MECH 6323 - HW 2

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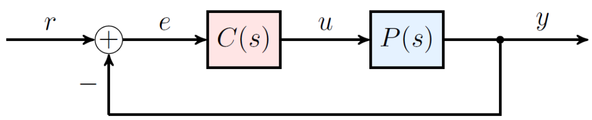
1 Problem 1

Consider the negative feedback interconnection:



1.1

Problem: If possible, give an example of P and C transfer functions such that $\frac{1}{1+PC}$ and $\frac{P}{1+PC}$ are stable, but $\frac{C}{1+PC}$ is not.

Solution:

$$P(s) =$$

$$C(s) =$$

These then produces the following stable transfer functions:

$$\frac{1}{1+PC} =$$

$$\frac{P}{1+PC} =$$

However, then this transfer function is unstable:

$$\frac{C}{1+PC} =$$

2 Problem 2

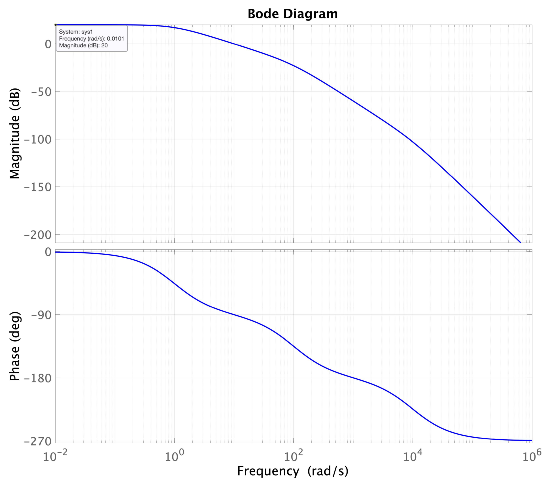
3 Problem 3

Problem: For each of the bode plots:

1. Determine the breakpoints and the transfer function.

2. Determine the gain cross-over frequency ω_c and the phase cross-over frequency ω_{180}.

3.1 Bode Plot 1:



3.1.1 Gain, Poles, and Zeros:

1. **Gain:** 20 db = 10

2. **Poles:**

   (a) $10^0 = 1$ rad/s

   (b) $10^2 = 100$ rad/s

   (c) $10^4 = 10,000$ rad/s

3. Zeros: (NA)

Transfer Function:

$$H(s) = \frac{10}{\left(1+\frac{s}{1}\right)\left(1+\frac{s}{100}\right)\left(1+\frac{s}{10000}\right)}$$

3.1.2 Cross-over Frequency:

1. $\omega_c = 10^1 = 10$ rad/s

2. $\omega_{180} = 10^3 = 100$ rad/s

3.2 Bode Plot 2:

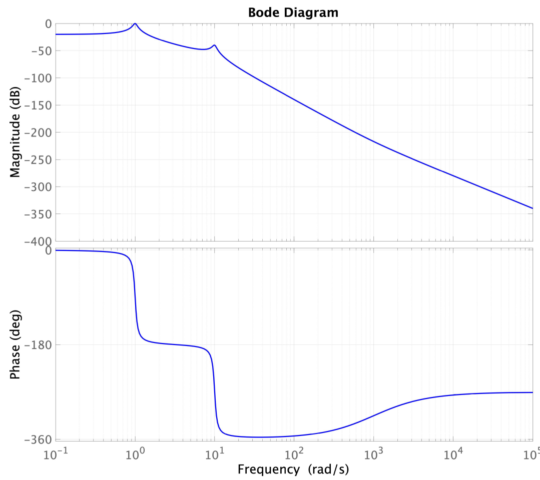


Figure 1: Bode Plot 2

3.2.1 Gain, Poles, and Zeros:

1. **Gain:** -20 db $= \frac{1}{10}$
2. **Poles:**
   (a) $10^0 = 1$ rad/s (complex)
   (b) $10^1 = 10$ rad/s (complex)
3. Zeros:
   (a) $10^3 = 1,000$ rad/s

3.2.2 Transfer Function:

$$H(s) = \frac{(1+\frac{s}{1000})}{10\big(\frac{1}{1}\big(s^2+2(\frac{1}{10})(1)s+(1)^2\big)\big)\big(\frac{1}{10}\big(s^2+2(\frac{1}{10})(10)s+(10)^2\big)\big)} = \frac{(s+1000)}{(s^2+0.2s+1)(s^2+2s+100)}$$

Assuming a Q-factor of around 10 to get the complex response.

3.2.3 Cross-over Frequency:

1. $\omega_c = 10^0 = 1$ rad/s
2. $\omega_{180} = 10^3 = 100$ rad/s

4 Problem 4

Consider the interconection of Problem 1 with the PI controller

$$C(s) = \frac{10(s+3)}{s}$$

and plant

$$P(s) = \frac{-0.5(s^2 - 2000)}{(s-3)(s^2 + 50s + 1000)}$$

4.1 Is the feedback system stable? Why?

$$\begin{aligned}
\frac{C(s)P(s)}{1 + C(s)P(s)} &= \frac{\frac{10(s+3)}{s}\frac{-0.5(s^2-2000)}{(s-3)(s^2+50s+1000)}}{1 + \frac{10(s+3)}{s}\frac{-0.5(s^2-2000)}{(s-3)(s^2+50s+1000)}} \\
&\approx \frac{-5s(s+3)(s-3)(s^2-2000)(s^2+50s+1000)}{s(s-3)(s^2+11.73s+73.9)(s^2+30.27s+406)(s^2+50s+1000)}
\end{aligned}$$

Yes and No. Internally it is not fully stable since it has a pole/zero pair at $s = 3$; however, if we only care about TF after cancellations, then it is stable.

4.2 Find phase and cross-over frequencies.

Problem: Use the Bode plot of the open loop transfer function $L(s) = C(s)P(s)$ to find the phase cross-over frequencies ω_0 such that $L(j\omega_0) = 180\,\text{deg}$. Use this information to compute the gain margin(s) of the feedback system. Check your answers using the *allmargin* command in MATLAB.

Solution: As marked in the Bode Plot seen in Figure 4.2, the gain cross-over frequency is $\omega_c = 10$ resulting in a phase margin around 25 deg. Similarly, the phase cross-over occurs around $\omega_{180} = 4$ or $\omega_{180} = 25$, resulting in gain margins of around ± 10 dB or around $g_0 = 0.3$ and $g_0 = 3$ respectively.

Verification with *allmargin* resulted in similar and likely more precise and accurate results:

1. Gain Margin(s)
   (a) $g_0 = 0.3585$ at $\omega_{180} = 3.5966$
   (b) $g_0 = 2.6490$ at $\omega_{180} = 26.3797$
2. Phase Margin
   (a) $PM = 27.5718$ at $\omega_c = 10.2049$

4.3 Gain Margin Closed-loop poles

Problem: For each gain margin g_0 obtained in the previous part, construct the closed-loop using the perturbed loop transfer function $g_0 L(s)$ and verify that the closed-loop has poles at $\pm\omega_0$.

Solution:

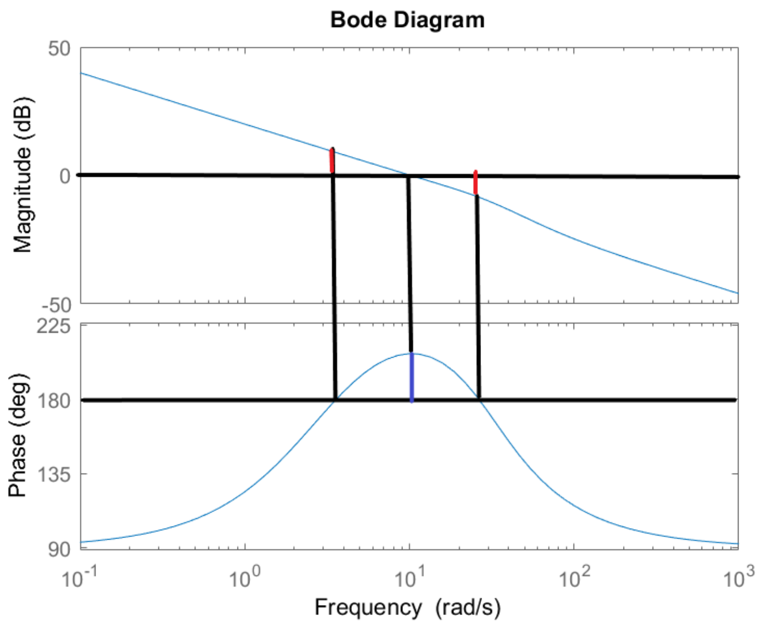


Figure 2: Open Loop Bode Plot of $L(S) = C(s)P(s)$

4.3.1 $g_0 = 0.3585$

Let $g_0 = 0.3585$,

$$g_0L(s) = \frac{-1.7927(s+3)(s+44.72)(s-44.72)}{s(s-3)(s^2+50s+1000)}$$

and the closed-loop becomes

$$H = \frac{-1.7927(s-44.72)(s+44.72)(s+3)}{(s^2+0.0008164s+12.93)(s^2+45.21s+831.7)}$$

This has complex poles located at $-0.0004 \pm j3.596$, which is essentially roots at $\pm j\omega_{180}$.

4.3.2 $g_0 = 2.6490$

Let $g_0 = 2.6490$,

$$g_0L(s) = \frac{-13.245(s+3)(s+44.72)(s-44.72)}{s(s-3)(s^2+50s+1000)}$$

and the closed-loop becomes

$$H = \frac{-13.245(s+3)(s+44.72)(s-44.72)}{(s+29.94)(s+3.814)(s^2+0.003402s+696)}$$

This has complex poles located at $-0.0017 \pm j26.3818$, which is essentially roots at $\pm j\omega_{180}$.

4.4 $\|S-T\|_\infty$

Problem: Compute $\|S-T\|_\infty$ and the corresponding frequency ω_p where the peak gain of $S-T$ is achieved.

Solution:

Sensitivity TF:

$$S(s) = \frac{1}{1+PC} = \frac{s(s-3)(s^2+50s+1000)}{(s^2+11.73s+73.9)(s^2+30.27s+406)}$$

Complementary Sensitivity TF:

$$T(s) = \frac{PC}{1+PC} = \frac{-5s(s-44.72)(s+44.72)(s+3)(s-3)(s^2+50s+1000)}{s(s-3)(s^2+11.73s+73.9)(s^2+30.27s+406)(s^2+50s+1000)}$$

$S(s) - T(s)$:

$$S(s)-T(s) = \frac{s(s-10.45)(s-3)(s+2.059)(s^2+11.73s+73.9)(s^2+30.27s+406)(s^2+50s+1000)(s^2+60.4s+1394)}{s(s-3)(s^2+11.73s+73.9)^2(s^2+30.27s+406)^2(s^2+50s+1000)}$$

$$= \frac{(s-10.45)(s+2.059)(s^2+60.4s+1394)}{(s^2+11.73s+73.9)(s^2+30.27s+406)}$$

Results:

$$\|S-T\|_\infty = 4.0763$$

$$\omega_p = 10.0798$$

4.5 Symmetric Disk Margin

Problem: What is the symmetric disk margin m for this plant and controller? Verify your answer using *dm = diskmargin(P*C)*. Note that the *diskmargin* command uses the convention m = *dm.DiskMargin / 2*.

Solution: By the Symmetric Disk Margin theorem, the disk margin defined for

$$\alpha \in \text{Disk}\left(\frac{1-m}{1+m}, \frac{1+m}{1-m}\right)$$

is the region that $C(s)$ stabilizes $\alpha P(s)$ for $m < 1$ satisfying

$$\|S-T\|_\infty \leq \frac{1}{m}$$

Therefore,

$$\overline{m}_{st} = \frac{1}{\|S-T\|_\infty} = \frac{1}{4.0763} = 0.453$$

4.6 α on Disk boundary

Problem: Construct an α on the boundary of Disk

$$\text{Disk}\left(\frac{1-m}{1+m}, \frac{1+m}{1-m}\right)$$

such that the perturbed closed-loop

$$S_\alpha = \frac{1}{1+\alpha L(s)}$$

has a pole at $j\omega_p$. Verify your construction by forming S_α and demonstrating that it has a pole at $j\omega_p$. Hint: Assume $\|S-T\|_\infty = \frac{1}{m}$ at frequency ω_p. Then there exists a complex number $S(j\omega_p) - T(j\omega_p) = \frac{1}{z}$ where $|z| = m$. Algebraically show that

$$\alpha = \frac{1+z}{1-z}$$

satisfies $1 + \alpha L(j\omega_p) = 0$ and this α is in the symmetric disk defined by m.

Solution: α can be constructed by first finding $z_0 = \frac{1}{S(j\omega_p) - T(j\omega_p)}$, calculating $z = m * \frac{z_0}{|z_0|}$, and then finding $\alpha = \frac{1+z}{1-z}$.

As demonstrated in MATLAB, this results in $\alpha = 0.8760 - j0.4572$. This is then verified as

$$\begin{aligned}
1 + \alpha L(j\omega_p) &\approx (0.8760 - j0.4572) \frac{-5(j10.0798 + 3)(j10.0798 + 44.72)(j10.0798 - 44.72)}{(j10.0798)(j10.0798 - 3)((j10.0798^2 + 50(j10.0798) + 1000)} \\
&= -2.7529e - 08 - j1.2029e - 06 \\
&\approx 0
\end{aligned}$$

A MATLAB Code:

All code I write in this course can be found on my GitHub repository:
https://github.com/jonaswagner2826/MECH6323