Final Project

Optimal Estimation and Kalman Filter (SYSE/MECH 6325)
Fall 2020
Due date: 11/30/2020

The purpose of this project is to examine your ability to design and simulate a Kalman filter for a variety of example systems. Perform all simulations in MATLAB, answer all the questions and submit the required information.

1. (20 points) Let p_k denote the wombat population at time k, and f_k denote the size of the wombat's food supply at time k. From one time step to the next, half of the existing wombat population dies, but the number of new wombats added to the population is equal to twice the food supply. The food supply is constant except for zero-mean random fluctuations with a variance of 10. At each time step the wombat population is counted with an error that has zero mean and a variance of 10. The initial state is

$$
\begin{aligned}
p_0 &= 650 \\
f_0 &= 250
\end{aligned}
$$

The initial state estimate and uncertainty is

$$
\begin{aligned}
\hat{p}_0 &= 600 \\
\mathbf{E}[(\hat{p}_0 - p_0)^2] &= 500 \\
\hat{f}_0 &= 200 \\
\mathbf{E}[(\hat{f}_0 - f_0)^2] &= 200
\end{aligned}
$$

Design a Kalman filter to estimate the population and food supply.

a) Simulate the system and the Kalman filter for 10 time step. Hand in the following.
   - Source code listing.
   - A plot showing the true population and the estimated population as a function of of time.
   - A plot showing the true food supply and the estimated food supply as a function of time.
   - A plot showing the standard deviation of the population and food supply estimation error as a function of time.
   - A plot showing the elements of the Kalman gain matrix as a function of time.

b) Compare the standard deviation of the estimation error of your simulation with the steady-state theoretical standard deviation based on P_k^+. Why is there such a discrepancy?

c) Run the simulation again for 1000 time steps and compare the experimental estimation error standard deviation with the theoretical standard deviation.

2. (20 points) Consider the RLC circuit in Fig. 1 with $R = 100$ and $L = C = 1$. Suppose the applied voltage is continuous-time, zero-mean white noise with a standard deviation of 3. The initial capacitor voltage and inductor current are both zero. Discretize the system with a time step of 0.1 s. The discrete-time measurements consist of the capacitor voltage and the inductor current, both measurements containing zero-mean unity variance noise. Implement a sequential Kalman filter for the system. Simulate the system for 2 seconds. Let the initial state estimate be equal to the initial state, and the initial estimation covariance be equal to $0.1I$ where I is the identity matrix. Hint: Set the discrete-time process noise covariance $Q = Q_c\Delta t$, where Q_c is the covariance of the continuous-time process noise, and Δt is the discretization step size.

a) Generate a plot showing the *a priori* variance of the capacitor voltage estimation error, and the two *a posteriori* variances of the capacitor voltage estimation error.

b) Generate a plot showing a typical trace of the true, *a posteriori* estimated, and measured capacitor voltage. What is the standard deviation of the capacitor voltage measurement error? What is the standard deviation of the capacitor voltage estimation error?

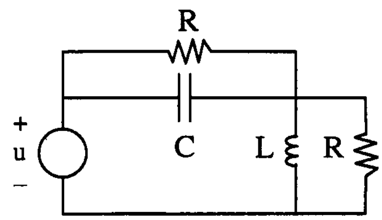

Figure 1: RLC circuit for Problem 2.

3. (20 points) Consider the following two-state system:

$$\dot{x} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} x + w$$
$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + v$$
$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}$$
$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

a) Integrate the Riccati equation with $a_1 = 1$, $a_2 = 2$, $q_{11} = q_{12} = q_{22} = 1$, and $P(0) = I$ where I is the identity matrix. Plot the Riccati equation solution as a function of time and verify that its steady-state value matches the result of MATLAB's CARE function.

b) Integrate the Riccati equation with $a_1 = a_2 = -1$, $q_{11} = 1$, $q_{12} = 2$, $q_{22} = 4$, and $P(0) = I$. Plot the Riccati equation solution as a function of time and verify that its steady-state value matches the result of MATLAB's CARE function.

c) Integrate the Riccati equation with $a_1 = a_2 = 1$, $q_{11} = 1$, $q_{12} = 2$, $q_{22} = 4$, and $P(0) = I$. Plot the Riccati equation solution as a function of time and verify that its steady-state value matches the result of MATLAB's CARE function.

d) Integrate the Riccati equation with $a_1 = a_2 = 1$, $q_{11} = 1$, $q_{12} = 2$, $q_{22} = 4$, and $P(0) = 0$. Plot the Riccati equation solution as a function of time. Dose the steady-state value match the result of MATLAB's CARE function? Does it result in a stable steady-state Kalman filter?

4. (20 points) Consider the second-order system

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

where $\omega = 6\,\text{rad/s}$ is the natural frequency of the system, and $\zeta = 0.16$ is the damping ratio. The input $w(t)$ is continuous-time white noise with a variance of 0.01. Measurements of the first state are taken every 0.5 s:

$$y(t_k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t_k) + v(t_k)$$

where $v(t_k)$ is discrete-time white noise with a variance of 10^{-4}. The initial state, estimate, and the covariance are

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\hat{x}(0) = x(0)$$
$$P(0) = \begin{bmatrix} 10^{-5} & 0 \\ 0 & 10^{-2} \end{bmatrix}$$

a) Discretize the system equation.

b) Implement the discrete-time Kalman filter, plot the variance of the estimation error of the two states, and compare the states and their estimates.

5. (20 points) A planar model for a satellite orbiting around the earth can be modeled as

$$\ddot{r} = r\dot{\theta}^2 - \frac{GM}{r^2} + w$$
$$\ddot{\theta} = \frac{-2\dot{\theta}\dot{r}}{r}$$

where r is the distance of the satellite from the center of the earth, θ is the angualr position of the satellite in its orbit, $G = 6.6742 \times 10^{-11}\,\mathrm{m}^3/\mathrm{kg}/\mathrm{s}^2$ is the universal gravitational constant, $M = 5.98 \times 10^{24}\,\mathrm{kg}$ is the mass of the earth, and $w \sim (0, 10^{-6})$ is random noise due to space debris, atmospheric drag, outgassing, and so on.

a) Write a state-space model for this system with $x_1 = r$, $x_2 = \dot{r}$, $x_3 = \theta$, and $x_4 = \dot{\theta}$.

b) What must $\dot{\theta}$ be equal to in order for the orbit to have a constant radius when $w = 0$?

c) Linearize the model around the point $r = r_0$, $\dot{r} = 0$, $\theta = \omega_0 T$, $\dot{\theta} = \omega_0$. What are the eigenvalues of the system matrix for the linearized system when $r_0 = 6.57 \times 10^6\,\mathrm{m}$? What would you estimate to be the largest integration step size that could be used to simulate the system? (Hint: determine the constant time of the system.)

d) Suppose that measurements of the satellite radius and angular position are obtained every minute, with error standard deviations of 100 meters and 0.1 radians, respectively. Simulate the linearized Kalman filter for three hours. Initialize the system with $x(0) = \begin{bmatrix} r_0 & 0 & 0 & 1.1\omega_0 \end{bmatrix}$, $\hat{x}(0) = x(0)$, and $P(0) = \mathrm{diag}(0,0,0,0)$. Plot the radius estimation error as a function of time. Why is the performance so poor? How could you modify the linearized Kalman filter to get better performance?

e) Implement an extended Kalman filter and plot the radius estimation error as a function of time. How does the performance compare with the linearized Kalman filter?