MECH 6v29.002 – Model Predictive Control

L8 – MPC Feasibility

Outline

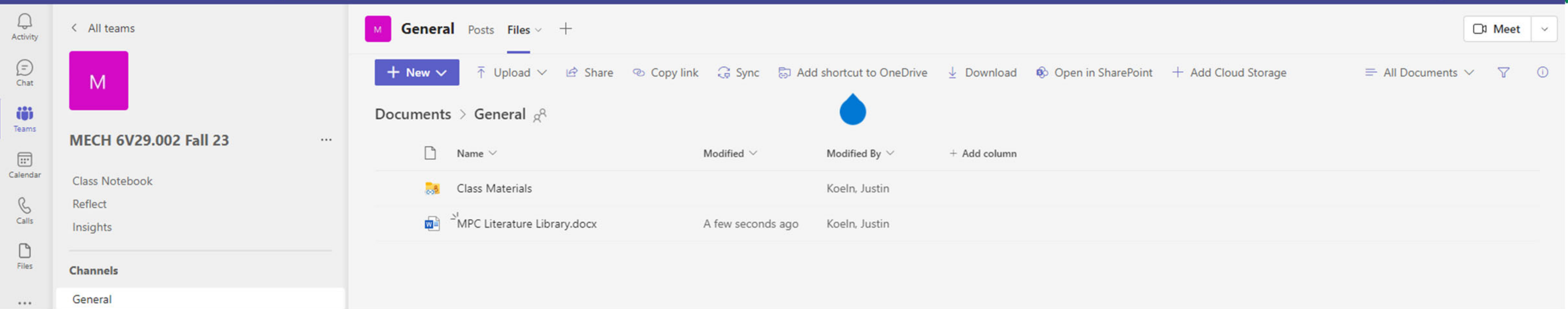
- Literature Review (Participation Assignment)
- Feasibility Analysis
  - Controllable and Reachable Sets
  - Precursor Set
  - Successor Set
  - *N*-step Controllable Set
  - *N*-step Reachable Set
  - Examples
- HW #2

Literature Review

- We should have a list of important MPC research papers to serve as supplementary material for the class
- Let's create this library of papers together
- Participation Assignment (Due Sept. 22)
    - Identify 1 (or more) seminal works of theoretical MPC development in 1 (or more) of the following areas
    - Identify 1 (or more) papers that reference this seminal work that has an interesting application example
    - Provide the references to these papers to the shared file on MS Teams
    - This will help prepare you for the class project where you will choose a specific MPC approach to study further and demonstrate through numerical simulation
    - First come, first served
        - Put your name in the topic area to which you plan to contribute
        - If more than one person wants to contribute to a single topic area, work together so that you do not provide the same references

Linear MPC
Nonlinear MPC
Robust MPC
Stochastic MPC
Decentralized MPC
Distributed MPC
Hierarchical MPC
Economic MPC
Hybrid MPC
Explicit MPC
Solvers for MPC
Other

Literature Review



MPC Literature Library

Use IEEE citation format: https://pitt.libguides.com/citationhelp/ieee

Textbooks:

Added by Justin Koeln

- J. B. Rawlings, D. Q. Mayne, and M. M. Diehl. *Model Predictive Control: Theory, Computation, and Design*. 2nd Edition. Nob Hill Publishing, 2019.
- F. Borrelli, A. Bemporad, and M. Morari. *Predictive Control for Linear and Hybrid Systems*. Cambridge University Press, 2017.

Nonlinear MPC:

Added by Justin Koeln

- Seminal – D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, pp. 789-814, 2000.
  - Application – P. Falcone, F. Borrelli, J. Asgari, H. E. Tseng, and D. Hrovat, "Predictive Active Steering Control for Autonomous Vehicle Systems," *IEEE Transactions on Control Systems Technology*, vol. 15, no. 3, pp. 566-580, 2007.

Constrained Linear Quadratic MPC

- For MPC with input and state/output constraints, it is important to analyze the feasibility of the optimization problem
  - Specifically, what is the set of initial states for which the constrained MPC problem is feasible?

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k + x_N^T P x_N$$

$$s.t.$$

$$x_{k+1} = Ax_k + Bu_k, \ k \in \{0,1,\dots,N-1\}$$

$$x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, \quad k \in \{0,1,\dots,N-1\}$$

$$x_N \in \mathcal{X}_f$$

$$x_0 = x(0)$$

- Note, feasibility is independent of cost function (just based on constraints)
- First we need to study controllable and reachable sets

Controllable and Reachable Sets

- Consider the autonomous systems

Nonlinear

$$x_{k+1} = g(x_k)$$

Linear

$$x_{k+1} = Ax_k$$

- And the systems with external inputs

Nonlinear

$$x_{k+1} = g(x_k, u_k)$$

Linear

$$x_{k+1} = Ax_k + Bu_k$$

- Each system is subject to state and input constraints at each discrete point in time

$$x_k \in \mathcal{X}, \;\; u_k \in \mathcal{U}, \quad \forall k \geq 0$$

- We are interested in quantifying (as a set) where the systems can go (both forward and backward) in time while satisfying these constraints

Constraint Sets

- We will assume the constraints are defined as convex polyhedra

$$x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, \quad \forall k \geq 0$$

- Convex Polyhedron:
  - Defined as the set of solutions to a system of linear inequalities
  - Most naturally represented in Halfspace-Representation (H-Rep)

$$\mathcal{X} = \left\{ x \in \mathbb{R}^n \mid H_x x \leq f_x \right\} \qquad H_x \in \mathbb{R}^{h_x \times n} \qquad f_x \in \mathbb{R}^{h_x}$$

$$\mathcal{U} = \left\{ u \in \mathbb{R}^m \mid H_u u \leq f_u \right\}$$

h_x = # of halfspaces (inequalities)

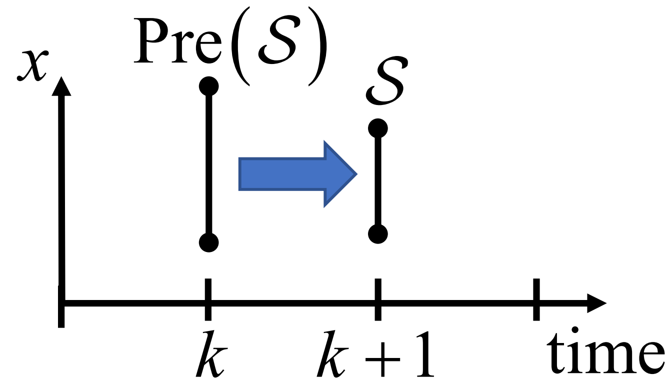
- Convex Polytope:
  - A bounded polyhedron
- Other representations can also be used
  - Vertex-Representation (V-Rep)
  - Constrained Zonotopes (CG-Rep)

Precursor Sets

- The precursor set to the set S is the set of states which evolve into the target set S in one discrete time step
- For the autonomous systems, the precursor set is defined as

$$\operatorname{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid x_{k+1} = g(x_k) \in \mathcal{S} \right\}$$

$$\operatorname{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid x_{k+1} = Ax_k \in \mathcal{S} \right\}$$



- For the systems in inputs, the precursor set is defined as

$$\operatorname{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid \exists u_k \in \mathcal{U} \ \ s.t. \ x_{k+1} = g(x_k, u_k) \in \mathcal{S} \right\}$$

$$\operatorname{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid \exists u_k \in \mathcal{U} \ \ s.t. \ x_{k+1} = Ax_k + Bu_k \in \mathcal{S} \right\}$$

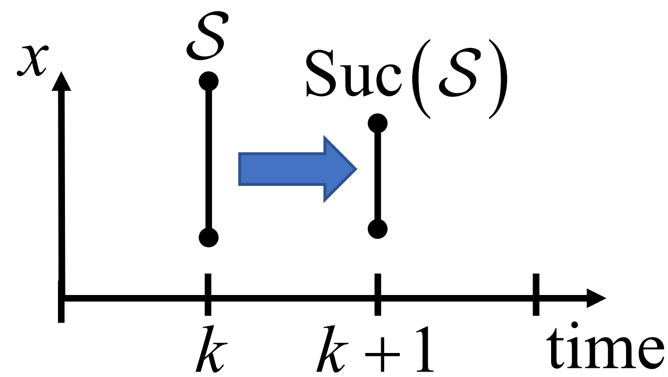
- Also called the one-step backward-reachable set

Successor Sets

- The successor set to the set S is the set of states that can be reached from S in one discrete-time step.
- For the autonomous systems, the successor set is defined as

$$\mathrm{Suc}(\mathcal{S}) = \left\{ x_{k+1} \in \mathbb{R}^n \mid \exists x_k \in \mathcal{S} \ s.t. \ x_{k+1} = g(x_k) \right\}$$

$$\mathrm{Suc}(\mathcal{S}) = \left\{ x_{k+1} \in \mathbb{R}^n \mid \exists x_k \in \mathcal{S} \ s.t. \ x_{k+1} = A x_k \right\}$$



- For the systems in inputs, the successor set is defined as

$$\mathrm{Suc}(\mathcal{S}) = \left\{ x_{k+1} \in \mathbb{R}^n \mid \exists x_k \in \mathcal{S}, \exists u_k \in \mathcal{U} \ s.t. \ x_{k+1} = g(x_k, u_k) \right\}$$

$$\mathrm{Suc}(\mathcal{S}) = \left\{ x_{k+1} \in \mathbb{R}^n \mid \exists x_k \in \mathcal{S}, \exists u_k \in \mathcal{U} \ s.t. \ x_{k+1} = A x_k + B u_k \right\}$$

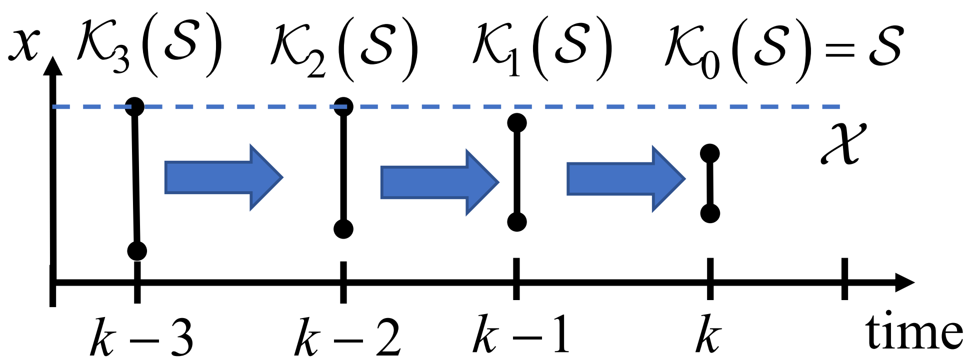
- Also called the one-step forward-reachable set

N-step Controllable Set

- The idea of the precursor set can be applied iteratively to determine the set of states which can evolve into the target set S in N discrete time step
- For a given target set $\mathcal{S} \subseteq \mathcal{X}$, the N-step Controllable Set $\mathcal{K}_N(\mathcal{S})$ for a discrete-time dynamic system subject to input and state constraints is defined recursively as

$$\mathcal{K}_0(\mathcal{S}) = \mathcal{S}$$

$$\mathcal{K}_j(\mathcal{S}) = \mathrm{Pre}\left(\mathcal{K}_{j-1}(\mathcal{S})\right) \cap \mathcal{X}, \quad j \in \{1, \ldots, N\}$$



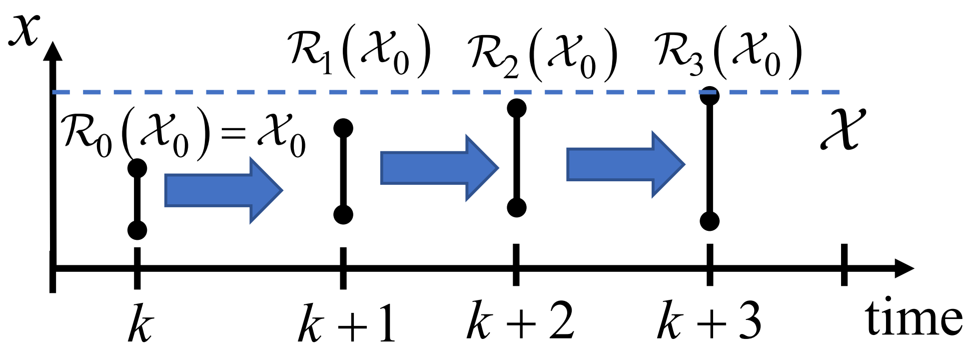
- For autonomous systems, all states in the N-step Controllable Set will evolve to the target set in N steps, while satisfying all state constraints
- For system with inputs, all states in the N-step Controllable Set can be driven to the target set in N steps, while satisfying all state constraints, with a suitable control sequence that satisfies all input constraints

N-step Reachable Set

- The idea of the successor set can be applied iteratively to determine the set of states that can be reached from S in N discrete-time steps
- For a given initial set $\mathcal{X}_0 \subseteq \mathcal{X}$, the N-step Reachable Set $\mathcal{R}_N(\mathcal{X}_0)$ for a discrete-time dynamic system subject to input and state constraints is defined recursively as

$$\mathcal{R}_0(\mathcal{X}_0) = \mathcal{X}_0$$

$$\mathcal{R}_{j+1}(\mathcal{X}_0) = \mathrm{Suc}\big(\mathcal{R}_j(\mathcal{X}_0)\big) \cap \mathcal{X}, \quad j \in \{0, \ldots, N-1\}$$



- For autonomous systems, all states in the initial set will evolve to the N-step Controllable Set in N steps, while satisfying all state constraints
- For system with inputs, all states in the initial set will evolve to the N-step Controllable Set in N steps, while satisfying all state constraints, with a suitable control sequence that satisfies all input constraints

Precursor Example

- Consider the autonomous stable 2nd order system

$$x_{k+1} = Ax_k = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x_k$$

- Subject to state constraints (box constraints, upper- and lower-bounds)

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}$$

- Collect constraints in H-Rep

$$\underline{x} \le x \le \overline{x} \quad \Rightarrow \begin{bmatrix} I \\ -I \end{bmatrix} x \le \begin{bmatrix} \overline{x} \\ -\underline{x} \end{bmatrix}$$

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid H_x x \le f_x \right\}$$

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} x \le \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\}$$

Precursor Example (cont.)

- Consider the autonomous stable 2nd order system

$$x_{k+1} = Ax_k = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x_k$$

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid H_x x \le f_x \right\}$$

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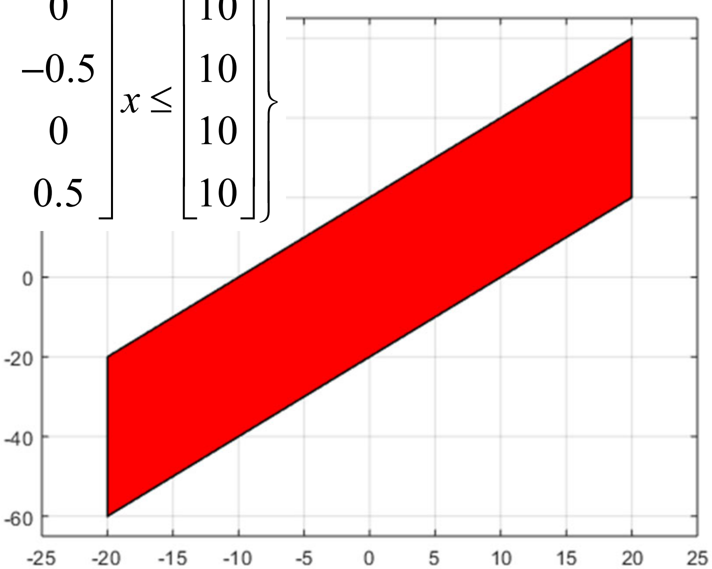
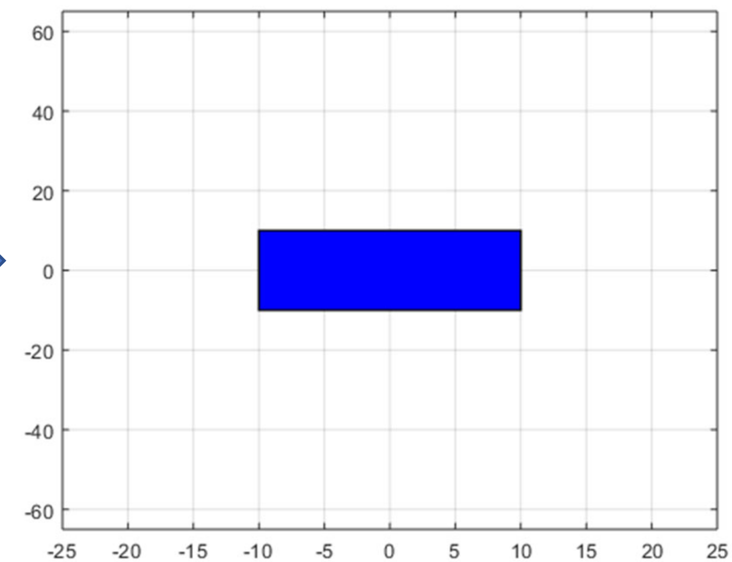
- Compute the precursor set

$$\mathrm{Pre}(\mathcal{X}) = \left\{ x_k \in \mathbb{R}^2 \mid x_{k+1} = Ax_k \in \mathcal{X} \right\}$$

$$x_{k+1} = Ax_k \in \mathcal{X} \;\Rightarrow H_x A x_k \le f_x \;\Rightarrow \mathrm{Pre}(\mathcal{X}) = \left\{ x_k \in \mathbb{R}^2 \mid H_x A x_k \le f_x \right\}$$

$$\mathrm{Pre}(\mathcal{X}) = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \\ -0.5 & 0 \\ -1 & 0.5 \end{bmatrix} x \le \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\}$$





Precursor Example (cont.)

- Consider the autonomous stable 2nd order system

$$x_{k+1} = Ax_k = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x_k$$

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid H_x x \le f_x \right\}$$

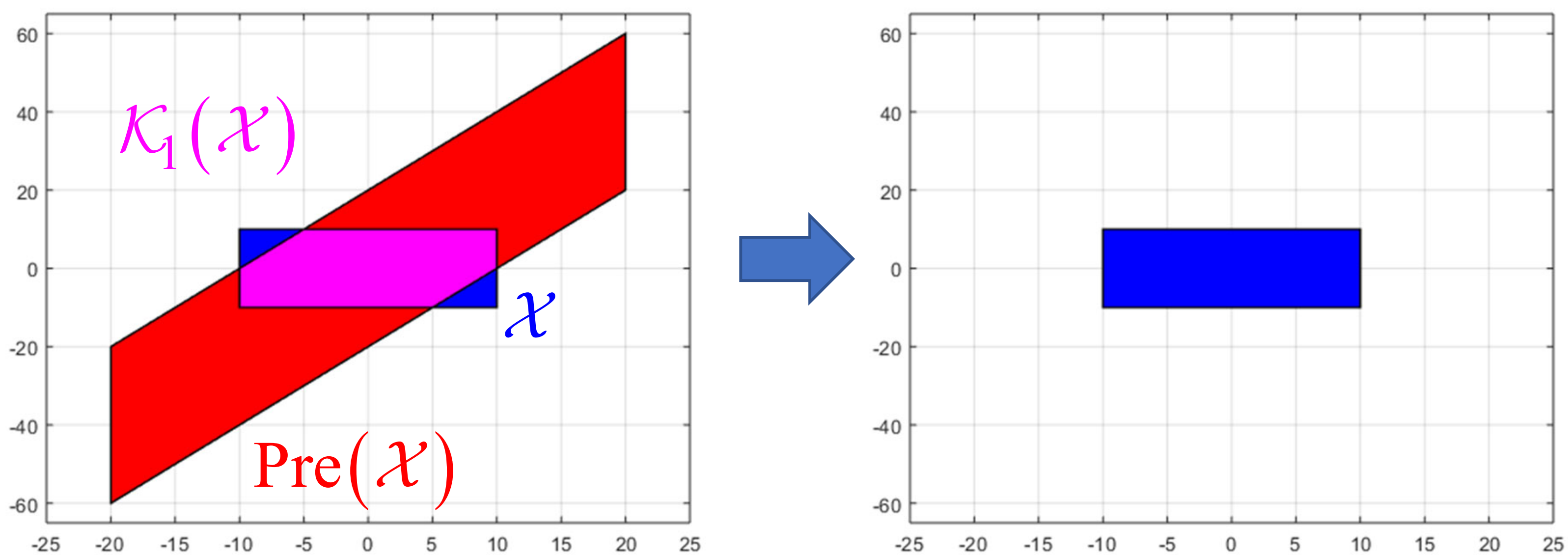
$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} x \le \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\}$$

- Compute the 1 step controllable set

$$\mathcal{K}_1(\mathcal{X}) = \mathrm{Pre}(\mathcal{X}) \cap \mathcal{X}$$

$$\mathcal{K}_1(\mathcal{X}) = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} H_x A \\ H_x \end{bmatrix} x \le \begin{bmatrix} f_x \\ f_x \end{bmatrix} \right\}$$

In H-Rep, intersection is computed by concatenating inequalities



Precursor Example (cont.)

- Consider the autonomous stable 2nd order system

$$x_{k+1} = Ax_k = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x_k$$

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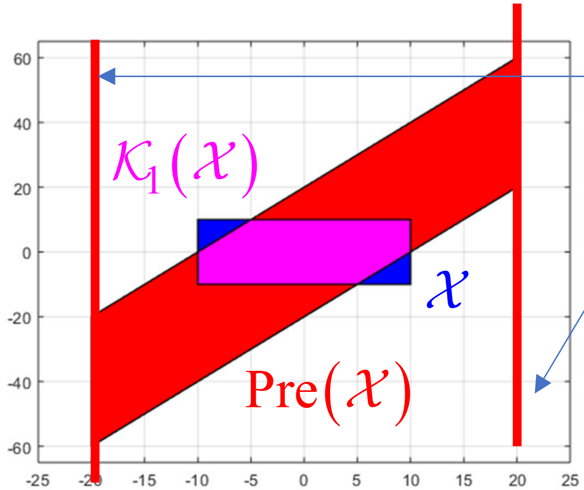
In H-Rep, intersection is computed by concatenating inequalities

- This can lead to redundant inequalities



Don't need the inequalities that define these faces of the precursor set

Precursor Example (cont.)

- Consider the autonomous stable 2nd order system

$$x_{k+1} = Ax_k = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x_k$$

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid H_x x \le f_x \right\}$$

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} x \le \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\}$$

- Compute the 1 step controllable set

$$\mathcal{K}_1(\mathcal{X}) = \mathrm{Pre}(\mathcal{X}) \cap \mathcal{X}$$

$$\mathcal{K}_1(\mathcal{X}) = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} H_x A \\ H_x \end{bmatrix} x \le \begin{bmatrix} f_x \\ f_x \end{bmatrix} \right\}$$

In H-Rep, intersection is computed by concatenating inequalities

- This can lead to redundant inequalities
- Remove redundant inequalities
  - minHRep command in MPT
  - Generally requires solving h Linear Programs (corresponding to the h inequalities)
  - Idea: Remove a constraint and see if you can exceed the constraint while subject to remaining constraints

Successor Example

- Consider the autonomous stable 2nd order system

$$x_{k+1} = Ax_k = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x_k$$

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid H_x x \le f_x \right\}$$

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} x \le \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\}$$

- Compute the successor set

$$\operatorname{Suc}(\mathcal{X}) = \left\{ x_{k+1} \in \mathbb{R}^2 \mid \exists x_k \in \mathcal{X} \ s.t. \ x_{k+1} = Ax_k \right\}$$

$$\operatorname{Suc}(\mathcal{X}) = A\mathcal{X}$$

Affine transformation of set X

- Let X be expressed in V-Rep

$$x_k \in \mathcal{X} = \operatorname{conv}(V) = CH(V) \qquad V = \left\{ V^i \right\}_{i=1}^{N_v}$$

Set of points

Convex hull

- Then, $\operatorname{Suc}(\mathcal{X}) = A\mathcal{X} = \operatorname{conv}(AV)$

Just map each of the vertices

Successor Example (cont.)

- Consider the autonomous stable 2nd order system

$$x_{k+1} = Ax_k = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x_k$$

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid H_x x \le f_x \right\}$$

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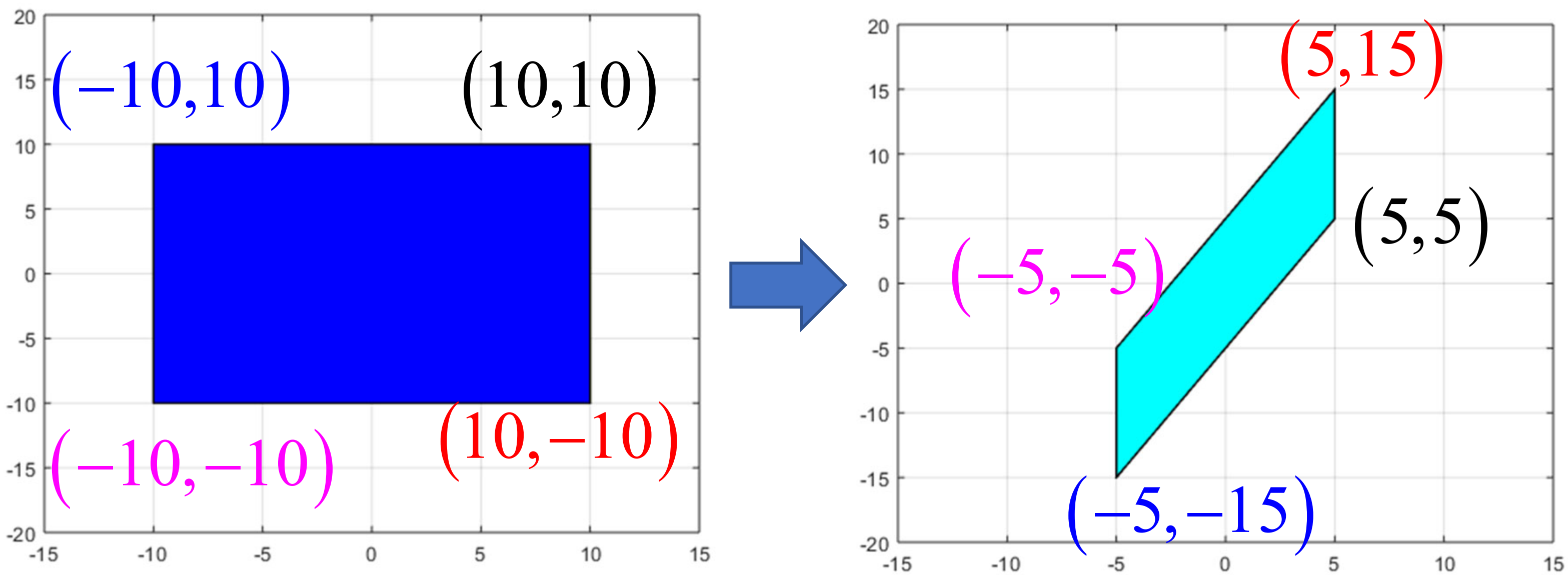
- Compute the successor set

$$\text{Suc}(\mathcal{X}) = A\mathcal{X} = \text{conv}(AV)$$



Successor Example (cont.)

- Consider the autonomous stable 2nd order system

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- Compute the successor set

$$\mathrm{Suc}(\mathcal{X}) = A\mathcal{X} = \mathrm{conv}(AV)$$

- Conversion from H-Rep to/from V-Rep has complexity exponential in n

- Alternatively, H-Rep can be directly used if
  - A is invertible $x_{k+1} = Ax_k \Rightarrow x_k = A^{-1} x_{k+1}$

$$\mathrm{Suc}(\mathcal{X}) = \left\{ x_{k+1} \in \mathbb{R}^2 \mid H_x A^{-1} x_{k+1} \le f_x \right\}$$

  - A is not invertible using QR decomposition

Homework #2 (Due: Sept. 29)

- Two problems
- Problem 1:
  - Assess closed-loop stability with and without output constraints and terminal constraints.
- Problem 2:
  - Explore closed-loop stability and the region of attraction when using a terminal constraint.

- In a single PDF, type your responses to the various questions, provide well formatted Matlab plots, and you Matlab code
  - All of this helps me provide you will more feedback