1)
$$x_k = F x_{k-1} + w_{k-1} \qquad w \sim (0, Q)$$
$$y_k = H x_k$$

$V = 0 \rightarrow R = 0$

$G = 0$

Prove if $(HQH^T)Q = QH^THQ$,
Then $P_\infty = 0$.

Steady-state:

Let $P_k^+ = P_{k-1}^+ = P_\infty = 0$

$K_k = P_k^- H^T (H P_k^- H^T + \cancel{R}^{0})^{-1}$

$K_\infty = P_k^- H^T H^{T^{-1}} P_k^{-1} H^{-1}$

$\boxed{K_\infty = H^{-1}}$

$$P_k^+ = (I - K_k H) P_k^- (I - K_k H)^T + \cancel{K_k R_k K_k^T}^{0}$$

$$P_k^- = F P_{k-1}^+ F^T + Q$$

$$\cancel{P_k^+}^{0} = (I - K_k H)(\cancel{F P_{k-1}^+ F}^{0} + Q_{k-1})(I - K_k H)^T$$

$$0 = (Q_{k-1} - K_k H Q_{k-1})(I - H^T K_k^T)$$

$$0 = Q_{k-1} - K_k H Q_{k-1} - Q_{k-1} H^T K_k^T + K_k H Q_{k-1} H^T K_k^T$$

$$K_k H Q_{k-1} H^T K_k^T = K_k H Q_{k-1} + Q_{k-1} H^T K_k^T - Q_{k-1}$$

K_∞

$$\cancel{H^{-1}} \cancel{H} Q \cancel{H^T} \cancel{H^{-T}} = \cancel{H^{-1}} \cancel{H} Q + Q \cancel{H^T} \cancel{H^{-T}} - Q$$

$$Q \stackrel{?}{=} Q$$

✓

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2)
$$x_k = x_{k-1} + w_{k-1}$$
$$y_k = x_k + v_k$$

$w_k \sim (0,1) \rightarrow Q = 1$

$v_k \sim (0,1) \rightarrow R = 1$

$G = 0$

$E(w_k v_{k+1}) = 1$

$M = 1$

a) $\underline{F = 1 \quad H = 1}$

$$K_k = P_k^+ H^T R_k^{-1}$$

$$P_k^+ = (I - K_k H)(F P_{k-1}^+ F^T + Q)$$

$\hat{x}_k^+ = (I - K_k H)(F \hat{x}_{k-1}^+) + K_k y_k$

$\hat{x}_k^+ = \hat{x}_{k-1}^+ - P_k^+ \hat{x}_{k-1} + P_k^+ y_k$

steady-state: $P_k^+ = P_{k-1}^+ = P^+$

$$P^+ = (I - P^+ H^T R^{-1})(F P^+ F^T + Q)$$

$$P^+ = (1 - P^+)(P^+ + 1) = 1 - (P^+)^2$$

$$(P^+)^2 + P^+ - 1 = 0 \rightarrow P^+ = \frac{-1}{2} \pm \frac{\sqrt{5}}{2}$$

$$\boxed{P^+ = \frac{\sqrt{5} - 1}{2}}$$

use ... K and in terms of e_k and e_{k-1}

b) ~~$\hat{x}_k^+ = (1 - P_k^+)\hat{x}_{k-1} + P_k^+ y_k = \hat{x}_{k-1} - P_k^+ \hat{x}_{k-1} + P_k^+ x_k + P_k^+ v_k$~~

~~$e_k = x_k - \hat{x}_k^+ = x_k - \left[(1 - P_k^+)\hat{x}_{k-1}^+ + P_k^+(x_k + v_k)\right]$~~

~~$\boxed{e_k = (1 - P_k^+)(x_k - \hat{x}_{k-1}^+) + P_k^+ v_k}$~~

~~steady state~~

~~$E[e_k^2] = E[(1-P^+)^2 e_k^2 + 2(1-P^+)v_k + (P^+ v_k)^2]$~~

~~$E[e_k^2] = \text{cov}(x, \hat{x}^+) - [(1-P^+)\ldots] + (P^+ v_k)^2$~~

~~$e^2 = (P^+)^2 \ldots$~~

$e_k = e_{k-1}(1 - K_k) + w_k(1 - K_k) - K_k v_k$

2) b) $\hat{x}_k^+ = (I - K_k H)(F\hat{x}_{k-1}^+ + \cancel{G u}^{\,0}) + K_k y_k \qquad y_k = Hx_k + v_k$

$= (I - K_k H) F \hat{x}_{k-1}^+ + K_k (H x_k) + K_k v_k$

$y_k = H x_{k-1} + w_{k-1}$

$e_k = x_k - \hat{x}_k = x_k - [(I - K_k H) F \hat{x}_{k-1}^+ - K_k H x_k - K_k v_k]$

$= (I - K_k \cancel{H})(\cancel{H} x_{k-1} + w_{k-1}) - (I - K_k \cancel{H}) \cancel{F} \hat{x}_{k-1}^+ - K_k v_k$

$= (I - K_k)(x_{k-1} - \hat{x}_{k-1}^+) + (I + K_k) w_{k-1} - K_k v_k$

$$\boxed{e_k = (I - K_k) e_{k-1} + (I + K_k) w_{k-1} - K_k v_k}$$

Scaler case →

$e_k^2 = (1-K_k)^2 e_{k-1}^2 + 2(1-K_k)(1+K_k) e_{k-1} w_{k-1} - 2K_k(1-K_k) v_k e_{k-1}$
$+ (1+K_k)^2 w_{k-1}^2 - 2K_k(1+K_k) w_{k-1} v_k + K_k^2 v_k^2$

$E[e_k^2]^{P_k} = (1-K_k)^2 E[e_{k-1}^2]^{\to P_{k-1}} + 2(1-K_k)^2 E[\cancel{w_{k-1} e_{k-1}}]^{\to 0} - 2K_k(1-K_k) E[\cancel{v_k e_{k-1}}]^{\to 0}$
$+ (1+K_k) E[\cancel{w_{k-1}^2}]^{\to Q} - 2K_k(1+K_k) E[\cancel{w_{k-1} v_k}]^{\to M} + K_k^2 E[\cancel{v_k^2}]^{\to R}$

Steady state: $P_k = P_{k-1} = P_\infty$

$P_\infty = (1-K_k)^2 P_\infty + (1+K_k)^2 \cancel{Q}^{\to 1} - 2K_k(1+K_k)(\cancel{M}^{\to 0}) + K_k^2 \cancel{R}^{\to 1}$

$K_\infty = P_\infty$

$\cancel{P_\infty} = \cancel{P_\infty} - 2K_k P_\infty + K_k^2 P_\infty + 1 - 2K_k + K_k^2 \cancel{-2K_k + 2K_k^2} + K_k^2$

$P_\infty (K_k^2 - 2K_k) = -1 - 2K_k^2 + 2K_k \quad -2K_k^2 = 2K_k(2-K_k)) + 1$

$$\boxed{P_\infty = \frac{2K_k^2 - 2K_k + 1}{K_k(2-K_k)}}$$

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2) b) cont.

assuming $K_\infty = P_\infty \cancel{H^T} \cancel{R}^{-1}$

$$P_\infty(K_K(2-K_K)) = 2K_K^2 - 2K_K + 1$$

$$P_\infty(P_\infty(2-P_\infty)) = 2P_\infty^2 - 2P_\infty + 1$$

$$2P_\infty^2 - P_\infty^3 = 2P_\infty^2 - 2P_\infty + 1$$

$$P_\infty^3 - 2P_\infty + 1 = 0$$

$$(P_\infty - 1)(x^2 + x - 1)$$

$$P_\infty = 1, \frac{\pm\sqrt{5} - 1}{2}$$

$$\boxed{P_\infty = \frac{\sqrt{5} - 1}{2}}$$

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2) c) $F = 1$ $\quad Q = 1$

$H = 1$ $\quad R = 1$

$M = 1$

(7.44) $$K_k = (P_k^- \cancel{H^T} + \cancel{M})(\cancel{H} P_k^- \cancel{H^T} + \cancel{HM} + \cancel{M^T H^T} + \cancel{R})^{-1}$$

$$= (P_k^- + 1)(P_k^- + 3)^{-1}$$

$$\boxed{K_k = \frac{1 + P_k^-}{3 + P_k^-}}$$

$$P_k^- = \cancel{F} P_{k-1}^+ \cancel{F^T} + \cancel{Q}$$
$$= 1 + P_{k-1}^+$$

$$\boxed{K_k = \frac{2 + P_{k-1}^+}{4 + P_{k-1}^+}}$$

$$P_k^+ = P_k^- - K_k(\cancel{H} P_k^- + \cancel{M^T})$$

$$= (1 + P_{k-1}^+) - (2 + P_{k-1}^+)(4 + P_{k-1}^+)^{-1}(\cancel{1+1} + P_{k-1}^+)$$

$$= \left[(1 + P_{k-1}^+)(4 + P_{k-1}^+) - (2 + P_{k-1}^+)(2 + P_{k-1}^+)\right](4 + P_{k-1}^+)$$

$$= \left[\cancel{4} + 5 P_{k-1}^+ + \cancel{(P_{k-1}^+)^2} - (\cancel{4} + 4 P_{k-1}^+ + \cancel{(P_{k-1}^+)^2})\right](4 + P_{k-1}^+)$$

$$\boxed{P_k^+ = \frac{P_{k-1}^+}{4 + P_{k-1}^+}}$$

Steady-state: $P_k^+ = P_{k-1}^+ = P^+$

$$P^+ = \frac{P^+}{4 + P^+}$$

$$4P^+ + (P^+)^2 = P^+$$

$$P^+(P^+ + 3) = 0 \longrightarrow P^+ = 0 \quad \cancel{(\text{or } -3)}$$

d) This Kalman filter design incorporates the covariance between the two to create an unbiased estimator. This also means as $t \to \infty$, the variance approaches 0. //

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3) $x_k = \frac{1}{2} x_{k-1} + w_{k-1}$ $\quad w_k \sim (0,1) \to Q_k = 1$

$F = \frac{1}{2}$ $\quad X_k = x_k + v_k$

$H = 1$ $\quad v_k = \frac{1}{2} v_{k-1} + \xi_{k-1}$ $\quad \xi_k \sim (0,1) \to Q_{\xi k} = 1$

$\psi = \frac{1}{2}$

assuming $E[\,]=0$

a) assume $v_k \sim (0,1) \to R_k = 1$

$$\boxed{K_k = P_k^+} \; \cancel{H^T}^{\to 1} \cancel{R_k^{-1}}^{\to 1}$$

$$P_k^+ = (\cancel{I}^{\to 1} - K_k \cancel{H}^{\to 1})(\cancel{F}^{\to \frac{1}{2}} P_{k-1}^+ \cancel{F^T}^{\to \frac{1}{2}} + \cancel{Q_k}^{\to 1})$$

$$= (1 - P_k^+)\left(\frac{1}{4} P_{k-1}^+ + 1\right)$$

steady-state: $P_k^+ = P_{k-1}^+ = P^+$

$$P^+ = \frac{1}{4} P^+ - \frac{1}{4}(P^+)^2 - P^+ + 1$$

$$\frac{1}{4}(P^+)^2 + \frac{7}{4} P^+ - 1 = 0 \qquad P^+ = \frac{-7 \pm \sqrt{65}}{2}$$

$$\boxed{P^+ = \frac{\sqrt{65} - 7}{2}}$$

b) $\hat{x}_k^+ = (I - K_k H)(F \hat{x}_{k-1}) + K_k (x_k + v_k)$

$$= (I - K_k H) F \hat{x}_{k-1} + K_k (F x_{k-1} + w_{k-1}) + K_k (\psi v_{k-1} + \xi_{k-1})$$

$$e_k = x_k - \hat{x}_k^+ = (F x_{k-1} + w_{k-1}) - ((I - K_k H) F \hat{x}_{k-1} - K_k F x_{k-1} + K_k w_{k-1} - K_k \psi v_{k-1} - K_k \xi_{k-1}$$

$$e_k = (I - K_k) F \underbrace{(x_{k-1} - \cancel{H}^{\to 1} \hat{x}_{k-1}^+)}_{e_{k-1}} + (I - K_k) w_{k-1} - K_k \psi v_{k-1} - K_k \xi_{k-1}$$

0.7329

1

$$\boxed{e_k = (I - K_k) F e_{k-1} + (I - K_k) w_{k-1} - K_k \psi v_{k-1} - K_k \xi_{k-1}}$$

$$e_k = \frac{1}{2}(1 - K) e_{k-1} + (1 - K) w_{k-1} - \frac{1}{2} K v_{k-1} - K \xi_{k-1}$$

3) b)

$$e_{k}^2 = \frac{1}{4}(1-K)^2 e_{k-1}^2 + \cancelto{1}{\frac{2}{2}}(1-K)^2 e_{k-1}v_{k-1} - \cancelto{\frac{1}{2}}{\frac{2}{4}}K(1-K)e_{k-1}v_{k-1} - \cancelto{1}{\frac{2}{2}}K(1-K)e_{k-1}\xi_{k-1}$$

$$+ (1-K)^2 w_{k-1}^2 - \cancelto{1}{\frac{2}{2}}K(1-K)w_{k-1}v_{k-1} - 2K(1-K)w_{k-1}\xi_{k-1}$$

$$+ \frac{1}{4}K^2 v_{k-1}^2 + \cancelto{1}{\frac{2}{2}}K^2 v_{k-1}\xi_{k-1} + K^2 \xi_{k-1}^2$$

$$\overbrace{E[e_k^2]}^{P_k} = \frac{1}{4}(1-K)^2 \cancelto{P_{k-1}}{E[e_{k-1}^2]} + (1-K)^2 \cancelto{0}{E[e_{k-1}w_{k-1}]} - \frac{1}{2}K(1-K)\cancelto{0}{E[e_{k-1}v_{k-1}]}$$

$$- K(1-K)\cancelto{0}{E[e_{k-1}\xi_{k-1}]} + (1-K)^2 \cancelto{Q_w}{E[w_{k-1}^2]} - K(1-K)\cancelto{0}{E[w_{k-1}v_{k-1}]}$$

$$- 2K(1-K)\cancelto{0}{E[w_{k-1}\xi_{k-1}]} + \frac{1}{4}K^2 \cancelto{R_v}{E[v_{k-1}^2]} + K^2 \cancelto{0}{E[v_{k-1}\xi_{k-1}]}$$

$$+ K^2 \cancelto{Q_\xi}{E[\xi_{k-1}^2]}$$

$$P_k = \frac{1}{4}(1-K)^2 P_{k-1} + (1-K)^2 \cancelto{1}{Q_w} + \frac{1}{4}K^2 \cancelto{1}{R_v} + K^2 \cancelto{1}{Q_\xi}$$

steady-state: $P_k = P_{k-1} = P_\infty \qquad K_k = P_k = P_\infty$

$$P_\infty = \frac{1}{4}(1-K)^2 P_\infty + (1-K)^2 + \frac{1}{4}K^2 + K^2$$

$$P_\infty = \left(\frac{1}{4} - \frac{1}{2}K + \frac{1}{4}K^2\right)P_\infty + 1 - 2K + K^2 + \frac{5}{4}K^2$$

$$P_\infty = \frac{1}{4}P_\infty - \frac{1}{2}P_\infty^2 + \frac{1}{4}P_\infty^3 + 1 - 2P_\infty + P_\infty^2 + \frac{5}{4}P_\infty^2$$

$$\frac{1}{4}P_\infty^3 + \frac{5}{4}P_\infty^2 - \frac{11}{4}P_\infty + 1 = 0$$

$$P_\infty^3 + 5P_\infty^2 - 11P_\infty + 4 = 0$$

$$\Rightarrow P_\infty = 0.531$$

< recheck w/ Matlab

3) c) $F = \frac{1}{2}$ $H = 1$ $\psi = \frac{1}{2}$ $w_k \sim (0, 1)$ $\xi \sim (0,1)$

$Q_w = 1$ $Q_\xi = 1$

$$\underbrace{\begin{bmatrix} x_k \\ v_k \end{bmatrix}}_{x'_k} = \underbrace{\begin{bmatrix} F & 0 \\ 0 & \psi \end{bmatrix}}_{F'} \underbrace{\begin{bmatrix} x_{k-1} \\ v_{k-1} \end{bmatrix}}_{x'_{k-1}} + \underbrace{\begin{bmatrix} w_{k-1} \\ \xi_{k-1} \end{bmatrix}}_{w'_{k-1}} \qquad Q' = \begin{bmatrix} Q_w & 0 \\ 0 & Q_\xi \end{bmatrix}$$

$$y_k = \underbrace{\begin{bmatrix} H_k & I \end{bmatrix}}_{H'} \begin{bmatrix} x_k \\ v_k \end{bmatrix} + 0 \qquad \begin{aligned} x'_k &= F' x'_{k-1} + w'_{k-1} \\ y_k &= H' x'_k \end{aligned}$$

$$P'_\infty = F' P'_\infty F'^T - F'(P'_\infty H'^T)(H' P'_\infty H'^T + \cancel{R'})^{-1}(H' P'_\infty) F'^T + Q'$$

$$= \frac{1}{2}(I) P_\infty \left(\frac{1}{2}\right)(I) - \frac{1}{2} I \left(\cancel{P_\infty}\, \cancel{H^T}\, \cancel{H^{T^{-1}}}\, \cancel{P_\infty^{-1}}\, H^{-1}\right)\left(\cancel{H}\, \cancel{P_\infty}\right) \frac{1}{2}(I) + Q'$$

$$= \frac{1}{4} P_\infty - \frac{1}{4} P_\infty + Q' = Q'$$

$$\boxed{P'_\infty = \begin{bmatrix} Q_w & 0 \\ 0 & Q_\xi \end{bmatrix} = I_2}$$

$$\hat{x}'^+_k = (I - K_k H')(F' \hat{x}'^+_{k-1}) + K_k y_k$$

$$P'^+_k = (I - K_k H')(F' P'^+_{k-1} F'^T + Q')$$

$$K'_k = P_k^- H_k^T (H_k P_k^- H_k^T)^{-1}$$

$$= \cancel{P_k^-}\, \cancel{H_k^T}\, \cancel{H_k^{T^{-1}}}\, \cancel{P_k^{-1}}\, H^{-1}$$

$$K'_k = (H')^{-1}$$

4)
$$\hat{x}_k^+ \overset{\circ}{=} (I - KH) F \hat{x}_{k-1}^+ + K y_k$$
$$\hat{y}_k = H \hat{x}_k^+$$

Note: $H(I-KH) = (I-HK)H$

Let (F, H) be observable, $\rho(I-HK) = n$ (full rank)

Test if $((I-KH)F, H)$ is observable...

Proof

$$\rho(V) = n \quad \text{w/ } V \triangleq \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}$$

$$\tilde{V} = \begin{bmatrix} H \\ H(I-KH)F \\ \vdots \\ H((I-KH)F)^{n-1} \end{bmatrix} \quad \text{w/ since } \rho(I-HK) = n,$$

$$\rho(\tilde{V}) = \rho(V) \rightarrow \text{full rank}$$

$$\rho \begin{bmatrix} H \\ H(I-KH)F \\ \vdots \\ H((I-KH)F)^{n-1} \end{bmatrix} = \rho \begin{bmatrix} H \\ (I-HK)HF \\ \vdots \\ (I-HK)^{n-1} HF^{n-1} \end{bmatrix} = \rho \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}$$

Thus, The Kalman Filter is an observable system.