MECH 6v29.002 – Model Predictive Control

L12 – Robustness

Outline

- Course Schedule Reminder
- Project Introduction
- Working towards Robust MPC
  - Robust Controllable and Reachable Sets
  - Robust Precursor and Successor Sets
  - N-Step Controllable and Reachable Sets
  - Minkowski Sum and Pontryagin Difference

Updated Schedule

Week of:	Topic	Due
8/21	Introduction to MPC – Key Concepts	
8/28	Mathematical Background and Dynamic Systems	
9/04	MPC Theory – Stability	
9/11	Unconstrained MPC and Extensions	HW #1
9/18	MPC Theory – Feasibility	
9/25	MPC Theory – Invariant Sets and Persistent Feasibility	HW #2
10/02	Robust MPC	
10/09	Robust MPC	Project Proposal
10/16	MPC Development and Applications	HW #3
10/23	Project Discussions	
10/30	Nonlinear MPC	
11/06	Decentralized and Distributed MPC	HW #4
11/13	Explicit and Hybrid MPC	
11/20	No Lectures (Fall Break)	
11/27	Project Presentations	
12/04	No Lectures (Last Week of classes)	Project Report

Project

- Full description posted on eLearning
  - Read this carefully before Thursday
  - Come to next class with questions
- Project Deliverables and Timeline:
  - 10/13 - Project Proposal: Submitted electronically by 5pm.
  - 10/24 and 10/26 - Project Discussions: 15 minute in-class one-on-one meetings.
  - 11/28 and 11/30 - Project Presentations: 15 minute in-class presentations.
  - 12/08 - Project Report: Submitted electronically by 5pm.
- Project can be **theory-driven** or **application-driven**
- Project Proposal (over the next two weeks)
  - Think of a high-level aspect of MPC or control application
  - Conduct a literature review on this idea to see what has been done already
  - Identify which aspects of your chosen reference you plan to use and how you might extend or deviate
  - Identify your scope or final goal
  - Think about the key steps – break the project down into manageable chunks

Robust Controllable and Reachable Sets

- Consider the autonomous systems

Nonlinear

$$x_{k+1} = g\left(x_k, w_k\right)$$

Linear

$$x_{k+1} = Ax_k + w_k$$

- And the systems with external inputs

Nonlinear

$$x_{k+1} = g\left(x_k, u_k, w_k\right)$$

Linear

$$x_{k+1} = Ax_k + Bu_k + w_k$$

- Each system is subject to state and input constraints at each discrete point in time

$$x_k \in \mathcal{X}, \;\; u_k \in \mathcal{U}, \quad \forall k \geq 0$$

- Each system is subject to unknown, yet bounded, disturbances

$$w_k \in \mathcal{W}, \quad \forall k \geq 0$$

- We are interested in quantifying (as a set) where the systems can go (both forward and backward) in time while satisfying these constraints subject to these unknown disturbances
  - Assume polyhedral constraint sets
  - Focus on similarities and difference with nominal case

Robust Precursor Sets

- The robust precursor set to the set S is the set of states which evolve into the target set S in one discrete time step for all possible disturbances $w_k \in \mathcal{W}$
- For the autonomous systems, the precursor set is defined as

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \left\{x_k \in \mathbb{R}^n \mid x_{k+1} = g(x_k, w_k) \in \mathcal{S},\ \ \forall w_k \in \mathcal{W}\right\}$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \left\{x_k \in \mathbb{R}^n \mid x_{k+1} = Ax_k + w_k \in \mathcal{S},\ \ \forall w_k \in \mathcal{W}\right\}$$

- For the systems in inputs, the precursor set is defined as

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \left\{x_k \in \mathbb{R}^n \mid \exists u_k \in \mathcal{U}\ \ s.t.\ x_{k+1} = g(x_k, u_k, w_k) \in \mathcal{S},\ \ \forall w_k \in \mathcal{W}\right\}$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \left\{x_k \in \mathbb{R}^n \mid \exists u_k \in \mathcal{U}\ \ s.t.\ x_{k+1} = Ax_k + Bu_k + w_k \in \mathcal{S},\ \ \forall w_k \in \mathcal{W}\right\}$$

- Also called the one-step robust backward-reachable set

Robust Successor Sets

- The robust successor set to the set S is the set of states that can be reached from S in one discrete-time step for all possible disturbances $w_k \in \mathcal{W}$
- For the autonomous systems, the successor set is defined as

$$\mathrm{Suc}(\mathcal{S},\mathcal{W}) = \left\{ x_{k+1} \in \mathbb{R}^n \mid \exists x_k \in \mathcal{S}, \exists w_k \in \mathcal{W} \;\; s.t. \; x_{k+1} = g(x_k, w_k) \right\}$$

$$\mathrm{Suc}(\mathcal{S},\mathcal{W}) = \left\{ x_{k+1} \in \mathbb{R}^n \mid \exists x_k \in \mathcal{S}, \exists w_k \in \mathcal{W} \; s.t. \; x_{k+1} = Ax_k + w_k \right\}$$

- For the systems in inputs, the successor set is defined as

$$\mathrm{Suc}(\mathcal{S},\mathcal{W}) = \left\{ x_{k+1} \in \mathbb{R}^n \mid \exists x_k \in \mathcal{S}, \exists u_k \in \mathcal{U}, \exists w_k \in \mathcal{W} \; s.t. \; x_{k+1} = g(x_k, u_k w_k) \right\}$$

$$\mathrm{Suc}(\mathcal{S},\mathcal{W}) = \left\{ x_{k+1} \in \mathbb{R}^n \mid \exists x_k \in \mathcal{S}, \exists u_k \in \mathcal{U}, \exists w_k \in \mathcal{W} \; s.t. \; x_{k+1} = Ax_k + Bu_k + w_k \right\}$$

- Also called the one-step robust forward-reachable set

N-step Robust Controllable Set

- For a given target set $\mathcal{S} \subseteq \mathcal{X}$, the N-step Robust Controllable Set $\mathcal{K}_N(\mathcal{S},\mathcal{W})$ for a discrete-time dynamic system subject to input and state constraints is defined recursively as

$$\mathcal{K}_0(\mathcal{S},\mathcal{W}) = \mathcal{S}$$

$$\mathcal{K}_j(\mathcal{S},\mathcal{W}) = \mathrm{Pre}\big(\mathcal{K}_{j-1}(\mathcal{S},\mathcal{W}),\mathcal{W}\big) \cap \mathcal{X}, \quad j \in \{1,\ldots,N\}$$

- Same algorithm as nominal case
- For autonomous systems, all states in the N-step Robust Controllable Set will evolve to the target set in N steps for all possible disturbances, while satisfying all state constraints
- For system with inputs, all states in the N-step Robust Controllable Set can be robustly driven to the target set in N steps for all possible disturbances, while satisfying all state constraints, with a suitable control sequence that satisfies all input constraints

N-step Robust Reachable Set

- For a given initial set $\mathcal{X}_0 \subseteq \mathcal{X}$, the N-step Robust Reachable Set $\mathcal{R}_N(\mathcal{X}_0, \mathcal{W})$ for a discrete-time dynamic system subject to input and state constraints is defined recursively as

$$\mathcal{R}_0(\mathcal{X}_0, \mathcal{W}) = \mathcal{X}_0$$
$$\mathcal{R}_{j+1}(\mathcal{X}_0, \mathcal{W}) = \mathrm{Suc}\big(\mathcal{R}_j(\mathcal{X}_0, \mathcal{W}), \mathcal{W}\big) \cap \mathcal{X}, \quad j \in \{0, \ldots, N-1\}$$

- For autonomous systems, all states in the initial set will evolve to the N-step Robust Controllable Set in N steps for all possible disturbances, while satisfying all state constraints
- For system with inputs, all states in the initial set will evolve to the N-step Controllable Set in N steps for all possible disturbances, while satisfying all state constraints, with a suitable control sequence that satisfies all input constraints

Robust Precursor Example

- Consider the autonomous stable 2nd order system

$$x_{k+1} = Ax_k = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x_k + w_k$$

- Subject to state constraints (box constraints, upper- and lower-bounds)

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}$$

- And bounded additive disturbances

$$w_k \in \mathcal{W} = \left\{ w \in \mathbb{R}^2 \mid \begin{bmatrix} -1 \\ -1 \end{bmatrix} \le w \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

- Convert constraints to H-Rep

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid H_x x \le f_x \right\} \qquad w_k \in \mathcal{W} = \left\{ w \in \mathbb{R}^2 \mid H_w w \le f_w \right\}$$

$$H_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \; f_x = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \qquad H_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \; f_w = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Robust Precursor Example (cont.)

- Consider the autonomous stable 2nd order system

$$x_{k+1} = Ax_k = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x_k + w_k \qquad \begin{aligned} x_k \in \mathcal{X} &= \left\{ x \in \mathbb{R}^2 \mid H_x x \le f_x \right\} \\ w_k \in \mathcal{W} &= \left\{ w \in \mathbb{R}^2 \mid H_w w \le f_w \right\} \end{aligned}$$

- Compute the robust precursor set

$$\mathrm{Pre}(\mathcal{X},\mathcal{W}) = \left\{ x_k \in \mathbb{R}^2 \mid x_{k+1} = Ax_k + w_k \in \mathcal{X},\ \forall w_k \in \mathcal{W} \right\}$$

$$x_{k+1} = Ax_k + w_k \in \mathcal{X},\ \forall w_k \in \mathcal{W}$$

$$\Rightarrow H_x\left(Ax_k + w_k\right) \le f_x,\ \forall w_k \in \mathcal{W}$$

$$\Rightarrow \mathrm{Pre}(\mathcal{X},\mathcal{W}) = \left\{ x_k \in \mathbb{R}^2 \mid H_x A x_k \le f_x - H_x w_k,\ \forall w_k \in \mathcal{W} \right\}$$

$$\Rightarrow \mathrm{Pre}(\mathcal{X},\mathcal{W}) = \left\{ x_k \in \mathbb{R}^2 \mid H_x A x_k \le \tilde{f} \right\}$$

i^{th} row of H matrix

$$\tilde{f} = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \\ \tilde{f}_4 \end{bmatrix} \qquad \tilde{f}_i = \min_{w \in \mathcal{W}} \left(f_{x,i} - H_{x,i} w \right) = \min \left(f_{x,i} - H_{x,i} w \right) \quad \text{s.t. } H_w w \le f_w$$

In general, equal to the number of halfspaces need to define $\mathcal{X}$

Robust Precursor Example (cont.)

- Consider the autonomous stable 2nd order system

$$x_{k+1} = Ax_k = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x_k + w_k$$

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid H_x x \le f_x \right\}$$
$$w_k \in \mathcal{W} = \left\{ w \in \mathbb{R}^2 \mid H_w w \le f_w \right\}$$

- Compute the robust precursor set

$$\mathrm{Pre}(\mathcal{X}, \mathcal{W}) = \left\{ x_k \in \mathbb{R}^2 \mid H_x A x_k \le \tilde{f} \right\} \qquad \tilde{f}_i = \min_{w \in \mathcal{W}} \left(f_{x,i} - H_{x,i} w \right)$$

- Requires solving 4 linear programs

$$H_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \; f_x = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}$$

$$\tilde{f}_1 = \min \left(10 - \begin{bmatrix} 1 & 0 \end{bmatrix} w \right) \quad s.t. \begin{bmatrix} -1 \\ -1 \end{bmatrix} \le w \le \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \tilde{f} = \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \end{bmatrix}$$

Robust Precursor Example (cont.)

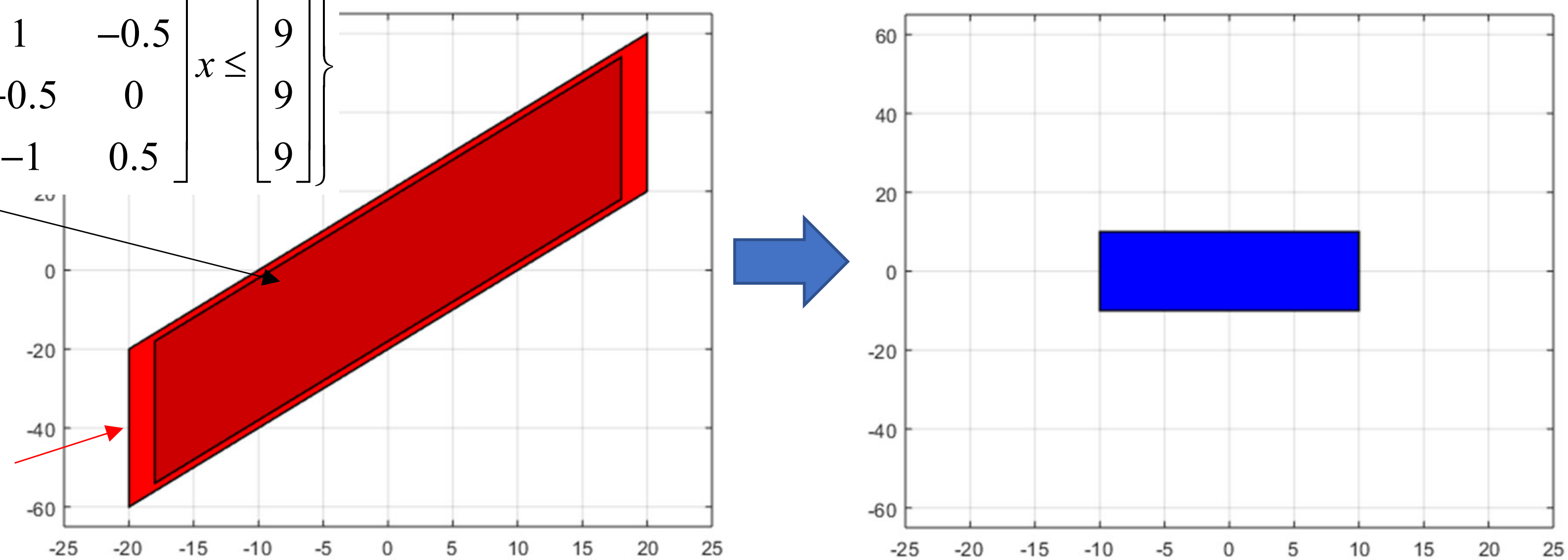
- Consider the autonomous stable 2nd order system

$$x_{k+1} = Ax_k = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x_k + w_k \qquad \begin{aligned} x_k \in \mathcal{X} &= \{x \in \mathbb{R}^2 \mid H_x x \le f_x\} \\ w_k \in \mathcal{W} &= \{w \in \mathbb{R}^2 \mid H_w w \le f_w\} \end{aligned}$$

- Compute the robust precursor set

$$\mathrm{Pre}(\mathcal{X}, \mathcal{W}) = \{x_k \in \mathbb{R}^2 \mid H_x A x_k \le \tilde{f}\} \qquad \tilde{f}_i = \min_{w \in \mathcal{W}} (f_{x,i} - H_{x,i} w)$$

$$\mathrm{Pre}(\mathcal{X}, \mathcal{W}) = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \\ -0.5 & 0 \\ -1 & 0.5 \end{bmatrix} x \le \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \end{bmatrix} \right\}$$



$\mathrm{Pre}(\mathcal{X})$

Robust Precursor Example (cont.)

- Consider the autonomous stable 2nd order system

$$x_{k+1} = Ax_k = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x_k + w_k$$

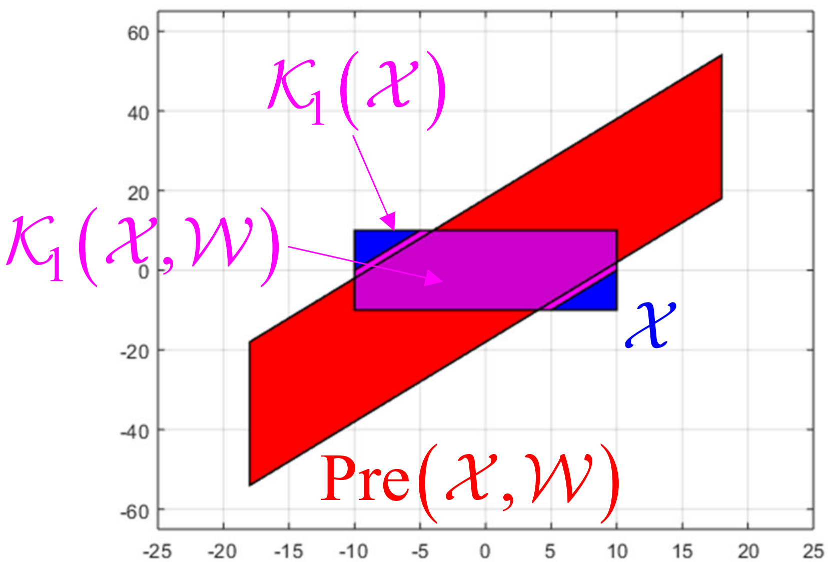
$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid H_x x \le f_x \right\}$$

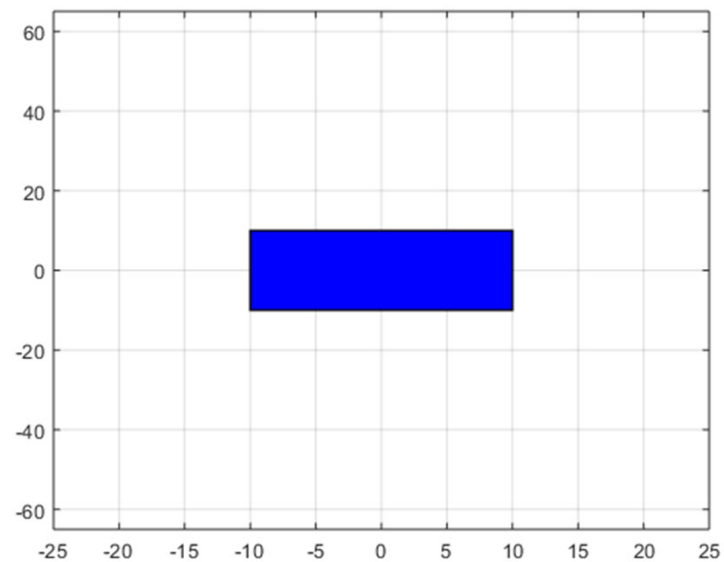
$$w_k \in \mathcal{W} = \left\{ w \in \mathbb{R}^2 \mid H_w w \le f_w \right\}$$

- Compute the 1 step robust controllable set

$$\mathcal{K}_1(\mathcal{X},\mathcal{W}) = \mathrm{Pre}(\mathcal{X},\mathcal{W}) \cap \mathcal{X}$$

$$\mathcal{K}_1(\mathcal{X},\mathcal{W}) = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} H_x A \\ H_x \end{bmatrix} x \le \begin{bmatrix} \tilde{f} \\ f_x \end{bmatrix} \right\}$$





Robust Successor Example

- Consider the autonomous stable 2nd order system

$$x_{k+1} = Ax_k = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x_k + w_k \qquad \begin{aligned} x_k &\in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid H_x x \leq f_x \right\} \\ w_k &\in \mathcal{W} = \left\{ w \in \mathbb{R}^2 \mid H_w w \leq f_w \right\} \end{aligned}$$

- Compute the robust successor set

$$\mathrm{Suc}(\mathcal{X}, \mathcal{W}) = \left\{ x_{k+1} \in \mathbb{R}^2 \mid \exists x_k \in \mathcal{X}, \exists w_k \in \mathcal{W} \ s.t. \ x_{k+1} = Ax_k + w_k \right\}$$

$$\mathrm{Suc}(\mathcal{X}, \mathcal{W}) = A\mathcal{X} \oplus \mathcal{W}$$

Minkowski sum

Linear transformation of set X

- We have already seen how linear transformations can be computed using V-Rep and convex hulls

$$x_k \in \mathcal{X} = \mathrm{conv}(V) \qquad V = \left\{ V^i \right\}_{i=1}^{N_v} \qquad A\mathcal{X} = \mathrm{conv}(AV)$$

- Now let's look at the Minkowski sum in more detail

Minkowski Sum

- The Minkowski sum of two polytopes is a polytope

$$\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y} = \left\{ z = x + y \in \mathbb{R}^n \mid x \in \mathcal{X}, y \in \mathcal{Y} \right\}$$

- Typically, computationally expensive
  - either requires vertex enumeration and convex hull, or
  - Projection from 2n down to n
- Projection approach $\mathcal{X} = \left\{ x \in \mathbb{R}^n \mid H_x x \leq f_x \right\}$ $\quad \mathcal{Y} = \left\{ y \in \mathbb{R}^n \mid H_y y \leq f_y \right\}$

$$\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y} = \left\{ z \in \mathbb{R}^n \mid z = x + y,\ H_x x \leq f_x, H_y y \leq f_y \right\}$$

$$\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y} = \left\{ z \in \mathbb{R}^n \mid \exists x,\ H_x x \leq f_x, H_y \left(z - x \right) \leq f_y \right\}$$

$$\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y} = \left\{ z \in \mathbb{R}^n \mid \exists x, \begin{bmatrix} 0 & H_x \\ H_y & -H_y \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} \leq \begin{bmatrix} f_x \\ f_y \end{bmatrix} \right\}$$

$$\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y} = \mathrm{proj}_{1:n} \left(\left\{ \begin{bmatrix} z \\ x \end{bmatrix} \in \mathbb{R}^{2n} \mid \begin{bmatrix} 0 & H_x \\ H_y & -H_y \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} \leq \begin{bmatrix} f_x \\ f_y \end{bmatrix} \right\} \right)$$

Pontryagin Difference

- The Pontryagin difference of two polytopes is a polytope

$$\mathcal{Z} = \mathcal{X} \ominus \mathcal{Y} = \left\{ x \in \mathbb{R}^n \mid x + y \in \mathcal{X},\ \forall y \in \mathcal{Y} \right\}$$

Also known as the Minkowski difference

$$\mathcal{Z} = \mathcal{X} \ominus \mathcal{Y} = \left\{ x \in \mathbb{R}^n \mid x \oplus \mathcal{Y} \subseteq \mathcal{X} \right\}$$

- Requires solving linear programs

$$\mathcal{X} = \left\{ x \in \mathbb{R}^n \mid H_x x \le f_x \right\} \qquad \mathcal{Y} = \left\{ y \in \mathbb{R}^n \mid H_y y \le f_y \right\}$$

$$\mathcal{Z} = \mathcal{X} \ominus \mathcal{Y} = \left\{ x \in \mathbb{R}^n \mid H_x(x + y) \le f_x,\ \forall y \in \mathcal{Y} \right\}$$

$$\mathcal{Z} = \mathcal{X} \ominus \mathcal{Y} = \left\{ x \in \mathbb{R}^n \mid H_x x \le f_x - H_x y,\ \forall y \in \mathcal{Y} \right\}$$

$$\mathcal{Z} = \mathcal{X} \ominus \mathcal{Y} = \left\{ x \in \mathbb{R}^n \mid H_x x \le \tilde{f} \right\} \qquad \tilde{f}_i = \min_{y \in \mathcal{Y}} \left(f_{x,i} - H_{x,i} y \right) = \begin{array}{l} \min \left(f_{x,i} - H_{x,i} y \right) \\ s.t. \quad H_y y \le f_y \end{array}$$

- Note that Minkowski sum and Pontryagin difference are different than addition and subtraction

$$\left(\mathcal{X} \ominus \mathcal{Y} \right) \oplus \mathcal{Y} \subseteq \mathcal{X}$$

Revisit Successor Set

- Nominal cases (no disturbances)

$$x_{k+1} = Ax_k \qquad \text{Suc}(\mathcal{S}) = \{x_{k+1} \in \mathbb{R}^n \mid \exists x_k \in \mathcal{S} \ s.t.\ x_{k+1} = Ax_k\}$$

$$x_{k+1} = Ax_k + Bu_k \qquad \text{Suc}(\mathcal{S}) = \{x_{k+1} \in \mathbb{R}^n \mid \exists x_k \in \mathcal{S}, \exists u_k \in \mathcal{U} \ s.t.\ x_{k+1} = Ax_k + Bu_k\}$$

$$\text{Suc}(\mathcal{S}) = A\mathcal{S} \oplus B\mathcal{U}$$

- Robust case

$$x_{k+1} = Ax_k + Bu_k + w_k$$

$$\text{Suc}(\mathcal{S}, \mathcal{W}) = \{x_{k+1} \in \mathbb{R}^n \mid \exists x_k \in \mathcal{S}, \exists u_k \in \mathcal{U}, \exists w_k \in \mathcal{W} \ s.t.\ x_{k+1} = Ax_k + Bu_k + w_k\}$$

$$\text{Suc}(\mathcal{S}, \mathcal{W}) = A\mathcal{S} \oplus B\mathcal{U} \oplus \mathcal{W}$$

Revisit Precursor Sets

- Nominal cases (no disturbances)

$$x_{k+1} = Ax_k \qquad \mathrm{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid Ax_k \in \mathcal{S} \right\}$$

$$x_{k+1} = Ax_k + Bu_k \qquad \mathrm{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid \exists u_k \in \mathcal{U} \;\; s.t.\; Ax_k + Bu_k \in \mathcal{S} \right\}$$

$$\mathrm{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid y_k = Ax_k + Bu_k,\; y_k \in \mathcal{S},\; u_k \in \mathcal{U} \right\}$$

$$\mathrm{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid Ax_k = y_k + (-Bu_k),\; y_k \in \mathcal{S},\; u_k \in \mathcal{U} \right\}$$

$$\mathrm{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid Ax_k \in \mathcal{C}, \quad \mathcal{C} = \mathcal{S} \oplus (-B)\mathcal{U} \right\}$$

- Robust case

$$x_{k+1} = Ax_k + Bu_k + w_k$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \left\{ x_k \in \mathbb{R}^n \mid \exists u_k \in \mathcal{U} \;\; s.t.\; Ax_k + Bu_k + w_k \in \mathcal{S}, \;\; \forall w_k \in \mathcal{W} \right\}$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \left\{ x_k \in \mathbb{R}^n \mid \exists y_k \in \mathcal{S}, \exists u_k \in \mathcal{U} \;\; s.t.\; y_k = Ax_k + Bu_k + w_k, \;\; \forall w_k \in \mathcal{W} \right\}$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \left\{ x_k \in \mathbb{R}^n \mid \exists y_k \in \mathcal{S}, \exists u_k \in \mathcal{U} \;\; s.t.\; Ax_k = y_k + \left(-Bu_k\right) - w_k, \;\; \forall w_k \in \mathcal{W} \right\}$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \left\{ x_k \in \mathbb{R}^n \mid Ax_k \in \mathcal{C}, \mathcal{C} = \mathcal{S} \oplus (-B)\mathcal{U} \ominus \mathcal{W} \right\}$$

Revisit Precursor Sets (cont.)

- Nominal cases (no disturbances)

$$x_{k+1} = Ax_k \qquad \mathrm{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid Ax_k \in \mathcal{S} \right\}$$

$$x_{k+1} = Ax_k + Bu_k \qquad \mathrm{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid \exists u_k \in \mathcal{U} \;\; s.t. \; Ax_k + Bu_k \in \mathcal{S} \right\}$$

$$\mathrm{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid Ax_k \in \mathcal{C}, \quad \mathcal{C} = \mathcal{S} \oplus (-B)\mathcal{U} \right\}$$

If A is invertible $\quad \mathrm{Pre}(\mathcal{S}) = A^{-1}\mathcal{S} \oplus (-A^{-1}B)\mathcal{U}$

- Robust case

$$x_{k+1} = Ax_k + Bu_k + w_k$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \left\{ x_k \in \mathbb{R}^n \mid \exists u_k \in \mathcal{U} \;\; s.t. \; Ax_k + Bu_k + w_k \in \mathcal{S}, \;\; \forall w_k \in \mathcal{W} \right\}$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \left\{ x_k \in \mathbb{R}^n \mid Ax_k \in \mathcal{C}, \mathcal{C} = \mathcal{S} \oplus (-B)\mathcal{U} \ominus \mathcal{W} \right\}$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = A^{-1}\mathcal{S} \oplus (-A^{-1}B)\mathcal{U} \ominus A^{-1}\mathcal{W}$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = A^{-1}(\mathcal{S} \ominus \mathcal{W}) \oplus (-A^{-1}B)\mathcal{U}$$