Assignment 4

Optimal Estimation and Kalman Filter
Fall 2020
Due date: October 30, 2020 at 1:00 PM

Problem 1. Show that the Kalman filter for an LTI system with a noise-free scalar measurement that satisfies $(HQH^T)Q = QH^THQ$ has a steady-state *a posteriori* covariance of zero.

Problem 2. Consider the scalar system

$$\begin{aligned} x_k &= x_{k-1} + w_{k-1} \\ y_k &= x_k + v_k \end{aligned}$$

where $w_k \sim (0, 1)$ and $v_k \sim (0, 1)$ are white noise processes. Suppose that $E(w_k v_{k+1}) = 1$.

a. Design a Kalman filter that ignores the correlation between w_k and v_{k+1}. What is the steady-state *a posteriori* estimation covariance with this incorrect Kalman filter?

b. For the Kalman filter designed above, write a recursive equation for the *a posteriori* estimation error $e_k = x_k - \hat{x}_k^+$. Use this equation to find the steady-state solution to $E(e_k^2)$.

c. Design a Kalman filter that takes into account the correlation between w_k and v_{k+1}. Show that the steady-state *a posteriori* estimation covariance is zero.

d. Explain why the estimation covariance goes to zero in spite of the existence of the process and measurement noise.

Problem 3. Consider the scalar system

$$\begin{aligned} x_k &= \frac{1}{2} x_{k-1} + w_{k-1} \\ y_k &= x_k + v_k \\ v_k &= \frac{1}{2} v_{k-1} + \zeta_{k-1} \end{aligned}$$

where $w_k \sim (0, 1)$ and $\zeta_k \sim (0, 1)$.

a. Design a Kalman filter that ignores the dynamics of the measurement noise v_k, but assumes that $v_k \sim (0, 1)$. What is the steady-state *a posteriori* estimation covariance with this incorrect Kalman filter?

b. With the above incorrect Kalman filter, what is the true steady-state *a posteriori* estimation covariance $E(e_k^2)$?

c. Design a Kalman filter that takes into account the dynamics of the measurement noise. What is the steady-state *a posteriori* estimation covariance?

d. Simulate in MATLAB both Kalman Filters. Numerically calculate *a posteriori* estimation covariance and compare with the analytically calculated values above.

Problem 4. The steady-state, zero-input, one-step formulation for the a posteriori Kalman filter is

$$\begin{aligned}\hat{x}_k^+ &= (I - KH)F\hat{x}_{k-1}^+ + Ky_k \\ \hat{y}_k &= H\hat{x}_k^+\end{aligned}$$

Prove that if (F, H) is observable and $(I - HK)$ is full rank, then the Kalman filter is an observable system. Hint: $H(I - KH) = (I - HK)H$.