MECH 6v29.002 – Model Predictive Control

L10 – Invariant Sets

Outline

- Invariant Sets
  - Positive Invariant Sets
    - Maximal Positive Invariant Sets
  - Control Invariant Sets
    - Maximal Control Invariant Sets
- Determinedness
- Maximal Controllable Sets
- Maximal Stabilizable Sets
- Examples

Invariant Sets

- Consider the autonomous systems

Nonlinear

$$x_{k+1} = g\left(x_k\right)$$

Linear

$$x_{k+1} = Ax_k$$

- And the systems with external inputs

Nonlinear

$$x_{k+1} = g\left(x_k, u_k\right)$$

Linear

$$x_{k+1} = Ax_k + Bu_k$$

- Each system is subject to state and input constraints at each discrete point in time

$$x_k \in \mathcal{X}, \;\; u_k \in \mathcal{U}, \quad \forall k \geq 0$$

- General idea: Find the set of initial states whose trajectory will never violate the state and input constraints.
    - No longer thinking about a target/terminal set

Types of Invariant Sets

- For autonomous systems:

Systems with inputs using a candidate feedback control law

$$x_{k+1} = g(x_k) \qquad \text{or} \qquad x_{k+1} = g(x_k, u_k), \quad u_k = k(x_k) \;\Rightarrow\; x_{k+1} = g(x_k, k(x_k))$$

$$x_{k+1} = Ax_k \qquad\qquad x_{k+1} = Ax_k + Bu_k, \quad u_k = Kx_k \;\Rightarrow\; x_{k+1} = (A+BK)x_k$$

- Positive Invariant Set:

  A set $\mathcal{O} \subseteq \mathcal{X}$ is said to be a positive invariant set for a constrained autonomous system if

$$x_0 \in \mathcal{O} \;\Rightarrow\; x_k \in \mathcal{O} \quad \forall k > 0$$

- Maximal Positive Invariant Set:

  A set $\mathcal{O}_\infty \subseteq \mathcal{X}$ is the maximal invariant set if it is invariant and contains all the invariant sets.

  - Also referred to as Maximal Admissible Set or Maximal Output Admissible Set, depending if the states or outputs are constrained
  - For nonlinear systems with multiple equilibria, the maximal positive invariant set may be the union of disconnected sets, each containing one of the equilibrium

Types of Invariant Sets (cont.)

- For systems with control inputs:

$$x_{k+1} = g(x_k, u_k)$$

$$x_{k+1} = Ax_k + Bu_k$$

  - Control Invariant Set:

    A set $\mathcal{C} \subseteq \mathcal{X}$ is said to be a control invariant set for a constrained autonomous system if

$$x_0 \in \mathcal{C} \quad \Rightarrow \exists u_k \in \mathcal{U},\ s.t.\ \ x_k \in \mathcal{C} \quad \forall k > 0$$

  - Maximal Control Invariant Set:

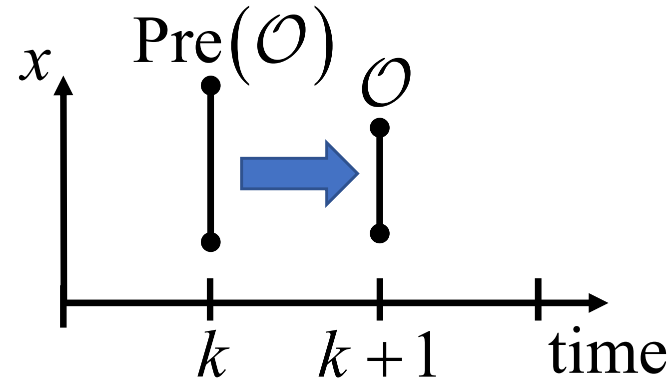
    A set $\mathcal{C}_\infty \subseteq \mathcal{X}$ is the maximal control invariant set if it is control invariant and contains all the control invariant sets.

Invariant Set Conditions

- Geometric Condition
  A set $\mathcal{O} \subseteq \mathcal{X}$ is positive invariant if and only if $\mathcal{O} \subseteq \operatorname{Pre}(\mathcal{O})$

$$\operatorname{Pre}(\mathcal{O}) = \left\{ x_k \in \mathbb{R}^n \mid x_{k+1} = g(x_k) \in \mathcal{O} \right\}$$

All the points that map to O in the next time step



$\mathcal{O} \subseteq \operatorname{Pre}(\mathcal{O})$ ⇒ All the points in O are points that map to O in the next time step ⇒ If you are in O, then you will stay in O

- Equivalent condition: $\operatorname{Pre}(\mathcal{O}) \cap \mathcal{O} = \mathcal{O}$

There are no points in O that are not in the precursor of O

Maximal Positive Invariant Set

- Algorithm: $x_{k+1} = g(x_k) \qquad x_k \in \mathcal{X}$

  Inputs: $g(x),\ \mathcal{X}$

  Outputs: $\mathcal{O}_\infty$

  $\Omega_0 \leftarrow \mathcal{X},\ k \leftarrow -1$

  Repeat

  $k \leftarrow k+1$

  $\Omega_{k+1} \leftarrow \mathrm{Pre}(\Omega_k) \cap \Omega_k$

  Until $\Omega_{k+1} = \Omega_k$

  $\mathcal{O}_\infty = \Omega_k$

- Generates a sequence such that $\Omega_{k+1} \subseteq \Omega_k$
- Not guaranteed to terminate, but can prove $\mathcal{O}_\infty = \lim_{k \to \infty} \Omega_k$
- Sufficient condition for finite termination
  - Linear, stable system
  - State constraint set is bounded and contains the origin

Example

- Consider the constrained autonomous stable 2nd order system

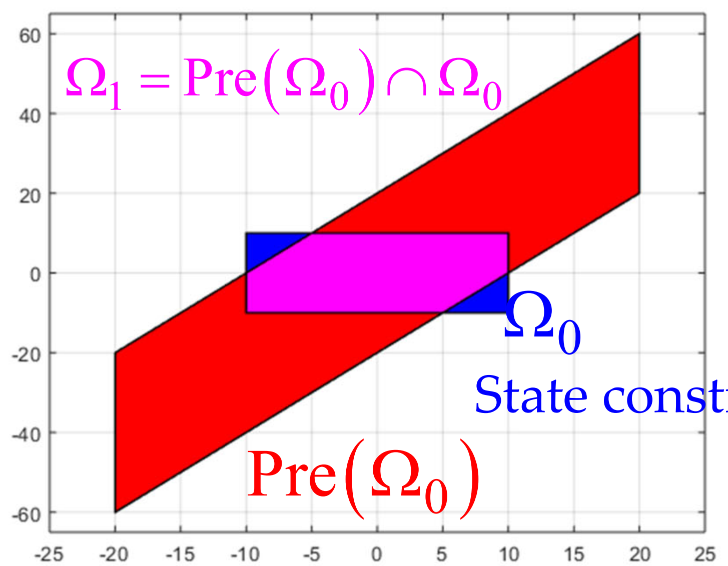
$$x_{k+1} = Ax_k = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x_k \qquad x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \,\middle|\, \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}$$

- It is unreasonable to expect that we can start anywhere within these constraints and remain within this set at every future point in time (even for stable systems)

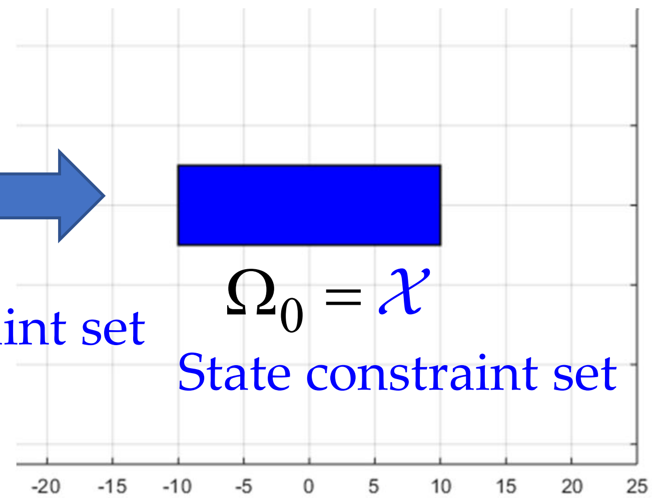
Inputs: $g(x)$, $\mathcal{X}$
Outputs: $\mathcal{O}_\infty$

$\quad \Omega_0 \leftarrow \mathcal{X},\ k \leftarrow -1$
$\quad$ Repeat
$\quad\quad k \leftarrow k+1$
$\quad\quad \Omega_{k+1} \leftarrow \mathrm{Pre}(\Omega_k) \cap \Omega_k$
$\quad$ Until $\Omega_{k+1} = \Omega_k$
$\quad \mathcal{O}_\infty = \Omega_k$

$$\Omega_2 = \mathrm{Pre}(\Omega_1) \cap \Omega_1 = \Omega_1$$



Precursor set

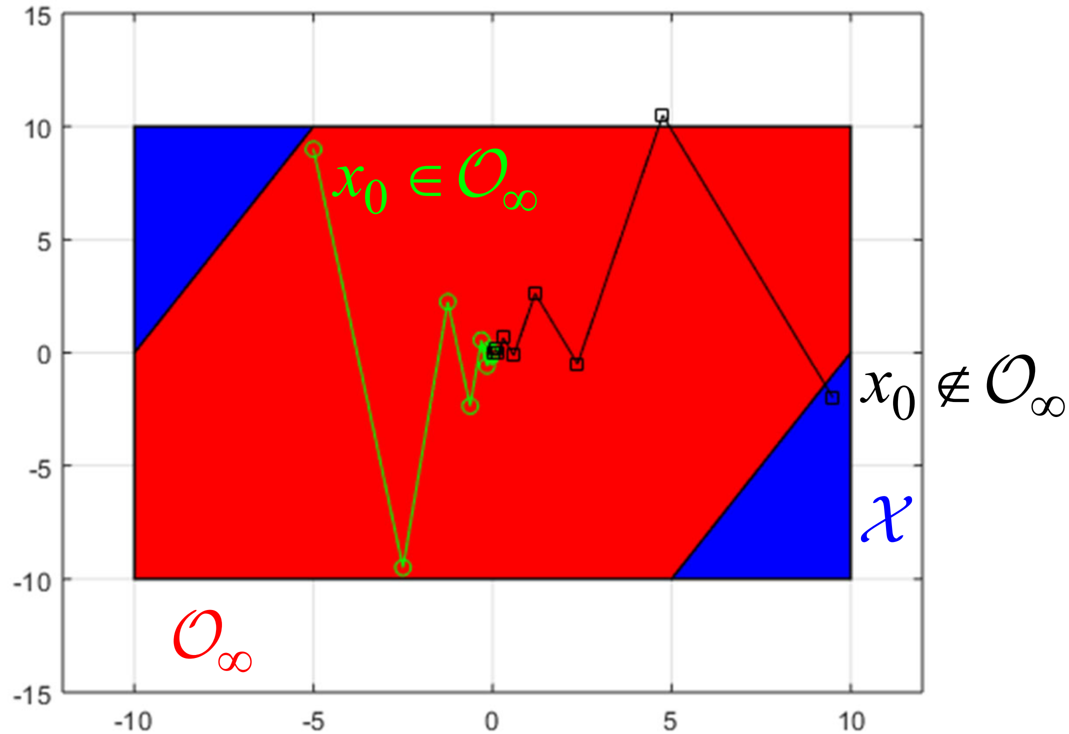


Example (cont.)

- Consider the constrained autonomous stable 2nd order system

$$x_{k+1} = Ax_k = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x_k \qquad x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}$$

- It is unreasonable to expect that we can start anywhere within these constraints and remain within this set at every future point in time (even for stable systems)



Maximal Control Invariant Set

- Same geometric conditions from invariant sets apply to control invariant sets
- Algorithm: $x_{k+1} = g(x_k, u_k) \qquad x_k \in \mathcal{X},\ u_k \in \mathcal{U}$

Inputs: $g(x,u),\ \mathcal{X},\ \mathcal{U}$

Outputs: $\mathcal{C}_\infty$

$$\Omega_0 \leftarrow \mathcal{X},\ k \leftarrow -1$$

Repeat

$$k \leftarrow k+1$$

$$\Omega_{k+1} \leftarrow \mathrm{Pre}(\Omega_k) \cap \Omega_k$$

Until $\Omega_{k+1} = \Omega_k$

$$C_\infty = \Omega_k$$

Same steps as invariant sets

$$\mathrm{Pre}(\Omega_k) = \left\{ x_k \in \mathbb{R}^n \mid \exists u_k \in \mathcal{U}\ \ s.t.\ x_{k+1} = g(x_k, u_k) \in \Omega_k \right\}$$

Requires projection operator

Maximal Control Invariant Set

- Algorithm: $x_{k+1} = g\left(x_k, u_k\right) \qquad x_k \in \mathcal{X},\ u_k \in \mathcal{U}$

  Inputs: $g(x,u),\ \mathcal{X},\ \mathcal{U}$

  Outputs: $\mathcal{C}_\infty$

  $\Omega_0 \leftarrow \mathcal{X},\ k \leftarrow -1$

  Repeat

  $k \leftarrow k+1$

  $\Omega_{k+1} \leftarrow \operatorname{Pre}\left(\Omega_k\right) \cap \Omega_k$

  Until $\Omega_{k+1} = \Omega_k$

  $C_\infty = \Omega_k$

Same steps as invariant sets

- Generates a sequence such that $\Omega_{k+1} \subseteq \Omega_k$
- Not guaranteed to terminate, and $\mathcal{C}_\infty \neq \lim_{k \to \infty} \Omega_k$
- Sufficient condition for convergence
  - System is continuous
  - Polyhedral constraint sets are bounded

Determinedness

- Algorithm: $x_{k+1} = g\left(x_k, u_k\right)$ $x_k \in \mathcal{X},\ u_k \in \mathcal{U}$

  Inputs: $g\left(x,u\right),\ \mathcal{X},\ \mathcal{U}$

  Outputs: $\mathcal{C}_\infty$

  $\Omega_0 \leftarrow \mathcal{X},\ k \leftarrow -1$

  Repeat

  $k \leftarrow k+1$

  $\Omega_{k+1} \leftarrow \mathrm{Pre}\left(\Omega_k\right) \cap \Omega_k$

  Until $\Omega_{k+1} = \Omega_k$

  $C_\infty = \Omega_k$

  Same steps as invariant sets

- The set O_∞ *or* C_∞ is finitely determined if and only if there exists i such that $\Omega_{i+1} = \Omega_i$

- The smallest i such that $\Omega_{i+1} = \Omega_i$ is called the determinedness index.

Example

- Consider the unstable 2nd order system

$$x_{k+1} = Ax_k + Bu_k = \begin{bmatrix} 1.5 & 0 \\ 1 & -1.5 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k$$

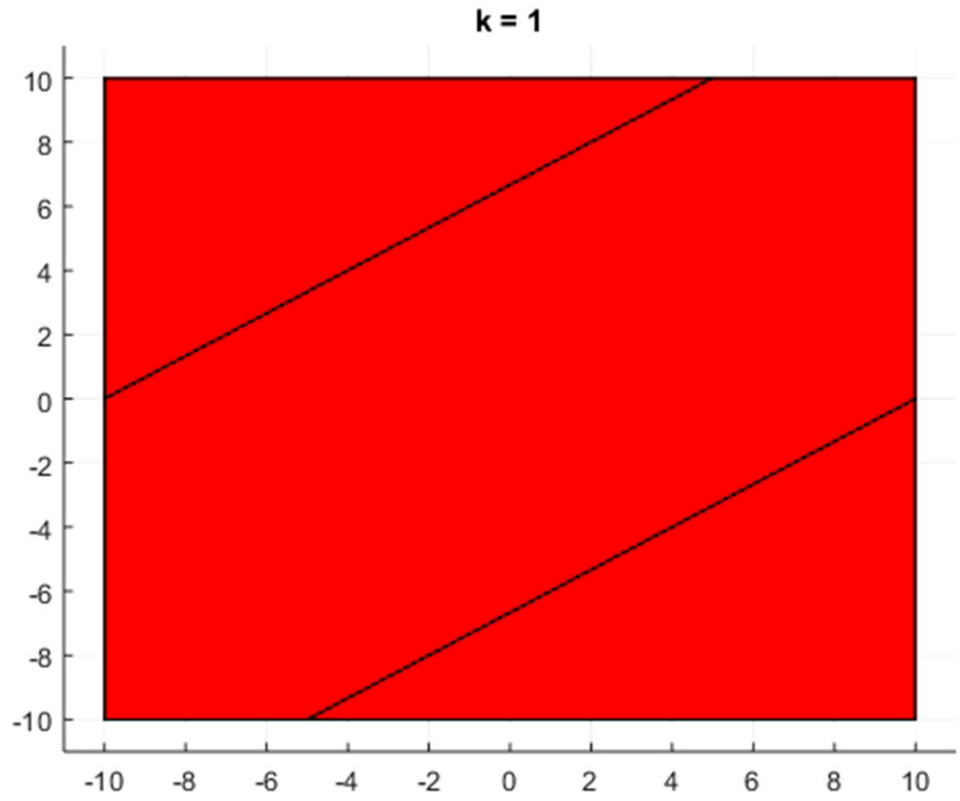
$$u_k \in U = \{u \in \mathbb{R} \mid -5 \le u \le 5\}$$

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \mid \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}$$

- Convergence to compute maximal control invariant set

$$\Omega_{k+1} \leftarrow \mathrm{Pre}(\Omega_k) \cap \Omega_k$$

$$\Omega_{k+1} = \Omega_k$$



Maximal Controllable Sets

- With the maximal positive invariant set $\mathcal{O}_\infty$ and maximal control invariant set $\mathcal{C}_\infty$, there is no notion of a target set
- Now we introduce target sets
- Maximal Controllable Set:

For a given target set $\mathcal{S} \subseteq \mathcal{X}$, the maximal controllable set $\mathcal{K}_\infty(\mathcal{S})$ for a constrained system with inputs is the union of all N-step controllable sets $\mathcal{K}_N(\mathcal{S})$

$$\mathcal{K}_N = \left\{ \begin{array}{r} x_0 \in \mathbb{R}^n \mid \exists U_0 \ s.t. \ x_k \in \mathcal{X}, u_k \in \mathcal{U}, \ \forall k = 0, \ldots, N-1 \\ x_N \in \mathcal{S}, \ x_{k+1} = A x_k + B u_k, \forall k = 0, \ldots, N-1 \end{array} \right\}$$

- We will often choose the target set to be a control invariant set so that once we drive the system to the target set, we know we can stay there

Maximal Stabilizable Sets

- If we choose the target set to be invariant, we get
- Maximal Stabilizable Set:

  For a given control invariant set $\mathcal{O} \subseteq \mathcal{X}$, the maximal stabilizable set for a constrained system with inputs is the maximal controllable set (the only difference is now our target set is invariant) $\mathcal{K}_\infty(\mathcal{O})$
- Same relationship is true for the N-step stabilizable and N-step controllable sets $\mathcal{K}_N(\mathcal{O})$
- Algorithm:

Inputs: $g(x,u)$, $\mathcal{X}$, $\mathcal{U}$

Control invariant set

Outputs: $\mathcal{K}_\infty(\mathcal{O})$

$\mathcal{K}_0 \leftarrow \mathcal{O},\ k \leftarrow -1$

Repeat

$k \leftarrow k+1$

State constraint set

$\mathcal{K}_{k+1} \leftarrow \mathrm{Pre}(\mathcal{K}_k) \cap \mathcal{X}$

Until $\mathcal{K}_{k+1} = \mathcal{K}_k$

$\mathcal{K}_\infty(\mathcal{O}) = \mathcal{K}_k$

Summary

- Positive Invariant $\mathcal{O} \subseteq \mathcal{X}$ $\qquad x_0 \in \mathcal{O} \;\Rightarrow\; x_k \in \mathcal{O} \quad \forall k > 0$
- Maximal Invariant Set $\mathcal{O}_\infty \subseteq \mathcal{X}$ (Union of all invariant sets)
- Control Invariant $\mathcal{C} \subseteq \mathcal{X}$ $\qquad x_0 \in \mathcal{C} \;\Rightarrow\; \exists u_k \in \mathcal{U},\ s.t.\ \ x_k \in \mathcal{C} \quad \forall k > 0$
- Maximal Control Invariant Set $\mathcal{C}_\infty \subseteq \mathcal{X}$ (Union of all ctrl. inv. sets)
- Determinedness index – number of steps for max controllable invariant set algorithm to converge (if it does)
- Maximal Controllable Sets $\mathcal{K}_\infty(\mathcal{S})$
  - Union of all N-step controllable sets that drive system to a target set
- Maximal Stabilizable Sets $\mathcal{K}_\infty(\mathcal{O})$
  - Same as Maximal Controllable Sets, but now the target is invariant

- Next steps:
  - Use all of this information to prove persistent feasibility of MPC