Due via email on Monday, February 1, 2021. Please use the subject line "MECH 6327: Homework 1 Submission", submit all solutions in a single pdf file, and name your file "Yourlastname_HW1.pdf".

Reminder: Please read through the Syllabus and BV Chapter 1 & Appendix A.

1 Optimization Problem Formulation

Find and research an example (ideally from your own research experience/interests) of an optimization problem in the real world. Write a short narrative of around 300 words to describe the optimization variables, objective function(s), constraint function(s), and formulate the problem as a mathematical optimization problem. Briefly discuss your knowledge of how computationally easy or difficult the problem is, the number of variables and constraints in typical problem instances, available algorithms or software for solving the problem, etc. (need not be formal, this is just to assess and get you to think about the current state of your knowledge about the problem).

2 Math Review

1. Prove the Cauchy-Schwarz inequality, which states that $\forall x, y \in \mathbf{R}^n$, there holds
$$|\langle x, y\rangle| \leq \|x\|\|y\|,$$
where $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ is the norm induced by the inner product. *Hint: There are several ways to show this, here is one possible approach. First, assume that $\|x\| = \|y\| = 1$ and consider $\|x-y\|^2$. Then use that observation to handle general $x, y \in \mathbf{R}^n$.*

2. Show the following dual norm relationships:
   - $\|x\|_{1*} = \|x\|_\infty$
   - $\|x\|_{\infty *} = \|x\|_1$
   - $\|x\|_{2*} = \|x\|_2$ *Hint: Use Cauchy-Schwarz.*

3. Sketch or plot the unit norm balls $\{x \in \mathbf{R}^2 \mid \|x\|_p \leq 1\}$ for $p = 1, 2, \infty$.

4. Show that the eigenvalues of $A \in \mathbf{S}^n$ are real.

5. Show that for $A \in \mathbf{S}^n$, $A \succeq 0$ if and only if all eigenvalues of A are nonnegative.

6. Derive the solution to the finite horizon open-loop optimal control problem
$$\text{minimize} \quad \sum_{t=0}^{N-1} (\|z_t\|_2^2 + \|u_t\|_2^2) + \|z_N\|_2^2,$$
where $z_{t+1} = Az_t + Bu_t$ with z_0 given and optimization variables $z_t \in \mathbf{R}^n$ for $t = 1, ..., N$ and $u_t \in \mathbf{R}^m$ for $t = 0, ..., N-1$. *Hint: Define concatenated vectors $\mathbf{z} = [z_1^T, ..., z_N^T]^T$ and $\mathbf{u} = [u_0^T, ..., u_{N-1}^T]^T$, write $\mathbf{z}$ in terms of $\mathbf{u}$ and z_0, and eliminate $\mathbf{z}$ in the objective function. Then take derivatives, set them to zero, and solve for $\mathbf{u}$.*