Due Date: Submit electronically to eLearning by 5:00pm on Friday, March 31.
Your solution must be typed.

1 Markov Chains

1. *Epidemic model simulation.* Consider an epidemic model where each individual can be in one of four states: susceptible (S), infected (I), deceased (D), or protected (or immune) (P). An individual can transition from S to I (they become infected), from I to D (they die), from I to S (they recover, but without immunity), or from I to P (they recover, and now have immunity). The states D and P are absorbing: once an individual is in either of these states, they never leave. The transition probabilities for each individual are, with infection states ordered (S,I,D,P),

$$\begin{bmatrix} 0.6 & 0.4 & 0 & 0 \\ 0.2 & 0.6 & 0.1 & 0.1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{with an infected neighbor}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 0.6 & 0.1 & 0.1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{with no infected neighbors.}$$

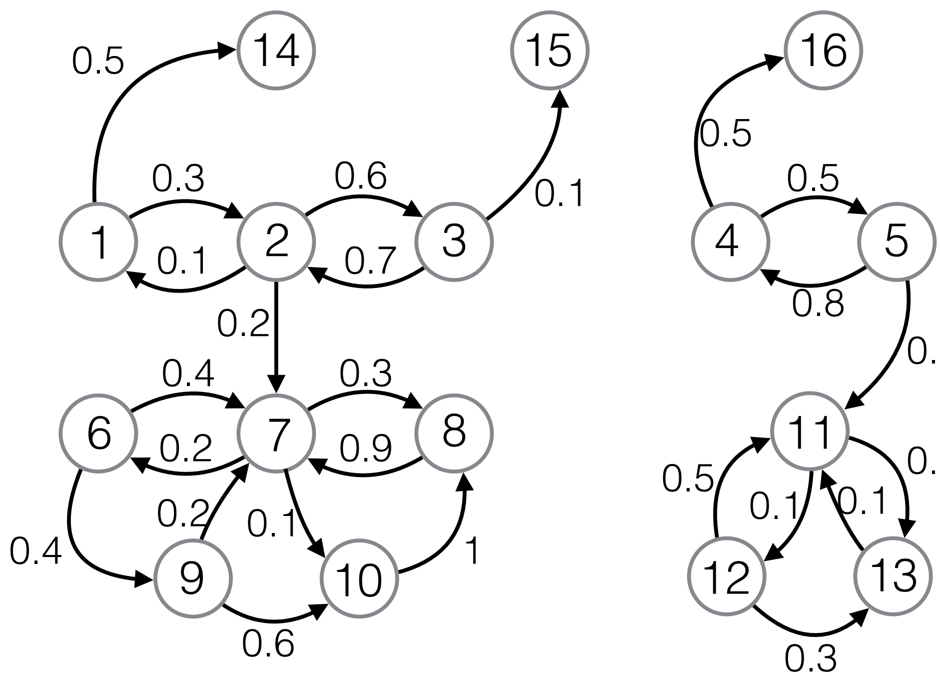
A typical individual has four neighbors: one above, one below, one to the left and one to the right; however, individuals on the boundary may be missing one or more of these neighbors. Use Monte Carlo simulation to estimate the expected number of individuals who are dead at $T = 50$, for a population of 100 individuals on a 10×10 grid, starting from each of the following initial distributions:

- Two individuals are infected, one at the $(2,2)$ location, and the other at the $(9,9)$ location. The rest of the population is susceptible.
- Two individuals are infected as above, but there are also eight protected individuals, at the following locations:

$$(2,3),(3,2),(3,7),(4,6),(6,4),(7,3),(8,9),(9,8).$$

  (These individuals could be immunized, for example.) All others are susceptible.

2. *Computing limiting transition matrix power and steady state distributions.* Consider the following Markov Chain transition graph (self-loops are not shown and can be determined from outgoing edges):



(a) Identify and classify the communicating classes as transient, ergodic, absorbing, or periodic.

(b) Do transition matrix powers for this chain converge? If so, compute the limiting value $L = \lim_{t\to\infty} P^t$.

(c) Do state distributions converge to a unique value for any initial distribution? Compute steady state distributions for the following initial distributions:

- $d_0 = [1, 0, ...0]$
- $d_0 = [0, 0, 0, 1, 0, ..., 0]$
- $d_0 = [0, 0.5, 0, 0, 0.5, 0, ..., 0]$

3. *Absorbing Markov Chains.* You and 7 of your friends find a $100 bill on the floor. No one has change to allow splitting it evenly, so you decide to play a game of chance to divide the money probabilistically. You all sit around a table. The game is played in turns. Each turn, one of three things happens with equal probability: the bill can move one position to the left, one position to the right, or the game ends and the person with the bill in front of him or her wins the game. You are the oldest in the group, so the bill starts in front of you. What are the chances you win the money?

4. *Hitting Times.* Consider a Markov Chain with states $\mathcal{X} = \{1, 2, 3, 4\}$. The transition matrix is

$$P = \begin{bmatrix} 0 & 0 & 0.4 & 0.6 \\ 0.3 & 0 & 0.3 & 0.4 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0.7 & 0.3 & 0 \end{bmatrix}.$$

Each part below defines a type of hitting time τ. For each part, plot the distribution of τ given that the chain starts in state $x_0 = 1$, and also determine $\mathbf{P}(\tau = 10 \mid x_0 = 1\}$.

(a) The hitting time τ_E, where $E = \{4\}$.

(b) The hitting time that is the smallest time such that $x_t = 4$ and $x_s = 2$ for some $s < t$, i.e., the first time state 4 is visiting *after* state 2 has been visited. *Hint*: Augment the chain with a copy of each state that encodes whether or not state 2 has been visited yet.