MECH 6325 HW7

1) $n=3$ $\quad p=\frac{1}{6}$

$k=2$

$$P(k=2) = \binom{3}{2}\left(\frac{1}{6}\right)^2\left(1-\frac{1}{6}\right) = (3)\left(\frac{(1)(1)(5)}{(6)(6)(6)}\right) = \frac{15}{216}$$

$$\boxed{P(k=2) = \frac{5}{72}}$$

2)

$$\boxed{P(k=k) = b_k^n = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!\, p^k (1-p)^{n-k}}{(n-k)!\, k!}}$$

3) $f_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}$, $\quad r \geq 0$

a) $$F_R(r) = \int_{\infty}^{r} f_R(r)\,dr \overset{\checkmark}{=} \int_0^r \frac{r}{\sigma^2} e^{\frac{-r^2}{2\sigma^2}}$$

$$= \cancel{\frac{2\sigma^2}{-2r}}\,\cancel{\frac{r}{\sigma^2}}\, e^{\frac{-r^2}{2\sigma^2}}\Big|_0^r = 1 - e^{\frac{-r^2}{2\sigma^2}}$$

$$\boxed{F_r(r) = 1 - e^{\frac{-r^2}{2\sigma^2}}}$$

3) $f_R(r) = \dfrac{r}{\sigma^2} e^{-r^2/2\sigma^2}$, $r \geq 0$

a) $F_R(r) = \displaystyle\int_{-\infty}^{r} f_R(r)\,dr = \int_0^r \frac{r}{\sigma^2} e^{\frac{-r^2}{2\sigma^2}}$

$= \cancel{\dfrac{2\sigma^2}{-2r}} \cancel{\dfrac{r}{\sigma^2}} e^{\frac{-r^2}{2\sigma^2}} \Big|_0^r = 1 - e^{\frac{-r^2}{2\sigma^2}}$

$$\boxed{F_r(r) = 1 - e^{\frac{-r^2}{2\sigma^2}},\ r \geq 0}$$

MECH 6325 HW1

3) b)

$$\boxed{\begin{aligned} \bar{r} &= \sigma\sqrt{\frac{\pi}{2}} \\ E\left[(r-\bar{r})^2\right] &= \frac{4-\pi}{2}\,\sigma^2 \end{aligned}}$$

c)

$$\mathrm{med}\left(f_R(r)\right) = \sigma\sqrt{2\ln(2)}$$

$$\mathrm{mode}\left(f_R(r)\right) = \sigma$$

MECH 6325 HW7

4) $f_x(x) = \begin{cases} \frac{x}{2}, & x = [0, 2] \\ 0, & \text{otherwise} \end{cases}$

$F_x(x) = \int_{-\infty}^{x} f_x(t)\,dt = \int_0^x \frac{t}{2}\,dt$

a) $\boxed{F_x(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{4}, & x = [0, 2] \\ 1, & x > 2 \end{cases}}$

b) $\bar{X} = E[X] = \int_{-\infty}^{\infty} x f_x(x)\,dx = \int_0^2 \frac{x^2}{2}\,dx = \frac{x^3}{6}\Big|_0^2$

$\boxed{\bar{X} = \frac{8}{6} = \frac{4}{3}}$

$var(x) = E[x^2] - \bar{x}^2$

$E[x^2] = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx = \int_0^2 \frac{x^3}{2}\,dx = \frac{x^4}{8}\Big|_0^2 = \frac{16}{8} = 2$

$var(x) = 2 - \left(\frac{4}{3}\right)^2 = \frac{18}{9} - \frac{16}{9} = \frac{2}{9}$

$\boxed{Var(x) = \frac{2}{9}}$

MECH 6325 HW1

5)

a) $$f_{XY}(x,y) = \begin{cases} c(y-x+1), & 0 \leq y \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$1 = \int_{x=0}^{1}\int_{y=0}^{x} c(y-x+1)\,dy\,dx$$

$$= c\int_0^1 \left(\frac{y^2}{2} + y(1-x)\Big|_0^x\right)dx$$

$$= c\int_0^1 \frac{x^2}{2} + x - x^2\,dx = c\int_0^1 x - \frac{x^2}{2}\,dx$$

$$= c\left(\frac{x^2}{2} - \frac{x^3}{6}\Big|_0^1\right) = c\underbrace{\left(\frac{1}{2} - \frac{1}{6}\right)}_{\frac{1}{3}} = 1$$

$$\boxed{c = 3}$$

MECH 6325 HW1

5)

b) $f_{XY}(x,y) = \begin{cases} 3(y-x+1), & 0 \le y \le x < 1 \\ 0, & \text{otherwise} \end{cases}$

$$f_X(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy = \int_0^x 3(y-x+1)\,dy$$

$$= 3\left(\frac{y^2}{2} + (1-x)y \Big|_0^x\right)$$

$$= \frac{3x^2}{2} + 3x - 3x^2$$

$$\boxed{f_x(x) = 3x - \frac{3x^2}{2}}$$

$$f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dx = \int_0^1 3(y-x+1)\,dx$$

$$= 3\left(xy - \frac{x^2}{2} + x \Big|_0^1\right) = 3\left(y - \frac{1}{2} + 1\right)$$

$$\boxed{f_y(y) = 3y + \frac{3}{2}}$$

MECH 6325 HW1

6) $f_{XY}(x,y) = \begin{cases} 6e^{-2x}e^{-3y}, & x>0, y>0 \\ 0, & \text{otherwise} \end{cases}$

$$\bar{X} = E[x] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{xy}(x,y)\,dx\,dy$$

$$= \int_0^{\infty}\int_0^{\infty} 6x e^{-2x} e^{-3y}\,dx\,dy$$

$$= 6\int_0^{\infty} e^{-3y}\left[\frac{-1}{4}(2x+1)e^{-2x}\Big|_0^{\infty}\right]dy$$

$$= \frac{6}{4}\int_0^{\infty} e^{-3y}\,dy$$

$$\boxed{\bar{X} = \frac{1}{2}} \qquad = \frac{\cancel{6}2}{\cancel{4}2}\left(\frac{-1}{\cancel{3}}e^{-3y}\Big|_0^{\infty}\right) = \frac{1}{2}$$

$$\bar{Y} = \int_0^{\infty}\int_0^{\infty} 6y e^{-2x} e^{-3y}\,dx\,dy = 6\underbrace{\int_0^{\infty} e^{-2x}\,dx}_{\frac{1}{2}}\underbrace{\int_0^{\infty} y e^{-3y}\,dy}_{\frac{1}{9}}$$

$$\boxed{\bar{Y} = \frac{1}{3}}$$

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$$f_{XY}(x,y) = \begin{cases} 6e^{-2x}e^{-3y}, & x>0, y>0 \\ 0, & \text{otherwise} \end{cases}$$

$$E(x^2) = \int_0^\infty \int_0^\infty 6x^2 e^{-2x} e^{-3y}\,dx\,dy$$

$$= \cancel{6}^{3} \underbrace{\int_0^\infty x^2 e^{-2x}\,dx}_{\frac{1}{\cancel{4}2}} \underbrace{\int_0^\infty e^{-3y}\,dy}_{\frac{1}{\cancel{3}}}$$

$$\boxed{E(x^2) = \frac{1}{2}}$$

$$E(y^2) = \int_0^\infty \int_0^\infty 6y^2 e^{-2x} e^{-3y}\,dx\,dy$$

$$= \cancel{6}^{3} \underbrace{\int_0^\infty e^{-2x}\,dx}_{\frac{1}{\cancel{2}}} \underbrace{\int_0^\infty y^2 e^{-3y}}_{\frac{2}{\cancel{27}9}}$$

$$\boxed{E(y^2) = \frac{2}{9}}$$

$$E(XY) = \int_0^\infty \int_0^\infty 6xy\, e^{-2x} e^{-3y}\,dx\,dy$$

$$= \cancel{6} \underbrace{\int_0^\infty x e^{-2x}\,dx}_{\frac{1}{\cancel{4}2}} \underbrace{\int_0^\infty y e^{-3y}\,dy}_{\frac{1}{\cancel{9}3}}$$

$$\boxed{E(XY) = \frac{1}{6}}$$

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b) $f_{XY}(x,y) = \begin{cases} 6e^{-2x}e^{-3y}, & x>0, y>0 \\ 0, & \text{otherwise} \end{cases}$

c) $R_{XY} = \begin{bmatrix} E[X^2] & E[XY] \\ E[YX] & E[Y^2] \end{bmatrix}$

$$\boxed{R_{XY} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{9} \end{bmatrix}}$$

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b) $f_{XY}(x,y) = \begin{cases} 6e^{-2x}e^{-3y}, & x>0, y>0 \\ 0, & \text{otherwise} \end{cases}$

d) $\bar{x} = \frac{1}{2}$ $\quad E[x^2] = \frac{1}{2}$ $\quad E[XY] = \frac{1}{6}$

$\bar{y} = \frac{1}{3}$ $\quad E[y^2] = \frac{2}{9}$

$$\sigma_x^2 = E[x^2] - (\bar{x})^2 = \frac{1}{2} - \cancelto{\frac{1}{4}}{\left(\frac{1}{2}\right)^2} = \frac{1}{4}$$

$$\boxed{\sigma_x^2 = \frac{1}{4}}$$

$$\sigma_y^2 = E[\boxed{y^2}] - \bar{y}^2 = \frac{2}{9} - \cancelto{\frac{1}{9}}{\left(\frac{1}{3}\right)^2} = \frac{1}{9}$$

$$\boxed{\sigma_y^2 = \frac{1}{9}}$$

$$C_{xy} = E[XY] - \bar{x}\bar{y} = \frac{1}{6} - \cancelto{\frac{1}{6}}{\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}$$

$$\boxed{C_{xy} = 0}$$

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b) $f_{XY}(x,y) = \begin{cases} 6e^{-2x} e^{-3y}, & x>0, y>0 \\ 0, & \text{otherwise} \end{cases}$

e) $C = \begin{bmatrix} \sigma_x^2 & C_{XY} \\ C_{YX} & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{9} \end{bmatrix}$

f) $\boxed{\rho = \frac{C_{XY}}{\sigma_x \sigma_y} = 0}$

MECH 6325 HW 1

7) $X = at + Y$ $\quad a = \mathbb{R}$

$Y = \mathcal{N}(0,\sigma^2)$

a)

$$E[X(t)] = \int_{-\infty}^{\infty} x f(x,t)\,dx$$

$$\bar{X}(t) = \int_{-\infty}^{\infty} x(at+Y)\,dx = (at+Y)\int_{-T}^{T} \cancel{x\,dx} \to 0$$

$$\cancel{\frac{x^2}{2}} \to \cancel{\frac{T^2}{2} - \frac{T^2}{2}} \to 0$$

$$\boxed{\bar{X}(t) = 0}$$

b)

$$A[X(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (at+Y)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left[\frac{at^2}{2} + Yt\right]_{-T}^{T} = \cancel{\frac{aT^2}{2}} + YT \cancel{-\frac{aT^2}{2}} + YT$$

$$= \lim_{T\to\infty} \cancel{\frac{1}{2T}}\, \cancel{2}Y\cancel{T}$$

$$\boxed{A[X(t)] = Y = \mathcal{N}(0,\sigma^2)}$$

c) Not stationary. nor Ergotic

$$A[X(t)] \neq E[X]$$

MECH 6325 HW 1

2) $X = \mathcal{N}(0,4) \quad Y = \mathcal{N}(1,9)$

$$Z = X + 2Y \qquad z = g(x,y)$$

$$Z = \mathcal{N}\left(\mu_x + 2\mu_y,\ (1\sigma_x)^2 + (2\sigma_y)^2\right)$$

$$= \mathcal{N}\left(0 + 2(1),\ 2^2 + (2(3))^2\right)$$

$$\boxed{Z = \mathcal{N}(2, 40)}$$

$$\boxed{f_Z(z) = \frac{1}{\sqrt{80\pi}}\, e^{-\frac{(z-2)^2}{80}}}$$

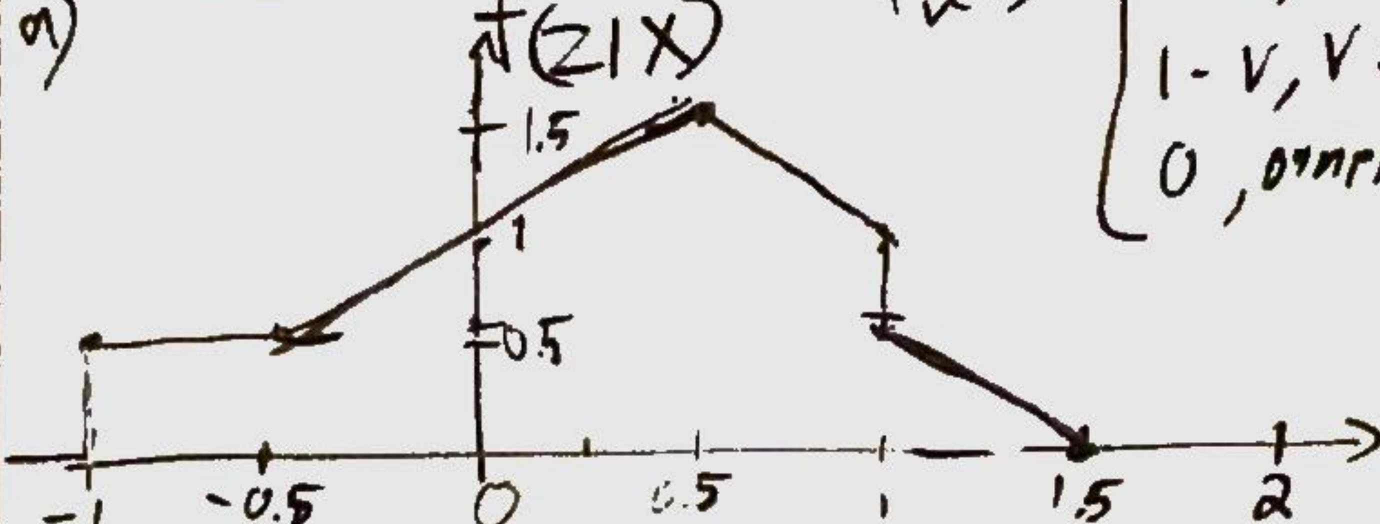
MECH 6325 HW 1

9) $Z = X + V$

$$f_x(x) = \begin{cases} 0.5, & x = [-1, 1] \\ 0, & \text{otherwise} \end{cases}$$

$$f_v(v) = \begin{cases} 1+v, & v = [-1, 0] \\ 1-v, & v = [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

a)



$\underline{Z = 0.5}$

b) $E[X \mid z=0.5] = \int x \underbrace{f_x(x \mid z=0.5)}\,dx$

$X = 0.5 - \cancel{V}$ $\bar{V} = 0$

$$\boxed{X = 0.5}$$

MECH 6325 HW 1

10) $S_x(w) = \dfrac{6w^2+12}{(w^2+4)(w^2+1)}$

$$= \frac{6(w^2+2)}{(w^2+4)(w^2+1)}$$

$$E[X^2] = \mathrm{Var}(X) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(w)\,dw$$

$$= \frac{1}{2\pi}\cancel{\int_{-\infty}^{\infty} \frac{6(w^2+2)}{(w^2+4)(w^2+1)}\,dw} \to 4\pi$$

$$\boxed{E[x^2] = 2}$$

$$E[x] = A[x(t)] = \lim_{w\to 0} S_x(w) = \frac{12}{4}$$

$$\boxed{E[x] = 3}$$

ECH 6325 HW 1

1)

$$X(t) \sim N(\bar{x}, \sigma_x^2)$$

$$R_x(\tau) = 4e^{-|\tau|}$$

stationary $\rightarrow$ $\sigma_x^2 = \sigma_{x_1}^2 = \sigma_{x_2}^2$, $\bar{x} = \bar{x}_1 = \bar{x}_2$

$X_1 = X(t_1)$

$X_2 = X(t_1 + 1)$

$$\boxed{R_x(1) = 4e^{-1}}$$

$$C_{x_1x_2} = R_x - \bar{X}_1\bar{X}_2 = R_x - \bar{x}^2$$

$$\rho = \frac{C_{x_1x_2}}{\sigma_{x_1}\sigma_{x_2}} = \frac{R_x - \bar{x}^2}{\sigma_x^2}$$

$$\boxed{\rho = \frac{4e^{-1} - \bar{x}^2}{\sigma_x^2}}$$

$$f(x_1, x_2) = \frac{\exp\left(\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\bar{x})^2 - 2\rho(x_1-\bar{x})(x_2-\bar{x}) + (x_2-\bar{x})^2}{\sigma_x^2}\right]\right)}{2\pi\sigma_x^2\sqrt{1-\rho^2}}$$

MECH 6325 Hw 1

12) $R_x(\tau) = \begin{cases} 1-|\tau|, & \tau=[-1,1] \\ 0, & \text{otherwise} \end{cases}$

$w_0 = \mathbb{R}$

$Y(t) = X(t)\cos(w_0 t + \lambda)$ $\quad \lambda = \begin{cases} \frac{1}{2\pi}, & [0,2\pi] \\ 0, & \text{otherwise} \end{cases}$

$S_x(w) = \int_{-\infty}^{\infty} R_x(\tau) e^{-jw\tau} d\tau = \int_{-1}^{0} + \int_{0}^{1} (1-\tau) e^{-jw\tau} d\tau$

$\underbrace{A\cos(w_0 t + \lambda)}$ Uniformly Distributed across Phase

$R_{x_2} = \frac{A^2}{2}\cos(w_0\tau)$

$S_x(w) = \frac{2(1-\cos(w))}{w^2}$

$S_{x_2} = \frac{1}{2}\delta(w-w_0) + \frac{1}{2}\delta(w+w_0)$

$$\boxed{S_Y(Y) = \frac{1-\cos(w)}{w^2}\left[\delta(w-w_0)+\delta(w+w_0)\right]}$$

MECH 6325 HW 1

13) $[X\ Y]^T$

$C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \sigma_x^2 & \alpha_{xy} \\ \alpha_{yx} & \sigma_y^2 \end{bmatrix}$

$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$\lambda = 1, 3$

$X = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\boxed{Z = P[X\ Y]^T}$

$C' = P C P^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$\boxed{\begin{aligned} \sigma_{x'}^2 &= 1 \\ \sigma_{y'}^2 &= 3 \end{aligned}}$

MECH 6325 HW 1

14) $\underbrace{X = Y}_{\text{Independent}} = \begin{cases} 1, & x = [0,1], y = [0,1] \\ 0, & \text{otherwise} \end{cases}$

$$Z = |X - Y|$$

$$f_Z(z) = \iint f_{XY}(x,y)\,dx\,dy = \int_{y=0}^{1}\int_{x=0}^{1} |X - Y|\,dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=0}^{y} (y - x)\,dx\,dy + \int_{x=0}^{1}\int_{y=0}^{x} (x - y)\,dy\,dx$$

$\cancel{yx - \frac{x^2}{2}} \rightarrow \frac{y^2}{2}$ $\quad$ $\cancel{xy - \frac{y^2}{2}} \rightarrow \frac{x^2}{2}$

$$= \cancel{\int_0^1 \frac{y^2}{2}\,dy} \rightarrow 1 + \cancel{\int_0^1 \frac{x^2}{2}\,dx} \rightarrow \frac{1}{2}$$

$$\boxed{f_Z(z) = \begin{cases} 1, & z = [0,1] \\ 0, & \text{otherwise} \end{cases}}$$

MECH 6325 HW 1 Ind. $\theta_1, \theta_2 = U[0, 2\pi]$

15)
a) $X(t) = \cos(w_0 t + \theta_1)\cos(w_0 t + \theta_2)$

$$E[X(t)] = \int_0^{2\pi}\int_0^{2\pi} \cos(w_0 t + \theta_1)\cos(w_0 t + \theta_2)\left(\frac{d\theta_1}{2\pi}\right)\left(\frac{d\theta_2}{2\pi}\right)$$

$$= \left(\underbrace{\frac{1}{2\pi}\int_0^{2\pi} \cos(w_0 t + \theta_1)\,d\theta_1}_{\downarrow}\right)\left(\underbrace{\frac{1}{2\pi}\int_0^{2\pi} \cos(w_0 t + \theta_2)\,d\theta_2}_{\downarrow}\right)$$

$$\cancel{\sin(w_0 t + \theta_1)\Big|_0^{2\pi}} \rightarrow 0 \qquad \cancel{\sin(w_0 t + \theta_2)\Big|_0^{2\pi}} \rightarrow 0$$

$$\boxed{E[x(t)] = \bar{x} = 0}$$

$$R_x(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$= E[\cos(w_0 t_1 + \theta_1)\cos(w_0 t_1 + \theta_2)\cos(w_0 t_2 + \theta_1)\cos(w_0 t_2 + \theta_2)]$$

$$= E[\cos(w_0 t_1 + \theta_1)\cos(w_0 t_2 + \theta_1)] + \ldots$$

$$= \underbrace{\int_0^{2\pi} \cos(w_0 t_1 + \theta_1)\cos(w_0 t_2 + \theta_1)\left(\frac{1}{2\pi}\right)d\theta_1}_{\downarrow} + \ldots$$

$$\boxed{R_x(t_1, t_2) = 2\pi \cos(w_0(t_1 - t_2))} + \ldots$$

MECH 6325 HW 1 Ind. $\theta_1, \theta_2 = U[0, 2\pi]$

15) b) $R_x(\tau) = 2\pi \cos(-w_0 \tau)$

$$S_x(w) = 2\pi^2 \left[\delta(w - w_0) + \delta(w + w_0) \right]$$

c)

$$A[x(t)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \cos(w_0 t + \theta_1) \cos(w_0 t + \theta_2) \, dt$$

$$A[x(t)] = \frac{\cos(\theta_1 - \theta_2)}{2} \neq E[x(t)]$$

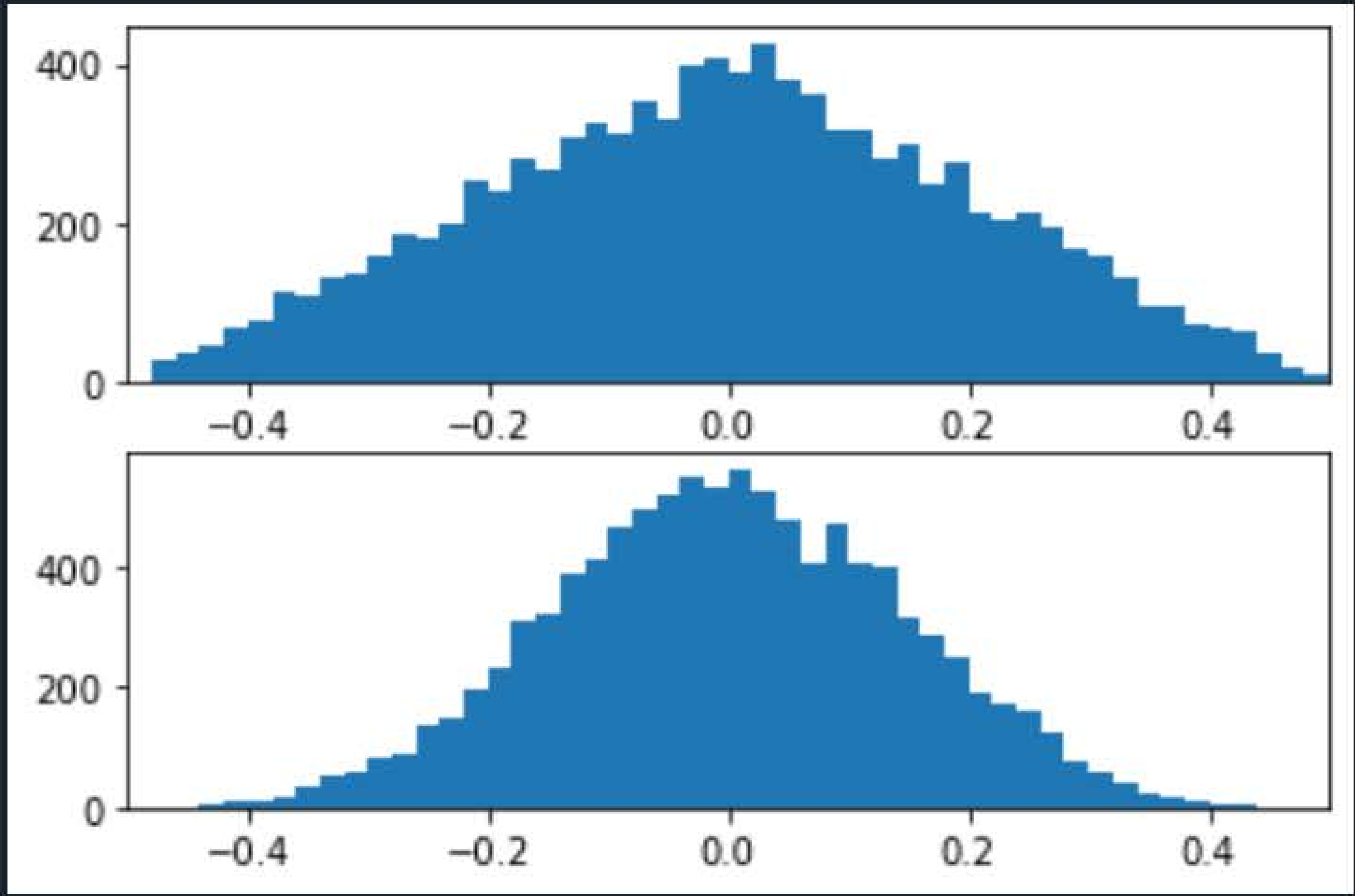
Not ergodic

...Drive - The University of Texas at Dallas (2020_Fall\MECH6325\Assignments\MECH6325-HW\MECH6325-HW1.py

MECH6325-HW1.py

```python
"""
Created on Fri Sep 11 07:17:41 2020

@author: Jonas
"""

import numpy as np
import matplotlib.pyplot as plt


# Problem 16


k = 2
n = 10000
X_1 = np.zeros(n)


for i, x in enumerate(X_1):
    j = 0
    temp = 0
    while j < k:
        temp += np.random.uniform(-0.5, 0.5)
        j += 1
    X_1[i] = temp / k


k = 4
n = 10000
X_2 = np.zeros(n)

for i, x in enumerate(X_2):
    j = 0
    temp = 0
    while j < k:
        temp += np.random.uniform(-0.5, 0.5)
        j += 1
    X_2[i] = temp / k


fig, (ax1, ax2) = plt.subplots(2, 1)

x_min = -0.5
x_max = 0.5

ax1.set_xlim(x_min, x_max)
ax2.set_xlim(x_min, x_max)

ax1.hist(X_1, 50, range = (x_min, x_max))
ax2.hist(X_2, 50, range = (x_min, x_max))
```

192 %



The main difference is the closer resemblence to a gaussian distribution. This makes sense as distributions grow they tend towards a normal curve.

Variable explorer Plots

Source Console Object

Usage

Here you can get help of any object by pressing **Ctrl+I** in front of it, either on the Editor or the Console.