1)

$$(HQH^T)Q = QH^THQ$$

$$x_k = F x_{k-1} + w_{k-1} \qquad w \sim (0, Q)$$

$$y_k = H x_k$$

$$V = 0 \rightarrow R = 0$$

$$G = 0$$

$$K_k = P_k^- H^T (H P_k^- H^T + \cancel{R}^{\,0})^{-1}$$

$$P_k^- = F P_{k-1}^+ F^T + Q$$

$$P_k^+ = (I - K_k H) P_k^-$$

$$P_k^+ = \left(I - P_k^- H^T (H P_k^- H^T)^{-1} H\right) P_k^-$$

$$= \left(I - (F P_{k-1}^+ F^T + Q) H^T \left(H (F P_{k-1} F^T + Q) H^T\right)^{-1} H\right)\left(F P_{k-1}^+ F + Q\right)$$

$$P = P - PH(HPH^T)^{-1}HP$$

2)
$$x_k = x_{k-1} + w_{k-1}$$
$$y_k = x_k + v_k$$

$w_k \sim (0,1) \rightarrow Q = 1$

$v_k \sim (0,1) \rightarrow R = 1$

$E(w_k v_{k+1}) = 1 \rightarrow M = 1$

a) $\underline{F = 1 \quad H = 1}$

$$K_k = \underbrace{P_k^+ H^T R_k^{-1}}$$

$$P_k^+ = (I - K_k H)(F P_{k-1}^+ F^T + Q)$$

$$\hat{x}_k^+ = (\overset{1}{I} - \overset{P_k^+}{K_k} \overset{1}{H})(\overset{1}{F} \hat{x}_{k-1}^+) + \overset{P_k^+}{K_k} y_k$$

$$\hat{x}_k^+ = \hat{x}_{k-1}^+ - P_k^+ \hat{x}_{k-1} + P_k^+ y_k$$

steady-state: $P_k^+ = P_{k-1}^+ = P^+$

$$P^+ = (\overset{1}{I} - P^+ \overset{1}{H^T} \overset{1}{R^{-1}})(\overset{1}{F} P^+ \overset{1}{F^T} + \overset{1}{Q})$$

$$P^+ = (1 - P^+)(P^+ + 1) = 1 - (P^+)^2$$

$$(P^+)^2 + P^+ - 1 = 0 \rightarrow P^+ = \frac{-1}{2} \pm \frac{\sqrt{5}}{2}$$

$$\boxed{P^+ = \frac{\sqrt{5} - 1}{2}}$$

b)
$$\hat{x}_k^+ = (1 - P_k^+)\hat{x}_{k-1} + P_k^+ y_k = \hat{x}_{k-1} - P_k^+ \hat{x}_{k-1} + P_k^+ x_k + P_k^+ v_k$$

$$e_k = x_k - \hat{x}_k^+ = x_k - \left[(1 - P_k^+)\hat{x}_{k-1}^+ + P_k^+ (x_k + v_k)\right]$$

$$\boxed{e_k = (1 - P_k^+)(x_k - \hat{x}_{k-1}^+) + P_k^+ v_k}$$

steady state: $e = \ldots \; x - \hat{x} = \tilde{e}_k$

$$E[e_k^2] = E\left[(1 - P^+)^2 \tilde{e}_k^2 + 2(1 - P^+) v_k + (P^+ v_k)^2\right]$$

$$= E\left[(1 - P^+)^2 \tilde{e}_k^2\right] + E\left[\cancel{2(1 - P^+) v_k}\right]^{0} + E\left[\cancel{(P^+ v_k)^2}\right]^{0}$$

$$\tilde{e}^2 = (1 - P^+)^2 \tilde{e}^2$$

a) b) cont.

$$E[e_K^2] = E\left[\left((1-P_K^+)(x_K - \hat{x}_{K-1}^+) + P_K^+ v_K\right)^2\right]$$

$$= E\left[\left[(1-P_K^+)(x_K - \hat{x}_{K-1}^+)\right]^2\right] + \cancel{E[2 v P_K^+ v_K]}^{\,0} + \cancel{E[(P_K^+ v_K)^2]}^{\,0}$$

$$= (1-P_K^+)^2 E\left[x_K^2 - 2x_K \hat{x}_{K-1}^+ + \hat{x}_{K-1}^{+2}\right]$$

$$x_K^2 + 2x_K\hat{x}_K^+ + \hat{x}_K^{2} = E[e_K^2] = (1-P_K^+)^2 (x_K^2 - 2x_K \hat{x}_{K-1}^+ + \hat{x}_{K-1}^{+2})$$

Steady state: $x_K = x_{K-1} = x$, $\hat{x}_K^+ = \hat{x}_{K-1}^+ = \hat{x}^+$, $P_K^+ = P^+$

$$x^2 - 2x\hat{x}^+ + \hat{x}^{+2} = (1-P^+)(x_K^2 - 2x\hat{x}^+ + \hat{x}^{+2})$$

? $e_K^2 \neq (1-P^+) e_K^2$

Not true ... Diverges?

2) c) $F = 1$ $\quad Q = 1$

$H = 1$ $\quad R = 1$

$M = 1$

$$K_k = \left(P_k^- \cancel{H^T} + \cancel{M}\right)\left(\cancel{H} P_k^- \cancel{H^T} + \cancel{HM} + \cancel{M^T H^T} + \cancel{R}\right)^{-1}$$

$$= \left(P_k^- + 1\right)\left(P_k^- + 3\right)^{-1}$$

$$\boxed{K_k = \frac{1 + P_k^-}{3 + P_k^-}}$$

$$P_k^- = \cancel{F} P_{k-1}^+ \cancel{F^T} + \cancel{Q}$$
$$= 1 + P_{k-1}^+$$

$$\boxed{K_k = \frac{2 + P_{k-1}^+}{4 + P_{k-1}^+}}$$

$$P_k^+ = P_k^- - K_k\left(\cancel{H} P_k^- + \cancel{M^T}\right)$$

$$= \left(1 + P_{k-1}^+\right) - \left(2 + P_{k-1}^+\right)\left(4 + P_{k-1}^+\right)^{-1}\left(\cancel{1+1} + P_{k-1}^+\right)$$

$$= \left[\left(1 + P_{k-1}^+\right)\left(4 + P_{k-1}^+\right) - \left(2 + P_{k-1}^+\right)\left(2 + P_{k-1}^+\right)\right]\left(4 + P_{k-1}^+\right)$$

$$= \left[\cancel{4} + 5 P_{k-1}^+ + \cancel{\left(P_{k-1}^+\right)^2} - \left(\cancel{4} + 4 P_{k-1}^+ + \cancel{\left(P_{k-1}^+\right)^2}\right)\right]\left(4 + P_{k-1}^+\right)$$

$$\boxed{P_k^+ = \frac{P_{k-1}^+}{4 + P_{k-1}^+}}$$

steady-state: $P_k^+ = P_{k-1}^+ = P^+$

$$P^+ = \frac{P^+}{4 + P^+}$$

$$4P^+ + \left(P^+\right)^2 = P^+$$

$$P^+\left(P^+ + 3\right) = 0 \longrightarrow P^+ = 0 \quad \cancel{(\text{or } -3)}$$

d) This Kalman filter design incorporates the covariance between the two to create an unbiased estimator. This also means as $t \to \infty$, the variance approaches 0

3) $x_k = \frac{1}{2} x_{k-1} + w_{k-1}$

$F = \frac{1}{2}$ $\quad y_k = x_k + v_k$

$H = 1$

$\psi = \frac{1}{2}$ $\quad v_k = \frac{1}{2} v_{k-1} + \xi_{k-1}$

$w_k \sim (0,1) \rightarrow Q_k = 1$

$\xi_k \sim (0,1) \rightarrow Q_{\xi,k} = 1$

assuming $E = 0$

a) assume $v_k \sim (0,1) \rightarrow R_k = 1$

$$\boxed{K_k = P_k^+} \cancel{H^T}^{\to 1} \cancel{R_k}^{\to 1}$$

$$P_k^+ = (\cancel{I}^{\to 1} - K_k \cancel{H}^{\to 1})(\cancel{F}^{\to \frac{1}{2}} P_{k-1}^+ \cancel{F^T}^{\to \frac{1}{2}} + \cancel{Q_k}^{\to 1})$$

$$= (1 - P_k^+)\left(\frac{1}{4} P_{k-1}^+ + 1\right)$$

steady-state: $P_k^+ = P_{k-1}^+ = P^+$

$$P^+ = \frac{1}{4} P^+ - \frac{1}{4}(P^+)^2 - P^+ + 1$$

$$\frac{1}{4}(P^+)^2 + \frac{7}{8} P^+ - 1 = 0 \qquad P^+ = \frac{-7 \pm \sqrt{53}}{2}$$

$$\boxed{P^+ = \frac{\sqrt{53} - 7}{2}}$$

b)

3) c)

$$F = \frac{1}{2} \qquad Q_k = 1$$
$$H = 1$$
$$\psi = \frac{1}{2} \qquad Q_{\xi,k} = 1$$

$$y'_k = y_k - \psi y_{k-1} = y_k - \frac{1}{2} y_{k-1}$$

$$H'_k = \cancel{H} \cancel{F} - \cancel{\psi} \cancel{H} = 0 \ ?$$

$$K_k = P_k^- \cancel{H'} (\cancel{H'} P_k^- \cancel{H'_k} + \cancel{R_k})^{-1} = 0 \ ?$$

$$M_k = \cancel{Q_k} \cancel{H} = 1$$

$$C_k = \cancel{M_k} (\cancel{H'_k} P_k^- \cancel{H'_k} + \cancel{R_k})^{-1} = 1$$

$$P_k^+ = (I - \cancel{K_k H'}) P_k^- (I - \cancel{K_k H'_k})^T + \cancel{K_k} R_k \cancel{K_k}$$

$$P_k^+ = P_k^- \ ?$$

4) $$x_k^+ \overset{?}{=} (I-KH)F x_{k-1}^+ + K y_k$$

Note: $H(I-KH) = (I-HK)H$

$$\hat{y}_k = H \hat{x}_k^+$$

Let $(F,H)$ be observable, $\rho(I-HK) = n$ (full rank)

Test if $((I-KH)F, H)$ is observable...

Proof

$\rho(V) = n$ w/ $V = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}$

$\tilde{V} = \begin{bmatrix} H \\ H(I-KH)F \\ \vdots \\ H((I-KH)F)^{n-1} \end{bmatrix}$ w/ since $\rho(I-HK) = n$,

$$\rho(\tilde{V}) = \rho(V) \rightarrow \text{full rank}$$

$$\rho\begin{bmatrix} H \\ H(I-KH)F \\ \vdots \\ H((I-HK)F)^{n-1} \end{bmatrix} = \rho\begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}$$

Thus, The Kalman Filter is an observable system.//

Alternative Method: Prove $P_\infty$ exists?

(would also need to prove $((I-KH)F, K)$ is controllable