

MECH 6327 - Homework 2

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(Probably not necessary... but its long)

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1 Problem Set 1: Convex Sets

1.1 Problem 2.5

Problem:

What is the distance between two parallel hyperplanes: $\{x \in \mathbb{R}^n | a^T x = b_1\}$ and $\{x \in \mathbb{R}^n | a^T x = b_2\}$?

Solution:

Under the assumption that $a \in \mathbb{R}^n$ and $b_1, b_2 \in \mathbb{R}$, the quantity $a^T x_0$ represents the component of x_0 in the normal direction. Similarly, the quantities b_1 and b_2 represent the euclidean distance of the hyperplane from the origin (in the normal direction). Since the hyperplanes are parallel, the distance between them is the difference between their offsets:

$$\text{Distance between hyperplanes: } b_1 - b_2 \quad (1)$$

1.2 Problem 2.7

Problem:

Voronoi description of halfspace. Let a and b be distinct points in \mathbb{R}^n . Show that the set of all points that are closer to a than b via the euclidean norm is a halfspace. Describe it explicitly as an inequality and draw a picture.

Solution:

The set of all points closer to a than b can be defined as:

$$\{x \in \mathbb{R}^n \mid \|x - a\|_2 \leq \|x - b\|_2\} \quad (2)$$

The boundary defining this halfspace will be a plane defined by the normal vector c representing the distance between a and b , and the offset coefficient d describing intersection of the plane through the half-way point between a and b . The quantities c and d can therefore be defined by:

$$\begin{aligned} c &= b - a \\ d &= \frac{c^T a + c^T b}{2} \\ &= \frac{1}{2} c^T (a + b) \end{aligned} \quad (3)$$

The halfspace, that is equivalent to x , can be described by the following:

$$\{x \in \mathbb{R}^n \mid c^T x \leq d\} \quad (4)$$

This can be visualized in two dimensions for $a = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $b = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$. The boundary (the red line) is calculated in the standard form using

$$x_2 = \frac{-1}{c_2} (c_1 * x_1 - d)$$

and then plotted. The half-space itself is the region below the boundary.

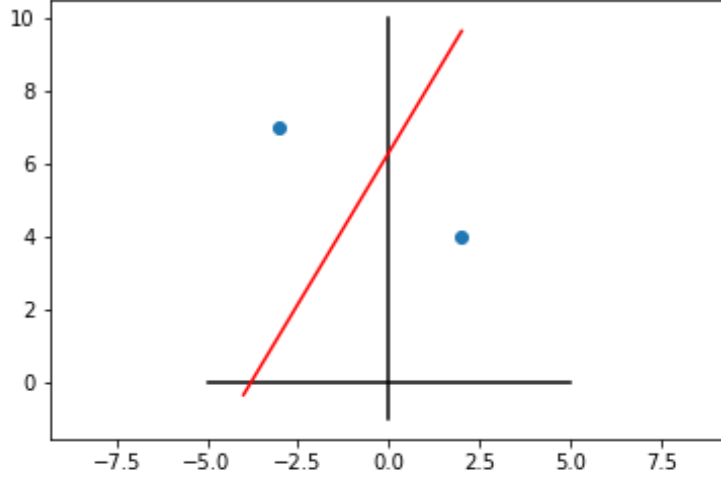


Figure 1: Visualization of the boundary for the halfspace.

1.3 Problem 2.12

Problem:

Which of the following sets are convex?

Solution:

1.3.1 (a) - Slab

A slab defined as

$$\{x \in \mathbb{R}^n \mid \alpha \leq a^T x \leq \beta\}$$

is convex as it consists of the intersection of two halfspaces which themselves are convex.

1.3.2 (b) - Rectangle/Hyperrectangle

A rectangle set defined as

$$\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, \dots, n\}$$

is convex as it is composed of the intersections of half spaces which are themselves convex. This is similar to the polyhedrons/polytopes that by definition are also convex.

1.3.3 (c) - Wedge

A wedge set given as

$$\{x \in \mathbb{R}^n \mid a_1^T x \leq b_1, a_2^T x \leq b_2\}$$

is convex as it is just an intersection of two halfspaces (a polyhedron).

1.3.4 (d) - Closer to a point than a set

A set of points closer to a given point than a given set is defined as

$$\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \ \forall y \in S\}$$

where $S \subset \mathbb{R}^n$ **is not convex** in general. This is because there is not enough information about y for a conclusion to be made whether it is convex or not. A counter example would be if y is a point in orbit around a convex shape S that would end up generating a concave x .

1.3.5 (e) - Closer to a set than another set

A set of points closer to a given set than another given set is defined as

$$\{x \mid \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\}$$

where $S, T \subset \mathbb{R}^n$, and

$$\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}$$

is not convex in general. This is because there is not enough information about S and T for a conclusion to be made whether it is convex or not. A counter example includes if S or T themselves are a concave shape that causes the set x to also be concave and therefore not convex.

1.3.6 (f) - Set of the sum being within a convex set

The set defined as

$$\{x \mid x + S_2 \subseteq S_1\}$$

with S_1 being convex **is**

????????????????????????????????????

1.3.7 (g) - Set with weighted distances to two points

The set of all points that is closer to a than b by at least a factor of θ , defined as

$$\{x \in \mathbb{R}^n \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}$$

with $a \neq b$ and $0 \leq \theta \leq 1$ **is convex**. This is known because, as proven in a previous problem, a hyperplane is formed for a similarity stated problem which itself is convex. When the distance to a must be less than a portion of the distance to b it will cause the pseudo-hyperplane to curve inwards and untimely remain convex.

1.4 Problem 2.28

Problem:

Define the positive semi-definite cone (S_+^n) for $n = 1, 2, 3$ in terms of ordinary inequalities with the matrix coefficients themselves.

Solution:

The positive semi-definite cone is defined for size n as the set of all symmetric matrices that are positive semi-definite:

$$S_+^n \equiv \{x \in S^n \mid x \succeq 0\} \quad (5)$$

One method to ensure that a matrix is positive semi-definite is to ensure that its leading principle minors are all non-negative (strictly positive for positive definite).

For $n = 1$ the required inequalities are simple,

$$X = \begin{bmatrix} x_1 \end{bmatrix} \in S_+^1 \iff x_1 \geq 0 \quad (6)$$

For $n = 2$ the inequalities can be found by ensuring the leading principle minors are all non negative:

$$\begin{aligned} m_1 &= \det[x_1] \\ &= x_1 \\ m_2 &= \det \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \\ &= x_1 x_3 - x_2^2 \end{aligned} \quad (7)$$

These definitions of the minors can be then be used to construct inequalities such that all the minors are positive:

$$X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \in S_+^2 \iff \begin{aligned} x_1 &\geq 0 \\ x_1 x_3 &\geq x_2^2 \end{aligned} \quad (8)$$

For $n = 3$ the inequalities can be found by ensuring the leading principle minors are all non negative:

$$\begin{aligned} m_1 &= \det[x_1] \\ &= x_1 \\ m_2 &= \det \begin{bmatrix} x_1 & x_2 \\ x_2 & x_4 \end{bmatrix} \\ &= x_1 x_4 - x_2^2 \\ m_3 &= \det \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix} \\ &= x_1(x_1 x_4 - x_2^2) - x_2(x_2 x_6 - x_3 x_5) + x_3(x_2 x_5 - x_3 x_4) \\ &= x_1^2 x_4 - x_1 x_2^2 - x_2^2 x_6 + x_2 x_3 x_5 + x_2 x_3 x_5 - x_3^2 x_4 \end{aligned} \quad (9)$$

These definitions of the minors can be then be used to construct inequalities such that all the minors are positive:

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix} \in S_+^3 \iff \begin{aligned} x_1 &\geq 0 \\ x_1 x_4 &\geq x_2^2 \\ x_1 x_2^2 + x_2^2 x_6 + x_3^2 x_4 &\geq x_1^2 x_4 + 2x_2 x_3 x_5 \end{aligned} \quad (10)$$

1.5 Problem 2.33

The monotone non-negative cone is defined as all the nonnegative vectors with components sorted in non-increasing order:

$$K_{m+} = \{x \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\} \quad (11)$$

1.5.1 Part a

Problem:

Show that K_{m+} is a proper cone.

Solution:

A set, $C \subseteq \mathbb{R}^n$, is considered a cone if

$$\theta x \in C \quad \forall x \in C \text{ and } \theta \geq 0 \quad (12)$$

It can be easily seen that the set K_{m+} satisfies this condition as scaling each element of the matrix $x \in K_{m+}$ will equally be scaled by the same amount and the conditions of non-increasing order will still apply. This guarantees that K_{m+} is in fact a cone.

To ensure convexity, the definition of convexity and of a cone can be incorporated into the following test:
The set $C \subseteq \mathbb{R}^n$ is a convex cone iff

$$\theta_1 x_1 + \theta_2 x_2 \in C \quad \forall x_1, x_2 \in C, \theta_1, \theta_2 \geq 0 \quad (13)$$

It is also clear that K_{m+} will satisfy as if each element element of one of the matrices is scaled it maintains the nonincreasing order. The same is true for summing two K_{m+} matrices as the nonincreasing order will be maintained.

It is also known by definition ***** ask about this....

1.5.2 Part b

Problem:

Find the dual cone, K_{m+}^* .

Solution:

A dual cone for K is defined as:

$$K^* = \{y \in \mathbb{R}^n \mid x^T y \geq 0 \quad \forall x \in K\} \quad (14)$$

??

It is known that the following is equivalent:

$$\begin{aligned} \sum_{i=1}^n x_i y_i &= (x_1 - x_2)y_1 + (x_2 - x_3)(y_1 + y_2) + \cdots \\ &= +(x_{n-1} - x_n)(y_1 + \cdots + y_{n-1}) + x_n(y_1 + \cdots + y_n) \end{aligned} \quad (15)$$

2 Problem Set 2: Convex Functions

2.1 Problem 3.6

Problem:

What is the epigraph for the following convex functions?

Solution:

The epigraph is defined as

$$\text{epi}f = \{(x, t) \mid x \in \text{dom}f, f(x) \leq t\} \quad (16)$$

where $\text{epi}f \subset \mathbb{R}^{n+1}$. This is equivalent to saying the space 'above' the function.

2.1.1 (a) - Epigraph of a halfspace

The epigraph of a halfspace in n dimensions is a halfspace in $n + 1$ dimensions.

2.1.2 (b) - Epigraph of a convex cone

The epigraph of a convex cone is another convex cone? maybe a triangular prism type thing...

2.1.3 (c) - Epigraph of a polyhedron

The epigraph of a polyhedron is another polyhedron. similar to convex cone... it isn't originally bounded

2.2 Problem 3.16

Problem:

Determine if the following functions are convex, concave, quasiconvex, or quasiconcave.

Solution:

2.2.1 (a) - $e^x - 1$

Let

$$f(x) = e^x - 1$$

on \mathbb{R} .

$f(x)$ is **convex** and can be proven in multiple ways. For instance, visualizing the epigraph of the function, it is clear that a convex set is produced. Additionally, $f(x)$ can be constructed by putting the convex function e^x through the convex affine function $x - 1$.

2.2.2 (b) - x_1x_2

Let

$$f(x_1, x_2) = x_1x_2$$

over the domain \mathbb{R}_{++}^2 .

This function is **convex** and can be demonstrated using the definition:

$$f(\theta x + (1 - \theta)y) = \theta f(x) + (1 - \theta)f(y) \quad (17)$$

$$\begin{aligned}
& (\theta x_1 + (1 - \theta)y_1)(\theta x_2 + (1 - \theta)y_2) = \theta x_1 x_2 + (1 - \theta)y_1 y_2 \\
& \theta^2 x_1 x_2 + \theta(1 - \theta)y_1 x_2 + \theta(1 - \theta)x_1 y_2 + (1 - \theta)^2 y_1 y_2 = \theta x_1 x_2 + (1 - \theta)y_1 y_2 \\
& \theta^2 x_1 x_2 + (\theta - \theta^2)(x_1 y_2 + x_2 y_1) + (1 - 2\theta + \theta^2)y_1 y_2 = \theta x_1 x_2 + (1 - \theta)y_1 y_2
\end{aligned} \tag{18}$$

If we analyze each set of terms it not explicitly clear by the result, but within the domain \mathfrak{R}_{++}^2 the equality holds true.

add plot...

2.2.3 (c) - $1/(x_1 x_2)$

Let

$$f(x_1, x_2) = 1/(x_1 x_2)$$

be defined over the domain R_{++}^2 .

2.2.4 (d) - x_1/x_2

Let

$$f(x_1, x_2) = x_1/x_2$$

be defined over the domain R_{++}^2 .

2.3 (e) - x_1^2/x_2

Let

$$f(x_1, x_2) = x_1^2/x_2$$

be defined over the domain $R \times R_+$.

2.4 (f) - $x_1^\alpha x_2^{1-\alpha}$

Let

$$f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$$

with $0 \leq \alpha \leq 1$ be defined over the domain \mathfrak{R}_{++}^2 .

2.5 Problem 3.18a

Problem:

Using the proof of concavity of the log-determinant function to show that

$$f(X) = \text{tr}(X^{-1})$$

is convex over the domain S_{++}^n .

Solution:

2.6 Problem 3.22

Problem:

Use various composition rules to show that the following functions are convex.

Solution:

2.6.1 (a) - double log functions

Let

$$f(x) = -\log\left(-\log\left(\sum_{i=1}^m e^{a_i^T x + b_i}\right)\right)$$

be defined over the domain $\{x \mid \sum_{i=1}^m e^{a_i^T x + b_i} < 1\}$.

It is known that

$$\log\left(\sum_{i=1}^n e^{y_i}\right)$$

is convex. Since all compositions of convex functions with affine functions are also convex, it can be said that

$$\sum_{i=1}^m e^{a_i^T x + b_i}$$

is also convex.

Additionally, it is known that the $\log(\cdot)$ function is concave, but when the sign changes it becomes convex, thus the composition of $-g(-g(h(x)))$ is convex for the concave function $g(x) = \log(x)$.

Therefore, the function $f(x)$ is convex.

2.6.2 (b) - square root of some product sum

Let

$$f(x, u, v) = -\sqrt{uv - x^T x}$$

be defined over the domain $\{(x, u, v) \mid uv > x^T x, u, v > 0\}$.

It is known that $g(x) = x^x/u$ is convex for $u > 0$ and that $h(x) = \sqrt{x_1 x_2}$ is convex on \mathbb{R}_{++}^2 .

The function $f(x, u, v)$ can be manipulated as follows:

$$\begin{aligned} f(x, u, v) &= -\sqrt{uv - x^T x} \\ &= -\sqrt{u\left(v - \frac{x^T x}{u}\right)} \end{aligned}$$

From what we know about the underlying functions, it can be said that the convex function $g(x)$ is summed with v (a convex combination) and then used as an input to the convex function $h(x)$, resulting in an overall convex function.

2.6.3 (c) - log of some product sum

Let

$$f(x, u, v) = -\log(uv - x^T x)$$

be defined over the domain $\{(x, u, v) \mid uv > x^T x, u, v > 0\}$.

It is known that $g(x) = x^x/u$ is convex for $u > 0$ as well as that the $h(x) = \log(x)$ function is concave.

Performing the same manipulation as in the previous problem, $f(x, u, v)$ can be written as:

$$f(x, u, v) = -\log\left(u\left(v - \frac{x^T x}{u}\right)\right)$$

From this it can be derived that the convex function $g(x)$ is summed with v (a convex combination) and then used as an input to the concave function $h(x)$ but is then negated to result in an overall convex function.

2.6.4 (d) - complicated root of a powered sum and norm

Let

$$f(x, t) = -\left(t^p - \|x\|_p^p\right)^{1/p}$$

be defined with $p > 1$ over the domain $\{(x, t) \mid t \geq \|x\|_p\}$.

It is known that $g(x, u) = \|x\|_p^p/u^{p-1}$ is convex for $u > 0$ and that $h(x, y) = -x^{1/p}y^{1-1/p}$ is convex over \mathbb{R}_{++}^2 .

??

2.6.5 (e) - complicated log of a powered sum and norm

Let

$$f(x, t) = -\log\left(t^p - \|x\|_p^p\right)$$

with $p > 1$ be defined over the domain $\{(x, t) \mid t \geq \|x\|_p\}$.

It is known that $g(x, u) = \|x\|_p^p/u^{p-1}$ is convex for $u > 0$.