Problem03

Saturday, April 2, 2022 06:02

3. [15 points] Let $A \in \mathbb{C}^{n\times m}$ where $n > m$. The singular value decomposition (SVD) of A is given by

$$A = U\Sigma V^* = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \hat{\Sigma} \\ 0_{n-m,m} \end{bmatrix} V^*$$

where U and V are unitary matrices and $\hat{\Sigma} = \mathrm{diag}(\sigma_1, \ldots, \sigma_m)$. Assume A is full rank and hence $\sigma_i > 0$ for $i = 1, \ldots, m$.

(a) What is the SVD of $(A^*A)^{-1}A^*$?

$$A = U\Sigma V^*$$
$$A^* = V\Sigma^* U^*$$

$$A^*A = V\Sigma^* U^* U \Sigma V^*$$

$$U^*U = I \Downarrow$$

$$A^*A = V\Sigma^*\Sigma V^*$$
$$(A^*A)^{-1} = (V^*)^{-1}(\Sigma^*\Sigma)^{-1}V^{-1}$$
$$(A^*A)^{-1}A^* = (V^*)^{-1}{\Sigma^*}^{-1}\Sigma^{-1}\underbrace{\cancel{(V)^{-1}V}}_{I}\Sigma^* U^*$$
$$= (V^*)^{-1}\cancel{{\Sigma^*}}^{-1}\Sigma^{-1}\cancel{\Sigma^*}U^*$$

comutable bc diagonal

$$\boxed{(A^*A)^{-1}A^* = (V^*)^{-1}\Sigma^{-1}U^*}$$

(b) Show that for any vectors $x \in \mathbb{C}^m$ and $b \in \mathbb{C}^n$,

$$\|Ax - b\|_2^2 = \|\hat{\Sigma}V^*x - U_1^*b\|_2^2 + \|U_2^*b\|_2^2. \quad (1)$$

$$\|Ax-b\|_2^2 = (Ax-b)^*(Ax-b) = (x^*A^*-b^*)(Ax-b)$$

$$= x^*A^*Ax - x^*A^*b - b^*Ax + b^*b$$

$$A^*A = V\Sigma^*\Sigma V^*$$

$$= x^*V\Sigma^*\Sigma Vx - x^*V\Sigma^*U^*b - b^*U\Sigma V^*x + b^*b$$

$$= x^*V\Sigma^*\Sigma Vx - x^*V\Sigma^*U^*b - b^*U\Sigma V^*x + b^*(U_1U_1^*+U_2U_2^*)b \quad (*)$$

$\|\hat{\Sigma} V^*x - U_1^*b\|_2^2 + \|U_2^*b\|_2^2.$ $= (\hat{\Sigma}V^*x - U_1^*b)(x^*V\hat{\Sigma}^* - b^*U_1) + b^*U_2U_2^*b$

$$= x^*V\Sigma^*\Sigma Vx - x^*V\Sigma^*U_1^*b - b^*U_1\Sigma V^*x + b^*U_1U_1^*b + b^*U_2U_2^*b \quad (2)$$

$$(1) \Leftrightarrow (2)$$

(c) Use Equation (1) to specify the vector x that solves

$$\min_x \|Ax-b\|_2$$

How is the solution related to the matrix in part (a)?

$$\frac{d}{dx} = 0 = 2\Sigma Vx - 2V\Sigma^*U_1b + 0 + 0$$

$$\cancel{2}V\Sigma^*U_1b = \cancel{2}\Sigma Vx$$

$$x = V^{-1}\Sigma^{-1}(V\Sigma^*U_1)b \rightarrow (A^*A)^{-1}A$$

This least-squares problem has a psudo-inverse solution which is the equivelent to the one in (a)