MECH 6v29.002 – Model Predictive Control

L18 – Nonlinear MPC

Outline

- Schedule
- MPC of a Constrained Nonlinear System
- Origin Terminal Constraint
- Relax the Terminal Constraint
- Nonlinear MPC
- Finding the Sublevel Set

Schedule

10/30	Nonlinear MPC	
11/06	Decentralized and Distributed MPC	HW #4
11/13	Explicit and Hybrid MPC	
11/20	No Lectures (Fall Break)	
11/27	Project Presentations	
12/04	No Lectures (Last Week of classes)	Project Report

- Goal of remaining 3 week of lectures:
  - Reinforce what we have learned so far by expanding to new control problems and MPC formulations
  - Highlight key features and general approaches
  - Provide background on some common topics that may be used in some projects

MPC of a Constrained Nonlinear System

- The goal is to analyze (and guarantee) the closed-loop stability (and feasibility) of MPC for a constrained nonlinear system
- Primarily focus on terminal cost and terminal constraint design (as well as assumptions on types of nonlinearity in dynamics, state and input constraints, and stage cost)

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} q(x_k, u_k) + p(x_N)$$

$$\begin{aligned}
&s.t.\\
&x_{k+1} = f(x_k, u_k),\ k \in \{0,1,\ldots,N-1\}\\
&h(x_k, u_k) \le 0, \quad k \in \{0,1,\ldots,N-1\}\\
&x_N = \mathcal{X}_f\\
&x_0 = x(0)
\end{aligned}$$

$$f(0) = 0$$

$x = 0$ is an equilibrium

Origin Terminal Constraint

- We have already proven closed-loop asymptotic stability of MPC using a terminal constraint ($x_N = 0$)

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} q(x_k, u_k) + p(x_N)$$

$$\begin{aligned}
&s.t.\\
&x_{k+1} = f(x_k, u_k),\ k \in \{0,1,\ldots,N-1\}\\
&h(x_k, u_k) \le 0,\quad k \in \{0,1,\ldots,N-1\}\\
&x_N = 0\\
&x_0 = x(0)
\end{aligned}$$

$$f(0) = 0$$

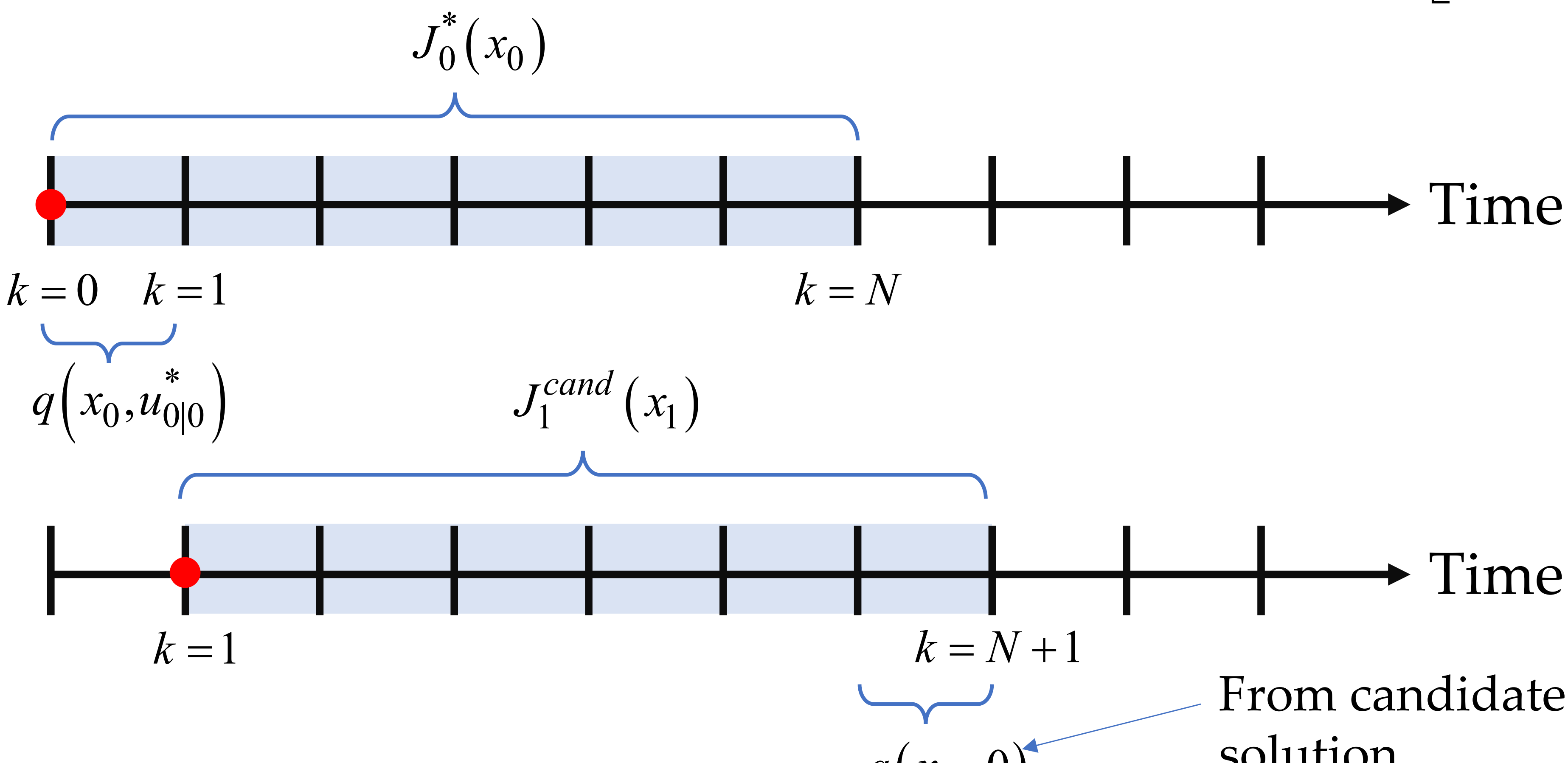
$x = 0$ is an equilibrium

- Assume feasible at time step 0
- Solve for $u_{0|0}^*$ based on $x(0)$
- Optimal (minimal) cost is $J_0^*(x_0)$
- System evolves to $x(1) = f\left(x(0), u_{0|0}^*\right)$

Origin Terminal Constraint (cont.)

- Key ideas:
  - Take solution at $k = 0$, to form a candidate solution at $k = 1$
  - Can analyze the cost of this candidate solution

$$U_1 = \begin{bmatrix} u^*_{1|0} \\ \vdots \\ u^*_{N-1|0} \\ \color{red}{0} \end{bmatrix}$$



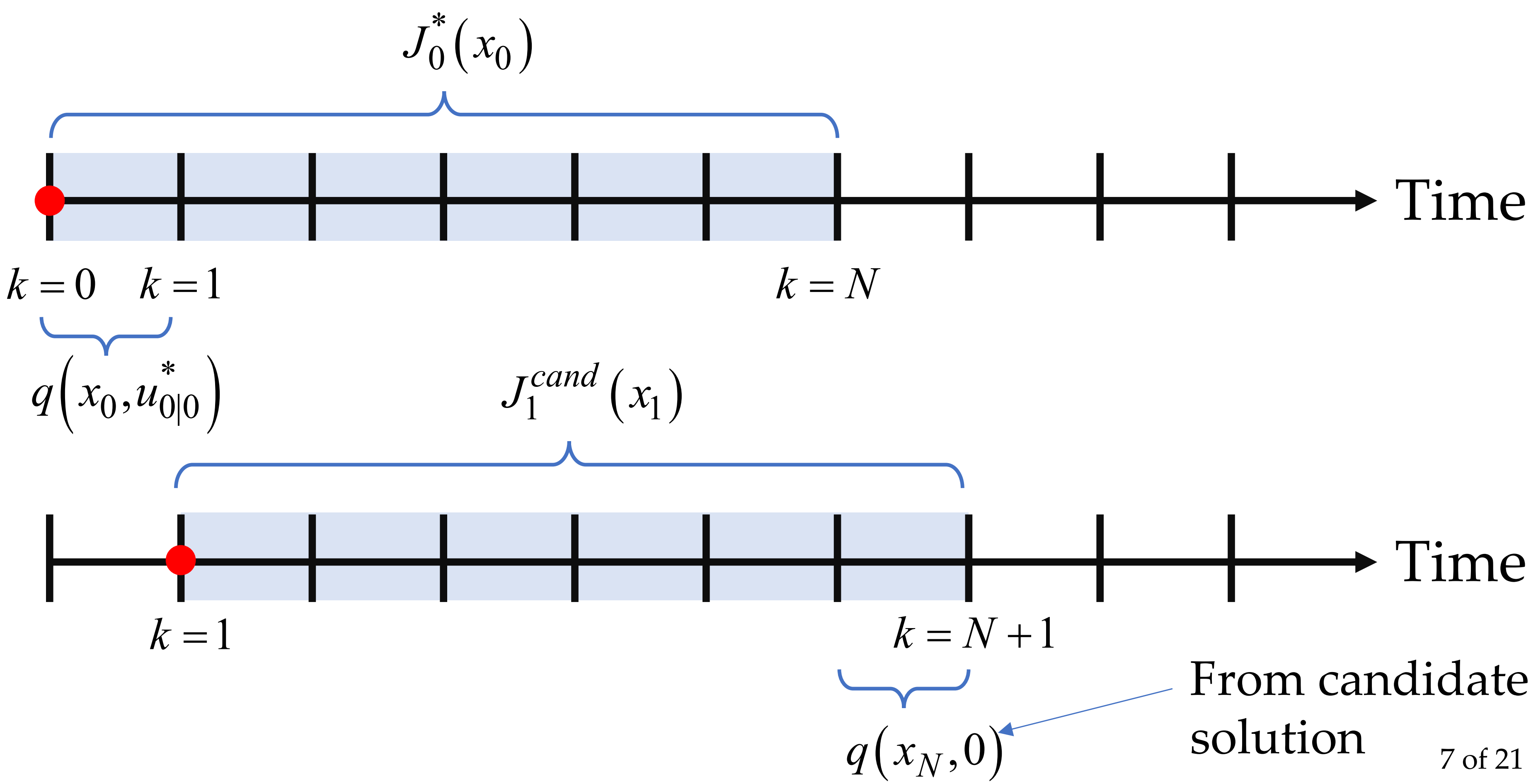
Origin Terminal Constraint (cont.)

- Key ideas (cont.):
  - Cost of candidate solution
  - This is a suboptimal solution

By terminal constraint

$$J_1^{cand}(x_1) = J_0^*(x_0) - q\left(x_0, u_{0|0}^*\right) + \cancel{q(x_N, 0)}$$

$$J_1^*(x_1) \le J_1^{cand}(x_1) = J_0^*(x_0) - q\left(x_0, u_{0|0}^*\right)$$



Origin Terminal Constraint (cont.)

- Key ideas (cont.):

By terminal constraint

  - Cost of candidate solution $J_1^{cand}(x_1) = J_0^*(x_0) - q\left(x_0, u_{0|0}^*\right) + \cancel{q(x_N, 0)}$
  - This is a suboptimal solution

$$J_1^*(x_1) \le J_1^{cand}(x_1) = J_0^*(x_0) - q\left(x_0, u_{0|0}^*\right)$$

  - Since the system dynamics and cost function are time invariant

$$J_1^*(x_1) = J_0^*(x_1) \qquad \Rightarrow J_0^*(x_1) \le J_0^*(x_0) - \underbrace{q\left(x_0, u_{0|0}^*\right)}_{\text{Always positive by design}} \qquad \Rightarrow J_0^*(x_1) \le J_0^*(x_0)$$

  - Can use $J_0^*(x)$ as a Lyapunov function for the closed-loop system

- Lyapunov Stability Theorem

From Lecture 7

  - Consider the equilibrium $x = 0$ of $x_{k+1} = f(x_k)$.
  - Let $\Omega \in \mathbb{R}^n$ be a closed and bounded set containing the origin.
  - Let $V : \mathbb{R}^n \to \mathbb{R}$ be a function, continuous at the origin, s.t.

1) $V(0) = 0$
2) $V(x) > 0, \ \forall x \in \Omega \setminus \{0\}$
3) $V(x_{k+1}) < V(x_k), \ \forall x_k \in \Omega \setminus \{0\}$

  - Then $x = 0$ is asymptotically stable in Ω.

Relax the Terminal Constraint

- We know $x_N = 0$ is one potential way to enforce closed-loop stability
- But we have also seen that this can be restrictive. How?
  - Can restrict the size of the feasible set $\mathcal{X}_0$
    - Set of initial states $x(0)$ for which the optimal control problem is feasible
    - Equivalent to the N-step Controllable Set

$$\mathcal{X}_0 = \left\{ \begin{array}{r} x(0) \in \mathbb{R}^n \mid \exists U_0 \ s.t.\ x_k \in \mathcal{X}, u_k \in \mathcal{U}, \ \forall k = 0,\ldots,N-1 \\ x_N \in \mathcal{X}_f, \ x_{k+1} = f(x_k, u_k), \forall k = 0,\ldots,N-1 \end{array} \right\}$$

- For linear systems, we have seen how to compute the Maximal Control Invariant Set and use this as a terminal constraint to increase the size of the feasible set (Lecture 11)
  - Alternatively, we can use a Maximial Positive Invariant Set for some predetermined candidate feedback control law
  - Use MPC to get to this set and then candidate feedback control law to stay in the set (and asymptotically converge to the origin)
- Can we do something similar for nonlinear systems? How?

Nonlinear MPC

Assumptions:

- Nonlinear dynamics $x_{k+1} = f(x_k, u_k)$
  - $x = 0$, $u = 0$ is an equilibrium
    $$f(0,0) = 0$$
  - $f(.)$ is twice continuously differentiable
- State and input constraints
  $$h(x_k, u_k) \le 0 \quad \Rightarrow x_k \in \mathcal{X},\ u_k \in \mathcal{U}$$
- Stage cost
  $$q(x_k, u_k) = \frac{1}{2}\left(x_k^T Q x_k + u_k^T R u_k\right) = \frac{1}{2}\left(\|x_k\|_Q^2 + \|u_k\|_R^2\right)$$
  $Q, R > 0$
  Positive definite

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} q(x_k, u_k) + p(x_N)$$
$$s.t.$$
$$x_{k+1} = f(x_k, u_k),\ k \in \{0, 1, \ldots, N-1\}$$
$$h(x_k, u_k) \le 0, \quad k \in \{0, 1, \ldots, N-1\}$$
$$x_N = \mathcal{X}_f$$
$$x_0 = x(0)$$

- Design terminal cost and terminal constraint to guarantee asymptotic stability of the origin for the closed-loop system [1-3]
  - We will use a linearization of the nonlinear system about the origin to do this

[1] J. Rawlings, D. Mayne, M. Diehl. “Model Predictive Control: Theory, Computation, and Design,” Nob Hill Publishing, 2nd Edition, 2019.

[2] P. Scokaert, D. Mayne, J. Rawlings. “Suboptimal Model Predictive Control,” IEEE TAC, 40(3), 1999.

[3] James Rawlings. “Tutorial: Model Predictive Control Technology,” ACC, 1999.

Nonlinear MPC (cont.)

- Linearization of nonlinear system about the origin

$$x_{k+1} = f(x_k, u_k) \quad \Rightarrow \quad x_{k+1} = Ax_k + Bu_k$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x,u=0} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{x,u=0}$$

- Assume linear system is stabilizable by a static feedback control law

$$u_k = Kx_k$$

  - Closed-loop linear system is globally exponentially stable

$$x_{k+1} = (A + BK)x_k \qquad x_{k+1} = A_K x_k \qquad A_K = A + BK$$

- Under this static feedback control law, stage cost becomes

$$q(x_k, u_k) = \frac{1}{2}\left(x_k^T Q x_k + u_k^T R u_k\right) \quad \Rightarrow \quad q(x_k, Kx_k) = \frac{1}{2}\left(x_k^T Q x_k + x_k^T K^T R K x_k\right)$$

$$q(x_k, Kx_k) = \frac{1}{2} x_k^T Q_K x_k$$

$$Q_K = Q + K^T R K$$

Nonlinear MPC (cont.)

- Define the matrix P that satisfies the Lyapunov equation

$$A_K^T P A_K - P + \mu Q_K = 0 \qquad \mu > 1 \leftarrow \text{We will see why later}$$

- Since Q_K is positive definite and A_K is stable, P is positive definite (also symmetric)
- This is how we will define our terminal cost function

$$p(x_N) = \frac{1}{2} x_N^T P x_N$$

- This terminal cost function is a global Control Lyapunov Function (CLF) for the linear system

$$x_{k+1} = Ax_k + Bu_k \qquad u_k = Kx_k \qquad x_{k+1} = A_K x_k$$

- From this, we know that this cost function decreases along solutions of the closed-loop linear system (use Lyapunov equation)

$$A_K^T P A_K - P + \mu Q_K = 0 \Rightarrow \underbrace{x^T A_K^T P A_K x}_{2p(A_K x)} - \underbrace{x^T P x}_{2p(x)} + \mu \underbrace{x^T Q_K x}_{>0} = 0$$

$$p(A_K x) = p(x) - \frac{\mu}{2} x^T Q_K x \Leftarrow p(A_K x) - p(x) + \frac{\mu}{2} x^T Q_K x = 0$$

$p(A_K x) < p(x) \leftarrow$ Cost decreases along closed-loop state trajectory

Nonlinear MPC (cont.)

- Now consider the nonlinear system with this same static feedback control law

$$x_{k+1} = f\left(x_k, u_k\right) \qquad u_k = Kx_k$$

$$x_{k+1} = f\left(x_k, Kx_k\right)$$

- Want to show that our choice of terminal cost function is also a Control Lyapunov Function for the nonlinear system, but in a restricted neighborhood of the origin (where the linear model is a good approximation)
- For the linear system, globally, we had

$$p\left(A_K x\right) - p\left(x\right) + \frac{\mu}{2} x^T Q_K x = 0 \quad \Rightarrow \quad p\left(A_K x\right) < p\left(x\right)$$

- Now, locally, we would like

$$p\left(f\left(x, Kx\right)\right) - p\left(x\right) + \frac{1}{2} x^T Q_K x \le 0 \qquad \forall x \in \mathrm{lev}_a\, p\left(x\right)$$

Sublevel set of the terminal cost function

- Sublevel set of a function

$$\mathrm{lev}_a\, p\left(x\right) \triangleq \left\{x \in \mathbb{R}^n \mid p\left(x\right) \le a\right\} \qquad p\left(x\right) = \frac{1}{2} x^T P x$$

  - Since P is positive definite, sublevel set is an ellipsoid centered at the origin

Nonlinear MPC (cont.)

- We have (1) $p(A_K x) - p(x) + \frac{\mu}{2} x^T Q_K x = 0 \quad \forall x \in \mathbb{R}^n$
- We would like (2) $p(f(x, Kx)) - p(x) + \frac{1}{2} x^T Q_K x \leq 0 \quad \forall x \in \mathrm{lev}_a\, p(x)$
- Assume difference in cost is "small", we have (3)

$$p(f(x, Kx)) - p(A_K x) \leq \frac{\mu - 1}{2} x^T Q_K x \quad \forall x \in \mathrm{lev}_a\, p(x)$$

- Then, start with (3) $p(f(x, Kx)) - p(A_K x) \leq \frac{\mu - 1}{2} x^T Q_K x$
- Plug in (1) $p(f(x, Kx)) - \left(p(x) - \frac{\mu}{2} x^T Q_K x \right) \leq \frac{\mu - 1}{2} x^T Q_K x$
- Rearrange to get (2) $p(f(x, Kx)) - p(x) + \frac{1}{2} x^T Q_K x \leq 0$
- Key now is to show that there exists $a > 0$ such that (3) is true

Nonlinear MPC (cont.)

- For now, assume that there exists $a > 0$ such that (3) is true
- Summary:

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} q(x_k, u_k) + p(x_N)$$

$$p(x_N) = \frac{1}{2} x_N^T P x_N$$

$$A_K^T P A_K - P + \mu Q_K = 0$$

$$\begin{aligned} s.t.& \\ & x_{k+1} = f(x_k, u_k),\ k \in \{0, 1, \ldots, N-1\} \\ & h(x_k, u_k) \le 0, \quad k \in \{0, 1, \ldots, N-1\} \\ & x_N = \mathcal{X}_f \\ & x_0 = x(0) \end{aligned}$$

$$\mathcal{X}_f = \operatorname{lev}_a p(x_N) = \{x_N \in \mathbb{R}^n \mid p(x_N) \le a\} = \left\{x_N \in \mathbb{R}^n \mid \frac{1}{2} x_N^T P x_N \le a\right\}$$

Ellipsoid centered at the origin

- Assuming initial condition is feasible, the origin is exponentially stable under this MPC controller formulation

Finding the Sublevel Set

- How do we find $a > 0$ to define the terminal constraint such that the following is true?

$$p\left(f\left(x, Kx\right)\right) - p\left(x\right) + \frac{1}{2} x^T Q_K x \le 0 \quad \forall x \in \mathcal{X}_f = \mathrm{lev}_a\, p\left(x_N\right) = \left\{x_N \in \mathbb{R}^n \mid p\left(x_N\right) \le a\right\}$$

- Could take a direct approach by solving the following nonlinear program.
- Choose $a > 0$ such that

$$\max_{x \in \mathrm{lev}_a\, p(x_N)} p\left(f\left(x, Kx\right)\right) - p\left(x\right) + \frac{1}{2} x^T Q_K x \le 0$$

- Would likely want to maximize terminal set by maximizing a

$$\max_{a>0} \left(\max_{x \in \mathrm{lev}_a\, p(x_N)} p\left(f\left(x, Kx\right)\right) - p\left(x\right) + \frac{1}{2} x^T Q_K x \le 0 \right)$$

$$\begin{aligned} s.t. \quad & \mathrm{lev}_a\, p\left(x_N\right) \subseteq \mathcal{X} \\ & K\ \mathrm{lev}_a\, p\left(x_N\right) \subseteq \mathcal{U} \end{aligned}$$

Could be hard to solve directly.

Finding the Sublevel Set (cont.)

- Alternatively, we could try to analytically find $a > 0$ such that

$$p\left(f\left(x, Kx\right)\right) - p\left(x\right) + \frac{1}{2}x^T Q_K x \leq 0 \quad \forall x \in \mathcal{X}_f = \text{lev}_a\, p\left(x_N\right) = \left\{x_N \in \mathbb{R}^n \mid p\left(x_N\right) \leq a\right\}$$

- We have already seen that this is achieved if $\left(p\left(x\right) = \frac{1}{2}x^T P x\right)$

$$\Delta p \triangleq p\left(f\left(x, Kx\right)\right) - p\left(A_K x\right) \leq \frac{\mu - 1}{2}x^T Q_K x \quad \forall x \in \text{lev}_a\, p\left(x\right)$$

- To find when this is true, define the error between the nonlinear and linear models $e\left(x\right) = f\left(x, Kx\right) - A_K x$
- Algebraic manipulations

$$p\left(f\left(x, Kx\right)\right) = p\left(e\left(x\right) - A_k x\right) = \frac{1}{2}\left(e\left(x\right) - A_k x\right)^T P\left(e\left(x\right) - A_k x\right)$$

$$= \frac{1}{2}e\left(x\right)^T Pe\left(x\right) + \frac{1}{2}x^T A_K^T P A_K x - x^T A_K^T Pe\left(x\right)$$

$$p\left(A_K x\right) = \frac{1}{2}x^T A_K^T P A_K x$$

$$\Delta p \triangleq p\left(f\left(x, Kx\right)\right) - p\left(A_K x\right) = \frac{1}{2}e\left(x\right)^T Pe\left(x\right) - x^T A_K^T Pe\left(x\right)$$

Finding the Sublevel Set (cont.)

- We need to find $a > 0$ such that

$$\Delta p = \frac{1}{2} e(x)^T P e(x) - x^T A_K^T P e(x) \leq \frac{\mu - 1}{2} x^T Q_K x \qquad \forall x \in \text{lev}_a\, p(x)$$

- Goal is to write every term as a function of $\|x\|_2$
- Note that

$$e(x)^T P e(x) \leq \lambda_{\max}(P) \|e(x)\|_2^2 \qquad\qquad \lambda_{\min}(Q_K) \|x\|_2^2 \leq x^T Q_K x$$

- Then

$$\Delta p = \frac{1}{2} e(x)^T P e(x) - x^T A_K^T P e(x)$$

$$\Delta p \leq \frac{1}{2} \lambda_{\max}(P) \|e(x)\|_2^2 - x^T A_K^T P e(x)$$

- Note that $-x^T A_K^T P e(x) \leq \|P A_K x\|_2 \|e(x)\|_2 \leq \|P A_K\|_2 \|x\|_2 \|e(x)\|_2$
- Therefore

$$\Delta p \leq \frac{1}{2} \lambda_{\max}(P) \|e(x)\|_2^2 + \|P A_K\|_2 \|x\|_2 \|e(x)\|_2$$

Finding the Sublevel Set (cont.)

- We have $$\Delta p \leq \frac{1}{2}\lambda_{\max}(P)\|e(x)\|_2^2 + \|PA_K\|_2 \|x\|_2 \|e(x)\|_2$$
- Let's assume the error is small (we will prove this later)

$$\|e(x)\|_2 \leq \frac{1}{2}c_\delta \|x\|_2^2 \qquad \forall x \in \delta\mathcal{B} = \left\{ x \in \mathbb{R}^n \mid x^T x = \|x\|_2^2 \leq \delta^2 \right\}$$

- Then $\Delta p \leq \frac{1}{8}c_\delta^2 \lambda_{\max}(P)\|x\|_2^4 + \frac{1}{2}c_\delta \|PA_K\|_2 \|x\|_2^3$
- Since we want $\Delta p \leq \frac{\mu-1}{2}x^T Q_K x$
- And we have

$$\Delta p \leq \frac{1}{8}c_\delta^2 \lambda_{\max}(P)\|x\|_2^4 + \frac{1}{2}c_\delta \|PA_K\|_2 \|x\|_2^3 \qquad \lambda_{\min}(Q_K)\|x\|_2^2 \leq x^T Q_K x$$

- We need

$$\frac{1}{8}c_\delta^2 \lambda_{\max}(P)\|x\|_2^4 + \frac{1}{2}c_\delta \|PA_K\|_2 \|x\|_2^3 \leq \frac{\mu-1}{2}\lambda_{\min}(Q_K)\|x\|_2^2$$

Finding the Sublevel Set (cont.)

- We need

$$\frac{1}{8}c_\delta^2\lambda_{\max}(P)\|x\|_2^4+\frac{1}{2}c_\delta\|PA_K\|_2\|x\|_2^3\le\frac{\mu-1}{2}\lambda_{\min}(Q_K)\|x\|_2^2$$

- Since $\|x\|_2>0\ \ \forall x\neq 0$

$$\frac{1}{8}c_\delta^2\lambda_{\max}(P)\|x\|_2^2+\frac{1}{2}c_\delta\|PA_K\|_2\|x\|_2\le\frac{\mu-1}{2}\lambda_{\min}(Q_K)$$

- We currently have the condition that $x\in\delta\mathcal{B}=\left\{x\in\mathbb{R}^n \mid x^Tx=\|x\|_2^2\le\delta^2\right\}$
- We want to impose the condition

$$x\in\mathcal{X}_f=\operatorname{lev}_a p(x)=\left\{x\in\mathbb{R}^n \mid p(x)\le a\right\}=\left\{x\in\mathbb{R}^n \mid \frac{1}{2}x^TPx\le a\right\}$$

- Since $\frac{1}{2}\lambda_{\min}(P)\|x\|_2^2\le\frac{1}{2}x^TPx$

$$x\in\mathcal{X}_f\Rightarrow\frac{1}{2}\lambda_{\min}(P)\|x\|_2^2\le a\ \ \Rightarrow\|x\|_2\le\sqrt{\frac{2a}{\lambda_{\min}(P)}}$$

Finding the Sublevel Set (cont.)

- Now we can finally choose $a > 0$ such that x being in the terminal constraint set results in

$$\|x\|_2 \leq \sqrt{\frac{2a}{\lambda_{\min}(P)}} \leq \delta$$

Since $x \in \delta\mathcal{B} = \left\{ x \in \mathbb{R}^n \mid x^T x = \|x\|_2^2 \leq \delta^2 \right\}$ is required for the linearization error to be small

- and $\frac{1}{8} c_\delta^2 \lambda_{\max}(P) \|x\|_2^2 + \frac{1}{2} c_\delta \|PA_K\|_2 \|x\|_2 \leq \frac{\mu - 1}{2} \lambda_{\min}(Q_K)$
- which results in the following condition

$$\frac{1}{8} c_\delta^2 \lambda_{\max}(P) \frac{2a}{\lambda_{\min}(P)} + \frac{1}{2} c_\delta \|PA_K\|_2 \sqrt{\frac{2a}{\lambda_{\min}(P)}} \leq \frac{\mu - 1}{2} \lambda_{\min}(Q_K)$$

To be continued…