# Assignment 2

Optimal Estimation and Kalman Filter
Fall 2020
Due date: October 9, 2020 at 1:00 PM

**Problem 1.** A random process with the PSD of

$$S_v = \frac{\omega^2 + 1}{\omega^2 + 4}$$

is to be filtered with an LTI system to produce a white noise random process with variance $\sigma_u^2 = 4$ at the output. What should be the differential equation of the LTI system.

**Problem 2.** Consider the system in Fig. 1 where the signal and noise power spectral density are given as

$$S_x = \frac{2}{\omega^2 + 2},$$
$$S_v = 1.$$

Assuming zero mean estimation error, calculate and compare the mean square errors of the causal and non-causal Wiener filter.

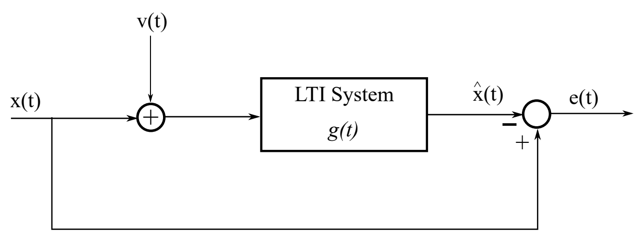


Figure 1:

**Problem 3.** An estimator of a parameter is called unbiased if its expected value is equal to the true value of the parameter. Let $\bar{X}$ and $S_x^2$ be the sample mean and variance for a set of random variables $\{x_1, x_2, \ldots, x_n\}$ of size $n$ with mean $\mu_x$ and variance $\sigma_x^2$. Similarly, let $\bar{Y}$ and $S_y^2$ be the sample mean and variance for a set of random variables $\{y_1, y_2, \ldots, y_m\}$ of size $m$ with mean $\mu_y$ and variance $\sigma_y^2$. Further suppose that $\mathbf{E}[(x_i - \mu_x)(x_j - \mu_x)] = 0$ and $\mathbf{E}[(y_i - \mu_y)(y_j - \mu_y)] = 0$ for $i \neq j$.

a. Is $\bar{X}^2 - \bar{Y}^2$ an unbiased estimator for the parameter $\mu_x^2 - \mu_y^2$? What happens to the bias as the sample sizes of $n$ and $m$ increase to $\infty$?

b. Assuming $\sigma_x^2 = \sigma_y^2 = \sigma^2$, show that

$$S_p = \frac{(n-1)S_x^2 + (m-1)S_y^2}{n+m-2}$$

is an unbiased estimator of $\sigma^2$.
(Hint: $S_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{X})^2$ and $S_y^2 = \frac{1}{m-1}\sum_{i=1}^{m}(y_i - \bar{Y})^2$)

**Problem 4.** A system has the transfer function

$$G(s) = \frac{1}{s-3}$$

If the input is an impulse, there are two solutions for the output $x(t)$ that satisfy the transfer function. One solution is causal and unstable, the other solution is anti-causal and stable. Find the two solutions.

**Problem 5.** Suppose $\bar{X}_1$ and $\bar{X}_2$ are the sample means of random normal samples of size $n$ such that $\bar{X}_1 \sim \mathcal{N}(\mu, \frac{\sigma_1^2}{n})$ and $\bar{X}_2 \sim \mathcal{N}(\mu, \frac{\sigma_2^2}{n})$. Furthermore suppose that $\bar{X}_1$ and $\bar{X}_2$ are independent.

a. Show that $T = \omega\bar{X}_1 + (1-\omega)\bar{X}_2$ is an unbiased estimator for $\mu$. i.e. $\mathbf{E}[T] = \mu$.

b. Find $\omega$ such that $T$ has the minimum variance among all the estimators.

**Problem 6.** Suppose a signal $x(t)$ with power spectral density

$$S_x(s) = \frac{1-s^2}{s^4 - 5s^2 + 4}$$

is corrupted with additive white noise $v(t)$ with a power spectral density $S_v(s) = 1$. Find the optimal causal Wiener filter to extract the signal from the noise corrupted signal.