Due via email on Wednesday, March 24, 2021. Please use the subject line "MECH 6327: Homework 3 Submission", submit all solutions in a single pdf file, and name your file "Yourlastname_HW3.pdf".

Reminder: Please read through BV Chapters 4 and 5.

Convex Optimization Problems

- Do problems 4.11, 4.16, 4.28(a), 4.43(a)-(c) from the BV textbook.

1. **Open-loop optimal control with $1-$ and $\infty-$norms.** Consider the following open-loop optimal regulation problem

$$
\begin{array}{ll}
\text{minimize} & \|x_T\|_p + \displaystyle\sum_{t=0}^{T-1} \|x_t\|_p + \gamma \|u_t\|_q \\
\text{subject to} & x_{t+1} = Ax_t + Bu_t, \quad t = 0, ..., T-1 \\
& \|x_t\|_\infty \leq \bar{x}, \quad t = 0, ..., T \\
& \|u_t\|_\infty \leq \bar{u}, \quad t = 0, ..., T
\end{array}
$$

with system state $x_t \in \mathbf{R}^n$, control input $u_t \in \mathbf{R}^m$, and parameter $\gamma > 0$ that governs the tradeoff between actuator effort and state regulation performance. Express the problem as a linear program (LP) for the following cases:

- $p = q = \infty$
- $p = q = 1$

Code these cases in CVX (or your chosen modeling language), and for the problem data in the Homework folder on eLearning, verify that the solution of your equivalent linear program matches with the solution obtained using the built-in norm functions. (You may find the function `norms` useful for computing norms of columns (or rows) of matrix variables). Plot the optimal state and input trajectories.

2. **Minimum time state transfer via quasiconvex optimization.** Consider the linear dynamical system

$$x_{t+1} = Ax_t + Bu_t, \quad t = 0, ..., T$$

with initial state x_0 given and subject to the input constraints

$$\underline{u} \leq u_t \leq \bar{u}, \quad t = 0, ..., T$$

Argue that the minimum time required to transfer the system from the initial state to a given desired state x_{des}

$$f(u_0, ..., u_T) = \min\{\tau \mid x_t = x_{\text{des}} \text{ for } \tau \leq t \leq T+1\}$$

is a quasiconvex function of the control input sequence. Implement a bisection algorithm to solve the problem for the problem data in the Homework folder on eLearning.

3. **State feedback control design via SDP.** Many feedback control design problems can be formulated using semidefinite programming (we will discuss this more in upcoming lectures). Use CVX (or your chosen modeling language) to solve the following SDP

$$\begin{array}{ll} \text{maximize} & \textbf{trace}(P) \\ \text{subject to} & \begin{bmatrix} R + B^T PB & B^T PA \\ A^T PB & Q + A^T PA - P \end{bmatrix} \succeq 0 \\ & P \succeq 0 \end{array} \tag{1}$$

with variable $P \in \mathbf{S}^n$ and problem data $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m}, Q \in \mathbf{S}^n_+, R \in \mathbf{S}^m_{++}$ (you can randomly generate the problem data). The solution of this problem corresponds to the optimal cost matrix of the infinite-horizon linear quadratic optimal control problem

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{t=0}^{\infty} x_t^T Q x_t + u_t^T R u_t \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t \end{array} \tag{2}$$

whose solution can be computed in Matlab via the function `dare(A,B,Q,R)`, and the optimal state feedback controller is given by $u_t = Kx_t$ with $K = -(R + B^T P^\star B)^{-1} B^T P^\star A$. Confirm that the solution of the SDP matches with the output of the `dare` function for several randomly generated problem instances.