MECH 6327 - Homework 2

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1 Problem Set 1: Convex Sets

1.1 Problem 2.5

Problem:
What is the distance between two parallel hyperplanes: $\{x \in \Re^n | a^T x = b_1\}$ and $\{x \in \Re^n | a^T x = b_2\}$?

Solution:
Under the assumption that $a \in \Re^n$ and $b_1, b_2 \in \Re$, the quantity $a^T x_0$ represents the component of x_0 in the normal direction. Similarly, the quantities b_1 and b_2 represent the euclidean distance of the hyperplane from the origin (in the normal direction). Since the hyperplanes are parrellel, the distance between them is the difference between their offsets:

$$\text{Distance between hyperplanes: } b_1 - b_2 \tag{1}$$

1.2 Problem 2.7

Problem:
Voronoi description of halfspace. Let a and b be distinct points in $\Re^n$. Show that the set of all points that are closer to a than b via the euclidean norm is a halfspace. Describe it explicitly as an inequality and draw a picture.

Solution:
The set of all points closer to a then b can be defined as:

$$\{x \in \Re^n \mid \|x - a\|_2 \leq \|x - b\|_2\} \tag{2}$$

The boundary defining this halfspace will be a plane defined by the normal vector c representing the distance between a and b, and the offset coefficient d describing intersection of the plane through the half-way point between a and b. The quantities c and d can therefore be defined by:

$$\begin{aligned} c &= b - a \\ d &= \frac{c^T a + c^T b}{2} \\ &= \frac{1}{2} c^T (a + b) \end{aligned} \tag{3}$$

The halfspace, that is equivenlent to x, can be described by the following:

$$\{x \in \Re^n \mid c^T x \leq d\} \tag{4}$$

This can be visualized in two dimensions for $a = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $b = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$. The boundary (the red line) is calculated in the standard form using

$$x_2 = \frac{-1}{c_2}(c_1 * x_1 - d)$$

and then plotted. The half-space itself is the region below the boundary.

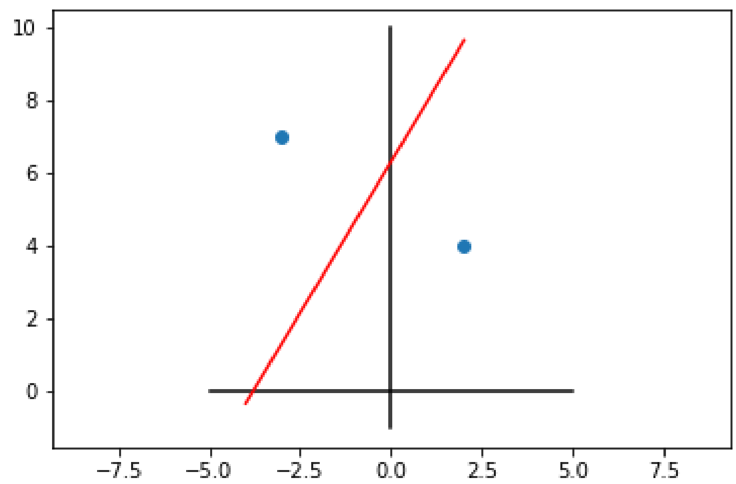

Figure 1: Visualization of the boundary for the halfspace.

1.3 Problem 2.12

Problem:
Which of the following sets are convex?

Solution:

1.3.1 (a) - Slab

A slab defined as

$$\{x \in \Re^n \mid \alpha \leq a^T x \leq \beta\}$$

is convex as it consists of the intersection of two halfspaces which themselves are complex.

1.3.2 (b) - Rectangle/Hyperrectangle

A rectangle set defined as

$$\{x \in \Re^n \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, \ldots, n\}$$

is convex as it is composed of the intersections of half spaces which are themselves convex. This is similar to the polyhedrons/polytopes that by definition are also convex.

1.3.3 (c) - Wedge

A wedge set given as

$$\{x \in \Re^n \mid a_1^T x \leq b_1, a_2^T x \leq b_2\}$$

is convex as it is just an intersection of two halfspaces (a polyhedron).

1.3.4 (d) - Closer to a point then a set

A set of points closer to a given point than a given set is defined as

$$\{x \| \|x - x_0\|_2 \leq \|x - y\|_2 \ \forall y \in S\}$$

where $S \subset \Re^n$ **is not convex** in general. This is because there is not enough information about y for a conclusion to be made whether it is convex or not. A counter example would be if y is a point in orbit around a convex shape S that would end up generating a concave x.

1.3.5 (e) - Closer to a set then another set

A set of points closer to a given set than another given set is defined as

$$\{x \mid \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\}$$

where $S, T \subset \Re^n$, and

$$\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}$$

is not convex in general. This is because there is not enough information about S and T for a conclusion to be made whether it is convex or not. A counter example includes if S or T themselves are a concave shave that causes the set x to also be concave and therefore not convex.

1.3.6 (f) - Set of the sum being within a convex set

The set defined as

$$\{x \mid x + S_2 \subset S_1\}$$

with S_1 being convex **is**

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1.3.7 (g) - Set with weighted distances to two points

The set of all points that is closer to a then b by at least a factor of θ, defined as

$$\{x \in \Re^n \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}$$

with $a \neq b$ and $0 \leq \theta \leq 1$ **is convex.** This is know because, as proven in a previous problem, a hyperplane is formed for a similarity stated problem which itself is convex. When the distance to a must be less then a portion of the distance to b it will cause the psudo-hyperplane to curve inwards and untimely remain convex.

1.4 Problem 2.28

Problem:
Define the positive semi-definite cone (S_+^n) for $n = 1, 2, 3$ in terms of ordinary inequalities with the matrix coefficients themselves.
Solution:
The positive semi-definite cone is defined for size n as the set of all symmetric matrices that are positive semi-definite:

$$S_+^n \equiv \{x \in S^n \mid x \succeq 0\} \tag{5}$$

One method to ensure that a matrix is positive semi-definite is to ensure that its leading principle minors are all non-negative (strictly positive for positive definite).

For $n = 1$ the required inequalities are simple,

$$X = \begin{bmatrix} x_1 \end{bmatrix} \in S_+^1 \iff x_1 \geq 0 \tag{6}$$

For $n = 2$ the inequalities can be found by ensuring the leading principle minors are all non negative:

$$\begin{aligned} m_1 &= \det[x_1] \\ &= x_1 \\ m_2 &= \det \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \\ &= x_1 x_3 - x_2^2 \end{aligned} \tag{7}$$

These definitions of the minors can be then be used to construct inequalities such that all the minors are positive:

$$X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \in S_+^2 \iff \begin{matrix} x_1 \geq 0 \\ x_1 x_3 \geq x_2^2 \end{matrix} \tag{8}$$

For $n = 3$ the inequalities can be found by ensuring the leading principle minors are all non negative:

$$\begin{aligned} m_1 &= \det[x_1] \\ &= x_1 \\ m_2 &= \det \begin{bmatrix} x_1 & x_2 \\ x_2 & x_4 \end{bmatrix} \\ &= x_1 x_4 - x_2^2 \\ m_3 &= \det \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix} \\ &= x_1(x_1 x_4 - x_2^2) - x_2(x_2 x_6 - x_3 x_5) + x_3(x_2 x_5 - x_3 x_4) \\ &= x_1^2 x_4 - x_1 x_2^2 - x_2^2 x_6 + x_2 x_3 x_5 + x_2 x_3 x_5 - x_3^2 x_4 \end{aligned} \tag{9}$$

These definitions of the minors can be then be used to construct inequalities such that all the minors are positive:

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix} \in S_+^2 \iff \begin{matrix} x_1 \geq 0 \\ x_1 x_4 \geq x_2^2 \\ x_1 x_2^2 + x_2^2 x_6 + x_3^2 x_4 \geq x_1^2 x_4 + 2 x_2 x_3 x_5 \end{matrix} \tag{10}$$

1.5 Problem 2.33

Problem:

Solution:

2 Problem Set 2: Convex Functions

2.1 Problem 3.6

Problem:

Solution:

2.2 Problem 3.16

Problem:

Solution:

2.3 Problem 3.18a

Problem:

Solution:

2.4 Problem 3.22

Problem:

Solution: