Due Date: Submit electronically to eLearning by 5:00pm on Friday, Feb. 10.
Your solution must be typed.

1 Stochastic Optimal Control Problem Formulation

Find and research an example (ideally from your own research experience/interests, or from an actual company/startup) of a problem of multi-step decision making under uncertainty in the real world. Write a short narrative that briefly describes i) the use (or potential use) of closed-loop feedback control/recourse due to uncertain factors affecting performance and operation and ii) the performance objectives and operation of the system, including the system dynamics, system states, inputs, outputs, and stochastic disturbances and other uncertainties. Comment specifically on any stochastic uncertainties and how you might go about collecting data to describe their probability distribution. Explain why an open-loop controller may not be sufficient and what are the advantages of using closed-loop feedback/recourse by utilizing knowledge of the system state and/or realizations of uncertain parameters at each time instant.

2 Some Linear Algebra and Probability Theory Review

Consider the discrete-time linear stochastic dynamical system

$$x_{t+1} = Ax_t + Bu_t + w_t,$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, and $w_t \in \mathbb{R}^n$ is a Gaussian random vector with zero mean $\mathbf{E}w_t = 0$ and covariance matrix $\mathbf{E}w_t w_t^\top = W \succeq 0$ (i.e., $w_t \sim \mathcal{N}(0, W)$), independent and identically distributed across time. Suppose the initial state is a zero mean Gaussian random vector with covariance matrix X_0 (i.e., $x_0 \sim \mathcal{N}(0, X_0)$) and independent from w_t, and the control input is given by a linear feedback control policy $u_t = Kx_t$, where $K \in \mathbb{R}^{m \times n}$.

a) Let the state mean be defined as $\mu_t := \mathbf{E}x_t$ and state second moment matrix be defined as $\Sigma_t := \mathbf{E}x_t x_t^\top$. Derive the dynamics for the state mean and second moment matrix, i.e., write μ_{t+1} and Σ_{t+1} in terms if μ_t and Σ_t and the given problem data (A, B, K, W, X_0).

b) Suppose $n = m = 1$. Derive an expression for the steady state second moment $\Sigma_\infty := \lim_{t\to\infty} \Sigma_t$. Under what conditions does this converge to a finite value? In this case, what is $\lim_{t\to\infty} \mathbf{P}(x_t \geq 3\sqrt{\Sigma_\infty})$, i.e., the probability that in steady state, the state takes a value greater than $3\sqrt{\Sigma_\infty}$?

c) Instead of a linear feedback control policy, derive an expression for an *open-loop control input sequence* $\mathbf{u} = [u_0^\top, u_1^\top, ..., u_{T-1}^\top]^\top$ that optimizes the expected finite-horizon cost for a fixed initial state x_0 with cost matrices $Q \succ 0$ and $R \succ 0$

$$\mathbf{E}\sum_{t=0}^{T-1}(x_t^\top Q x_t + u_t^\top R u_t) + x_T^\top Q x_T.$$

Hint: It may be useful to vectorize the dynamics and cost in terms of the sequences $\mathbf{x} = [x_1^\top, x_2^\top, ..., x_T^\top]^\top$, $\mathbf{u} = [u_0^\top, u_1^\top, ..., u_{T-1}^\top]^\top$, *and* $\mathbf{w} = [w_0^\top, w_1^\top, ..., w_{T-1}^\top]^\top$ *and eliminate the* *state sequence in the cost function. Remember that the disturbances are independent of the open-loop control sequence.* Comment on the effect that the open-loop control sequence has on how the disturbance affects the cost.

3 Value of Information/Recourse/Feedback

Optimal disposition of a stock. You must sell a total amount $B > 0$ of a stock in two rounds. In each round you can sell any nonnegative amount of the stock; by the second round all of the initial stock amount B must be sold. The (positive) prices in the two rounds are p_0 and p_1, respectively. These are independent log-normal variables:

$$\log p_0 \sim \mathcal{N}(\mu_0, \sigma_0^2), \quad \log p_1 \sim \mathcal{N}(\mu_1, \sigma_1^2).$$

The goal is to maximize the total expected revenue from the sales in the two rounds.

We consider three different information patterns.

- *Prescient.* You know p_0 and p_1 before you decide the amounts to sell in each period.
- *No knowledge.* You do not know the prices, but you know their distribution parameters.
- *Partial knowledge.* You are told the price p_0 before you decide how much to sell in period 0, and you are told the price p_1 before you decide how much to sell in period 1.

(a) Find the optimal policies for each of the three different information patterns. The amount sold in each period can depend on the problem data $(B, \mu_0, \mu_1, \sigma_0, \sigma_1)$ and of course the additional information available, which depends on the information pattern.

(b) Consider the specific case with

$$B = 100, \quad \mu_0 = 0, \quad \mu_1 = 0.1, \quad \sigma_0 = \sigma_1 = 0.4.$$

Plot the distribution of total revenue for the stochastic control problems for the three different information patterns, using (if necessary) Monte Carlo estimation. Give the expected values of total revenue in each case (again, computed by Monte Carlo estimation).

Hints.

- If $\log x \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbf{E}x = e^{\mu+\sigma^2/2}$.
- In Matlab you can plot the histogram of a vector v in n bins using the command `hist(v,n)`.

You don't need to know a general method for solving stochastic optimal control problems to solve this problem. You can solve it directly using basic and simple arguments.