Due Tuesday 03/01/22 (10:00pm)

1. (a) For $M \in \mathbb{C}^{n\times m}$, show that for all $x \in \mathbb{C}^m$,

$$\|Mx\|_2 \;\leq\; \|M\|_{2\to 2}\,\|x\|_2.$$

   (b) Let $\{\lambda_i\}_{i=1}^n$ denote the eigenvalues of matrix $A \in \mathbb{C}^{n\times n}$. Show that $\rho(A) \leq \|A\|_{2\to 2}$, where $\rho(A)$ is the spectral radius of matrix A, i.e., $\rho(A) := \max_i |\lambda_i(A)|$.

   (c) Let $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$. Prove the multiplicative property of the induced 2-norm:

$$\|AB\|_{2\to 2} \;\leq\; \|A\|_{2\to 2}\,\|B\|_{2\to 2}.$$

   (d) Let $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$. Show that if $\|y\|_2 \leq \|x\|_2$, then there exists a $\Delta \in \mathbb{C}^{n\times m}$ such that $y = \Delta x$ and $\bar{\sigma}(\Delta) \leq 1$. The choice of Δ should only be expressed in terms of x, y, and their norms. Conversely, show that if $\|y\|_2 > \|x\|_2$, then there is no $\Delta \in \mathbb{C}^{n\times m}$ such that $y = \Delta x$ and $\bar{\sigma}(\Delta) \leq 1$.

2. Consider the MIMO system P with state-space representation

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -2 & 5 \\ -5 & -3 \end{bmatrix} x + \begin{bmatrix} -2 & 4 \\ -2 & -2 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix} x. \end{aligned}$$

   (a) Is P stable? Why?

   (b) What is $\|P\|_\infty$? What is the frequency ω_p that achieves the peak gain ($\bar{\sigma}(P(j\omega_p)) = \|P\|_\infty$)?

   (c) What is the SVD of $P(j\omega_p)$? Verify that the computed largest singular value satisfies $\bar{\sigma}(P(j\omega_p)) = \|P\|_\infty$.

   (d) Generate and submit the σ-plot of P. Verify that the peak of the singular value plot agrees with the computed values for $\|P\|_\infty$ and ω_p.

   (e) Construct the vectors a, ϕ, b, and $\psi \in \mathbb{R}^2$ with $\|a\|_2 = 1$ and $\|b\|_2 = \|P\|_\infty$ such that the input signal

$$u(t) \;=\; \begin{bmatrix} a_1 \sin(\omega_p t + \phi_1) \\ a_2 \sin(\omega_p t + \phi_2) \\ \vdots \\ a_m \sin(\omega_p t + \phi_m) \end{bmatrix}$$

   gives the steady-state output

$$y(t) \;=\; \begin{bmatrix} b_1 \sin(\omega_p t + \psi_1) \\ b_2 \sin(\omega_p t + \psi_2) \\ \vdots \\ b_n \sin(\omega_p t + \psi_n) \end{bmatrix}.$$

Note that the amplitude vectors satisfy $\dfrac{\|b\|_2}{\|a\|_2} = \|P\|_\infty$, i.e., the peak gain is achieved by this pair of (real) input/output signals.

(f) Simulate the linear system P with the input signal constructed in the previous part, e.g., using the lsim command in MATLAB. Plot the steady-state output signal predicted from your construction in the previous part. Plot the simulated output on the same plot. Verify that both results agree after the initial transient decays to zero.

3. Let S and T denote the sensitivity and complementary sensitivity closed-loop transfer functions. Prove that

$$\|S\|_\infty \;\geq\; \|T\|_\infty \;-\; 1.$$