MECH 6v29.002 – Model Predictive Control

L14 – Robust MPC

Outline

- Robust invariant sets
- Robust MPC
  - Constraint tightening
  - Need for feedback
  - Uncertainty analysis
  - Nilpotent candidate controller

Robust Invariant Sets

- Nothing conceptually changes now that we have bounded uncertainties
- Robust Positive Invariant (RPI) Set:

  A set $\mathcal{O} \subseteq \mathcal{X}$ is said to be a robust positive invariant set for a constrained autonomous system if

$$x_0 \in \mathcal{O} \;\Rightarrow\; x_k \in \mathcal{O} \quad \forall w_k \in \mathcal{W}, \quad \forall k > 0$$

- Refer back to Lecture 10 to make slight modifications to other sets
  - Maximal Robust Positive Invariant Set
  - Robust Control Invariant Set
  - Maximal Robust Control Invariant Set
- Computational algorithms are the same too
  - (just compute the robust precursor set)
- Will use this to formulate robust MPC this week!

Inputs: $g(x,w),\ \mathcal{X},\ \mathcal{W}$

Outputs: $\mathcal{O}_\infty$

$\Omega_0 \leftarrow \mathcal{X},\ k \leftarrow -1$

Repeat

$k \leftarrow k+1$

$\Omega_{k+1} \leftarrow \text{Pre}(\Omega_k, \mathcal{W}) \cap \Omega_k$

Until $\Omega_{k+1} = \Omega_k$

$\mathcal{O}_\infty = \Omega_k$

Robust MPC

- Consider an (linear) system with bounded disturbances, e.g.

$$x_{k+1} = Ax_k + Bu_k + w_k \qquad w_k \in \mathcal{W}, \quad \forall k \geq 0$$

Compact convex set with the origin in its interior

- With state and input constraints

$$x_k \in \mathcal{X}, \ \ u_k \in \mathcal{U}, \quad \forall k \geq 0.$$

- Our goal is to design an MPC controller based on a nominal model of the system

$$\hat{x}_{k+1} = A\hat{x}_k + B\hat{u}_k$$

- Such that the state and input constraints are satisfied for all possible disturbance trajectories.
- Many different methods to solve this robust MPC problem in the literature
    - We will focus on two that leverage our recent work on set-based reachability and feasibility analysis

Robust MPC (cont.)

- Central idea
  - Use the nominal system model to make predictions
  - Use constraint tightening such that if the nominal system trajectories satisfy the tightened constraints, then the system with disturbances will satisfy the original constraints

$$\hat{x}_{k+1} = A\hat{x}_k + B\hat{u}_k \qquad\qquad x_{k+1} = Ax_k + Bu_k + w_k$$

$$\begin{bmatrix} \hat{x}_k \\ \hat{u}_k \end{bmatrix} \in \hat{\mathcal{X}} \times \hat{\mathcal{U}} \quad \Rightarrow \quad \begin{bmatrix} x_k \\ u_k \end{bmatrix} \in \mathcal{X} \times \mathcal{U}$$

$$\hat{\mathcal{X}} \subseteq \mathcal{X}$$

$$\hat{\mathcal{U}} \subseteq \mathcal{U}$$

Constraint Tightening

- At time step $k = 0$, lets assume that we have measured the state and are predicting one step into the future $\hat{x}_0 = x_0$

Where we think we will be $\longrightarrow$ $\hat{x}_1 = Ax_0 + Bu_0$

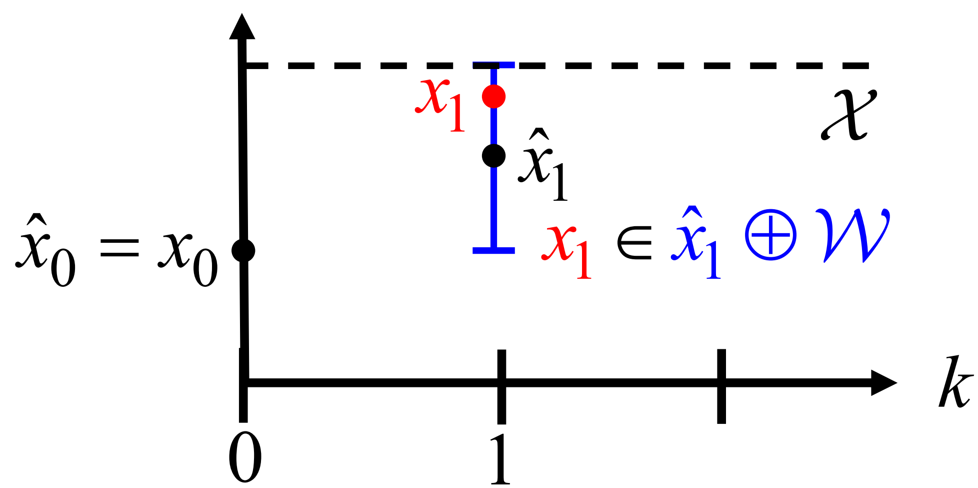
Where we will actually be $\longrightarrow$ $x_1 = Ax_0 + Bu_0 + w_0 \qquad w_0 \in \mathcal{W}$

- Difference between predicted and actual state is bounded

$$x_1 - \hat{x}_1 = w_0 \qquad w_0 \in \mathcal{W} \qquad x_1 - \hat{x}_1 \in \mathcal{W}$$

- This gives us some insight
  - If we want $x_1 \in \mathcal{X}$
  - And $x_1 \in \hat{x}_1 \oplus \mathcal{W}$
  - Then

    $\hat{x}_1 \oplus \mathcal{W} \subseteq \mathcal{X} \;\Rightarrow x_1 \in \mathcal{X}$
  - Note



closing

$$\hat{x}_1 \oplus \mathcal{W} \ominus \mathcal{W} \subseteq \mathcal{X} \ominus \mathcal{W}$$ $\longleftarrow$ Increasing property of Pontryagin difference

Extensive property of the closing

$$\hat{x}_1 \subseteq \hat{x}_1 \oplus \mathcal{W} \ominus \mathcal{W} \subseteq \mathcal{X} \ominus \mathcal{W}$$

$$\hat{x}_1 \in \mathcal{X} \ominus \mathcal{W} \Rightarrow x_1 \in \mathcal{X}$$

Need for Feedback

- We can repeat this process to determine how much the constraints need to be tightened at each future step

$$\hat{x}_1 = Ax_0 + Bu_0$$
$$x_1 = Ax_0 + Bu_0 + w_0$$

$$\Rightarrow \quad x_1 - \hat{x}_1 = w_0$$
$$x_1 - \hat{x}_1 \in \mathcal{W}$$

$$\hat{x}_2 = A\hat{x}_1 + Bu_1$$
$$x_2 = Ax_1 + Bu_1 + w_1$$

$$\Rightarrow \quad x_2 - \hat{x}_2 = A\left(x_1 - \hat{x}_1\right) + w_1 = Aw_0 + w_1$$
$$x_2 - \hat{x}_2 \in A\mathcal{W} \oplus \mathcal{W}$$

$$\Downarrow$$

$$x_k - \hat{x}_k \in \bigoplus_{i=0}^{k-1} A^i \mathcal{W}$$

$$\Downarrow$$

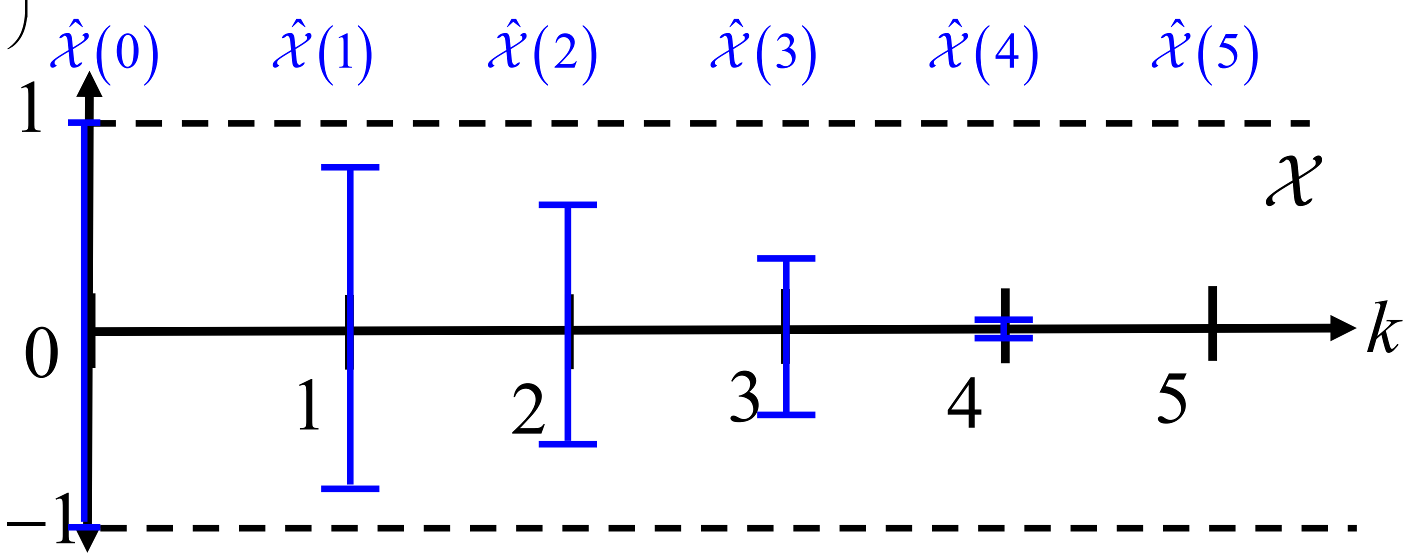
$$\hat{x}_k \in \mathcal{X} \ominus \left(\bigoplus_{i=0}^{k-1} A^i \mathcal{W} \right) \Rightarrow x_k \in \mathcal{X}$$

Need for Feedback (cont.)

- However, this creates a problem!
  - Uncertainty set can get too big
  - Resulting in the tightened constraint set to be empty
- Simple example $x_{k+1} = x_k + u_k + w_k \quad x_k, u_k, w_k \in \mathbb{R}$

$$x_k \in \mathcal{X} = [-1,1], \;\; u_k \in \mathcal{U} = [-1,1], \;\; w_k \in \mathcal{W} = [-0.25, 0.25]$$

$\bigoplus_{i=0}^{k-1} A^i \mathcal{W}$	$[-0.25,0.25]$	$[-0.5,0.5]$	$[-0.75,0.75]$	$[-1,1]$	$[-1.25,1.25]$
$\hat{\mathcal{X}}(k) = \mathcal{X} \ominus \left(\bigoplus_{i=0}^{k-1} A^i \mathcal{W} \right)$	$[-0.75,0.75]$	$[-0.5,0.5]$	$[-0.25,0.25]$	$\{0\}$	$\varnothing$



Need for Feedback (cont.)

- Note that we are currently trying to make our MPC controller robust by restricting the set that the controller can use for planning nominal state trajectories

$$\hat{x}(k) \in \hat{\mathcal{X}}(k) = \mathcal{X} \ominus \left(\bigoplus_{i=0}^{k-1} A^i \mathcal{W} \right)$$

- But we are still allowing the controller to use all of the possible inputs $u(k) \in \mathcal{U}$
- Perhaps it makes sense to give up a little of both
- This will also help solve the problem from the previous slide by preventing the uncertainty from getting too big
- The key idea is that, while we are planning an open-loop trajectory, we can leverage the fact that we can implement a closed-loop controller that reacts to the unknown disturbances
- Let's choose a candidate control law

Input to the system

$$u_k = \hat{u}_k + K\left(x_k - \hat{x}_k\right)$$

Nominal input chosen by MPC

Measured system state

Nominal system state used in MPC predictions

Uncertainty Analysis

- Real and nominal systems

$$x_{k+1} = Ax_k + Bu_k + w_k \qquad w_k \in \mathcal{W}, \quad \forall k \geq 0$$

$$\hat{x}_{k+1} = A\hat{x}_k + B\hat{u}_k$$

- Candidate control law

$$u_k = \hat{u}_k + K\left(x_k - \hat{x}_k\right)$$

- Prediction error $e_k = x_k - \hat{x}_k$
- Prediction error dynamics

$$\begin{aligned} e_k^+ = x_k^+ - \hat{x}_k^+ &= \left(Ax_k + Bu_k + w_k\right) - \left(A\hat{x}_k + B\hat{u}_k\right) \\ &= A\left(x_k - \hat{x}_k\right) + B\left(u_k - \hat{u}_k\right) + w_k \\ &= Ae_k + BKe_k + w_k \\ &= \left(A + BK\right)e_k + w_k \\ &= A_K e_k + w_k \end{aligned}$$

- Error is an autonomous system driven by the bounded disturbance

Uncertainty Analysis (cont.)

- Without a candidate feedback control law, we originally had

$$e_k = x_k - \hat{x}_k \in \bigoplus_{i=0}^{k-1} A^i \mathcal{W} \qquad \hat{x}_k \in \mathcal{X} \ominus \left(\bigoplus_{i=0}^{k-1} A^i \mathcal{W} \right) \Rightarrow x_k \in \mathcal{X} \qquad u_k \in \mathcal{U}$$

- With a feedback control law, we now have

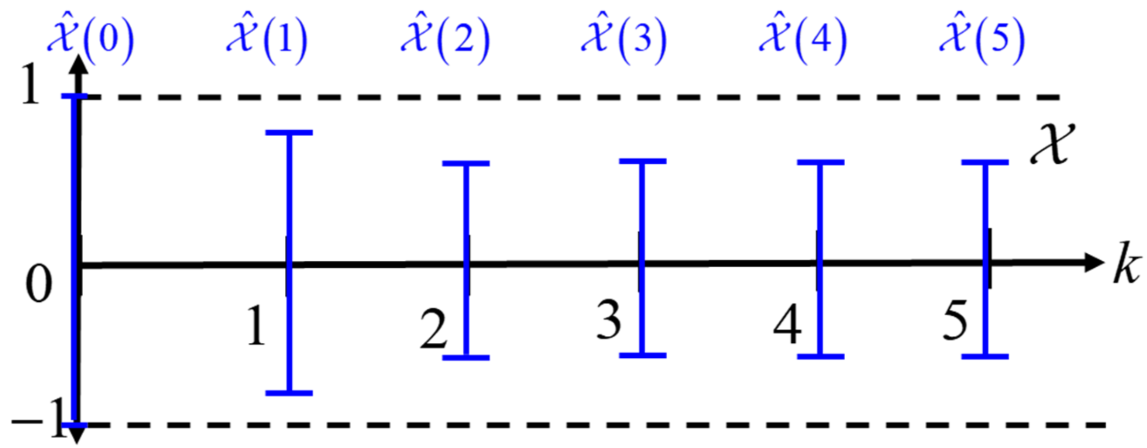
$$e_k = x_k - \hat{x}_k \in \bigoplus_{i=0}^{k-1} A_K^i \mathcal{W} \qquad \hat{x}_k \in \mathcal{X} \ominus \left(\bigoplus_{i=0}^{k-1} A_K^i \mathcal{W} \right) \Rightarrow x_k \in \mathcal{X}$$

$$\hat{u}_k \in \mathcal{U} \ominus K \left(\bigoplus_{i=0}^{k-1} A_K^i \mathcal{W} \right) \Rightarrow u_k \in \mathcal{U}$$

- Through the design of K we can choose how much we want to tighten X at the cost of tightening U
- This is particularly valuable for marginally stable or unstable open-loop systems where the error gets very large without feedback

Nilpotent Candidate Controller

- The proper design of K can prevent the error set from growing indefinitely
  - Stops the tightened constraints from continuously shrinking



- Design K to place the closed-loop eigenvalues at the origin

$$eig\left(A_K\right) = eig\left(A + BK\right) = 0$$

- Closed-loop system is nilpotent $A_K \in \mathbb{R}^{n \times n},\ \left(A_K\right)^n = 0 \Rightarrow \bigoplus_{i=0}^{\infty} A_K^i \mathcal{W} = \bigoplus_{i=0}^{n-1} A_K^i \mathcal{W}$
- Connect in terms of Cayley-Hamilton theorem
  - All eigenvalues at the origin results in the characteristic equation

$$(\lambda - 0)(\lambda - 0)....(\lambda - 0) = 0$$

$$\lambda^n = 0 \xrightarrow{\text{Cayley-Hamilton}} \left(A_K\right)^n = 0$$

Robust MPC Formulations

- Next class we will present two difference robust MPC formulations based on this uncertainty analysis
  - Time-varying constraint tightening [1]
  - Tube-based MPC [2]
- In preparation, carefully read at least the following
  - Sections 2.2, 2.3, and 2.5.1 of [1]
  - Sections 2, 3, and first part of 4 up until Proposition 2 in [2]
- You will be implementing both of these in HW #3
  - Specifically, you will be implementing the numerical example from 2.5.1 to recreate Fig. 2-2 in [1] using both of the methods from [1] and [2]
- Also, please complete the Mid-semester feedback survey on eLearning by this Friday

[1] Arthur Richards, “Robust Constrained Model Predictive Control,” Ph.D. Dissertation, MIT, 2002.
[2] D.Q. Mayne, M.M. Seron, S.V. Rakovic, “Robust Model Predictive Control of Constrained Linear Systems with Bounded Disturbances,” Automatica, 2005.

Time-varying Constraint Tightening

- Approach from Arthur Richards [1] (Chapter 2)

$$J^*\left(x(k)\right) = \min_{U_k} \sum_{j=0}^{N-1} \hat{x}_{k+j|k}^T Q \hat{x}_{k+j|k} + \hat{u}_{k+j|k}^T R \hat{u}_{k+j|k} + \hat{x}_{k+N|k}^T P \hat{x}_{k+N|k}$$

$$s.t.$$

$$\hat{x}_{k+j+1|k} = A\hat{x}_{k+j|k} + B\hat{u}_{k+j|k}, \;\; j \in \{0,1,\ldots,N-1\}$$

$$\hat{y}_{k+j|k} = C\hat{x}_{k+j|k} + D\hat{u}_{k+j|k} \in \hat{\mathcal{Y}}(j), \quad j \in \{0,1,\ldots,N-1\}$$

$$\hat{x}_{k+N|k} \in \mathcal{X}_f$$

$$x_{k|k} = x(k)$$

- We will derive the following output constraint tightening

$$\hat{\mathcal{Y}}(0) = \mathcal{Y} \qquad\qquad L(0) = I_n$$

$$\hat{\mathcal{Y}}(j+1) = \hat{\mathcal{Y}}(j) \ominus (C+DK)L(j)\mathcal{W} \qquad\qquad L(j+1) = (A+BK)L(j)$$

[1] Arthur Richards, "Robust Constrained Model Predictive Control," Ph.D. Dissertation, MIT, 2002.

Time-varying Constraint Tightening (cont.)

- Derive the constraint tightening

$$\hat{y}_{k+j|k} = C\hat{x}_{k+j|k} + D\hat{u}_{k+j|k} \in \hat{\mathcal{Y}}(j)$$

$$\hat{\mathcal{Y}}(0) = \mathcal{Y} \qquad L(0) = I_n$$

$$\hat{\mathcal{Y}}(j+1) = \hat{\mathcal{Y}}(j) \ominus (C+DK)L(j)\mathcal{W} \qquad L(j+1) = (A+BK)L(j)$$

- Don't need to tighten at current time step $\hat{\mathcal{Y}}(0) = \mathcal{Y}$
  - State is perfectly measured
  - Get to use entire input set
- Using the candidate control law, we get

$$x_{j+1} = Ax_j + Bu_j + w_j \qquad e_j = x_j + \hat{x}_j \qquad e_{j+1} = (A+BK)e_j + w_j$$

$$y_j = Cx_j + Du_j \qquad e_j^y = y_j + \hat{y}_j \qquad e_j^y = (C+DK)e_j$$

- At time step 1, we tighten the output constraints based on the prediction error

$$e_1 = x_1 - \hat{x}_1 \in \mathcal{W} \quad \Rightarrow e_1^y \in (C+DK)\mathcal{W}$$

$$\hat{\mathcal{Y}}(1) = \hat{\mathcal{Y}}(0) \ominus (C+DK)\mathcal{W}$$

$$\Rightarrow$$

$$\hat{\mathcal{Y}}(1) = \hat{\mathcal{Y}}(0) \ominus (C+DK)L(0)\mathcal{W}$$

$$\hat{\mathcal{Y}}(0) = \mathcal{Y} \qquad L(0) = I_n$$

Time-varying Constraint Tightening (cont.)

- Derive the constraint tightening

$$\hat{y}_{k+j|k} = C\hat{x}_{k+j|k} + D\hat{u}_{k+j|k} \in \hat{\mathcal{Y}}(j)$$

$$\hat{\mathcal{Y}}(0) = \mathcal{Y} \qquad L(0) = I_n$$

$$\hat{\mathcal{Y}}(j+1) = \hat{\mathcal{Y}}(j) \ominus (C+DK)L(j)\mathcal{W} \qquad L(j+1) = (A+BK)L(j)$$

- At time step 2, we tighten the output constraints based on the prediction error

$$e_{j+1} = (A+BK)e_j + w_j$$

$$e_j^y = (C+DK)e_j$$

$$e_2 = x_2 - \hat{x}_2 \in (A+BK)\mathcal{W} \oplus \mathcal{W}$$

$$\Rightarrow e_1^y \in (C+DK)\big((A+BK)\mathcal{W} \oplus \mathcal{W}\big)$$

$$\hat{\mathcal{Y}}(2) = \hat{\mathcal{Y}}(0) \ominus (C+DK)\big((A+BK)\mathcal{W} \oplus \mathcal{W}\big)$$

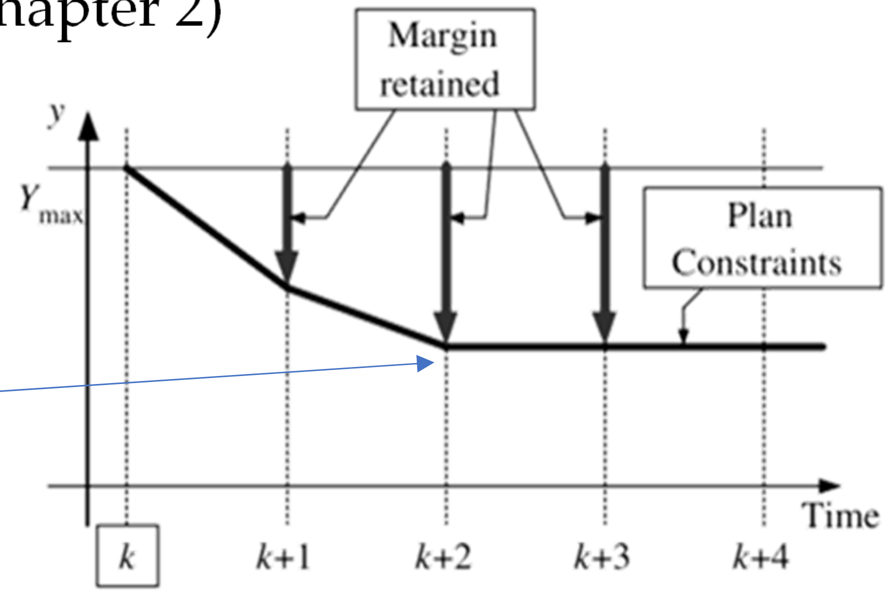
$$\hat{\mathcal{Y}}(2) = \hat{\mathcal{Y}}(1) \ominus (C+DK)\big((A+BK)\mathcal{W}\big)$$

$$\Rightarrow \hat{\mathcal{Y}}(2) = \hat{\mathcal{Y}}(1) \ominus (C+DK)L(1)\mathcal{W}$$

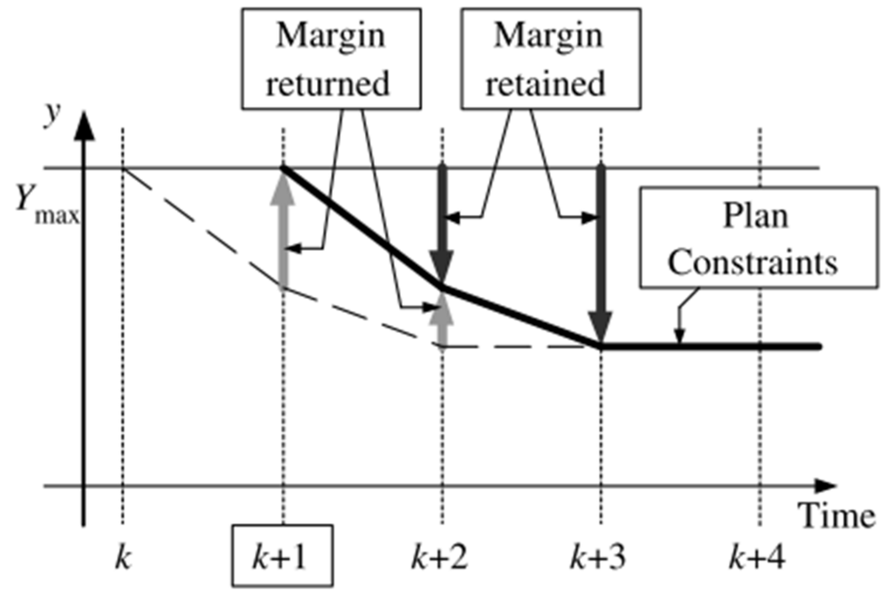
$$\hat{\mathcal{Y}}(1) = \hat{\mathcal{Y}}(0) \ominus (C+DK)L(0)\mathcal{W} \qquad L(1) = A+BK$$

Time-varying Constraint Tightening (cont.)

- Approach from Arthur Richards [1] (Chapter 2)



- Nilpotent candidate feedback controller was important to allow constraints to stop shrinking



- In this approach, you never actually implement the candidate control law
  - The input from MPC is applied directly to the system
  - The candidate control law is used for constraint tightening and guarantees that the MPC optimization problem has a least one feasible solution

Tube-based MPC

- The time-varying constraint tightening has its benefits, as you will see in HW #3
  - But the challenge is that you need a different set of constraints at every time step – complicating control design and analysis
  - It would be simpler if the constraint tightening was the same at each time step
- Approach from [2]
  - Based on the idea of robust positive invariant sets
  - Main idea:
    - We have already see that the prediction error using a candidate feedback control law is bounded by

$$e_k = x_k - \hat{x}_k \in \bigoplus_{i=0}^{k-1} A_K^i \mathcal{W}$$

    - This set gets bigger and bigger as k increases
    - But if A_K is stable (or ideally nilpotent), the size of this set will converge
    - If it converges, this is known as the minimal Robust Positively Invariant (minRPI) set

[2] D.Q. Mayne, M.M. Seron, S.V. Rakovic, “Robust Model Predictive Control of Constrained Linear Systems with Bounded Disturbances,” Automatica, 2005.

minRPI set

- We have previously talked about positive invariant sets for $x_{k+1} = Ax_k$

A set $\mathcal{O} \subseteq \mathcal{X}$ is said to be a positive invariant set for a constrained autonomous system if

$$x_0 \in \mathcal{O} \quad \Rightarrow \quad x_k \in \mathcal{O} \quad \forall k > 0$$

- With bounded additive disturbances: $x_{k+1} = Ax_k + w_k \quad w_k \in \mathcal{W}$

A set $\mathcal{O} \subseteq \mathcal{X}$ is said to be a robust positive invariant set for a constrained autonomous system if

$$x_0 \in \mathcal{O} \quad \Rightarrow \quad x_k \in \mathcal{O} \quad \forall w_k \in \mathcal{W},\ \forall k > 0$$

- In terms of Minkowski sum

$$\begin{aligned} x_{k+1} &= Ax_k + w_k \quad w_k \in \mathcal{W} \\ x_k \in \mathcal{O} \quad &\Rightarrow \quad x_{k+1} \in \mathcal{O} \quad \forall w_k \in \mathcal{W} \end{aligned} \quad \Longrightarrow \quad A\mathcal{O} \oplus \mathcal{W} \subseteq \mathcal{O}$$

- We have talked about the maximal positive invariant set that contains all invariant sets
- Now we have the minimal robust positive invariant set that is contained in every robust positive invariant set.

minRPI set (cont.)

- Lots of details on computing the minRPI set provided in [1]
  - I have provided my implementation of the algorithm in [1] to help with HW #3
- Key ideas for computing minRPI set
  - State prediction error (can do this for any system)

$$e_{k+1} = (A+BK)e_k + w_k \qquad e_k \in \bigoplus_{i=0}^{k-1} A_K^i \mathcal{W}$$

  - If as some point $A_K^{i+1} = 0$, then

$$e_k \in \bigoplus_{i=0}^{k-1} A_K^i \mathcal{W} = \mathcal{O} \qquad e_{k+1} \in \bigoplus_{i=0}^{k} A_K^i \mathcal{W} = \bigoplus_{i=0}^{k-1} A_K^i \mathcal{W} = \mathcal{O}$$

$$e_k \in \mathcal{O} \Rightarrow e_{k+1} \in \mathcal{O}$$ ← Definition of robust positive invariant!

- Turns out that this is the way to compute the smallest RPI set
- Also works if A_K^{i+1} never equals 0
  - Set might no longer be a polytope and may need to be approximated

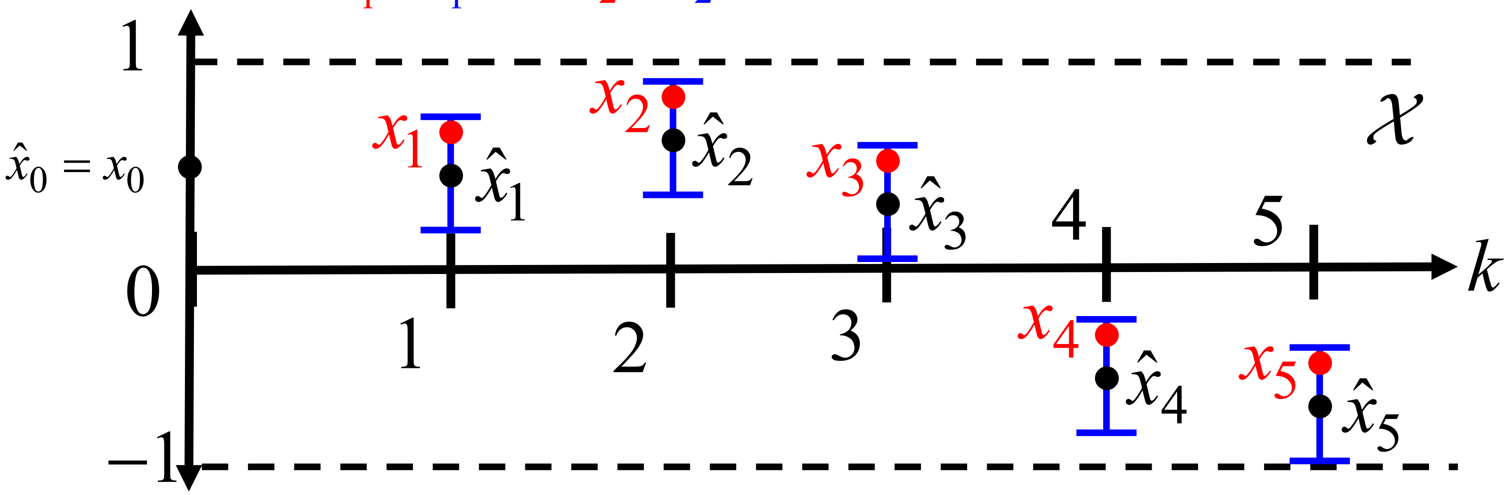
[1] S.V. Rakovic, E.C. Kerrigan, K.I. Kouranas, D.Q. Mayne, "Invariant Approximations of the Minimal Robust Positively Invariant Set," IEE TAC, 2005.

Tube-based MPC (cont.)

- Now that we can compute the minRPI set, we know that if the state prediction error starts "small" it will always stay small
- Thus, the actual state trajectory will always be in a "tube" around the nominal state trajectory – and this tube is the minRPI set

$x_0 - \hat{x}_0 = 0 \in \mathcal{O}$ $\quad x_1 - \hat{x}_1 \in \mathcal{O}$ $\quad x_2 - \hat{x}_2 \in \mathcal{O}$

$x_1 \in \hat{x}_1 \oplus \mathcal{O}$ $\quad x_2 \in \hat{x}_2 \oplus \mathcal{O}$



- Main result:

$$x_k - \hat{x}_k \in \mathcal{O}, \ \forall k$$

$$u_k - \hat{u}_k = K(x_k - \hat{x}_k) \in K\mathcal{O}, \ \forall k$$

$$\hat{x}_k \in \mathcal{X} \ominus \mathcal{O} \quad \Rightarrow \quad x_k \in \mathcal{X} \quad \forall k$$

$$\hat{u}_k \in \mathcal{U} \ominus K\mathcal{O} \quad \Rightarrow \quad u_k \in \mathcal{U} \quad \forall k$$

Tube-based MPC (cont.)

- Approach from [2]

$$J^*\left(x(k)\right) = \min_{U_k} \sum_{j=0}^{N-1} \hat{x}_{k+j|k}^T Q \hat{x}_{k+j|k} + \hat{u}_{k+j|k}^T R \hat{u}_{k+j|k} + \hat{x}_{k+N|k}^T P \hat{x}_{k+N|k}$$

$$s.t.$$

$$\hat{x}_{k+j+1|k} = A\hat{x}_{k+j|k} + B\hat{u}_{k+j|k}, \;\; j \in \{0,1,\ldots,N-1\}$$

$$\hat{x}_{k+j|k} \in \hat{\mathcal{X}} = \mathcal{X} \ominus \mathcal{O}, \quad j \in \{0,1,\ldots,N-1\}$$

$$\hat{u}_{k+j|k} \in \hat{\mathcal{U}} = \mathcal{U} \ominus K\mathcal{O}, \quad j \in \{0,1,\ldots,N-1\}$$

$$\hat{x}_{k+N|k} \in \mathcal{X}_f \subseteq \mathcal{X} \ominus \mathcal{O}$$

Approximation of minRPI set (referred to as Z in [2])

$$x(k) - x_{k|k} \in \mathcal{O}$$

Choose initial condition error to start in the minRPI set

- Actually implement candidate feedback control law

$$u_k = \hat{u}_{k|k}^* + K\left(x(k) - \hat{x}_{k|k}^*\right)$$

[2] D.Q. Mayne, M.M. Seron, S.V. Rakovic, “Robust Model Predictive Control of Constrained Linear Systems with Bounded Disturbances,” Automatica, 2005.