Assignment 5

Optimal Estimation and Kalman Filter
Fall 2020
Due date: November 13, 2020 at 1:00 PM

Problem 1. A stationary Gaussian random process has a power spectrum of the form

$$S_y(\omega) = \frac{\omega^2 + 1}{\omega^4 + 20\omega^2 + 64} \tag{1}$$

1. Determine the spectral factorization of $S_y(\omega)$ to produce the shaping filter that converts unity white noise into the $y(t)$ process.
2. Perform a partial fraction expansion of the shaping filter. This should result in the parallel combination of two first-order systems.
3. Define the states x_1 and x_2 as the outputs of these first-order systems, and develop a (continuous-time) Kalman filter model.

Problem 2. Consider the continuous-time scalar system

$$\begin{aligned} \dot{x} &= -2x + w \\ y &= x + v \end{aligned}$$

where $w(t)$ and $v(t)$ are continuous-time white noise with variances $Q_c = 2$ and $R_c = 1$ respectively. Design a Kalman filter to estimate x.

1. What is the theoretical steady-state variance of the estimation error?
2. Simulate the system for 1000 s with discretization step sizes of 0.4, 0.2 and 0.1 s. What are the resulting simulated estimation-error variances?

Problem 3. Consider the system

$$\begin{aligned} z_{k+1} &= az_k + w_k, \quad w_k \sim (0, Q) \\ y_k &= z_k + v_k, \quad v_k \sim (0, R) \end{aligned}$$

with unknown parameter a. Suppose that an EKF is used to estimate the state z_k and the parameter a. Further suppose that the artificial noise term used in the estimation of a is zero, and the EKF converges to the correct value of a with zero variance. Show that the EKF in this situation is equivalent to the standard Kalman filter for the scalar system when a is known.

Problem 4. Consider the scalar system

$$\dot{x} = -x + u + w$$

w is zero-mean process noise with a variance of Q. The control has a mean value of u_0, an uncertainty of 2 (one standard deviation), and is uncorrelated with w. Rewrite the system equations to obtain an equivalent system with normalized control that is perfectly known. What is the variance of the new process noise term in the transformed system equation?

Problem 5. Consider the system

$$\begin{aligned} x_{k+1} &= \phi x_k + w_k \\ y_k &= x_k \end{aligned}$$

where $w_k \sim (0, 1)$, and $\phi = 0.9$ is an unknown constant. Design an extended Kalman filter to estimate ϕ. Simulate the filter for 100 time steps with $z_0 = 1$, $P_0 = I$, $\hat{x}_0 = 0$, and $\hat{\phi}_0 = 0$. Hand in your source code and a plot showing $\hat{\phi}$ as a function of time.