1)

$$x_{k+1} = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_k \qquad w_k \sim (0,1)$$

$$P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x_{k+1} = F x_k + G u + W_k \qquad W_k = \begin{bmatrix} w_k \\ w_k \end{bmatrix}$$

$$u = 0$$

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$Q = 1$$

$$\tilde{Q} = L Q L^T$$

a)

$$\bar{x} = (I - F)^{-1} G \cancel{Q}\, 0$$

$$\boxed{\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

b)

$$P = F P F^T + Q$$

ss)

$$\boxed{\bar{P} = \text{dlyap}(F, Q) = \begin{bmatrix} 1.06 & 0.99 \\ 0.99 & 0.24 \end{bmatrix}}$$

explicit)

$$P_1 = F P_0 F^T + Q = F F^T + Q \qquad (P_0 = I)$$

$$P_2 = F P_1 F^T + Q = F(F F^T + Q) F^T + Q = F F F^T F^T + F Q F^T + Q$$

$$P_3 = F P_2 F^T + Q = F^{(3)} F^{T(3)} + F^{(2)} Q F^{T(2)} + F Q F + Q$$

$$\boxed{P_i = F^{(i)} F^{T(i)} + \sum_{k=0}^{i-1} F^{(k)} Q F^{T(k)}}$$

2) a)

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \underbrace{\begin{bmatrix} 5 \\ 1 \end{bmatrix}}_{L} w(t) \qquad w \sim (0,1)$$

$$Q = 1 \qquad P(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tilde{Q} = L Q L^T = \begin{bmatrix} 25 & 5 \\ 5 & 1 \end{bmatrix}$$

$$\underbrace{0 = AP + PA^T + Q}_{\downarrow\ \mathrm{lyap}(A,Q)}$$

$$P = \begin{bmatrix} 12.5 & 2.5 \\ 2.5 & 0.5 \end{bmatrix}$$

b)

$$x_{k+1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} x_k + w_k \qquad w_1 \sim N(0,Q)$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underbrace{P = FPF^T + Q}_{\downarrow\ \mathrm{dlyap}(F,Q)}$$

$$P = \begin{bmatrix} 1.33 & 0 \\ 0 & 1.33 \end{bmatrix}$$

3)

$$x_k = \frac{-1}{2} x_{k-1} + w_{k-1}$$

$$E(w_{k-1}) = 0$$

$$\cancel{E(w_k w_j) = e^{-|k-j|}}$$

$$F = \frac{-1}{2}$$

$$P_0 = 1$$

$$Q = 1$$

$$P_k = F P_{k-1} F^T + Q$$

$$P_k = F^{2k} + \sum_{i=0}^{k-1} F^{2i} Q = \left(-\frac{1}{2}\right)^{2k} + \sum_{i=0}^{k-1} \left(-\frac{1}{2}\right)^{2k} (1)$$

$$= \left(\frac{1}{4}\right)^{k} + \sum_{i=0}^{k-1} \left(\frac{1}{4}\right)^{k} = \sum_{i=0}^{k} \left(\frac{1}{4}\right)^{k}$$

$$\bar{P} = \underbrace{F \bar{P} F + Q}_{\downarrow\, \mathrm{dlyap}(F, Q)}$$

$$\boxed{\bar{P} = 1.33}$$

$$\boxed{P_k = \sum_{i=0}^{k} \left(\frac{1}{4}\right)^{k}}$$

4) $y_K = z_K + z_{K-1}$ $\qquad z \sim (0,1)$

a) $x_K = \begin{bmatrix} z_{K-1} \\ z_K \end{bmatrix}$ $\qquad x_{K-1} = \begin{bmatrix} z_{K-2} \\ z_{K-1} \end{bmatrix}$

$$x_{K} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{F} \begin{bmatrix} z_{K-1} \\ z_{K} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{L} z_{K-1} \qquad w_K \sim (0,1) \quad Q = 1$$

$G = 0$ $\qquad \tilde{Q} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$y_K = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{H} x_K$$

$P_K^- = F P_K^+ F^T + Q$

b) Let $P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad \underline{R = 0}$

$P_K^+ = (I - K_K H_K)(P_K^-)$

$= P_K^- - P_K^- H_K^T (H_K P_K^- H_K^T)^{-1} H_K P_K^-$

$K = P_K^- H_K^T (H_K P_K^- H_K^T + \cancel{R_K})^{-1}$

$K = P_K^- H_K^T (H_K P_K^- H_K^T)^{-1}$

$H_K P_K^- H_K^T = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} P_{K11}^- & P_{K12}^- \\ P_{K21}^- & P_{K22}^- \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= (P_{K11}^- + P_{K21}^-) + (P_{K21}^- + P_{K22})$

$P_1^+ = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$K = \frac{1}{(P_{K11}^- + P_{K12}^- + P_{K21}^- + P_{K22}^-)} \begin{bmatrix} P_{K11}^- + P_{K12} \\ P_{K21} + P_{K22} \end{bmatrix}$

$P_K^- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_{K11}^+ & P_{K12}^+ \\ P_{K21}^+ & P_{K22}^+ \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} P_{K22}^+ & 0 \\ 0 & 1 \end{bmatrix}$

$K_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$K = \frac{1}{P_{(K-1)22}^+ + 1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\boxed{P_K^+ = \frac{1}{P_{(K-1)22}^+ + 1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{K+1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

4) cont.

c) $E\left[\|x_k - \hat{x}_k^+\|_2^2\right] = Tr\left(P_k^+\right) = Tr\left(\frac{1}{k+1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right)$

$$= \boxed{\frac{2}{k+1}}$$

---

5) $R_k = R$ $\quad x_{k+1} = F x_k + \cancel{Gu}^{\,0}$ $\quad R_\Sigma = R + \Delta R$ $\quad K_k = P_k^+ H_k^T R_k^{-1}$

$P_k^+ = P_k^+$ $\quad y_k = H x_k + v_k$ $\quad P_{k\Sigma}^+ = \Sigma_k^+$

$P_k^- = P_k^-$ $\quad v_k \sim (0, R)$ $\quad P_{k\Sigma}^- = \Sigma_k^-$ $\quad v_k \sim (0, R+\Delta R)$

$$\hat{x}_{k+1}^- = F(I - K_k H_k)\hat{x}_k^- + F K_k \cancel{y_k}^{(Hx_k + v_k)}$$

$$\hat{x}_{k+1}^- - x_{k+1} = F(I - K_k H)\hat{x}_k + F K_k H x_k + F K_k v_k - F x_k$$

$$= F\hat{x}_k - F K_k H(\hat{x}_k - x_k) - F x_k + F K_k v_k$$

$$\hat{x}_{k+1}^- - x_{k+1} = F(I - K_k H)(\hat{x}_k - x_k) + F K_k v_k$$

$$P_{k+1}^- = E\left[(\hat{x}_{k+1} - x_{k+1})(\hat{x}_{k+1} - x_{k+1})^T\right]$$

$$= E\left[\left(F(I - K_k H)(\hat{x}_k - x_k) + F K_k v_k\right)\left(F(I - K_k H)(\hat{x}_k - x_k) + F K_k v_k\right)^T\right]$$

$$= E\left[F\left((I - K_k H)(\hat{x}_k - x_k) + K_k v_k\right)\left((\hat{x}_k - x_k)^T (I - K_k H)^T + (K_k v_k)^T\right)F^T\right]$$

$$= F\left(E\left[(I - K_k H)(\hat{x}_k - x_k)(\hat{x}_k - x_k)^T (I - K_k H)^T + (I - K_k H)(\hat{x}_k - x_k)(K_k v_k)^T + (K_k v_k)(K_k v_k)^T + (K_k v_k)(\hat{x}_k - x_k)^T (I - K_k H)^T\right]\right)F^T$$

5) cont.

$$P_{k+1}^- = F\begin{bmatrix} (I-K_kH)\,\underbrace{E\left[(\hat{x}_k - x_k)(\hat{x}_k - x_k)^T\right]}_{P_k^-}(I-K_kH)^T \\ + (I-K_kH)\,\underbrace{E\left[(\hat{x}_k - x_k)\right]}_{0}(K_kV_k)^T \\ + (K_kV_k)\,\underbrace{E\left[(\hat{x}_k - x_k)^T\right]}_{0}(I-K_kH)^T \\ + K_k\,\underbrace{E(V_kV_k^T)}_{R}\,K_k^T \end{bmatrix}F^T$$

$$P_{k+1}^- = F\left((I-K_kH)P_k^-(I-K_kH)^T + K_kRK_k^T\right)F^T$$

similarily,

$$\Sigma_{k+1}^- = F\left((I-K_kH)\Sigma_k^-(I-K_kH)^T + K_k(R+\Delta R)K_k^T\right)F^T$$

Let $\Delta_{k+1}^- = \left(\Sigma_k^- - P_k^-\right)$

Clearly,

$$\Delta_{k+1}^- = F\left((I-K_kH)\Delta_k^-(I-K_kH) + K_k(\Delta R)K_k^T\right)F^T$$

It can be said that $\forall\ \Delta R = \Delta R^T > 0$,

$$\Delta_k^- = \Delta_k^{-T} > 0$$

6)
$$x_K = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_{K-1} + w_{K-1} \qquad v_K \sim (0, Q)$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$y_K = \begin{bmatrix} 1 & 0 \end{bmatrix} x_K + v_K$$

$$v_K \sim (0, R)$$

$$P_0 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \qquad R_K = 2 + (-1)^K$$

P_∞^+ does not exist... It flips between two steady states as R constantly changes.

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7) $$x_k = x_{k-1} + w_{k-1}$$

$Q = 4$ $P_0^+ = 100$

$R = 1$

i) $$y_k = 0.5\, x_k + v_k$$

$w_{k-1} \sim (0, Q)$

$v_k \sim (0, R)$

Error decends to a steady-state value around 2.5.

ii) $$y_k = (\cos(1 + k/120))\, x_k + v_k$$

The Error fluctuates as a function of time as H fluctuates. The error remains low until the time step aproaches 60 (when the Cos goes to zero) and the error jumps up above 50. This is likely due to the large size of measurement noise compared to measurement signal.

8) $$\dot{x} = -x(t) + w(t)$$

$$A = -1$$

$x(0) = 1$

$Q(t) = e^{-2}\delta(t)$

$P(0) = 1$

b) ss)

$$0 = \underbrace{AP + PA^T + Q}_{\downarrow\ \text{lyap}(A,Q)}$$

$$\bar{P} = 0.0677$$

```
% MECH 6325 - Homework 3
close all

% Problem 1
F = [0      1/2
    -1/2    2];
G = [0  0]';
L = [1  1]';
q = 1;
Q = L * q * L';

% Part 1a
x_bar_1a = inv(eye(2) - F)*G*0

% Part 1b
P_1b = dlyap(F,Q)


% Problem 2
% Part 2a
A = [-1 0
    0   -1];
L = [5  1]';
q = 1;
Q = L * q * L';

P_2a = lyap(A,Q)

% Part 2b
F = [0.5    0
    0       0.5];
Q = [1  0
    0   1];

P_2b = dlyap(F,Q)

% Problem 3
F = -1/2;
Q = 1;

P_3 = dlyap(F,Q)


% Problem 4
F = [0  1
    0   0];
L = [0  1]';
q = 1;
Q = L * q * L';

H = [1  1];
R = 0;
```

```
syms pr11 pr12 pr21 pr22
Pr =  [pr11 pr12; pr21 pr22];

K = Pr * H' * inv(H * Pr * H' + R);

Po = (eye(2) - K * H) * Pr;

Pr = F * Po * F' + Q;

%-----------------


% Problem 6
F = [1  1
    0   1];
Q = eye(2);
H = [1  0];
syms k
R_sym = 2 + (-1)^k;

n = 2;
N = 10;
P_pri = zeros(n,n,N);
P_post = zeros(n,n,N);
K_all = zeros(n,1,N);

P_0 = 10 * eye(1);

for i = 1:N
    R = subs(R_sym,k,i);
    if (i == 1)
        Pr = P_0;
        Po = P_0;
    else
        Pr = P_pri(:,:,i-1);
        Po = P_post(:,:,i-1);
    end
    K = Pr * H' * inv(H * Pr * H' + R);
    Po = (eye(2) - K * H) * Pr;
    Pr = F * Po * F' + Q;

    P_pri(:,:,i) = Pr;
    P_post(:,:,i) = Po;
    K_all(:,:,i) = K;
end

P_post

% Problem 7
F = 1;

Q = 4;
R = 1;
```

```
P_0 = 100;

% Part 7i
H = 0.5;

n = 1;
N = 200;
P_pri = zeros(n,n,N);
P_post = zeros(n,n,N);
K_all = zeros(n,1,N);

for i = 1:N
    if (i == 1)
        Pr = P_0;
        Po = P_0;
    else
        Pr = P_pri(:,:,i-1);
        Po = P_post(:,:,i-1);
    end
    K = Pr * H' * inv(H * Pr * H' + R);
    Po = (eye(n) - K * H) * Pr;
    Pr = F * Po * F' + Q;

    P_pri(:,:,i) = Pr;
    P_post(:,:,i) = Po;
    K_all(:,:,i) = K;
end

figure()
y = reshape(P_post,1,[]);
plot(y)
title("Problem 7 a")



% Part 7i
syms k
H_sym = cos(1 + k/120);

n = 1;
N = 200;
P_pri = zeros(n,n,N);
P_post = zeros(n,n,N);
K_all = zeros(n,1,N);

for i = 1:N
    H = subs(H_sym,k,i);
    if (i == 1)
        Pr = P_0;
        Po = P_0;
    else
        Pr = P_pri(:,:,i-1);
        Po = P_post(:,:,i-1);
```

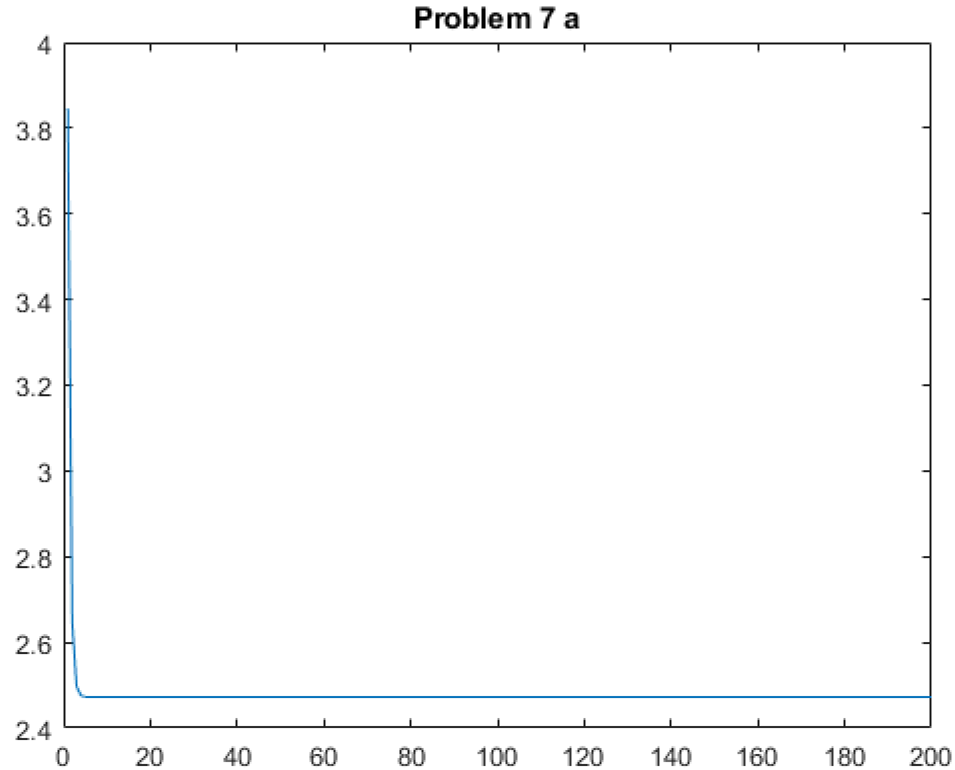
```
    end
    K = Pr * H' * inv(H * Pr * H' + R);
    Po = (eye(n) - K * H) * Pr;
    Pr = F * Po * F' + Q;

    P_pri(:,:,i) = Pr;
    P_post(:,:,i) = Po;
    K_all(:,:,i) = K;
end

figure()
y = reshape(P_post,1,[]);
plot(y)
title("Problem 7 b")

%--------------------

% Problem 8
A = -1;
Q = exp(-2);

n = 1;
N = 100;
tMax = 5;
T = tMax / N;
t = linspace(0,tMax,N);
X = zeros(n,n,N);
P = zeros(n,n,N);

x_0 = 1;
P_0 = 1;

for i = 1:N
    if i == 1
        x = x_0;
        p = P_0;
    else
        x = X(i-1);
        p = P(i-1);
    end

    x = x + (A * x) * T;
    p = p + (A * p + p * A' + Q) * T;

    X(i) = x;
    P(i) = p;
end

x = t;
y1 = reshape(X, 1, []);
y2 = reshape(P, 1, []);

figure()
```

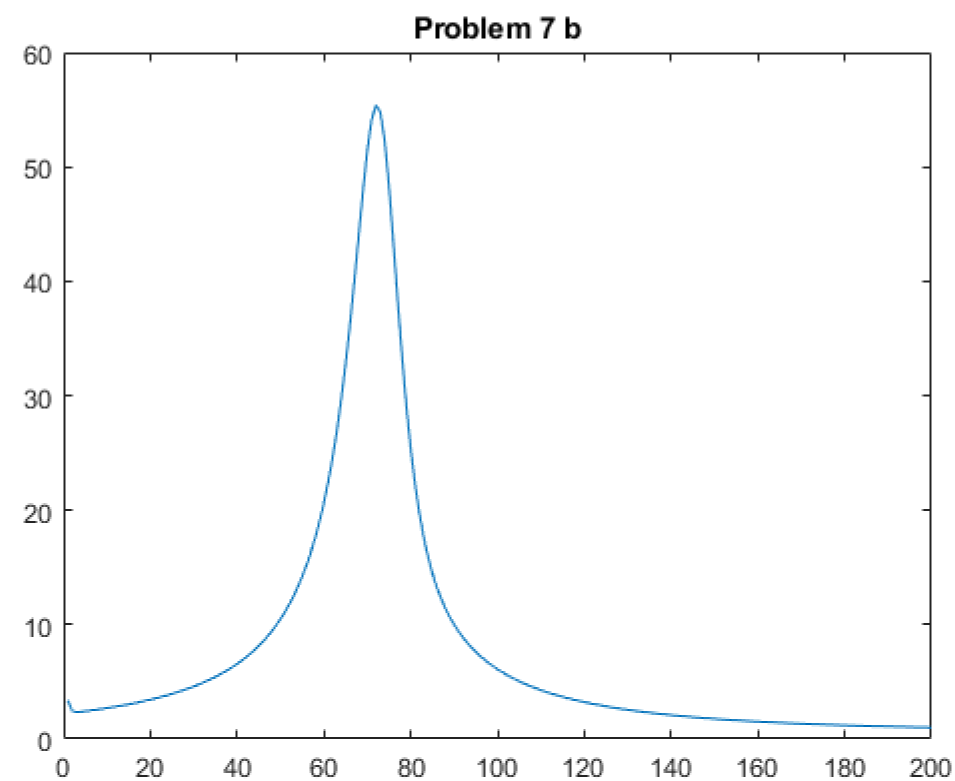
```
subplot(1,2,1);
title("Problem 8 - States")
hold on
plot(x,y1)

subplot(1,2,2);
title("Problem 8 - Covariance")
hold on
plot(x,y2)


x_0 = 1;
P_0 = 0;

for i = 1:N
    if i == 1
        x = x_0;
        p = P_0;
    else
        x = X(i-1);
        p = P(i-1);
    end

    x = x + (A * x) * T;
    p = p + (A * p + p * A' + Q) * T;

    X(i) = x;
    P(i) = p;
end

x = t;
y1 = reshape(X, 1, []);
y2 = reshape(P, 1, []);

subplot(1,2,1);
plot(x,y1)
hold off

subplot(1,2,2);
plot(x,y2)

p_8 = lyap(A,Q)
```

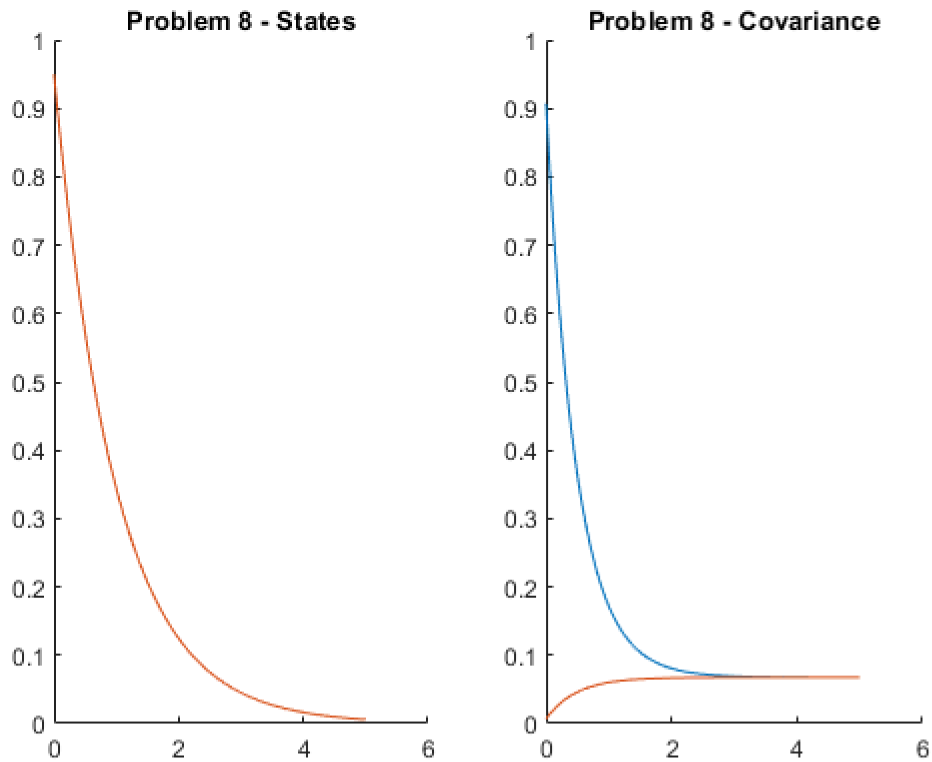
```
x_bar_1a =

     0
     0


P_1b =

    1.0598    0.9915
    0.9915    0.2393
```

```
P_2a =

   12.5000    2.5000
    2.5000    0.5000


P_2b =

    1.3333         0
         0    1.3333


P_3 =

    1.3333


P_post(:,:,1) =

    0.9091         0
         0   10.0000


P_post(:,:,2) =

    2.3963    2.0122
    2.0122    4.2927


P_post(:,:,3) =

    0.9213    0.4959
    0.4959    2.1659


P_post(:,:,4) =

    1.8860    0.9884
    0.9884    2.2889


P_post(:,:,5) =

    0.8773    0.4020
    0.4020    1.9713


P_post(:,:,6) =

    1.8239    0.9304
    0.9304    2.2353
```

```
P_post(:,:,7) =

    0.8737    0.3997
    0.3997    1.9699


P_post(:,:,8) =

    1.8225    0.9301
    0.9301    2.2353


P_post(:,:,9) =

    0.8737    0.3998
    0.3998    1.9698


P_post(:,:,10) =

    1.8225    0.9301
    0.9301    2.2352


p_8 =

    0.0677
```


Problem 7 a
4
3.8
3.6
3.4
3.2
3
2.8
2.6
2.4
0
20
40
60
80
100
120
140
160
180
200


Problem 7 b
60
50
40
30
20
10
0
0
20
40
60
80
100
120
140
160
180
200



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