MECH 6v29.002 – Model Predictive Control

L6b – MPC Formulation and Extensions

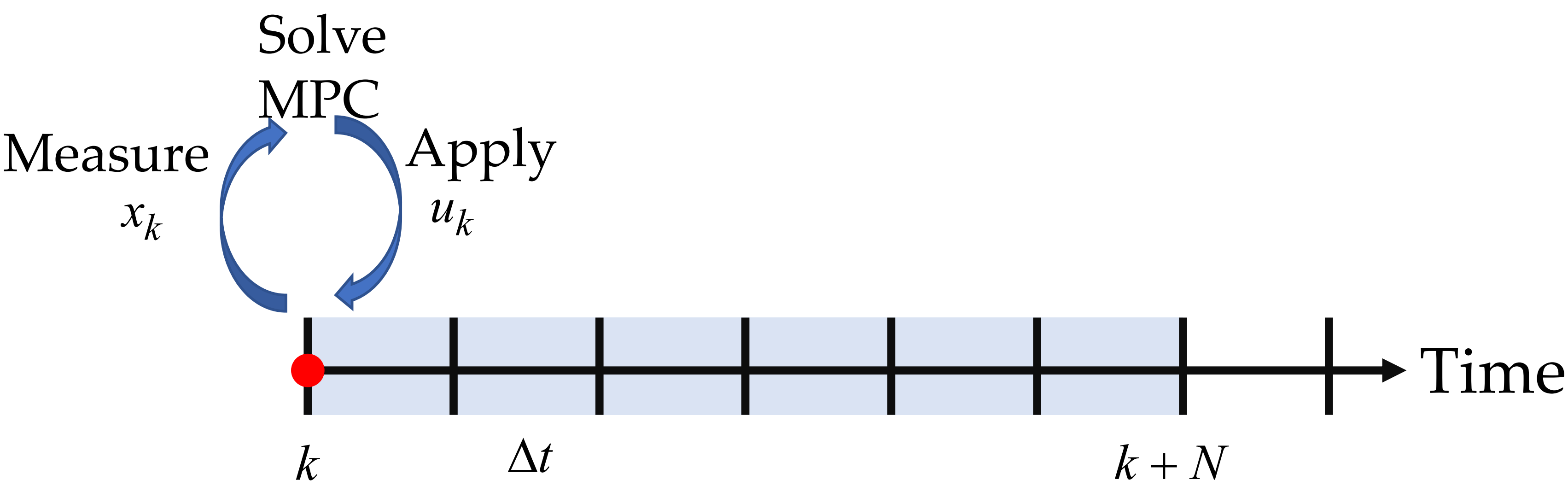
Outline

- Extensions
  - Constraints
  - Soft Constraints
  - Prediction Horizons
  - Reference Tracking
  - Preview
  - Rejection of Measured Disturbances
  - Rejection of Unmeasured Disturbances
  - Time-delays
    - Computational
    - Input
  - Lecture 3 Assignment Review
  - Theoretical Results
    - Feasibility

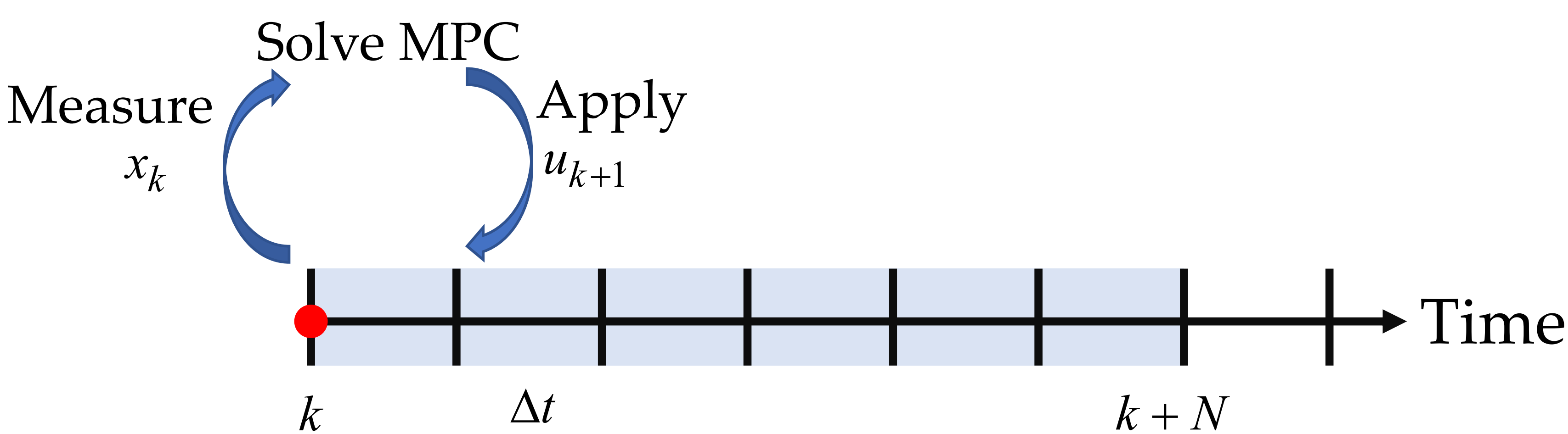
Alberto Bemporad. "Model Predictive Control Design: New Trends and Tools." CDC, 2006.

Time Delay – Computational

- In general, we assume instantaneous calculations



- However, in practice we solve optimization problem between discrete updates



Time Delay – Computational

- Modify MPC formulation

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k + x_N^T P x_N$$

$$s.t.$$

$$x_{k+1} = Ax_k + Bu_k,\ k \in \{0,1,\ldots,N-1\}$$

$$x_0 = x(0)$$

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k + x_N^T P x_N$$

$$s.t.$$

$$x_{k+1} = Ax_k + Bu_k,\ k \in \{0,1,\ldots,N-1\}$$

$$x_0 = x(0)$$

$$u_0 = u(0)$$

- Initial input is fixed and provided as an input to the optimization problem
    - This input is what is effecting the system while the optimization problem is being solved
    - This input corresponds to the optimal solution at the previous time step $u(0) = u_{0|-1}^*$
- Output of the controller is the optimal input at the second time-step $u_{1|0}^*$

Time Delay – Input Delay

- Physical systems can have delays between when an input changes and when this change effects the states
- Often due to transport delay
- Let the input delay be $0 \leq \tau$ seconds
- In discrete time, we can approximate this as a finite number of steps n_d

$$\tau \approx \Delta t \; n_d$$

- Input-delayed, discrete-time state-space model

$$x_{k+1} = Ax_k + Bu_{k-n_d}$$
$$y_k = Cx_k$$

- Can augment state-state space to account for this delay

$$x_{k+1} = Ax_k + Bx_k^1$$
$$x_{k+1}^1 = x_k^2$$
$$\vdots$$
$$x_{k+1}^{n_d-1} = x_{k+1}^{n_d}$$
$$x_{k+1}^{n_d} = u_k$$

Example: $n_d = 3$

$$\begin{bmatrix} x_{k+1} \\ x_{k+1}^1 \\ x_{k+1}^2 \\ x_{k+1}^3 \end{bmatrix} = \begin{bmatrix} A & B & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ x_k^1 \\ x_k^2 \\ x_k^3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} u_k$$

Time Delay – 1 step Input Delay

- Uncompensated

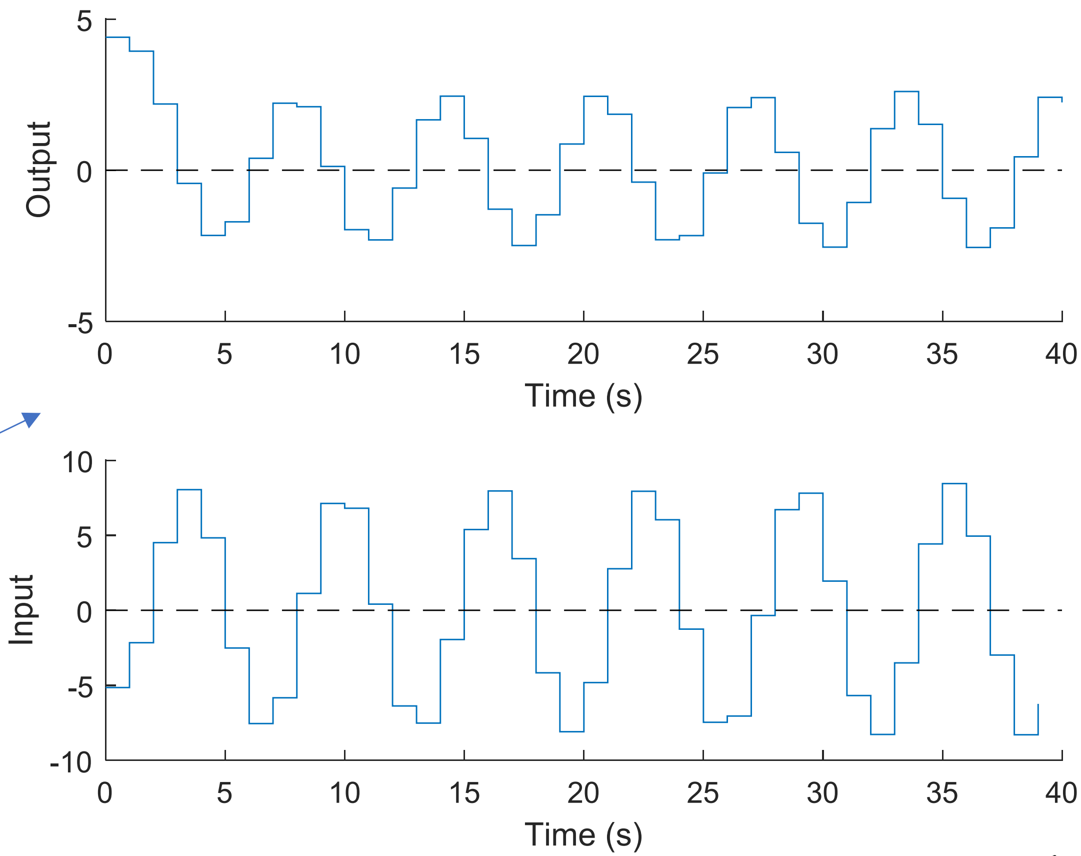
$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k + x_N^T P x_N$$

$$s.t.$$

$$x_{k+1} = Ax_k + Bu_k, \; k \in \{0,1,\ldots,N-1\}$$

$$x_0 = x(0)$$

```
% Simulate Closed-loop System
% Define simulation length
N_sim = 40;
% Initialize state and input trajectories
x0 = [10;10];
x_sim = [x0];
u_sim = [];
uOld = 0;
% Step through simulation
for i = 1:N_sim
    x0 = x_sim(:,i);
    inputs = {x0};
    u = controller{inputs};
    u_sim = [u_sim u];
    x_sim = [x_sim A*x0+B*uOld];
    uOld = u;
end
```



Marginally stable, any more time delay would result in a unstable CL system

Time Delay – 1 step Input Delay

- Compensated

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k + x_N^T P x_N$$

$s.t.$

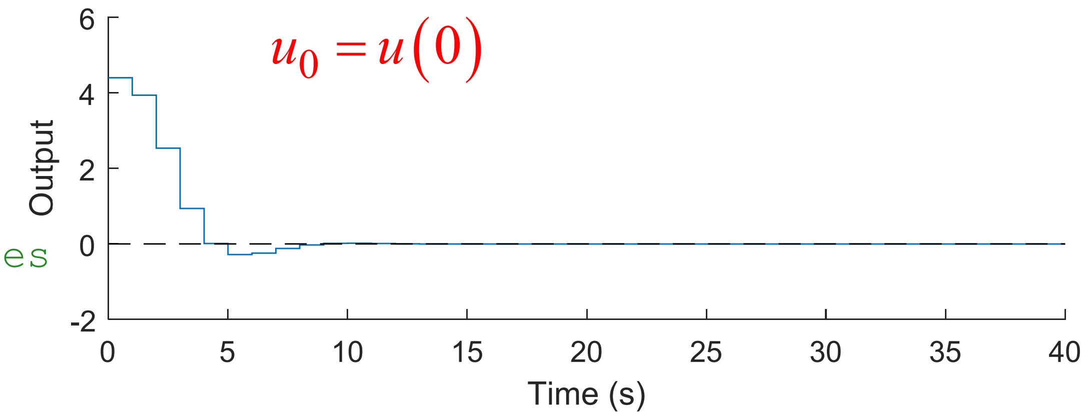
$$x_{k+1} = Ax_k + Bu_k,\ k \in \{0,1,\ldots,N-1\}$$

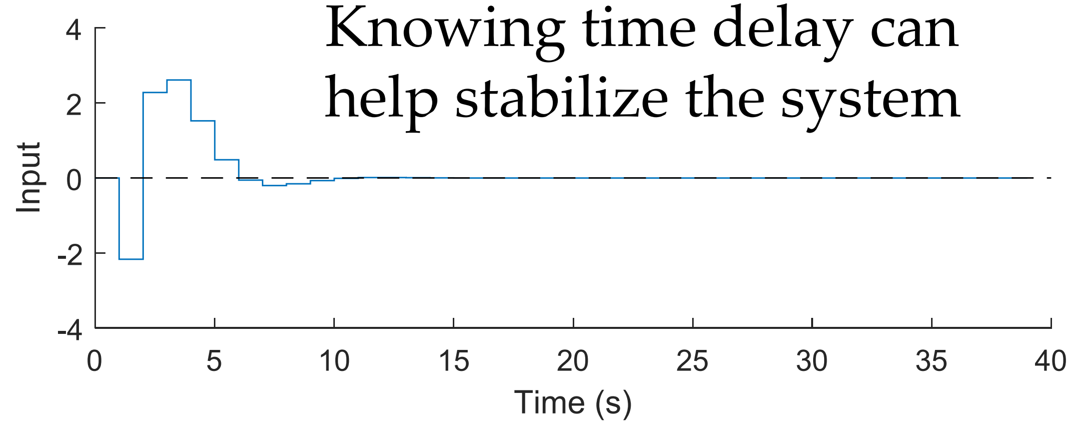
$$x_0 = x(0)$$

$$u_0 = u(0)$$

```
% Specify Controller Inputs and Outputs
% Add first input as input to controller
inputs = {x_{1},u_{1}};
% Output the second input in the trajectory
outputs = {[u_{2}]};

% Initialize state and input trajectories
x0 = [10;10];
x_sim = [x0];
uOld = 0;
u_sim = [uOld];
% Step through simulation
for i = 1:N_sim
    x0 = x_sim(:,i);
    inputs = {x0,uOld};
    u = controller{inputs};
    u_sim = [u_sim u];
    x_sim = [x_sim A*x0+B*uOld];
    uOld = u;
end
```





Knowing time delay can help stabilize the system

Lecture 3 Assignment Review

- Prove one of the following

$$Q > 0 \text{ if and only if } \lambda(Q) > 0,\ \ \lambda \in eig(Q)$$
$$Q \geq 0 \text{ if and only if } \lambda(Q) \geq 0,\ \ \lambda \in eig(Q)$$

- Remember to go both directions, i.e.
  - Assume $Q > 0$ and show $\lambda(Q) > 0,\ \lambda \in eig(Q)$
  - Then assume $\lambda(Q) > 0,\ \lambda \in eig(Q)$ and show $Q > 0$

Key ideas

- First direction
  - Start with definition of eigenvalues/eigenvectors
  - Multiply both sides by x^T
  - See that all eigenvalues must be positive
- Second direction
  - Eigenvalue decomposition (similarity transform) of Q to get diagonal matrix
  - Then considered $x^TQx = y^TDy$, where D is diagonal matrix of eigenvalues
  - Results in definition of positive definite

Types of Theoretical Results

- Successful practical application has motivated a wide variety of research into specific control formulations and analysis that produce theoretical guarantees
  - Linear MPC – linear prediction model
  - Nonlinear MPC – nonlinear prediction model
  - Robust MPC – uncertain disturbances or parameters in model
  - Stochastic MPC – stochastic variables in prediction model
  - Decentralized MPC – multiple controllers, no communication
  - Distributed MPC – multiple controller, with communication
  - Hierarchical MPC – multiple controller with comm. hierarchy
  - Economic MPC – cost function with economic objectives
  - Hybrid MPC – model with discrete and continuous dynamics
  - Explicit MPC – off-line computation of MPC control law
  - Solvers for MPC – on-line algorithms specifically for MPC
- Main theoretical outcomes:
  - Feasibility (Constraint satisfaction)
  - Stability (Convergence)

Feasibility

- We have seen from the batch approach that the standard MPC formulation can be rearranged as

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k + x_N^T P x_N$$

$$\begin{aligned} &s.t.\\ &x_{k+1} = Ax_k + Bu_k,\ k \in \{0,1,\ldots,N-1\}\\ &u_{\min} \le u_k \le u_{\max},\quad k \in \{0,1,\ldots,N-1\}\\ &y_{\min} \le Cx_k \le y_{\max},\ k \in \{1,2,\ldots,N\}\\ &x_0 = x(0)\end{aligned}$$

$$\Rightarrow$$

$$J_0^*(x_0) = \min_{U_0}\ U_0^T H U_0 + 2x_0^T F U_0 + x_0^T Y x_0$$

$$\begin{aligned} &s.t.\\ &GU_0 \le W + Sx_0\end{aligned}$$

- This optimization problem is feasible if there exists $\tilde{U}_0$ such that $G\tilde{U}_0 \le W + Sx_0$
  - Alternatively, define the set of input trajectories that satisfy all constraints $\mathcal{S}(x_0) = \{U_0 \mid GU_0 \le W + Sx_0\}$
  - Feasible if $\mathcal{S}(x_0) \ne \varnothing$

Empty set

Feasibility (cont.)

- Only input constraints
- When is this feasible?
  - Always
  - As long as input constraints are consistent

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k + x_N^T P x_N$$

$$s.t.$$

$$x_{k+1} = Ax_k + Bu_k, \; k \in \{0,1,\dots,N-1\}$$

$$u_{\min} \le u_k \le u_{\max}, \quad k \in \{0,1,\dots,N-1\}$$

$$\Delta u_{\min} \le u_k - u_{k-1} \le \Delta u_{\max}, \quad k \in \{0,1,\dots,N-1$$

$$x_0 = x(0)$$

$$u_{-1} = u(-1)$$

- Input constraint and soft output constraints
- When is this feasible?
  - Always
  - As long as input constraints are consistent

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k + \lambda s_k^2 + x_N^T P x_N$$

$$s.t.$$

$$x_{k+1} = Ax_k + Bu_k, \; k \in \{0,1,\dots,N-1\}$$

$$u_{\min} \le u_k \le u_{\max}, \quad k \in \{0,1,\dots,N-1\}$$

$$y_{\min} - W_{\min} s_k \le Cx_k \le y_{\max} + W_{\max} s_k, \; k \in \{1,2,\dots,N\}$$

$$0 \le s_k, \quad k \in \{1,2,\dots,N\}$$

$$x_0 = x(0)$$

Feasibility (cont.)

- Input and output constraints (hard constraints)
- When is this feasible?
  - It depends (not always)

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k + x_N^T P x_N$$

$$\begin{aligned}
&s.t. \\
&x_{k+1} = Ax_k + Bu_k, \ k \in \{0,1,\ldots,N-1\} \\
&u_{\min} \le u_k \le u_{\max}, \quad k \in \{0,1,\ldots,N-1\} \\
&y_{\min} \le Cx_k \le y_{\max}, \ k \in \{1,2,\ldots,N\} \\
&x_0 = x(0)
\end{aligned}$$

- If we assume that it is feasible at time k = 0, can we guarantee it will be feasible for all k > 0?
  - In general, no, assuming $N < \infty$
  - Yes, if $N = \infty$
    - But then we have an infinite number of constraints that must be satisfied at $k = 0$.
- We will study this more in future lectures.

Feasibility - Example

- A simplified vehicle driving close to a wall.
  - Double integrator
    - State 1 – position
    - State 2 – velocity
    - Input - force

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u_k$$

$$J_0^*(x_0) = \min_{U_0} \sum_{k=0}^{N-1} x_k^T x_k + 0.1 u_k^T u_k + x_N^T x_N$$

Drive to the origin

$$s.t.$$

$$x_{k+1} = Ax_k + Bu_k, \ k \in \{0,1,\ldots,N-1\}$$

$$-1 \le u_k \le 1, \quad k \in \{0,1,\ldots,N-1\}$$

Limited acceleration and deceleration

$$\begin{bmatrix} 1 & 0 \end{bmatrix} x_k \le 0, \ k \in \{1,2,\ldots,N\}$$

Wall at the origin

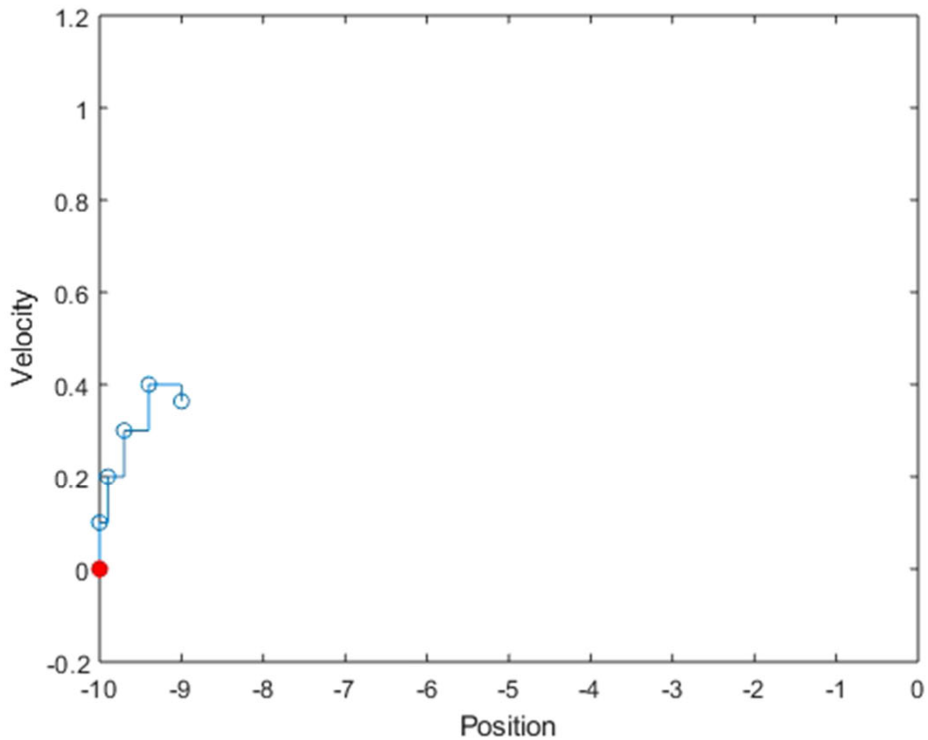
$$x_0 = x(0)$$

$$x(0) = \begin{bmatrix} -10 \\ 0 \end{bmatrix}$$

Start at rest to the left of the wall

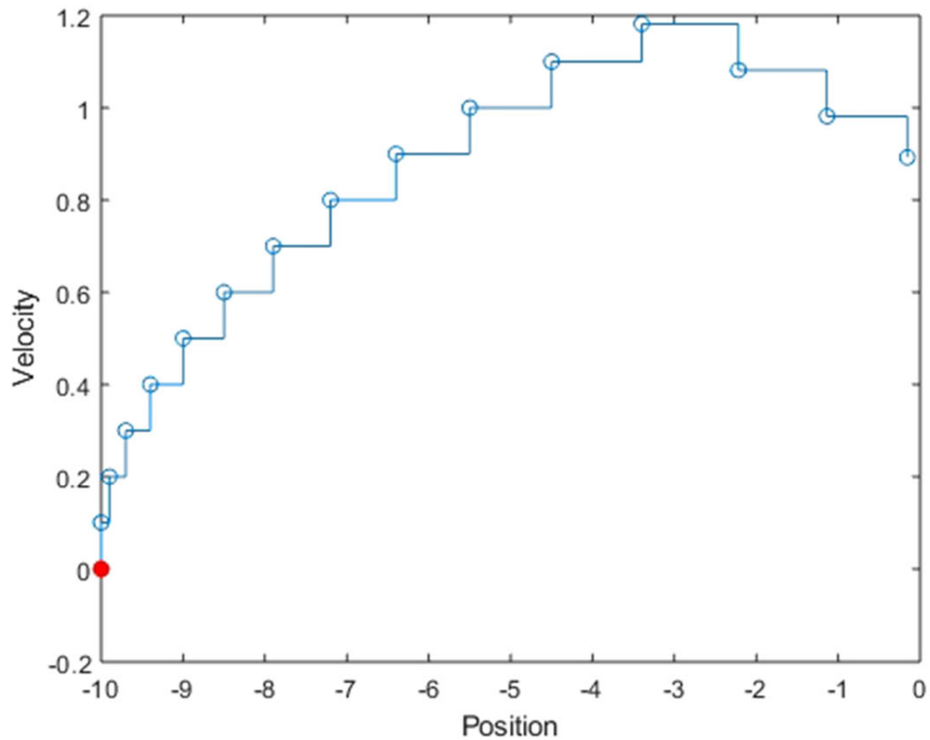
Feasibility - Example

$N = 5$



Velocity goes too high and the vehicle cannot slow down fast enough to avoid hitting the wall (infeasible optimization problem)

$N = 15$



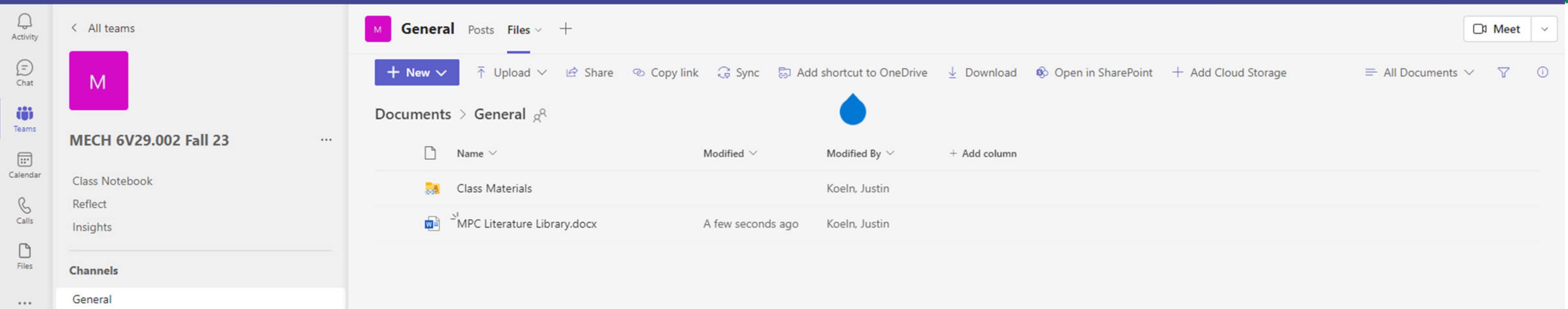
Longer prediction horizon prevents velocity from going too high and vehicle arrives at origin without hitting the wall

Literature Review

- We should have a list of important MPC research papers to serve as supplementary material for the class
- Let's create this library of papers together
- Participation Assignment (Due Sept. 22)
  - Identify 1 (or more) seminal works of theoretical MPC development in 1 (or more) of the following areas
  - Identify 1 (or more) papers that reference this seminal work that has an interesting application example
  - Provide the references to these papers to the shared file on MS Teams
  - This will help prepare you for the class project where you will choose a specific MPC approach to study further and demonstrate through numerical simulation
  - First come, first served
    - Put your name in the topic area to which you plan to contribute
    - If more than one person wants to contribute to a single topic area, work together so that you do not provide the same references

Linear MPC
Nonlinear MPC
Robust MPC
Stochastic MPC
Decentralized MPC
Distributed MPC
Hierarchical MPC
Economic MPC
Hybrid MPC
Explicit MPC
Solvers for MPC
Other

Literature Review



MPC Literature Library

Use IEEE citation format: https://pitt.libguides.com/citationhelp/ieee

Textbooks:

Added by Justin Koeln

- J. B. Rawlings, D. Q. Mayne, and M. M. Diehl. *Model Predictive Control: Theory, Computation, and Design*. 2nd Edition. Nob Hill Publishing, 2019.
- F. Borrelli, A. Bemporad, and M. Morari. *Predictive Control for Linear and Hybrid Systems*. Cambridge University Press, 2017.

Nonlinear MPC:

Added by Justin Koeln

- Seminal – D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, pp. 789-814, 2000.
  - Application – P. Falcone, F. Borrelli, J. Asgari, H. E. Tseng, and D. Hrovat, "Predictive Active Steering Control for Autonomous Vehicle Systems," *IEEE Transactions on Control Systems Technology*, vol. 15, no. 3, pp. 566-580, 2007.