Assignment 1

Optimal Estimation and Kalman Filter
Fall 2020
Due date: September 11, 2020 at 1:00 PM

Problem 1. Three fair dice are tossed. Find the probability that 2 of the numbers will be the same and other will be different.

Problem 2. A binary sequence, consisting of zeros and ones, is n bits in length. If the probability of making an error in the transmission of a single bit is p, and if the error probability does not depend on the outcome of any previous transmission, find the probability of occurrence of exactly k bit errors in a message.

Problem 3. The Rayleigh probability density function is defined as

$$f_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} \quad r \geq 0$$

where σ^2 is the parameter of distribution.

a. Find the probability distribution function.

b. Find the mean and variance of a Rayleigh distributed random variable (R).

c. Find the median and mode of R.

Problem 4. A random variable has a probability density function shown in Fig. 1.

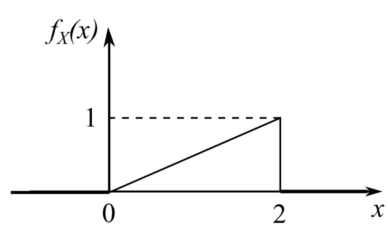

Figure 1:

a. Find and sketch the probability distribution function of this RV.

b. What is the variance of X?

Problem 5. The random variables X and Y have the density function

$$f_{X,Y}(x,y) = \begin{cases} c(y-x+1) & 0 \le y \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

a. Find the value of c so that $f_{X,Y}(x,y)$ is a valid joint pdf.

b. Obtain two marginal density functions.

Problem 6. Consider the following density function

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-2x}e^{-3y} & x > 0\,,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

a. Calculate $\bar{x}$ and $\bar{y}$.

b. Calculate $\mathbf{E}(X^2)$, $\mathbf{E}(Y^2)$, and $\mathbf{E}(XY)$.

c. Calculate autocorrelation matrix of the random vector $[X\ Y]^T$.

d. Calculate the variance σ_x^2, σ_y^2, and the covariance C_{XY}.

e. Calculate the autocovariance matrix of the random vector $[X\ Y]^T$.

f. Calculate the correlation coefficient between X and Y.

Problem 7. A random process $X(t)$ has sample realization of the form

$$X = at + Y$$

where a is a known constant and Y is a random variable whose distribution is $\mathcal{N}(0, \sigma^2)$.

a. Find the expected value of $X(t)$.

b. Find the time average of $X(t)$, i.e. $A[X(t)]$.

c. Is the process stationary and ergodic?

Problem 8. Assume $X \sim \mathcal{N}(0, 4)$ and $Y \sim \mathcal{N}(1, 9)$. We also know that X and Y are independent. Let us define

$$Z = X + 2Y$$

Write the explicit expression for pdf of Z.

Problem 9. Consider the equation $Z = X + V$. The pdfs of X and V are given in Fig. 2.

a. Plot the pdf of $(Z|X)$ as a function of X for $Z = 0.5$.

b. Given $Z = 0.5$, what is the conditional expectation of X, i.e. $\mathbf{E}(X|Z = 0.5)$?

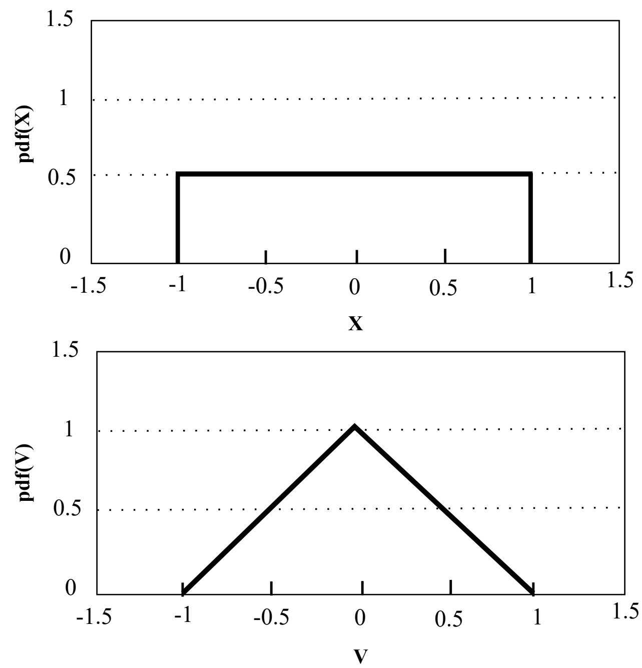

Figure 2:

Problem 10. A stationary random process $X(t)$ has a spectral density function of the form

$$S_X(\omega) = \frac{6\omega^2 + 12}{(\omega^2 + 4)(\omega^2 + 1)}$$

Find $\mathbf{E}(X)$ and $\mathbf{E}(X^2)$.

Problem 11. A stationary process $X(t)$ is Gaussian and has an autocorrelation function of the form

$$R_X(\tau) = 4e^{-|\tau|}$$

Let the random variable X_1 denote $X(t_1)$ and X_2 denote $X(t_1 + 1)$. Write the expression for the joint probability density function $f_{X1X2}(x_1, x_2)$.

Problem 12. Assume $X(t)$ is a stationary random process with autocorrelation function

$$R_X(\tau) = \begin{cases} 1 - |\tau| & -1 \leq \tau \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the spectral density $S_Y(\omega)$ for $Y(t) = X(t)\cos(\omega_0 t + \lambda)$ when ω_0 is a constant and λ is a RV uniformly distributed on the interval $[0,\ 2\pi]$.

Problem 13. The random vector $[X\ Y]^T$ has a covariance matrix

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Find a linear transformation to decorrelate the random vector. What are the variances of uncorrelated random variables?

Problem 14. Two random variables X and Y are independent and uniformly distributed on $[0,\, 1]$. Find the pdf of $Z = |X - Y|$.

Problem 15. Consider the random process

$$X(t) = \cos(\omega_0 t + \theta_1)\cos(\omega_0 t + \theta_2)$$

where θ_1 and θ_2 are independent RVs uniformly distributed on $[0,\, 2\pi]$.

a. Show that $X(t)$ is wide sense stationary.

b. Calculate $R_X(\tau)$ and $S_X(\omega)$.

c. Is $X(t)$ ergodic?

Problem 16. Generate 10000 samples of $\frac{x_1+x_2}{2}$, where each x_i are random number uniformly distributed over $[-\frac{1}{2},\, \frac{1}{2}]$. Plot the 50-bin histogram. Repeat for $\frac{x_1+x_2+x_3+x_4}{4}$. Describe the difference between the two histograms.