# Lecture 12 : Lyapunov Theory

goals :

- intro to Lyapunov theory
- understand how basic underlying ideas are useful far beyond just stability analysis
- automate search for Lyapunov functions using SOS programming + SDP

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## What is Lyapunov theory?

- used to make conclusions about properties of dynamical systems, without explicitly finding the trajectories

- classical case : study stability of autonomous system

  Lyapunov (1890s)

$$\dot{x}(t) = f(x(t)) \qquad x(t) \in \mathbb{R}^n, \; x(t_0) = x_0, \quad f : \mathbb{R}^n \to \mathbb{R}^n$$

- analytical solution to ODE only in a few special cases

- typical Lyapunov theorem:
  - if there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying some conditions on $V$ and $\dot{V}$
  - then the system satisfies some property (e.g. stability)

- useful way beyond stability
  - bounds on performance indices
  - rates of convergence or growth
  - regions of attraction
  - robustness to uncertain dynamics, disturbances
  - bounds on reachable sets
  - set invariance
    - safety, collision avoidance, constraint satisfaction
  - input/output analysis (passivity, dissipativity)
  - feedback control design
  - etc.

# Stability Definitions

Consider the dynamical system

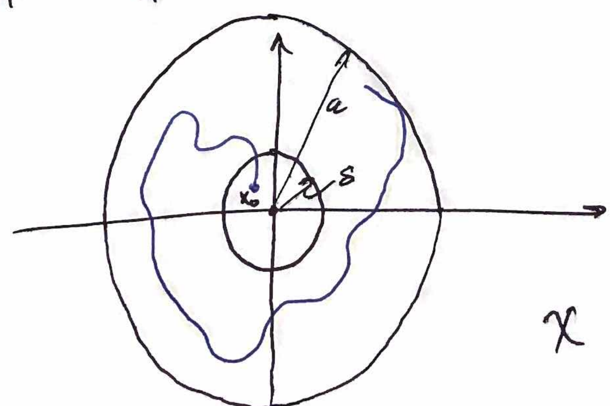
$$\dot{x}(t) = f(x(t)) \qquad (*)$$

- assume $f$ satisfies standard conditions for existence + uniqueness of solutions (e.g. Lipschitz continuity)

- a point $\bar{x} \in \mathbb{R}^n$ is called an equilibrium if $f(\bar{x}) = 0$
  - for analysis can assume $\bar{x} = 0$ wlog with a simple coordinate transformation

Definition: An equilibrium point $\bar{x} = 0$ of $(*)$ is called stable (in the sense of Lyapunov) if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\|x(t_0)\| < \delta \implies \|x(t)\| < \varepsilon \quad \forall t \geq t_0$$



**Definition** An equilibrium point $\bar{x}=0$ of $(*)$ is called asymptotically stable if

① $\bar{x}$ is stable

② $\exists\, \alpha > 0$ such that

$$\|x(t_0)\| < \alpha \implies \lim_{t\to\infty} x(t) = 0$$

i.e., $\bar{x}=0$ is locally attractive.

$\bar{x}$ is called globally asymptotically stable if

$\lim_{t\to\infty} x(t) = 0$ for every initial state.

**Definition** An equilibrium point $\bar{x}=0$ is called exponentially stable if $\exists\, m>0,\ \gamma>0,\ \delta>0$ s.t.

$$\|x(t_0)\| < \delta \implies \|x(t)\| \leq m e^{-\gamma(t-t_0)} \|x(t_0)\| \quad \forall t \geq t_0$$

• analogous defn. for global exponential stability

**Definition:** An equilibrium $\bar{x} = 0$ is called **unstable** if it is not stable, i.e. if

$\exists\, \varepsilon > 0$ such that $\forall\, \delta > 0$ with $\|x(t_0)\| < \delta$,

$\exists$ finite $t^* \geq t_0$ such that $\|x(t^*)\| \geq \varepsilon$

**Definition** The solution $x(t)$ of $(*)$ is called **bounded** if $\exists\, \beta(x_0)$ such that $\|x(t) - x(t_0)\| < \beta \;\; \forall t$

- useful when $(*)$ has no equilibria, or has limit cycles

- many variants of stability for different types of systems

## Generalized Energy Functions

**Definition:** A function $V: \mathbb{R}^n \to \mathbb{R}$ is called **positive definite** (PD) if

- $V(z) \geq 0 \quad \forall z \in \mathbb{R}^n$
- $V(z) = 0 \iff z = 0$
- $V(z) \to \infty$ as $\|z\| \to \infty$ (radial unboundedness)
  - $\Rightarrow$ sublevel sets of $V$ are bounded

Ex $V(z) = z^T P z$ is PD iff $P \succ 0$

**Definition** A function $V:\mathbb{R}^n \to \mathbb{R}$ is *locally positive definite* (LPD) if $\exists \varepsilon > 0$ such that $V$ is PD on $\{x \mid \|x\| \le \varepsilon\}$.

- Consider $\dot{x} = f(x)$ w/ equilibrium $\bar{x} = 0$ and a differentiable function $V:\mathbb{R}^n \to \mathbb{R}$. The derivative of $V$ along the system trajectories is given by

$$\dot{V}(x) = \nabla V(x)^T f(x)$$

$\boxed{\text{Basic Lyapunov Theorem}}$

① If $V$ is LPD and $\dot{V} \le 0$ locally, then $\bar{x} = 0$ is stable (in the sense of Lyapunov)

② If $V$ is LPD and $-\dot{V}$ is LPD, then $\bar{x} = 0$ is locally asymptotically stable

③ If $V$ is PD and $-\dot{V}$ is PD, then $\bar{x} = 0$ is globally asymptotically stable

④ If $V$ is PD and $\dot{V}(x) \le -\alpha V(x)$ for some $\alpha > 0$, then $\bar{x} = 0$ is globally exponentially stable

- not necessary to solve $\dot{x} = f(x)$, just need to find a Lyapunov function (satisfying the conditions)
  - remarkable!

- can interpret $V$ as a <u>generalized</u> energy function, that dissipates energy along all system trajectories
  - e.g. total energy of a mechanical system with friction losses
  - Lyapunov's brilliant idea: $V$ doesn't have to come from physics!

- converse also holds: if an equilibrium is stable, then there's a Lyapunov function that proves it

<u>How to find $V$?</u>

- classical: choose a form (typically quadratic) and try to verify properties by hand

- modern: search for parameters of $V$ using <u>convex optimization</u>!

- When $V$ and $f$ are polynomials, all conditions can be expressed as polynomial nonnegativity, replaced w/ SOS constraints

<u>Ex</u> Suppose we can find a polynomial $V$ s.t.

$$V(0) = 0$$

$$V(x) - \varepsilon x^T x \text{ is SOS for some } \varepsilon > 0$$

$$-\nabla V(x)^T f(x) \text{ is SOS}$$

Then $\bar{x} = 0$ is stable.

Searching for coefficients of $V$ for a given $f$ is an SDP feasibility problem!

- Essentially identical results hold for <u>discrete time</u> dynamical systems

$$x_{t+1} = f(x_t) \qquad t = 0, 1, \ldots$$

if we interpret $\dot{V}$ as

$$V(x_{t+1}) - V(x_t) = V(f(x_t)) - V(x_t)$$

<u>Ex</u> If $V$ is LPD and $V(f(x)) \le V(x)$ locally, then equilibrium $\bar{x} = 0$ of $x_{t+1} = f(x_t)$ is stable

$\hookrightarrow \bar{x} = f(\bar{x})$ for DT systems

Ex Global Asymptotic / Exponential Stability of Linear Systems

- Consider $\dot{x} = Ax$ and Lyapunov function candidate
$V(x) = x^T P x \,, \quad P = P^T$

  - $V$ PD $\iff P \succ 0$
  - $\dot{V} = \dot{x}^T P x + x^T P \dot{x}$

$$= x^T A^T P x + x^T P A x$$

$$= x^T (A^T P + PA) x$$

$\implies -\dot{V}$ PD $\iff A^T P + PA \prec 0$

  - $\dot{x} = Ax$ globally asymptotically stable $\iff$
$\exists P \succ 0$ such that $A^T P + PA \prec 0$ (LMIs in $P$!)
  - $\iff \forall Q = Q^T \succ 0 \,, \exists P = P^T \succ 0 : A^T P + PA + Q = 0$
$\iff \mathrm{Real}(\lambda_i(A)) < 0 \quad \forall i = 1, \dots, n$

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- Consider $x_{t+1} = A x_t$, Lyapunov func. candidate $V(x) = x^T P x$
  - "$\dot{V}$" $= V(x_{t+1}) - V(x_t) = (Ax_t)^T P (Ax_t) - x_t^T P x_t$

$$= x_t^T (A^T P A - P) x_t$$

$\implies -$"$\dot{V}$" PD $\iff A^T P A - P \prec 0$

  - $x_{t+1} = A x_t$ GAS $\iff \exists P \succ 0 : A^T P A - P \prec 0$ (LMIs in P!)
  - $\iff \forall Q = Q^T \succ 0 \,, \exists P = P^T \succ 0 : A^T P A - P + Q = 0$
$\iff |\lambda_i(A)| < 1 \quad \forall i = 1, \dots, n$

Ex Jet engine compressor model (Moore + Greitzer 1986)

- compressor stall (disruption of air flow to engine) can cause engine failure, crashes

- dynamic model of pressure related variables

$$\left.\begin{aligned} \dot{x} &= -y - \tfrac{3}{2}x^2 - \tfrac{1}{2}x^3 \\ \dot{y} &= 3x - y \end{aligned}\right\} \ \underbrace{\text{nonlinear}}_{f(x,y)} \ \text{ODE}$$

- Is the origin stable? Can we find $V$ s.t.

$$V(0) = 0$$

$$V(x,y) - \varepsilon\,\phi(x,y) \in SOS \quad \text{for some } \varepsilon > 0,\ \phi \text{ LPD}$$

$$-\nabla V(x,y)^T f(x,y) \in SOS$$

- Using SOSTOOLS/YALMIP to transform to SDP, we obtain Lyapunov function

$$\begin{aligned} V(x,y) = {} & 4.58\,x^2 - 1.578\,xy + 1.78\,y^2 - 0.13\,x^3 + 2.52\,x^2y \\ & - 0.34\,xy^2 + 0.61\,y^3 + 0.48x^4 - 0.05\,x^3y + 0.44\,x^2y^2 \\ & + 0.0000019\,xy^3 + 0.09\,y^4 \end{aligned}$$

$\Longrightarrow$ origin (globally asymptotically) stable

Ex Robust stability of linear systems

- Consider the system $x_{t+1} = A x_t$, where $A$ is unknown but assumed to lie in a set

$$A \in \bar{A} = \mathrm{conv}(A_1, \ldots, A_m)$$

We'd like to know if the system is robustly stable, i.e. stable for every matrix in $\bar{A}$.

- No analytical solution!
  - set of stable dynamics matrices non-convex generally, cannot simply check stability of $A_i$ $i=1,\ldots,m$

Ex $A_1 = \begin{bmatrix} 0.2 & 0.3 & 0.7 \\ 0.9 & 0 & 0 \\ 0 & 0.8 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.3 & 0.9 & 0.4 \\ 0.5 & 0 & 0 \\ 0 & 0.9 & 0 \end{bmatrix}$ stable

but $\frac{3}{5} A_1 + \frac{2}{5} A_2$ not stable!

- Determining when a system is robustly stable is hard (NP) in general, but Lyapunov theory gives an efficiently checkable sufficient condition:

Theorem Let $A_1, \ldots, A_m \in \mathbb{R}^{n \times n}$. If $\exists P \succ 0$ such that

$$A_i^T P A_i - P \prec 0 \quad \forall i = 1, \ldots, m$$

then $\rho(A) < 1$ $\forall A \in \bar{A} = \mathrm{conv}(A_1, \ldots, A_m)$

spectral radius

i.e. robust stability

Proof: Consider an arbitrary point in $\mathrm{conv}(A_1, \dots, A_m)$

$$A = \sum_i \alpha_i A_i \quad , \quad \alpha_i \geq 0 \quad , \quad \sum_i \alpha_i = 1.$$

Suppose we can find a quadratic Lyapunov function that simultaneously certifies stability of each matrix generating the convex hull, i.e. $\exists P \succ 0$:

$$A_i^T P A_i - P \prec 0 \qquad \forall i = 1, \dots, m.$$

Taking the Schur complement gives

$$\begin{bmatrix} P & A_i^T \\ A_i & P^{-1} \end{bmatrix} \succ 0 \qquad \forall i = 1, \dots, m$$

Multiplying by $\alpha_i \geq 0$ and summing gives

$$\begin{bmatrix} P & A^T \\ A & P^{-1} \end{bmatrix} \succ 0$$

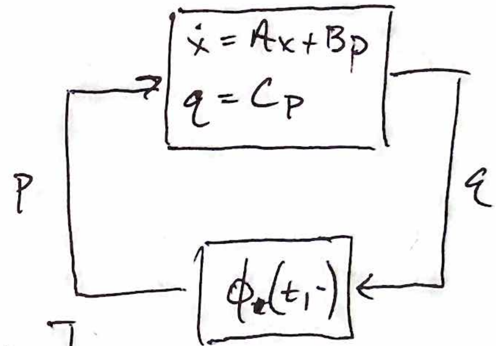
and via Schur complement again $A^T P A - P \prec 0$.

Hence $A$ is stable, which implies robust stability. ∎

- powerful + remarkable result, since we just proved stability for an infinite # of systems!

## Ex Sector bounded nonlinearities

- Consider the system



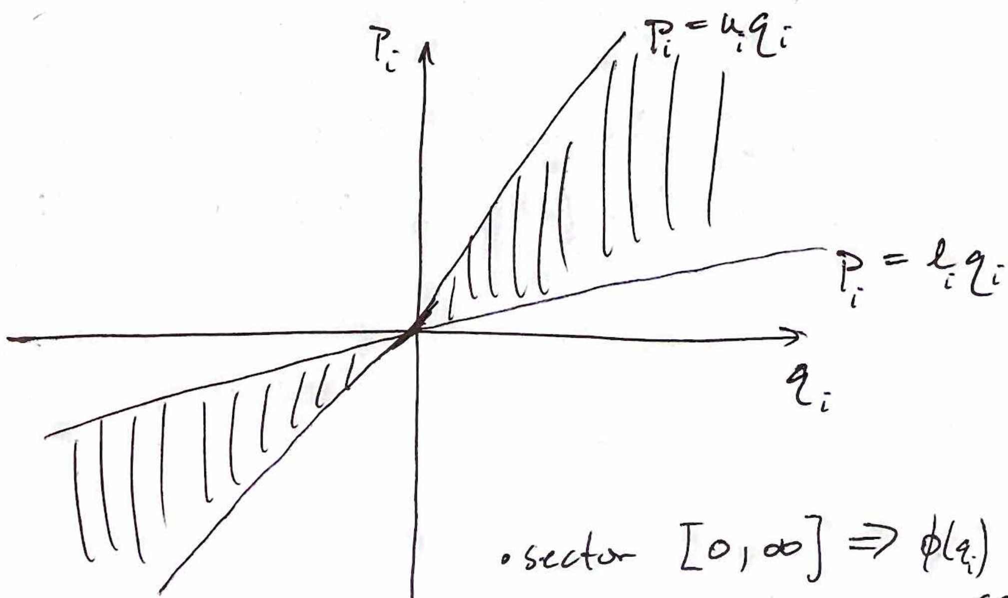

$$\dot{x} = Ax + B\phi(t, Cx)$$

where $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $\phi = \begin{bmatrix} \phi_1(t, c_1^T x) \\ \vdots \\ \phi_m(t, c_m^T x) \end{bmatrix}$

- separates linear + nonlinear, time-varying parts

- Assume that $\phi_i(t, \cdot)$ is sector bounded in sector $[l_i, u_i]$ with $p_i = \phi_i(t, q_i)$

$$(p_i - u_i q_i)(p_i - l_i q_i) \leq 0 \quad \forall q_i \in \mathbb{R} \text{ with } p_i = \phi_i(t, q_i)$$




- sector $[0, \infty] \Rightarrow \phi(q_i)$ and $q_i$ have same sign

- sector $[-1, 1] \Longrightarrow |\phi_i(t, q_i)| \leq |q_i| \quad \forall q_i$

- Is the system stable for every possible nonlinearity in the sector bound?

- Let's look for a quadratic Lyapunov function that establishes global exponential stability

$$V(x) = x^T P x \qquad P \succ 0$$

we want: $\dot{V}(x) \le -\alpha V(x) \quad \forall x$, for some given $\alpha > 0$

$$\dot{V}(x) + \alpha V(x) = \dot{x}^T P x + x^T P \dot{x} + \alpha x^T P x$$

$$= (Ax + Bp)^T P x + x^T P (Ax + Bp) + \alpha x^T P x$$

with $p = \phi(t, Cx)$

$$= \begin{bmatrix} x \\ p \end{bmatrix}^T \begin{bmatrix} A^T P + PA + \alpha P & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}$$

we want this expression $\le 0$ whenever the nonlinearities satisfy $(p_i - u_i q_i)(p_i - l_i q_i) \le 0$ $\quad i = 1, \dots, m$

$q_i = c_i^T x$

$$Q_i(x,p) = \begin{bmatrix} x \\ p \end{bmatrix}^T \begin{bmatrix} \sigma_i c_i c_i^T & -\nu_i c_i e_i^T \\ -\nu_i e_i c_i^T & e_i e_i^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \le 0 \qquad i = 1, \dots, m$$

where $\sigma_i = l_i u_i$, $\nu_i = \dfrac{l_i + u_i}{2}$, $e_i$ = $i$th standard basis vector

- Now we can use the <u>S-Procedure</u> to get a sufficient condition:

$$\dot{V}(x) + \alpha V(x) \le 0 \qquad \text{whenever} \quad Q_i(x,p) \le 0 \qquad i=1,\dots,m$$

if $\exists\ \tau_1,\dots,\tau_m \ge 0$ such that

$$\begin{bmatrix} A^T P + PA + \alpha P - \sum_{i=1}^{m} \tau_i \sigma_i c_i c_i^T & PB + \sum_{i=1}^{m} \tau_i \nu_i c_i e_i^T \\ B^T P + \sum_{i=1}^{m} \tau_i \nu_i e_i c_i^T & -\sum_{i=1}^{m} \tau_i e_i e_i^T \end{bmatrix} \preceq 0$$

- an LMI in variables $P = P^T \succ 0$, $\tau_1, \dots, \tau_m$, problem data $A, B, \alpha, l_i, u_i, c_i$ $i=1,\dots,m$

- if we find feasible $P, \tau_i$, we certify global exponential stability for a large class of nonlinear, time-varying systems, a very strong result!

- Historical note: when there's only one nonlinearity, the exact S-Procedure applies, known as the <u>Lur'e problem</u>