MECH 6v29.002 – Model Predictive Control

Tuesday and Thursday 8:30 – 9:45am

L3 – Math Background

Outline

- Mathematical Background for MPC
  - Vectors
  - Matrices
  - Sets
  - Functions

- Note:
  - Not intended to be complete, focus on common assumptions and operations for MPC
  - Notation is not always consistent in the literature, so ask questions throughout the semester
  - Let me know what topics are new to you so that I can improve this review in the future!

MPC-motivated Math Review

Functions:

- Continuous
- Convex
- Lipschitz
- Comparison

Cost function

$$\min_{U_t} \sum_{k=t}^{t+N-1} J(x_k, u_k) + J_T(x_{t+N}, u_{t+N})$$

$$s.t.\ \forall k = t, \ldots, t+N-1$$

Model

$$x_{k+1} = Ax_k + Bu_k$$

Constraints

$$u_k \in \mathcal{U},\ x_k \in \mathcal{X},\ y_k = g(x_k, u_k) \in \mathcal{Y}$$

$$x_t = x(t)$$

Vectors and Matrices:

- Notation
- Norms
- Spaces

Sets

- Compact
- Convex
- Representations
- Operations

For more mathematical background, see:

- Appendix A of *Model Predictive Control: Theory, Computation, and Design 2nd Edition* by Rawlings, Mayne, and Diehl, 2019.
- Appendix A of *Convex Optimization* by Boyd and Vandenberghe, 2009.

Vectors

- Vector notation
  - x is a vector, x_i are scalars

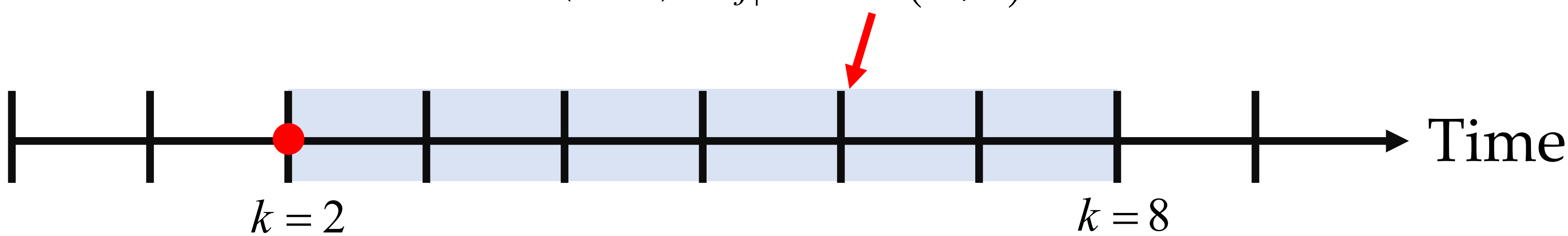
    $x \in \mathbb{R}^n \qquad x_1, x_2, \ldots, x_n \in \mathbb{R}$

    $$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$$

    $$x = [x_i],\ i \in 1, \ldots, n$$

  - State at time t – $x(t)$
  - Predicted state at time step k – $x(k)$, $x[k]$, x_k
  - Predicted state at time step j determined at time step k

(double-index notation) $x(j\,|\,k)$, $x_{j|k}$ $\qquad x(6\,|\,2)$



Vectors

- Vector spaces
  - Mostly dealing with Euclidean space $\mathbb{R}^n$
  - Frequently use n-dimensional state space and m-dimensional input space

$$x_k \in \mathbb{R}^n \qquad u_k \in \mathbb{R}^m$$

  - Vector spaces must satisfy a set of axioms
    - https://en.wikipedia.org/wiki/Vector_space
    - Most importantly, if $\mathcal{V}$ is a vector space and x and y are elements of this vector space $x, y \in \mathcal{V}$, then

$$\alpha x + \beta y \in \mathcal{V} \text{ for all scalars } \alpha, \beta \in \mathbb{R}$$

  - Not to be confused with set, which can be finite
  - (Linear) subspaces contained in a vector space $\mathcal{S} \subset \mathcal{V}$

$$\alpha x + \beta y \in \mathcal{S} \quad \text{for all scalars } \alpha, \beta \in \mathbb{R}$$

    - Examples?
    - Line, plane, origin, empty set

Vectors

- Absolute value for a scalar $x \in \mathbb{R}$ $\quad |x| \in \mathbb{R}_+$
- Absolute value of a vector $x \in \mathbb{R}^n$
  - Apply element-wise
  - Need to be careful with notation

$$|x| = \begin{bmatrix} |x_1| \\ \vdots \\ |x_n| \end{bmatrix} \in \mathbb{R}_+^n$$

- Norm of a vector $x \in \mathbb{R}^n$ $\quad \|x\| \in \mathbb{R}$
  - All norms are:
    - Nonnegative $\quad \|x\| \geq 0, \ \forall x \in \mathbb{R}^n$
    - Definite $\quad \|x\| = 0$ only if $x = 0$
    - Homogeneous $\quad \|\alpha x\| = |\alpha| \|x\| \ \forall x \in \mathbb{R}^n, \alpha \in \mathbb{R}$
    - Satisfy triangle inequality

$$\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^n$$

Vectors

- Vector norms (continued)
  - Common types of norms

$$\ell_1 - norm: \quad \|x\|_1 = |x_1| + |x_2| + \ldots + |x_n|$$

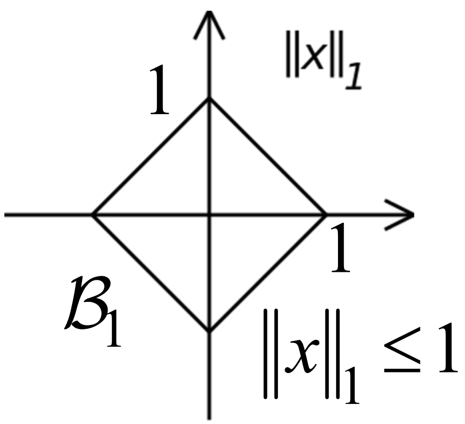
sum-absolute-value

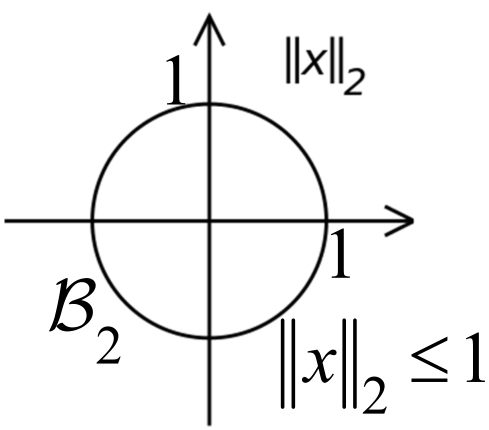
$$\ell_2 - norm: \quad \|x\|_2 = \left(x^T x\right)^{1/2} = \left(x_1^2 + x_2^2 + \ldots + x_n^2\right)^{1/2}$$

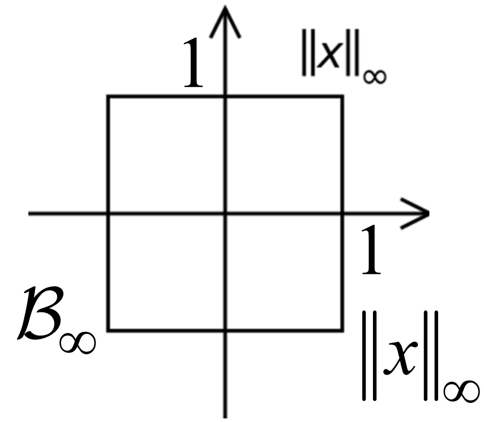
Root-sum-square

$$\ell_p - norm: \quad \|x\|_p = \left(|x_1|^p + |x_2|^p \ldots + |x_n|^p\right)^{1/p} \qquad p \geq 1$$

$$\ell_\infty - norm: \quad \|x\|_\infty = \max\left\{|x_1|, |x_2|, \ldots, |x_n|\right\}$$







Different versions of the unit ball

Vectors

- Vector norms (continued)
  - Common types of norms

$$\ell_1 - norm: \quad \|x\|_1 = |x_1| + |x_2| + \ldots + |x_n|$$

$$\ell_2 - norm: \quad \|x\|_2 = \left(x^T x\right)^{1/2} = \left(x_1^2 + x_2^2 + \ldots + x_n^2\right)^{1/2}$$

$$\ell_p - norm: \quad \|x\|_p = \left(|x_1|^p + |x_2|^p \ldots + |x_n|^p\right)^{1/p} \qquad p \geq 1$$

$$\ell_\infty - norm: \quad \|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$$

- Norm equivalence
  - For any two norms, there exists positive constants such that

$$\alpha\|x\|_a \leq \|x\|_b \leq \beta\|x\|_a$$

  - Common relationships

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_\infty$$

Matrices

- Matrix notation
  - M is a matrix

$$M \in \mathbb{R}^{m\times n} \qquad M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & \ddots & & \vdots \\ \vdots & & & \\ m_{m1} & \cdots & & m_{mn} \end{bmatrix} \qquad M = \left[x_{ij}\right],\ i \in 1,\ldots,m,\quad j \in 1,\ldots,n$$

  - Matrices are linear transformations (linear mappings)

$$\mathbb{R}^n \to \mathbb{R}^m$$

    - Maps each vector $x \in \mathbb{R}^n$ to $y = Mx \in \mathbb{R}^m$

Matrices

- Matrix notation $M \in \mathbb{R}^{m \times n}$ $x \in \mathbb{R}^n$ $y = Mx \in \mathbb{R}^m$
  - It makes since to think about where this mapping can take you (and where it cannot)
  - Range of M : $\mathcal{R}(M) = \left\{ y = Mx \mid x \in \mathbb{R}^n \right\}$
    - Set of vectors that can be written as linear combinations of the columns of M

$\mathcal{R}(M)$ is a subspace of $\mathbb{R}^m$
with dimension equal to the # of linearly independent columns of M ($=$ rank of M)

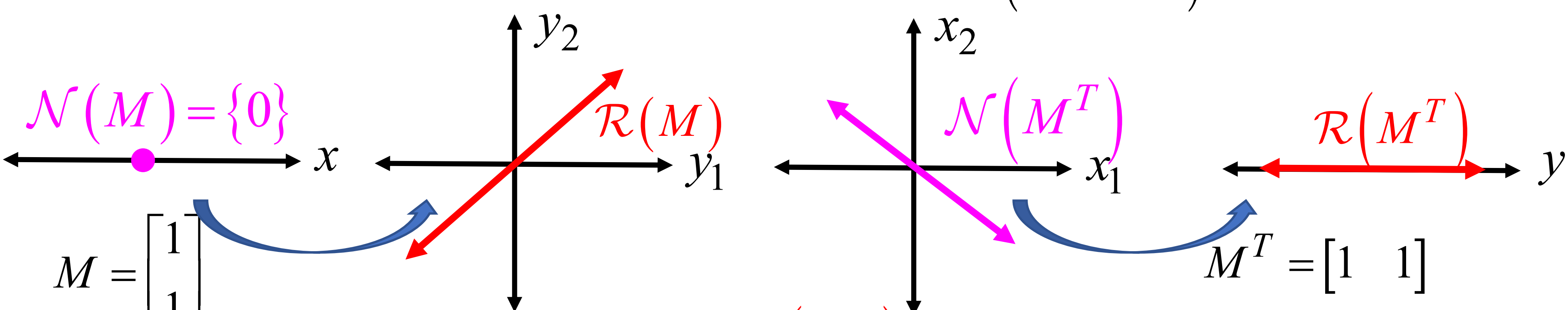
  - Nullspace of M : $\mathcal{N}(M) = \{ x \mid Mx = 0 \}$
    - Set for vectors x that are mapped to zero by M

$\mathcal{N}(M)$ is a subspace of $\mathbb{R}^n$
with dimension equal to $n - \text{rank}(M)$

i.e. sum of the dimensions of $\mathcal{R}(M)$ and $\mathcal{N}(M) = n$

Matrices

- The four fundamental subspaces [1] $M \in \mathbb{R}^{m \times n}$ $x \in \mathbb{R}^n$ $y = Mx \in \mathbb{R}^m$
  - Column space (Range): $\mathcal{R}(M) \subset \mathbb{R}^m$
    - All linear combination of columns of M
  - Nullspace: $\mathcal{N}(M) \subset \mathbb{R}^n$
    - All solutions x such that $Mx = 0$
  - Row space (Range of transpose): $\mathcal{R}(M^T) \subset \mathbb{R}^n$
    - All linear combination of rows of M
  - Left nullspace (Nullspace of transpose): $\mathcal{N}(M^T) \subset \mathbb{R}^m$
    - All solutions x such that $M^T x = 0$ $\left(x^T M = 0\right)$



$M = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $M^T = \begin{bmatrix} 1 & 1 \end{bmatrix}$

$\mathcal{R}(M) \oplus \mathcal{N}(M^T) = \mathbb{R}^m$ $\mathcal{R}(M^T) \oplus \mathcal{N}(M) = \mathbb{R}^n$

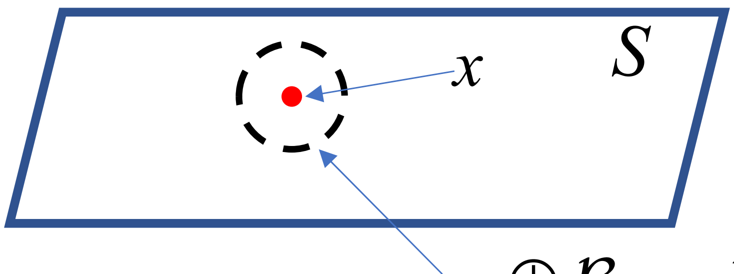
[1] https://ocw.mit.edu/courses/mathematics/18-06sc-linear-algebra-fall-2011/ax-b-and-the-four-subspaces/the-four-fundamental-subspaces/MIT18_06SCF11_Ses1.10sum.pdf

Matrices

- Some square matrices can be positive definite or positive semidefinite
- Let Q be a real, symmetric matrix
  - Then Q is positive definite $Q>0$ if
  $$x^T Qx > 0, \quad \forall x \in \mathbb{R}^n$$
  - Then Q is positive semidefinite $Q \geq 0$ if
  $$x^T Qx \geq 0, \quad \forall x \in \mathbb{R}^n$$
- Lecture Assignment:
  - Prove one of the following
  $$Q > 0 \text{ if and only if } \lambda(Q) > 0, \;\; \lambda \in eig(Q)$$
  $$Q \geq 0 \text{ if and only if } \lambda(Q) \geq 0, \;\; \lambda \in eig(Q)$$
  - Remember to go both directions, i.e.
    - Assume $Q>0$ and show $\lambda(Q)>0, \;\; \lambda \in eig(Q)$
    - Then assume $\lambda(Q)>0, \;\; \lambda \in eig(Q)$ and show $Q>0$

Sets

- Consider a set $S \in \mathbb{R}^n$
  - An element of this set, $x \in S$, is an interior point if the set of "nearby points" are also elements of this set
  - Formally, $\exists \ \ \epsilon > 0$ s.t. $\left\{ y \mid \left\| y - x \right\|_2 \leq \epsilon \right\} \subseteq S$
  - Alternatively, we can think about the existence of an epsilon-ball centered at x that lies entirely in the set

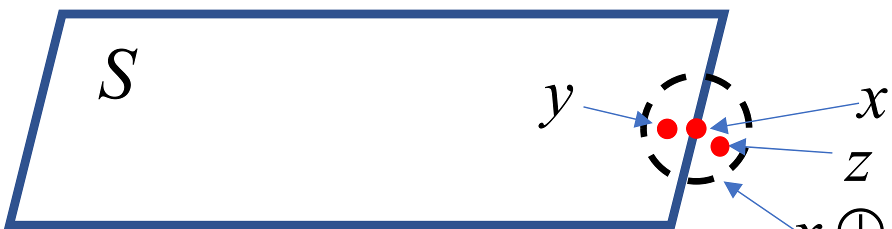


$x \oplus \mathcal{B}, \qquad \mathcal{B} = \left\{ y \mid \left\| y \right\|_2 \leq \epsilon \right\}$

Sets

- Consider a set $S \in \mathbb{R}^n$
  - An element of this set, $x \in S$, is a boundary point if there are both "nearby points" that are in and not in the set
  - Formally, $\forall \;\; \epsilon > 0 \;\; \exists \; y \in S$ and $z \notin S$ such that

$$\|y - x\|_2 \leq \epsilon \qquad \|z - x\|_2 \leq \epsilon$$



$$x \oplus \mathcal{B}, \; \mathcal{B} = \left\{ y \mid \|y\|_2 \leq \epsilon \right\}$$

$$\forall \epsilon > 0$$

- A set is closed if it contains its boundary
- A set is open if it contains no boundary points
- A set is bounded if there exists a finite number that bounds the norm of every element of the set

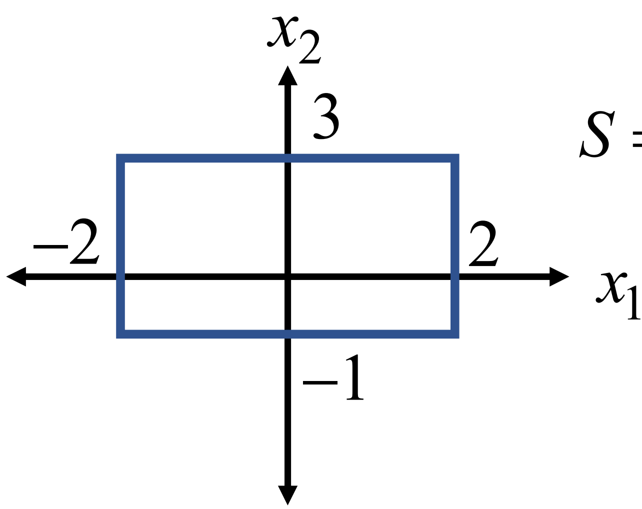
$$\exists M < \infty, \;\; s.t. \;\; \|x\| \leq M, \;\; \forall x \in S$$

- A set is compact if it is closed and bounded

Sets

- A set is convex if for any two elements in the set, all of the points on a straight line connecting these elements are also in the set
- Formally, $\forall x, y \in S$ and $\lambda \in [0,1], \quad \big(\lambda x + (1-\lambda) y\big) \in S$
- We will primarily deal with convex sets
- Sets can be defined using different representations
  - Intervals:

$$S = \left[\underline{x}, \overline{x}\right] = \left\{ x \in \mathbb{R}^n \mid \underline{x} \le x \le \overline{x} \right\}$$



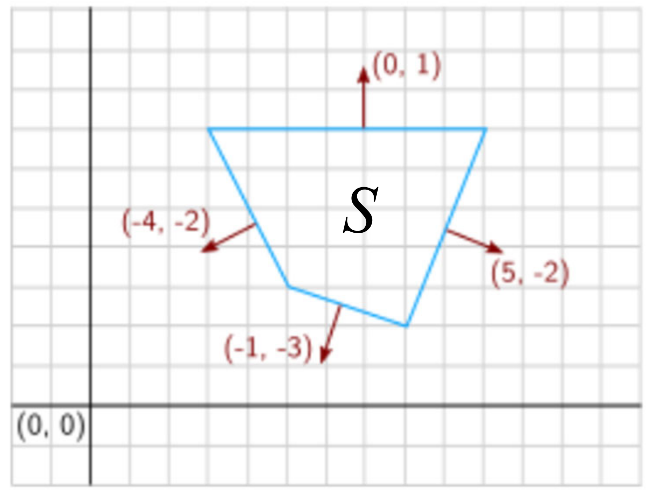
$$S = \left[\underline{x}, \overline{x}\right], \quad \underline{x} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \overline{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Sets

- Sets can be defined using different representations
  - Halfspace-Representation (H-Rep):

$$S=\left\{x\in\mathbb{R}^n \mid Ax\le b\right\},\quad A\in\mathbb{R}^{n_h\times n},\ \ b\in\mathbb{R}^{n_h}\qquad n_h=\#\text{ of halfspaces}$$

$$\begin{bmatrix} 0 & 1 \\ 5 & -2 \\ -1 & -3 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} 7 \\ 36 \\ -14 \\ -26 \end{bmatrix}$$



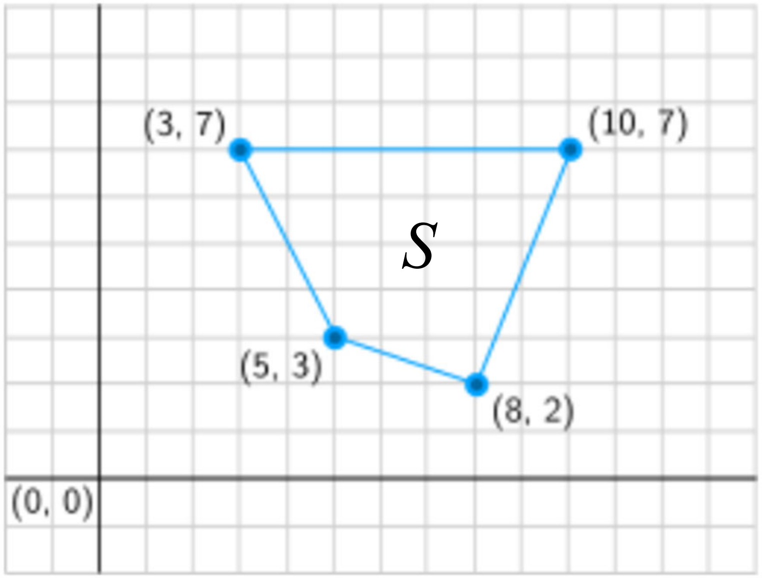
[1]

[1] https://scaron.info/teaching/polyhedra-and-polytopes.html

Sets

- Sets can be defined using different representations
  - Vertex-Representation (V-Rep): (convex hull of vertices)

$$S = \left\{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^{n_v} \lambda_i v_i,\ \lambda_i \geq 0,\ \sum_{i=1}^{n_v} \lambda_i = 1 \right\}, \qquad n_v = \#\text{ of vertices}$$



[1]

- H-rep and V-rep can be used to represent any convex polytope
  - A polytope is a geometric object with "flat" sides
  - Also a bounded cases of a more general (potentially unbounded) polyhedron

[1] https://scaron.info/teaching/polyhedra-and-polytopes.html

Sets

- Common set operations
  - Linear transformation: $MS = \{Mx \mid x \in S\}, \quad M \in \mathbb{R}^{m \times n}$
  - Minkowski sum:

$$S_1 \oplus S_2 = \{x_1 + x_2 \mid x_1 \in S_1, x_2 \in S_2\}$$

  - Pontryagin difference:

$$\begin{aligned} S_1 \ominus S_2 &= \{x_1 \in S_1 \mid x_1 \oplus S_2 \subseteq S_1\} \\ &= \{x_1 \in S_1 \mid x_1 + x_2 \in S_1, \forall x_2 \in S_2\} \end{aligned}$$

  - Intersection:

$$S_1 \cap S_2 = \{x_1 \in S_1 \mid x_1 \in S_2\}$$

  - Convex hull:

$$CH(S_1, S_2) = \{\lambda x_1 + (1-\lambda) x_2 \mid x_1 \in S_1, x_2 \in S_2, \lambda \in [0,1]\}$$

Sets

- Lecture Assignment:
  - Explore the computational geometry features of the Multi-Parametric Toolbox (MPT)
    - https://www.mpt3.org/

Multi-Parametric Toolbox 3

The Multi-Parametric Toolbox (or MPT for short) is an open source, Matlab-based toolbox f

Installation

- Installation & updating instructions
- License
- How to cite MPT3

Contact

Questions and comments should be posted via the MPT forum at Google Groups.

First steps with MPT3

- Quick start using demos
- Parametric optimization
- Computational geometry features
- MPC synthesis (regulation, tracking)
  - Modeling of dynamical systems
  - Closed-loop simulations
  - Additional constraints (move blocking, soft & rate constraints, terminal sets, etc.)
  - Fine-tuning MPC setups via YALMIP
  - Code generation
  - Low-complexity explicit MPC algorithms
- Computation of invariant sets
- Construction of Lyapunov functions

Particularly these ones

https://www.mpt3.org/Geometry/Geometry

Tour on the computational geometry

Objects for representing sets and functions

- Construction and properties of basic sets
- Construction of function objects

Objects for representing unions of sets

- Construction and properties of unions of sets

Objects for representing functions over sets

- Construction of function over sets

Geometric operations

- Geometric operations with general convex sets
- Geometric operations with polyhedra

See question on last slide.

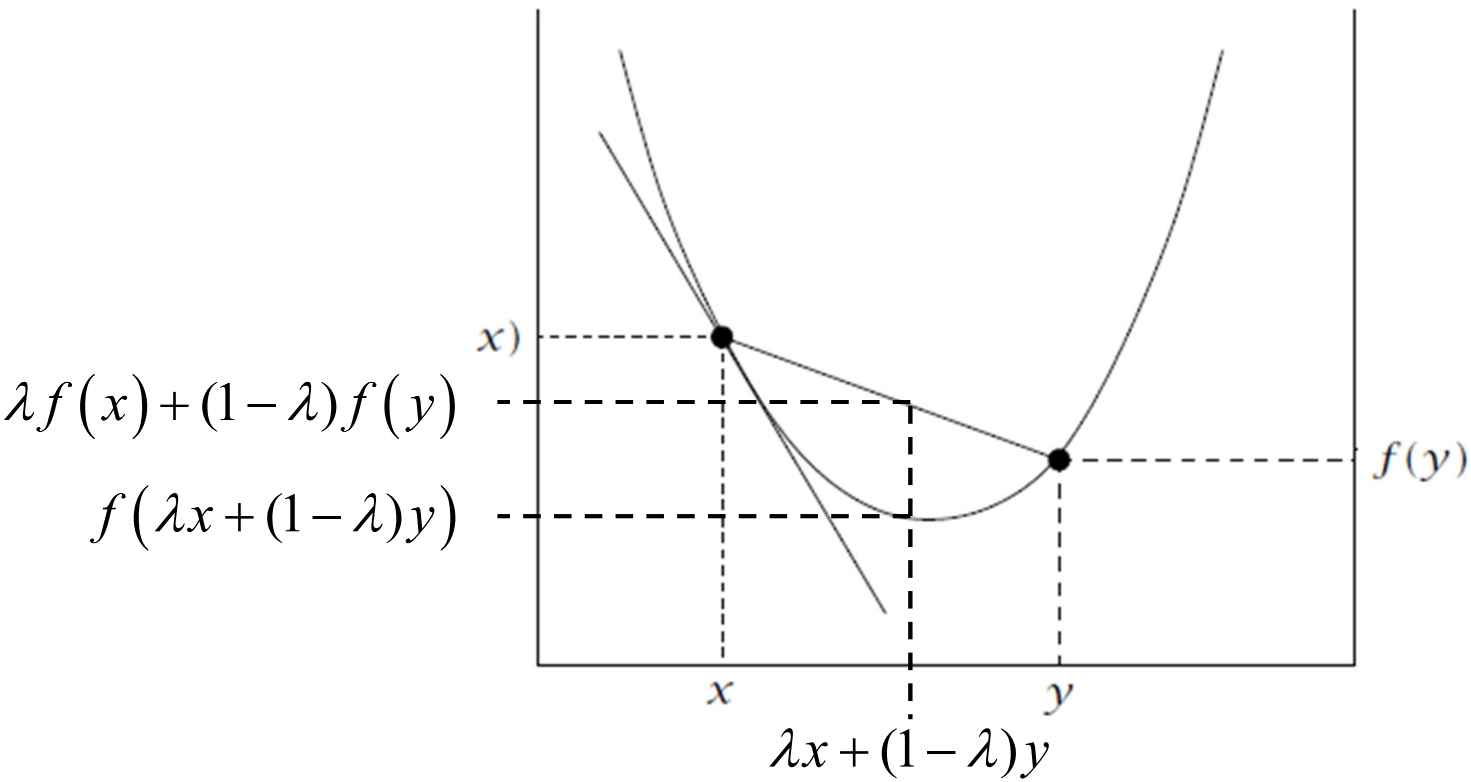
Functions

- A function $f:\mathbb{R}^n \to \mathbb{R}^m$ is continuous if $\forall x \in \mathbb{R}^n$

$$\exists \delta \;\; s.t. \;\; \|y-x\|_2 \le \delta \Rightarrow \|f(y)-f(x)\|_2 \le \epsilon, \;\; \forall \epsilon > 0$$

- A function $f:\mathbb{R}^n \to \mathbb{R}$ is convex if $\forall x, y \in \mathbb{R}^n, \lambda \in [0,1]$

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$



Functions

- The level set of a function $f:\mathbb{R}^n \rightarrow \mathbb{R}$ is

$$\left\{ x \in \mathbb{R}^n \mid f(x) = \alpha \right\} \quad \alpha \in \mathcal{R}$$

- The sublevel set is similar with

$$\left\{ x \in \mathbb{R}^n \mid f(x) \leq \alpha \right\} \quad \alpha \in \mathcal{R}$$

- A function is Lipschitz, with Lipschitz constant *L*, if

$$\forall x, y \in \mathbb{R}^n, \quad \left\| f(y) - f(x) \right\|_2 \leq L \left\| y - x \right\|_2$$

- Lecture Assignment:
  - Provide an example of a function that is continuous but not Lipschitz

Functions

- A function $f:\mathbb{R}^n \to \mathbb{R}$ is positive definite if

$$f(0)=0 \quad \text{and} \quad f(x)>0,\ \forall x \neq 0$$

- A function $f:\mathbb{R}^n \to \mathbb{R}$ is positive semidefinite if

$$f(0)=0 \quad \text{and} \quad f(x)\geq 0,\ \forall x$$

- A quadratic function $f(x)=x^T Qx$, where Q is symmetric, is positive (semi)definite if Q is a positive (semi)definite matrix

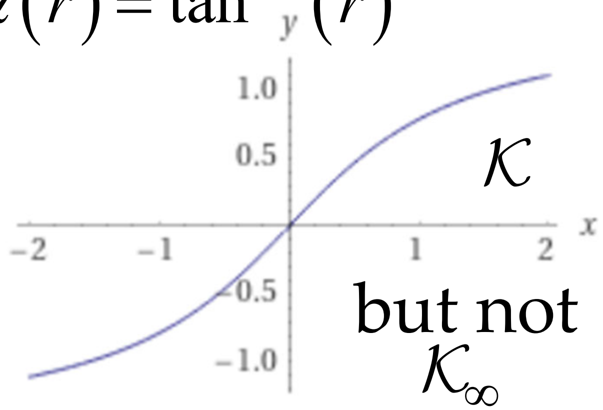
Comparison Functions

- Class $\mathcal{K}$: A scalar continuous function $\alpha(r):[0,a)\to\mathbb{R}$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0)=0$
- Class $\mathcal{K}_\infty$: A scalar continuous function $\alpha(r):[0,\infty)\to\mathbb{R}$ belongs to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and $\alpha(r)\to\infty$ as $r\to\infty$
- Class $\mathcal{KL}$: A scalar continuous function $\beta(r,s):[0,a)\times[0,\infty)\to\mathbb{R}$ belongs to class $\mathcal{KL}$ if for each fixed value of s, $\beta(r,s)$ belongs to class $\mathcal{K}$ with respect to r and for each fixed value of r, $\beta(r,s)$ is decreasing with respect to s and

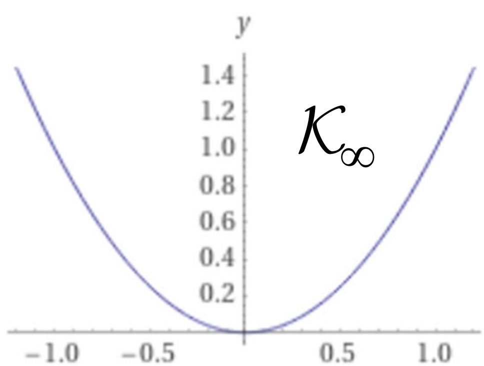
$$\beta(r,s)\to 0 \text{ as } s\to\infty$$

- Examples:

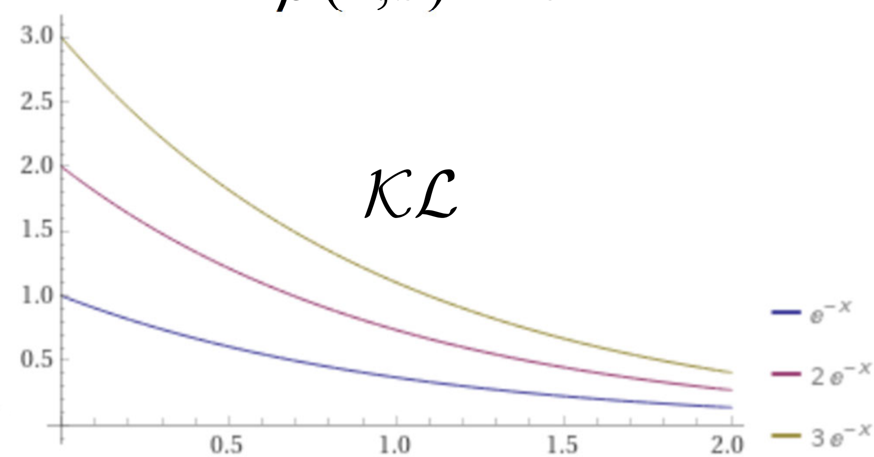
$\alpha(r)=\tan^{-1}(r)$



$\alpha(r)=r^2$



$\beta(r,s)=re^{-s}$



Summary of Lecture Assignments

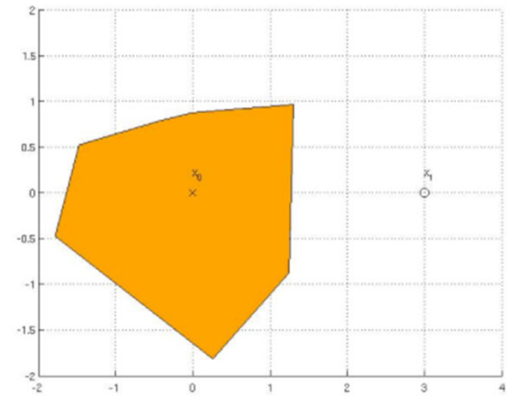
1) • Prove one of the following

$$Q > 0 \text{ if and only if } \lambda(Q) > 0, \;\; \lambda \in eig(Q)$$

$$Q \geq 0 \text{ if and only if } \lambda(Q) \geq 0, \;\; \lambda \in eig(Q)$$

• Remember to go both directions, i.e.
  - Assume $Q > 0$ and show $\lambda(Q) > 0, \;\; \lambda \in eig(Q)$
  - Then assume $\lambda(Q) > 0, \;\; \lambda \in eig(Q)$ and show $Q > 0$

2) MPT Toolbox
   - Review computational geometry features
   - Determine H-rep of orange set on page
   - https://www.mpt3.org/Geometry/OperationsWithPolyhedra



3) • Provide an example of a function that is continuous but not Lipschitz