MECH 6325 - HW5 | Jonas Wagner | 2020-11-11 | 1

1)

$$S_x(w) = \frac{w^2+1}{w^4+20w^2+64}$$

1.1)

$$S_y(w) = \frac{w^2+1}{(w^2+4)(w^2+16)}$$

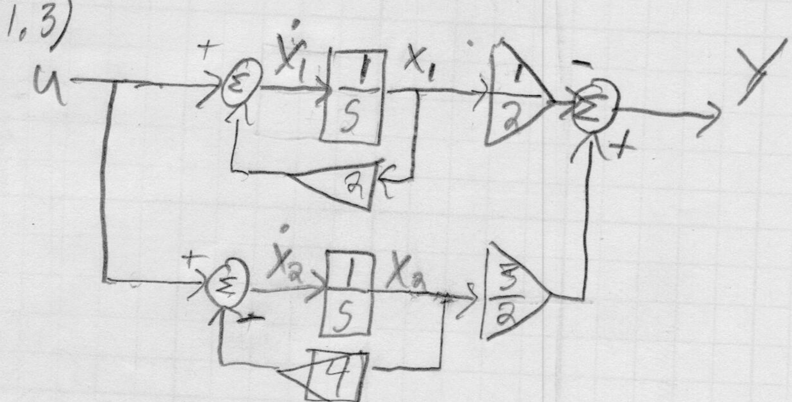
$$G(w)\,G(-w) = S_y(w) = \frac{(1+jw)(1+j(-w))}{(2+jw)(2+j(-w))(4+jw)(4+j(-w))}$$

$$\boxed{G(s) = \frac{s+1}{(s+2)(s+4)}}$$

1.2)

$$\boxed{G(s) = \frac{-1}{2(s+2)} + \frac{3}{2(s+4)}}$$

1.3)



$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \end{bmatrix} x$$

2) d) cont.

Defining $\dot{x} = -2x + w$

$y = x + v$

$A = -2 \quad L - Q_c = 2$

$C = 1 \quad G - R_c = 1$

$\hat{x}(0) = E[x(0)]$

$P(0) = E[(x(0) - \hat{x}(0))(x(0) - \hat{x}(0))^T]$

$K = P C^T R^{-1} = P(1)(1)^{-1} = P$

$\dot{\hat{x}} = A\hat{x} + \cancel{Bu} + K(y - C\hat{x}) = -2\hat{x} + K(y - \hat{x})$

$\dot{P} = -P \cancel{C^T R_c^{-1} C} P + AP + PA^T + Q_c$

$\boxed{\dot{P} = -P^2 - 4P + 2}$

$\boxed{\dot{\hat{x}} = -2\hat{x} + K(y - \hat{x})}$

2.1) SS:

$0 = P^2 + 4P - 2$

$P = \frac{-4 \pm \sqrt{4^2 - 4(-2)}}{2} = -2 \pm \sqrt{6}$

$\boxed{P = \sqrt{6} - 2 \cong 0.449}$

2.2) $F = I + AT \quad Q = Q_c(T)$

$H = C \quad R = \frac{R_c}{T}$

$x_{k+1} = F x_{k-1} + w_{k-1}$

$y_k = H x_k + v_k$

3) $z_{k+1} = a z_k + w_k \qquad w_k \sim (0, Q)$

$y_k = z_k + v_k \qquad v_k \sim (0, R)$

Let,

$$x_k = \begin{bmatrix} z_k \\ a_k \end{bmatrix}$$

$$x_{k+1} = \begin{bmatrix} a_k z_k + w_k \\ a_k + w_{pk} \end{bmatrix} \leftarrow f(a_k, z_k, w_k, w_{pk})$$

$w_{pk} \sim (0, 0)$

$$y_k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} v_k \\ 0 \end{bmatrix}$$

EKF:

Initialize

$$\hat{x}_0^+ = E[x_0]$$

$$P_0^+ = E\left[(x_0 - \hat{x}_0^+)(x_0 - \hat{x}_0^+)^T\right]$$

3-a)

$$F_{k-1} = \frac{\partial f_{k-1}}{\partial x}\bigg|_{\hat{x}^+_{k-1}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{\hat{x}^+_{k-1}} = \begin{bmatrix} a_k & z_k \\ 0 & 1 \end{bmatrix}\Bigg|_{\hat{x}^+_{k-1}} = \begin{bmatrix} \hat{a}^+_{k-1} & \hat{z}^+_{k-1} \\ 0 & 1 \end{bmatrix}$$

$$L = I_2$$

b)

$$P_k^- = F_{k-1} P^+_{k-1} F^T_{k-1} + \cancel{L}\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}\cancel{L^T} = \begin{bmatrix} \hat{a}^{+2}_{k-1} P^+_{k|1,1-1} + Q & \\ \hat{a}_{k-1} P^+_{k|2-1} & \end{bmatrix}$$

$$\hat{x}_k^- = f(\hat{x}^+_{k-1}, 0) = \begin{bmatrix} \hat{a}^+_{k-1} \hat{z}^+_{k-1} \\ \hat{a}_{k-1} \end{bmatrix}$$

c)

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

3) cont.

EKF: cont

3-d)

$$K_K = P_K^- H^T (H P_K^- H^T + M R M^T)^{-1}$$

$$= P_K^- \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} P_K^- \begin{bmatrix} 1 \\ 0 \end{bmatrix} + R \right)^{-1}$$

$$= P_K^- \begin{bmatrix} (P_{K11}^- + R)^{-1} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} P_{K11}^- & \sim \\ \hat{a}_{K-1} P_{K12}^- & \sim \end{bmatrix} \begin{bmatrix} (P_{K11}^- + R)^{-1} \\ 0 \end{bmatrix} = \begin{bmatrix} P_{K11}^- (P_{K11}^- + R)^{-1} & \sim \\ \hat{a}_{K-1} P_{K12}^- (P_{K11}^- + R)^{-1} & \sim \end{bmatrix}$$

$$\cancel{K_K = \frac{P_{K11}^-}{P_{K11}^- + R} \begin{bmatrix} 1 & \sim \\ P_{K12}^- & \sim \end{bmatrix}}$$

$$\hat{x}_K^+ = \hat{x}_K^- + K_K (y_K - h(\hat{x}_K^-, 0))$$

$$= \begin{bmatrix} \hat{z}_K^- \\ \hat{a}_K^- \end{bmatrix} + K_K \begin{bmatrix} y_K - \hat{z}_K \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \hat{z}_K^+ \\ \hat{a}_K^+ \end{bmatrix} = \begin{bmatrix} (1 - K_{K11}) \hat{z}_K^- + K_{K11} (y_K) \\ (1 - K_{K12}) \hat{z}_K^- + K_{K12} (y_K) \end{bmatrix}$$

$$P_K^+ = (I - K_K H_K) P_K^-$$

$$= \begin{bmatrix} (1 - K_{K11}) & 0 \\ 0 & 1 \end{bmatrix} P_K^- = \begin{bmatrix} (1 - K_{K11}) P_{K11}^- & (1 - K_{K11}) P_{K11}^- \\ P_{K12}^- & P_{K22}^- \end{bmatrix}$$

same...

only $\hat{z}_K$

$$P_K^- = a^2 P_K^+ + Q$$

$$K_K = P_K^- (P_{K11}^- + R)^{-1} \qquad P_K^+ = (1 - K) P_K^-$$

$$\hat{z}_K^- = a z_K^+$$

$$\hat{z}_K^+ = (1 - K) \hat{z}_K^- + K(y)$$

3) cont.

KF:

$$F = a \qquad w_K \sim (0, Q)$$
$$H = 1 \qquad v_K \sim (0, R)$$

$$\hat{z}_K^+ = E[\hat{z}_K^+]$$
$$P_0^+ = E[(z_K - \hat{z}_K^+)(z_K - \hat{z}_K^+)^T]$$

$$P_K^- = \cancel{F} P_{K-1}^+ \cancel{F}^T + Q = a^2 P_{K-1}^+ + Q$$

$$K_K = P_K^- \cancel{H}^T (\cancel{H} P_K^- \cancel{H}^T + R)^{-1} = P_K^- (P_K^- + R)^{-1}$$

$$\hat{z}_K^+ = \hat{z}_K^- + K(y_K - \cancel{H}_K \hat{z}_K^-)$$
$$= (1 - K)\hat{z}_K^- + K y_K$$

$$\hat{z}_K^- = F z_{K-1}^+ = a F_{K-1}^+$$

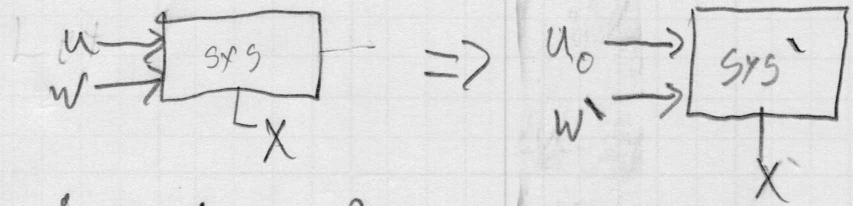
$$P_K^+ = (I - K_K H) P_K^- = (1 - K) P_K^-$$

They are the same KF filter for a deterministic a_K and artificial a_K error $= 0$

4) $\dot{X} = -X + u + W$ $\qquad w \sim (0, Q)$

$A = -1 \quad B = 1 \quad L = 1$ $\qquad u \sim (0, \sigma^2)$

$\sigma = 2$



$\dot{x} = Ax + Bu + Lw$

$\dot{x}' = Ax + B'u_0 + L'w'$

$B' = B \qquad L' = [L \quad B] \qquad w' \sim (0, Q')$

$Q' = \begin{bmatrix} Q & 0 \\ 0 & \sigma^2 \end{bmatrix}$

$w' \sim (0, \overset{1}{L^T} Q + \overset{1}{B^T} \overset{2^2}{\sigma^2})$

$\boxed{w' \sim (0, Q+4)}$

$\boxed{\dot{X} = -X + u_0 + w'}$

5) $$x_{k+1} = \phi x_k + w_k \qquad w_k \sim (0,1)$$
$$y_k = x_k \qquad \phi = 0.9$$

unkown...

Let,
$$\grave{x}_k = \begin{bmatrix} x_k \\ \phi_k \end{bmatrix} \qquad y_k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \grave{x}_k$$

$$\grave{x}_{k+1} = \begin{bmatrix} \phi_k x_k + w_k \\ \phi_k + w_{kP} \end{bmatrix} = \begin{bmatrix} \phi_k x_k \\ \phi_k \end{bmatrix} + \begin{bmatrix} w_k \\ w_{kP} \end{bmatrix}$$

$$\hat{\grave{x}}_0 = E[\grave{x}_k] = \begin{bmatrix} \hat{x}_0 \\ \hat{\phi}_0 \end{bmatrix} \qquad w' \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad w_{kP} \sim (0,0)$$

$$P_0^+ = E[(\grave{x}_k - \hat{\grave{x}}_k)(\grave{x}_k - \hat{\grave{x}}_k)^T] = \begin{bmatrix} P_{x0} & 0 \\ 0 & P_{\phi 0} \end{bmatrix}$$

$$F_{k-1} = \left.\frac{\partial f_{k-1}}{\partial x}\right|_{\hat{x}^+_{k-1}} = \begin{bmatrix} \phi_{k-1} & x_{k-1} \\ 0 & 1 \end{bmatrix}$$

$$L_{k-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$P_k^- = F_{k-1} P_{k-1}^+ F_{k-1}^T + \cancel{L_{k-1}}^{I} Q_{k-1} \cancel{L_{k-1}^T}^{I}$$
$$= \begin{bmatrix} \hat{\phi}^+_{k-1} & \hat{x}^+_{k-1} \\ 0 & 1 \end{bmatrix} P^+_{k-1} \begin{bmatrix} \hat{\phi}^+_{k-1} & \hat{x}^+_{k-1} \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} Q_k & 0 \\ 0 & 0 \end{bmatrix}$$

$$\hat{x}_k^- = \begin{bmatrix} \hat{\phi}^+_{k-1} \hat{x}^+_{k-1} \\ \hat{\phi}^+_{k-1} \end{bmatrix}$$

5) cont.

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_k = 0$$

$$K_k = P_k^- H^T (H P_k^- H^T + \cancel{M R M^T}^{\,0})^{-1}$$

$$\cancel{= \begin{bmatrix} P_{k11}^- & (P_{k11}^-) \\ P_{k12}^- & P_{k11}^- \end{bmatrix} = P_{k11}^- \begin{bmatrix} P_{k11}^- \\ P_{k12}^- \end{bmatrix}}$$

$$\hat{x}_k^+ = \hat{x}_k^- + K_k \left(y_k - \begin{bmatrix} \hat{x}_{k-1}^- \\ 0 \end{bmatrix} \right) = \qquad 1 - K_{k11}$$

$$P_k^+ = (I - K_k H) P_k^-$$