

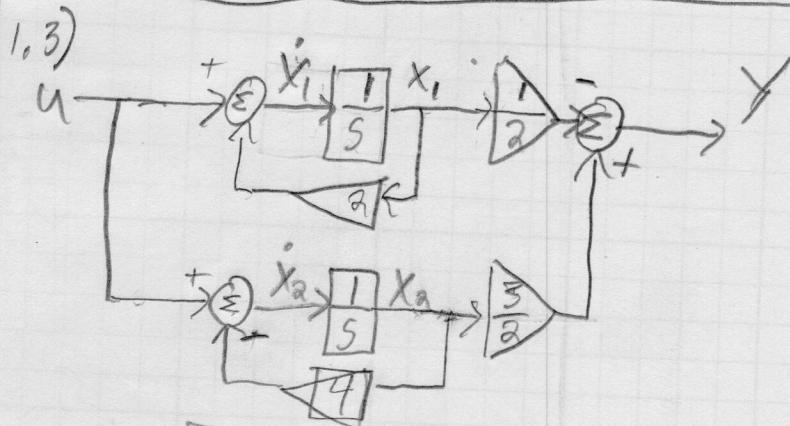
$$1) S_x(w) = \frac{w^2 + 1}{w^4 + 20w^2 + 64}$$

$$1.1) S_y(w) = \frac{w^2 + 1}{(w^2 + 4)(w^2 + 16)}$$

$$G(w) G(-w) = S_y(w) = \frac{(1+jw)(1+j(-w))}{(2+jw)(2+j(-w))(4+jw)(4+j(-w))}$$

$$1.2) G(s) = \boxed{\frac{s+1}{s+2(s+4)}}$$

$$G(s) = \frac{-1}{2(s+2)} + \frac{3}{2(s+4)}$$



$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}x + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \end{bmatrix}x$$

2)d) cont.

$$\text{defining } \tilde{x} = ne^{-2x} + w$$

$$\tilde{x} = x - v$$

$$A = -2 \quad L = Q_C = 2$$

$$E = 1 \quad G = R_C = 1$$

$$\hat{x}(0) = E[x(0)]$$

$$P(0) = E[(x(0) - \hat{x}(0))(x(0) - \hat{x}(0))^T]$$

$$K = P C^T R^{-1} = P(I)(I)^{-1} = P$$

$$\dot{\hat{x}} = \tilde{A}\hat{x} + \tilde{B}u + K(y - \tilde{C}\hat{x}) = -2\hat{x} + K(y - \hat{x})$$

$$\dot{P} = -P C^T R^{-1} C P + \tilde{A}P + PA^T + Q$$

$$\dot{P} = -P^2 - 4P + 2$$

$$\dot{\hat{x}} = -2\hat{x} + K(y - \hat{x})$$

$$2.1) \text{ SS: } 0 = P^2 + 4P - 2$$

$$P = \frac{-4 \pm \sqrt{16 - 4(-2)}}{2} = -2 \pm \sqrt{16}$$

$$P = \sqrt{16} - 2 \approx 0.449$$

$$2.2) \quad F = I + AT, \quad Q = Q_C(T),$$

$$H = C \quad R = \frac{R_C}{T}$$

$$X_{k+1} = F X_k + W_{k+1}$$

$$Y_k = H X_k + V_k$$

$$X = F X_k + G Y_k$$

$$= \underbrace{e^{(C(I - e^{-2KT}) - 1)}_G}_{G}$$

$$P =$$

$$3) \quad z_{k+1} = a z_k + w_k \quad w_k \sim \mathcal{N}(0, Q)$$

$$y_k = z_k + v_k \quad v_k \sim \mathcal{N}(0, R)$$

Let,

$$\text{Let } x_k = \begin{bmatrix} z_k \\ a_k \end{bmatrix} \quad f(a_k, z_k, w_k, v_k)$$

$$x_{k+1} = \begin{bmatrix} a_k z_k + w_k \\ a_k + v_k \end{bmatrix} \quad v_k \sim \mathcal{N}(0, 0)$$

$$y_k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} v_k \\ 0 \end{bmatrix}$$

EKF:

$$\text{Initialize } \hat{x}_0^+ = E[x_0]$$

$$P_0^+ = E[(\hat{x}_0 - \hat{x}_0^+)(\hat{x}_0 - \hat{x}_0^+)^T]$$

$$3-a) \quad F_{k-1} = \frac{\partial f_{k-1}}{\partial x} \Big|_{\hat{x}_{k-1}^+} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} a_k & z_k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{a}_{k-1}^+ & \hat{z}_{k-1}^+ \\ 0 & 1 \end{bmatrix}$$

$$L = I_a$$

$$b) \quad P_k^- = F_{k-1} P_{k-1}^+ F_{k-1}^T + L \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} L^T = \begin{bmatrix} \hat{a}_{k-1}^{+2} P_{k-1}^+ \hat{a}_{k-1}^{+1}, Q \\ \hat{a}_{k-1}^{+1} P_{k-1}^+ \hat{a}_{k-1}^{+2}, R \end{bmatrix}$$

$$\hat{x}_k^- = f(\hat{x}_{k-1}^+, 0) = \begin{bmatrix} \hat{a}_{k-1}^+ & \hat{z}_{k-1}^+ \\ \hat{a}_{k-1} \end{bmatrix}$$

$$c) \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

3) cont.

EKF: cont

3-d)

$$K_K = P_K^- H^T (H P_K^- H^T + M' R M')^{-1}$$

$$= P_K^- \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + R)^{-1}$$

$$= P_K^- \begin{bmatrix} (P_{KII}^- + R)^{-1} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} P_{KII}^- & 0 \\ \tilde{a}_{K12}^- P_{K12}^- & 0 \end{bmatrix} \begin{bmatrix} (P_{KII}^- + R)^{-1} \\ 0 \end{bmatrix} = \begin{bmatrix} P_{KII}^- (P_{KII}^- + R)^{-1} & 0 \\ \tilde{a}_{K12}^- P_{K12}^- (P_{KII}^- + R)^{-1} & 0 \end{bmatrix}$$

~~$$K_K = P_{KII}^- \begin{bmatrix} 1 & 0 \\ P_{KII}^- + R & P_{K12}^- \end{bmatrix}$$~~

$$\begin{aligned} \hat{x}_K^+ &= \hat{x}_K^- + K_K (y_K - h(\hat{x}_K^-, 0)) \\ &= \begin{bmatrix} \hat{z}_K^- \\ \hat{q}_K^- \end{bmatrix} + K_K \begin{bmatrix} z_K y_K - \hat{z}_K^- \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} \hat{z}_K^+ \\ \hat{q}_K^+ \end{bmatrix} = \begin{bmatrix} (I - K_{KII}) \hat{z}_K^- + K_{KII} (y_K) \\ (I - K_{K12}) \hat{z}_K^- + K_{K12} (y_K) \end{bmatrix}$$

$$P_K^+ = (I - K_K H_K) P_K^-$$

$$P_K^+ = \begin{bmatrix} (I - K_{KII}) & 0 \\ 0 & 1 \end{bmatrix} \cdot P_K^- = \begin{bmatrix} (I - K_{KII}) P_{KII}^- & (I - K_{KII}) P_{K12}^- \\ P_{K12}^- & P_{K22}^- \end{bmatrix}$$

$$P_K^- = -\alpha \tilde{a} P_K^+ + Q$$

$$K_K = P_K^- (P_{KII}^- + R)^{-1} \quad P_K^+ = (I - K) P_K^-$$

$$\hat{z}_K^- = \alpha \hat{z}_K^+$$

$$\hat{z}_K^+ = (I - K) \hat{z}_K^- + K(y_K - \hat{z}_K^-)$$

Solve...

3) cont.

KF:

$$F = a$$

$$H = I$$

$$w_k \sim (0, Q)$$

$$v_k \sim (0, R)$$

$$\hat{z}_k^+ = E[\hat{z}_k^+]$$

$$P_k^+ = E[(\hat{z}_k - \hat{z}_k^+) (\hat{z}_k - \hat{z}_k^+)^T]$$

$$P_k^- = F P_{k-1}^+ F^T + Q \approx a^2 P_{k-1}^+ + Q$$

$$P_k^- = P_{k-1}^- H^T (A^T P_{k-1}^- A + R)^{-1} = P_{k-1}^- (P_{k-1}^- + R)^{-1}$$

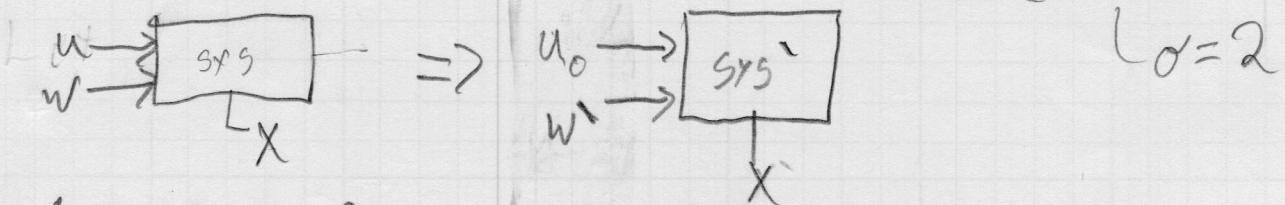
$$\begin{aligned} \tilde{z}_k &= F \hat{z}_{k-1}^+ \\ &= a \hat{z}_{k-1}^+ \\ &= \tilde{z}_k^- + K(Y_k - H \tilde{z}_k^-) \\ &= (1 - K) \hat{z}_k^- + K Y_k \end{aligned}$$

$$P_k^+ = (I - K H) P_k^- = (I - K) P_k^-$$

They are the same KF filter  
 for a deterministic  $a_x$   
 and artificial  $a_x$  error = 0

$$4) \dot{x} = -x + u + w \quad w \sim (0, \alpha)$$

$$A = -1 \quad B = 1 \quad L = 1 \quad u \sim (0, \alpha)$$



$$\dot{x} = Ax + Bu + Lw$$

$$\dot{x} = Ax + \underbrace{B^1 u_0}_{B' = B} + \underbrace{L^1 w_1}_{L' = [L \ B]} \quad w' \sim (0, \alpha)$$

$$w' \sim (0, \sqrt{\alpha} + \sqrt{B^1 \alpha^2})$$

$w' \sim (0, Q+4)$

$$\boxed{\dot{x} = -x + u_0 + w'}$$

$$5) \quad \begin{aligned} x_{k+1} &= \phi x_k + w_k & w_k \sim \mathcal{N}(0, I) \\ y_k &= x_k & \phi = 0.9 \end{aligned}$$

Let,

$$\hat{x}_k = \begin{bmatrix} x_k \\ \phi_k \end{bmatrix} \quad y_k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \hat{x}_k \quad \text{Unknown...}$$

$$\hat{x}_{k+1} = \begin{bmatrix} \phi x_k + w_k \\ \phi_k + w_{kp} \end{bmatrix} = \begin{bmatrix} \phi x_k \\ \phi_k \end{bmatrix} + \begin{bmatrix} w_k \\ w_{kp} \end{bmatrix}$$

$$\hat{x}_0 = E[\hat{x}_k] = \begin{bmatrix} \hat{x}_0 \\ \hat{\phi}_0 \end{bmatrix} \quad w' \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad w_{kp} \sim \mathcal{N}(0, 0)$$

$$P_0^+ = E[(\hat{x}_k - \hat{x}_0)(\hat{x}_k - \hat{x}_0)^T] = \begin{bmatrix} P_{x0} & 0 \\ 0 & P_{\phi 0} \end{bmatrix}$$

$$F_{k-1} = \left. \frac{\partial f_{k-1}}{\partial x} \right|_{\hat{x}_k^+} = \begin{bmatrix} \phi_{k-1} & x_{k-1} \\ 0 & 1 \end{bmatrix}$$

$$L_{k-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\begin{aligned} P_k^- &= F_{k-1} P_{k-1}^+ F_{k-1}^T + L_{k-1}^T Q_{k-1} L_{k-1} \\ &= \begin{bmatrix} \hat{\phi}_{k-1}^+ & \hat{x}_{k-1}^+ \\ 0 & 1 \end{bmatrix} P_{k-1}^+ \begin{bmatrix} \hat{\phi}_{k-1}^+ & \hat{x}_{k-1}^+ \\ 0 & 1 \end{bmatrix}^T + \begin{bmatrix} Q_k & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$\hat{x}_k^- = \begin{bmatrix} \hat{\phi}_{k-1}^+ & \hat{x}_{k-1}^+ \\ \hat{\phi}_{k-1}^+ & \end{bmatrix}$$

5) cont.

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad M_K = 0$$

$$K_{IK} = P_K^- H^T (H P_{IK}^- H^T + M_R \tilde{M}^T)^{-1}$$

$$= \cancel{\begin{bmatrix} P_{KU}^- & P_{KH}^- \\ P_{KU}^- & P_{KH}^- \end{bmatrix}} = \begin{bmatrix} P_{KU}^- & P_{KH}^- \\ P_{KU}^- & P_{KH}^- \end{bmatrix}$$

$$\hat{x}_{IK}^+ = \hat{x}_{IK}^- + K_{IK} (x_{IK} - \begin{bmatrix} \hat{x}_{KU}^- \\ 0 \end{bmatrix}) = I - K_{KH}$$

$$P_{IK}^+ = (I - K_{IK} H) P_{IK}^-$$

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% MECH 6325 - HW 5
%Jonas Wagner
%2020-11-11

clear
close all;

% Problem 2
n=1;
A = -2;
C = 1;

Q_c = 2;
R_c = 1;

x_0 = 100;
P_0 = 2;
x_hat_0 = 7;

T_max = 1000;

% T = 0.4
T = 0.4;

x = x_0;
y = C * x + R_c * randn;
p = P_0;
x_hat = x_hat_0;

k = p * C' * inv(R_c);

X = [];
Y = [];
P = [];

K = [];
X_hat = [];

for t = T:T:T_max
    x_dot = A * x + Q_c * randn;
    x = x + x_dot * T;
    y = C * x + R_c * randn;
    p_dot = - p * C' * inv(R_c) * C * p + A * p + p * A' + Q_c;
    p = p + p_dot * T;

    k = p * C' * inv(R_c);
    x_hat_dot = A * x_hat + K * (y - C * x_hat);
    x_hat = x_hat + x_hat_dot * T;

    X = [X x];
    P = [P p];
    Y = [Y y];
end

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x_hat = [x_hat x_hat];
end

X_04 = X;
P_04 = P;
Y_04 = Y;
X_hat_04 = x_hat;

% T = 0.2
T = 0.2;

x = x_0;
y = C * x + R_c * randn;
p = P_0;
x_hat = x_hat_0;

k = p * C' * inv(R_c);

X = [];
Y = [];
P = [];

K = [];
X_hat = [];

for t = T:T:T_max
    x_dot = A * x + Q_c * randn;
    x = x + x_dot * T;
    y = C * x + R_c * randn;
    p_dot = - p * C' * inv(R_c) * C * p + A * p + p * A' + Q_c;
    p = p + p_dot * T;

    k = p * C' * inv(R_c);
    x_hat_dot = A * x_hat + K * (y - C * x_hat);
    x_hat = x_hat + x_hat_dot * T;

    X = [X x];
    P = [P p];
    Y = [Y y];

    X_hat = [x_hat x_hat];
end

X_02 = X;
P_02 = P;
Y_02 = Y;
X_hat_02 = x_hat;

% T = 0.1
T = 0.1;

x = x_0;

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y = C * x + R_c * randn;
p = P_0;
x_hat = x_hat_0;

k = p * C' * inv(R_c);

X = [];
Y = [];
P = [];

K = [];
X_hat = [];

for t = T:T:T_max
    x_dot = A * x + Q_c * randn;
    x = x + x_dot * T;
    y = C * x + R_c * randn;
    p_dot = - p * C' * inv(R_c) * C * p + A * p + p * A' + Q_c;
    p = p + p_dot * T;

    k = p * C' * inv(R_c);
    x_hat_dot = A * x_hat + K * (y - C * x_hat);
    x_hat = x_hat + x_hat_dot * T;

    X = [X x];
    P = [P p];
    Y = [Y y];

    X_hat = [x_hat x_hat];
end

X_01 = X;
P_01 = P;
Y_01 = Y;
X_hat_01 = X_hat;

% Descretization Mehtod

% T = 0.4
T = 0.4;

F = 1 + A * T;
H = C;

Q = Q_c * T;
R = R_c / T;

x = x_0;
y = C * x + R_c * randn;
p_pre = P_0;
p_post = p_pre;

x_hat_pre = x_hat_0;

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k = p * C' * inv(R_c);
x_hat_post = x_hat_pre;

X = [];
Y = [];

K = [];
X_hat_pre= [];
X_hat_post = [];
P_pre = [];
P_post = [];

for i = 1:(T_max / T)
    x = F * x + Q * randn;
    y = H * x + R * randn;

    p_pre = F * p_post * F' + Q;
    k = p_pre * H' * inv(H * p_pre * H' + R);
    x_hat_pre = F * x_hat_post;
    x_hat_post = x_hat_pre + k * (y - H * x_hat_pre);
    p_post = (eye(n) - k * H) * p_pre * (eye(n) - k * H)' + k * R * k';

    X = [X x];
    Y = [Y y];

    K = [K k];
    X_hat_pre = [X_hat_pre x_hat_pre];
    X_hat_post = [X_hat_post x_hat_post];
    P_pre = [P_pre p_pre];
    P_post = [P_post p_post];
end

X_04 = X;
Y_04 = Y;

K_04 = K;
X_hat_pre_04 = X_hat_pre;
X_hat_post_04 = X_hat_post;
P_pre_04 = P_pre;
P_post_04 = P_post;

p_ss_04 = dlyap(F,Q);

% T = 0.2
T = 0.2;

F = 1 + A * T;
H = C;

Q = Q_c * T;
R = R_c / T;

x = x_0;

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y = C * x + R_c * randn;
p_pre = P_0;
p_post = p_pre;

x_hat_pre = x_hat_0;
k = p * C' * inv(R_c);
x_hat_post = x_hat_pre;

X = [];
Y = [];

K = [];
X_hat_pre= [];
X_hat_post = [];
P_pre = [];
P_post = [];

for i = 1:(T_max / T)
    x = F * x + Q * randn;
    y = H * x + R * randn;

    p_pre = F * p_post * F' + Q;
    k = p_pre * H' * inv(H * p_pre * H' + R);
    x_hat_pre = F * x_hat_post;
    x_hat_post = x_hat_pre + k * (y - H * x_hat_pre);
    p_post = (eye(n) - k * H) * p_pre * (eye(n) - k * H)' + k * R * k';

    X = [X x];
    Y = [Y y];

    K = [K k];
    X_hat_pre = [x_hat_pre x_hat_pre];
    X_hat_post = [x_hat_post x_hat_post];
    P_pre = [P_pre p_pre];
    P_post = [P_post p_post];
end

X_02 = X;
Y_02 = Y;

K_02 = K;
X_hat_pre_02 = X_hat_pre;
X_hat_post_02 = X_hat_post;
P_pre_02 = P_pre;
P_post_02 = P_post;

p_ss_02 = dlyap(F,Q);

% T = 0.1
T = 0.1;

F = 1 + A * T;
H = C;

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Q = Q_c * T;
R = R_c / T;

x = x_0;
y = C * x + R_c * randn;
p_pre = P_0;
p_post = p_pre;

x_hat_pre = x_hat_0;
k = p * C' * inv(R_c);
x_hat_post = x_hat_pre;

X = [];
Y = [];

K = [];
X_hat_pre= [];
X_hat_post = [];
P_pre = [];
P_post = [];

for i = 1:(T_max / T)
    x = F * x + Q * randn;
    y = H * x + R * randn;

    p_pre = F * p_post * F' + Q;
    k = p_pre * H' * inv(H * p_pre * H' + R);
    x_hat_pre = F * x_hat_post;
    x_hat_post = x_hat_pre + k * (y - H * x_hat_pre);
    p_post = (eye(n) - k * H) * p_pre * (eye(n) - k * H)' + k * R * k';

    X = [X x];
    Y = [Y y];

    K = [K k];
    X_hat_pre = [x_hat_pre x_hat_pre];
    X_hat_post = [X_hat_post x_hat_post];
    P_pre = [P_pre p_pre];
    P_post = [P_post p_post];
end

X_01 = X;
Y_01 = Y;

K_01 = K;
X_hat_pre_01 = X_hat_pre;
X_hat_post_01 = X_hat_post;
P_pre_01 = P_pre;
P_post_01 = P_post;

p_ss_01 = dlyap(F,Q);

% Plotting Comparisons
t_04 = 0.4:(0.4):T_max;

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t_02 = 0.2:(0.2):T_max;
t_01 = 0.1:(0.1):T_max;

figure()
hold on
plot(t_04, P_04)
plot(t_02, P_02)
plot(t_01, P_01)
title('Integrationg KF')
legend('0.4','0.2','0.1')

figure()
hold on
plot(t_04, P_pre_04);
plot(t_02, P_pre_02);
plot(t_01, P_pre_01);
title('Discretization Method (A Priori)');
legend('0.4','0.2','0.1')

figure()
hold on
plot(t_04, P_post_04);
plot(t_02, P_post_02);
plot(t_01, P_post_01);
title('Discretization Method (A Postari)');
legend('0.4','0.2','0.1')

P_ss_from_plots = ...
    [ "T"      "A Priori"   "A Postari";
     "0.4"    "0.825"      "0.620";
     "0.2"    "0.590"      "0.527";
     "0.1"    "0.511"      "0.486" ];

clear
% close all
% Problem 5
n=2;
T_max = 10000;
T = 1;
phi = 0.9;

F = phi;

Q_1 = 1;
Q_2 = 0;
Q = diag([Q_1, Q_2]);
R = zeros(n);

x_0 = 1;
p_0 = eye(n);

x_hat_0 = zeros(2);

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x = x_0;
%y = H * x + R_c * randn;
p_pre = p_0;
p_post = p_pre;

x_hat_pre = x_hat_0;
x_hat_post = x_hat_pre;

X = [];
Y = [];

K = [];
X_hat_pre= [];
X_hat_post = [];
P_pre = [];
P_post = [];

for i = 1:(T_max / T)
    % System itself
    x = phi * x + Q_1 * randn;
    y = x;

    % Kalmen Filter
    F = [x_hat_post(2) x_hat_post(1);
          0           1];
    L = eye(n);

    p_pre = F * p_post * F' + L * Q * L';
    x_hat_pre = [x_hat_post(2) * x_hat_post(1);
                 x_hat_post(2)];

    H = diag([1,0]);
    M = zeros(2);

    k = p_pre * H' * inv(p_pre(1,1)); %cheating for special case
    %inv(H * p_pre * H' + M * R * M');
    x_hat_post = x_hat_pre + k * ([y; 0] - (H * x_hat_pre));
    p_post = (eye(n) - k * H) * p_pre;

    X = [X x];
    Y = [Y y];

    K = [K k];
    X_hat_pre = [X_hat_pre x_hat_pre];
    X_hat_post = [X_hat_post x_hat_post];
    P_pre = [P_pre p_pre];
    P_post = [P_post p_post];
end

figure()
plot(X_hat_post(2,1:100))
title('Phi Estimate')

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phi_hat_ss = mean(X_hat_post(2,0.8*T_max:T_max))
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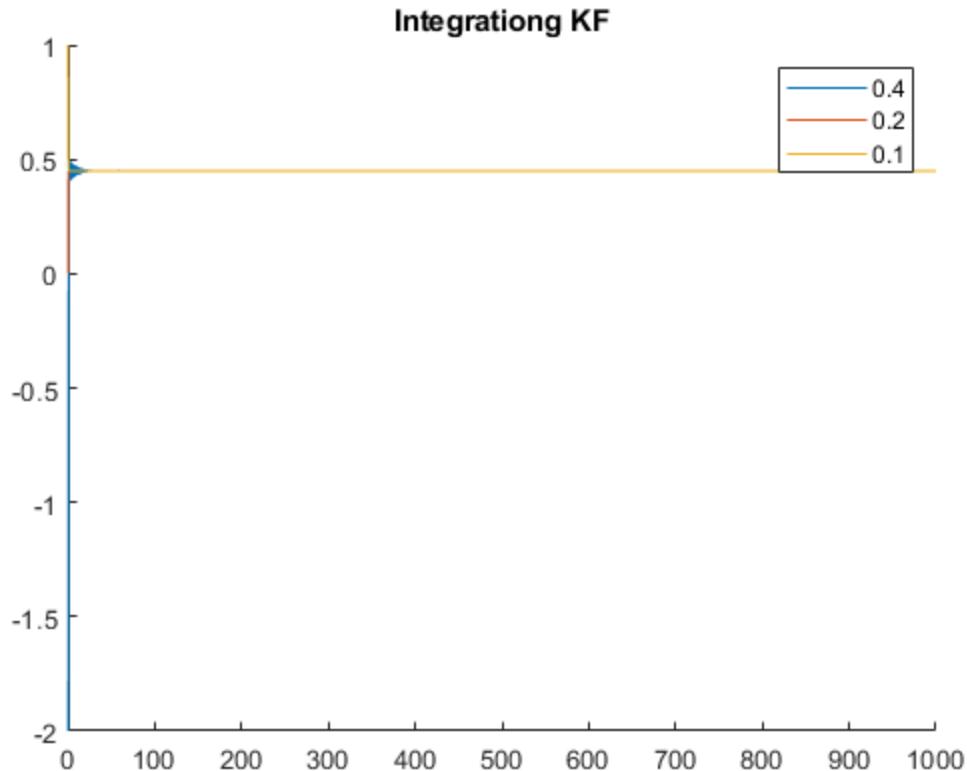
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P_ss_from_plots =
```

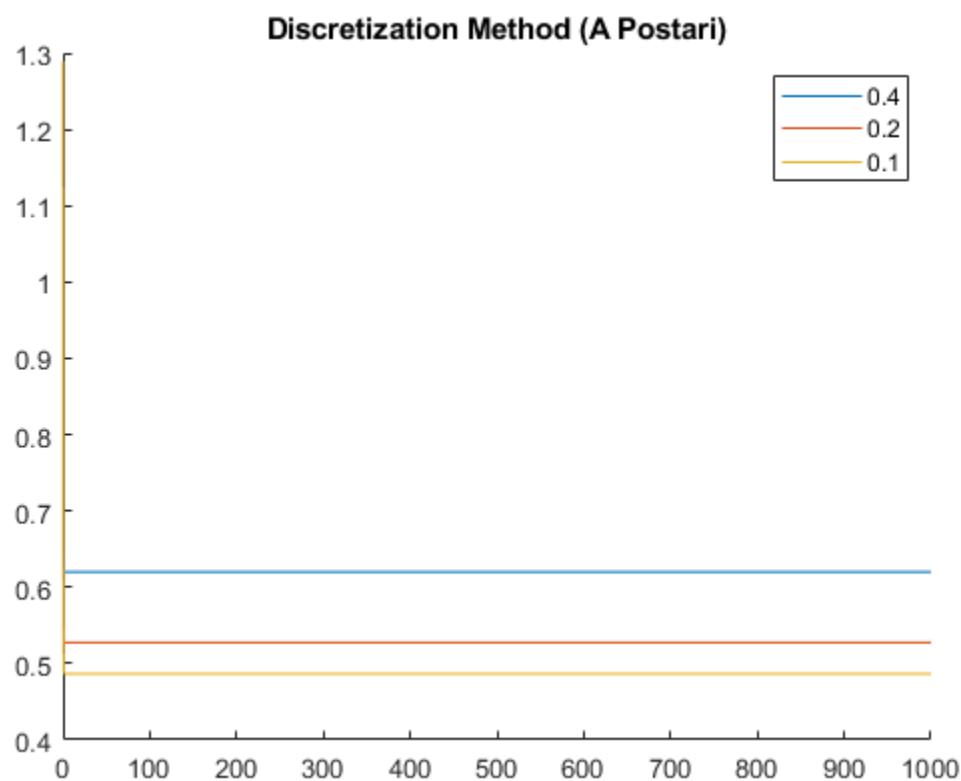
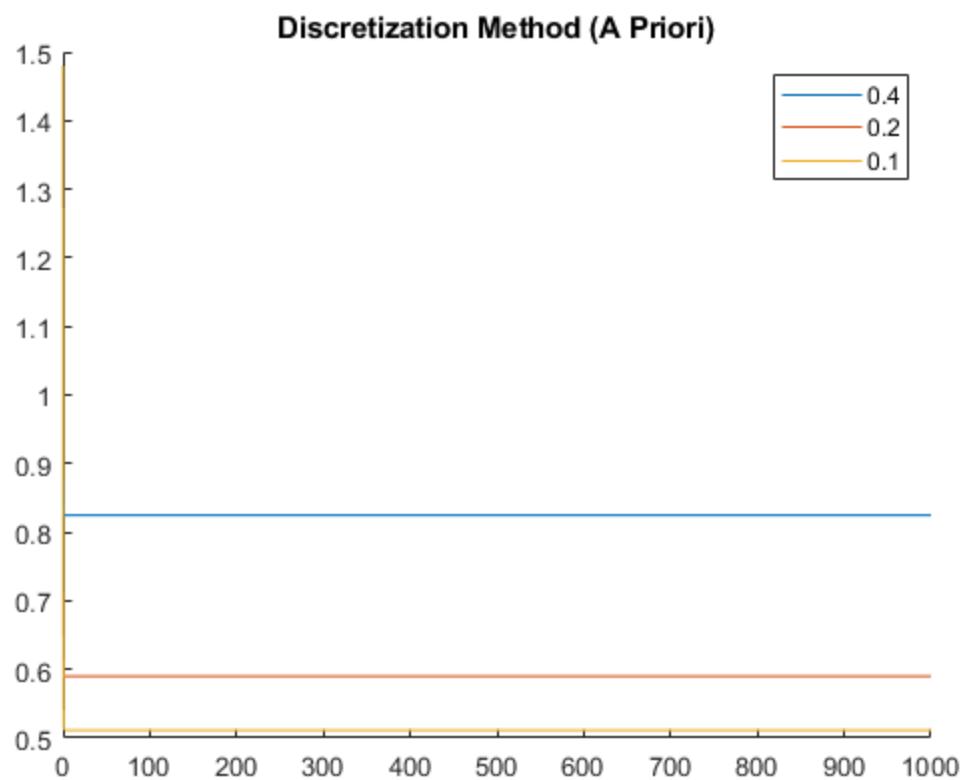
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4x3 string array
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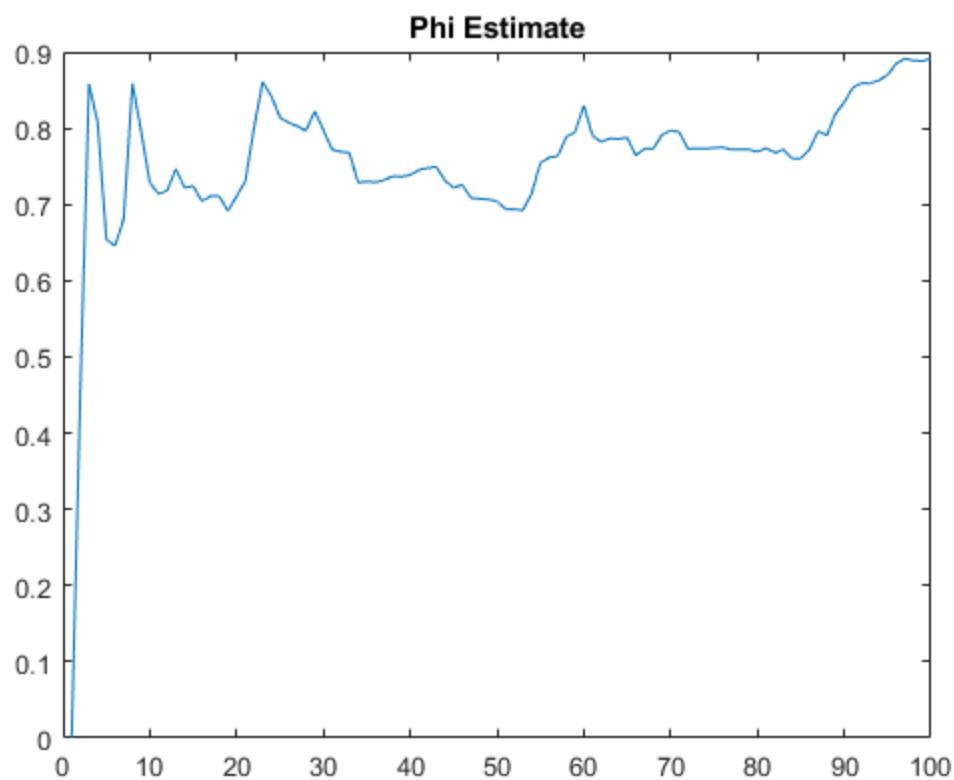
|       |            |             |
|-------|------------|-------------|
| "T"   | "A Priori" | "A Postari" |
| "0.4" | "0.825"    | "0.620"     |
| "0.2" | "0.590"    | "0.527"     |
| "0.1" | "0.511"    | "0.486"     |

```
phi_hat_ss =
```

```
0.8958
```







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