MECH 6323 - HW 5

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Contents

Preliminaries

Definition 1. ***Matrix Basics:*** *For $A \in \mathbb{C}^{n \times m}$ and $x \in \mathbb{C}^m$,*

1. *The* Eigenvalues *(λ_i) and* Eigenvectors *(x_i) of A are defined as the solutions to*

$$\lambda_i A = \lambda_i x_i$$

2. *The* Spectral Radius *of A is defined as*

$$\rho(A) := \max_i |\lambda_i(A)|$$

3. *The* Complex Conjugate Transpose *of A, denoted as A^*, is defined so that*

$$\Re(A^*) = \Re(A^T)$$

and

$$\Im(A^*) = -\Im(A^T)$$

Definition 2. ***Vector Norms:*** *For $x \in \mathbb{C}^n$,*

1. *The* 2-norm*, or* Euclidean norm*, is defined as*

$$\|x\|_2 := \sqrt{\sum_{i=1}^{n} x_i^2}$$

2. *The* 1-norm *is defined as*

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|$$

3. *The* ∞-norm *is defined as*

$$\|x\|_\infty := \max i = 1, \ldots, n |x_i|$$

4. *The* p-norm *is defined as*

$$\|x\|_p := \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}}$$

Definition 3. ***Matrix Norms:*** *For $A \in \mathbb{C}^{n \times m}$ and $x \in \mathbb{C}^m$,*

1. *The* Induced 2-norm *is defined as*

$$\|A\|_{2 \to 2} := \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

and is also known as the spectral norm *and has the additional properties of*

(a) $\|A\|_{2 \to 2} = \sqrt{\lambda_{max}(A^* A)} = \overline{\sigma}(A)$

(b) $\|A^* A\|_{2 \to 2} = \|A A^*\|_{2 \to 2} = \|A\|_2^2$

Definition 4. ***Closed-loop Transfer Functions:*** *Let P and C represent the plant and controller transfer functions respectively. Within a standard unity feedback system,*

1. *the* <u>sensitivity closed-loop transfer function</u> *is defined as:*

$$S = \frac{1}{1 + PC}$$

2. *the* <u>complementary sensitivity closed-loop transfer function</u> *is defined as:*

$$T = \frac{PC}{1 + PC}$$

Theorem 1. ***Singular Value Decomposition:*** *Any matrix $M \in \mathbb{C}^{m \times n}$ can be decomposed as*

$$M = U \Sigma V^*$$

with $U \in \mathbb{C}^{n \times n}$ and $V \in \mathbb{C}^{m \times m}$ are unitary. Additionally, $\Sigma \in \mathbb{R}^{n \times m}$ was rectangular diagonal with

$$\Sigma = \begin{bmatrix} \hat{\Sigma} & 0_{n \times (m-n)} \end{bmatrix}, \ m \geq n \qquad\qquad \Sigma = \begin{bmatrix} \hat{\Sigma} \\ 0_{n \times (m-n)} \end{bmatrix}, \ n > m$$

and

$$\hat{\Sigma} = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_k \end{bmatrix} \in \mathbb{R}^{k \times k}, \quad k = \min(n, m), \quad (\sigma_1 \geq \sigma_2 \geq \cdots \geq 0)$$

Additionally, U and V are orthonormal, $UU^* = I = VV^*$.

1. *We have*

$$M v_i = \sigma_i u_i$$

and

$$M = \sum_{i=1}^{k} \sigma_i u_i v_i^*$$

2. *SVD is related to the Eigenvalues and Eigenvectors.*

$$\lambda_i(M^* M) = \sigma_i^2(M)$$

with corresponding eigenvectors v_i and

$$\lambda_i(M M^*) = \sigma_i^2(M)$$

with corresponding eigenvectors u_i.

Theorem 2. ***Small-gain Theorem:*** *Consider an $n_y \times n_u$ LTI system M and $\Delta \in \mathbb{C}^{n_u \times n_y}$.*

Assuming M is stable, then the feedback system in Figure 1 is stable for all $\overline{\sigma}(\Delta) < m$ if and only if

$$\|M\|_\infty \leq \frac{1}{m}$$

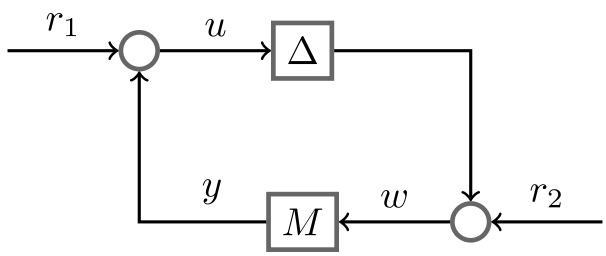

Figure 1: Small-gain theorem feedback system

1 Problem 1

Consider the feedback configuration in the figure below.

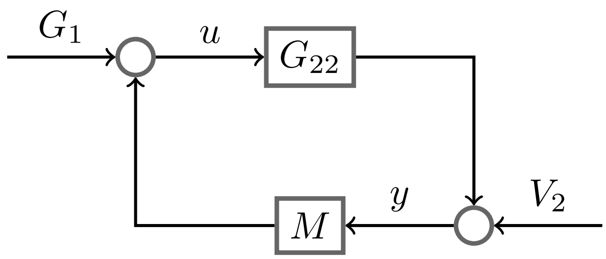

Figure 2: Problem 1 Feedback Configuration

Prove that

$$\begin{bmatrix} I & -K \\ -G_{22} & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - KG_{22})^{-1} & (I - KG_{22})^{-1}K \\ (I - G_{22}K)^{-1}G_22 & (I - G_{22}K)^{-1} \end{bmatrix} = H(G_{22}, K)$$

2 Problem 2

Consider the block diagram in the previous exercise. Suppose G_{22} and K have minimal state space realizations $\begin{bmatrix} A & B_2 \\ C_2 & D_{22} \end{bmatrix}$ and $\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$. Let

$$T = \begin{bmatrix} A_1 = \begin{bmatrix} A & \\ & A_K \end{bmatrix} & B = \begin{bmatrix} B_2 & \\ & B_k \end{bmatrix} \\ -C = \begin{bmatrix} 0 & -C_K \\ -C_2 & 0 \end{bmatrix} & D = \begin{bmatrix} I & -D_K \\ -D_2 2 & I \end{bmatrix} \end{bmatrix}$$

Thus,

$$T^{-1} = \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} := \begin{bmatrix} A_1 + BD^{-1}C & BD^{-1} \\ D^{-1}C & D^{-1} \end{bmatrix}$$

where

$$\begin{aligned} \overline{D} = D^{-1} &= \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} I + (I - D_{22}D_K)^{-1}D_{22} & D_K(I - D_{22}D_K)^{-1} \\ (I - D_{22}D_K)^{-1}D_2 2 & (I - D_{22}D_K)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} (I - D_{22}D_K)^{-1}D_{22} & D_K(I - D_{22}D_K)^{-1} \\ (I - D_{22}D_K)^{-1}D_2 2 & (I - D_{22}D_K)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} D_K \\ I \end{bmatrix} (I - D_{22}D_K)^{-1} \begin{bmatrix} D_2 2 & I \end{bmatrix} \end{aligned}$$

Thus,

$$\overline{A} = A_1 + BD^{-1}C = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_2 D_K \\ B_K \end{bmatrix} (I - D_{22}D_K)^{-1} \begin{bmatrix} C_2 \\ D_2 2C_K \end{bmatrix}$$

Prove that the following are equivalent:

1. $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ is stabilizable and detectable.
2. (A, B_2, C_2, D_{22}) and (A_K, B_K, C_K, D_K) are stabilizable and detectable.

3 Problem 3

For the feedback configuration in Figure 2, prove that:

1. If K is stable then the closed loop interconnection is stable if and only if $G_{22}(I - KG_{22})^{-1}$ is stable.

2. If G_{22} is stable then the closed loop interconnection is stable if and only if $K(I - G_{22}K)^{-1}$ is stable.

4 Problem 4

Consider the standard negative feedback loop with the nominal plant dynamics $P(s) = \frac{2}{s+1}$ and controller $K(s) = 20$. Assume the "true" dynamics lie within the following multiplicative uncertainty set:

$$\mathcal{M} := \left\{ \hat{P} = P(1 + W_u\Delta) : \|\Delta\|_\infty < 1 \text{ and } \Delta \text{ stable} \right\}$$

Assume the uncertainty weight is $W_u(s) = \frac{2s+1}{s+10}$.

1. Provide an interpretation for the uncertainty describe by the weight W_u.

2. Is the nominal feedback system stable? What are the gain and phase margins of the nominal loop $L = PK$?

3. The robust stability condition for this type of multiplicative uncertainty is stated as: K stabilizes all $\hat{P} \in \mathcal{M}$ if and only if $\|W_u T\|_\infty \leq 1$. Does K robustly stabilize all models in $\mathcal{M}$ based on this condition?

4. We can construct the uncertainty set M in MATLAB using the following commands:

   $>>$ Delta = ultidyn(0Delta0; [1 1]);
   $>>$ Phat = P * (1 + Wu * Delta);
   $>>$ Lhat = Phat * K;
   $>>$ That = feedback(Lhat; 1);

   The ultidyn command constructs an uncertain, LTI transfer function object that satisfies $\|\Delta\|_\infty < 1$. The function inputs are the name and input/output size of the object. Construct the uncertain model Phat using the commands above. Generate a Bode magnitude plot with 10 samples drawn from the uncertainty set and draw the nominal response P on the same plot. Note: The command bodemag(Phat) will automatically generate 10 samples and draw their plots.

5. Finally, we can perform the robustness test using the following command:

   $>>$ [stabmarg; destabunc; report] = robstab(That)

   Refer to the function documentation for robstab for a short description of the input/output arguments. Does the result obtained with robstab agree with your conclusions in part (c)?

A MATLAB Code:

See attached. Additionally, all the code I write in this course can be found on my GitHub repository:
https://github.com/jonaswagner2826/MECH6323