Lecture 17: MPC: Feasibility + Stability

goals:
- identify + discuss feasibility + stability issues in MPC
- discuss sufficient conditions for guaranteeing recursive feasibility + stability of MPC

What can go wrong w/ "standard" MPC?

① Infeasibility: at some time the optimization problem does not have a solution

② Instability: trajectories fail to converge to the origin from some initial states

Ex in Matlab

- feasibility and stability properties depend (in a complex way) on certain parameters (horizon, cost, constraints)
- problems originate from use of a "short sighted" strategy
  - ideally have long/infinite lookahead, but becomes computationally intractable

How to guarantee feasibility + stability?

- Main Idea: try to choose key parameters to mimic infinite horizon, encode future feasibility + stability into problem

Horizon $\longrightarrow$ Terminal Cost

$$\text{minimize} \quad \sum_{t=0}^{\boxed{T_h - 1}} c(x_t, u_t) + \boxed{p(x_{T_h})}$$

$$\text{subject to} \quad x_{t+1} = A x_t + B u_t \qquad t = 0, \ldots, T_h - 1$$

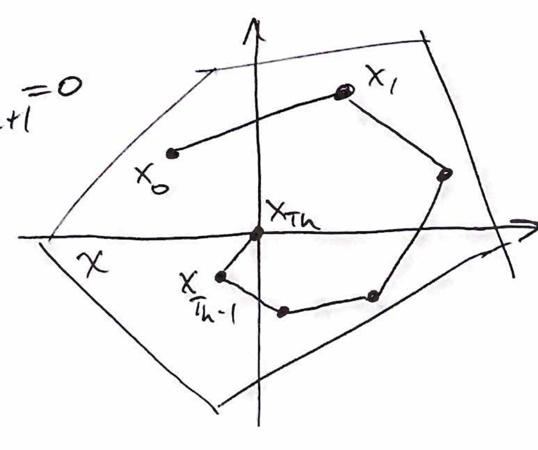
$$x_t \in \mathcal{X}, \; u_t \in \mathcal{U} \qquad \forall t$$

$$\boxed{x_{T_h} \in \mathcal{X}_{T_h}} \quad \text{Terminal Constraint Set}$$

Case 1: $\mathcal{X}_{T_h} = \{0\}$

- suppose x_0 is feasible. let $\{u_0^*, u_1^*, \ldots, u_{T_h-1}^*\}$ and $\{x_1^*, x_2^*, \ldots, x_{T_h}^*\}$ be an optimal (thus feasible) solution to the MPC problem with $\mathcal{X}_{T_h} = \{0\}$ ($\Leftrightarrow x_{T_h}^* = 0$)
- apply u_0^* and let system evolve to $x_1 = A x_0 + B u_0^*$
- from state x_1 the control sequence $\{u_1^*, u_2^*, \ldots, u_{T_h-1}^*, 0\}$ is feasible $\Rightarrow x_{T_h+1}^* = 0$
- and so on

$$\boxed{\Longrightarrow \text{Recursive Feasibility!}}$$



- for stability, we'll show that the optimal cost $\overset{J_0^*(x_0)}{\wedge}$ of the MPC problem (w/ $\mathcal{X}_{T_h} = \{0\}$) is a Lyapunov function for the closed-loop system
    - show $J_0^*(x_1) < J_0^*(x_0) \quad \forall x_0 \neq 0$

- assume stage costs are positive definite

- note: $J_0^*(x_0) = \sum_{t=0}^{T_h-1} c(x_t^*, u_t^*) + \overset{0}{\cancel{p(x_{T_h}^*)}} = \sum_{t=0}^{T_h-1} c(x_t^*, u_t^*)$

$$J_1^*(x_1) = \sum_{t=1}^{T_h} c(x_t^*, u_t^*) + \overset{0}{\cancel{p(x_{T_h+1}^*)}}$$

$$= \underbrace{\sum_{t=0}^{T_h-1} c(x_t^*, u_t^*)}_{J_0^*(x_0)} \underbrace{- c(x_0, u_0^*)}_{<0 \ \forall x_0 \neq 0} + \overset{0}{\cancel{c(x_{T_h}^*, u_{T_h}^*)}}$$

$$< J_0^*(x_0)$$

$\Rightarrow J_0^*(x)$ is a (local) Lyapunov function w/ $\mathcal{X}_{T_h} = \{0\}$

$$\boxed{\Rightarrow \text{MPC locally asymptotically stable!}}$$

Any problem w/ $\mathcal{X}_{T_h} = \{0\}$?

- may need long horizon to ensure initial feasibility
- for fixed horizon enforcing $x_{T_h} = 0$ reduces size of feasible set

Idea: use more general terminal set to increase region of feasibility + attraction

Case 2: $\mathcal{X}_{Th}$ = general (convex) set

Definition: A set $C \subset \mathbb{R}^n$ is called positively invariant for system $x_{t+1} = f_{cl}(x_t)$ if

$$x_0 \in C \implies x_t \in C \quad \forall\, t = 1, 2, \ldots$$

- the maximal positively invariant set is the positively invariant set that contains every closed pos. inv. set

- Assume:

① Stage costs positive definite

② Terminal set is positively invariant under a local control law $\pi(x)$

$$Ax + B\pi(x) \in \mathcal{X}_{Th} \quad \forall\, x \in \mathcal{X}_{Th}$$

and all state and input constraints satisfied in $\mathcal{X}_{Th}$

$$\mathcal{X}_{Th} \subseteq \mathcal{X}, \quad \pi(x) \in \mathcal{U} \quad \forall x \in \mathcal{X}_{Th}$$

③ Terminal cost is a continuous Lyapunov function for closed-loop w/ $\pi(x)$ in $\mathcal{X}_{Th}$ and satisfies

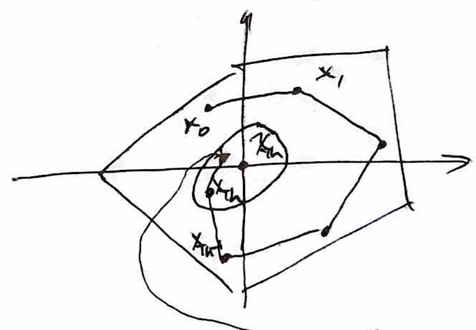
$$p(x_{t+1}) - p(x_t) \leq -c(x_t, \pi(x_t)) \quad \forall x_t \in \mathcal{X}_{Th}$$

Theorem Under these assumptions, the MPC control law $u_0^*(x)$ is locally asymptotically stable and the terminal set $\mathcal{X}_{Th}$ is positively invariant for $x_{t+1} = Ax_t + Bu_0^*(x)$

Proof: • Assume x_0 feasible, let $\{u_0^*, u_1^*, \dots, u_{Th-1}^*\}$ and $\{x_1^*, x_2^*, \dots, x_{Th}^*\}$ be an optimal (thus feasible) solution to MPC problem w/ general terminal set $\mathcal{X}_{Th}$

- from x_1 the control sequence $\{u_1^*, u_2^*, \dots, \pi(x_{Th}^*)\}$ is feasible by assumption, and also $x_{Th+1} = Ax_{Th} + B\pi(x_{Th}^*) \in \mathcal{X}_{Th}$
- and so on

$\Longrightarrow$ Recursive Feasibility



- $J_0^*(x_0) = \sum_{t=0}^{Th-1} c(x_t^*, u_t^*) + p(x_{Th}^*)$

- A feasible sub-optimal sequence for x_1 is $\{u_1^*, u_2^*, \dots, \pi(x_{Th}^*)\}$

$$\Rightarrow J_0^*(x_1) \le \sum_{t=1}^{Th} c(x_t^*, u_t^*) + p(Ax_{Th}^* + B\pi(x_{Th}^*))$$

$$= \sum_{t=0}^{Th-1} c(x_t^*, u_t^*) + p(x_{Th}^*) - c(x_0, u_0^*) + c(x_{Th}^*, \pi(x_{Th}^*))$$
$$- p(x_{Th}^*) + p(Ax_{Th}^* + B\pi(x_{Th}^*))$$

$$= \underbrace{J_0^*(x_0) - c(x_0, u_0^*)}_{<0} + \underbrace{p(Ax_{Th}^* + B\pi(x_{Th}^*)) - p(x_{Th}^*) + c(x_{Th}^*, \pi(x_{Th}^*))}_{\le 0 \text{ by assumption on terminal cost}}$$

$$\Longrightarrow J_0^*(x_1) < J_0^*(x_0) \quad \forall x_0 \neq 0$$

$\Longrightarrow$ local asymptotic stability

Linear Quadratic MPC

- Compute unconstrained LQR control law

$$K = -(R + B^T P B)^{-1} B^T P A$$

matlab: <u>dare</u>

where P solves algebraic Riccati equation

$$P = A^T P A + Q - A^T P B (R + B^T P B)^{-1} B^T P A$$

- Set terminal cost $\quad p(x) = x^T P x$

- Compute terminal set $\mathcal{X}_{Th}$ to be max. pos. invariant set for closed-loop $x_{t+1} = (A+BK)x_t$

$$x \in \mathcal{X}_{Th} \implies (A+BK)x \in \mathcal{X}_{Th}$$

and all constraints satisfied

$$x \in \mathcal{X}_{Th} \implies x \in \mathcal{X}, \quad Kx \in \mathcal{U}$$

Ex Compute $\mathcal{X}_{Th}$ via LMIs. Suppose $\mathcal{X} = \{x \mid Hx \le g\}$, $\mathcal{U} = \{u \mid Fu \le e\}$
Consider ellipsoidal $\mathcal{X}_{Th} = \{x \mid x^T W x \le 1\}$ $\quad W \in S^n_{++}$

$$\left[x^T W x \le 1 \implies x(A+BK)^T W (A+BK) x \le 1 \right] \Longleftarrow (A+BK)^T W (A+BK) \preceq W$$

$$\left[x^T W x \le 1 \implies \begin{bmatrix} H \\ FK \end{bmatrix} x \le \begin{bmatrix} g \\ e \end{bmatrix} \right] \Longleftarrow \begin{bmatrix} W & \frac{1}{2} K^T f_i \\ \frac{1}{2} f_i^T K & e_i \end{bmatrix} \succeq 0 \quad i = 1, \dots, n_u$$

$$\begin{bmatrix} W & \frac{1}{2} h_i \\ \frac{1}{2} h_i^T & g_i \end{bmatrix} \succeq 0 \quad i = 1, \dots, n_x$$

S-Procedure

LMI in W

Data: A, B, K, H, g, F, e

Comments: (Feasibility + Stability)

- Properly chosen terminal set, terminal cost, horizon sufficient for closed-loop stability + recursive feasibility
  - terminal constraint reduces region of feasibility + attraction, but can enlargen by increasing horizon

- Terminal sets generally not used in practice
  - difficult to understand + compute by end users (need polyhedral / LMI computations)
  - reduces region of attraction
  - often unnecessary (e.g. for stable systems w/ long horizon)

- General approach can be directly extended to nonlinear systems
  - via Lyapunov Theory!