MECH 6327 - Homework 3

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(Probably not necessary... but its long)

Contents

BV Textobook Problems

0.1 Problem 4.11

Formulate each problem as a LP and explained the relationship between the optimal solution of the problems and the solution of its LP.

0.1.1 Part a: Minimize $\|Ax - b\|_\infty$

0.1.2 Part b: Minimize $\|Ax - b\|_1$

0.1.3 Part c: Minimize $\|Ax - b\|_1$ subject to $\|\infty\| \leq 1$

0.1.4 Part d: Minimize $\|x\|_1$ subject to $\|Ax - b\|_\infty \leq 1$

0.1.5 Part e: Minimize $\|Ax - b\|_1 + \|x\|_\infty$

0.2 Problem 4.16

Consider the system given as

$$x(t + 1) = Ax(t) + bu(t), \ t = 0, \ldots, N - 1 \tag{1}$$

with $x(t) \in \Re^n, u(t) \in \Re, \forall t = 0, \ldots, N - 1$ and $A \in \Re^{n \times n}, b \in \Re^n$, and $x(0) = 0$.

The minimum fuel optimal control problem is to select the minimum amount of inputs to minimize the amount of fuel used, given as

$$
\begin{aligned}
\text{minimize} \quad & F = \sum_{t=1}^{N-1} f(u(t)) \\
\text{subject to} \quad & x(N) = x_{des}
\end{aligned}
\tag{2}
$$

with N as the time-horizon, $x_{des} \in \Re^n$ as the desired final state, and $f : \Re \to \Re$ given as

$$
f(a) = \begin{cases} |a| & |a| \leq 1 \\ 2|a| - 1 & |a| > 1 \end{cases}
\tag{3}
$$

Problem: Formulate this problem as a Linear Program.
Solution:

0.3 Problem 4.28

Consider the convex quadratic program given as

$$\text{minimize} \quad \frac{1}{2}x^T P x + q^T x + r$$
$$\text{subject to} \quad Ax \preceq b$$

(4)

with a robust equivalent defined as

$$\text{minimize} \quad \sup_{P \in \mathcal{E}} \frac{1}{2}x^T P x + q^T x + r$$
$$\text{subject to} \quad Ax \preceq b$$

(5)

where $\mathcal{E}$ is the set of all possible matrices of P.

0.3.1 Part a

Problem: Express the robust QP as a convex problem given $\mathcal{E} = P_1, \ldots, P_k$ where $P_i \in S_+^n, i = 1, \ldots, K$.

   Solution:

0.4 Problem 4.43

Suppose $A : \Re^n \rightarrow S^m$ is affine such that

$$A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n \tag{6}$$

where $A_i \in S^m$. Let $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_m(x)$ be the eigenvalues of $A(x)$.

For each of the following minimization criteria, formulate the problem as an SDP.

0.4.1 Part a

Problem: Minimize the maximum eigenvalue of A:

$$\text{minimize} \quad \lambda_1(x)$$

Solution:

0.4.2 Part b

Problem: Minimize the spread of the eigenvalues of A:

$$\text{minimize} \quad \lambda_1(x) - \lambda_m(x)$$

Solution:

0.4.3 Part c

Problem: Minimize the conditional number of A while remaining postive definite:

$$\text{minimize} \quad k(A(x)) = \frac{\lambda_1(x)}{\lambda_m(x)} \quad \forall \ x \in \{x \mid A(x) \succ 0\}$$
$$\text{subject to} \quad A(x) \succ 0$$

Solution:

1 Problem 1: Open-loop optimal control with $1-$ and $\infty-$ norms.

The following open-loop optimal regulation problem is given as:

$$\begin{aligned}
\text{minimze} \quad & \|x_T\|_p + \sum_{t=0}^{T-1} \|x_t\|_p + \gamma \|u_t\|_q \\
\text{subject to} \quad & x_{t+1} = Ax_t + Bu_t, \ t = 0, \ldots, T-1 \\
& \|x_t\|_\infty \leq \bar{x}, \ t = 0, \ldots, T \\
& \|u_t\|_\infty \leq \bar{u}, \ t = 0, \ldots, T
\end{aligned} \quad (7)$$

with $x_t \in \Re^n$ and $u_t \in \Re^m$ as the system state and control input respectively and parameter $\gamma > 0$ governing the actuator and state regulation performance.

Problem: Express this problem as a linear program for (i) $p = q = \infty$ and (ii) $p = q = 1$. Code both in CVX and for the problem data provided. Verify the equivalence between the original optimization problem and transformed linear program obtained and plot the optimal state and input trajectories for each.

Solution:

1.1 Linear program for $p = q = \infty$

1.2 Linear program for $p = q = 1$

1.3 CVX Formulation and Results:

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2 Problem 2: Minimum time state transfer via quasiconvex optimization.

Consider the LTI system:

$$x_{t+1} = Ax_t + Bu_t, \ t = 0, \ldots, T \underline{u} \leq u_t \leq \bar{u}, t = 0, \ldots, T \tag{8}$$

with x_0 as the initial state.

Problem: Show that the minimum time required to transfer the system from x_0 to $x_{desired}$, given as

$$f(u_0, \ldots, u_T) = \min \tau \mid x_t = x_{desired} \text{for} \tau \leq t \leq T + 1 \tag{9}$$

is a quasiconvex function. Impliment a bisection algorithm to solve the problem for the given data.

Solution:

3 Problem 3: State feedback control design via SDP

Feedback control problems can be formulated using a semidefinite program, such as

$$\text{minimze} \qquad \text{tr } P \text{subject to} \qquad \begin{bmatrix} R + B^T P B & B^T P A \\ A^T P B & Q + A^T P A - P \end{bmatrix} \succeq 0 \qquad (10)$$

$$P \succeq 0$$

with variable $P \in S^n$ and problem data $A \in \Re^{n \times n}, B \in \Re^{n \times m}, Q \in S_+^n, \Re \in S_{++}^m$.

This problem is equivalent to the solution to the optimal solution to the infinite-horizon LQR problem:

$$\text{minimze} \qquad \sum_{t=0}^{\infty} x_t^T Q x_t + u_t^T R u_t \text{subject to} \qquad x_{t+1} = A x_t + B u_t, \ t \geq 0, \ x(t = 0) = x_0 \qquad (11)$$

This is also equivelent to the solution the the discrete-time richotte equation (DARE) and can be solved in matlab with dare(A,B,Q,R). The solution to the feedback controller is

$$u_t = K x_t K = -\left(R + B^T B\right)^{-1} B^T P^* A \qquad (12)$$

Problem: Confirm the solution to the SDP given in (10) is equivalent to the LQR problem given in (11) for multiple randomly generated problems.

Solution: The following results are provided for various randomly generated problems and solutions. This was generated using the code in AppendixB.

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contents....
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A MATLAB Code:

All code I write in this course can be found on my GitHub repository:

https://github.com/jonaswagner2826/MECH6327

B Problem 3 MATLAB Code:

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