Lecture 8 : Examples + Code

goals:

- more example problems
- look at some CVX code

Ex Constrained ∞-norm minimization

minimize $\|x\|_\infty$

subject to $Ax \le b$

Note: $\|x\|_\infty = \max_i |x_i|$

$= \max_i \{x_i, -x_i\}$

$= \max_i \{x_1, \dots, x_n, -x_1, \dots, -x_n\}$

$\Updownarrow$

minimize t

subject to $x_i \le t$ $\quad i=1,\dots,n$

$-x_i \le t$

$Ax \le b$

$\Longleftrightarrow$

minimize t

subject to

$-\mathbf{1}t \le x \le \mathbf{1}t$

$Ax \le b$

- CVX code

Ex Constrained 1-norm minimization

minimize $\|Ax-b\|_1$

subject to $Fx \preceq g$

Note: $\|Ax-b\|_1 = \sum_i |(Ax-b)_i|$

$= \sum_i \max\{(Ax-b)_i, -(Ax-b)_i\}$

- epigraph transformation works for each term in a sum separately

$$\Updownarrow$$

minimize $t_1 + \cdots + t_m$

subject to $(Ax-b)_i \le t_i$

$-(Ax-b)_i \le t_i$

$Fx \le g$

$$\Longleftrightarrow$$

minimize $\mathbf{1}^T t$

subject to

$-t \le Ax-b \le t$

$Fx \le g$

- CVX code

Ex Robust LP as an SOCP

- CVX code

- Schur Complements

    - an important tool for manipulating matrix inequalities

Consider $X \in S^n$ partitioned as

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

If $\det A \neq 0$, then

$$S = C - B^T A^{-1} B$$

is called the Schur complement of X.

We have the following definiteness properties:

① If $A \succ 0$, then $X \succeq 0 \iff S \succeq 0$

② $X \succ 0 \iff A \succ 0$ and $S \succ 0$

- Stability + Stabilization via SDP

Consider the discrete-time linear dynamical system

$$x_{t+1} = A x_t$$

- global asymptotic stability (GAS):

  $\forall x_0 \in \mathbb{R}^n, \quad x_t \to 0$ as $t \to \infty$

- standard linear algebra:

$$x_{t+1} = A x_t \text{ GAS} \iff |\lambda_i(A)| < 1 \quad \forall i$$

$$\iff \rho(A) = \max_i |\lambda_i(A)| < 1$$

spectral radius

- we'll discuss a different characterization related to Lyapunov theory (much more later) and semidefinite programming
  - much more useful for going beyond basic stability questions for linear systems

Theorem $x_{t+1} = A x_t$ GAS

$$\Longleftrightarrow$$

$\exists P \in S^n$ such that $P \succ 0$ and $P \succ A^T P A$

- Given A, finding such a P is an SDP!
- Proof based on fundamental concept of a <u>Lyapunov function</u>
  - $V(x) = x^T P x$
  - Conditions imply:
    - $V(0) = 0$ and $V(x) > 0 \quad \forall x \neq 0$
    - $V(Ax) < V(x) \quad \forall x \neq 0$
  - This means V decreases monotonically along all system trajectories, leads to GAS

- <u>Note</u>: In this simple case, we don't need SDP since we can also solve the <u>Lyapunov Equation</u>

$$P = Q + A^T P A \quad \text{for } Q \succ 0$$

- Stabilization via state feedback

Consider $x_{t+1} = Ax_t + Bu_t$, $u_t = Kx_t$

$$\Longrightarrow x_{t+1} = (A+BK)x_t$$

Does there exist a stabilizing gain matrix $K \in \mathbb{R}^{m \times n}$?

$$\exists K \in \mathbb{R}^{m \times n} : \rho(A+BK) < 1 \text{ ?}$$

Lyapunov: $A+BK$ GAS $\Longleftrightarrow$

$$\exists P \succ 0 : (A+BK)^T P (A+BK) \prec P$$

or $P - (A+BK)^T P (A+BK) \succ 0$

$$\Big\Updownarrow \text{ Schur complement}$$

$$\begin{bmatrix} P^{-1} & (A+BK)^T \\ A+BK & P \end{bmatrix} \succ 0 \quad \underset{\substack{\text{Mult.} \\ \text{L and R} \\ \text{by } \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}}}{\Longleftrightarrow} \quad \begin{bmatrix} P & P(A+BK)^T \\ (A+BK)P & P \end{bmatrix} \succ 0$$

Let $L = KP$. Then we have

$$P \succ 0, \begin{bmatrix} P & PA^T + L^T B^T \\ AP + BL & P \end{bmatrix} \succ 0$$

An LMI/SDP in variables L and P!

Solve via convex optimization, recover stabilizing controller

$$K = LP^{-1}$$

- CVX code