Lecture 14: Lyapunov Theory Beyond Stability (cont.)

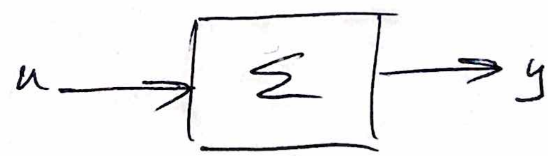
goals:

- more examples of Lyapunov ideas beyond stability
  - H_2 norm computation + control design
  - Approximate Dynamic Programming
  - Robust safety guarantees via Barrier Certificates

System H_2 norm computation + design via SDP

- Consider the system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$



- The system H_2 norm is defined by

$$\|\Sigma\|_2^2 = \int_0^\infty y(t)^T y(t)\, dt$$ when $x(0) = 0$ and $u(t)$ is a <u>unit</u> impulse signal

- <u>Fact</u>: $\|\Sigma\|_2^2 = \operatorname{trace}(B^T X_o B) = \operatorname{trace}(C X_c C^T)$

where $A^T X_o + X_o A + C^T C = 0$, $A X_c + X_c A^T + BB^T = 0$

- Lyapunov equations for controllability + observability <u>Gramians</u>

- LMI Characterization of $\mathcal{H}_2$ norm

<u>FACT</u>: A is asymp. stable and $\|\Sigma\|_2^2 < \gamma \iff$

$$\boxed{\exists\, X = X^T \succ 0 \text{ such that } AX + XA^T + BB^T \prec 0, \quad \mathrm{trace}(CXC^T) < \gamma}$$

$\iff$ $\exists\, X = X^T \succ 0,\ W = W^T \succ 0$ s.t.

$$AX + XA^T + BB^T \prec 0, \quad \begin{bmatrix} W & CX \\ XC^T & X \end{bmatrix} \succ 0 \quad \text{and} \quad \mathrm{trace}(W) < \gamma$$

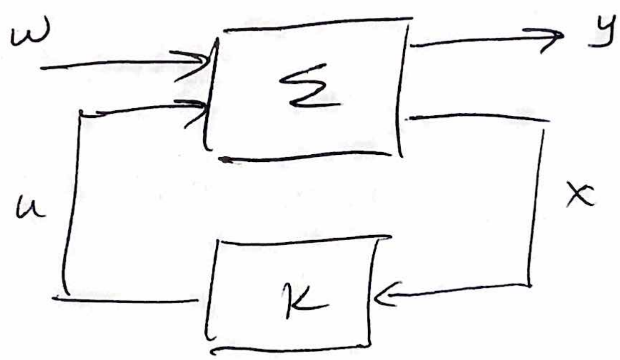
$$\int_0^\infty \Big(x^T \underbrace{C^TC}_{Q} x + 2x^T C^T D u + u^T \underbrace{D^TD}_{R} u \Big)\, dt$$

- $\mathcal{H}_2$ control design

- Consider the system

$$\dot{x} = Ax + Bu + Fw$$

$$y = Cx + Du$$



- Goal: Design full state feedback controller $u(t) = Kx(t)$ that minimizes the $\mathcal{H}_2$ system norm from disturbance input w to performance output y (in closed-loop)

- Let's use the $\mathcal{H}_2$ LMI above for closed-loop system

$$\dot{x} = (A + BK)\,x + Fw$$

$$y = (C + DK)\,x$$

$$(A+BK)X + X(A+BK)^T + FF^T \prec 0$$

$$\begin{bmatrix} W & (C+DK)X \\ X(C+DK)^T & X \end{bmatrix} \succ 0 \qquad \text{trace}(W) < \gamma$$

- Let $L = KX$. Can find optimal H_2 controller (that gives smallest closed-loop γ) by solving the SDP in variables γ, X, L, W

$$\boxed{\begin{array}{ll} \underset{\gamma, X, L, W}{\text{minimize}} & \gamma \\ \text{subject to} & \text{trace}(W) < \gamma \\ & AX + BL + XA^T + L^TB^T + FF^T \prec 0 \\ & \begin{bmatrix} W & CX + DL \\ XC^T + L^TD^T & X \end{bmatrix} \succ 0 \end{array}}$$

- Recover H_2 optimal controller via

$$K = LX^{-1}$$

- X, W give <u>certificate</u> that closed-loop H_2 norm is less than γ

Approximate Dynamic Programming

- Consider the system (stochastic + nonlinear)

$$x_{t+1} = f(x_t, u_t, w_t)$$

where disturbance $w_t \sim \mathbb{P}_w$ i.i.d.

state $x_t \in \mathbb{R}^n$, with $x_0 \sim \mathbb{P}_0$

input $u_t \in \mathbb{R}^m$, with constraint $u_t \in \mathcal{U} \subseteq \mathbb{R}^m$

- We want to find a state feedback policy $u_t = \pi(x_t)$ to minimize a cost function

$\gamma \in (0,1)$ - discount factor

$$J_\pi(x_0) = \mathbb{E}_{w_t} \sum_{t=0}^{\infty} \gamma^t g(x_t, \pi(x_t), w_t)$$

- Denote optimal value function $J^*(x_0) = \min_\pi J_\pi(x_0)$

and optimal value $\bar{J}^* = \mathbb{E}_{x_0} J^*(x_0)$.

  - an infinite-dimensional nonlinear, stochastic optimal control problem (π is a function, not a vector)

  - extremely difficult in general, but can be often effectively approximated using convex optimization!

- In principle solution via dynamic programming: optimal value function satisfies Bellman equation

$$V(x) = \min_{u \in \mathcal{U}} \mathbb{E}_{w}\left[g(x,u,w) + \gamma V(f(x,u,w)) \right]$$

then optimal policy is

$$u_t^* = \pi^*(x_t) = \underset{u_t \in \mathcal{U}}{\operatorname{argmin}} \mathbb{E}_{w_t}\left[g(x_t,u_t,w_t) + \gamma V(f(x_t,u_t,w_t)) \right]$$

- but difficult/impossible to compute in general

- Let's relax the Bellman equation to an inequality

$$V(x) \leq \min_{u \in \mathcal{U}} \mathbb{E}_{w}\left[g(x,u,w) + \gamma V(f(x,u,w)) \right] \quad \forall x \in \mathbb{R}^n$$

$$\iff V(x) \leq \mathbb{E}_{w}\left[g(x,u,w) + \gamma V(f(x,u,w)) \right] \quad \forall x \in \mathbb{R}^n, \forall u \in \mathcal{U}$$

- <u>FACT</u>: Any function $\hat{V}(x)$ satisfying this gives a <u>lower bound</u> on optimal value function

$$\longrightarrow \hat{V}(x) \leq J^*(x) \quad \forall x \in \mathbb{R}^n$$

- useful performance bound (for any policy)
- can often generate very good suboptimal policies

$$\hat{u}_t = \hat{\pi}(x_t) = \underset{u_t \in \mathcal{U}}{\operatorname{argmin}} \mathbb{E}_{w_t}\left[g(x_t,u_t,w_t) + \gamma \hat{V}(f(x_t,u_t,w_t)) \right]$$

- To find <u>best</u> lower bound, we could solve

$$\begin{array}{ll} \underset{V}{\text{maximize}} & \underset{x_0}{E}\, V(x) \\ \text{subject to} & \underbrace{\underset{w}{E}\left[g(x,u,w) + \gamma V(f(x,u,w)) \right] - V(x)}_{p(x,u)} \geq 0 \quad \forall x \in \mathbb{R}^n \\ & \forall u \in \mathcal{U} \end{array}$$

- Suppose f and g are polynomials, $\mathcal{U}$ is a basic semialgebraic set, all moments of $\mathbb{P}_w$ are assumed known/given, and we look for polynomial V (with coefficient vector α):

use SOS S-Procedure to restrict to $\mathcal{U}$

$$\begin{array}{ll} \underset{\alpha}{\text{maximize}} & c^T \alpha \\ \text{subject to} & p(x,u) \in SOS \quad \forall x \in \mathbb{R}^n, u \in \mathcal{U} \end{array}$$

- an SDP in variable α!

<u>Ex</u> Suppose $f = Ax + Bu + w$, $g = x^T Q x + u^T R u$
$Ew = 0$, $Eww^T = W$, $\mathcal{U} = \mathbb{R}^m$, and $V = x^T P x + r$
$Ex_0 = 0$, $E x_0 x_0^T = X_0$

$$\Longrightarrow \quad \begin{array}{ll} \text{maximize} & tr(P X_0) \\ \text{subject to} & \begin{bmatrix} R + \gamma B^T P B & \gamma B^T P A \\ \gamma A^T P B & Q + \gamma A^T P A - P \end{bmatrix} \succeq 0 \end{array}$$

$r = \frac{\gamma}{1-\gamma} tr(W)$

Barrier Functions for Safety

- Consider the system

$$\dot{x} = f(x, d)$$

where $x(t) \in \mathbb{R}^n$ is the state
$d(t) \in D \subset \mathbb{R}^m$ is an <u>unknown</u> disturbance input

D ↑ defines bounds assumed on d

- Let $X_s \subset \mathbb{R}^n$ denote a set of <u>safe</u> states

Let $X_u \subset \mathbb{R}^n$ denote a set of <u>unsafe</u> states

  - encoding e.g. obstacles, collisions, or any constraints on the system state

Ex Drone w/ sense + avoid capabilities flying through cluttered environment w/ unknown but bounded wind

- How could we guarantee that if the system starts in X_s, its trajectories would <u>never</u> end up in X_u, regardless of the disturbance d?
  - <u>NOT</u> easy for nonlinear systems, unknown d

- Let's use a Lyapunov approach to obtain a "Barrier Certificate" that guarantees safety

Theorem Suppose there exists a differentiable function $B : \mathbb{R}^n \to \mathbb{R}$ s.t.

$$B(x) > 0 \quad \forall x \in \mathcal{X}_u$$

$$B(x) \leq 0 \quad \forall x \in \mathcal{X}_s$$

$$\dot{B}(x) = \nabla B(x)^T f(x,d) \leq 0 \quad \forall x \in \mathcal{X}_s, \ \forall d \in D.$$

Then if $x(0) \in \mathcal{X}_s$ then $x(t) \in \mathcal{X}_s$, $x(t) \notin \mathcal{X}_u$ $\forall t \geq 0$.
(i.e., robust safety is guaranteed).

- Once again, if f is polynomial, D, $\mathcal{X}_s$, and $\mathcal{X}_u$ are basic semialgebraic sets, and B is a polynomial barrier certificate candidate, we can automate search for coefficients of B via convex optimization (w/ SOS/SDP)

- The zero level set of $B(x)$ provides a "barrier" b/t possible system trajectories and the unsafe set

- Many extensions + variations in recent literature:
  - Safety verification for stochastic hybrid systems (Prajna, Jadbabaie, Pappas TAC 2007)
  - Control Barrier Functions + QP (Ames, Xu, Grizzle, Tabuada TAC 2017)
  - Real-time planning w/ Barrier Functions (Ahmadi + Majumdar 2017)