Lecture 13: Lyapunov Theory Beyond Stability

goals:

- discuss the use of underlying Lyapunov ideas beyond stability
  - reachable sets
  - input-output properties
  - control design

Systems with Inputs

- Consider $\dot{x} = f(x, u)$, $x(t) \in \mathbb{R}^n$, $\underset{\text{input}}{u(t)} \in \mathbb{R}^m$

- For a function $V : \mathbb{R}^n \to \mathbb{R}$ we have

$$\dot{V} = \nabla V(x)^T \dot{x} = \nabla V(x)^T f(x, u)$$

  - $\dot{V}$ depends on <u>both</u> state and input

- Could we simultaneously search for a Lyapunov function and a feedback controller $u = \pi(x)$ that stabilizes an equilibrium point?
  - related to Control Lyapunov functions (CLF)

- In general, joint search for Lyapunov function coefficients and feedback controller coefficients (gains) is non-convex; however, convex optimization techniques can be used effectively (more later)

    - alternating optimization
    - fixing a CLF and computing control online
    - in some cases, exact reformulation by change of variables

Reachable Sets

- Suppose $u(t) \in \mathcal{U} \quad \forall t$

- Define reachable set

$$R_T = \{ x(T) \mid \dot{x} = f(x,u),\ x(0) = 0,\ u(t) \in \mathcal{U},\ T > 0 \}$$

i.e. the set of states that can be hit by a trajectory at time T with some admissible input signal

- If u is a control input, R_T contains states we can reach (so big R_T is good)
- If u is a disturbance/noise/adversarial input, R_T describes its worst-case effect on the state (so big R_T is bad)

Lyapunov bound on reachable set

- Suppose $\exists\, V : \mathbb{R}^n \longrightarrow \mathbb{R}$ PD and $a > 0$ such that

$$V(x) = b \text{ and } u \in \mathcal{U} \implies \dot{V}(x,u) \le -a$$

and define $C = \{ x \mid V(x) \le b \}$.

Then $R_T \subseteq C$ (i.e. C outer approximates R_T)

- <u>Idea</u>: On the boundary of C, every trajectory goes <u>into</u> C, no matter what input is applied

  - C is a <u>robust invariant set</u>: every trajectory starting in C stays in C for any admissable input

- Can use SOS and SOS S-Procedure to automatically search for polynomial V when f is polynomial and $\mathcal{U}$ is a basic semialgebraic set: $\mathcal{U} = \{ u \in \mathbb{R}^m \mid g_i(u) \le 0 \}_{i=1,\dots,p}$

find coefficients of polynomials V, τ, λ_i

subject to

$$V(0) = 0$$

$$V(x) - \varepsilon \phi(x) \in SOS \qquad \text{some } \varepsilon > 0,\ \phi \text{ PD}$$

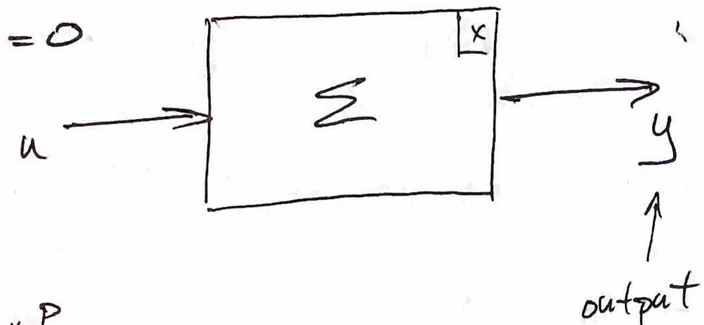
$$\nabla V(x)^T f(x,u) + a \le \sum_{i=1}^{p} \lambda_i(u) g_i(u) + \tau(x)\big(V(x) - b\big) \qquad \forall x, \forall u$$

$$\lambda_i(u) \in SOS$$

Lyapunov Method for bounding system H_∞ norm

- Consider the system

$$\dot{x} = f(x,u), \quad x(0) = 0$$
$$\underline{y} = g(x,u)$$



$x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$

- Define system H_∞ norm (aka induced L_2 gain)

$$\|\Sigma\|_\infty = \sup_{\|u\| \neq 0} \frac{\|y\|_2}{\|u\|_2} \qquad \text{where } \|\cdot\|_2^2 = \int_0^\infty \|u(t)\|_2^2 \, dt$$

  - signal norm

  - system norm

- Suppose $\gamma \geq 0$ and $\exists V : \mathbb{R}^n \to \mathbb{R}$ such that
  - $V \geq 0$, $V(0) = 0$
  - $\dot{V}(x,u) \leq \gamma^2 u^T u - y^T y = \gamma^2 u^T u - g(x,u)^T g(x,u) \quad \forall x, u$

Then $\|\Sigma\|_\infty \leq \gamma$

- V certifies bound on H_∞ system norm

Proof: Suppose we've found such a V. For any $T > 0$

$$V(x(T)) - V(x(0)) = \int_0^T \dot{V}(x(t), u(t))\, dt$$

$$\leq \int_0^T \left(\gamma^2 u(t)^T u(t) - y(t)^T y(t)\right) dt$$

$$\forall x, y$$

Since $V(x(0)) = V(0) = 0$ and $V(x(T)) \geq 0$

$$\int_0^T y(t)^T y(t)\, dt \leq \gamma^2 \int_0^T u(t)^T u(t)\, dt \quad \forall u, y$$

Taking the limit as $T \to \infty$ gives

$$\|y\|_2^2 \leq \gamma^2 \|u\|_2^2 \Rightarrow \|\Sigma\|_\infty \leq \gamma$$

$$\forall u, y$$

∎

- Once again, can automatically search for polynomial V when f and g are polynomial using SOS programming

$$\boxed{\begin{array}{l} \text{find coefficients of polynomial } V \\ \text{subject to } V(x) \in SOS \\ \gamma^2 u^T u - g(x,u)^T g(x,u) - \nabla V(x)^T f(x,u) \in SOS \end{array}}$$

Bounded Real Lemma

- Let's consider a quadratic Lyapunov function $V(x) = x^T P x$ and linear system

$$\dot{x} = Ax + Bu \qquad x(0) = 0$$
$$y = Cx$$

- The $\dot{V}$ condition becomes

$$\dot{V}(x,u) - \gamma^2 u^T u + y^T y \leq 0$$

$$(Ax + Bu)^T P x + x^T P (Ax + Bu) - \gamma^2 u^T u + x^T C^T C x \leq 0$$

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T P + PA + C^T C & PB \\ B^T P & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq 0 \quad \forall x, u$$

$$\Longleftrightarrow \begin{bmatrix} A^T P + PA + C^T C & PB \\ B^T P & -\gamma^2 I \end{bmatrix} \preceq 0 \qquad \text{LMI in } P!$$

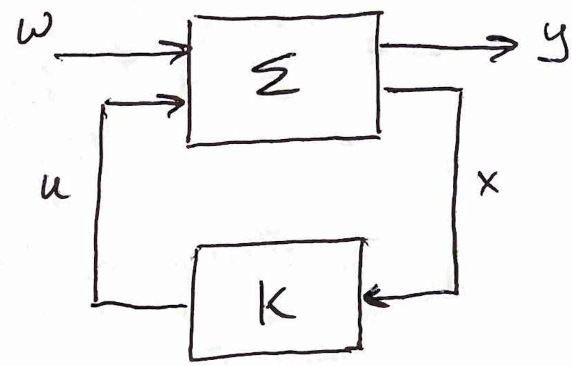
$$(\exists P \succeq 0)$$

- Turns out for linear systems this condition is both necessary + sufficient for $\|\Sigma\|_\infty \leq \gamma$
  - called Bounded Real Lemma

H_∞ Control Design via SDP

- Consider the system

$$\dot{x} = Ax + Bu + Fw$$
$$y = Cx + Du$$



- Goal: Design a full state feedback controller $u(t) = Kx(t)$ that minimizes the H_∞ system norm from disturbance input w to performance output y (in closed-loop)
    - i.e. minimize the worst-case effect of w on y
    - closely related to robust control of uncertain systems
    - also closely related to the dynamic game problem

$$\min_u \max_w \int_0^\infty \left(y(t)^T y(t) - \gamma^2 w(t)^T w(t) \right) dt$$

subject to
$$\dot{x}(t) = Ax(t) + Bu(t) + Fw(t)$$
$$y(t) = Cx(t) + Du(t)$$

- Let's use the Bounded Real LMI for the <u>closed</u>-<u>loop</u> with $u = Kx$

$$\Longrightarrow \quad \begin{aligned} \dot{x} &= (A+BK)x + Fw \\ y &= (C+DK)x \end{aligned}$$

$$\begin{bmatrix} (A+BK)^T P + P(A+BK) + (C+DK)^T(C+DK) & PF \\ F^T P & -\gamma^2 I \end{bmatrix} \preceq 0$$

- <u>not</u> an LMI in P and K, but can reformulate via Schur complement + variable substitutions

- Assume $P \succ 0$ and let $Q = P^{-1}$, then multiply LHS on left and right by $\begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}$

$$\begin{bmatrix} Q(A+BK)^T + (A+BK)Q + Q(C+DK)^T(C+DK)Q & F \\ F^T & -\gamma^2 I \end{bmatrix} \preceq 0$$

- Now take the Schur complement of upper left block and define $L = KQ$

$$\begin{bmatrix} QA^T + L^T B^T + AQ + BL & F & QC^T + L^T D^T \\ F & -\gamma^2 I & 0 \\ CQ + DL & 0 & -I \end{bmatrix} \preceq 0$$

- LMI in variables Q and L!

problem data: A, B, F, C, D, γ

- If feasible, can recover controller gain by

$$K = L Q^{-1}$$

which gives closed-loop H_∞ norm of γ

- Can find optimal H_∞ controller (one that gives smallest γ) by defining variable $\eta = \gamma^2$ and solving the SDP

$$\begin{aligned}
\underset{\eta,\, Q,\, L}{\text{minimize}} \quad & \eta \\
\text{subject to} \quad & \begin{bmatrix} QA^T + L^T B^T + AQ + BL & F & QC^T + L^T D^T \\ F & -\eta I & 0 \\ CQ + DL & 0 & -I \end{bmatrix} \preceq 0 \\
& Q \succ 0
\end{aligned}$$