MECH 6v29.002 – Model Predictive Control

L7 – MPC Theory

Outline

- Stability
  - Lyapunov Stability
  - Closed-loop MPC Stability

Stability

- The closed-loop stability of a system under MPC is a complex function of the
  - Model: A, B, C
  - Control parameters: $N, Q \geq 0, R > 0, P \geq 0$
  - Constraints: $u_{\min}, u_{\max}, y_{\min}, y_{\max}$

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k + x_N^T P x_N$$

$$s.t.$$

$$x_{k+1} = Ax_k + Bu_k, \; k \in \{0,1,\ldots,N-1\}$$

$$u_{\min} \leq u_k \leq u_{\max}, \quad k \in \{0,1,\ldots,N-1\}$$

$$y_{\min} \leq Cx_k \leq y_{\max}, \; k \in \{1,2,\ldots,N\}$$

$$x_0 = x(0)$$

- Multiple approaches to guarantee stability
  - Terminal constraint: $x_{k+N} = 0$
  - Relaxed terminal constraint: $x_{k+N} = \Omega$
  - Terminal cost: $x_N^T P x_N$, P satisfies DARE
  - Contraction constraints: $\|x_{k+1}\| \leq \alpha \|x_k\|, \; \alpha < 1$
- We will explore some of these

Lyapunov Stability

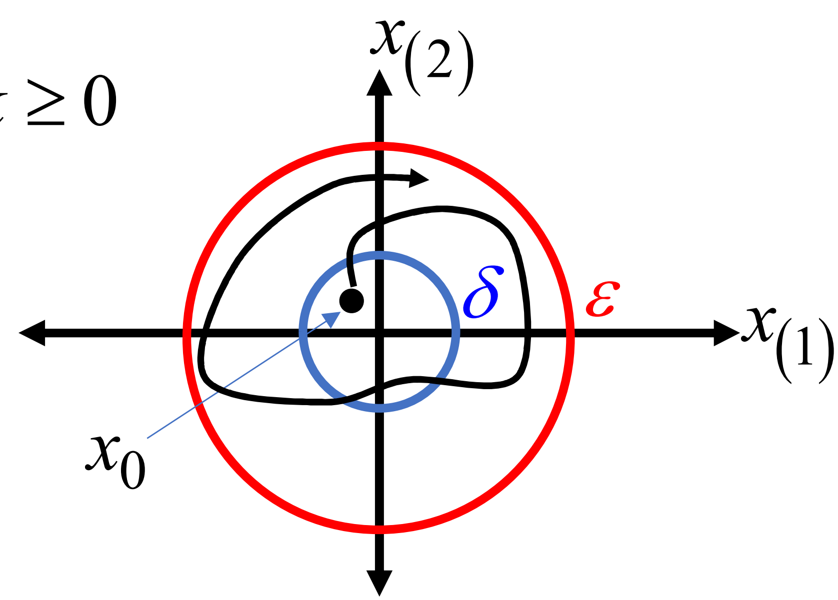
- While there are multiple ways of defining and analyzing stability, we will focus on Lyapunov stability
- Consider the generic, autonomous system

$$x_{k+1} = f\left(x_k\right) \qquad f(0) = 0$$

$x = 0$ is an equilibrium

- The equilibrium is
  - Stable $-$ if $\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0$ s.t.

$$\|x_0\| < \delta \Rightarrow \|x_k\| < \varepsilon,\ \forall k \geq 0$$



  - Unstable - if it is not stable

Lyapunov Stability

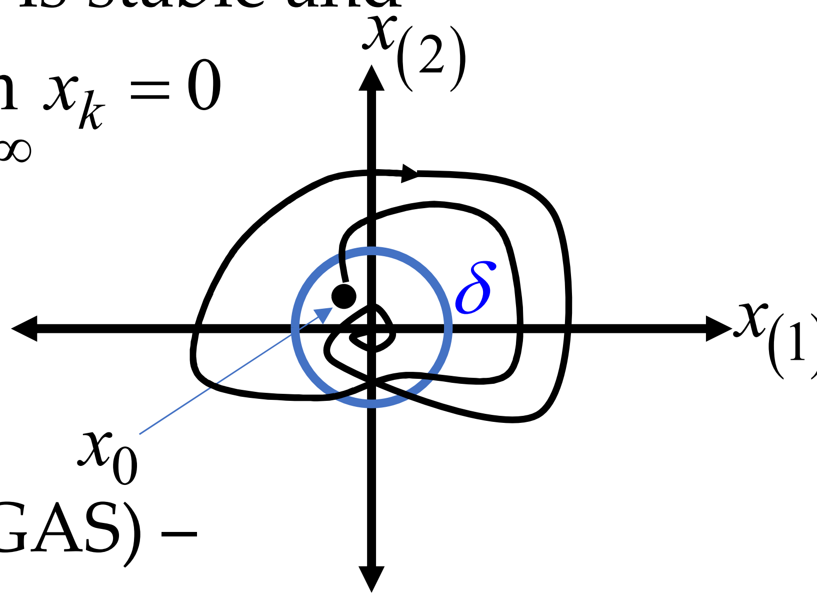
- Consider the generic, autonomous system

$$x_{k+1} = f\left(x_k\right) \qquad\qquad f\left(0\right) = 0$$

$x = 0$ is an equilibrium

- The equilibrium is
  - Asymptotically stable (AS) – if it is stable and

$$\exists \delta \ \text{ s.t. } \ \left\|x_0\right\| < \delta \Rightarrow \lim_{k\to\infty} x_k = 0$$



  - Globally asymptotically stable (GAS) – if it is AS $\forall x_0 \in \mathbb{R}^n$
  - Exponentially stable – if it is stable and $\exists \alpha > 0, \gamma \in \left(0,1\right)$ s.t.

$$\left\|x_0\right\| < \delta \Rightarrow \left\|x_k\right\| \le \alpha \left\|x_0\right\| \gamma^k, \ \forall k \ge 0$$

    - Norm is bounded by an exponentially decaying envelope

Lyapunov Function

- Lyapunov stability is determined based on the existence of a Lyapunov function

- Lyapunov Stability Theorem
  - Consider the equilibrium $x = 0$ of $x_{k+1} = f\left(x_k\right)$.
  - Let $\Omega \in \mathbb{R}^n$ be a closed and bounded set containing the origin.
  - Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function, continuous at the origin, s.t.

1) $V\left(0\right) = 0$
2) $V\left(x\right) > 0, \ \forall x \in \Omega \setminus \{0\}$
3) $V\left(x_{k+1}\right) < V\left(x_k\right), \ \forall x_k \in \Omega \setminus \{0\}$

  - Then $x = 0$ is asymptotically stable in Ω.

- A function $V(x)$ satisfying the above conditions is a Lyapunov function for our system

Lyapunov Stability Details

- Conceptually, it is often useful to think about the Lyapunov function as a measure of energy in the system
  - Energy must be zero at the equilibrium $V(0)=0$
  - Energy must be positive $V(x)>0,\ \forall x \in \Omega \setminus \{0\}$
  - Energy must decrease as the system evolves

$$V(x_{k+1}) < V(x_k),\ \forall x_k \in \Omega \setminus \{0\}$$

- The Lyapunov stability theory is only sufficient (not necessary)
  - just because you can't find a Lyapunov function, doesn't mean the system is not stable

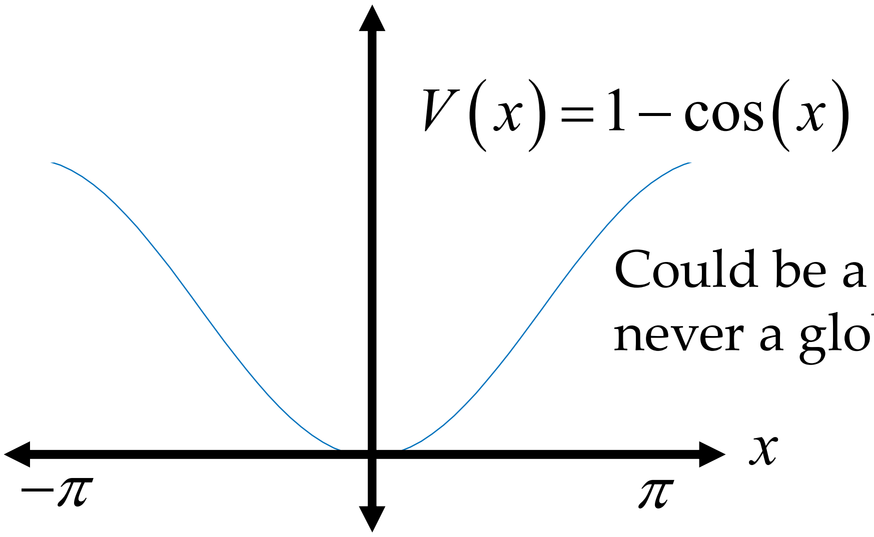
Global Lyapunov Stability

- Global Lyapunov Stability Theorem
  - Consider the equilibrium $x = 0$ of $x_{k+1} = f\left(x_k\right)$.
  - Let $V : \mathbb{R}^n \to \mathbb{R}$ be a function, continuous at the origin, s.t.

1) $V\left(0\right) = 0$
2) $V\left(x\right) > 0, \ \forall x \neq 0$
3) $V\left(x_{k+1}\right) < V\left(x_k\right), \ \forall x_k \neq 0$
4) $\|x\| \to \infty \ \Rightarrow V\left(x\right) \to \infty$ ← Radially unbounded

  - Then $x = 0$ is globally asymptotically stable.

- Example:



Could be a Lyapunov function but never a global Lyapunov function

Lyapunov Stability – Linear Systems

- Stability Theorem
  - A linear system $x_{k+1} = Ax_k$ is globally asymptotically stable in the sense of Lyapunov if and only if all eigenvalues of A are inside the unit circle.
  - Note that stable linear systems are always globally stable

- Proof:
  - Try the candidate quadratic Lyapunov function

$$V(x) = x^T Px, \quad P > 0$$

1) $V(0) = 0$
2) $V(x) > 0, \ \forall x \neq 0$
3) $V(x_{k+1}) < V(x_k), \ \forall x_k \neq 0$
4) $\|x\| \to \infty \ \Rightarrow V(x) \to \infty$

$$V(x_{k+1}) < V(x_k)$$

$$V(x_{k+1}) - V(x_k) < 0$$

$$x_{k+1}^T P x_{k+1} - x_k^T P x_k < 0$$

$$x_k^T A^T P A x_k - x_k^T P x_k < 0$$

$$x_k^T \left(A^T PA - P \right) x_k < 0$$

$$-x_k^T Q x_k < 0 \Rightarrow x_k^T Q x_k > 0$$

Discrete-time Lyapunov equation

$$A^T PA - P = -Q, \quad Q > 0$$

Existence of P is guaranteed for any Q

MPC Stability Proof

- Use Lyapunov Stability to guarantee closed-loop asymptotic stability of MPC with terminal constraints

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} q(x_k, u_k) + p(x_N)$$

$$\begin{aligned} & s.t. \\ & x_{k+1} = f(x_k, u_k), \ k \in \{0,1,\ldots,N-1\} \\ & h(x_k, u_k) \le 0, \quad k \in \{0,1,\ldots,N-1\} \\ & x_N = 0 \\ & x_0 = x(0) \end{aligned}$$

$f(0) = 0$

$x = 0$ is an equilibrium

- Assume feasible at time step 0
- Solve for $u_{0|0}^*$ based on $x(0)$
- Optimal (minimal) cost is $J_0^*(x_0)$
- System evolves to $x(1) = f\left(x(0), u_{0|0}^*\right)$

MPC Stability Proof

- At time step 1, consider the (suboptimal) input trajectory

$$U_1 = \begin{bmatrix} u^*_{1|0} \\ \vdots \\ u^*_{N-1|0} \\ 0 \end{bmatrix}$$

$$J^*_0(x_0) = \min_{U_0} \sum_{k=0}^{N-1} q(x_k, u_k) + p(x_N)$$

$$\begin{aligned} & s.t. \\ & x_{k+1} = f(x_k, u_k),\ k \in \{0,1,\ldots,N-1\} \\ & h(x_k, u_k) \le 0, \quad k \in \{0,1,\ldots,N-1\} \\ & x_N = 0 \\ & x_0 = x(0) \end{aligned}$$

- Cost of this candidate solution is

$$J^*_0(x_0) - q\left(x_0, u^*_{0|0}\right) + \underbrace{q(x_{N+1}, 0)}_{=0}$$

Because of terminal constraint at previous time step

- Since this is a suboptimal solution

$$J^*_1(x_1) \le J^*_0(x_0) - q\left(x_0, u^*_{0|0}\right)$$

- Since the system dynamics and cost function are time invariant

$$J^*_1(x_1) = J^*_0(x_1) \qquad \Rightarrow J^*_0(x_1) \le J^*_0(x_0) - q\left(x_0, u^*_{0|0}\right) \quad \Rightarrow J^*_0(x_1) \le J^*_0(x_0)$$

- Can now show that $J^*_0(x)$ is a Lyapunov function