MECH 6327 - Homework 3

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******Test Change***********

BV Textobook Problems

0.1 Problem 4.11

Problem: Formulate each problem as a LP and explained the relationship between the optimal solution of the problems and the solution of its LP.
Solution:

0.1.1 Part a: Minimize $\|Ax - b\|_\infty$

Define the following minimization problem:

$$\begin{array}{ll} \text{minimize} & \|Ax-b\|_\infty \\ \text{subject to} & \text{math} \end{array} \tag{1}$$

From the definition of an ∞-norm as

$$\|x\|_\infty = \max_i |x_i|$$

the following can be derived:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & (Ax-b)_i \leq t,\ \forall i = 1, \ldots, n \\ & -(Ax-b)_i \leq t,\ \forall i = 1, \ldots, n \end{array} \tag{2}$$

Which is equivalent to the following linear program

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & -\mathbf{1}t \leq Ax - b \leq \mathbf{1}t \end{array} \tag{3}$$

This minimum is related to the original minimization problem by the following transformation: ************************** fill in details

0.1.2 Part b: Minimize $\|Ax - b\|_1$

Define the following minimization problem:

$$\begin{aligned} \text{minimize} \quad & \|Ax - b\|_1 \\ \text{subject to} \quad & \text{math} \end{aligned} \tag{4}$$

From the definition of an 1-norm as

$$\|x\|_1 = \sum_i |x_i|$$

the following can be derived:

$$\begin{aligned} \text{minimize} \quad & t_1 + \cdots + t_n \\ \text{subject to} \quad & (Ax - b)_i \leq t_i, \ \forall i = 1, \ldots, n \\ & -(Ax - b)_i \leq t_i, \ \forall i = 1, \ldots, n \end{aligned} \tag{5}$$

Which is equivalent to the following linear program

$$\begin{aligned} \text{minimize} \quad & \mathbf{1}^T t \\ \text{subject to} \quad & -t \leq Ax - b \leq t \end{aligned} \tag{6}$$

This minimum is related to the original minimization problem by the following transformation: ** fill in details

0.1.3 Part c: Minimize $\|Ax - b\|_1$ subject to $\|x\|_\infty \le 1$

Define the following minimization problem:

$$\begin{aligned} \text{minimize} \quad & \|Ax - b\|_1 \\ \text{subject to} \quad & \|x\|_\infty \le 1 \end{aligned} \tag{7}$$

From the definition of an 1-norm as

$$\|x\|_1 = \sum_i |x_i|$$

and the definition of an ∞-norm as

$$\|x\|_\infty = \max_i |x_i|$$

the following can be derived:

$$\begin{aligned} \text{minimize} \quad & t_1 + \cdots + t_n \\ \text{subject to} \quad & (Ax - b)_i \le t_i, \ \forall i = 1, \dots, n \\ & -(Ax - b)_i \le t_i, \ \forall i = 1, \dots, n \\ & x_i \le 1, \forall i = 1, \dots, n \\ & -x_i \le 1, \forall i = 1, \dots, n \end{aligned} \tag{8}$$

Which is equivalent to the following linear program

$$\begin{aligned} \text{minimize} \quad & \mathbf{1}^T t \\ \text{subject to} \quad & -t \le Ax - b \le t \\ & -\mathbf{1} \le x \le \mathbf{1} \end{aligned} \tag{9}$$

This minimum is related to the original minimization problem by the following transformation:
** fill in details

0.1.4 Part d: Minimize $\|x\|_1$ subject to $\|Ax-b\|_\infty \le 1$

Define the following minimization problem:

$$\begin{aligned} \text{minimize} \quad & \|x\|_1 \\ \text{subject to} \quad & \|Ax-b\|_\infty \le 1 \end{aligned} \tag{10}$$

From the definition of an 1-norm as

$$\|x\|_1 = \sum_i |x_i|$$

and the definition of an ∞-norm as

$$\|x\|_\infty = \max_i |x_i|$$

the following can be derived:

$$\begin{aligned} \text{minimize} \quad & t_1 + \cdots + t_n \\ \text{subject to} \quad & x_i \le t_i,\ \forall i = 1, \ldots, n \\ & -x_i \le t_i,\ \forall i = 1, \ldots, n \\ & (Ax-b)_i \le 1,\ \forall i = 1, \ldots, n \end{aligned} \tag{11}$$

From this a linear program can be defined as:

$$\begin{aligned} \text{minimize} \quad & \mathbf{1}^T t \\ \text{subject to} \quad & -t \le x \le t \\ & Ax - b \le \mathbf{1} \end{aligned} \tag{12}$$

This minimum is related to the original minimization problem by the following transformation:
** fill in details

0.1.5 Part e: Minimize $\|Ax - b\|_1 + \|x\|_\infty$

Define the following minimization problem:

$$\begin{aligned} \text{minimize} \quad & \|Ax - b\|_1 + \|x\|_\infty \\ \text{subject to} \quad & math \end{aligned} \tag{13}$$

From the definition of an 1-norm as

$$\|x\|_1 = \sum_i |x_i|$$

and the definition of an ∞-norm as

$$\|x\|_\infty = \max_i |x_i|$$

the following can be derived:

$$\begin{aligned} \text{minimize} \quad & t_1 + \cdots + t_n + s \\ \text{subject to} \quad & (Ax - b)_i \leq t_i, \ \forall i = 1, \ldots, n \\ & -(Ax - b)_i \leq t_i, \ \forall i = 1, \ldots, n \\ & x_i \leq s, \ \forall i = 1, \ldots, n \\ & -x_i \leq s, \ \forall i = 1, \ldots, n \end{aligned} \tag{14}$$

This can be written as a standard linear program as:

$$\begin{aligned} \text{minimize} \quad & \mathbf{1}^T t + s \\ \text{subject to} \quad & -t \leq Ax - b \leq t \\ & -\mathbf{1}s \leq x \leq \mathbf{1}s \end{aligned} \tag{15}$$

This minimum is related to the original minimization problem by the following transformation:
** fill in details

0.2 Problem 4.16

Consider the system given as

$$x(t+1) = Ax(t) + bu(t),\ t = 0, \dots, N-1 \tag{16}$$

with $x(t) \in \Re^n, u(t) \in \Re, \forall t = 0, \dots, N-1$ and $A \in \Re^{n \times n}, b \in \Re^n$, and $x(0) = 0$.

The minimum fuel optimal control problem is to select the minimum amount of inputs to minimize the amount of fuel used, given as

$$\begin{aligned} \text{minimize} \quad & F = \sum_{t=1}^{N-1} f(u(t)) \\ \text{subject to} \quad & x(t+1) = Ax(t) + bu(t),\ t = 0, \dots, N-1 \\ & x(N) = x_{des} \end{aligned} \tag{17}$$

with N as the time-horizon, $x_{des} \in \Re^n$ as the desired final state, and $f : \Re \to \Re$ given as

$$f(a) = \begin{cases} |a| & |a| \leq 1 \\ 2|a| - 1 & |a| > 1 \end{cases} \tag{18}$$

Problem: Formulate this problem as a Linear Program.

Solution: First, 17 can be rewritten in an epigraph form (with the additional assumption that $f(u(t))$ is always positive):

$$\begin{aligned} \text{minimize} \quad & F_1 + \cdots + F_{N-1} \\ \text{subject to} \quad & f(u(t)) = F_t,\ \forall t = 1, \dots, N-1 \\ & x(t+1) = Ax(t) + bu(t),\ \forall t = 0, \dots, N-1 \\ & x(N) = x_{des} \end{aligned} \tag{19}$$

Now looking at the nonlinear component, fuel usage as defined by (18), can be equated to:

$$\begin{aligned} |a| &\leq g \\ 2|a| - 1 &\leq g \end{aligned} \tag{20}$$

or equivalently,

$$\begin{aligned} -g \leq a &\leq g \\ -g \leq 2a - 1 &\leq g \end{aligned} \tag{21}$$

This represents an intersection of two half-spaces which is a simpliler convex restriction.
This can now be combined with (19) to produce the linear program:

$$\begin{aligned} \text{minimize} \quad & F_1 + \cdots + F_{N-1} \\ \text{subject to} \quad & -F_t \leq u(t) \leq F_t,\ \forall t = 1, \dots, N-1 \\ & -F_t \leq 2u(t) - 1 \leq F_t,\ \forall t = 1, \dots, N-1 \\ & x(t+1) = Ax(t) + bu(t),\ \forall t = 0, \dots, N-1 \\ & x(N) = x_{des} \end{aligned} \tag{22}$$

Which can then be rewritten as:

$$\begin{aligned}
\text{minimize} \quad & \mathbf{1}^T F \\
\text{subject to} \quad & -F \leq \mathbf{u} \leq F \\
& x(t+1) = Ax(t) + bu(t), \ \forall t = 0, \ldots, N-1 \\
& x(N) = x_{des}
\end{aligned} \tag{23}$$

0.3 Problem 4.28

Consider the convex quadratic program given as

$$\begin{aligned} &\text{minimize} \quad \frac{1}{2}x^T P x + q^T x + r \\ &\text{subject to} \quad Ax \preceq b \end{aligned} \tag{24}$$

with a robust equivalent defined as

$$\begin{aligned} &\text{minimize} \quad \sup_{P \in \mathcal{E}} \frac{1}{2}x^T P x + q^T x + r \\ &\text{subject to} \quad Ax \preceq b \end{aligned} \tag{25}$$

where $\mathcal{E}$ is the set of all possible matrices of P.

0.3.1 Part a

Problem: Express the robust QP as a convex problem given $\mathcal{E} = P_1, \ldots, P_k$ where $P_i \in S^n_+, i = 1, \ldots, K$.

Solution:

0.4 Problem 4.43

Suppose $A : \Re^n \to S^m$ is affine such that

$$A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n \tag{26}$$

where $A_i \in S^m$. Let $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_m(x)$ be the eigenvalues of $A(x)$.

For each of the following minimization criteria, formulate the problem as an SDP.

0.4.1 Part a

Problem: Minimize the maximum eigenvalue of A:

$$\text{minimize} \;\; \lambda_1(x)$$

Solution:

0.4.2 Part b

Problem: Minimize the spread of the eigenvalues of A:

$$\text{minimize} \;\; \lambda_1(x) - \lambda_m(x)$$

Solution:

0.4.3 Part c

Problem: Minimize the conditional number of A while remaining postive definite:

$$\begin{array}{rl} \text{minimize} & k(A(x)) = \dfrac{\lambda_1(x)}{\lambda_m(x)} \; \forall \; x \in \{x \mid A(x) \succ 0\} \\ \text{subject to} & A(x) \succ 0 \end{array}$$

Solution:

1 Problem 1: Open-loop optimal control with $1-$ and $\infty-$ norms.

The following open-loop optimal regulation problem is given as:

$$\begin{aligned}
\text{minimze} \qquad & \|x_T\|_p + \sum_{t=0}^{T-1} \|x_t\|_p + \gamma \|u_t\|_q \\
\text{subject to} \qquad & x_{t+1} = Ax_t + Bu_t,\ t = 0, \dots, T-1 \\
& \|x_t\|_\infty \leq \bar{x},\ t = 0, \dots, T \\
& \|u_t\|_\infty \leq \bar{u},\ t = 0, \dots, T
\end{aligned} \tag{27}$$

with $x_t \in \Re^n$ and $u_t \in \Re^m$ as the system state and control input respectively and parameter $\gamma > 0$ governing the actuator and state regulation performance.

Problem: Express this problem as a linear program for (i) $p = q = \infty$ and (ii) $p = q = 1$. Code both in CVX and for the problem data provided. Verify the equivalence between the original optimization problem and transformed linear program obtained and plot the optimal state and input trajectories for each.

Solution:

1.1 Linear program for $p = q = \infty$

1.2 Linear program for $p = q = 1$

1.3 CVX Formulation and Results:

2 Problem 2: Minimum time state transfer via quasiconvex optimization.

Consider the LTI system:

$$x_{t+1} = Ax_t + Bu_t,\ t = 0, \dots, T\underline{u} \leq u_t \leq \bar{u}, t = 0, \dots, T \tag{28}$$

with x_0 as the initial state.

Problem: Show that the minimum time required to transfer the system from x_0 to $x_{desired}$, given as

$$f(u_0, \dots, u_T) = \min \tau \mid x_t = x_{desired} \text{for} \tau \leq t \leq T + 1 \tag{29}$$

is a quasiconvex function. Impliment a bisection algorithm to solve the problem for the given data.

Solution:

3 Problem 3: State feedback control design via SDP

Feedback control problems can be formulated using a semidefinite program, such as

$$\begin{array}{lll} \text{minimze} & \operatorname{tr} P \text{subject to} & \begin{bmatrix} R + B^T PB & B^T PA \\ A^T PB & Q + A^T PA - P \end{bmatrix} \succeq 0 \\ & P \succeq 0 & \end{array} \tag{30}$$

with variable $P \in S^n$ and problem data $A \in \Re^{n \times n}, B \in \Re^{n \times m}, Q \in S^n_+, \Re \in S^m_{++}$.

This problem is equivalent to the solution to the optimal solution to the infinite-horizon LQR problem:

$$\text{minimze} \quad \sum_{t=0}^{\infty} x_t^T Q x_t + u_t^T R u_t \text{subject to} \quad x_{t+1} = Ax_t + Bu_t,\ t \geq 0,\ x(t=0) = x_0 \tag{31}$$

This is also equivelent to the solution the the discrete-time richotte equation (DARE) and can be solved in matlab with dare(A,B,Q,R). The solution to the feedback controller is

$$u_t = Kx_t K = -\big(R + B^T B\big)^{-1} B^T P^* A \tag{32}$$

Problem: Confirm the solution to the SDP given in (30) is equivalent to the LQR problem given in (31) for multiple randomly generated problems.

Solution: The following results are provided for various randomly generated problems and solutions. This was generated using the code in AppendixB.

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contents....
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A MATLAB Code:

All code I write in this course can be found on my GitHub repository:
https://github.com/jonaswagner2826/MECH6327

B Problem 3 MATLAB Code:

All code I write in this course can be found on my GitHub repository:
https://github.com/jonaswagner2826/MECH6327