MECH 6323 - Homework 4

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```
clear
close all
```

Problem 1

Part a

a) $H(s) = \frac{1}{s+a}$, $a > 0 \Rightarrow$ stab

$$H_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{|H(j\omega)|^2}_{} d\omega}$$

$\rightarrow$ $H(j\omega)$

$H^* H =$

$$= \sqrt{\frac{1}{2\pi} \underbrace{\int_{-\infty}^{\infty} \frac{d\omega}{\omega^2 + a^2}}_{}}$$ $\rightarrow$ $\tan^{-}$

$$\boxed{H_2 = \sqrt{\frac{1}{2a}}}$$

Part b

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0$$

$$G(s) = C(sI - A)^{-1}B + D$$

B

$$(sI - A)^{-1} = \frac{1}{s^2+1}\begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}$$

$$\boxed{G(s) = \frac{-1}{s^2+1}}$$

$$H_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|^2 \, d\omega}$$

$$H(j\omega) = \frac{-1}{(j\omega)^2+1} = \frac{-1}{1-\omega^2}$$

$$H^*(j\omega) = \frac{-1}{1-\omega^2} = \frac{1}{\omega^2-1}$$

$$H^* H = \frac{1}{(\omega^2-1)^2}$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{(\omega^2-1)^2} = \frac{1}{4}\ln\frac{|\omega+1|}{|\omega-1|} - \frac{1}{2(\omega+1)}\Big|$$

$$H_2 = \sqrt{\frac{1}{2\pi}\left(\frac{1}{4}\right)\cancel{\left(\ln(1) - \ln(1) + 0\right)}} \; 0 =$$

$$H_\infty = \max_{\omega} |H(j\omega)| \rightarrow \text{unbounded}$$

$$H_\infty = \infty$$

Problem 2

Part a

```
% simple written down already
k = sym('k', 'real');
R = sym('R', 'real'); % should be positive... will show

A = [
    -(1 + k^2)/R    0
    k               -(2+k^2)/R]
```

A =

$$\begin{pmatrix} -\frac{k^2+1}{R} & 0 \\ k & -\frac{k^2+2}{R} \end{pmatrix}$$

```
Lambda = eig(A)
```

Lambda =

$$\begin{pmatrix} -\frac{k^2+1}{R} \\ -\frac{k^2+2}{R} \end{pmatrix}$$

This means that the system is stable $\forall k \in \mathscr{R}$ and $\forall R > 0$

Part b

```
R = sym('R', 'positive');
S = svd(A)
```

S =

$$\begin{pmatrix} \frac{\sqrt{\sigma_1 + 3\,k^2 + k^4 + \frac{R^2\,k^2}{2} + \frac{5}{2}}}{R} \\ \frac{\sqrt{3\,k^2 - \sigma_1 + k^4 + \frac{R^2\,k^2}{2} + \frac{5}{2}}}{R} \end{pmatrix}$$

where

$$\sigma_1 = \frac{\sqrt{(R^2\,k^2+1)\,(R^2\,k^2+4\,k^4+12\,k^2+9)}}{2}$$

```
H_2_norm_fun = matlabFunction(norm(S,2), 'Vars', [R, k]);
H_2_norm = H_2_norm_fun(k,R)
```

H_2_norm =

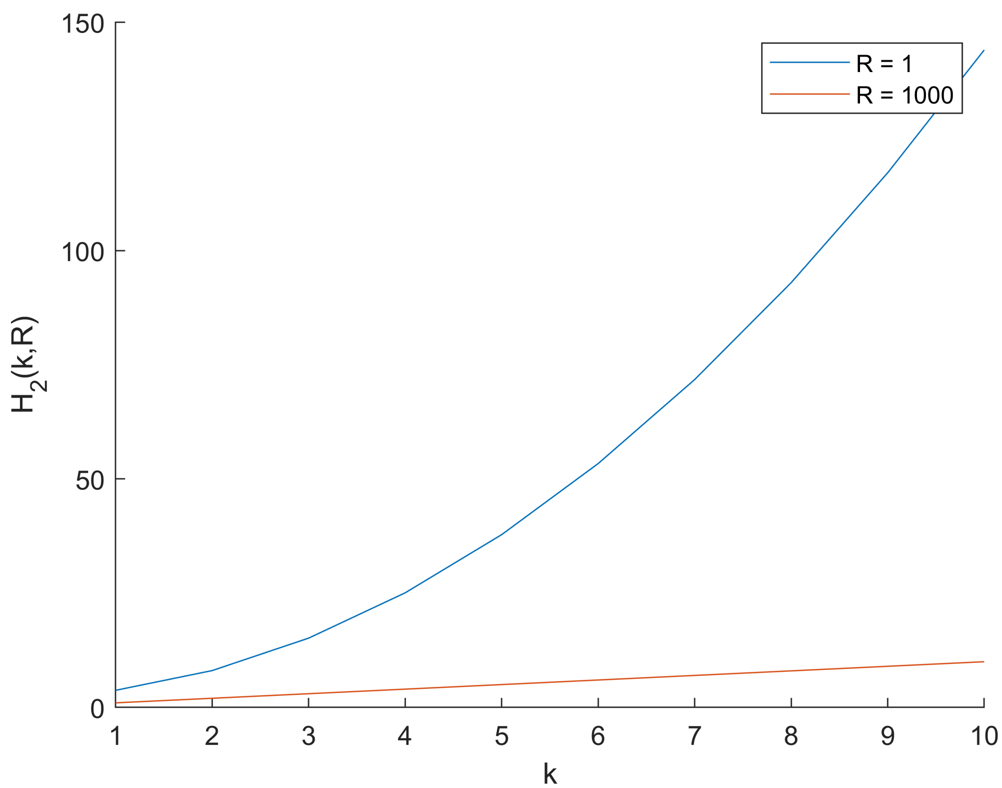
$$\sqrt{\frac{\left|3\,R^2-\sigma_1+R^4+\frac{R^2\,k^2}{2}+\frac{5}{2}\right|}{k^2}+\frac{\sigma_1+3\,R^2+R^4+\frac{R^2\,k^2}{2}+\frac{5}{2}}{k^2}}$$

where

$$\sigma_1=\frac{\sqrt{(R^2\,k^2+1)\,\left(4\,R^4+R^2\,k^2+12\,R^2+9\right)}}{2}$$

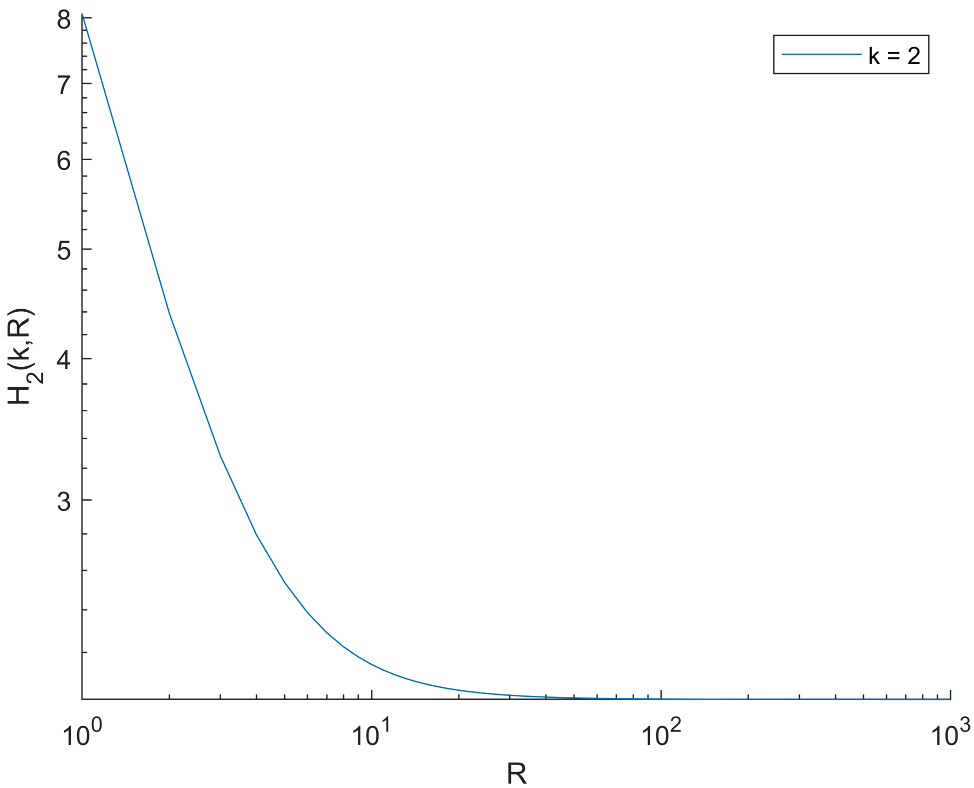
Ploting H_2(k,R) vs k

```
k_val = 1:10;
R_val = [1, 1000];

figure
hold on
for idx = 1:2
    plot(k_val, H_2_norm_fun(R_val(idx), k_val), ...
        'DisplayName', ['R = ', num2str(R_val(idx))]);
end
xlabel('k')
ylabel('H_2(k,R)')
legend
```



Ploting H_2(k,R) vs R

```
k_val = [2];
R_val = 1:1000;

figure
hold on
for idx = 1:1
    plot(R_val, H_2_norm_fun(R_val, k_val(idx)), ...
        'DisplayName', ['k = ', num2str(k_val(idx))]);
end
xlabel('R')
ylabel('H_2(k,R)')
legend
set(gca, 'YScale', 'log')
set(gca, 'XScale', 'log')
```



Part c

```
B = [1 0]';
C = [0 1];
D = 0;

G = partfrac(C * inv(sym('s')*eye(2) - A) * B + D, sym('s'), 'FactorMode', 'full')
```

G =

$$\frac{R\,k}{s+\frac{k^2+1}{R}} - \frac{R\,k}{s+\frac{k^2+2}{R}}$$

Therefore we have:

```
A_1 = R *k;      lambda_1 = -(k^2 + 1) / R;
A_2 = - R *k;    lambda_2 = -(k^2 + 2) / R;
G_t = A_1 * exp(lambda_1 * sym('t')) + A_2 * exp(lambda_2 * sym('t'))
```

G_t =

$$R\,k\,\mathrm{e}^{-\frac{t\,(k^2+1)}{R}} - R\,k\,\mathrm{e}^{-\frac{t\,(k^2+2)}{R}}$$

or directly

```
G_t = C * expm(A*sym('t')) * B + D
```

G_t =

$$R\,k\,\mathrm{e}^{-\frac{t\,k^2+t}{R}} - R\,k\,\mathrm{e}^{-\frac{t\,k^2+2\,t}{R}}$$

Part d

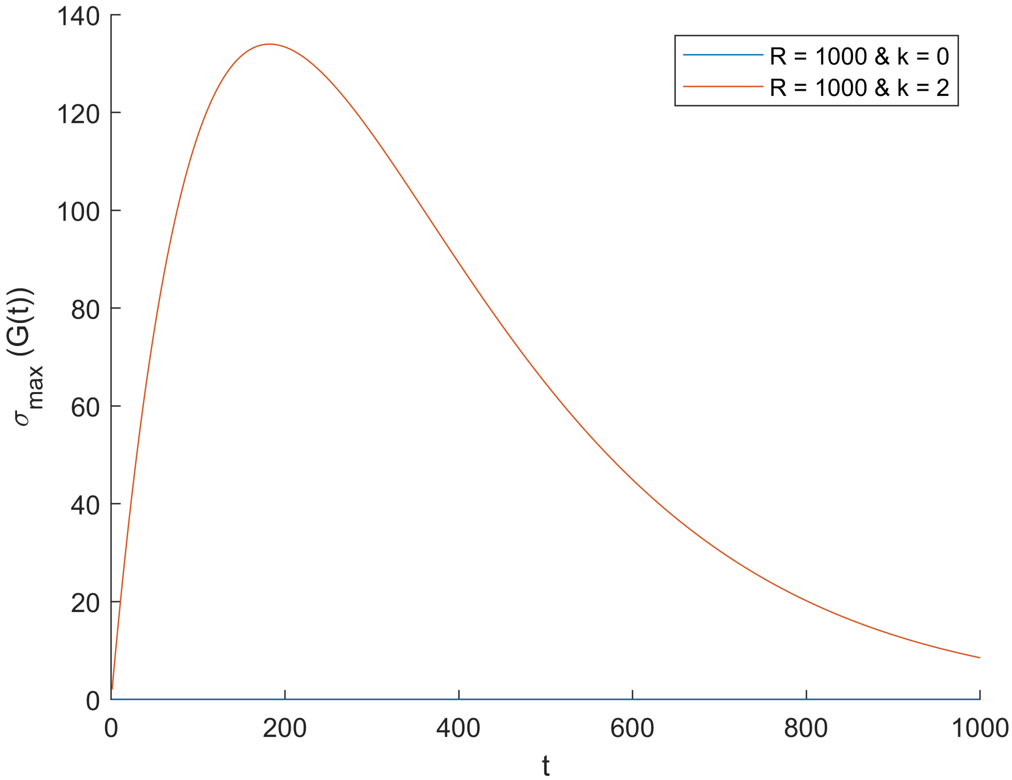
```
sigma_max_fun = matlabFunction(svd(G_t), 'Vars', [R, k, sym('t')]); % SISO system...
sigma_max = sigma_max_fun(R, k, sym('t'))
```

sigma_max =

$$\sqrt{-\left(R\,k\,\mathrm{e}^{-\frac{\bar{t}\,k^2+2\,\bar{t}}{R}} - R\,k\,\mathrm{e}^{-\frac{\bar{t}\,k^2+\bar{t}}{R}}\right)\left(R\,k\,\mathrm{e}^{-\frac{t\,k^2+t}{R}} - R\,k\,\mathrm{e}^{-\frac{t\,k^2+2\,t}{R}}\right)}$$

Ploting

```
T = 1:1000;
R_val = [1000, 1000];
k_val = [0, 2];

figure
hold on
for idx = 1:2
    plot(T, sigma_max_fun(R_val(idx), k_val(idx), T), ...
        'DisplayName', ['R = ', num2str(R_val(idx)), ' & k = ', num2str(k_val(idx))]);
end

ylabel('\sigma_{max} (G(t))')
xlabel('t')
legend
```



Problem 3

Part a

(a) Prove that $\underline{\sigma} = \min_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$.

(assuming A nonsingular)

Let $Ax = y$

$x = A^{-1} y$

$$\underline{\sigma} = \min \frac{\|Ax\|_2}{\|x\|_2}$$

a) $$\frac{1}{\underline{\sigma}} = \sup \frac{\|x\|_2}{\|Ax\|_2}$$

$$\frac{1}{\underline{\sigma}} = \sup \frac{\|A^{-1} y\|_2}{\|y\|_2} = \bar{\sigma}(A^{-1})$$

Part b

$$A = U \Sigma V^{-1}$$

$$A^{-1} = V \Sigma^{-1} U^{-1}$$

$$\sigma_i(A^{-1}) = \frac{1}{\sigma_j(A)}, \quad j + i = m + 1$$

$$\bar{\sigma}(A^{-1}) = \frac{1}{\underline{\sigma}(A)}$$

Part c

```
a = sym('a', 'positive');
b = sym('b', 'positive');
epsilon = sym('epsilon', 'positive');

A = [
    -a epsilon
    0   -b
    ]
```

A =

$$\begin{pmatrix} -a & \varepsilon \\ 0 & -b \end{pmatrix}$$

```
Lambda = eig(A)
```

Lambda =

$$\begin{pmatrix} -a \\ -b \end{pmatrix}$$

```
S = svd(A)
```

S =

$$\begin{pmatrix} \sqrt{\frac{a^2}{2} - \sigma_1 + \frac{b^2}{2} + \frac{\varepsilon^2}{2}} \\ \sqrt{\sigma_1 + \frac{a^2}{2} + \frac{b^2}{2} + \frac{\varepsilon^2}{2}} \end{pmatrix}$$

where

$$\sigma_1 = \frac{\sqrt{(a^2 - 2\,a\,b + b^2 + \varepsilon^2)\,(a^2 + 2\,a\,b + b^2 + \varepsilon^2)}}{2}$$

```
sigma_lim_inf = limit(S, epsilon, inf)
```

sigma_lim_inf =

$$\begin{pmatrix} 0 \\ \infty \end{pmatrix}$$

Part d

```
% a = sym('a')
% A =
B = sym('B', 2, 'real');
C = sym('C', 2, 'real');
D = sym('D', 2, 'real');

% TF
s = sym('s');
G = C \ (s * eye(2) - A) * B + D
```

G =

$$\begin{pmatrix} D_{1,1} + \frac{B_{1,1}\,(C_{2,2}\,a + C_{2,2}\,s)}{\sigma_1} - \frac{B_{2,1}\,\sigma_2}{\sigma_1} & D_{1,2} + \frac{B_{1,2}\,(C_{2,2}\,a + C_{2,2}\,s)}{\sigma_1} - \frac{B_{2,2}\,\sigma_2}{\sigma_1} \\ D_{2,1} - \frac{B_{1,1}\,(C_{2,1}\,a + C_{2,1}\,s)}{\sigma_1} + \frac{B_{2,1}\,\sigma_3}{\sigma_1} & D_{2,2} - \frac{B_{1,2}\,(C_{2,1}\,a + C_{2,1}\,s)}{\sigma_1} + \frac{B_{2,2}\,\sigma_3}{\sigma_1} \end{pmatrix}$$

where

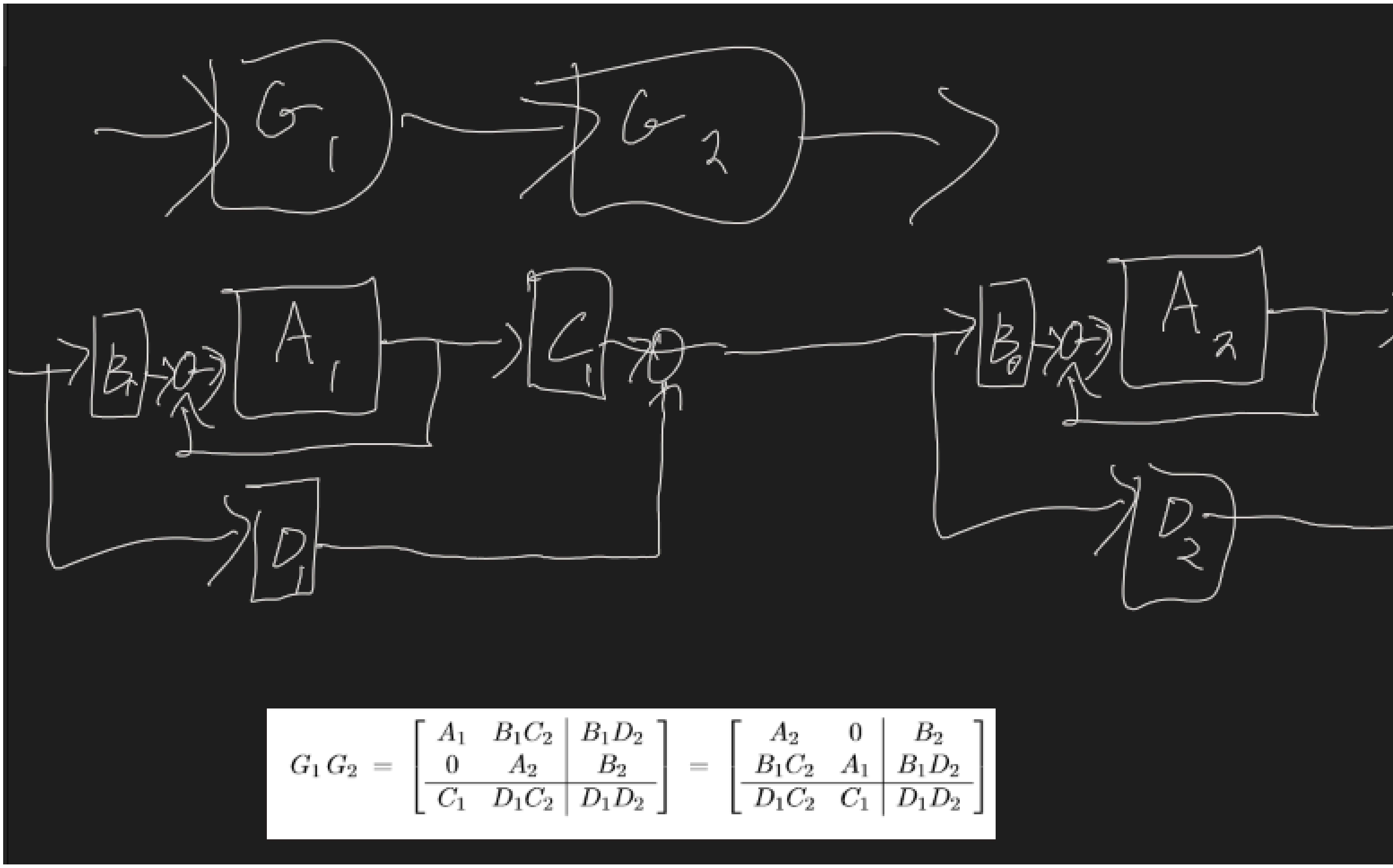
$$\sigma_1 = C_{1,1}\,C_{2,2} - C_{1,2}\,C_{2,1}$$

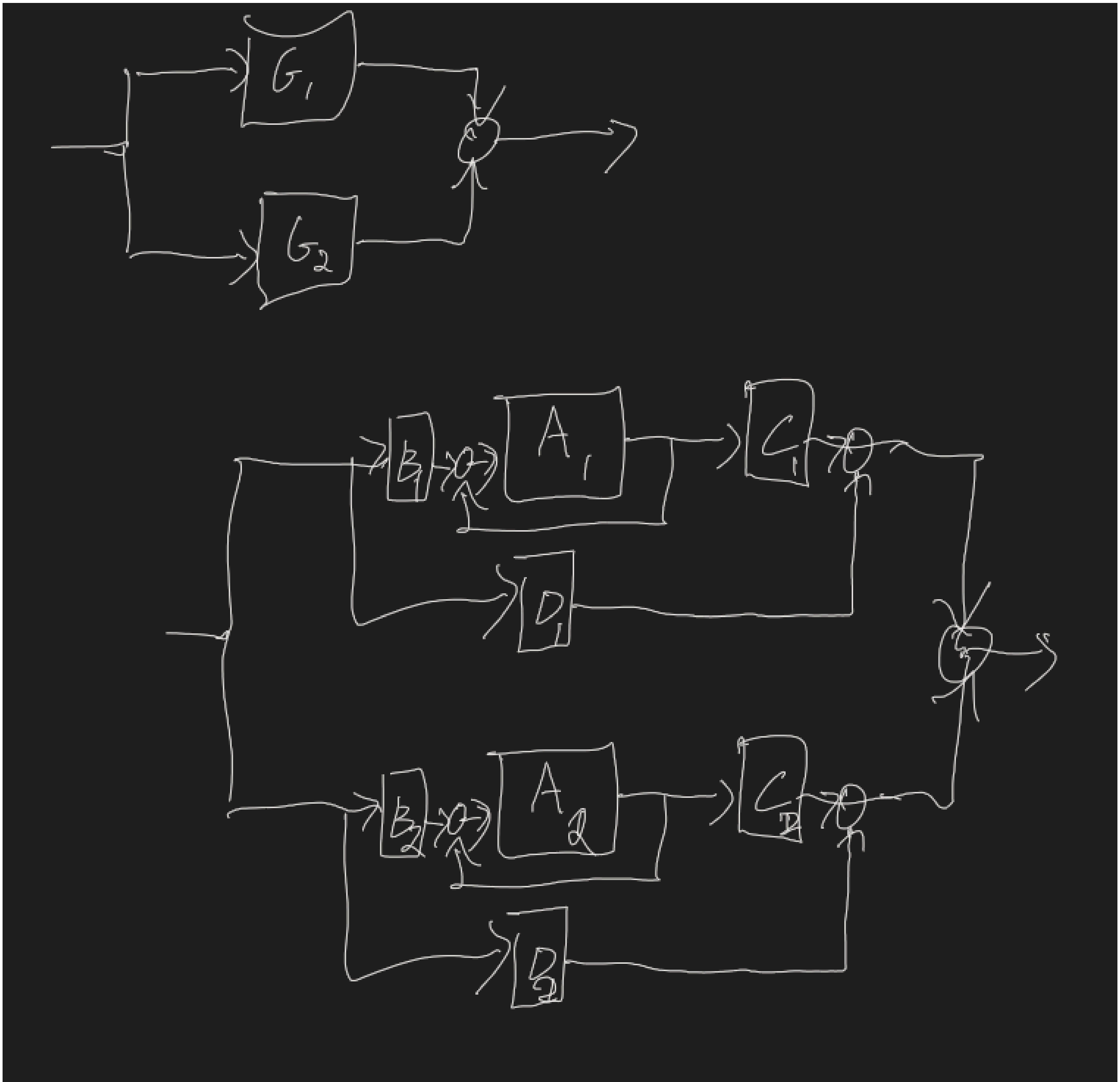
$$\sigma_2 = C_{1,2}\,b + C_{2,2}\,\varepsilon + C_{1,2}\,s$$

$$\sigma_3 = C_{1,1}\,b + C_{2,1}\,\varepsilon + C_{1,1}\,s$$

```
% zeros(G)
% Lambda = eig(G)
% epsilonInLambda = hasSymType(Lambda,'symfunOf', epsilon)
% S = svd(G)
```

Problem 4


G_1
G_2
B_1
A_1
C_1
D_1
B_2
A_2
D_2
$$G_1 G_2 = \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1C_2 & A_1 & B_1D_2 \\ \hline D_1C_2 & C_1 & D_1D_2 \end{array}\right]$$

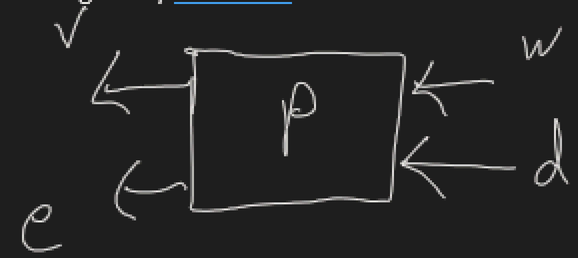


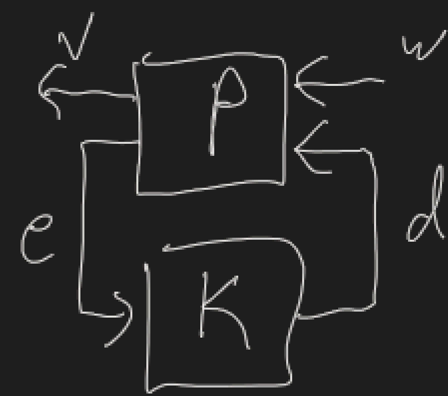
$$G_1 + G_2 = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

Note: Very simple... just ran out of time...

Problem 5

Referring to my LFT Notes





$$\begin{pmatrix} v \\ e \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} w \\ d \end{pmatrix}$$

$$T = F_\ell(P,K) = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}$$

$$= GK(I + GK)^{-1}$$

$$\Downarrow$$

$$P_{11} = 0$$

$$P_{12} = G$$

$$P_{22} = -G$$

$$P_{21} = I$$

$$\boxed{P = \begin{bmatrix} 0 & G \\ I & -G \end{bmatrix}}$$

Problem 6

Note: note great....

Problem 7

$$F_l\left(H, \frac{1}{s}\right) = C(sI - A)^{-1}B + D$$

$$= H_{11} + H_{12}\left(\frac{1}{s}\right)\left(I - H_{22}\left(\frac{1}{s}\right)\right)^{-1} H_{21}$$

$$= H_{12}\left(\frac{1}{s}\right)\left(\frac{1}{s}\left[sI - H_{22}\right]\right)^{-1} H_{21} + H_{11}$$

$$= H_{12}(sI - H_{22})^{-1} H_{21} + H_{11}$$

$$\Downarrow$$

$$H_{11} = D$$

$$H_{12} = C$$

$$H_{21} = B$$

$$H_{22} = A$$

$$\boxed{H = \begin{bmatrix} D & C \\ B & A \end{bmatrix}}$$