1) $S_v(w) = \frac{w^2+1}{w^2+4}$ $\qquad S_u = \sigma_u^2 = 4$

Let

$$G(w) = \frac{G_0(jw+a)}{jw+b}$$

$$G(w)\,G(-w) = \frac{G_0^2(w^2+a^2)}{(w^2+b^2)}$$

$$S_v\,G(w)\,G(-w) = S_u$$

$$\left(\frac{w^2+1}{w^2+4}\right)\frac{G_0^2(w^2+a^2)}{(w^2+b^2)} = 4 \Rightarrow \begin{matrix} G_0^2 = 4 \\ a^2 = 4 \\ b^2 = 1 \end{matrix} \quad \begin{matrix} G_0 = 2 \\ a = 2 \\ b = 1 \end{matrix}$$

$$G(s) = \frac{2(s+2)}{s+1} = \frac{U(s)}{V(s)}$$

$$(s+1)\,U(s) = (2s+4)\,V(s)$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$\boxed{\dot{u} + u = 2\dot{v} + 4v}$$

2) $S_x = \frac{2}{w^2+2}$ $\quad S_v = 1$

Non-Causal:

$$G(w) = \frac{S_x(w)}{S_x(w)+S_v(w)} = \frac{\frac{2}{w^2+2}}{\frac{2}{w^2+2}+1} = \frac{\frac{2}{\cancel{w^2+2}}}{\frac{w^2+4}{\cancel{w^2+2}}}$$

$$\boxed{G(w) = \frac{2}{w^2+4}}$$

$$S_e(w) = [1-G(w)][1-G(-w)]S_x(w) - G(w)G(-w)S_v(w)$$

$$= \left[\frac{w^2+4-2}{w^2+4}\right]\left[\frac{w^2+4-2}{w^2+4}\right]\left(\frac{2}{w^2+2}\right) - \left(\frac{2}{w^2+4}\right)^2(1)$$

$$= \frac{2(w^2+2)}{(w^2+4)^2} - \frac{2^2}{(w^2+4)^2}$$

$$\boxed{S_e(w) = \frac{2w^2+\cancel{4-4}}{(w^2+4)^2}}$$

$$E[e^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2w^2}{(w^2+4)^2}\,dw$$

integrate

$$\boxed{E[e^2(t)] = 0.25}$$

2) cont. $S_x = \frac{2}{w^2+2} \qquad S_v = 1$

Causaul:

$$S_{xv}(w) = S_x(w) + S_v(w) = \frac{2}{w^2+2} + 1$$

$$= \frac{w^2+4}{w^2+2} = \underbrace{\left(\frac{jw+\sqrt{4}}{jw+\sqrt{2}}\right)}_{S_{xv}^{+}(w)}\underbrace{\left(\frac{-jw+\sqrt{4}}{-jw+\sqrt{2}}\right)}_{S_{xv}^{-}(w)}$$

$$\frac{S_x(w)}{S_{xv}^{-}(w)} = \frac{\frac{2}{w^2+2}}{\frac{-jw+2}{-jw+\sqrt{2}}} = \frac{2(-jw+\sqrt{2})}{(w^2+2)(-jw+2)} = \frac{2\cancel{(-jw+\sqrt{2})}}{(jw+\sqrt{2})\cancel{(-jw+\sqrt{2})}(-jw+2)}$$

$$= \frac{2}{(jw+\sqrt{2})(-jw+2)}$$

PFE

$$= \underbrace{\left(\frac{2-\sqrt{2}}{jw+\sqrt{2}}\right)}_{\text{Causal}} + \left(\frac{2-\sqrt{2}}{-jw+2}\right)$$

$$\boxed{G(w) = \frac{\frac{2-\sqrt{2}}{jw+\sqrt{2}}}{\underbrace{\frac{jw+\sqrt{4}}{jw+\sqrt{2}}}} = \frac{2-\sqrt{2}}{jw+\sqrt{4}}}$$

$$\cancel{S_e(w) = [1-G(w)][1-G(-w)]S_x(w) - G(w)G(-w)S_v(w)}$$

$$\cancel{= \left[1-\frac{\sqrt{2}-2}{jw+\sqrt{4}}\right]\left[1-\frac{\sqrt{2}-2}{-jw+\sqrt{4}}\right]\frac{2}{w^2+2} - \frac{(\sqrt{2}-2)^2}{(jw+\sqrt{4})(-jw+\sqrt{4})}(1)}$$

$$\cancel{= \left[\frac{jw+2-\sqrt{2}+2}{jw+\sqrt{4}}\right]\left[\frac{-jw+2-\sqrt{2}+2}{-jw+\sqrt{4}}\right]\frac{2}{w^2+2} - \frac{2-(2)(2)(\sqrt{2})+4}{w^2+4}}$$

2) cont.

$$\boxed{G(\omega) = \frac{2-\sqrt{2}}{j\omega + \sqrt{4}}}$$

$$S_e(\omega) = [1 - G(\omega)][1 - G(-\omega)] S_x(\omega) - G(\omega)G(-\omega)S_v(\omega)$$

$$= \left(\frac{j\omega + \sqrt{4} - (2-\sqrt{2})}{j\omega + \sqrt{4}}\right)\left(\frac{-j\omega + \sqrt{4} - (2-\sqrt{2})}{-j\omega + \sqrt{4}}\right)\left(\frac{2}{\omega^2 + 2}\right) - \frac{(2-\sqrt{2})^2}{(j\omega + \sqrt{4})(-j\omega + \sqrt{4})}$$

$$= \frac{2(\omega^2 + 2)}{\omega^2 + 4} - \frac{4 - 4\sqrt{2} + 2}{\omega^2 + 4}$$

$$= \frac{2\omega^2 + 4 - 6 + 4\sqrt{2}}{\omega^2 + 4} = \frac{2\omega^2 - 2 + 4\sqrt{2}}{\omega^2 + 4}$$

$$= \frac{2(\omega^2 + (2\sqrt{2} - 1))}{\omega^2 + 4} \approx \frac{2(\omega^2 + 1.83)}{\omega^2 + 4}$$

$$\boxed{\begin{gathered} E[e^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2(\omega^2 + 1.83)}{\omega^2 + 4}\, d\omega \\ E[e^2(t)] \approx 0.42 \end{gathered}}$$

3) Unbiased: $E[X] = \mu_x$

$$\hat{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \hat{y} = \frac{1}{m}\sum_{i=1}^{m} y_i$$

a)
$$\hat{x}^2 - \hat{y}^2 = \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) - \left(\frac{1}{m}\sum_{i=1}^{m} y_i\right)\left(\frac{1}{m}\sum_{i=1}^{m} y_i\right)$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j\right) - \frac{1}{m^2}\left(\sum_{i=1}^{m}\sum_{i=1}^{n} y_i y_j\right)$$

$$E[\hat{x}^2] = E\left[\frac{1}{n^2}\sum_{i=1}^{n}\sum_{i=j}^{m} (x_i - \mu_x)(x_j - \mu_x) + \mu_x^2 + x_i\mu_x + x_j\mu_x\right]$$

$E[(x_i-\mu_x)(x_j-\mu_x)] = 0 \ \forall i \neq j$

$$E[\hat{x}^2] = \frac{1}{n^2}\left[\sum_{i=1}^{n} \underbrace{E[(x_i - \mu_x)^2]}_{\sigma_x^2} + \sum_{j=1}^{n} - \mu_x^2 + \underbrace{E[x_i\mu_x]}_{\mu_x^2} + \underbrace{E[x_j\mu_x]}_{\mu_x^2}\right]$$

$$= \frac{1}{n^2}\left[n\sigma_x^2 + n^2\mu_x^2\right]$$

$$E[\hat{x}^2] = \mu_x^2 + \frac{\sigma_x^2}{n}$$

$$\boxed{E[\hat{x}^2 - \hat{y}^2] = \mu_x^2 + \frac{\sigma_x^2}{n} - \mu_y^2 - \frac{\sigma_y^2}{m}}$$

Biased...

as n and $m \to \infty$ it becomes unbiased

3) cont.

b) $\sigma_x^2 = \sigma_y^2 = \sigma^2$ $\bigg\{ S_p = \frac{(n-1)S_x^2 + (m-1)S_y^2}{n+m-2}$

$$\hat{S}_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{x})^2 = \frac{1}{n-1}\sum_{i=1}^{n} x_i^2 - 2x_i\hat{x} + \hat{x}^2$$

$$\hat{S}_y^2 = \frac{1}{m-1}\sum_{i=1}^{m}(y_i - \hat{y})^2$$

$$E[S_x^2] = \frac{1}{n-1}\sum_{i=1}^{n} E[x_i^2] - E[2x_i\hat{x}] + E[\hat{x}^2]$$

$$= \frac{1}{n-1}\sum_{i=1}^{n} \cancel{\mu_x^2} - \cancel{2\mu_x^2} + \cancel{\mu_x^2} + \frac{\sigma_x^2}{n}$$

$$E[S_x^2] = \frac{1}{n-1}\frac{\cancel{n}\sigma_x^2}{\cancel{n}} = \frac{\sigma_x^2}{n-1} = \frac{\sigma^2}{n-1}$$

$$E[S_y^2] = \frac{\sigma_y^2}{n-1} = \frac{\sigma^2}{n-1}$$

$$E[S_p] = \frac{(n-1)E[S_x^2] + (m-1)E[S_y^2]}{n+m-2} = \frac{\cancel{\frac{(n-1)}{(n-1)}}\sigma^2 + \cancel{\frac{(m-1)}{(m-1)}}\sigma^2}{n+m-2}$$

$$= \frac{2\sigma^2}{n+m-2}$$

4) $G(s) = \frac{1}{s-3}$

Causal + unstable: $Re\{s\} > 3$

$g_1(t) = e^{3t} u(t)$

$G_1(s) = \int_{0}^{\infty} g_1(t) e^{-st} dt$

$= \mathcal{L}\{\}$ unilateral $\mathcal{L}$ table

$\frac{1}{s-3}$

Noncausal + stable: $Re\{s\} < 3$

$g_2(t) = -e^{3t} u(-t)$

$G_2(s) = \int_{-\infty}^{0} e^{-st} g_2(t)\, dt$

$\mathcal{L}\{\}$ Bi-lateral $\mathcal{L}$ table

$\frac{1}{s-3}$

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5) $\bar{X}_1 \sim N\left(\mu, \frac{\sigma_1^2}{n}\right)$ $\quad \bar{x}_1, x_2 \equiv$ independent

$\bar{X}_2 \sim N\left(\mu, \frac{\sigma_2^2}{n}\right)$

a) $T = w\bar{X}_1 + (1-w)\bar{X}_2$

$$E[T] = E[w\bar{X}_1] + E[(1-w)\bar{X}_2] = w\mu_x + (1-w)\mu_x$$

$$\boxed{E[T] = \mu_x}$$

b) $$\sigma_T^2 = w^2\left(\frac{\sigma_1^2}{n}\right)^2 + (1-w)^2\left(\frac{\sigma_2^2}{n}\right)^2 = w^2\left(\frac{\sigma_1^2}{n}\right)^2 + (1-2w+w^2)\left(\frac{\sigma_2^2}{n}\right)^2$$

$$\frac{\partial \sigma_T^2}{\partial w} = 2w\left(\frac{\sigma_1^2}{n}\right)^2 + (-2+2w)\left(\frac{\sigma_2^2}{n}\right)^2 = 0$$

$$\cancel{2}w\left[\left(\frac{\sigma_1^2}{\cancel{n}}\right)^2 + \left(\frac{\sigma_2^2}{\cancel{n}}\right)^2\right] = \cancel{2}\left(\frac{\sigma_2^2}{\cancel{n}}\right)^2$$

$$\boxed{w = \frac{\sigma_2^4}{\sigma_1^4 + \sigma_2^4}}$$

6)

$$S_x(s) = \frac{1-s^2}{s^4-5s^2+4} \qquad S_v(s) = 1$$

$$= \frac{1-s^2}{(s-2)(s-1)(s+1)(s+2)}$$

Let

$$\underbrace{S_{xv}(s) = S_x(s) + S_v(s)} = \frac{1-s^2}{s^4-5s^2+4} + 1$$

$$= \frac{(1-s^2)+(s^4-5s^2+4)}{s^4-5s^2+4}$$

$$= \frac{s^4-6s^2+5}{s^4-5s^2+4} = \frac{\cancel{(s-1)}\cancel{(s+1)}(s^2-5)}{(s-2)\cancel{(s-1)}\cancel{(s+1)}(s+2)}$$

$$= \frac{s^2-5}{(s-2)(s+2)} \xrightarrow{PFE} \underbrace{\frac{0.25}{s-2}}_{\substack{\text{anti-stable} \\ S_{xv}^-(s)}} + \underbrace{\frac{0.25}{s+2} + 1}_{\substack{\text{stable} \\ S_{xv}^+(s)}}$$

$$\frac{S_x(s)}{S_{xv}^-(s)} = \frac{\dfrac{1-s^2}{\cancel{(s-2)}(s-1)(s+1)(s+2)}}{\dfrac{0.25}{\cancel{(s-2)}}}$$

$$= \frac{4(1-s^2)}{(s-1)(s+1)(s+2)} = \frac{-4\cancel{(s-1)}\cancel{(s+1)}}{\cancel{(s-1)}\cancel{(s+1)}(s+2)}$$

$$\boxed{\frac{S_x(s)}{S_{xv}^-(s)} = \frac{-4}{s+2}}$$

6) cont.

$$G(s) = \frac{\text{Causal}\left(\frac{S_x(s)}{S_{xv}^-(s)}\right)}{S_{xv}^+(s)}$$

$$= \frac{\frac{-4}{\cancel{s+2}}}{\frac{1}{4\cancel{(s+2)}} + (4\cancel{(s+2)})} = \frac{-4(4)}{4s+9}$$

$$= \frac{-4\cancel{(4)}}{\cancel{4}(s+\frac{9}{4})}$$

$$\boxed{G(s) = \frac{-4}{s+\frac{9}{4}}}$$