1) P_k — wombat population

f_k — Wombat Food supply

$$P_{k+1} = \frac{1}{2} P_k + 2 f_k \qquad P_0 = 650$$

$$f_k = f_0 + \tilde{w} \qquad f_0 = 250$$

$$y = P_k + v \qquad \tilde{w} \sim (0, 10)_Q$$

$$v \sim (0, 10)_R$$

Let, $x_k = \begin{bmatrix} P_k \\ f_k \end{bmatrix}$ $\qquad \hat{P}_0 = 600 \quad E[(\hat{P}_0 - P_0)^2] = 500$

$\qquad \hat{f}_0 = 200 \quad E[(\hat{f}_0 - f_0)^2] = 200$

$$x_{k+1} = \begin{bmatrix} \frac{1}{2} & 2 \\ 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v$$

$$F = \begin{bmatrix} \frac{1}{2} & 2 \\ 0 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad L = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\hat{x}_0 = \begin{bmatrix} 600 \\ 200 \end{bmatrix} \qquad P_0 = \begin{bmatrix} 500 & 0 \\ 0 & 200 \end{bmatrix}$$

a) standard KF Process

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1)

$M = \emptyset$

b) $P_\infty = F P_\infty F^T - F P H^T (H P H^T + R)^{-1} H P F^T + Q$

$P_\infty = DARE(F^T, H^T, Q, R, 0)$

c) $M = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

wrong

$M \neq 0$

b) $P_\infty = DARE(F^T, H^T, Q, HM + M^T H^T + R, FM)$

$\underline{M = 0}$

c) $P_\infty^- = idare(F^T, H^T, Q, R)$

b)

$K_\infty = P_\infty^- H^T (H P_\infty^- H^T + R)^{-1}$

$P_\infty^+ = (I - K_\infty H) P_\infty^-$

$\hat{X}_K^+ = (I - K_\infty H) F \hat{X}_{K-1}^+ + K_\infty (H X_K + V)$ $\quad V \sim (0, R)$

$\hat{X}_K^+ = F \hat{X}_{K-1}^+ - K_\infty H F \hat{X}_{K-1}^+ + K_\infty H X_K + \underbrace{K_\infty V}$

$\hat{X}_K^-$ $\quad \hat{X}_K^-$

ss - $\hat{X}_K^+ = \hat{X}_K^- \rightsquigarrow K_\infty H (\hat{X}_K - X_K) = K_\infty V$

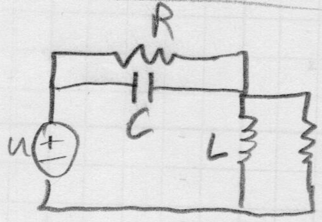
$\hat{X}_K - X_K = \underbrace{H^{-1} V}$

$(0, H^{-1} R)$

nondeterministic term

Ex. 1.8 in book...

a)



$$\dot{V}_C = \frac{-2}{RC} V_C + \frac{1}{C} i_L + \frac{1}{RC} u$$

$$\dot{i}_L = \frac{-1}{L} V_C + 0 + \frac{1}{L} u$$

$$\dot{x} = \begin{bmatrix} V_C \\ \dot{V}_L \end{bmatrix} \qquad \dot{x} = Ax + Lw \qquad u = w$$

$$A = \begin{bmatrix} \frac{-2}{RC} & \frac{1}{C} \\ \frac{-1}{L} & 0 \end{bmatrix} \qquad L = \begin{bmatrix} \frac{1}{RC} \\ \frac{1}{L} \end{bmatrix} \qquad w \sim (0, Q_c)$$

$$Q_c = 3^?$$

$$y(t_k) = H\, x(t_k) + v$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad v \sim (0, R), \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2) cont.

Let, $F = e^{A \Delta t}$ $\quad$ $\underline{\Delta t = 0.1}$

$Q = Q_c \Delta t$ $\quad$ $x_0 = 0$ $\quad$ $\tilde{w} = L w$

1) $x_{k+1} = F x_k + \tilde{w}$ — $\tilde{w} \sim (0, \tilde{Q})$

$y_k = H x_k + v$ — $v \sim (0, R)$

$R = I_2$

2) $\hat{x}_0^+ = 0$

$\hat{P}_0^+ = 0.1 I_4$

b) $\underline{\text{Just plot each trace}}$

std: error between for each

3) $P_k^- = F P_{k-1}^+ F^T + L Q L^T$

$\hat{x}_k^- = F \hat{x}_{k-1}^+ + \cancel{G u}^{0}$

4) For each sensor, $(i = 1, 2)$

$$K_{ik} = \frac{P_{i-1,k}^+ H_{ik}^T}{H_{ik} P_{i-1,k}^+ H_{ik}^T + R_{ik}} = \frac{P_{ik}^+ H_{ik}^T}{R_{ik}}$$

$$\hat{x}_{ik}^+ = \hat{x}_{i-1}^+ + K_{ik}(y_{ik} - H_{ik} \hat{x}_{i-1,k}^+)$$

$$P_{ik}^+ = (I - K_{ik} H_{ik}) P_{i-1,k}^+ (I - K_{ik} H_{ik})^T + K_{ik} R_{ik} K_{ik}^T$$

$$\hat{x}_k^+ = \hat{x}_{rk}^+$$

$$\hat{P}_k^+ = \hat{P}_{rk}^+$$

3) Riccati Equation Value comparisons

$$\dot{x} = Ax + w \quad A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \quad w \sim \left(0, \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}\right)$$

$$y = Hx + V \quad H = I_2, \ V \sim (0, I_2)$$

a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $P(0) = I$

b) $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $P(0) = I$

c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $P(0) = I$

☆d) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $P(0) = 0$

Extra Part

Does it result in a ss-KF? ← pg. 254

Check observer poles for stability

No, with the P_∞ from integration this isn't the case:

$$P_\infty = \begin{bmatrix} 0.69 & 1.38 \\ 1.38 & 2.76 \end{bmatrix}$$

$$K_\infty = P_\infty C^T R^{-1} = \begin{bmatrix} 0.69 & 1.37 \\ 1.38 & 2.76 \end{bmatrix}$$

$$(A - KC) = \begin{bmatrix} 0.31 & -1.38 \\ -1.38 & -1.76 \end{bmatrix} \longrightarrow \lambda(A-KC) = (-2.5, 1)$$

Not stable ✓

4)
$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{L} \tilde{w}(t)$$

$\omega = 6$ rad/s
$\xi = 0.16$

$$y(t_k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x + v(t_k) \qquad \tilde{w}(t) \sim (0, \underbrace{0.01}_{Q_c})$$

$$v(t_k) \sim (0, 10^{-4})$$

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\hat{x}(0) = x(0) \qquad P(0) = \begin{bmatrix} 10^{-5} & 0 \\ 0 & 10^{-2} \end{bmatrix}$$

a)
$$F = e^{A\Delta t} \qquad \Delta t = 0.5 \text{ s}$$

$$x_{k+1} = F x_k + w \quad w \sim \Big(0, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & Q_c \Delta t \end{bmatrix}}_{Q}\Big)$$

$$y_k = H_k + v \quad v \sim (0, 10^{-4})$$

b) Standard DT KF

- Plot $E[(x-\hat{x}_k)(x-\hat{x})^T]$ over time
- Compare states + Estimates

5)

$$\ddot{r} = r\dot{\theta}^2 - \frac{GM}{r^2} + \tilde{w}$$

$$\ddot{\theta} = -\frac{2\dot{\theta}\dot{r}}{r}$$

$G = 6.674 \times 10^{-11} \frac{m^3}{kg/s^2}$

$M = 5.98 \times 10^{24}$ kg

$\tilde{w} \sim (0, 10^{-6})$

a)

$$x = \begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \dot{r} \\ r\dot{\theta}^2 - \frac{GM}{r^2} + w \\ \dot{\theta} \\ -\frac{2\dot{\theta}\dot{r}}{r} \end{bmatrix} \cancel{+ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} w}$$

b) $\tilde{w} = 0$

$$0 = r\dot{\theta}^2 - \frac{GM}{r^2}$$

$$\frac{GM}{r^2} = r\dot{\theta}^2$$

$$\boxed{\dot{\theta} = \pm\sqrt{\frac{GM}{r^3}}}$$

5) cont.

c) $$X_0 = \begin{bmatrix} r_0 \\ 0 \\ \omega_0 T \\ \omega_0 \end{bmatrix}$$

$$A = \left.\frac{df}{dx}\right|_{X_0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dot{\theta}_0^2 + \frac{2GM}{r_0^3} & 0 & 0 & 2r_0\dot{\theta}_0 \\ 0 & 0 & 0 & 1 \\ 2\dot{\theta}_0 \cancel{\dot{r}_0}^{\to 0} & \frac{-2\dot{\theta}_0}{r_0} & 0 & \frac{-2\cancel{\dot{r}_0}^{\to 0}}{r_0} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega_0^2 + \frac{2GM}{r_0^3} & 0 & 0 & 2r_0\omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-2\omega_0}{r_0} & 0 & 0 \end{bmatrix}$$

$B = 0$

$L\tilde{w} \to \tilde{w} \sim (0, Q)$

$$\tilde{Q} = \begin{bmatrix} 0 & & 0 \\ & 10^{-6} & \\ 0 & 0 & 0 \end{bmatrix}$$

$\cancel{\lambda} =$ for $r_0 = 6.57 \times 10^6\ m$:

$$\begin{bmatrix} 0 \\ 0 + j1.2e^{-3} \\ 0 - j1.2e^{-3} \\ 0 \end{bmatrix} ???$$

$2\pi f_0 = 1.2 \times 10^{-3} = \frac{2\pi}{\tau}$

Largest Integration step: ???

$$\tau = \frac{2\pi}{1.2 \times 10^{-3}} \simeq 5.2 \times 10^3$$

$$\boxed{0.1\tau = 520\,s}$$

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5) cont.

d) $\Delta t = 1$ min $\quad \sigma_r = 100$ m, $\sigma_\theta = 0.1$ rad $\quad y = [I_{2} \; 0]x + v, \begin{bmatrix} 100^2 & 0 \\ 0 & 0.1^2 \end{bmatrix}$

$\begin{bmatrix} (100)^2 & 0 \\ 0 & (0.1)^2 \end{bmatrix}$

$$X(0) = \begin{bmatrix} r_0 \\ 0 \\ 0 \\ 1.1 w_0 \end{bmatrix} \qquad \hat{X}(0) = X(0)$$

$$P(0) = diag(0,0,0,0)$$

Linearized CT-KF. $\quad$ should do hybrid

$\hat{r} - r$ plot ... Modifications to improve?

e) EKF:

2) A ~ same

$$L = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{X}_0^+ = \begin{bmatrix} r_0 \\ 0 \\ 0 \\ 1.1 w_0 \end{bmatrix}$$

$$P_0^+ = diag(0,0,0,0)$$

3) $\hat{Q} = diag\{0, 10^{-6}, 0, 0\}$

$\hat{R} = diag\{100, 0.1\}$

3) a) Integrate steps:

$$\dot{\hat{x}} = f(\hat{x}, u, 0, t)$$

$$\dot{P} = AP + PA^T + LQL^T$$

b) standard KF measurment update

-> Compare with CT-KF