MECH 6327 - Homework 2

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1 Problem Set 1: Convex Sets

1.1 Problem 2.5

Problem:
What is the distance between two parallel hyperplanes: $\{x \in \Re^n | a^T x = b_1\}$ and $\{x \in \Re^n | a^T x = b_2\}$?

Solution:
Under the assumption that $a \in \Re^n$ and $b_1, b_2 \in \Re$, the quantity $a^T x_0$ represents the component of x_0 in the normal direction. Similarly, the quantities b_1 and b_2 represent the euclidean distance of the hyperplane from the origin (in the normal direction). Since the hyperplanes are parrellel, the distance between them is the difference between their offsets:

$$\text{Distance between hyperplanes: } b_1 - b_2 \tag{1}$$

1.2 Problem 2.7

Problem:
Voronoi description of halfspace. Let a and b be distinct points in $\Re^n$. Show that the set of all points that are closer to a than b via the euclidean norm is a halfspace. Describe it explicitly as an inequality and draw a picture.
Solution:
The set of all points closer to a then b can be defined as:

$$\{x \in \Re^n \mid \|x - a\|_2 \leq \|x - b\|_2\} \tag{2}$$

The boundary defining this halfspace will be a plane defined by the normal vector c representing the distance between a and b, and the offset coefficient d describing intersection of the plane through the half-way point between a and b. The quantities c and d can therefore be defined by:

$$\begin{aligned} c &= b - a \\ d &= \frac{c^T a + c^T b}{2} \\ &= \frac{1}{2} c^T (a + b) \end{aligned} \tag{3}$$

The halfspace, that is equivenlent to x, can be described by the following:

$$\{x \in \Re^n \mid c^T x \leq d\} \tag{4}$$

This can be visualized in two dimensions for $a = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $b = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$. The boundary (the red line) is calculated in the standard form using

$$x_2 = \frac{-1}{c_2}(c_1 * x_1 - d)$$

and then plotted. The half-space itself is the region below the boundary.

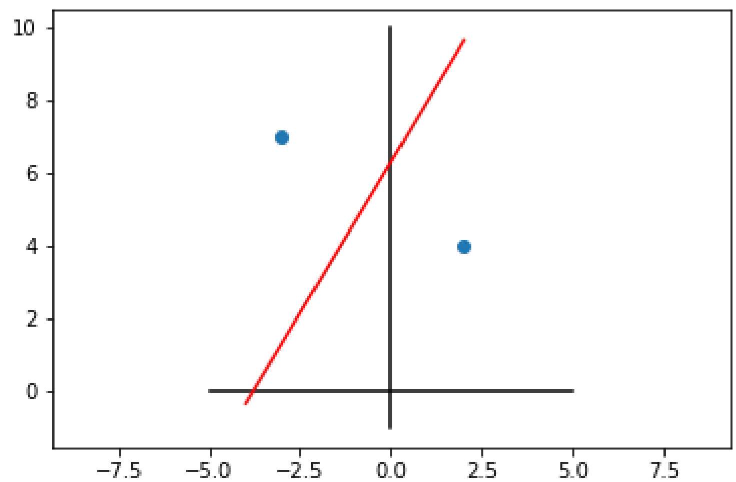

Figure 1: Visualization of the boundary for the halfspace.

1.3 Problem 2.12

Problem:
Which of the following sets are convex?

Solution:

1.3.1 (a) - Slab

A slab defined as

$$\{x \in \Re^n \mid \alpha \leq a^T x \leq \beta\}$$

is convex as it consists of the intersection of two halfspaces which themselves are complex.

1.3.2 (b) - Rectangle/Hyperrectangle

A rectangle set defined as

$$\{x \in \Re^n \mid \alpha_i \leq x_i \leq \beta_i, \ i = 1, \ldots, n\}$$

is convex as it is composed of the intersections of half spaces which are themselves convex. This is similar to the polyhedrons/polytopes that by definition are also convex.

1.3.3 (c) - Wedge

A wedge set given as

$$\{x \in \Re^n \mid a_1^T x \leq b_1, a_2^T x \leq b_2\}$$

is convex as it is just an intersection of two halfspaces (a polyhedron).

1.3.4 (d) - Closer to a point then a set

A set of points closer to a given point than a given set is defined as

$$\{x \| \|x - x_0\|_2 \leq \|x - y\|_2 \; \forall y \in S\}$$

where $S \subset \Re^n$ **is not convex** in general. This is because there is not enough information about y for a conclusion to be made whether it is convex or not. A counter example would be if y is a point in orbit around a convex shape S that would end up generating a concave x.

1.3.5 (e) - Closer to a set then another set

A set of points closer to a given set than another given set is defined as

$$\{x \mid \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\}$$

where $S, T \subset \Re^n$, and

$$\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}$$

is not convex in general. This is because there is not enough information about S and T for a conclusion to be made whether it is convex or not. A counter example includes if S or T themselves are a concave shave that causes the set x to also be concave and therefore not convex.

1.3.6 (f) - Set of the sum being within a convex set

The set defined as

$$\{x \mid x + S_2 \subseteq S_1\}$$

with S_1 being convex **is**

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1.3.7 (g) - Set with weighted distances to two points

The set of all points that is closer to a then b by at least a factor of θ, defined as

$$\{x \in \Re^n \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}$$

with $a \neq b$ and $0 \leq \theta \leq 1$ **is convex.** This is know because, as proven in a previous problem, a hyperplane is formed for a similarity stated problem which itself is convex. When the distance to a must be less then a portion of the distance to b it will cause the psudo-hyperplane to curve inwards and untimely remain convex.

1.4 Problem 2.28

Problem:
Define the positive semi-definite cone (S_+^n) for $n = 1, 2, 3$ in terms of ordinary inequalities with the matrix coefficients themselves.
Solution:
The positive semi-definite cone is defined for size n as the set of all symmetric matrices that are positive semi-definite:

$$S_+^n \equiv \{x \in S^n \mid x \succeq 0\} \tag{5}$$

One method to ensure that a matrix is positive semi-definite is to ensure that its leading principle minors are all non-negative (strictly positive for positive definite).

For $n = 1$ the required inequalities are simple,

$$X = \begin{bmatrix} x_1 \end{bmatrix} \in S_+^1 \iff x_1 \geq 0 \tag{6}$$

For $n = 2$ the inequalities can be found by ensuring the leading principle minors are all non negative:

$$\begin{aligned} m_1 &= \det[x_1] \\ &= x_1 \\ m_2 &= \det \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \\ &= x_1 x_3 - x_2^2 \end{aligned} \tag{7}$$

These definitions of the minors can be then be used to construct inequalities such that all the minors are positive:

$$X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \in S_+^2 \iff \begin{matrix} x_1 \geq 0 \\ x_1 x_3 \geq x_2^2 \end{matrix} \tag{8}$$

For $n = 3$ the inequalities can be found by ensuring the leading principle minors are all non negative:

$$\begin{aligned} m_1 &= \det[x_1] \\ &= x_1 \\ m_2 &= \det \begin{bmatrix} x_1 & x_2 \\ x_2 & x_4 \end{bmatrix} \\ &= x_1 x_4 - x_2^2 \\ m_3 &= \det \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix} \\ &= x_1(x_1 x_4 - x_2^2) - x_2(x_2 x_6 - x_3 x_5) + x_3(x_2 x_5 - x_3 x_4) \\ &= x_1^2 x_4 - x_1 x_2^2 - x_2^2 x_6 + x_2 x_3 x_5 + x_2 x_3 x_5 - x_3^2 x_4 \end{aligned} \tag{9}$$

These definitions of the minors can be then be used to construct inequalities such that all the minors are positive:

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix} \in S_+^2 \iff \begin{matrix} x_1 \geq 0 \\ x_1 x_4 \geq x_2^2 \\ x_1 x_2^2 + x_2^2 x_6 + x_3^2 x_4 \geq x_1^2 x_4 + 2 x_2 x_3 x_5 \end{matrix} \tag{10}$$

1.5 Problem 2.33

The monotone non-negative cone is defined as all the nonnegative vectors with components sorted in non-increasing order:

$$K_{m+} = \{x \in \Re^n \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq n\} \tag{11}$$

1.5.1 Part a

Problem:
Show that K_{m+} is a proper cone.
Solution:
A set, $C \subseteq \Re^n$, is considered a cone if

$$\theta x \in C \ \forall x \in C \text{ and } \theta \geq 0 \tag{12}$$

It can be easily seen that the set K_{m+} satisfies this condition as scaling each element of the matrix $x \in K_{m+}$ will equally be scaled by the same amount and the conditions of non-increasing order will still apply. This guarantees that K_{m+} is in fact a cone.

To ensure convexity, the definition of convexity and of a cone can be incorporated into the following test: The set $C \subseteq \Re^n$ is a convex cone iff

$$\theta_1 x_1 + \theta_2 x_2 \in C \ \forall x_1, x_2 \in C, \theta_1, \theta_2 \geq 0 \tag{13}$$

It is also clear that K_{m+} will satisfy as if each element element of one of the matrices is scaled it maintains the nonincreasing order. The same is true for summing two K_{m+} matrices as the nonincreasing order will be maintained.

It is also known by definition ************************************** ask about this....

1.5.2 Part b

Problem:
Find the dual cone, K^*_{m+}.
Solution:
A dual cone for K is defined as:

$$K^* = \{y \in \Re^n \mid x^T y \geq x^T y \geq 0 \ \forall x \in K\} \tag{14}$$

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It is known that the following is equivalent:

$$\begin{aligned} \sum_{i=1}^{n} x_i y_i &= (x_1 - x_2)y_1 + (x_2 - x_3)(y_1 + y_2) + \cdots \\ &= +(x_{n-1} - x_n)(y_1 + \cdots + y_{n-1}) + x_n(y_1 + \cdots + y_n) \end{aligned} \tag{15}$$

2 Problem Set 2: Convex Functions

2.1 Problem 3.6

Problem:
What is the epigraph for the following convex functions?
Solution:
The epigraph is defined as

$$\mathbf{epi}f = \{(x, t) \mid x \in \mathbf{dom}f, f(x) \leq t\} \tag{16}$$

where $\mathbf{epi}f \subset R^{n+1}$. This is equivalent to saying the space 'above' the function.

2.1.1 (a) - Epigraph of a halfspace

The epigraph of a halfspace in n dimensions is a halfspace in $n + 1$ dimensions.

2.1.2 (b) - Epigraph of a convex cone

The epigraph of a convex cone is another convex cone? mabye a riangular prism type thing...

2.1.3 (c) - Epigraph of a polyhedron

The epigraph of a polyhedron is another polyhedron. similar to convex cone... it isn't originally bounded

2.2 Problem 3.16

Problem:
Determine if the following functions are convex, concave, quasiconvex, or quasiconcave.
Solution:

2.2.1 (a) - $e^x - 1$

Let

$$f(x) = e^x - 1$$

on $\Re$.
$f(x)$ **is convex** and can be proven in multiple ways. For instance, visualizing the epigraph of the function, it is clear that a convex set is produced. Additionally, $f(x)$ can be constructed by putting the convex function e^x through the convex affine function $x - 1$.

2.2.2 (b) - $x_1 x_2$

Let

$$f(x_1, x_2) = x_1 x_2$$

over the domain $\Re^2_{++}$.
This function **is convex** and can be demonstrated using the definition:

$$f(\theta x + (1 - \theta)y) = \theta f(x) + (1 - \theta)f(y) \tag{17}$$

$$
\begin{aligned}
(\theta x_1 + (1-\theta)y_1)(\theta x_2 + (1-\theta)y_2) &= \theta x_1 x_2 + (1-\theta)y_1 y_2 \\
\theta^2 x_1 x_2 + \theta(1-\theta)y_1 x_2 + \theta(1-\theta)x_1 y_2 + (1-\theta)^2 y_1 y_2 &= \theta x_1 x_2 + (1-\theta)y_1 y_2 \\
\theta^2 x_1 x_2 + (\theta - \theta^2)(x_1 y_2 + x_2 y_1) + (1 - 2\theta + \theta^2) y_1 y_2 &= \theta x_1 x_2 + (1-\theta)y_1 y_2 \qquad (18)
\end{aligned}
$$

If we analyze each set of terms it not explicitly clear by the result, but within the domain $\Re^2_{++}$ the equality holds true.

——————— add plot...

2.2.3 (c) - $1/(x_1 x_2)$

Let

$$f(x_1, x_2) = 1/(x_1 x_2)$$

be defined over the domain R^2_{++}.

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2.2.4 (d) - x_1/x_2

Let

$$f(x_1, x_2) = x_1/x_2$$

be defined over the domain R^2_{++}.

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2.3 (e) - x_1^2/x_2

Let

$$f(x_1, x_2) = x_1^2/x_2$$

be defined over the domain $R \times R_+$.

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2.4 (f) - $x_1^\alpha x_2^{1-\alpha}$

Let

$$f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$$

with $0 \leq \alpha 1$ be defined over the domain $\Re^2_{++}$.

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2.5 Problem 3.18a

Problem:
Using the proof of concavity of the log-determinant function to show that

$$f(X) = \mathrm{tr}(X^{-1})$$

is convex over the domain S^n_{++}.

Solution:

2.6 Problem 3.22

Problem:
Use various composition rules to show that the following functions are convex.
Solution:

2.6.1 (a) - double log functions

Let

$$f(x) = -\log\left(-\log\left(\sum_{i=1}^{m} e^{a_i^T x + b_i}\right)\right)$$

be defined over the domain $\{x \mid \sum_{i=1}^{m} e^{a_i^T x + b_i} < 1\}$.
It is known that

$$\log\left(\sum_{i=1}^{n} e^{y_i}\right)$$

is convex. Since all compositions of convex functions with affine functions are also convex, it can be said that

$$\sum_{i=1}^{m} e^{a_i^T x + b_i}$$

is also convex.
Additionally, it is known that the log() function is concave, but when the sign changes it becomes convex, thus the composition of $-g(-g(h(x)))$ is convex for the concave function $g(x) = \log(x)$.
Therefore, the function $f(x)$ is convex.

2.6.2 (b) - square root of some product sum

Let

$$f(x, u, v) = -\sqrt{uv - x^T x}$$

be defined over the domain $\{(x, u, v) \mid uv > x^T x, u, v > 0\}$.
It is known that $g(x) = x^x/u$ is convex for $u > 0$ and that $h(x) - \sqrt{x_1 x_2}$ is convex on $\Re^2_{++}$.
The function $f(x, u, v)$ can be manipulated as follows:

$$\begin{aligned} f(x, u, v) &= -\sqrt{uv - x^T x} \\ &= -\sqrt{u\left(v - \frac{x^T x}{u}\right)} \end{aligned}$$

From what we know about the underling functions, it can be said that the convex function $g(x)$ is summed with v (a convex combination) and then used as an input to the convex function $h(x)$, resulting in an overall convex function.

2.6.3 (c) - log of some product sum

Let

$$f(x, u, v) = -\log\big(uv - x^T x\big)$$

be defined over the domain $\{(x, u, v) \mid uv > x^T x, u, v > 0\}$.
It is known that $g(x) = x^x/u$ is convex for $u > 0$ as well as that the $h(x) = \log(x)$ function is concave.
Performing the same manipulation as in the previous problem, $f(x, u, v)$ can be written as:

$$f(x, u, v) = -\log\left(u\left(v - \frac{x^T x}{u}\right)\right)$$

From this it can be derived that the convex function $g(x)$ is summed with v (a convex combination) and then used as an input to the concave function $h(x)$ but is then negated to result in an overall convex function.

2.6.4 (d) - complicated root of a powered sum and norm

Let

$$f(x, t) = -\Big(t^p - \|x\|_p^p\Big)^{1/p}$$

be defined with $p > 1$ over the domain $\{(x, t) \mid t \geq \|x\|_p\}$.
It is known that $g(x, u) = \|x\|_p^p/u^{p-1}$ is convex for $u > 0$ and that $h(x, y) = -x^{1/p}y^{1-1/p}$ is convex over $\Re^2_{++}$.

???

2.6.5 (e) - complicated log of a powered sum and norm

Let

$$f(x, t) = -\log\Big(t^p - \|x\|_p^p\Big)$$

with $p > 1$ be defined over the domain $\{(x, t) \mid t \geq \|x\|_p\}$.
It is known that $g(x, u) = \|x\|_p^p/u^{p-1}$ is convex for $u > 0$.