MECH 6327 - Homework 4

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BV Textobook Problems

0.1 Problem 5.43

The dual a SOCP defined as:

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \ i = 1, \ldots, m \end{array} \tag{1}$$

with $x \in \Re^n$ can be expressed as:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{m} \left(b_i^T u_i - d_i v_i \right) \\ \text{subject to} & \sum_{i=1}^{m} \left(A_i^T u_i - c_i v_i \right) + f = 0 \\ & \|u_i\|_2 \leq v_i, \ i = 1, \ldots, m \end{array} \tag{2}$$

with variables $u_i \in \Re_i^n, \ v_i \in \Re, \ i = 1, \ldots, m$ and problem data $f \in \Re^n, A_i \in \Re^{n_i \times n}, b_i \in \Re^{n_i}, c_i \in \Re, \ i = 1, \ldots, m$.

0.1.1 Part a

Problem: Derive the dual by defining $y_i \in \Re^{n_i}$ and $t_i \in \Re$ and the equalities $y_i = A_i x + b_i,\ t_i = c_i^T x + d_i$ then deriving the Lagrange dual.

Solution: The problem can first be written in a standard form as:

$$\begin{array}{ll} \text{minimize} & f^T \\ \text{subject to} & y_i = A_i x + b_i \\ & t_i = c_i^T x + d_i \\ & \|y_i\|_2 \le t_i,\ \forall i = 1, \ldots, m \end{array} \tag{3}$$

The lagrange can then be defined with

$$L(x, y, t, \lambda_i, \nu_i, \mu_i) = f^T + \sum_{i=1}^{m} \lambda_i(\|y_i\|_2 - t_i) + \sum_{i=1}^{m} \mu_i^T(t_i - C_i x + d_i) \tag{4}$$

$$\begin{aligned} &= \left(f + \sum_{i=1}^{m} \left(A_i^T \mu_i - c_i \nu_i\right)\right)^T x + \sum_{i=1}^{m} \lambda_i \|y_i\|_2 + \mu_i^T y_i \\ &+ \sum_{i=1}^{m} (-\lambda_i + \nu_i) t_i - \left(\sum_{i=1}^{m} b_i^T \mu_i - d_i \nu_i\right) \end{aligned} \tag{5}$$

Since the definition of the dual optimization problem is to maximize

$$g(\lambda_i, \nu_i, \mu_i) = \inf_{x, y_i, t_i} L(x, y, t, \lambda_i, \nu_i, \mu_i)$$

the inf can be found by determining when a min/max would occur for each of the variables.

For the critical point on x the direvative can be set to zero and thus the following equality must hold:

$$f + \sum_{i=1}^{m} \left(A_i^T \mu_i - c_i \nu_i\right) = 0 \tag{6}$$

For the y_i related term, it is known

$$\sum_{i=1}^{m} \lambda_i \|y_i\|_2 + \mu_i^T y_i$$

will be bounded below if it is within the cone defined by $\lambda_i \|y_i\|_2 \ge \|\mu_i\|_2 y_i$ which can be rewritten as:

$$\|\mu_i\|_2 \le \lambda_i$$

For the critical point over t_i the equality $\nu_i = \lambda_i$.

From this the dual problem can be obtained when the quantity $\left(\sum_{i=1}^{m} b_i^T \mu_i - d_i \nu_i\right)$ is maximized.

Thus the dual problem is defined as:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{m} \left(b_i^T u_i - d_i v_i\right) \\ \text{subject to} & \sum_{i=1}^{m} \left(A_i^T u_i - c_i v_i\right) + f = 0 \\ & \|u_i\|_2 \le v_i,\ i = 1, \ldots, m \end{array} \tag{7}$$

0.1.2 Part b

Problem: Start with the conic formulation of the SOCP and use the conic dual to prove the equivalence. Use the fact that the second-order dual is self-dual.

Solution: Starting with the SOCP given as

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i,\ i = 1, \dots, m \end{array} \tag{8}$$

a standard form can be defined by

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \left(A_i x + b_i, c_i^T x + d_i\right) \preceq_2 0 \end{array} \tag{9}$$

Since the conic dual transformation is known to transform

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & -\left(A^T x + b, c^T x + d\right) \preceq_K 0 \end{array} \tag{10}$$

into its dual according to its dual cone definition

$$\begin{array}{ll} \text{maximize} & b^T u + d\ v \\ \text{subject to} & A^T u + vc = f \\ & (u, v) \succeq_{K^*} 0,\ i = 1, \dots, m \end{array} \tag{11}$$

and from the fact that the 2-norm is self-dual, the dual program is given as

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{m} \left(b_i^T u_i - d_i v_i\right) \\ \text{subject to} & \displaystyle -\sum_{i=1}^{m} \left(A_i^T u_i - c_i v_i\right) = f \\ & (u_i, v_i) \succeq_2 0,\ i = 1, \dots, m \end{array} \tag{12}$$

or equivalently

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{m} \left(b_i^T u_i - d_i v_i\right) \\ \text{subject to} & \displaystyle\sum_{i=1}^{m} \left(A_i^T u_i - c_i v_i\right) + f = 0 \\ & \|u_i\|_2 \leq v_i,\ i = 1, \dots, m \end{array} \tag{13}$$

1 Problem 1: Robust control design

For the standard DT dynamical system defined as:

$$x_{t+1} = Ax_t + Bu_t \tag{14}$$

with dynamic matrix A unknown but assumed to belong to a set:

$$A \in \mathcal{A} = \mathrm{conv}(A_1, \ldots, A_m) \tag{15}$$

with A_i and B known.

Problem: For a state-feedback controller $u_t = Kx_t$ use Lyapunov techniques to design it so the system is globally asymptotically stable (GAS) by solving a semi-definite program (SDP).

Solution: The closed-loop system for the DT dynamical system can be defined by the dynamics

$$x_{t+1} = \hat{A}x = (A + BK)x \tag{16}$$

where $\hat{A} = A + BK$.

In order for the closed-loop system to be Globally Asymptotically Stable, a quadratic Lyapnov Function could be used to prove that if the following inequality is true then the system is GAS:

$$\hat{A}^T P \hat{A} - P \prec 0 \tag{17}$$

Since the system dynamics themselves are uncertain, this inequality will not be enough to prove GAS. This can be be address, however, by considering all $A \in \mathcal{A}$ to be a linear combination of the individual corner matrices. Since this is a convex hull, it is known that following this to its conclusion, GAS can be guaranteed for all $A \in \mathcal{A}$ if A_i is GAS $\forall i = 1, \ldots, m$.

Following this, a stabalizing gain can then be found as follows:

$$\hat{A}_i^T P \hat{A}_i - P \prec 0 \tag{18}$$

recognizing the Schur's compliment form, the following is true

$$\begin{bmatrix} P & \hat{A}_i^T \\ \hat{A}_i P^{-1} & \end{bmatrix} \succ 0 \tag{19}$$

$$\begin{bmatrix} P & (A_i + BK)^T \\ (A_i + BK) & P^{-1} \end{bmatrix} \succ 0 \tag{20}$$

A SDP feasibility problem can then be done to solve the problem such that

$$\begin{gathered} \begin{bmatrix} P & \hat{A}_i^T \\ \hat{A}_i P^{-1} & \end{bmatrix} \succ 0 \\ B^{-1}\left(\hat{A}_i - A_i\right) - K = 0, \ \forall i = 1, \ldots, m \end{gathered} \tag{21}$$

with variables P, $\hat{A}_i$, and K, along with problem data A_i and B. This is a problem that can now be easily implemented using CVX or YALMIP in MATLAB for given problem data.

2 Problem 2: Nonnegative and sum of squares polynomials

The Motzkin polynomial is defined as:

$$M(x, y) = x^2y^4 + x^4y^2 + 1 - 3x^2y^2 \quad (22)$$

Problem: Show that the Motzkin polynomial is nonnegative but can be expressed as sum of squares. It is sufficient to show this using numerical and/or symbolic solvers.

Solution: Nonegativity of the Motzkin polynomial can be proven using the AM-GM inequality using $n = 3$ with $x^4y^2, x^2y^4, 1$.

$$\sqrt[n]{\prod_{i=1}^{n} x_i} \leq \frac{1}{n}\sum_{i=1}^{n} x_i \quad (23)$$

$$\sqrt[3]{(x^4y^2)(x^2y^4)(1)} \leq \frac{1}{3}(x^4y^2 + x^2y^4 + 1) \quad (24)$$

$$x^2y^2 \leq \frac{1}{3}(x^4y^2 + x^2y^4 + 1) \quad (25)$$

$$3x^2y^2 \leq x^4y^2 + x^2y^4 + 1 \quad (26)$$

$$x^4y^2 + x^2y^4 + 1 - 3x^2y^2 \geq 0 \quad (27)$$

Therefore the Motzkin polynomial is non-negative.

However, it is not possible to put this into sum of square form using the standard solver as demonstrated by the infeasability result from the solvesos() command in yalmip (shown in AppendixA)

A MATLAB Code:

All code I write in this course can be found on my GitHub repository:
https://github.com/jonaswagner2826/MECH6327

Script 1: MECH6327_HW4

```
% MECH 6327 - HW 4

x = sdpvar(1,1);
y = sdpvar(1,1);
p = x^2 * y^4 + x^4 * y^2 + 1 - 3 * x^2 * y^2;
F = sos(p);
solvesos(F);
sdisplay(sosd(F))


% Results:
% ----------------------------
% -------------------------------------------------------------------------
% YALMIP SOS module started...
% -------------------------------------------------------------------------
% Detected 0 parametric variables and 2 independent variables.
% Detected 0 linear inequalities, 0 equality constraints and 0 LMIs.
% Using kernel representation (options.sos.model=1).
% Initially 8 monomials in R^2
% Newton polytope (2 LPs).........Keeping 4 monomials (0.20313sec)
% Finding symmetries..............Found 3 symmetries (0sec)
% Partitioning using symmetry.....1x1(4)
%
%
% Problem is unbounded.
%
%
% -> Solver reported unboundness of the dual problem.
% -> Your SOS problem is probably infeasible (SOS is dualized).
```

Refrences: * not bibtex becouse of time...
https://people.eecs.berkeley.edu/ elghaoui/Teaching/EE227A/lecture10.pdf
https://people.orie.cornell.edu/miketodd/iccopt.pdf