Lecture 16 : Model Predictive Control (MPC)

goals:

- intro to MPC
- discuss advantages + challenges
- MPC + convex optimization

Recall Trajectory Optimization (discrete time)

$$\text{minimize} \quad \sum_{t=0}^{T-1} g_t(x_t, u_t) + g_T(x_T) \qquad \text{(objective)}$$

$$\text{subject to} \quad x_{t+1} = f_t(x_t, u_t) \quad t = 0, \dots, T-1 \qquad \text{(system model)}$$

$$x_t \in \mathcal{X}_t \quad \forall t \qquad \text{(state constraints)}$$

$$u_t \in \mathcal{U}_t(x_t) \quad \forall t \qquad \text{(input constraints)}$$

w/ variables $u_0, \dots, u_{T-1}, x_1, \dots, x_T$

problem data $x_0, g_t, f_t, \mathcal{X}_t, \mathcal{U}_t$

Limitations

① open-loop input sequence, not feedback control law

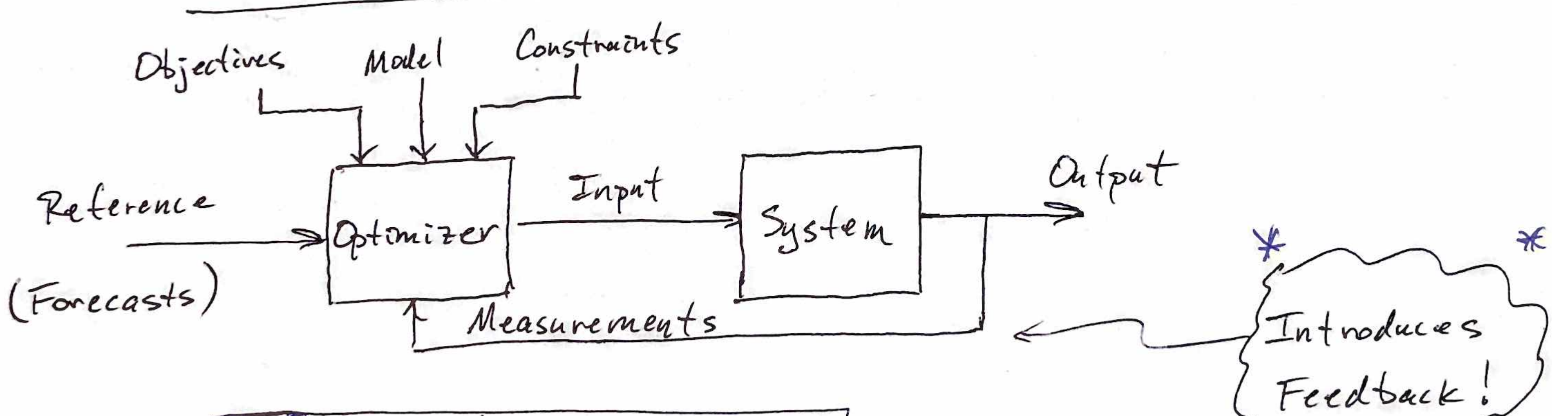
- lacks <u>robustness</u> to unknown disturbances
  unmodeled dynamics

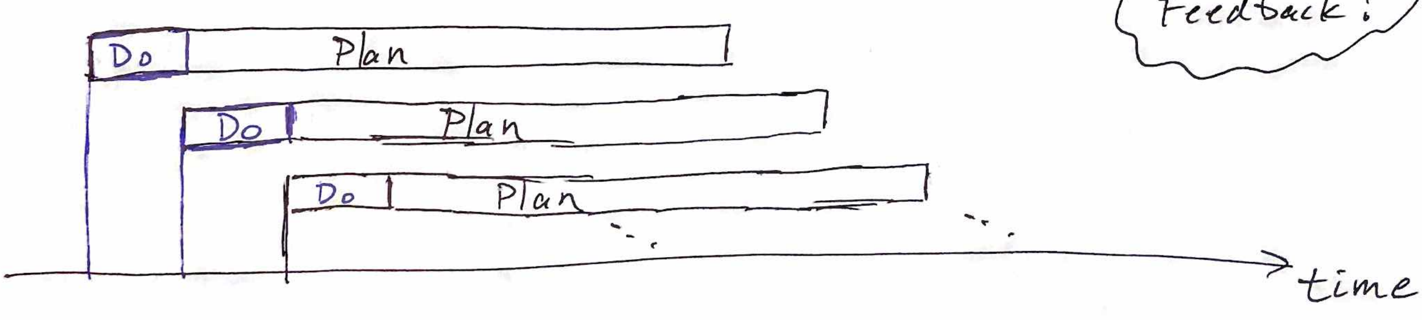
② horizon T may be very long or even infinite, making optimization problem difficult/impossible to solve

$\boxed{\text{IDEA}}$ Repeatedly solve problem over horizon T_h (with possibly $T_h < T$)

- implement only <u>first</u> control action (u_0^*)
- re-solve problem based on measurement of new state (possibly estimated from output measurements)

- called Model Predictive Control (MPC)
  or Receding Horizon Control (RHC)





Challenges:

- Implementation: requires real-time optimization often on limited embedded hardware; must be fast + reliable

- Stability, Robustness, Feasibility
  - requires careful design to ensure closed-loop stability, robustness to unknown disturbances, uncertainties, recursive feasibility of optimization problem
    - e.g. even for deterministic linear quadratic problems, stability not guaranteed if horizon T_h is too short

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MPC and Convex Optimization

- natural synergy: can exploit speed + reliability of modern convex optimization solvers for MPC,

- many variations, both convex + non convex (w/ convex subproblems): robust, stochastic, hybrid, mixed-integer, time-varying, nonlinear, etc.

- analysis + design methods often utilize convex optimization to study stability, robustness, feasibility, etc.

MPC/RHC (Optimization-based Control)

① **Measure/Estimate** state x_t at time t

② **Compute** optimal open-loop plan by solving

$$\begin{aligned}
\text{minimize} \quad & \sum_{\tau=t}^{t+T_h-1} g_\tau(x_\tau, u_\tau) + g_{t+T_h}(x_{t+T_h}) \\
\text{subject to} \quad & x_{\tau+1} = f_\tau(x_\tau, u_\tau) \qquad \tau = t, .., t+T_h-1 \\
& x_\tau \in \mathcal{X}_\tau \quad \forall \tau \\
& u_\tau \in \mathcal{U}_\tau(x_\tau) \quad \forall \tau
\end{aligned}$$

- need fast, reliable constrained optimization solver
- possibility of infeasibility (more later)

③ **Implement** first control action u_t^* on system (during period $[t, t+1]$)

④ **Wait** until new state/output measurements received at sampling time $t+1$, and go to step ①

Advantages:
- explicit, systematic incorporation of performance objectives and constraints
- very general methodology: works (in principle) for any type of system model + objective
- large + growing # of applications, enabled by rapid recent increases in computational power
  - see ETH slides

Convex Quadratic MPC

$$\text{minimize} \quad \sum_{\tau=t}^{t+T_h-1} x_\tau^T Q_\tau x_\tau + u_\tau^T R_\tau u_\tau \; + \; x_{t+T_h}^T Q_{t+T_h} x_{t+T_h}$$

$$\text{subject to} \quad x_{\tau+1} = A_\tau x_\tau + B_\tau u_\tau + c_\tau \qquad \tau = t, \dots, t+T_h-1$$

$$x_\tau \in \mathcal{X}_\tau = \{ x \in \mathbb{R}^n \mid F_\tau x \leq h_\tau \} \qquad \forall \tau$$

$$u_\tau \in \mathcal{U}_\tau = \{ u \in \mathbb{R}^m \mid G_\tau u \leq d_\tau \}$$

$\longrightarrow$ a convex QP $\quad (Q_\tau \succeq 0, \; R_\tau \succ 0)$

1- and ∞-norm MPC $\quad$ ($p = 1$ or ∞, or combo thereof)

$$\text{minimize} \quad \sum_{\tau=t}^{t+T_h-1} \|M_\tau x_\tau\|_p + \|N_\tau u_\tau\|_p \; + \; \|M_{t+T_h} x_{t+T_h}\|_p$$

$$\text{subject to} \quad x_{\tau+1} = A_\tau x_\tau + B_\tau u_\tau + c_\tau \qquad \tau = t, \dots, t+T_h-1$$

$$x_\tau \in \mathcal{X}_\tau = \{ x \in \mathbb{R}^n \mid F_\tau x \leq h_\tau \} \qquad \forall \tau$$

$$u_\tau \in \mathcal{U}_\tau = \{ u \in \mathbb{R}^m \mid G_\tau u \leq d_\tau \}$$

$\longrightarrow$ (equivalent to) a (convex) LP

$g_{t+T_h}(x_{t+T_h})$

Key design parameters: horizon length (T_h), terminal cost (Q_{t+T_h}), terminal constraint set $\mathcal{X}_{t+T_h}$

- Let $u_t^*(x_t)$ denote the optimal first input from the MPC optimization problem in state x_t

- Let $J_t^*(x_t)$ denote the optimal value of the MPC optimization problem in state x_t

Then we have the following:

① Convex Quadratic MPC:

- $u_t^* : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous, piecewise affine function on a polyhedral partition of the state space:

$$u_t^*(x) = K_t^i x + k_t^i \quad \text{if } x \in P_t^i, \quad i = 1, \dots, N_t^r$$

$$\text{where } P_t^i = \{x \in \mathbb{R}^n \mid Z_t^i x \leq z_t^i\}$$

- $J_t^* : \mathbb{R}^n \to \mathbb{R}$ is a convex, piecewise quadratic function on the same partition

Ex from ETH slides via MPT Toolbox

- these functions can be computed explicitly for low dimensional problems (known a Explicit MPC)

② 1- and ∞-norm MPC

- u_t^* is continuous, piecewise affine on polyhedral partition
- J_t^* is convex, piecewise linear on same partition

- if problem data time invariant, optimal control law time invariant
- cf. optimal unconstrained LQR controller, which is affine