MECH 6v29.002 – Model Predictive Control

L13 – Robustness (continued)

Outline

- Project Questions
- Minkowski Sum and Pontryagin Difference
  - Examples using MPT
- Parametric Uncertainty
- Robust Invariant Sets

Project Questions

- Project Deliverables and Timeline:
  - 10/13 - Project Proposal: Submitted electronically by 5pm.
  - 10/24 and 10/26 - Project Discussions: 15 minute in-class one-on-one meetings.
  - 11/28 and 11/30 - Project Presentations: 15 minute in-class presentations.
  - 12/08 - Project Report: Submitted electronically by 5pm.
- Project can be **theory-driven** or **application-driven**
- Project Proposal (over the next two weeks)
  - Think of a high-level aspect of MPC or control application
  - Conduct a literature review on this idea to see what has been done already
  - Identify which aspects of your chosen reference you plan to use and how you might extend or deviate
  - Identify your scope or final goal
  - Think about the key steps – break the project down into manageable chunks

Minkowski Sum

- The Minkowski sum of two polytopes is a polytope

$$\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y} = \left\{ z = x + y \in \mathbb{R}^n \mid x \in \mathcal{X}, y \in \mathcal{Y} \right\}$$

- Typically, computationally expensive
  - either requires vertex enumeration and convex hull, or
  - Projection from $2n$ down to n
- Projection approach $\mathcal{X} = \left\{ x \in \mathbb{R}^n \mid H_x x \le f_x \right\}$ $\quad \mathcal{Y} = \left\{ y \in \mathbb{R}^n \mid H_y y \le f_y \right\}$

$$\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y} = \left\{ z \in \mathbb{R}^n \mid z = x + y,\ H_x x \le f_x, H_y y \le f_y \right\}$$

$$\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y} = \left\{ z \in \mathbb{R}^n \mid \exists x,\ H_x x \le f_x, H_y (z - x) \le f_y \right\}$$

$$\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y} = \left\{ z \in \mathbb{R}^n \mid \exists x, \begin{bmatrix} 0 & H_x \\ H_y & -H_y \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} \le \begin{bmatrix} f_x \\ f_y \end{bmatrix} \right\}$$

$$\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y} = \mathrm{proj}_{1:n} \left(\left\{ \begin{bmatrix} z \\ x \end{bmatrix} \in \mathbb{R}^{2n} \mid \begin{bmatrix} 0 & H_x \\ H_y & -H_y \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} \le \begin{bmatrix} f_x \\ f_y \end{bmatrix} \right\} \right)$$

Pontryagin Difference

- The Pontryagin difference of two polytopes is a polytope

$$\mathcal{Z} = \mathcal{X} \ominus \mathcal{Y} = \left\{ x \in \mathbb{R}^n \mid x + y \in \mathcal{X},\ \forall y \in \mathcal{Y} \right\}$$

Also known as the Minkowski difference

$$\mathcal{Z} = \mathcal{X} \ominus \mathcal{Y} = \left\{ x \in \mathbb{R}^n \mid x \oplus \mathcal{Y} \subseteq \mathcal{X} \right\}$$

- Requires solving linear programs

$$\mathcal{X} = \left\{ x \in \mathbb{R}^n \mid H_x x \le f_x \right\} \qquad \mathcal{Y} = \left\{ y \in \mathbb{R}^n \mid H_y y \le f_y \right\}$$

$$\mathcal{Z} = \mathcal{X} \ominus \mathcal{Y} = \left\{ x \in \mathbb{R}^n \mid H_x (x + y) \le f_x,\ \forall y \in \mathcal{Y} \right\}$$

$$\mathcal{Z} = \mathcal{X} \ominus \mathcal{Y} = \left\{ x \in \mathbb{R}^n \mid H_x x \le f_x - H_x y,\ \forall y \in \mathcal{Y} \right\}$$

$$\mathcal{Z} = \mathcal{X} \ominus \mathcal{Y} = \left\{ x \in \mathbb{R}^n \mid H_x x \le \tilde{f} \right\} \qquad \tilde{f}_i = \min_{y \in \mathcal{Y}} \left(f_{x,i} - H_{x,i} y \right) = \min \left(f_{x,i} - H_{x,i} y \right) \quad s.t.\ \ H_y y \le f_y$$

- Note that Minkowski sum and Pontryagin difference are different than addition and subtraction

$$\left(\mathcal{X} \ominus \mathcal{Y} \right) \oplus \mathcal{Y} \subseteq \mathcal{X}$$

Revisit Successor Set

- Nominal cases (no disturbances)

$$x_{k+1} = Ax_k \qquad \mathrm{Suc}(\mathcal{S}) = \left\{x_{k+1} \in \mathbb{R}^n \mid \exists x_k \in \mathcal{S} \ s.t. \ x_{k+1} = Ax_k\right\}$$

$$x_{k+1} = Ax_k + Bu_k \qquad \mathrm{Suc}(\mathcal{S}) = \left\{x_{k+1} \in \mathbb{R}^n \mid \exists x_k \in \mathcal{S}, \exists u_k \in \mathcal{U} \ s.t. \ x_{k+1} = Ax_k + Bu_k\right\}$$

$$\mathrm{Suc}(\mathcal{S}) = A\mathcal{S} \oplus B\mathcal{U}$$

- Robust case

$$x_{k+1} = Ax_k + Bu_k + w_k$$

$$\mathrm{Suc}(\mathcal{S}, \mathcal{W}) = \left\{x_{k+1} \in \mathbb{R}^n \mid \exists x_k \in \mathcal{S}, \exists u_k \in \mathcal{U}, \exists w_k \in \mathcal{W} \ s.t. \ x_{k+1} = Ax_k + Bu_k + w_k\right\}$$

$$\mathrm{Suc}(\mathcal{S}, \mathcal{W}) = A\mathcal{S} \oplus B\mathcal{U} \oplus \mathcal{W}$$

Revisit Precursor Sets

- Nominal cases (no disturbances)

$$x_{k+1} = Ax_k \qquad \mathrm{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid Ax_k \in \mathcal{S} \right\}$$

$$x_{k+1} = Ax_k + Bu_k \qquad \mathrm{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid \exists u_k \in \mathcal{U} \;\; s.t.\; Ax_k + Bu_k \in \mathcal{S} \right\}$$

$$\mathrm{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid y_k = Ax_k + Bu_k,\; y_k \in \mathcal{S},\; u_k \in \mathcal{U} \right\}$$

$$\mathrm{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid Ax_k = y_k + (-Bu_k),\; y_k \in \mathcal{S},\; u_k \in \mathcal{U} \right\}$$

$$\mathrm{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid Ax_k \in \mathcal{C}, \quad \mathcal{C} = \mathcal{S} \oplus (-B)\mathcal{U} \right\}$$

- Robust case

$$x_{k+1} = Ax_k + Bu_k + w_k$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \left\{ x_k \in \mathbb{R}^n \mid \exists u_k \in \mathcal{U} \;\; s.t.\; Ax_k + Bu_k + w_k \in \mathcal{S}, \;\; \forall w_k \in \mathcal{W} \right\}$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \left\{ x_k \in \mathbb{R}^n \mid \exists y_k \in \mathcal{S}, \exists u_k \in \mathcal{U} \;\; s.t.\; y_k = Ax_k + Bu_k + w_k, \;\; \forall w_k \in \mathcal{W} \right\}$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \left\{ x_k \in \mathbb{R}^n \mid \exists y_k \in \mathcal{S}, \exists u_k \in \mathcal{U} \;\; s.t.\; Ax_k = y_k + \left(-Bu_k\right) - w_k, \;\; \forall w_k \in \mathcal{W} \right\}$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \left\{ x_k \in \mathbb{R}^n \mid Ax_k \in \mathcal{C}, \mathcal{C} = \mathcal{S} \oplus (-B)\mathcal{U} \ominus \mathcal{W} \right\}$$

Revisit Precursor Sets (cont.)

- Nominal cases (no disturbances)

$$x_{k+1} = Ax_k \qquad \mathrm{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid Ax_k \in \mathcal{S} \right\}$$

$$x_{k+1} = Ax_k + Bu_k \qquad \mathrm{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid \exists u_k \in \mathcal{U} \ \ s.t. \ Ax_k + Bu_k \in \mathcal{S} \right\}$$

$$\mathrm{Pre}(\mathcal{S}) = \left\{ x_k \in \mathbb{R}^n \mid Ax_k \in \mathcal{C}, \quad \mathcal{C} = \mathcal{S} \oplus (-B)\mathcal{U} \right\}$$

If A is invertible $\quad \mathrm{Pre}(\mathcal{S}) = A^{-1}\mathcal{S} \oplus (-A^{-1}B)\mathcal{U}$

- Robust case

$$x_{k+1} = Ax_k + Bu_k + w_k$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \left\{ x_k \in \mathbb{R}^n \mid \exists u_k \in \mathcal{U} \ \ s.t. \ Ax_k + Bu_k + w_k \in \mathcal{S}, \ \ \forall w_k \in \mathcal{W} \right\}$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \left\{ x_k \in \mathbb{R}^n \mid Ax_k \in \mathcal{C}, \mathcal{C} = \mathcal{S} \oplus (-B)\mathcal{U} \ominus \mathcal{W} \right\}$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = A^{-1}\mathcal{S} \oplus (-A^{-1}B)\mathcal{U} \ominus A^{-1}\mathcal{W}$$

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = A^{-1}(\mathcal{S} \ominus \mathcal{W}) \oplus (-A^{-1}B)\mathcal{U}$$

Example

- Consider the unstable 2nd order system

$$x_{k+1} = Ax_k + Bu_k + w_k = \begin{bmatrix} 1.5 & 0 \\ 1 & -1.5 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k + w_k$$

- Subject to input, state, and disturbance constraints

$$u_k \in U = \{ u \in \mathbb{R} \mid -5 \le u \le 5 \}$$

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}$$

$$w_k \in \mathcal{W} = \left\{ w \in \mathbb{R}^2 \mid \begin{bmatrix} -1 \\ -1 \end{bmatrix} \le w \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

- Compute, using Minkowski sum and Pontryagin difference, the
  - Precursor set
  - 1-step robust controllable set
  - Successor set

Example (Precursor Set)

- Consider the unstable 2nd order system

$$x_{k+1} = \begin{bmatrix} 1.5 & 0 \\ 1 & -1.5 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k + w_k$$

$$u_k \in U = \{ u \in \mathbb{R} \mid -5 \le u \le 5 \}$$

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}$$

$$w_k \in \mathcal{W} = \left\{ w \in \mathbb{R}^2 \mid \begin{bmatrix} -1 \\ -1 \end{bmatrix} \le w \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$\mathrm{Pre}(\mathcal{X}, \mathcal{W}) = \left\{ x_k \in \mathbb{R}^n \mid A x_k \in \mathcal{C},\ \mathcal{C} = \mathcal{X} \oplus (-B)\mathcal{U} \ominus \mathcal{W} \right\}$$

- Using MPT

```
A = [1.5 0; 1 -1.5];
B = [1; 0];

Hx = [eye(2);-eye(2)];
fx = 10*ones(4,1);
X = Polyhedron('H',[Hx fx]);

Hu = [1;-1];
fu = 5*ones(2,1);
U = Polyhedron('H',[Hu fu]);

Hw = [eye(2);-eye(2)];
fw = ones(4,1);
W = Polyhedron('H',[Hw fw]);
```

Matlab uses Operator Overloading to define different operations for different variable (object) types

Affine map

Minkowski sum

```
C = X + (-B)*U - W;
Pre = C*A;
```

Pontryagin diff.

Inverse Affine map

```
>> edit Polyhedron\plus
>> edit Polyhedron\minus
>> edit Polyhedron\mtimes
>> edit Polyhedron\affineMap
>> edit Polyhedron\invAffineMap
```

Example (Precursor Set)

- Consider the unstable 2nd order system

$$u_k \in U = \{u \in \mathbb{R} \mid -5 \le u \le 5\}$$

$$x_{k+1} = \begin{bmatrix} 1.5 & 0 \\ 1 & -1.5 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k + w_k$$

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}$$

$$w_k \in \mathcal{W} = \left\{ w \in \mathbb{R}^2 \mid \begin{bmatrix} -1 \\ -1 \end{bmatrix} \le w \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

If A is invertible

$$\mathrm{Pre}(\mathcal{X},\mathcal{W}) = A^{-1}(\mathcal{X} \ominus \mathcal{W}) \oplus (-A^{-1}B)\mathcal{U}$$

- Using MPT

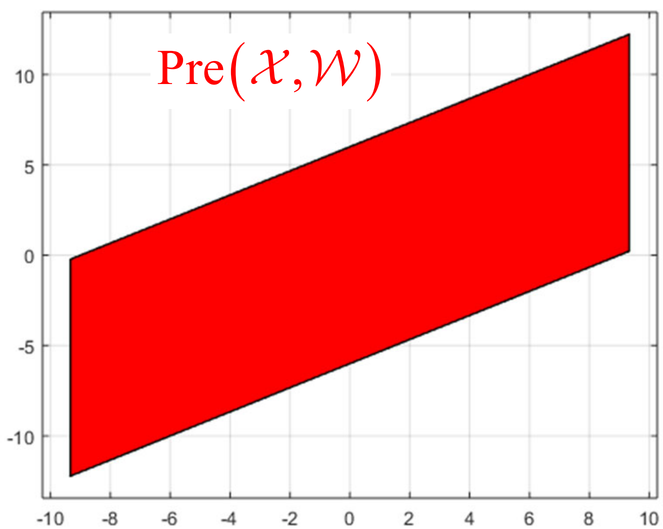
```
Pre = inv(A) * (X - W) + (-inv(A)*B)*U;
```

```
A = [1.5 0; 1 -1.5];
B = [1; 0];

Hx = [eye(2);-eye(2)];
fx = 10*ones(4,1);
X = Polyhedron('H',[Hx fx]);

Hu = [1;-1];
fu = 5*ones(2,1);
U = Polyhedron('H',[Hu fu]);

Hw = [eye(2);-eye(2)];
fw = ones(4,1);
W = Polyhedron('H',[Hw fw]);
```



Set of states at time k that can be driven (using the constrained input) into the state constraint set X at time k+1 for any disturbance w

Example (1-step Controllable Set)

- Consider the unstable 2nd order system

$$x_{k+1} = \begin{bmatrix} 1.5 & 0 \\ 1 & -1.5 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k + w_k$$

$$u_k \in U = \{u \in \mathbb{R} \mid -5 \le u \le 5\}$$

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}$$

$$w_k \in \mathcal{W} = \left\{ w \in \mathbb{R}^2 \mid \begin{bmatrix} -1 \\ -1 \end{bmatrix} \le w \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$\mathcal{K}_1(\mathcal{X}, \mathcal{W}) = \mathrm{Pre}(\mathcal{X}, \mathcal{W}) \cap \mathcal{X}$$

```
Pre = inv(A) * (X - W) + (-inv(A)*B)*U;
K1 = intersect(Pre,X);
```

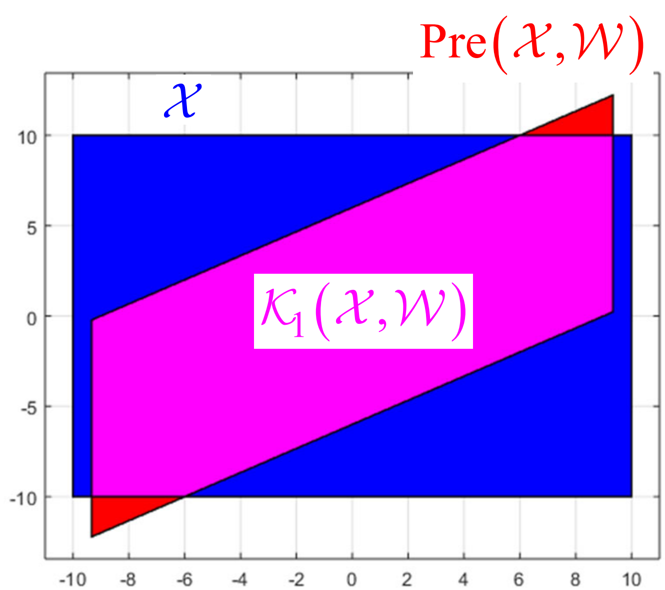
- Using MPT

```
A = [1.5 0; 1 -1.5];
B = [1; 0];

Hx = [eye(2);-eye(2)];
fx = 10*ones(4,1);
X = Polyhedron('H',[Hx fx]);

Hu = [1;-1];
fu = 5*ones(2,1);
U = Polyhedron('H',[Hu fu]);

Hw = [eye(2);-eye(2)];
fw = ones(4,1);
W = Polyhedron('H',[Hw fw]);
```



Questions from Last Lecture

- Pontryagin Difference (Minkowski Difference)

$$\mathcal{Z} = \mathcal{X} \ominus \mathcal{Y} = \left\{ x \in \mathbb{R}^n \mid x + y \in \mathcal{X},\ \forall y \in \mathcal{Y} \right\}$$

$$\mathcal{Z} = \mathcal{X} \ominus \mathcal{Y} = \left\{ x \in \mathbb{R}^n \mid x \oplus \mathcal{Y} \subseteq \mathcal{X} \right\}$$

$$\mathcal{Z} = \mathcal{X} \ominus \mathcal{Y} = \left(\mathcal{X}^c \oplus (-\mathcal{Y}) \right)^c$$

Wiki page was updated since Fall 2020

- Complement of a set

$$\mathcal{X} \subset \mathbb{R}^n \qquad \mathcal{X}^c = \mathbb{R}^n \setminus \mathcal{X}$$

Everything not in X

- Simple example

$$\mathcal{X} = [-2, 2] \qquad \mathcal{Y} = [-0.5, 0.2] \qquad \mathcal{Z} = \mathcal{X} \ominus \mathcal{Y} = [-1.5, 1.8]$$

- Good reminder that

$$\mathcal{X} \ominus \mathcal{Y} \neq \mathcal{X} \oplus (-\mathcal{Y})$$

$$-\mathcal{Y} = \left\{ -y \mid y \in \mathcal{Y} \right\} = [-0.2, 0.5] \qquad \mathcal{X} \oplus (-\mathcal{Y}) = [-2.2, 2.5]$$

Questions from Last Lecture (cont.)

- When can we do this?

$$\text{Pre}(\mathcal{S},\mathcal{W}) = A^{-1}\mathcal{S} \oplus (-A^{-1}B)\mathcal{U} \ominus A^{-1}\mathcal{W}$$
$$\text{Pre}(\mathcal{S},\mathcal{W}) = A^{-1}(\mathcal{S} \ominus \mathcal{W}) \oplus (-A^{-1}B)\mathcal{U}$$

- Minkowski sum and Pontryagin difference properties
  - Both are Increasing $\mathcal{X} \subseteq \mathcal{Y} \quad \Rightarrow \mathcal{X} \oplus \mathcal{Z} \subseteq \mathcal{Y} \oplus \mathcal{Z}$
    $\Rightarrow \mathcal{X} \ominus \mathcal{Z} \subseteq \mathcal{Y} \ominus \mathcal{Z}$
  - Sum is commutative $\mathcal{X} \oplus \mathcal{Y} = \mathcal{Y} \oplus \mathcal{X}$
  - Sum is associative $(\mathcal{X} \oplus \mathcal{Y}) \oplus \mathcal{Z} = \mathcal{Y} \oplus (\mathcal{X} \oplus \mathcal{Z})$
  - Difference is not associative
    $$(\mathcal{X} \ominus \mathcal{Y}) \ominus \mathcal{Z} \neq \mathcal{X} \ominus (\mathcal{Y} \ominus \mathcal{Z})$$
    - instead $(\mathcal{X} \ominus \mathcal{Y}) \ominus \mathcal{Z} = \mathcal{X} \ominus (\mathcal{Y} \oplus \mathcal{Z})$
- However, [1] refers to the associative property of the Pontryagin difference (Remark 10.8 page 201)

$$(\mathcal{X} \oplus \mathcal{Y}) \ominus \mathcal{Z} = (\mathcal{X} \ominus \mathcal{Z}) \oplus \mathcal{Y} \ ?$$

[1] Borrelli, Bemporad, Morari, "Predictive Control for Linear and Hybrid Systems," Cambridge University Press, 2017.

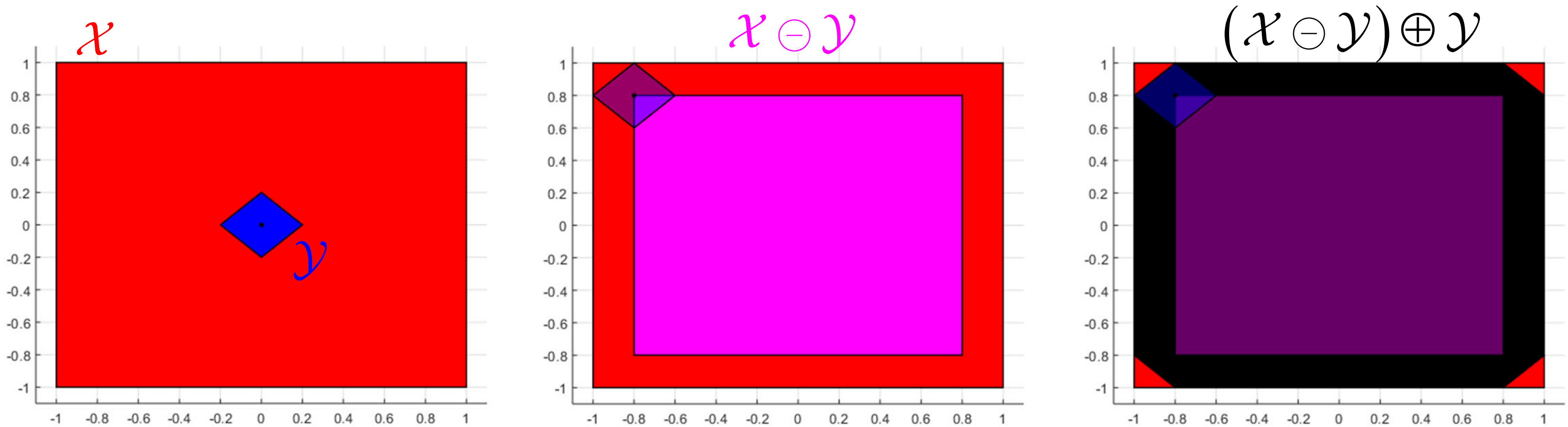
Opening

- The opening of set X by set Y is

$$\mathcal{X} \circ \mathcal{Y} = (\mathcal{X} \ominus \mathcal{Y}) \oplus \mathcal{Y}$$

- The opening is anti-extensive

$$\mathcal{X} \circ \mathcal{Y} = (\mathcal{X} \ominus \mathcal{Y}) \oplus \mathcal{Y} \subseteq \mathcal{X}$$



Closing

- The closing of set X by set Y is

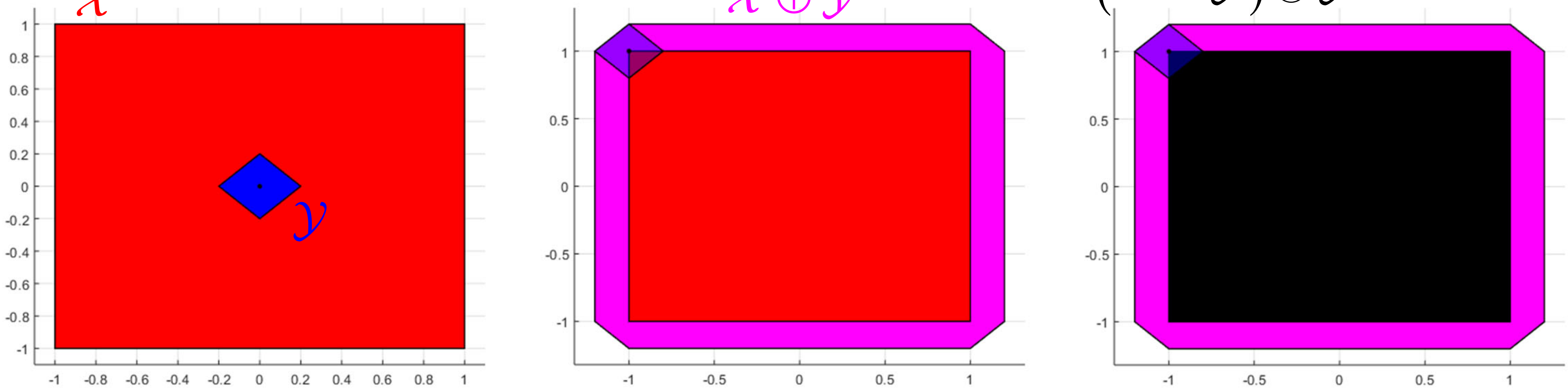
$$\mathcal{X} \bullet \mathcal{Y} = (\mathcal{X} \oplus \mathcal{Y}) \ominus \mathcal{Y}$$

- The closing is extensive

$$\mathcal{X} \subseteq \mathcal{X} \bullet \mathcal{Y} = (\mathcal{X} \oplus \mathcal{Y}) \ominus \mathcal{Y}$$

Not always true

$(\mathcal{X} \oplus \mathcal{Y}) \ominus \mathcal{Y} = \mathcal{X}$



Parametric Uncertainty

- We have seen how additive disturbances enter a system

$$x_{k+1} = Ax_k + Bu_k$$

- But what if there is uncertainty in the parameters of a system?
  - For example:

$$x_{k+1} = A\left(w_p\right)x_k = \begin{bmatrix} 0.5 + w_p & 0 \\ 1 & -0.5 \end{bmatrix} x_k$$

  - These uncertain parameters could affect stability
- We will not spend much time on this but there are a few interesting ideas/concepts to observe

Parametric Uncertainty (cont.)

- First, a couple of key insights
- Ignoring dynamic systems for now, consider a generic nonlinear function

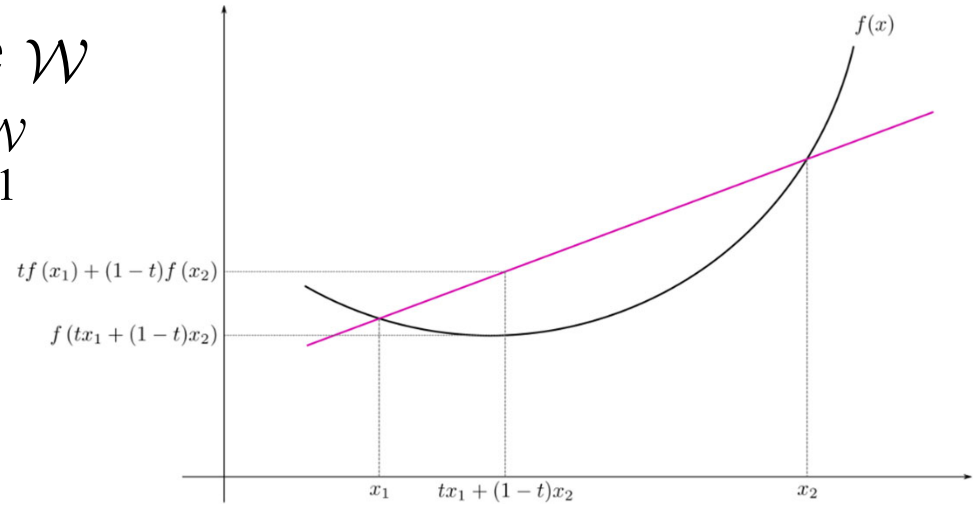
$$g(z,x,w) : \mathbb{R}^{n_z} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_g}$$

- Assume that this function is convex in w for each pair (z,x).
- Assume that w belongs to a polytope $\mathcal{W}$ defined in V-Rep with vertices $\{\bar{w}_i\}_{i=1}^{n_{\mathcal{W}}}$
- Then the constraint

$$g(z,x,w) \le 0, \;\; \forall w \in \mathcal{W}$$

- is satisfied if and only if

$$g(z,x,\bar{w}_i) \le 0, \;\; i = 1,\dots,n_{\mathcal{W}}$$



Can just consider extreme points (vertices) of the set

- Proof: use the fact that the maximum of a convex function over a compact set is attained at an extreme point of the set

Example (1-step robust controllable set)

- Consider the autonomous 2nd order system

$$x_{k+1} = A\left(w_k^p\right)x_k + w_k^a = \begin{bmatrix} 0.5 + w_k^p & 0 \\ 1 & -0.5 \end{bmatrix} x_k + w_k^a \qquad x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}$$

- Collect all uncertainty

$$w_k^a \in \mathcal{W}^a = \left\{ w^a \in \mathbb{R}^2 \mid \begin{bmatrix} -1 \\ -1 \end{bmatrix} \le w^a \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$w_k = \begin{bmatrix} w_k^a \\ w_k^p \end{bmatrix} \in \mathcal{W} = \mathcal{W}^a \times \mathcal{W}^p \qquad w_k^p \in \mathcal{W}^p = \left\{ w^p \in \mathbb{R} \mid 0 \le w^p \le 0.5 \right\}$$

- Convert sets to H-Rep
- Compute 1-step robust controllable set $\mathcal{K}_1(\mathcal{X},\mathcal{W}) = \mathrm{Pre}(\mathcal{X},\mathcal{W}) \cap \mathcal{X}$
  - Robust precursor set

$$\mathrm{Pre}(\mathcal{X},\mathcal{W}) = \left\{ x : H_x\left(A\left(w_k^p\right)x_k + w_k^a\right) \le f_x, \forall w_k^a \in \mathcal{W}^a, \forall w_k^p \in \mathcal{W}^p \right\}$$

  - First, handle additive uncertainty

$$\mathrm{Pre}(\mathcal{X},\mathcal{W}) = \left\{ x : H_x A\left(w_k^p\right)x_k \le f_x - H_x w_k^a,\ \forall w_k^a \in \mathcal{W}^a, \forall w_k^p \in \mathcal{W}^p \right\}$$

$$\mathrm{Pre}(\mathcal{X},\mathcal{W}) = \left\{ x : H_x A\left(w_k^p\right)x_k \le \tilde{f}_x,\ \forall w_k^p \in \mathcal{W}^p \right\} \qquad \tilde{f}_{x,i} = \min_{w_k^a \in \mathcal{W}^a} \left(f_{x,i} - H_{x,i} w_k^a \right)$$

Example (1-step robust controllable set)

- Consider the autonomous 2nd order system

$$x_{k+1} = A\left(w_k^p\right)x_k + w_k^a = \begin{bmatrix} 0.5 + w_k^p & 0 \\ 1 & -0.5 \end{bmatrix} x_k + w_k^a \qquad x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}$$

- Collect all uncertainty

$$w_k^a \in \mathcal{W}^a = \left\{ w^a \in \mathbb{R}^2 \mid \begin{bmatrix} -1 \\ -1 \end{bmatrix} \le w^a \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$w_k = \begin{bmatrix} w_k^a \\ w_k^p \end{bmatrix} \in \mathcal{W} = \mathcal{W}^a \times \mathcal{W}^p \qquad w_k^p \in \mathcal{W}^p = \left\{ w^p \in \mathbb{R} \mid 0 \le w^p \le 0.5 \right\}$$

- Convert sets to H-Rep
- Compute 1-step robust controllable set $\mathcal{K}_1\left(\mathcal{X},\mathcal{W}\right) = \text{Pre}\left(\mathcal{X},\mathcal{W}\right) \cap \mathcal{X}$
  - Robust precursor set

$$\text{Pre}\left(\mathcal{X},\mathcal{W}\right) = \left\{ x : H_x\left(A\left(w_k^p\right)x_k + w_k^a \right) \le f_x, \forall w_k^a \in \mathcal{W}^a, \forall w_k^p \in \mathcal{W}^p \right\}$$

  - Second, handle parameter uncertainty

$$\text{Pre}\left(\mathcal{X},\mathcal{W}\right) = \left\{ x : H_x A\left(w_k^p\right)x_k \le \tilde{f}_x,\ \forall w_k^p \in \mathcal{W}^p \right\}$$

$$\text{Pre}\left(\mathcal{X},\mathcal{W}\right) = \left\{ x : \begin{bmatrix} H_x A(0) \\ H_x A(0.5) \end{bmatrix} x_k \le \begin{bmatrix} \tilde{f}_x \\ \tilde{f}_x \end{bmatrix} \right\}$$

$$g\left(z, x, w\right) \le 0,\ \forall w \in \mathcal{W}$$
$$\Updownarrow$$
$$g\left(z, x, \overline{w}_i\right) \le 0,\ i = 1, \ldots, n_{\mathcal{W}}$$

Example (1-step robust controllable set)

- Consider the autonomous 2nd order system

$$x_{k+1} = A\left(w_k^p\right)x_k + w_k^a = \begin{bmatrix} 0.5 + w_k^p & 0 \\ 1 & -0.5 \end{bmatrix} x_k + w_k^a$$

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}$$

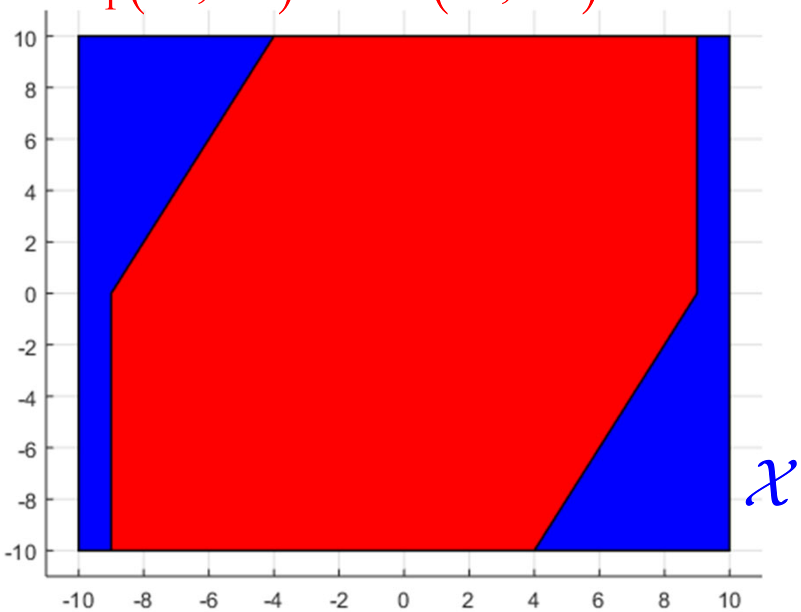
$$w_k^a \in \mathcal{W}^a = \left\{ w^a \in \mathbb{R}^2 \mid \begin{bmatrix} -1 \\ -1 \end{bmatrix} \le w^a \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$w_k^p \in \mathcal{W}^p = \left\{ w^p \in \mathbb{R} \mid 0 \le w^p \le 0.5 \right\}$$

$$\mathrm{Pre}(\mathcal{X},\mathcal{W}) = \left\{ x : \begin{bmatrix} H_x A(0) \\ H_x A(0.5) \end{bmatrix} x_k \le \begin{bmatrix} \tilde{f}_x \\ \tilde{f}_x \end{bmatrix} \right\}$$

- For uncertain parameters that enter the A matrix linearly, the robust precursor set will be convex

$$\mathcal{K}_1(\mathcal{X},\mathcal{W}) = \mathrm{Pre}(\mathcal{X},\mathcal{W}) \cap \mathcal{X}$$




```
%% 1-step robust controllable
A1 = [0.5 0; 1 -0.5];
A2 = [  1 0; 1 -0.5];

Hx = [eye(2);-eye(2)];
fx = 10*ones(4,1);
X = Polyhedron('H',[Hx fx]);

Hwa = [eye(2);-eye(2)];
fwa = ones(4,1);
Wa = Polyhedron('H',[Hwa fwa]);

Hwp = [1;-1];
fwp = [0.5;0];
Wp = Polyhedron('H',[Hwp fwp]);

fx_tilde = 9*ones(4,1);

Pre = Polyhedron('H',[Hx*A1 fx_tilde; Hx*A2 fx_tilde]);
OneStep = intersect(Pre,X);

figure;hold on
plot(X,'color','b')
plot(OneStep)
```

Example (successor set)

- Consider the autonomous 2nd order system

$$x_{k+1} = A\left(w_k^p\right)x_k = \begin{bmatrix} 0.5 + w_k^p & 0 \\ 1 & -0.5 \end{bmatrix} x_k$$

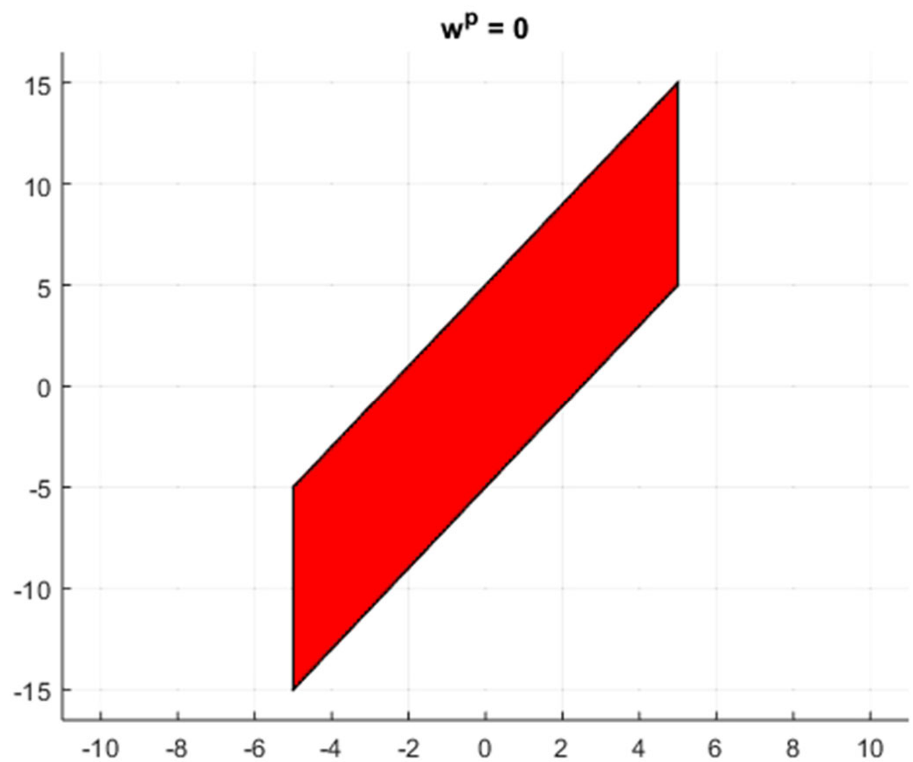
$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}$$

$$w_k^p \in \mathcal{W}^p = \left\{ w^p \in \mathbb{R} \mid 0 \leq w^p \leq 0.5 \right\}$$

- Compute the successor set
  - The successor set is an infinite union of reachable sets

$$\mathrm{Suc}(\mathcal{X}, \mathcal{W}) = \bigcup_{\overline{w} \in \mathcal{W}} \mathrm{Suc}(\mathcal{X}, \overline{w})$$

  - Unlike the precursor set, the successor set can be nonconvex for linear systems with bounded parameter uncertainty



Robust Invariant Sets

- Nothing conceptually changes now that we have bounded uncertainties
- Robust Positive Invariant (RPI) Set:

  A set $\mathcal{O} \subseteq \mathcal{X}$ is said to be a robust positive invariant set for a constrained autonomous system if

  $$x_0 \in \mathcal{O} \quad \Rightarrow \quad x_k \in \mathcal{O} \quad \forall w_k \in \mathcal{W}, \quad \forall k > 0$$

- Refer back to Lecture 10 to make slight modifications to other sets
  - Maximal Robust Positive Invariant Set
  - Robust Control Invariant Set
  - Maximal Robust Control Invariant Set
- Computational algorithms are the same too
  - (just compute the robust precursor set)
- Will use this to formulate robust MPC next week!

Inputs: $g(x,w)$, $\mathcal{X}$, $\mathcal{W}$

Outputs: $\mathcal{O}_\infty$

$\Omega_0 \leftarrow \mathcal{X},\ k \leftarrow -1$

Repeat

$k \leftarrow k+1$

$\Omega_{k+1} \leftarrow \text{Pre}(\Omega_k, \mathcal{W}) \cap \Omega_k$

Until $\Omega_{k+1} = \Omega_k$

$\mathcal{O}_\infty = \Omega_k$