# Assignment 3

Optimal Estimation and Kalman Filter
Fall 2020
Due date: October 23, 2020 at 1:00 PM

**Problem 1.** Consider the system

$$x_{k+1} = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_k \quad P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $w_k$ is white noise.

a. Find the steady-state values of the mean of $x_k$.

b. Find the covariance matrix of $x_k$ and its steady state values.

**Problem 2.** Find the steady-state covariance of the state vector for the following systems:

a. $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 5 \\ 1 \end{bmatrix} w(t), \quad w \sim \mathcal{N}(0,1)$ and white, $\quad P(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. $x_{k+1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} x_k + w_k, \quad w_k \sim \mathcal{N}(0, Q), \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**Problem 3.** Find the covariance of $x_k$ as a function of $k$ and its steady-state value for the system

$$x_k = -\frac{1}{2} x_{k-1} + w_{k-1}$$

where $\mathbf{E}(w_{k-1}) = 0$ and $\mathbf{E}(w_k w_j) = e^{-|k-j|}$. Assume the initial value of the covariance ($P_0$) is 1.

**Problem 4.** The measured output of a simple moving average process is $y_k = z_k + z_{k-1}$, where $\{z_j\}$ is zero mean white noise with a variance of one.

a. Generate a state-space description for this system with the first element of $x_k$ equal to $z_{k-1}$ and second element equal to $z_k$.

b. Suppose that the initial estimation-error covariance is equal to the identity matrix. Show that the *a posteriori* estimation-error covariance is given by

$$P_k^+ = \frac{1}{k+1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

c. Find $\mathbb{E}[||x_k - \hat{x}_k^+||_2^2]$ as a function of $k$.

**Problem 5.** Suppose that a Kalman filter is designed for a discrete LTI system with an assumed measurement noise covariance of $R$, but the actual measurement noise covariance is $(R+\Delta R)$. The output of the Kalman filter will indicate that the *a priori* estimation-error covariance is $P_k^-$, but the actual *a priori* estimation-error covariance will be $\Sigma_k^-$. Find a difference equation for $\Delta_k = (\Sigma_k^- - P_k^-)$. Will $\Delta_k$ always be positive definite?

**Problem 6.** A vector of discrete-time random sequence $x_k$ is given by

$$\begin{aligned} x_k &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_{k-1} + w_{k-1}, \\ w_k &\sim \mathcal{N}(0_{2\times 2}, I_{2\times 2}). \end{aligned}$$

The observation equation is given by

$$\begin{aligned} y_k &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k, \\ v_k &\sim \mathcal{N}(0, 2+(-1)^k), \end{aligned}$$

with the process and measurement noise both white. Calculate the values of $P_k^+$, $P_k^-$, and $K_k$ for $k = 1, 2, \ldots, 10$. In addition, determine the steady-state error covariance matrix ($P_\infty^+$) with

$$P_0 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}.$$

**Problem 7.** Consider the dynamic process

$$x_k = x_{k-1} + w_{k-1}.$$

Two measurements of this process are available:

Measurement (i):

$$y_k = 0.5 x_k + v_k$$

Measurement (ii) :

$$y_k = (\cos(1 + k/120)) x_k + v_k$$

Perform error covariance analysis for each of these measurements with $Q = 4$, $R = 1$ and $P_0^+ = 100$, for $k = 1, 2, \ldots, 200$. Plot the estimation error variance for the state against the time index $k$ for each case. Explain the difference observed between the two plots.

**Problem 8.** Write code for the propagation of the mean and variance of the state of the following system

$$\begin{aligned} \dot{x} &= -\,x(t) + w(t) \\ x(0) &= 1 \\ Q(\tau) &= e^{-2}\delta(\tau) \end{aligned}$$

Plot the mean and variance of $x$ for 5 seconds assuming $P(0) = 1$. Repeat for $P_0 = 0$. Based on the plots, what does the steady-state value of the variance appear to be? What is the analytically determined steady-state value of the variance?