MECH 6v29.002 – Model Predictive Control

L15 – Robust MPC (cont.)

Outline

- Homework #3
- Robust MPC
  - Constraint tightening
  - Need for feedback
  - Uncertainty analysis
  - Nilpotent candidate controller
  - Time-varying constraint tightening
  - Tube-based MPC
    - minRPI set

Homework #3

- Today, we will present two difference robust MPC formulations based on this uncertainty analysis
  - Time-varying constraint tightening [1]
  - Tube-based MPC [2]
- Hopefully you have read at least the following
  - Sections 2.2, 2.3, and 2.5.1 of [1]
  - Sections 2, 3, and first part of 4 up until Proposition 2 in [2]
- You will be implementing both of these in HW #3
  - Specifically, you will be implementing the numerical example from 2.5.1 to recreate Fig. 2-2 in [1] using both of the methods from [1] and [2]
- Also, please complete the Mid-semester feedback survey on eLearning by this Friday

[1] Arthur Richards, "Robust Constrained Model Predictive Control," Ph.D. Dissertation, MIT, 2002.
[2] D.Q. Mayne, M.M. Seron, S.V. Rakovic, "Robust Model Predictive Control of Constrained Linear Systems with Bounded Disturbances," Automatica, 2005.

Time-varying Constraint Tightening

- Approach from Arthur Richards [1] (Chapter 2)

$$J^*\left(x(k)\right)=\min_{U_k}\sum_{j=0}^{N-1}\hat{x}_{k+j|k}^T Q\hat{x}_{k+j|k}+\hat{u}_{k+j|k}^T R\hat{u}_{k+j|k}+\hat{x}_{k+N|k}^T P\hat{x}_{k+N|k}$$

$$s.t.$$

$$\hat{x}_{k+j+1|k}=A\hat{x}_{k+j|k}+B\hat{u}_{k+j|k},\ \ j\in\{0,1,\ldots,N-1\}$$

$$\hat{y}_{k+j|k}=C\hat{x}_{k+j|k}+D\hat{u}_{k+j|k}\in\hat{\mathcal{Y}}(j),\quad j\in\{0,1,\ldots,N-1\}$$

$$\hat{x}_{k+N|k}\in\mathcal{X}_f$$

$$x_{k|k}=x(k)$$

- We will derive the following output constraint tightening

$$\hat{\mathcal{Y}}(0)=\mathcal{Y} \qquad\qquad L(0)=I_n$$

$$\hat{\mathcal{Y}}(j+1)=\hat{\mathcal{Y}}(j)\ominus(C+DK)L(j)\mathcal{W} \qquad\qquad L(j+1)=(A+BK)L(j)$$

[1] Arthur Richards, "Robust Constrained Model Predictive Control," Ph.D. Dissertation, MIT, 2002.

Time-varying Constraint Tightening (cont.)

- Derive the constraint tightening

$$\hat{y}_{k+j|k} = C\hat{x}_{k+j|k} + D\hat{u}_{k+j|k} \in \hat{\mathcal{Y}}(j)$$

$$\hat{\mathcal{Y}}(0) = \mathcal{Y} \qquad L(0) = I_n$$

$$\hat{\mathcal{Y}}(j+1) = \hat{\mathcal{Y}}(j) \ominus (C+DK)L(j)\mathcal{W} \qquad L(j+1) = (A+BK)L(j)$$

- Don't need to tighten at current time step $\hat{\mathcal{Y}}(0) = \mathcal{Y}$
  - State is perfectly measured
  - Get to use entire input set
- Using the candidate control law, we get

$$x_{j+1} = Ax_j + Bu_j + w_j \qquad e_j = x_j + \hat{x}_j \qquad e_{j+1} = (A+BK)e_j + w_j$$

$$y_j = Cx_j + Du_j \qquad e_j^y = y_j + \hat{y}_j \qquad e_j^y = (C+DK)e_j$$

- At time step 1, we tighten the output constraints based on the prediction error

$$e_1 = x_1 - \hat{x}_1 \in \mathcal{W} \quad \Rightarrow e_1^y \in (C+DK)\mathcal{W}$$

$$\hat{\mathcal{Y}}(1) = \hat{\mathcal{Y}}(0) \ominus (C+DK)\mathcal{W}$$

$$\Rightarrow$$

$$\hat{\mathcal{Y}}(1) = \hat{\mathcal{Y}}(0) \ominus (C+DK)L(0)\mathcal{W}$$

$$\hat{\mathcal{Y}}(0) = \mathcal{Y} \qquad L(0) = I_n$$

Time-varying Constraint Tightening (cont.)

- Derive the constraint tightening

$$\hat{y}_{k+j|k} = C\hat{x}_{k+j|k} + D\hat{u}_{k+j|k} \in \hat{\mathcal{Y}}(j)$$

$$\hat{\mathcal{Y}}(0) = \mathcal{Y} \qquad L(0) = I_n$$

$$\hat{\mathcal{Y}}(j+1) = \hat{\mathcal{Y}}(j) \ominus (C+DK)L(j)\mathcal{W} \qquad L(j+1) = (A+BK)L(j)$$

- At time step 2, we tighten the output constraints based on the prediction error

$$e_{j+1} = (A+BK)e_j + w_j$$

$$e_j^y = (C+DK)e_j$$

$$e_2 = x_2 - \hat{x}_2 \in (A+BK)\mathcal{W} \oplus \mathcal{W}$$

$$\Rightarrow e_1^y \in (C+DK)\big((A+BK)\mathcal{W} \oplus \mathcal{W}\big)$$

$$\hat{\mathcal{Y}}(2) = \hat{\mathcal{Y}}(0) \ominus (C+DK)\big((A+BK)\mathcal{W} \oplus \mathcal{W}\big)$$

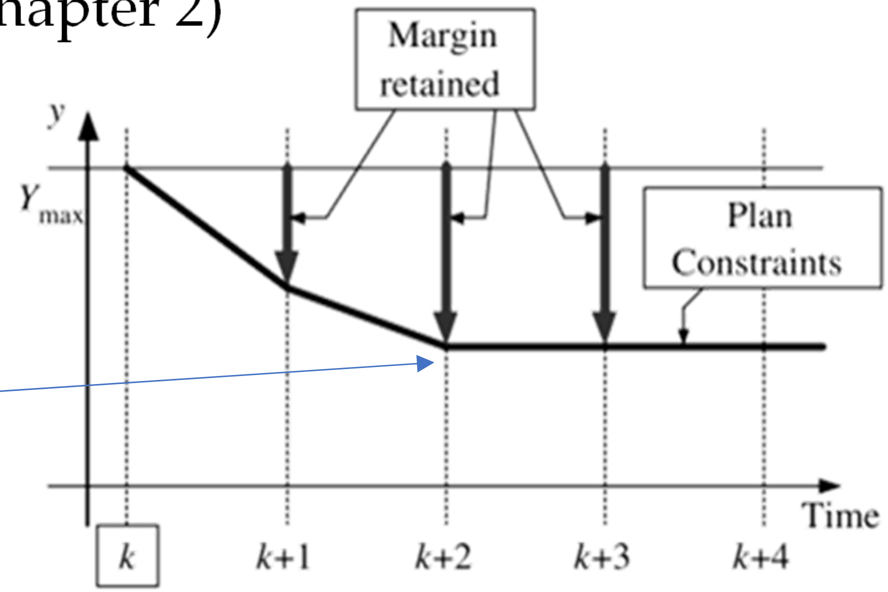
$$\hat{\mathcal{Y}}(2) = \hat{\mathcal{Y}}(1) \ominus (C+DK)\big((A+BK)\mathcal{W}\big)$$

$$\Rightarrow \quad \hat{\mathcal{Y}}(2) = \hat{\mathcal{Y}}(1) \ominus (C+DK)L(1)\mathcal{W}$$

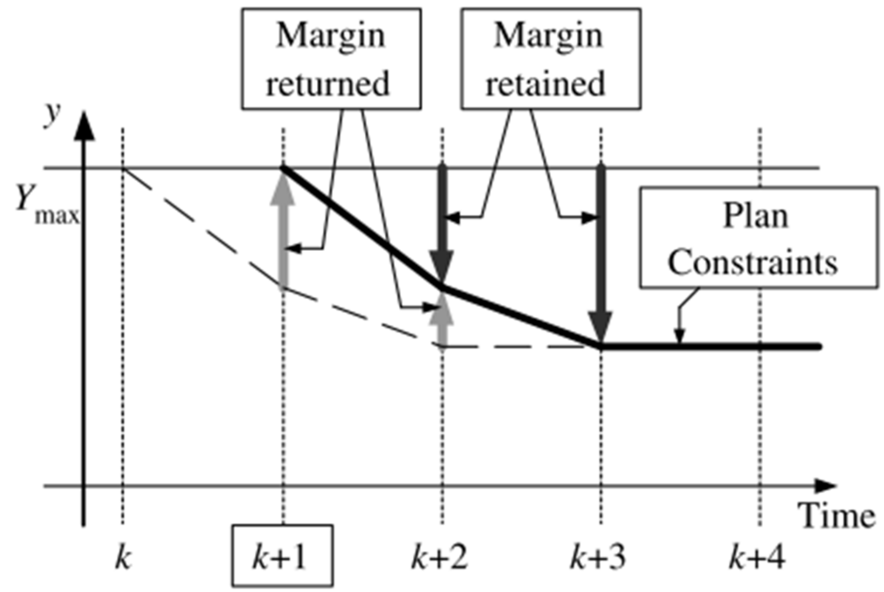
$$\hat{\mathcal{Y}}(1) = \hat{\mathcal{Y}}(0) \ominus (C+DK)L(0)\mathcal{W} \qquad L(1) = A+BK$$

Time-varying Constraint Tightening (cont.)

- Approach from Arthur Richards [1] (Chapter 2)



- Nilpotent candidate feedback controller was important to allow constraints to stop shrinking



- In this approach, you never actually implement the candidate control law
  - The input from MPC is applied directly to the system
  - The candidate control law is used for constraint tightening and guarantees that the MPC optimization problem has a least one feasible solution

Tube-based MPC

- The time-varying constraint tightening has its benefits, as you will see in HW #3
  - But the challenge is that you need a different set of constraints at every time step – complicating control design and analysis
  - It would be simpler if the constraint tightening was the same at each time step
- Approach from [2]
  - Based on the idea of robust positive invariant sets
  - Main idea:
    - We have already see that the prediction error using a candidate feedback control law is bounded by

$$e_k = x_k - \hat{x}_k \in \bigoplus_{i=0}^{k-1} A_K^i \mathcal{W}$$

    - This set gets bigger and bigger as k increases
    - But if A_K is stable (or ideally nilpotent), the size of this set will converge
    - If it converges, this is known as the minimal Robust Positively Invariant (minRPI) set

[2] D.Q. Mayne, M.M. Seron, S.V. Rakovic, "Robust Model Predictive Control of Constrained Linear Systems with Bounded Disturbances," Automatica, 2005.

minRPI set

- We have previously talked about positive invariant sets for $x_{k+1} = Ax_k$

A set $\mathcal{O} \subseteq \mathcal{X}$ is said to be a positive invariant set for a constrained autonomous system if

$$x_0 \in \mathcal{O} \quad \Rightarrow \quad x_k \in \mathcal{O} \quad \forall k > 0$$

- With bounded additive disturbances: $x_{k+1} = Ax_k + w_k \quad w_k \in \mathcal{W}$

A set $\mathcal{O} \subseteq \mathcal{X}$ is said to be a robust positive invariant set for a constrained autonomous system if

$$x_0 \in \mathcal{O} \quad \Rightarrow \quad x_k \in \mathcal{O} \quad \forall w_k \in \mathcal{W}, \ \forall k > 0$$

- In terms of Minkowski sum

$$\begin{array}{c} x_{k+1} = Ax_k + w_k \quad w_k \in \mathcal{W} \\ x_k \in \mathcal{O} \quad \Rightarrow \quad x_{k+1} \in \mathcal{O} \quad \forall w_k \in \mathcal{W} \end{array} \quad \Longrightarrow \quad A\mathcal{O} \oplus \mathcal{W} \subseteq \mathcal{O}$$

- We have talked about the maximal positive invariant set that contains all invariant sets
- Now we have the minimal robust positive invariant set that is contained in every robust positive invariant set.

minRPI set (cont.)

- Lots of details on computing the minRPI set provided in [1]
  - I have provided my implementation of the algorithm in [1] to help with HW #3
- Key ideas for computing minRPI set
  - State prediction error (can do this for any system)

$$e_{k+1} = (A + BK)e_k + w_k \qquad e_k \in \bigoplus_{i=0}^{k-1} A_K^i \mathcal{W}$$

  - If as some point $A_K^{i+1} = 0$, then

$$e_k \in \bigoplus_{i=0}^{k-1} A_K^i \mathcal{W} = \mathcal{O} \qquad e_{k+1} \in \bigoplus_{i=0}^{k} A_K^i \mathcal{W} = \bigoplus_{i=0}^{k-1} A_K^i \mathcal{W} = \mathcal{O}$$

$$e_k \in \mathcal{O} \Rightarrow e_{k+1} \in \mathcal{O}$$

← Definition of robust positive invariant!

- Turns out that this is the way to compute the smallest RPI set
- Also works if A_K^{i+1} never equals 0
  - Set might no longer be a polytope and may need to be approximated

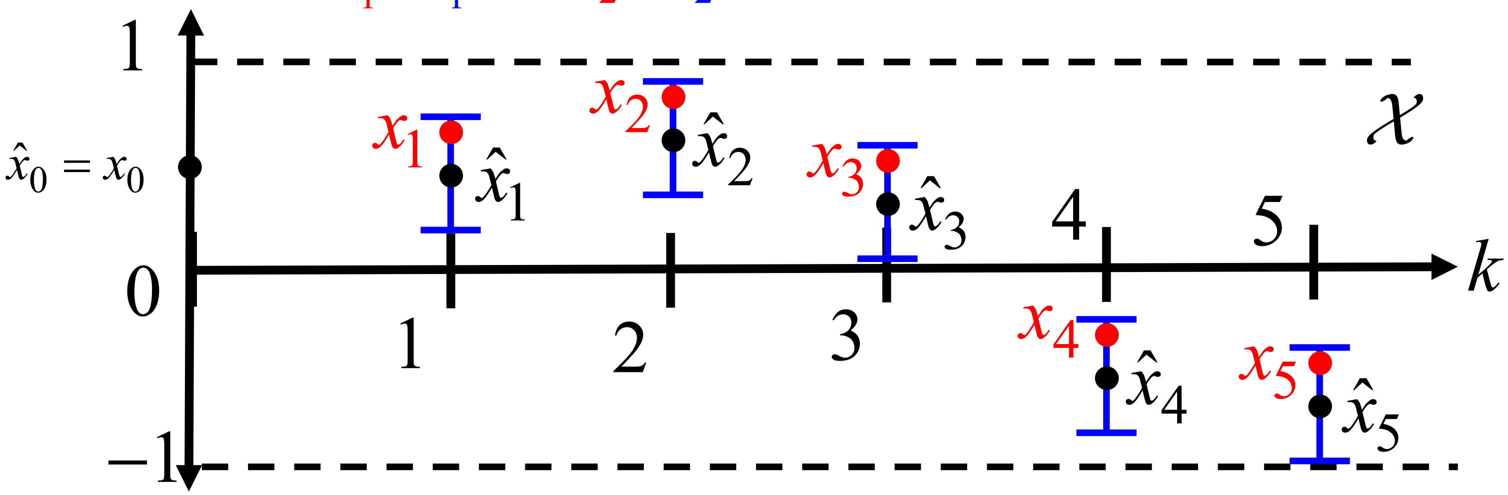
[1] S.V. Rakovic, E.C. Kerrigan, K.I. Kouranas, D.Q. Mayne, "Invariant Approximations of the Minimal Robust Positively Invariant Set," IEE TAC, 2005.

Tube-based MPC (cont.)

- Now that we can compute the minRPI set, we know that if the state prediction error starts "small" it will always stay small
- Thus, the actual state trajectory will always be in a "tube" around the nominal state trajectory – and this tube is the minRPI set

$x_0 - \hat{x}_0 = 0 \in \mathcal{O}$ $\quad x_1 - \hat{x}_1 \in \mathcal{O}$ $\quad x_2 - \hat{x}_2 \in \mathcal{O}$

$x_1 \in \hat{x}_1 \oplus \mathcal{O}$ $\quad x_2 \in \hat{x}_2 \oplus \mathcal{O}$



- Main result:

$$x_k - \hat{x}_k \in \mathcal{O}, \ \forall k$$

$$u_k - \hat{u}_k = K\left(x_k - \hat{x}_k\right) \in K\mathcal{O}, \ \forall k$$

$$\hat{x}_k \in \mathcal{X} \ominus \mathcal{O} \quad \Rightarrow \ x_k \in \mathcal{X} \quad \forall k$$

$$\hat{u}_k \in \mathcal{U} \ominus K\mathcal{O} \quad \Rightarrow \ u_k \in \mathcal{U} \quad \forall k$$

Tube-based MPC (cont.)

- Approach from [2]

$$J^*\left(x(k)\right) = \min_{U_k} \sum_{j=0}^{N-1} \hat{x}_{k+j|k}^T Q \hat{x}_{k+j|k} + \hat{u}_{k+j|k}^T R \hat{u}_{k+j|k} + \hat{x}_{k+N|k}^T P \hat{x}_{k+N|k}$$

$$s.t.$$

$$\hat{x}_{k+j+1|k} = A\hat{x}_{k+j|k} + B\hat{u}_{k+j|k}, \ \ j \in \{0,1,\ldots,N-1\}$$

$$\hat{x}_{k+j|k} \in \hat{\mathcal{X}} = \mathcal{X} \ominus \mathcal{O}, \quad j \in \{0,1,\ldots,N-1\}$$

$$\hat{u}_{k+j|k} \in \hat{\mathcal{U}} = \mathcal{U} \ominus K\mathcal{O}, \quad j \in \{0,1,\ldots,N-1\}$$

$$\hat{x}_{k+N|k} \in \mathcal{X}_f \subseteq \mathcal{X} \ominus \mathcal{O}$$

Approximation of minRPI set (referred to as Z in [2])

$$x(k) - x_{k|k} \in \mathcal{O}$$

Choose initial condition error to start in the minRPI set

- Actually implement candidate feedback control law

$$u_k = \hat{u}_{k|k}^* + K\left(x(k) - \hat{x}_{k|k}^*\right)$$

[2] D.Q. Mayne, M.M. Seron, S.V. Rakovic, “Robust Model Predictive Control of Constrained Linear Systems with Bounded Disturbances,” Automatica, 2005.