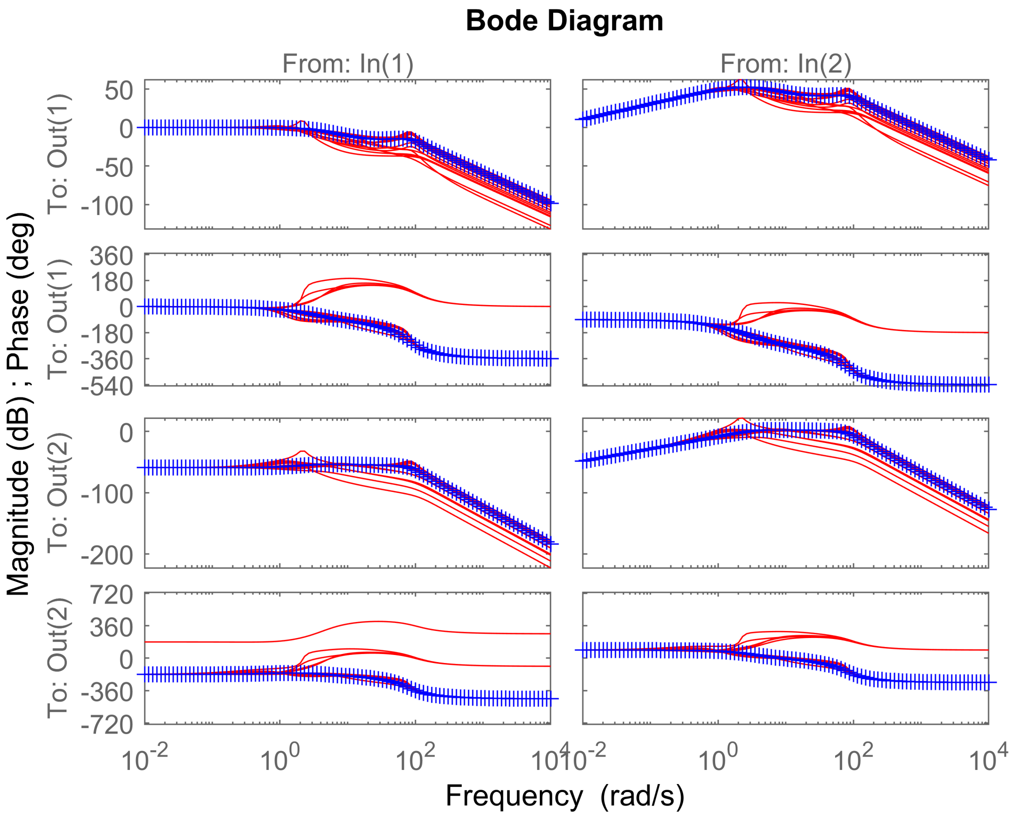
```
delta_1_u = ureal('delta_1', 0);
delta_2_u = ureal('delta_2', 0);
delta_3_u = ureal('delta_3', 0);
delta_4_u = ureal('delta_4', 0);

Delta_u = [
    delta_1_u
    delta_2_u
    delta_3_u
    delta_4_u
];
```

Uncertain SS system

```
sys_S_u = @(k) sys_S(k, Delta_u);
```

Uncertain System Frequency Respons

```
w_min = -2;
w_max = 4;
freqs = logspace(w_min,w_max,100);
sys_S_u_k1 = sys_S_u(1);
usysfrd = ufrd(sys_S_u_k1, freqs);
bode(usysfrd,'r', usysfrd.NominalValue, 'b+')
```



robostab

```
opts = robOptions(  'Display','on', ...
                    'VaryFrequency','on',...
                    'Sensitivity','on');
[stabmarg, destabunc, report] = robstab(sys_S_u_k1, opts)
```

```
Computing bounds... Points completed: 42/42
Computing peak...  Percent completed: 100/100
System is not robustly stable for the modeled uncertainty.
 -- It can tolerate up to 60.3% of the modeled uncertainty.
 -- There is a destabilizing perturbation amounting to 60.4% of the modeled uncertainty.
 -- This perturbation causes an instability at the frequency 3.44 rad/seconds.
 -- Sensitivity with respect to each uncertain element is:
      3% for delta_1. Increasing delta_1 by 25% decreases the margin by 0.75%.
      1% for delta_2. Increasing delta_2 by 25% decreases the margin by 0.25%.
      16% for delta_3. Increasing delta_3 by 25% decreases the margin by 4%.
      42% for delta_4. Increasing delta_4 by 25% decreases the margin by 10.5%.
stabmarg = struct with fields:
           LowerBound: 0.6035
           UpperBound: 0.6045
    CriticalFrequency: 3.4386
destabunc = struct with fields:
    delta_1: -0.6045
    delta_2: -0.6045
    delta_3: 0.6045
    delta_4: -0.6045
report = struct with fields:
                Model: 1
            Frequency: [44×1 double]
               Bounds: [44×2 double]
    WorstPerturbation: [44×1 struct]
          Sensitivity: [1×1 struct]
```

From this we have $k_{max} \approx 0.6045$ to result in $k * \Delta = 0.6045 * [-1 \quad -1 \quad 1 \quad -1]^T$.

This value is consistant with the numerical results as it is below, but not too small by comparrision to what is expected.

Additionally, we know that the critical frequency that this is occurring at is

---

(d)

```
k_max = 0.6045;
sys_S_critical = sys_S(k_max, [-1; -1; 1; -1])
```

```
sys_S_critical =

  A =
             x1          x2          x3          x4          x5          x6
   x1    -0.516           1    -0.08472           0           0           0
   x2     76.55           0      -41.46           0           0           0
   x3         0           0           0           1           0           0
   x4      2804        4086  -1.231e+04      -135.6       -4086   1.277e+04
   x5    -1.373           0     -0.2254           0           0           0
   x6     1.318        1.92      0.2164           0       -1.92           0
```

```
  B = 
            u1       u2
   x1        0        0
   x2        0        0
   x3        0        0
   x4    6.129    -4086
   x5   -0.003        0
   x6  0.00288    -1.92

  C = 
           x1      x2      x3      x4      x5      x6
   y1  -457.6       0  -75.12       0       0       0
   y2       0       1       0       0       0       0

  D = 
       u1  u2
   y1   0   0
   y2   0   0

Continuous-time state-space model.
```

```
tf_S_critical = tf(sys_S_critical)
```

```
tf_S_critical =
 
  From input 1 to output...
                  -460.4 s^4 - 3684 s^3 + 1.46e05 s^2 + 1.212e06 s + 1.818e06
   1:  -----------------------------------------------------------------------------------
       s^6 + 136.1 s^5 + 1.23e04 s^4 + 1.619e05 s^3 + 2.989e05 s^2 + 1.896e06 s + 1.818e06
 
                               -254.1 s^3 - 2204 s^2 - 4416 s - 2050
   2:  -----------------------------------------------------------------------------------
       s^6 + 136.1 s^5 + 1.23e04 s^4 + 1.619e05 s^3 + 2.989e05 s^2 + 1.896e06 s + 1.818e06
 
  From input 2 to output...
                     3.07e05 s^4 + 1.842e06 s^3 - 1.01e08 s^2 - 6.061e08 s
   1:  -----------------------------------------------------------------------------------
       s^6 + 136.1 s^5 + 1.23e04 s^4 + 1.619e05 s^3 + 2.989e05 s^2 + 1.896e06 s + 1.818e06
 
                      1.694e05 s^3 + 1.13e06 s^2 + 6.835e05 s - 5.387e-11
   2:  -----------------------------------------------------------------------------------
       s^6 + 136.1 s^5 + 1.23e04 s^4 + 1.619e05 s^3 + 2.989e05 s^2 + 1.896e06 s + 1.818e06
 
Continuous-time transfer function.
```

```
eig_S_critical = eig(sys_S_critical)
```

```
eig_S_critical = 6×1 complex
 -60.4635 +82.3580i
 -60.4635 -82.3580i
 -14.1488 + 0.0000i
   0.0006 + 3.4385i
   0.0006 - 3.4385i
  -1.0412 + 0.0000i
```

As can be seen by the eigenvalues at $\lambda_{4,5} = 0.00 \pm j3.44$, the poles of the system cross the $j\omega$-axis to become unstable at $k_{max} \approx 0.6045$ with $\omega \approx 3.44$. This confirms the robostab estimates to force the system to become unstable at the maximum perturbation of $\Delta = [-1 \quad -1 \;\; 1 \quad -1]^T$.