1)
$$x_{k+1} = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_k \qquad w_k \sim (0,1)$$

$$P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x_{k+1} = F x_k + \underbrace{G u}_{u=0} + W_k \qquad W_k = \begin{bmatrix} w_k \\ w_k \end{bmatrix}$$

a)

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\tilde{Q} = L Q L^T \qquad Q = 1$$

$$\bar{x} = (I - F)^{-1} G \cancel{u}^{0}$$

$$\boxed{\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

b)
$$P = F P F^T + Q$$

ss) $$\boxed{\bar{P} = \text{dlyap}(F, Q) = \begin{bmatrix} 1.06 & 0.99 \\ 0.99 & 0.24 \end{bmatrix}}$$

explicit)
$$P_1 = F P_0^{\,I} F^T + Q = F F^T + Q$$

$$P_2 = F P_1 F^T + Q = F(FF^T + Q)F^T + Q = FFF^TF^T + FQF^T + Q$$

$$P_3 = F P_2 F^T + Q = F^{(3)} F^{T(3)} + F^{(2)} Q F^{T(2)} + FQF + Q$$

$$\boxed{P_i = F^{(i)} F^{T(i)} + \sum_{k=0}^{i-1} F^{(k)} Q F^{T(k)}}$$

2) a)

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 5 \\ 1 \end{bmatrix} w(t) \qquad w \sim (0,1)$$

$$Q = 1 \qquad P(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tilde{Q} = L Q L^T = \begin{bmatrix} 25 & 5 \\ 5 & 1 \end{bmatrix}$$

$$\underbrace{0 = AP + PA^T + Q}_{\downarrow \text{ lyap}(A,Q)}$$

$$P = \begin{bmatrix} 12.5 & 2.5 \\ 2.5 & 0.5 \end{bmatrix}$$

b)

$$x_{k+1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} x_k + w_k \qquad w_k \sim \mathcal{N}(0,Q)$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underbrace{P = FPF^T + Q}_{\downarrow \text{ dlyap}(F,Q)}$$

$$P = \begin{bmatrix} 1.33 & 0 \\ 0 & 1.33 \end{bmatrix}$$

3)

$$x_K = \frac{-1}{2} x_{K-1} + w_{K-1}$$

$$E(w_{K-1}) = 0$$

$$E(w_K w_j) = e^{-|K-j|}$$

$$P_0 = 1$$

$$F = \frac{-1}{2}$$

$$Q = 1$$

$$P_K = F P_{K-1} F^T + Q$$

$$P_K = F^{2K} + \sum_{i=0}^{K-1} F^{2i} Q = \left(-\frac{1}{2}\right)^{2K} + \sum_{i=0}^{K-1} \left(-\frac{1}{2}\right)^{2K} (1)$$

$$= \left(\frac{1}{4}\right)^K + \sum_{i=0}^{K-1} \left(\frac{1}{4}\right)^K = \sum_{i=0}^{K} \left(\frac{1}{4}\right)^K$$

$$\bar{P} = \underbrace{F \bar{P} F + Q}_{\downarrow \text{dlyap}(F, Q)}$$

$$\boxed{\bar{P} = 1.33}$$

$$\boxed{P_K = \sum_{i=0}^{K} \left(\frac{1}{4}\right)^K}$$

4) $y_k = z_k + z_{k-1}$ $\quad z \sim (0,1)$

a) $x_k = \begin{bmatrix} z_{k-1} \\ z_k \end{bmatrix}$ $\quad x_{k-1} = \begin{bmatrix} z_{k-2} \\ z_{k-1} \end{bmatrix}$

$$x_{k} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{F} \begin{bmatrix} z_{k-1} \\ z_k \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{L} z_{k-1} \quad w_k \sim (0,1) \quad Q = 1$$

$G = 0$ $\quad \tilde{Q} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$y_k = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{H} x_k$$

$P_k^- = F P_k^+ F^T + Q$

b) Let $P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \underline{R = 0}$

$P_k^+ = (I - K_k H_k)(P_k^-)$ $\quad K = P_k^- H_k^T (H_k P_k^- H_k^T + \cancel{R_k})^{-1}$

$= P_k^- - P_k^- H_k^T (H_k P_k^- H_k^T)^{-1} H_k P_k^-$ $\quad K = P_k^- H_k^T (H_k P_k^- H_k^T)^{-1}$

$$H_k P_k^- H_k^T = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} P_{k11}^- & P_{k12}^- \\ P_{k21}^- & P_{k22}^- \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$P_1^+ = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= (P_{k11}^- + P_{k21}^-) + (P_{k21}^- + P_{k22})$$

$$P_k^- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_{k11}^+ & P_{k12}^+ \\ P_{k21}^+ & P_{k22}^+ \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K = \frac{1}{(P_{k11}^- + P_{k12}^- + P_{k21}^- + P_{k22}^-)} \begin{bmatrix} P_{k11}^- + P_{k12} \\ P_{k21} + P_{k22} \end{bmatrix}$$

$$= \begin{bmatrix} P_{k22}^+ & 0 \\ 0 & 1 \end{bmatrix}$$

$$K = \frac{1}{P_{(k-1)22}^+ + 1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad K_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{P_k^+ = \frac{1}{P_{(k-1)22}^+ + 1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{k+1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

4) cont.

c) $E\left[\|x_k - \hat{x}_k^+\|_2^2\right] = Tr\left(P_k^+\right) = Tr\left(\frac{1}{k+1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)$

$$= \boxed{\frac{2}{k+1}}$$

---

5) $R_k = R$ $\quad x_{k+1} = F x_k + \cancel{G u}^{0}$ $\quad R_\Sigma = R + \Delta R$ $\quad K_k = P_k^+ H_k^T R_k^{-1}$

$P_k^+ = P_k^+$ $\quad y_k = H x_k + v_k$ $\quad P_{k\Sigma}^+ = \Sigma_k^+$

$P_k^- = P_k^-$ $\quad v_k \sim (0, R)$ $\quad P_{k\Sigma}^- = \Sigma_k^-$ $\quad v_k \sim (0, R+\Delta R)$

$$\hat{x}_{k+1}^- = F(I - K_k H_k)\hat{x}_k^- + F K_k \underbrace{y_k}_{(Hx_k + v_k)}$$

$$\hat{x}_{k+1}^- - x_{k+1} = F(I - K_k H)\hat{x}_k + F K_k H x_k + F K_k v_k - F x_k$$

$$= F\hat{x}_k - F K_k H(\hat{x}_k - x_k) - F x_k + F K_k v_k$$

$$\hat{x}_{k+1}^- - x_{k+1} = F(I - K_k H)(\hat{x}_k - x_k) + F K_k v_k$$

$$P_{k+1}^- = E\left[(\hat{x}_{k+1} - x_{k+1})(\hat{x}_{k+1} - x_{k+1})^T\right]$$

$$= E\left[\left(F(I - K_k H)(\hat{x}_k - x_k) + F K_k v_k\right)\left(F(I - K_k H)(\hat{x}_k - x_k) + F K_k v_k\right)^T\right]$$

$$= E\left[F\left((I - K_k H)(\hat{x}_k - x_k) + K_k v_k\right)\left((\hat{x}_k - x_k)^T (I - K_k H)^T + (K_k v_k)^T\right)F^T\right]$$

$$= F\left(E\left[(I - K_k H)(\hat{x}_k - x_k)(\hat{x}_k - x_k)^T (I - K_k H)^T + (I - K_k H)(\hat{x}_k - x_k)(K_k v_k)^T + (K_k v_k)(K_k v_k)^T + (K_k v_k)(\hat{x}_k - x_k)^T (I - K_k H)^T\right]\right)F^T$$

5) cont.

$$P_{k+1}^- = F\left[\begin{array}{l}(I-K_kH)\underbrace{E[(\hat{x}_k-x_k)(\hat{x}_k-x_k)^T]}_{P_k^-}(I-K_kH)^T \\ +(I-K_kH)\underbrace{E[(\hat{x}_k-x_k)]}_{0}(K_kV_k)^T \\ +(K_kV_k)\underbrace{E[(\hat{x}_k-x_k)^T]}_{0}(I-K_kH)^T \\ +K_k\underbrace{E(V_kV_k^T)}_{R}K_k^T\end{array}\right]F^T$$

$$P_{k+1}^- = F\left((I-K_kH)P_k^-(I-K_kH)^T + K_kRK_k^T\right)F^T$$

similarily,

$$\Sigma_{k+1}^- = F\left((I-K_kH)\Sigma_k^-(I-K_kH)^T + K_k(R+\Delta R)K_k^T\right)F^T$$

Let $\Delta_{k+1}^- = \left(\Sigma_k^- - P_k^-\right)$

Clearly,

$$\Delta_{k+1}^- = F\left((I-K_kH)\Delta_k^-(I+K_kH) + K_k(\Delta R)K_k^T\right)F^T$$

It can be said that $\forall\ \Delta R = \Delta R^T > 0$,

$$\Delta_k^- = \Delta_k^{-T} > 0$$

6)
$$x_k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_{k-1} + w_{k-1} \qquad w_k \sim (0, Q)$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k$$

$$v_k \sim (0, R)$$

$$P_0 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$R_k = 2 + (-1)^k$$

P_∞^+ does not exist... It flips between two steady states as R constantly changes.

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7) $x_k = x_{k-1} + w_{k-1}$

$Q = 4$ $P_0^+ = 100$

$R = 1$

$w_{k-1} \sim (0, Q)$

$v_k \sim (0, R)$

i) $y_k = 0.5\, x_k + v_k$

Error decends to a steady-state value around 2.5.

ii) $y_k = \left(\cos\left(1 + k/120\right)\right) x_k + v_k$

The Error fluctuates as a function of time as H fluctuates. The error remains low until the time step aproaches 60 (when the cos goes to zero) and the error jumps up above 50. This is likely due to the large size of measurement noise compared to measurement signal.

8) $\dot{x} = -x(t) + w(t)$

$A = -1$

$x(0) = 1$

$Q(t) = e^{-2}\delta(t)$

$P(0) = 1$

ss) $0 = \underbrace{AP + PA^T + Q}_{\text{lyap}(A,Q)}$

$\bar{P} = 0.0677$