MECH 6v29.002 – Model Predictive Control

L11 – Persistent Feasibility

Outline

- Summary of Invariant Sets
- Vehicle/Wall Example
  - Effect of prediction horizon length on feasibility
  - Use of N-step stabilizable set terminal constraint
  - Use of maximal control invariant set terminal constraint
- Persistent Feasibility for MPC

Summary

- Positive Invariant $\mathcal{O} \subseteq \mathcal{X}$ $\quad x_0 \in \mathcal{O} \;\Rightarrow\; x_k \in \mathcal{O} \quad \forall k > 0$
- Maximal Invariant Set $\mathcal{O}_\infty \subseteq \mathcal{X}$ (Union of all invariant sets)
- Control Invariant $\mathcal{C} \subseteq \mathcal{X}$ $\quad x_0 \in \mathcal{C} \;\Rightarrow\; \exists u_k \in \mathcal{U},\ s.t.\ x_k \in \mathcal{C} \quad \forall k > 0$
- Maximal Control Invariant Set $\mathcal{C}_\infty \subseteq \mathcal{X}$ (Union of all ctrl. inv. sets)
- Determinedness index – number of steps for max controllable invariant set algorithm to converge (if it does)
- Maximal Controllable Sets $\mathcal{K}_\infty(\mathcal{S})$
  - Union of all N-step controllable sets that drive system to a target set
- Maximal Stabilizable Sets $\mathcal{K}_\infty(\mathcal{O})$
  - Same as Maximal Controllable Sets, but now the target is invariant

- Next steps:
  - Use all of this information to prove persistent feasibility of MPC

Example

- Recall the vehicle example from Lecture 7 (driving close to a wall)
- Double integrator
  - State 1 – position
  - State 2 – velocity
  - Input - force

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u_k$$

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T x_k + 0.1 u_k^T u_k + x_N^T x_N$$

Drive to the origin

$$
\begin{aligned}
& s.t. \\
& x_{k+1} = A x_k + B u_k,\ k \in \{0,1,\ldots,N-1\} \\
& -1 \le u_k \le 1, \quad k \in \{0,1,\ldots,N-1\} \\
& \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \le 0,\ k \in \{1,2,\ldots,N\} \\
& x_0 = x(0)
\end{aligned}
$$

Limited acceleration and deceleration

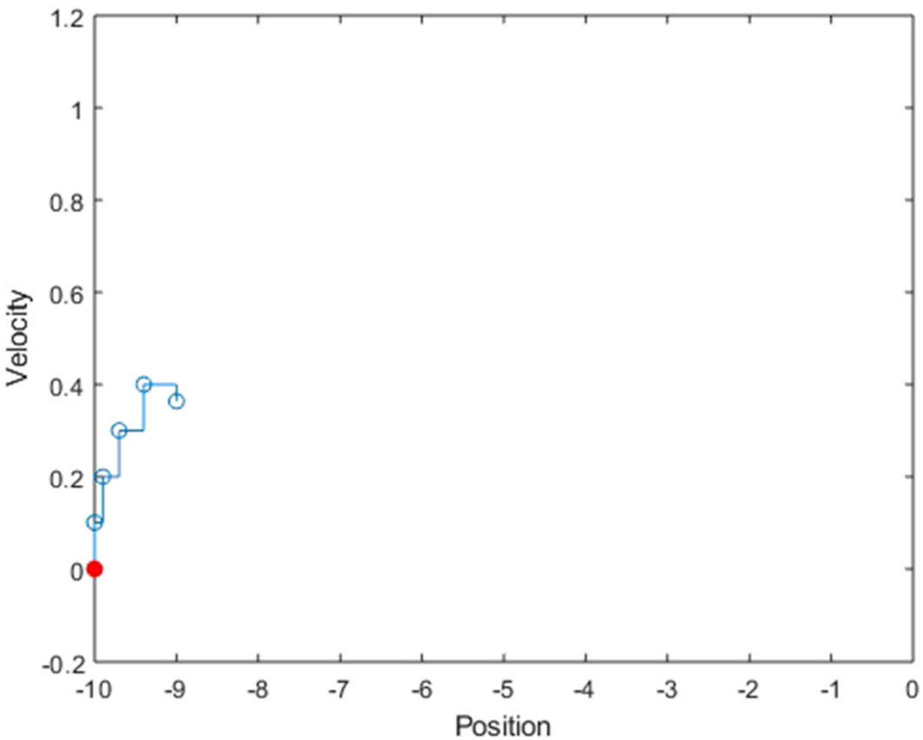
Wall at the origin

$$x(0) = \begin{bmatrix} -10 \\ 0 \end{bmatrix}$$

Start at rest to the left of the wall

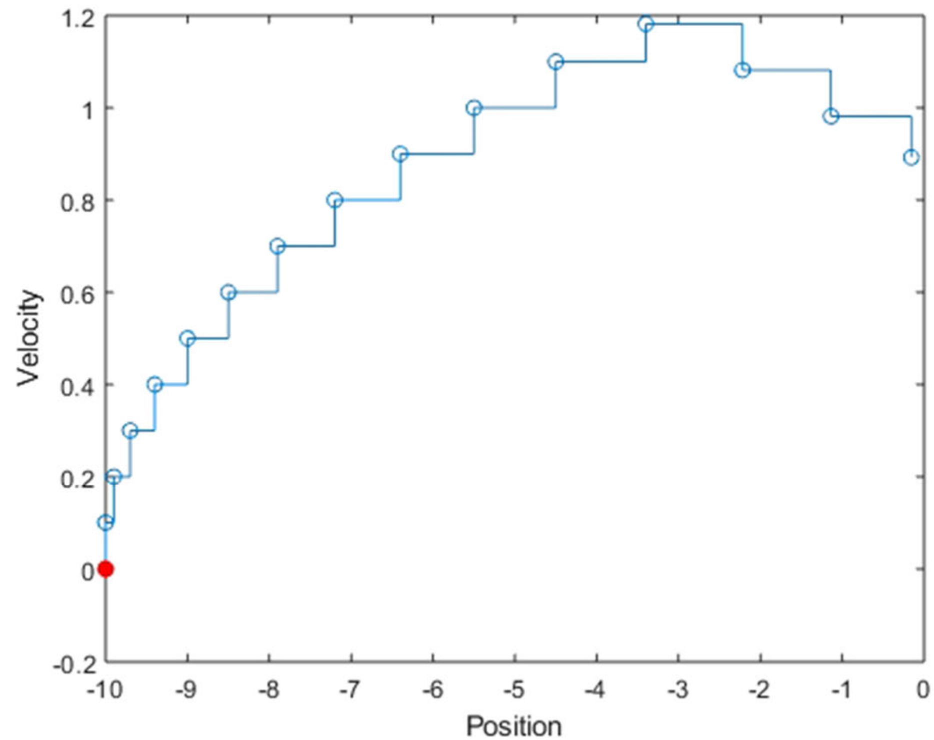
Example (cont.)

$N = 5$



Velocity goes too high and the vehicle cannot slow down fast enough to avoid hitting the wall (infeasible optimization problem)

$N = 15$



Longer prediction horizon prevents velocity from going too high and vehicle arrives at origin without hitting the wall

Example (cont.)

- Add terminal constraint to guarantee feasibility (and stability)
- Problem:
  Initial condition becomes infeasible for $N = 5$ or 15

$$x(0) = \begin{bmatrix} -10 \\ 0 \end{bmatrix} \notin \mathcal{X}_0$$

- Since, $x_N = 0$ is an invariant set, we can think about computing the N-step stabilizable set $\mathcal{K}_N(\mathcal{O})$ $\quad \mathcal{O} = \{0\}$

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T x_k + 0.1 u_k^T u_k + x_N^T x_N$$

$$s.t.$$

$$x_{k+1} = Ax_k + Bu_k, \ k \in \{0,1,\ldots,N-1\}$$

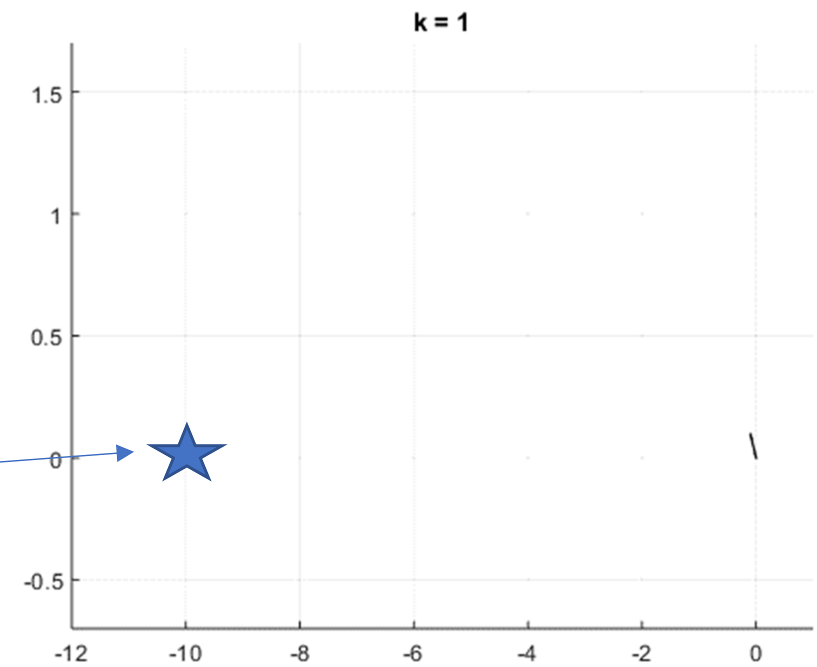
$$-1 \le u_k \le 1, \quad k \in \{0,1,\ldots,N-1\}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} x_k \le 0, \ k \in \{1,2,\ldots,N\}$$

$$x_N = 0$$

$$x_0 = x(0)$$

We wanted to start here



Example (cont.)

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- Problem:
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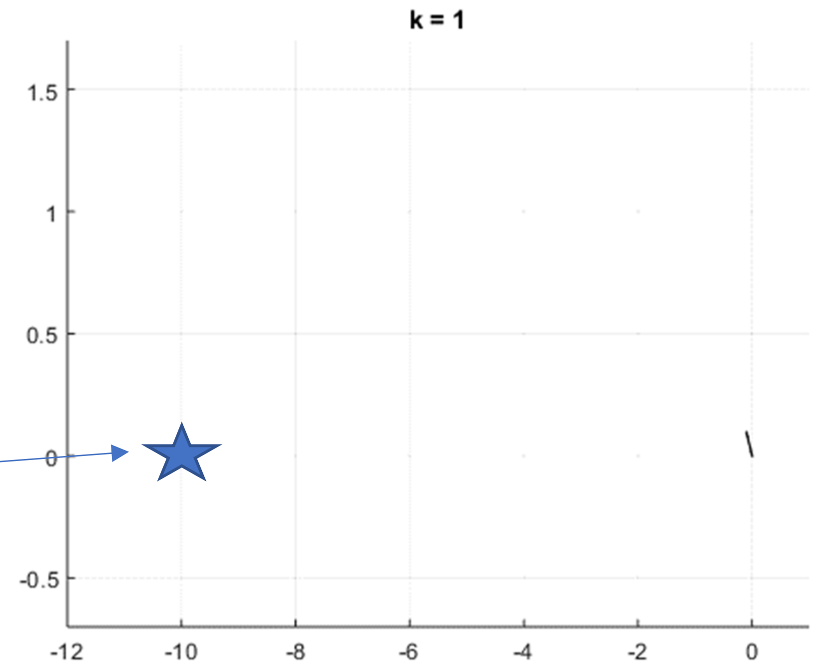
$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T x_k + 0.1 u_k^T u_k + x_N^T x_N$$

$$\begin{aligned} &s.t. \\ &x_{k+1} = Ax_k + Bu_k,\ k \in \{0,1,\ldots,N-1\} \\ &-1 \le u_k \le 1, \quad k \in \{0,1,\ldots,N-1\} \\ &[1 \quad 0] x_k \le 0,\ k \in \{1,2,\ldots,N\} \\ &x_N = 0 \\ &x_0 = x(0) \end{aligned}$$

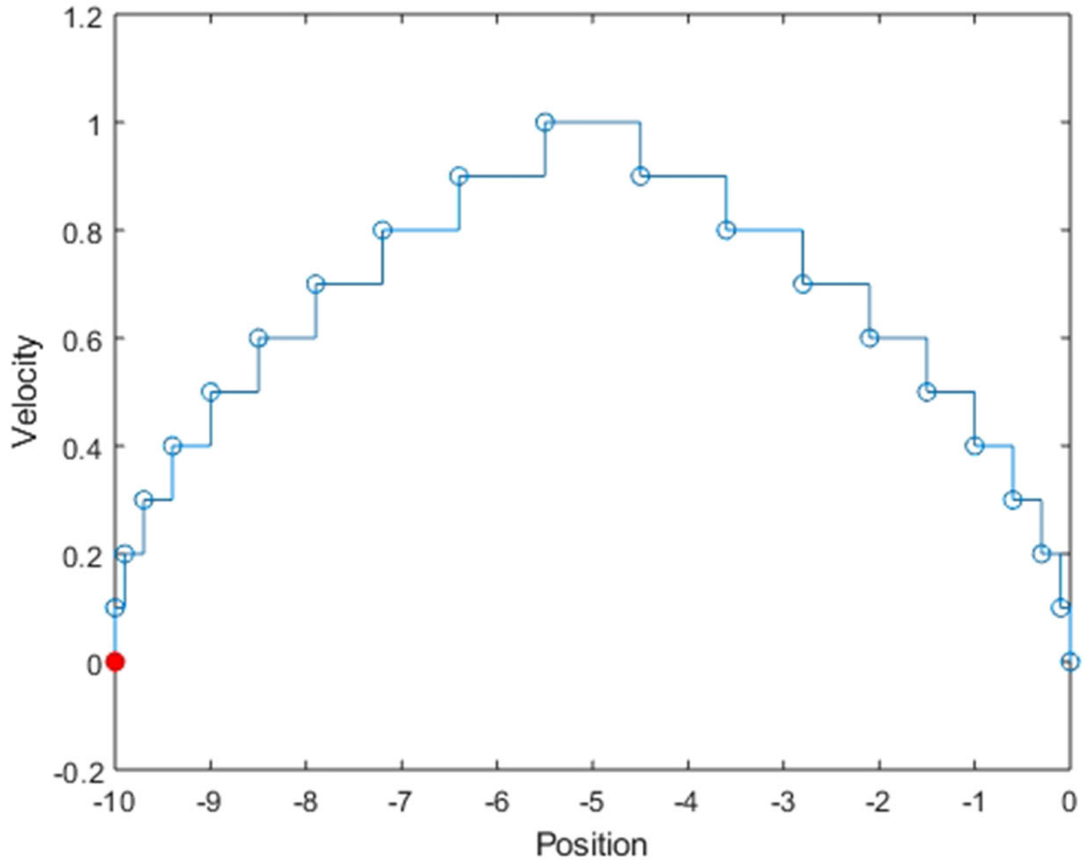


We wanted to start here

Need 20 steps

Example (cont.)

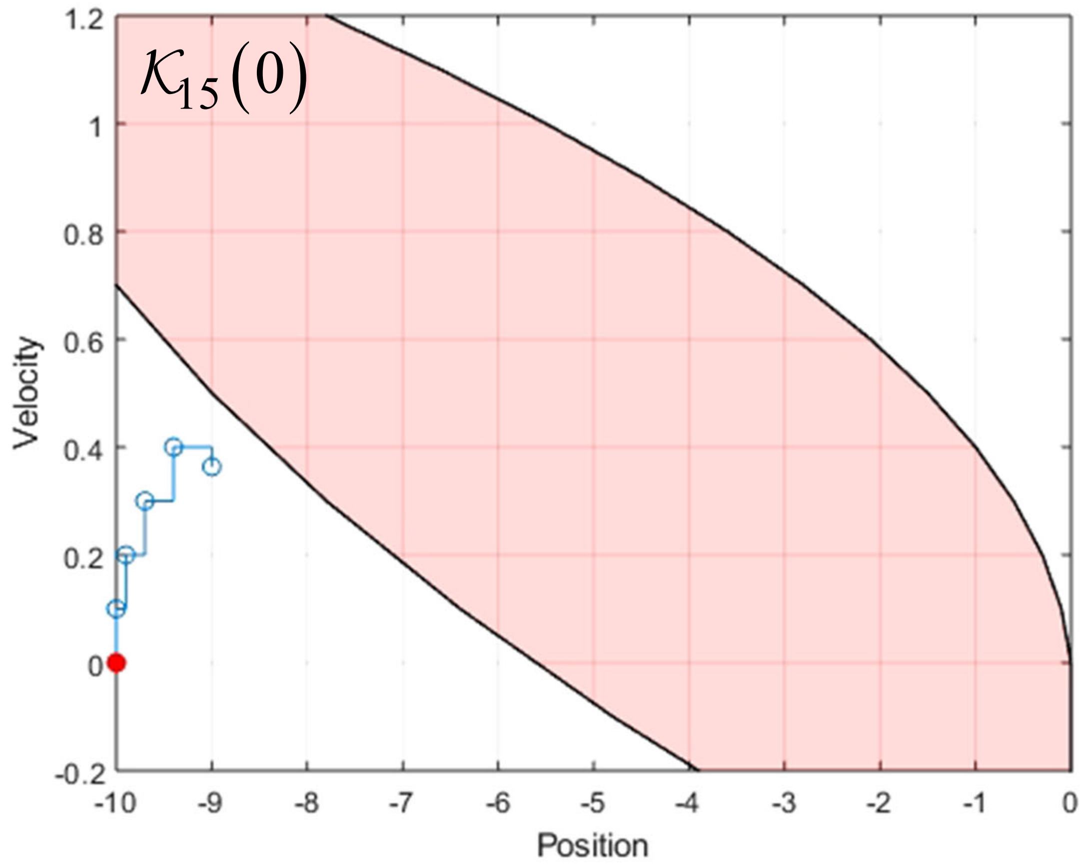
- Redesign MPC with prediction horizon $N = 20$



Example (cont.)

- What if we want to use a shorter prediction horizon of $N = 5$?

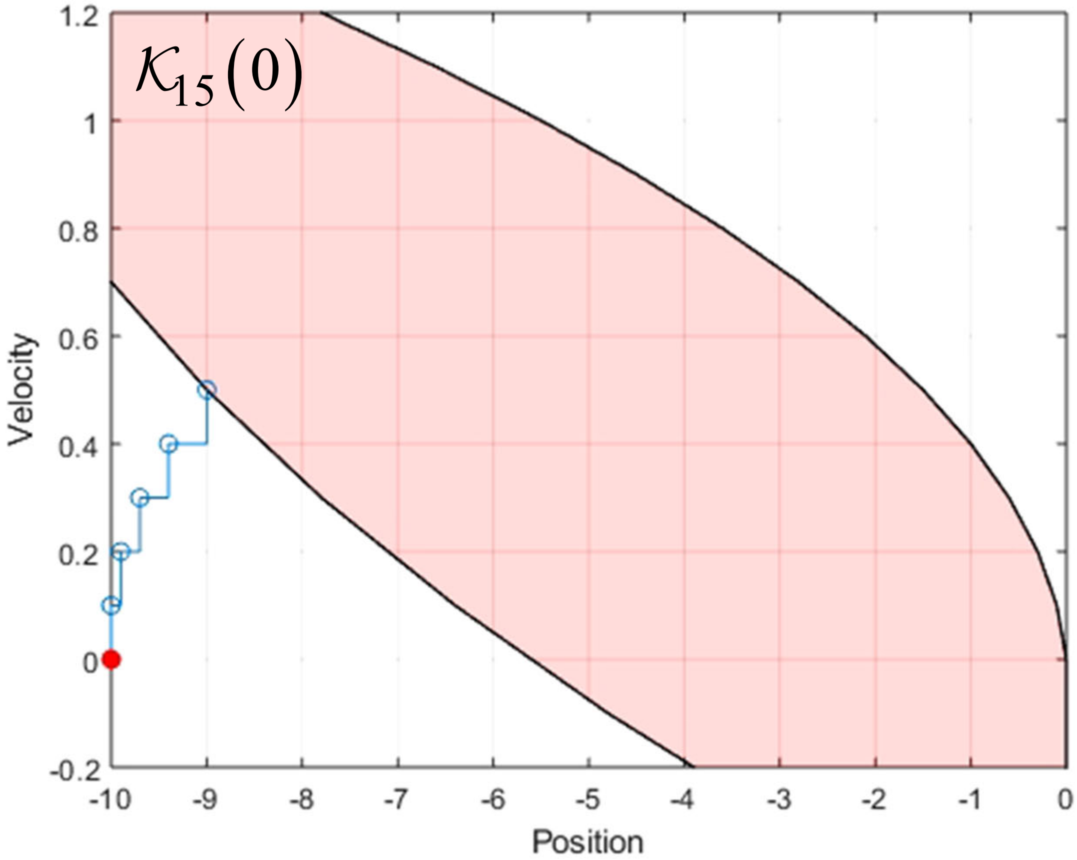
- Enters and then exists the N-step reachable set (becoming infeasible)



Example (cont.)

- What if we want to use a shorter prediction horizon of $N = 5$?

- Add terminal constraint (similar to increasing the prediction horizon from a feasibility point of view with little computational cost)



$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T x_k + 0.1 u_k^T u_k + x_N^T x_N$$

$$\begin{aligned}
&s.t. \\
&x_{k+1} = Ax_k + Bu_k,\ k \in \{0,1,\dots,N-1\} \\
&-1 \le u_k \le 1,\quad k \in \{0,1,\dots,N-1\} \\
&[1 \quad 0] x_k \le 0,\ k \in \{1,2,\dots,N\} \\
&x_N \in \mathcal{K}_{15}(0) \\
&x_0 = x(0)
\end{aligned}$$

Example (cont.)

- Alternative approach:
  - Add terminal constraint corresponding to the Maximal Control Invariant Set
- First, let's compute the Maximal Control Invariant Set

- We saw last class that part of a sufficient condition for the finite termination of the Maximal Control Invariant Set computation algorithm is that the state constraint set is bounded
  - So let's define some additional constraints

$$\begin{bmatrix} 1 & 0 \end{bmatrix} x_k \le 0 \quad \Longrightarrow \quad \begin{bmatrix} -10 \\ -2 \end{bmatrix} \le x_k \le \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Example (cont.)

- Alternative approach:
  - Add terminal constraint corresponding to the Maximal Control Invariant Set
- First, let's compute the Maximal Control Invariant Set $\begin{bmatrix} -10 \\ -2 \end{bmatrix} \le x_k \le \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Inputs: $g(x,u)$, $\mathcal{X}$, $\mathcal{U}$

Outputs: $\mathcal{C}_\infty$

$$\Omega_0 \leftarrow \mathcal{X},\ k \leftarrow -1$$

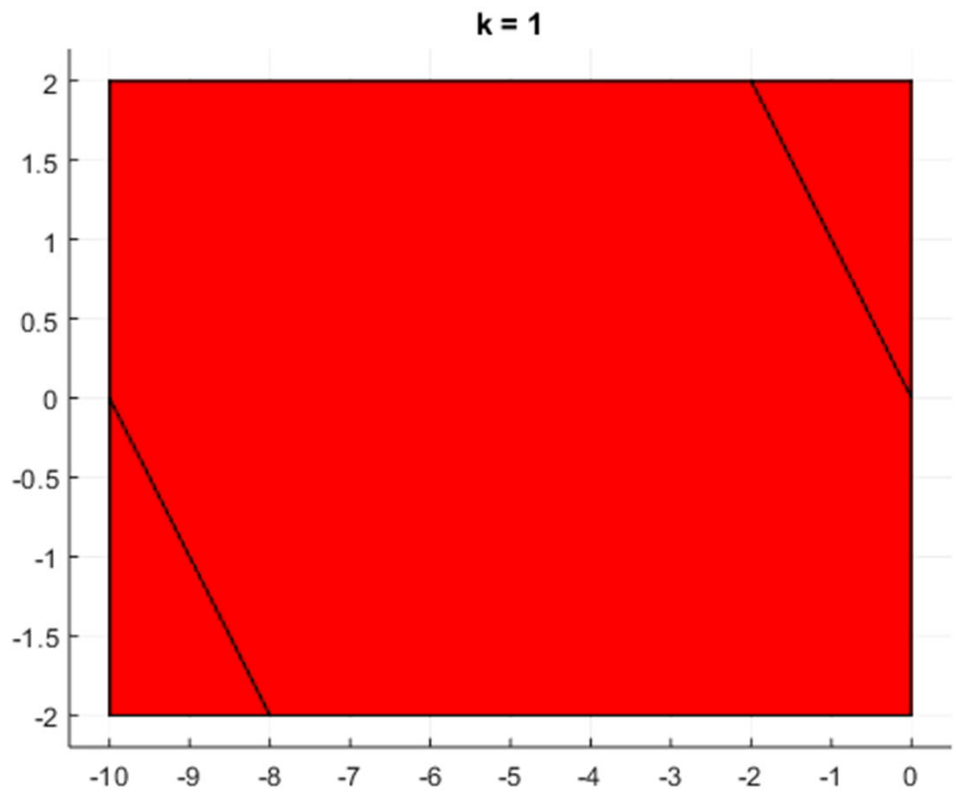
Repeat

$$k \leftarrow k+1$$

$$\Omega_{k+1} \leftarrow \mathrm{Pre}(\Omega_k) \cap \Omega_k$$

Until $\Omega_{k+1} = \Omega_k$

$$C_\infty = \Omega_k$$



Example (cont.)

- Alternative approach:
  - Add terminal constraint corresponding to the Maximal Control Invariant Set
- First, let's compute the Maximal Control Invariant Set

$$\begin{bmatrix}-10\\-2\end{bmatrix} \le x_k \le \begin{bmatrix}0\\2\end{bmatrix}$$

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T x_k + 0.1 u_k^T u_k + x_N^T x_N$$

$$s.t.$$

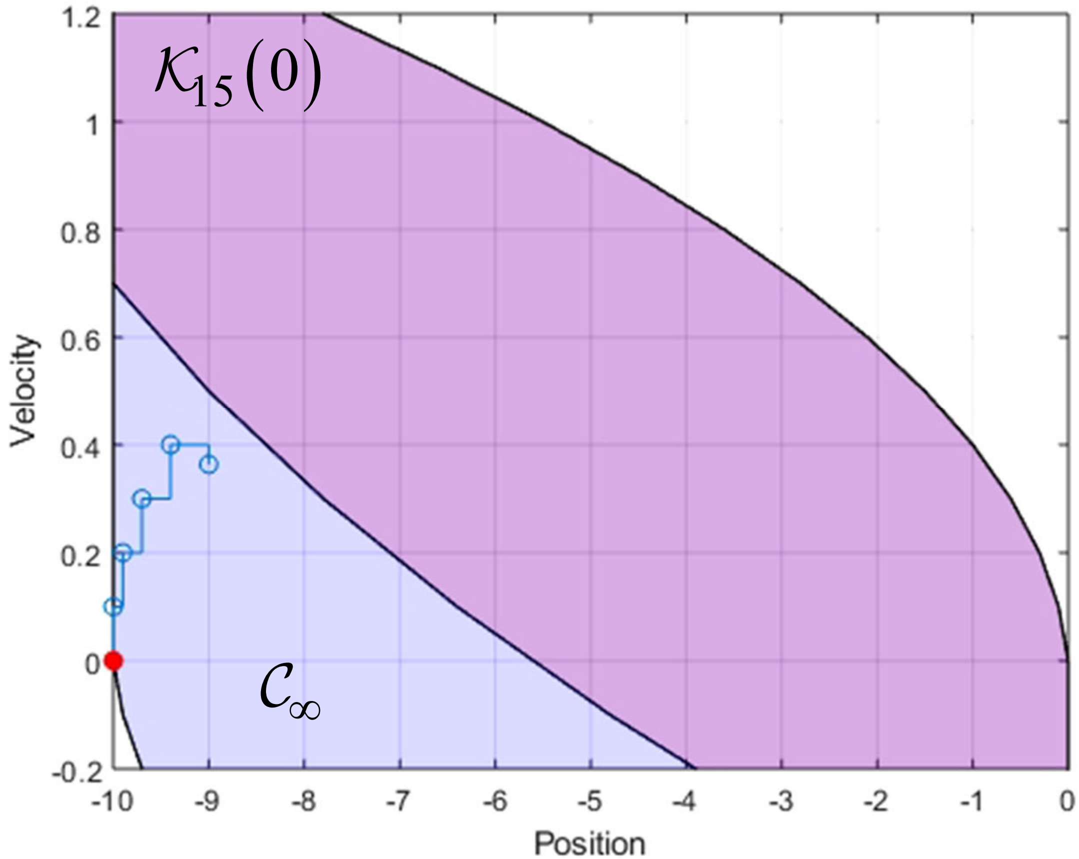
$$x_{k+1} = Ax_k + Bu_k,\ k \in \{0,1,\dots,N-1\}$$

$$-1 \le u_k \le 1, \quad k \in \{0,1,\dots,N-1\}$$

$$\begin{bmatrix}1 & 0\end{bmatrix} x_k \le 0,\ k \in \{1,2,\dots,N\}$$

$$x_N \in \mathcal{C}_\infty$$

$$x_0 = x(0)$$



No longer dependent on specific choice of prediction horizon N or number of steps in N-step stabilizable set $\mathcal{K}_{15}(0)$

Persistent Feasibility

- Now let's apply all of this to MPC
- We have already seen how to compute the set of initial condition for which the optimization problem is feasible

$$x(0) \in \mathcal{X}_0 = \mathcal{K}_N\left(\mathcal{X}_f\right)$$

- Same as N-step controllable set for the given terminal constraint set
- Now, we want to prove:

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k + x_N^T P x_N$$

$$s.t.$$

$$x_{k+1} = Ax_k + Bu_k, \; k \in \{0,1,\ldots,N-1\}$$

$$x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, \quad k \in \{0,1,\ldots,N-1\}$$

$$x_N \in \mathcal{X}_f$$

$$x_0 = x(0)$$

- Persistent Feasibility

  The MPC problem is persistently feasible if for all initial states $x(0) \in \mathcal{X}_0$ feasibility for all future times is guaranteed.

- Recursive Feasibility

  The MPC problem is recursively feasible if feasibility at time step k guarantees feasibility at time step k+1

Persistent Feasibility (cont.)

- We have seen cases where the controller starts of feasible $x(0) \in \mathcal{X}_0$ but then becomes infeasible
  - We want to prevent this!
- First, let's think about some of the various sets we have defined previously, now in the context of MPC

Persistent Feasibility (cont.)

- Maximal Control Invariant set $\mathcal{C}_\infty$
  - Only depends on dynamics and state/input constraint sets

$$x_{k+1} = Ax_k + Bu_k \quad \mathcal{X} \quad \mathcal{U}$$

  - Largest set of state that we can expect *any* controller to work
- Feasible set $\mathcal{X}_0$
  - Depends on dynamics, state/input constraint sets, prediction horizon, and terminal set

$$x_{k+1} = Ax_k + Bu_k \quad \mathcal{X} \quad \mathcal{U} \quad N \quad \mathcal{X}_f$$

  - Does not depend on objective function and has no relation with $\mathcal{C}_\infty$
- Maximal Positive Invariant set $\mathcal{O}_\infty$
  - For closed-loop system under MPC control
  - Depends on everything, including cost function

$$x_{k+1} = Ax_k + Bu_k \quad \mathcal{X} \quad \mathcal{U} \quad N \quad \mathcal{X}_f \quad Q \quad R \quad P$$

  - Subset of feasible set $\mathcal{O}_\infty \subseteq \mathcal{X}_0$
  - Invariant → persistently feasible
  - Subset of maximal control invariant set $\mathcal{O}_\infty \subseteq \mathcal{C}_\infty$
  - Very hard to compute!

Persistent Feasibility (cont.)

- Necessary and sufficient conditions for persistent feasibility
  - Let $\mathcal{O}_\infty$ be the maximal positive invariant set of the closed-loop system under MPC control. The optimization problem is persistently feasible if and only if $\mathcal{X}_0 = \mathcal{O}_\infty$.

- Implication:
  - Since $\mathcal{X}_0$ does not depend on the cost function but $\mathcal{O}_\infty$ does, persistent feasibility is only achieved for some designs of Q, R, P
  - This is one of the reasons it can be hard to tune a controller in a way that guarantees persistent feasibility
  - Which motivates the use of control invariant terminal constraints

Persistent Feasibility (cont.)

- Sufficient condition for persistent feasibility
  - If $\mathcal{X}_f$ is a control invariant set for the constrained system with inputs, then the MPC optimization problem is persistently feasible.

- Outline of Proof:
  - Determine the optimal input trajectory at time step 0 based on the assumed feasibility of the initial condition.
  - Construct a candidate solution at time step 1 based on this optimal input trajectory and the properties of a control invariant terminal set

- Implication:
  - Now persistent feasibility does not depend on the cost function design
  - Can focus on control performance without affecting feasibility

Persistent Feasibility (cont.)

- Alternative condition based on determinedness
  - For any choice of $\mathcal{X}_f$ (no longer necessarily invariant), if the prediction horizon N is greater than the determinedness index $\bar{N}$ of the maximal controllable set $\mathcal{K}_\infty(\mathcal{X}_f)$, then MPC optimization problem is persistently feasible.

- Outline of Proof:
  - The maximal controllable set is a control invariant set
  - If it is finitely determined then $\mathcal{K}_N(\mathcal{X}_f) = \mathcal{K}_\infty(\mathcal{X}_f), \;\; \forall N \geq \bar{N}$
  - The feasible set is $\mathcal{X}_0 = \mathcal{K}_N(\mathcal{X}_f) = \mathcal{K}_\infty(\mathcal{X}_f)$ which is control invariant
- Implication:
  - Choosing a large enough prediction horizon gives you more options in terms of terminal constraints.

Summary

- Using invariant sets allows us to design specific MPC formulations to achieve persistent feasibility
- This ensures that all state and input constraints will be satisfied at all discrete time steps
- We have assumed a perfect model of the system with no disturbances
  - We will soon discuss how to handle uncertainty
- Persistent feasibility does not guarantee that the closed-loop trajectories converge to a desired equilibrium
  - This is where we need to think about designing the cost function to be a Lyapunov function for our system and analyzing to make sure our operational cost is monotonically decreasing
  - See Lecture 7