MECH 6v29 - Model Predictive Control
Homework 3

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Contents

Problem 1

Problem Data:

1a)

Problem Data:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{B} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \quad \mathcal{W} = \{\mathbf{w} = \mathbf{B}z \ : \ |z| \leq 0.3\}$$
$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{D} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathcal{Y} = \{\mathbf{y} \in \mathbb{R}^3 \ : \ \|\mathbf{y}\|_\infty \leq 1\} \tag{1}$$

Prediction Horizon: $N = 10$
Initial Condition: $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

1b) Nilpotent candidate controller

$\Lambda(\mathbf{A} + \mathbf{BK}) = 0$
Result w/ Acker: $\mathbf{K} = \begin{bmatrix} -1 & -1.5 \end{bmatrix}$
Same as in [**?**].

1c) Output constraint tightening

From reference (using different notation):

$$\begin{aligned} \mathcal{Y}_0 &= \mathcal{Y} \\ \mathcal{Y}_{j+1} &= \mathcal{Y}_j \ominus (\mathbf{C} + \mathbf{DK})\mathbf{L}_j \mathcal{W}, \quad \forall_{j \in \{0,\ldots,N-1\}} \end{aligned} \tag{2}$$

where $\mathbf{L}_j = (\mathbf{A} + \mathbf{BK})^j$.[1]
Or equivalently, using the time-invarience of K and some version of the Cayley-Hamilton theorem,

$$\mathcal{Y}_j = \mathcal{Y} \ominus \bigoplus_{i=1,\ldots,n} (\mathbf{C} + \mathbf{DK})(\mathbf{A} + \mathbf{BK})^{i-1} \tag{3}$$

(eliminating if the power is negative...)
TODO: double check this... (pretty sure this falls under some distributed property...)
For this system, $(\mathbf{C} + \mathbf{DK}) = \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{K} \end{bmatrix}$

$$\begin{aligned} \mathcal{Y}_0 &= \{\mathbf{y} \in \mathbb{R}^3 \ : \ \|\mathbf{y}\|_\infty \leq 1\} \\ &= \{\mathbf{y} \in \mathbb{R}^3 \ : \ |y_1| \leq 1, |y_2| \leq 1, |y_3| \leq 1\} \\ \mathcal{Y}_1 &= \mathcal{Y}_0 \ominus (\mathbf{C} + \mathbf{DK})\mathcal{W} \\ &= \mathcal{Y}_0 \ominus \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{K} \end{bmatrix} \{\mathbf{B}w \in \mathbb{R} \ : \ |w| \leq 0.3\} \\ &= \{\mathbf{y} \in \mathbb{R}^3 \ : \ |y_1| \leq 0.85, |y_2| \leq 0.7, |y_3| \leq 0.4\} \\ \mathcal{Y}_1 &= \mathcal{Y}_1 \ominus (\mathbf{C} + \mathbf{DK})(\mathbf{A} + \mathbf{BK})\mathcal{W} \\ &= \mathcal{Y}_1 \ominus \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{K} \end{bmatrix} \{\mathbf{B}w \in \mathbb{R} \ : \ |w| \leq 0.3\} \\ &= \{\mathbf{y} \in \mathbb{R}^3 \ : \ |y_1| \leq 0.7, |y_2| \leq 0.4, |y_3| \leq 0.1\} \end{aligned} \tag{4}$$

which is the same for the remaining since $(\mathbf{A} + \mathbf{BK})^2 = \mathbf{0}$.

[1] not explicitly, but eliminating time-variance that's what it is...

Problem 2

Problem Data:

$$\begin{aligned}
\mathbf{A} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{B} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \quad \mathcal{W} = \{\mathbf{w} = \mathbf{B}z \ : \ |z| \leq 0.3\} \\
\mathbf{C} &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{D} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathcal{Y} = \{\mathbf{y} \in \mathbb{R}^3 \ : \ \|\mathbf{y}\|_\infty \leq 1\}
\end{aligned} \tag{5}$$

Prediction Horizon: $N = 10$
Initial Condition: $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
Candidate Feedback Controller: $K = \begin{bmatrix} -1 & -1.5 \end{bmatrix}$

2a) Set Definitions

State/Input Constraints: From the output constraints the individual state/input constraints can be found by satisfying $C\mathcal{X} = \mathcal{Y} \ominus D\mathcal{U}$ and $D\mathcal{U} = \mathcal{Y} \ominus C\mathcal{X}$; but C and D would have to be invertible for a direct solution, so I'm not sure if it's good.
TODO: double check this...
Alternatively, we can look at the individual dimensions/values and do it simply by observation as we can decompose the dimensions:

$$\begin{aligned}
\mathcal{X} &= \{x \in \mathbb{R}^n \ : \ \|Cx\|_\infty \leq 1\} = \{x \in \mathbb{R}^n \ : \ |x_1| \leq 1, \ |x_2| \leq 1\} \\
\mathcal{U} &= \{u \in \mathbb{R}^m \ : \ \|Du\|_\infty \leq 1\} = \{u \in \mathbb{R}^m \ : \ |u_1| \leq 1\}
\end{aligned} \tag{6}$$

In H-rep this becomes:

$$\begin{aligned}
\mathcal{X} &= \left\{ A = \begin{bmatrix} \mathbf{I}_2 \\ -\mathbf{I}_2 \end{bmatrix}, \ b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \\
\mathcal{U} &= \left\{ A = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}
\end{aligned} \tag{7}$$

Disturbance Set: $\mathcal{W}$ is defined by

$$\mathcal{W} = \{\mathbf{w} = \mathbf{B}z \ : \ |z| \leq 0.3\} \tag{8}$$

and in H-rep it becomes

$$\mathcal{W} = \left\{ A = \begin{bmatrix} B \\ -B \end{bmatrix}, \ b = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix} \right\} \tag{9}$$

2b) RPI Sets

Constructing the RPI set was done in MATLAB following process in provided code (as instructed):

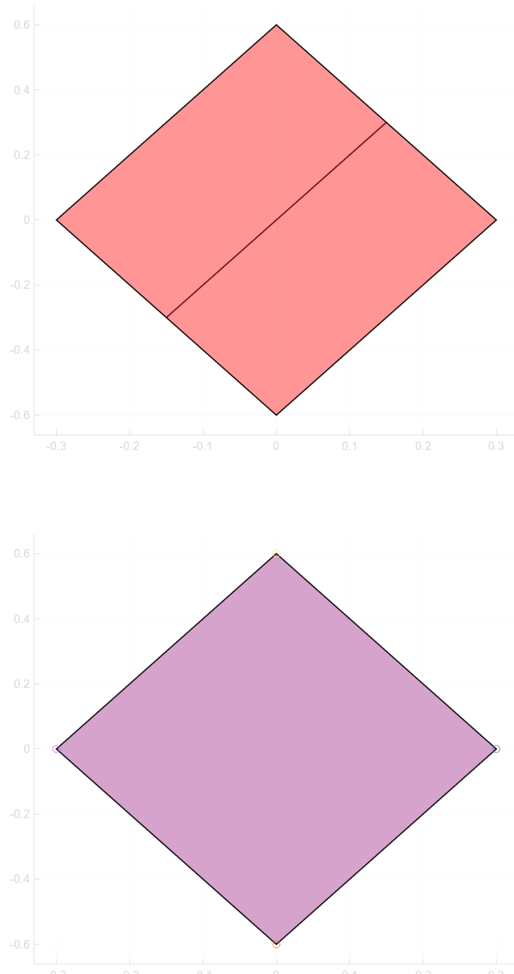

Figure 1: First one is the iterative portion and the second one is the one using the provided Approx_RPI() function

Testing different epsilon values resulted in an RPI set that didn't look all that different and the vertices all appeared to line up. (Tested many epsilon values all less than 1)
The result of Z is calculated in MATLAB as:

$$Z = \begin{bmatrix} 0.8639 & -0.4319 & 0.2592 \\ 0.8639 & 0.4319 & 0.2592 \\ -0.8639 & 0.4319 & 0.2592 \\ -0.8639 & -0.4319 & 0.2592 \end{bmatrix} \tag{10}$$

2c) Tightened Input/Output Constraint Sets

From [**?**], the disturbance invarient set for $x_{k+1} = A_K x_k + w$ is defined by $A_K Z \oplus W \subseteq Z$, where $A_K = A + B * K$ is stable.
This Z is the RPI set for $x_{k+1} = A_K x_k + w$. The proposition from the reference says that if $x_k \in \bar{x}_k$ and $u_k \in \bar{u}_k + K(x - \bar{x})$, then $x_{k+1} \in \bar{x}_{k+1} \oplus Z \; \forall_{w_k \in W}$. Where $x_{k+1} = Ax_k + Bu_k + w_k$. [2]
Result makes it so that $u_k = \bar{u}_k + K(x_k = \bar{x}_k)$ keeps the state close to the nominal one. [3]
The new state and input constraints are based on Z_k for each time-step.
To ensure that $x_k^* = x_k \oplus Z$ satisfies the original constraint set, the new constraints of

$$\bar{\mathcal{X}} = \mathcal{X} \ominus Z = \left\{ x \in \mathbb{R}^2 \ : \ |x_1| \leq 0.7, |x_2| \leq 0.4 \right\} \tag{11}$$

For the input, the state constraints are satisfied as

$$\bar{\mathcal{U}} = \mathcal{U} \ominus KZ = \{ u \in \mathbb{R} \ : \ |u| \leq 0.1 \} \tag{12}$$

As expected, both constraint sets are smaller than the original sets.

2d) Controller Formulation

The controller is now described differently then before in a different way.

[2] some of the time-invarience isn't included

[3] this also means it can ensure that it satisfies state/input constraints as well

A Code

1a) Github

See my github repo for all my course related materials: `https://github.com/jonaswagner2826/MECH6v29_MPC`

1b) Matlab results

MATLAB code and results are attached.