MECH 6v29 - Model Predictive Control
Homework 2

Jonas Wagner
jonas.wagner@utdallas.edu

2023, September 29th

Contents

Problem 1

1a)

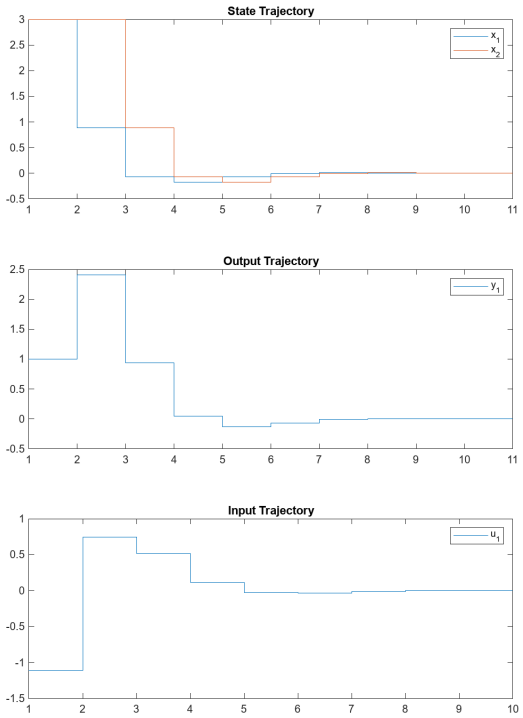


Figure 1: Problem 1a results

The transient behavior is pretty reasonable and has a mild overshoot. The maximum output is around 2.5 (2.409) and it appears to initially reach the origin after 4 timesteps but takes around 7 to settle.

1b)

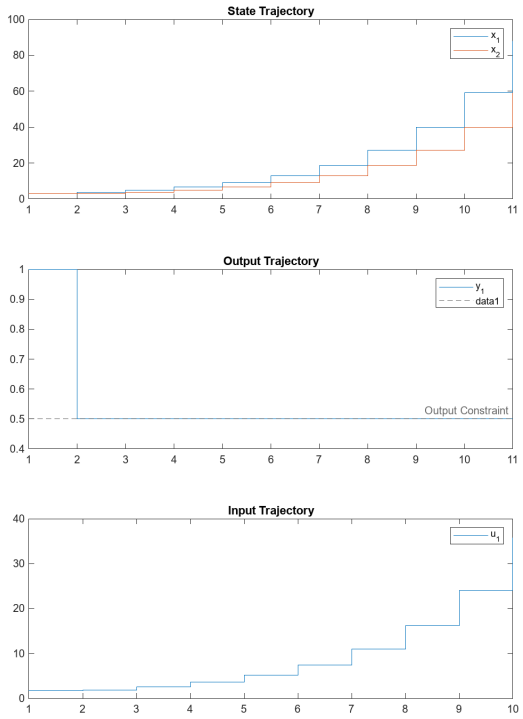


Figure 2: Problem 1b results

The controller is unable to stabilize the system.
Is it possible to tune the cost function matrices? Potentially, although I'm not sure how to explicitly prove this either way for every case that this controller would face. I did test multiple versions of the matrices but did not find any that satisfied it.

1c)

The first time step "solution" is 0. Looking at error code we find that the controller result is actually is infeasible or unbounded.
After it runs for a couple time steps with $u = 0$, it then becomes feasible and has a solution that ultimately does stabilize the system.

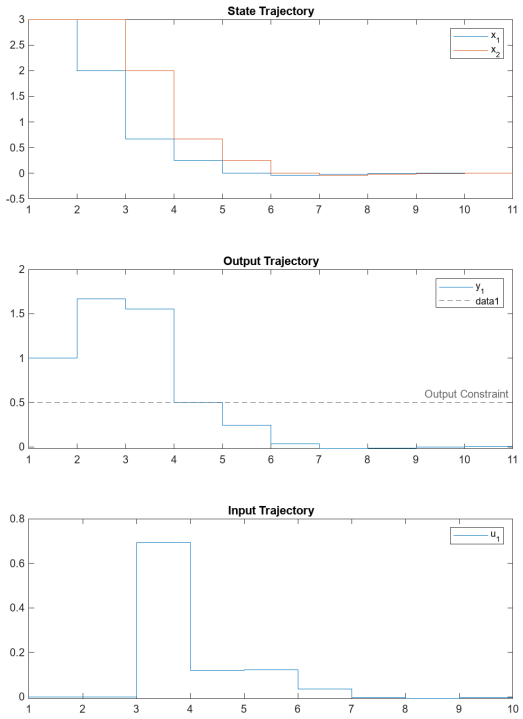
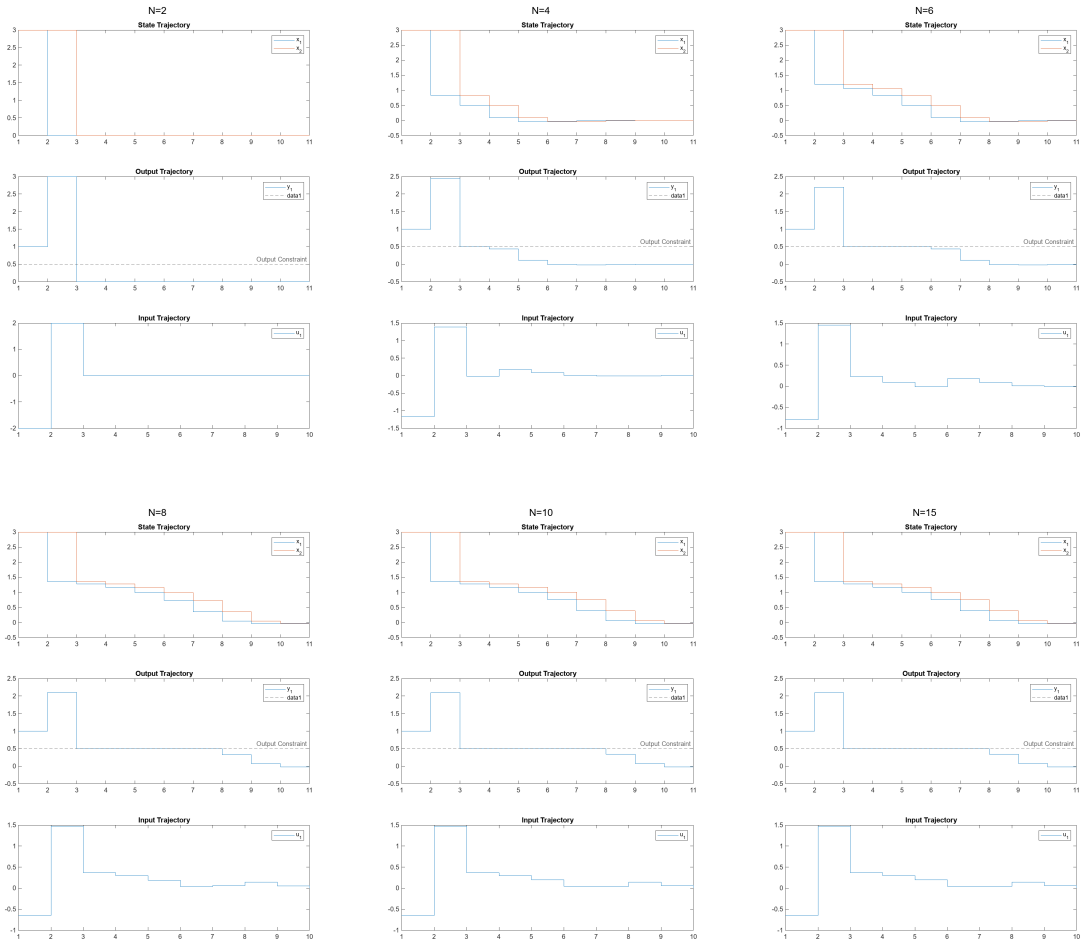


Figure 3: Problem 1c results

Increasing the time does not appear to solve this issue. Specifically, increasing it incrementally up to 10 saw no difference, and then running it at $N = 50$ also demonstrated no changes.

1d)

I ran this for many different time-horrizons (and the slack variable was weighted with a gain of 100). A few of these are included here (the rest available on github). Essential takeaway is that the extension of time-horizon allows for a less extreme peak output value and longer period that the output constraint is maintained. Additionally, the specific note of combining the two is that feasibility becomes a much larger concern but at a minimum, the constraint at $x_N = 0$ ensures stability (although a slack variable should also be added to ensure feasibility).



Problem 2

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathcal{X} = \{x \in \mathbb{R}^2 \ : \ \begin{bmatrix} 1 & 0 \end{bmatrix} x \leq 5\}$$

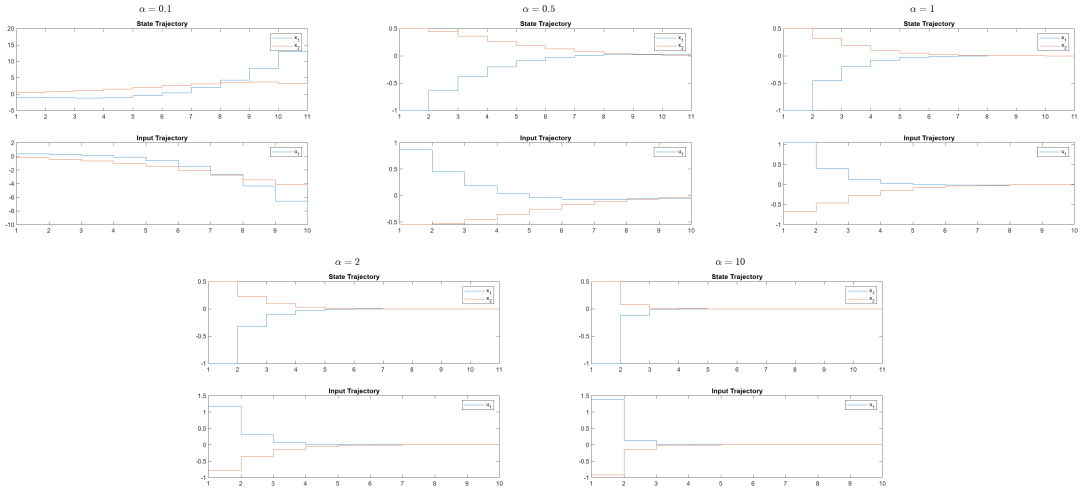
$$\mathcal{U} = \{u \in \mathbb{R}^2 \ : \ -\mathbf{1} \leq u \leq \mathbf{1}\}$$

Note: $\mathcal{X} = \{x \in \mathbb{R}^2 \ : \ x_1 \leq 5\}$ and $\mathcal{U} = \{u \in \mathbb{R}^2 \ : \ \|u\|_\infty \leq 1\}$

Or (in H-rep) $\mathcal{X} = \{\begin{bmatrix} 1 & 0 \end{bmatrix}, 5\}$ and $\mathcal{U} = \{\begin{bmatrix} \mathbf{I}_2 \\ -\mathbf{I}_2 \end{bmatrix}, \begin{bmatrix} \mathbf{1}_2 \\ \mathbf{1}_2 \end{bmatrix}\}$

2a)

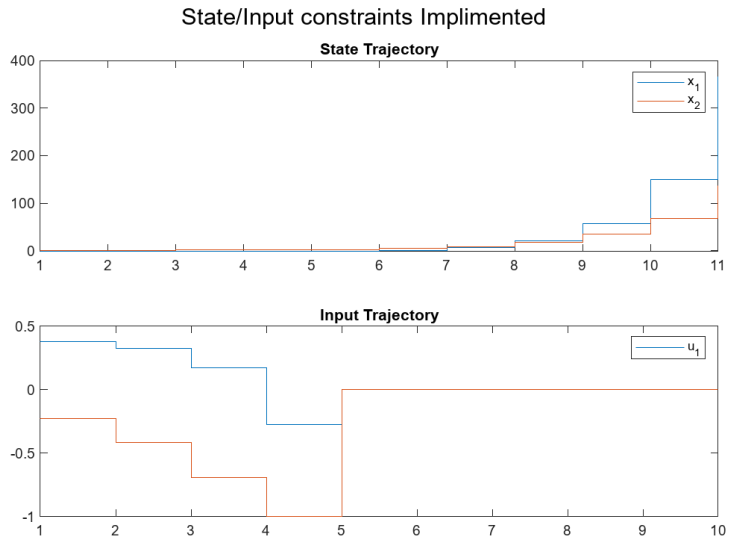
Running it for many different values of alpha resulted in the following:



When $\alpha = 0.1$ the system is unstable.

2b)

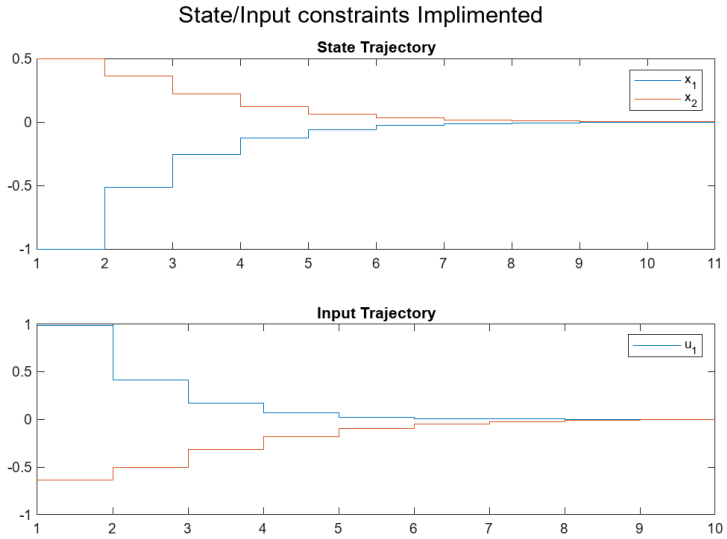
When implementing the state and input constraints, the closed loop system is still unstable. Additionally,



the results become infeasible once it diverges from the origin too much.

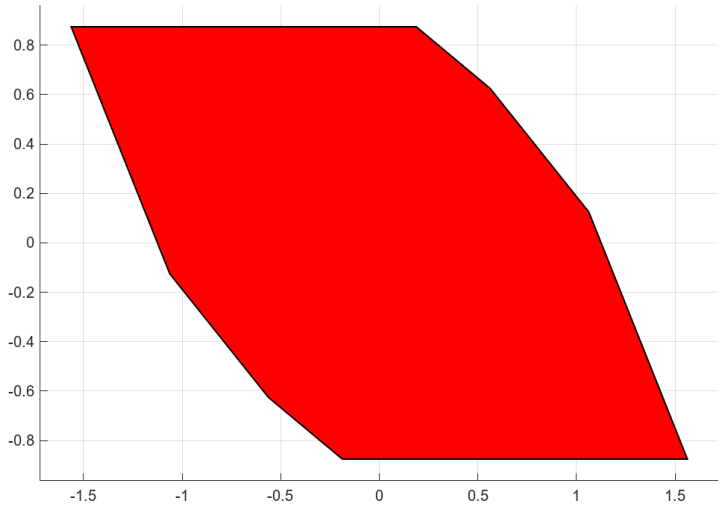
2c)

Implementing it with the $x_N = 0$ constraint allows it to be stable (and feasibility issues disappear).



2d)

I had a lot of difficulty getting the batch method to work correctly (see matlab)... The specific results requested are here:



The recursive approach was faster as expected (0.16s vs 0.38s).

A MATLAB Files

MATLAB files included as follows.

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MPC HW 2 - Problem 1

Author: Jonas Wagner Date: 2023-09-29

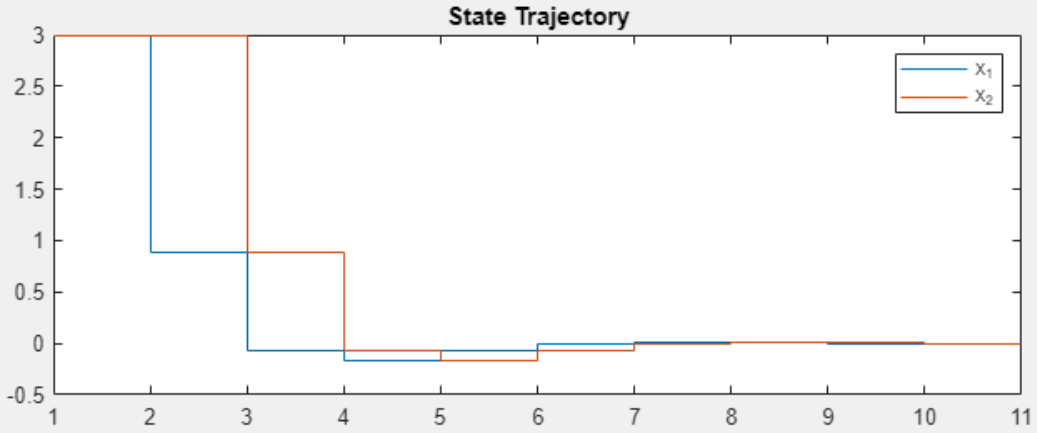
```
clear
close all
subfolder = fileparts(mfilename('fullpath'));
if ~isfolder(strcat(subfolder,filesep,'figs'))
    mkdir(strcat(subfolder,filesep,'figs'));
end

% Problem Information
A = [4/3, -2/3; 1, 0];
B = [1; 0];
C = [-2/3, 1];
D = 0;
dt = 1;
sys = ss(A,B,C,D,dt);

% Size parameters
nx = size(A,1);
nu = size(B,2);

% MPC Parameters
N = 5;
Q = eye(nx);
R = eye(nu);
P = 0;
```

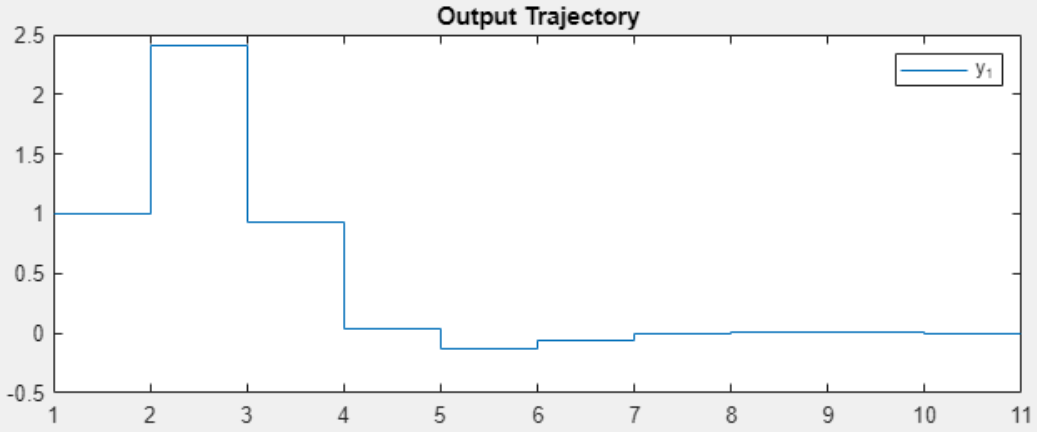
Part 1a

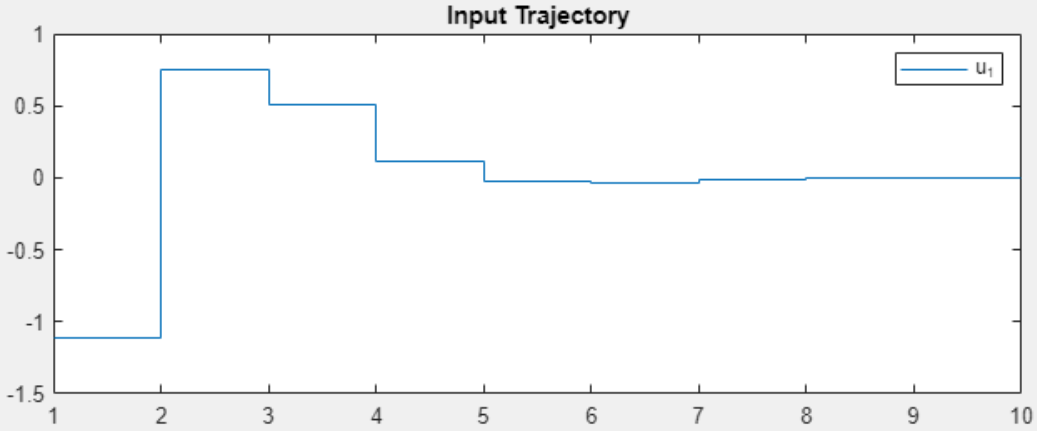
Controller setup

```
cons = @(x,u,s) [];
cons_f = @(x,u,s) [];
controller = mpc_yalmip_controller(A, B, P, Q, R, N, cons, cons_f);

% Simulation
x0 = [3;3]; tf = 10;
[X, U, ~] = run_sim(A,B,controller, x0, tf);
Y = C*X;
```

```
% Results
figName = 'pblm1a';
fig = plot_trajectory(X, Y, U);
saveas(fig,[subfolder,filesep,'figs',filesep,figName],'png')

maxOutput = max(Y);
fprintf('Max Output: %f\n',maxOutput);
```


State Trajectory
x1
x2

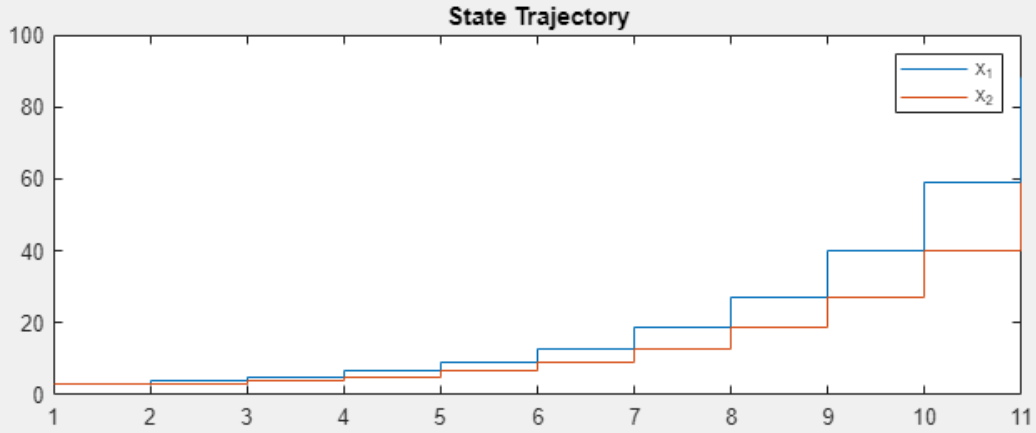

Output Trajectory
y1

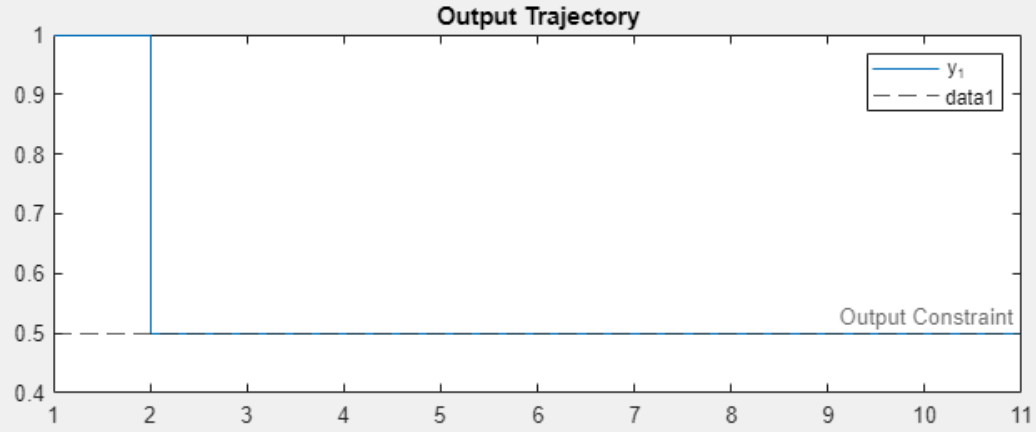

Input Trajectory
u1

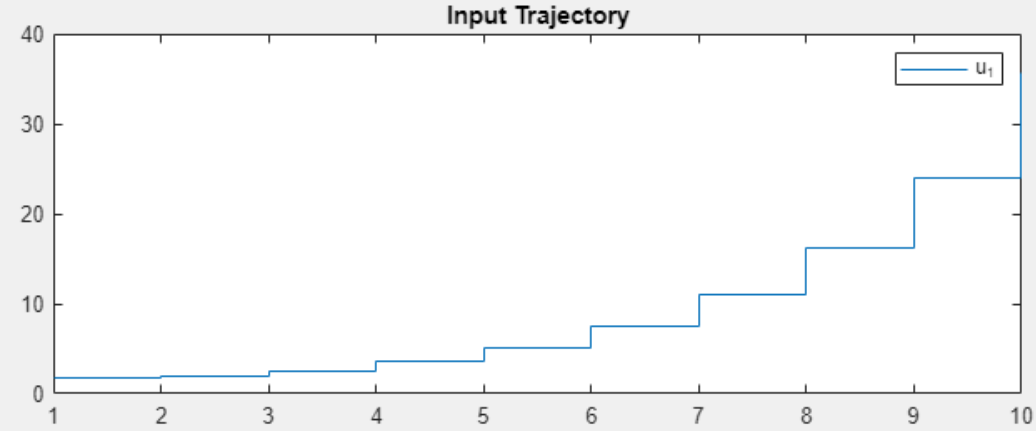
Part 1b

Controller setup

```
cons = @(x,u,s) (C*x)<=0.5;
cons_f = @(x,u,s) [];
controller = mpc_yalmip_controller(A, B, P, Q, R, N, cons, cons_f);

% Simulation
x0 = [3;3]; tf = 10;
[X, U, ~] = run_sim(A,B,controller, x0, tf);
Y = C*X;

% Results
figName = 'pblm1b';
fig = plot_trajectory(X, Y, U);
subplot(3,1,2); yline(0.5,'--','Output Constraint')
saveas(fig,[subfolder,filesep,'figs',filesep,figName],'png')

% % Changing Q/R
% Q = C'*C;
% R = 0;
% controller = mpc_yalmip_controller(A, B, P, Q, R, N, cons);
% [X, U] = run_sim(A,B,controller, x0, tf);
%
% figName = 'pblm1b_2';
% fig = plot_trajectory(X, Y, U);
% subplot(3,1,2); yline(0.5,'--','Output Constraint')
% saveas(fig,[subfolder,filesep,'figs',filesep,figName],'png')
```


State Trajectory
100
80
60
40
20
0
1
2
3
4
5
6
7
8
9
10
11
x1
x2


Output Trajectory
1
0.9
0.8
0.7
0.6
0.5
0.4
1
2
3
4
5
6
7
8
9
10
11
y1
data1
Output Constraint


Input Trajectory
40
30
20
10
0
1
2
3
4
5
6
7
8
9
10
u1

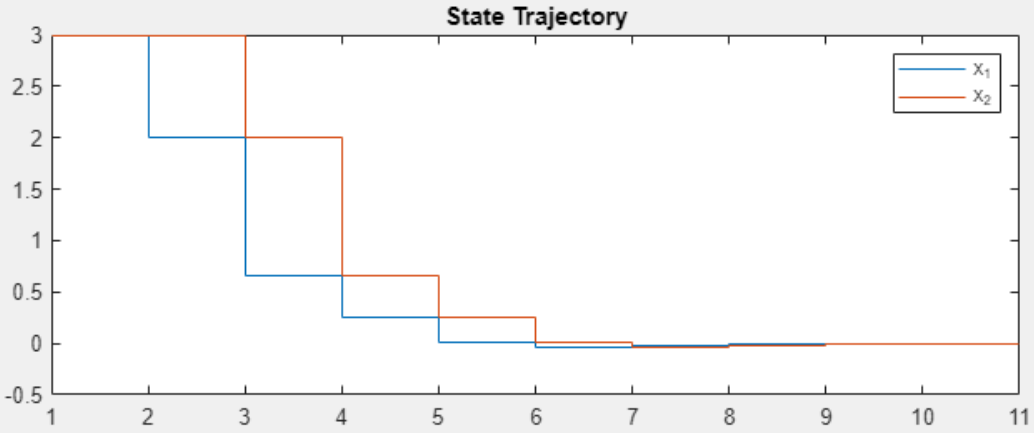
Part 1c

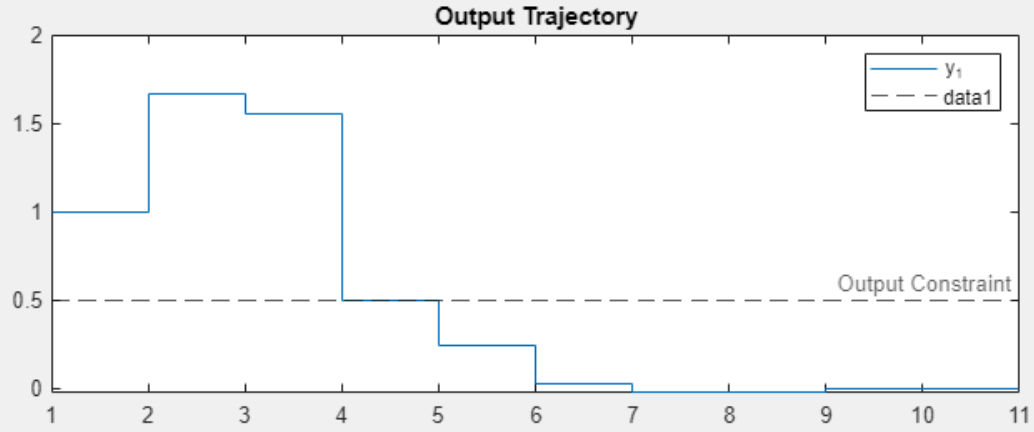
Controller setup

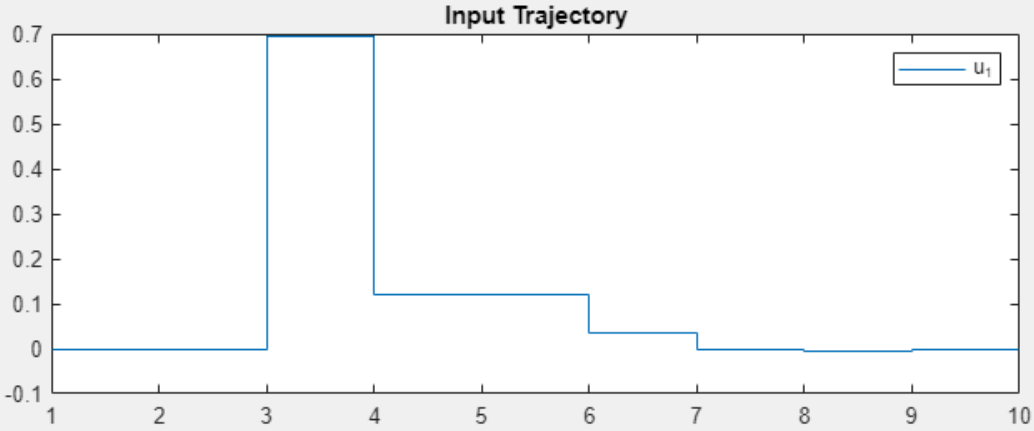
```
cons = @(x,u,s) (C*x)<=0.5;
cons_f = @(x,u,s) x==0;
controller = mpc_yalmip_controller(A, B, P, Q, R, N, cons, cons_f);

% Simulation
x0 = [3;3]; tf = 10;
[X,U, diagnostics_] = run_sim(A,B,controller, x0, tf);
Y = C*X;

% Results
figName = 'pblm1c';
fig = plot_trajectory(X, Y, U);
subplot(3,1,2); yline(0.5,'--','Output Constraint')
saveas(fig,[subfolder,filesep,'figs',filesep,figName],'png')

% U and Errors
for k = 1:tf
    fprintf('U_%d solution: %f\n',k,U(:,k))
    fprintf('Error (k=%d): %s\n',k,yalmiperror(diagnostics_{k}));
end

% %% 1c_2
% % Controller setup
% N = 50;
% controller = mpc_yalmip_controller(A, B, P, Q, R, N, cons, cons_f);
% [X,U, diagnostics_] = run_sim(A,B,controller, x0, tf);
% Y = C*X;
% N = 5
%
% % Results
% figName = 'pblm1c_2';
% fig = plot_trajectory(X, Y, U);
% subplot(3,1,2); yline(0.5,'--','Output Constraint')
% saveas(fig,[subfolder,filesep,'figs',filesep,figName],'png')
%
% % U and Errors
% for k = 1:tf
%     fprintf('U_%d solution: %f\n',k,U(:,k))
%     fprintf('Error (k=%d): %s\n',k,yalmiperror(diagnostics_{k}));
% end
```

```
U_1 solution: 0.000000
Error (k=1): Either infeasible or unbounded
U_2 solution: 0.000000
Error (k=2): Either infeasible or unbounded
U_3 solution: 0.694444
Error (k=3): Successfully solved
U_4 solution: 0.119462
Error (k=4): Successfully solved
```

```
U_5 solution: 0.120159
Error (k=5): Successfully solved
U_6 solution: 0.035475
Error (k=6): Successfully solved
U_7 solution: -0.002816
Error (k=7): Successfully solved
U_8 solution: -0.006824
Error (k=8): Successfully solved
U_9 solution: -0.002636
Error (k=9): Successfully solved
U_10 solution: -0.000120
Error (k=10): Successfully solved
```


State Trajectory
x1
x2


Output Trajectory
y1
data1
Output Constraint

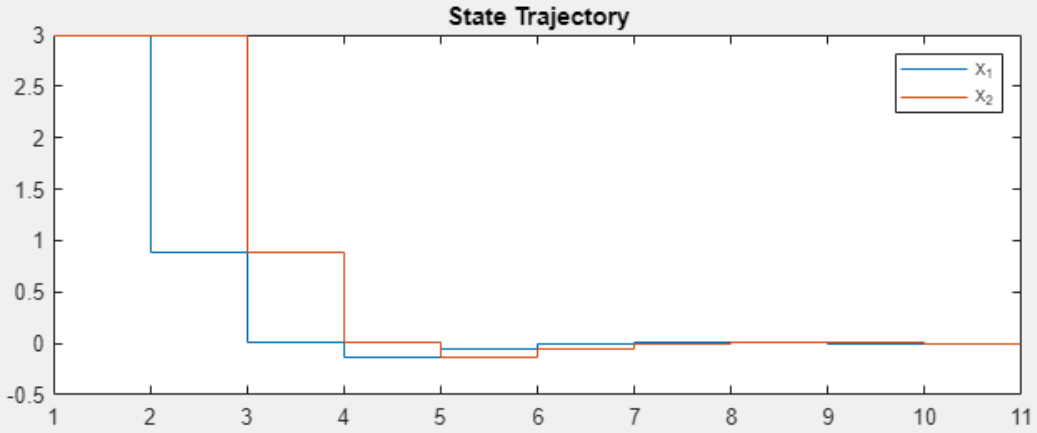

Input Trajectory
u1

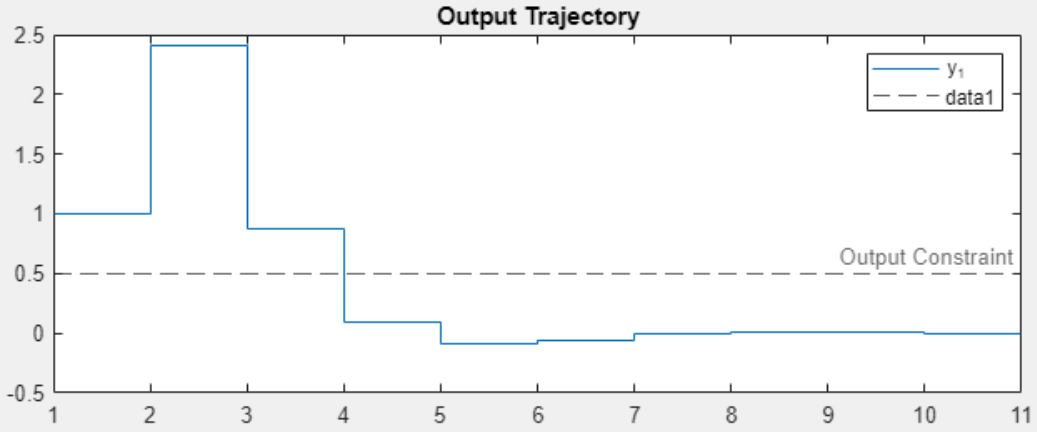
Part 1d

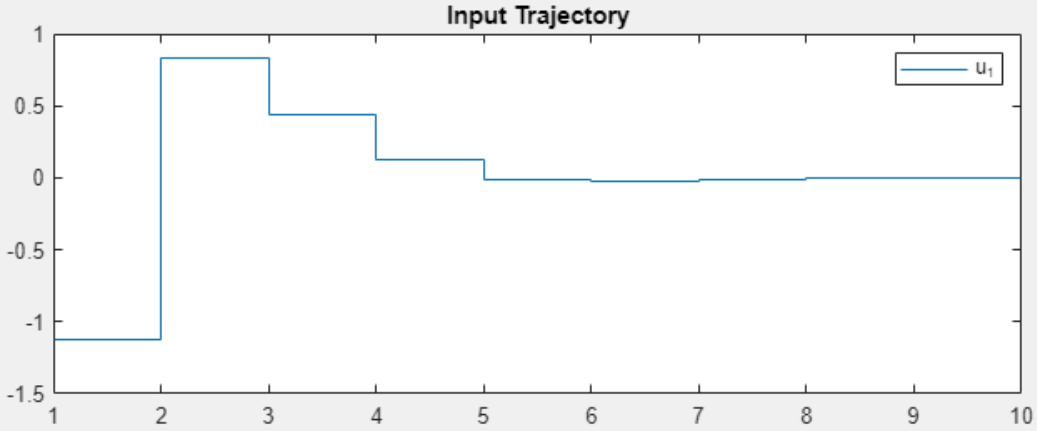
Controller setup

```
cons = @(x,u,s) (C*x)<=0.5+s;
cons_f = @(x,u,s) x==0;
controller = mpc_yalmip_controller(A, B, P, Q, R, N, cons, cons_f);

% Simulation
x0 = [3;3]; tf = 10;
[X,U, diagnostics_] = run_sim(A,B,controller, x0, tf);
Y = C*X;

% Results
figName = 'pblm1d';
fig = plot_trajectory(X, Y, U);
subplot(3,1,2); yline(0.5,'--','Output Constraint')
saveas(fig,[subfolder,filesep,'figs',filesep,figName],'png')

% U and Errors
for k = 1:tf
    fprintf('U_%d solution: %f\n',k,U(:,k))
    fprintf('Error (k=%d): %s\n',k,yalmiperror(diagnostics_{k}));
end
```

```
U_1 solution: -1.120675
Error (k=1): Successfully solved
U_2 solution: 0.832654
Error (k=2): Successfully solved
U_3 solution: 0.444066
Error (k=3): Successfully solved
U_4 solution: 0.122051
Error (k=4): Successfully solved
U_5 solution: -0.014490
Error (k=5): Successfully solved
U_6 solution: -0.025641
Error (k=6): Successfully solved
U_7 solution: -0.009292
Error (k=7): Successfully solved
U_8 solution: -0.000176
Error (k=8): Successfully solved
U_9 solution: 0.001376
Error (k=9): Successfully solved
U_10 solution: 0.000648
Error (k=10): Successfully solved
```


State Trajectory
x1
x2


Output Trajectory
y1
data1
Output Constraint


Input Trajectory
u1

1d - multiple

```
sigma = 100;
for i = 2:15
    N = i; cons = @(x,u,s) (C*x) <= 0.5 + (1/sigma)*s;
    controller = mpc_yalmip_controller(A, B, P, Q, R, N, cons, cons_f);
    [X,U, diagnostics_] = run_sim(A,B,controller, x0, tf); Y = C*X;
    figName = sprintf('pblm1d_N=%d',i);
    fig(i) = plot_trajectory(X, Y, U);
    subplot(3,1,2); yline(0.5,'--','Output Constraint')
    sgtitle(sprintf('N=%d',i));
    saveas(fig(i),[subfolder,filesep,'figs',filesep,figName],'png')
end
```

Local Functions

```
function controller = mpc_yalmip_controller(A,B,P,Q,R,N,cons,cons_f)
    yalmip('clear')
    nx = size(A,1);
    nu = size(B,2);

    u_ = sdpvar(repmat(nu,1,N),ones(1,N));
    x_ = sdpvar(repmat(nx,1,N+1),ones(1,N+1));
    s_ = sdpvar(ones(1,N+1),ones(1,N+1));

    constraints = [];
    objective = 0;
    for k = 1:N
        objective = objective + x_{k}'*Q*x_{k} + u_{k}'*R*u_{k} +
 s_{k}; %norm(Q*x_{k},2) + norm(R*u_{k},2);
        constraints = [constraints, s_{k} >= 0];
        constraints = [constraints, x_{k+1} == A*x_{k} + B*u_{k}];
        constraints = [constraints, cons(x_{k+1},u_{k},s_{k})];
    end
    constraints = [constraints,cons_f(x_{k+1},u_{k},s_{k})];
    objective = objective + x_{k+1}'*P*x_{k+1};

    opts = sdpsettings;
    controller = optimizer(constraints, objective,opts,x_{1},u_{1});
end

function [X,U,diagnostics_] = run_sim(A,B,controller,x0, tf)

    X_{tf+1} = []; U_{tf} = []; diagnostics_{tf} = [];
    X_{1} = x0;
    for k = 1:tf
        [U_{k},diagnostics_{k}] = controller{X_{k}};
        X_{k+1} = A*X_{k} + B*U_{k};
    end
    X = [X_{:}]; U = [U_{:}];
end
```

```
function fig = plot_trajectory(X, Y, U)
    fig = figure(...
        WindowStyle="normal",...
        Position=[0 0 750 1000]);
    hold on; grid on;
    subplot(3,1,1);
    stairs(X')
    title('State Trajectory')
    legend({'x_1','x_2'})
    subplot(3,1,2);
    stairs(Y');
    title('Output Trajectory')
    legend({'y_1'})
    subplot(3,1,3);
    stairs(U');
    title('Input Trajectory')
    legend({'u_1'})
end
```

```
Max Output: 2.408838
```

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MPC HW 2 - Problem 2

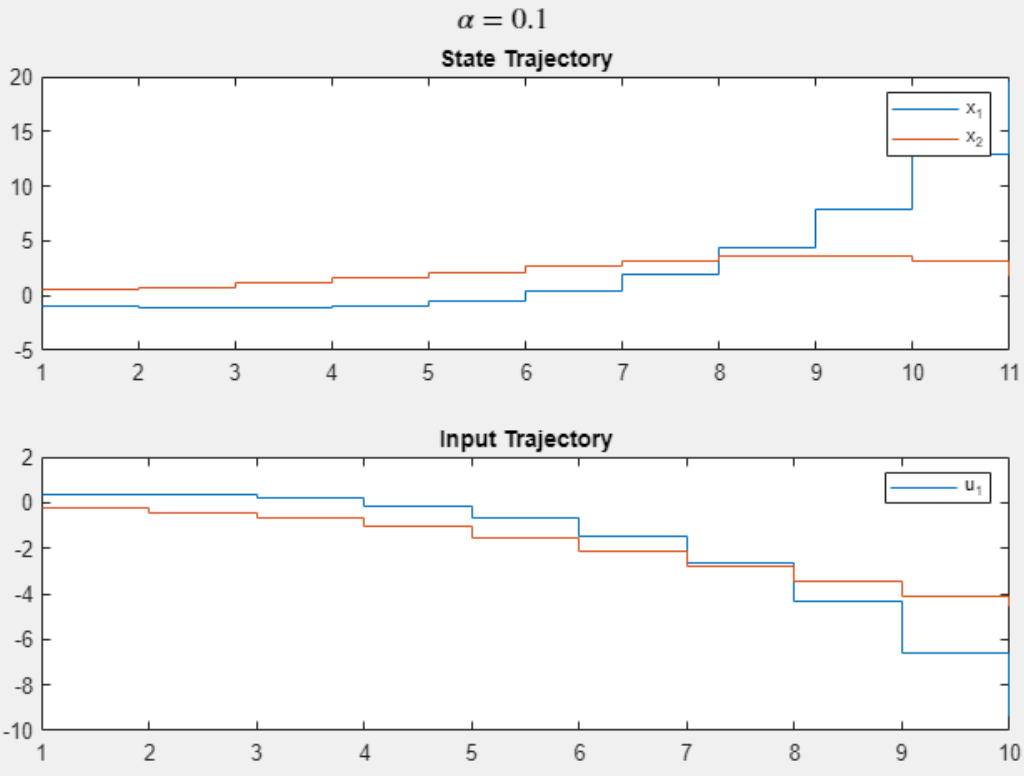
Author: Jonas Wagner Date: 2023-09-29

```
clear
close all
subfolder = fileparts(mfilename('fullpath'));
if ~isfolder(strcat(subfolder,filesep,'figs'))
    mkdir(strcat(subfolder,filesep,'figs'));
end

% Problem Information
A = [2, 1; 0, 2];
B = eye(2);
C = eye(2);
D = 0;
dt = 1;
sys = ss(A,B,C,D,dt);

% State/input constraints
cons_x = @(x) x(1) <= 5;
cons_u = @(u) norm(u,'inf') <= 1;

% Size parameters
nx = size(A,1);
nu = size(B,2);

% MPC Parameters
N = 3;
R = eye(nu);
P = 0;
```

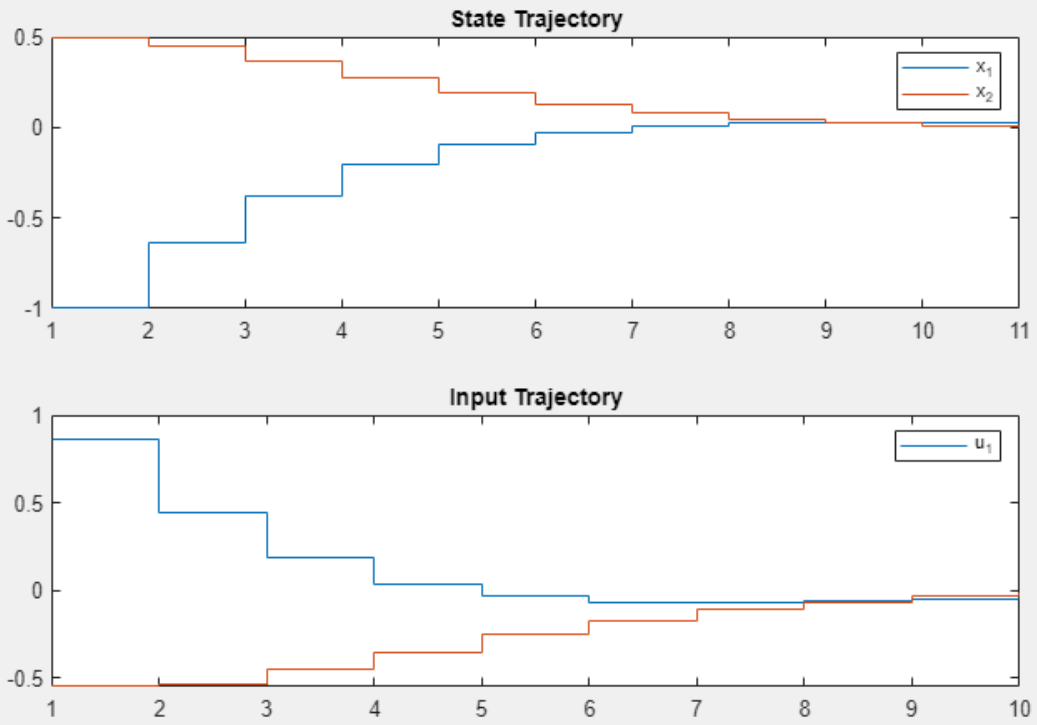
Part 2a

```
x0 = [-1; 0.5]; tf = 10;

Alpha = [0.1,0.5, 1, 2, 10];
for alpha = Alpha
    Q = alpha*eye(nx);
```

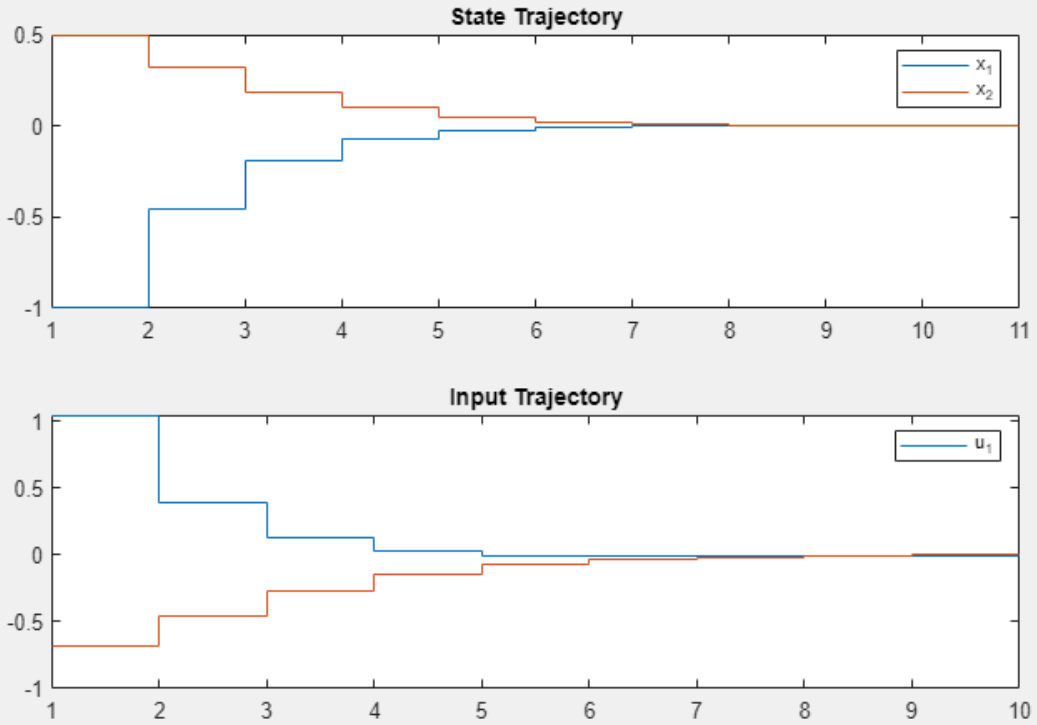
```
    % Controller setup
    cons = @(x,u,s) [];
    cons_f = @(x,u,s) [];
    controller = mpc_yalmip_controller(A, B, P, Q, R, N, cons, cons_f);

    % Simulation
    [X, U, ~] = run_sim(A,B,controller, x0, tf);

    % Results
    figName = sprintf('pblm2a_alpha=%.0e',alpha);
    fig = plot_trajectory(X, U);
    sgtitle(strcat('$\alpha = ',num2str(alpha),'$'),'Interpreter','latex');
    saveas(fig,[subfolder,filesep,'figs',filesep,figName],'png')
end
```



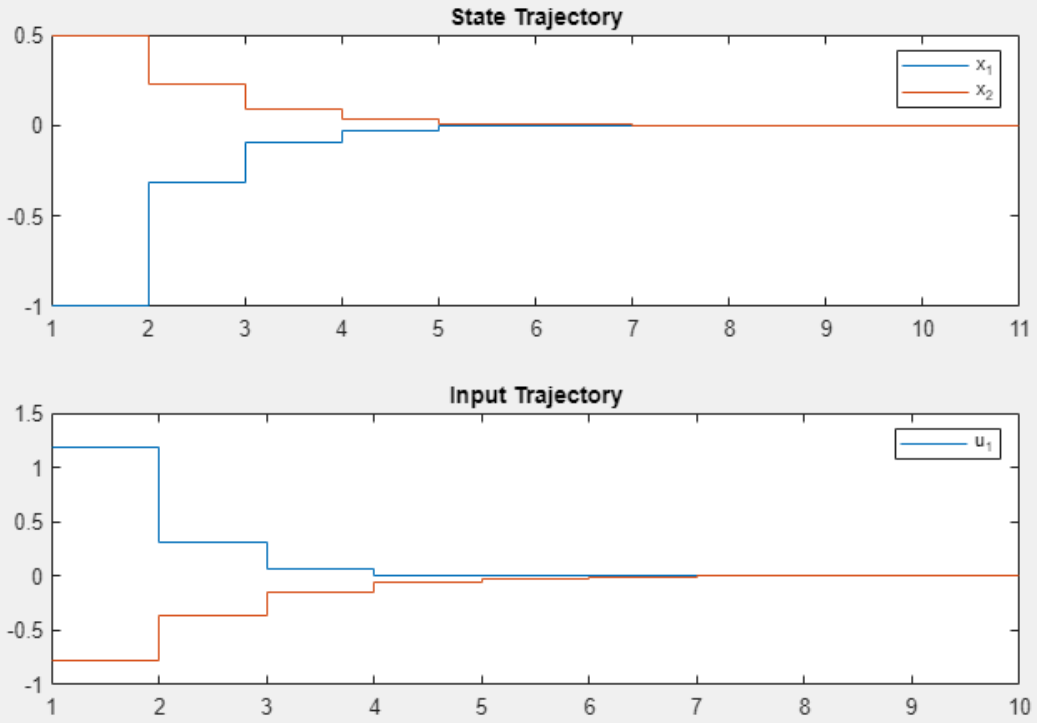
$\alpha = 0.5$



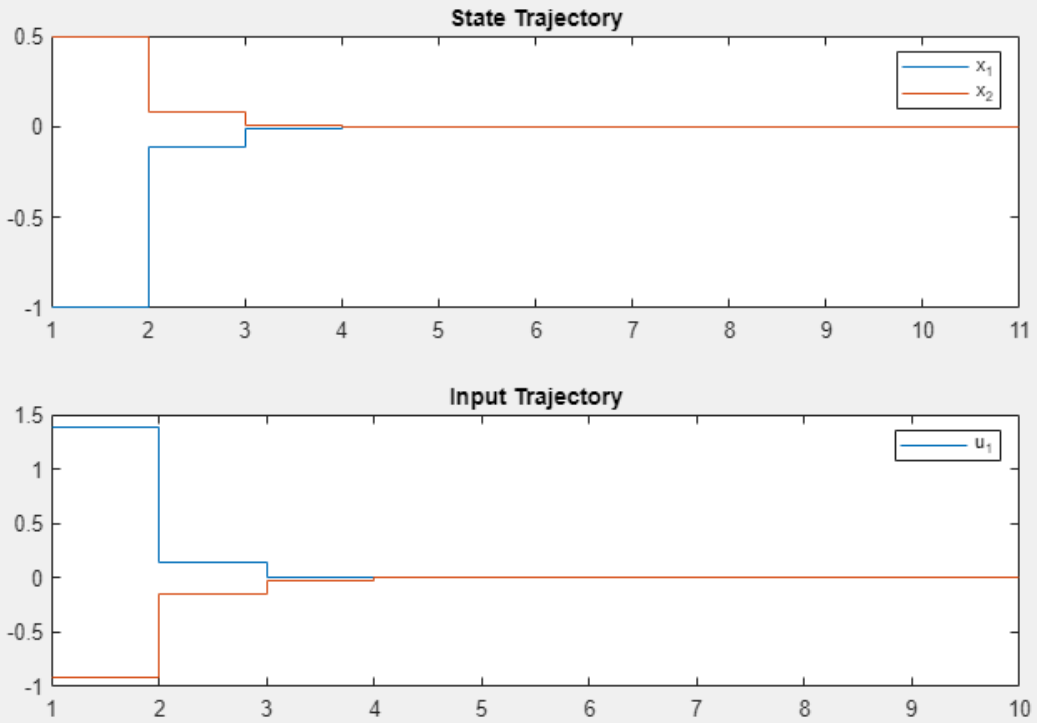
$\alpha = 1$



$\alpha = 2$



$\alpha = 10$

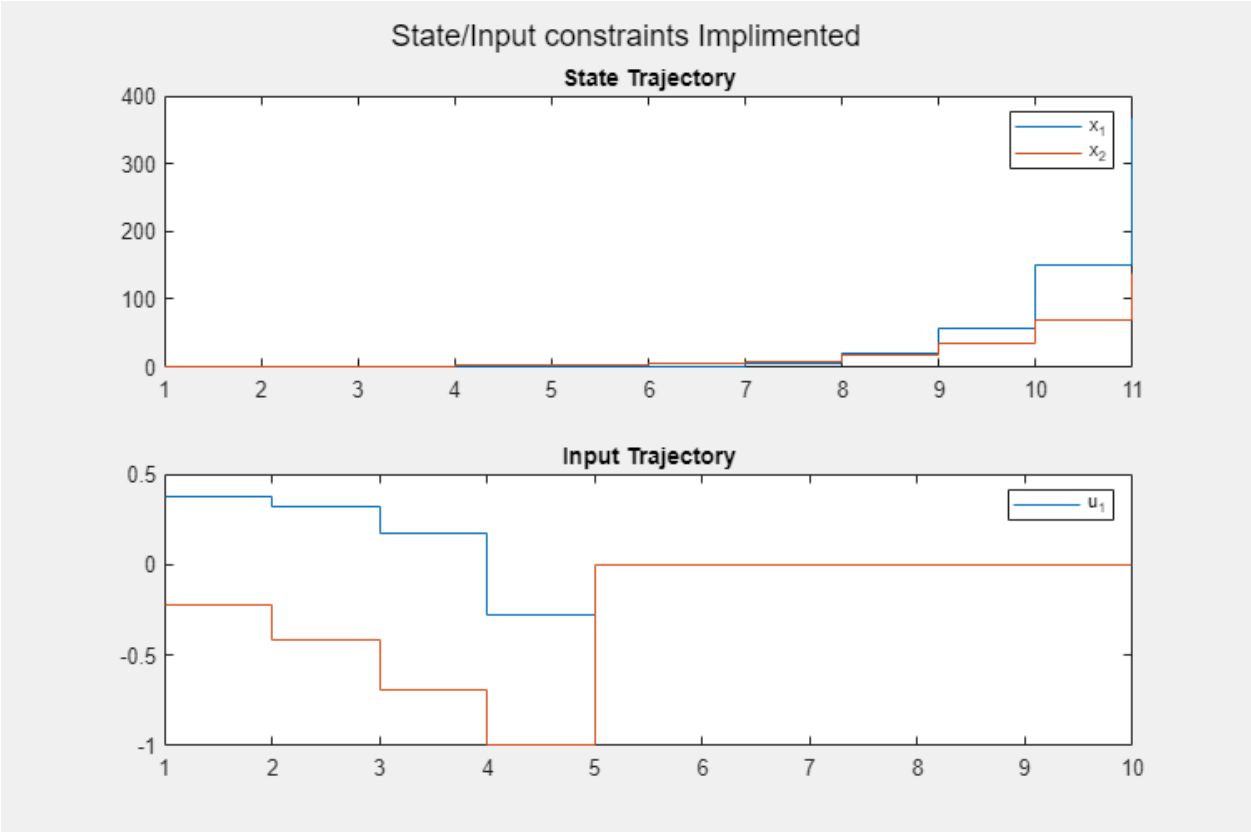


Part 2b

Setup and sim

```
alpha = 0.1; Q = alpha*eye(nx);
cons = @(x,u,s) [cons_x(x), cons_u(u)];
controller = mpc_yalmip_controller(A, B, P, Q, R, N, cons, cons_f);
[X, U, diagnostics_] = run_sim(A,B,controller, x0, tf);

% Results
figName = sprintf('pblm2b');
fig = plot_trajectory(X, U);
sgtitle('State/Input constraints Implimented')
saveas(fig,[subfolder,filesep,'figs',filesep,figName],'png')

% U and Errors
for k = 1:tf
    fprintf('U_%d solution: %f\n',k,U(:,k))
    fprintf('Error (k=%d): %s\n',k,yalmiperror(diagnostics_{k}));
end
```

```
U_1 solution: 0.378958
U_-2.256091e-01 solution: Error (k=1): Successfully solved
U_2 solution: 0.324550
U_-4.187884e-01 solution: Error (k=2): Successfully solved
U_3 solution: 0.171374
U_-6.909988e-01 solution: Error (k=3): Successfully solved
U_4 solution: -0.277110
U_-1.000000e+00 solution: Error (k=4): Successfully solved
U_5 solution: 0.000000
U_0 solution: Error (k=5): Either infeasible or unbounded
U_6 solution: 0.000000
U_0 solution: Error (k=6): Either infeasible or unbounded
U_7 solution: 0.000000
U_0 solution: Error (k=7): Either infeasible or unbounded
U_8 solution: 0.000000
U_0 solution: Error (k=8): Either infeasible or unbounded
U_9 solution: 0.000000
U_0 solution: Error (k=9): Either infeasible or unbounded
U_10 solution: 0.000000
U_0 solution: Error (k=10): Either infeasible or unbounded
```



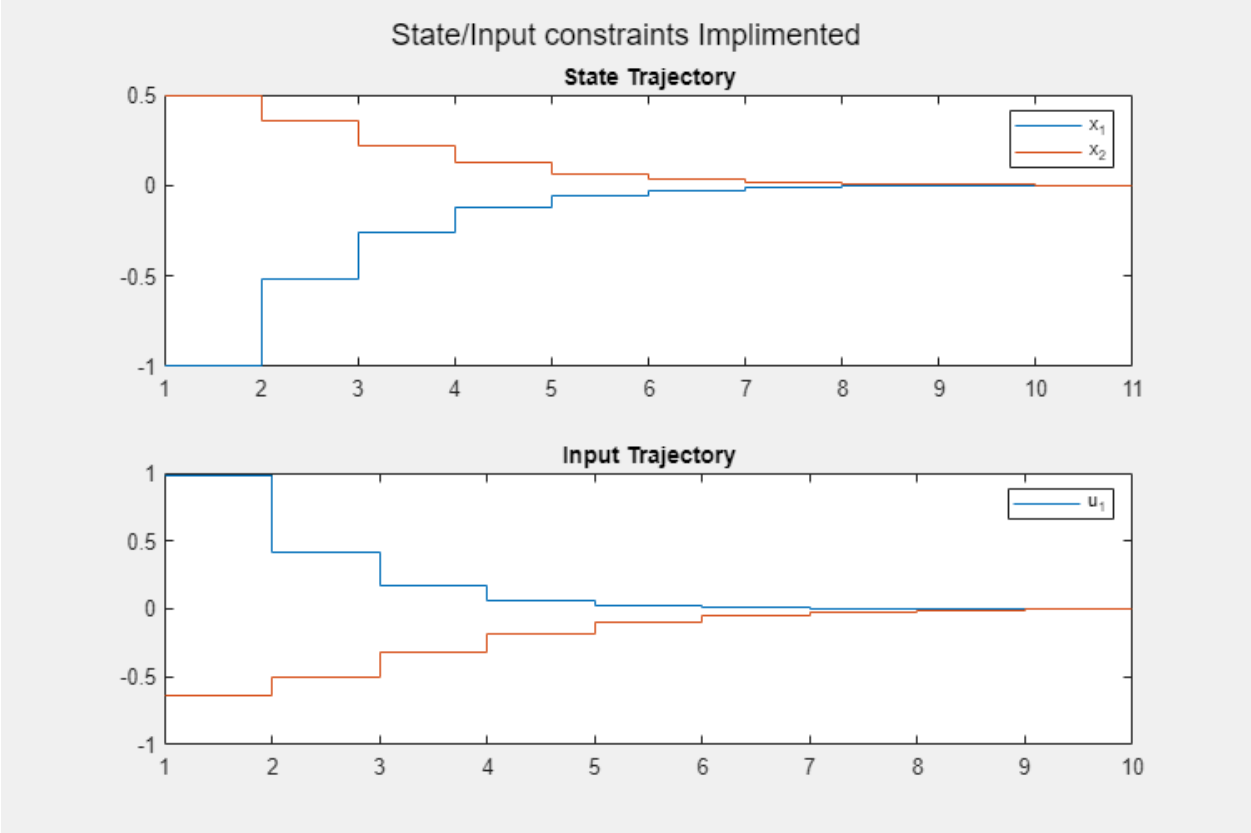
Part 2c

Setup and sim

```
alpha = 0.1; Q = alpha*eye(nx);
cons = @(x,u,s) [cons_x(x), cons_u(u)];
cons_f = @(x,u,s) x==0;
controller = mpc_yalmip_controller(A, B, P, Q, R, N, cons, cons_f);
[X, U, diagnostics_] = run_sim(A,B,controller, x0, tf);

% Results
figName = sprintf('pblm2c');
fig = plot_trajectory(X, U);
sgtitle('State/Input constraints Implimented')
saveas(fig,[subfolder,filesep,'figs',filesep,figName],'png')

% U and Errors
for k = 1:tf
    fprintf('U_%d solution: %f\n',k,U(:,k))
    fprintf('Error (k=%d): %s\n',k,yalmiperror(diagnostics_{k}));
end
```

```
U_1 solution: 0.984663
U_-6.399073e-01 solution: Error (k=1): Successfully solved
U_2 solution: 0.414154
U_-4.999188e-01 solution: Error (k=2): Successfully solved
```

```
U_3 solution: 0.168661
U_-3.170114e-01 solution: Error (k=3): Successfully solved
U_4 solution: 0.065967
U_-1.815646e-01 solution: Error (k=4): Successfully solved
U_5 solution: 0.024420
U_-9.764496e-02 solution: Error (k=5): Successfully solved
U_6 solution: 0.008310
U_-5.022334e-02 solution: Error (k=6): Successfully solved
U_7 solution: 0.002421
U_-2.495200e-02 solution: Error (k=7): Successfully solved
U_8 solution: 0.000458
U_-1.204290e-02 solution: Error (k=8): Successfully solved
U_9 solution: -0.000089
U_-5.665221e-03 solution: Error (k=9): Successfully solved
U_10 solution: -0.000175
U_-2.601983e-03 solution: Error (k=10): Successfully solved
```



Part 2d

```
tic
% Batch approch
[S_x, S_u] = construct_Sx_Su(A,B,N);

% State/Input constraints
A_x = [1 0]; b_x = 5;
A_xf = [eye(nx);-eye(nx)]; b_xf = zeros(2*nx,1); %<--- origin
```

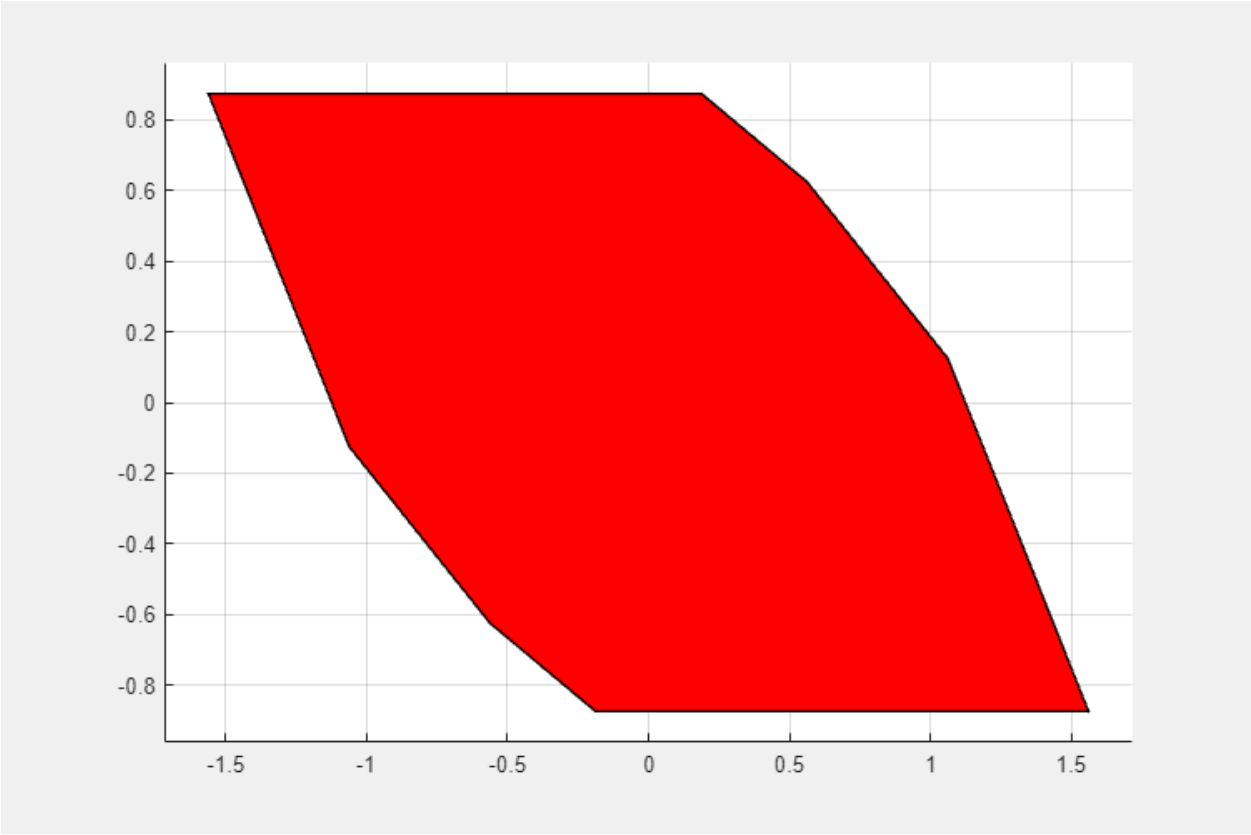
```
A_u = [eye(nu);-eye(nu)]; b_u = ones(2*nu,1);

for i = 1:N
    A_con_{i} = A_x;
    b_con_{i} = b_x;
end
A_con_{N+1} = A_xf;
b_con_{N+1} = b_xf;
for i = 1:N
    A_con_{i+N+1} = A_u;
    b_con_{i+N+1} = b_u;
end

A_con = blkdiag(A_con_{:}) * [S_x, S_u; zeros(N*nu,nx),eye(N*nu)];%
 zeros(size(S_u,2),size(S_x,2)), eye(size(S_u,2))];
b_con = vertcat(b_con_{:});

P = Polyhedron(A_con,b_con);
Q = P.projection(1:nx)
toc
```

```
Polyhedron in R^2 with representations:
    H-rep (redundant)   : Inequalities   9 | Equalities   0
    V-rep               : Unknown (call computeVRep() to compute)
Functions : none
Elapsed time is 0.385552 seconds.
```

Recursive Approch

```
tic
X_{N+1} = Polyhedron(A_xf,b_xf);
for k = N:-1:1
    P = Polyhedron( ...
        [X_{k+1}.A*A, X_{k+1}.A*B; zeros(size(A_u,1),nx),A_u],...
        [X_{k+1}.b; b_u]);
    X_{k} = P.projection(1:nx);
end

X_{1}
toc

% Ploting
figName = 'pblm2d';
fig = figure(WindowStyle="normal", Position=[0 0 750 500]);
X_{1}.plot
saveas(fig,[subfolder,filesep,'figs',filesep,figName],'png')
```

```
Polyhedron in R^2 with representations:
    H-rep (redundant)   : Inequalities   8 | Equalities   0
    V-rep               : Unknown (call computeVRep() to compute)
Functions : none
Elapsed time is 0.160652 seconds.
```



Local Functions

```
function controller = mpc_yalmip_controller(A,B,P,Q,R,N,cons,cons_f)
    yalmip('clear')
    nx = size(A,1);
    nu = size(B,2);

    u_ = sdpvar(repmat(nu,1,N),ones(1,N));
    x_ = sdpvar(repmat(nx,1,N+1),ones(1,N+1));
    s_ = sdpvar(ones(1,N+1),ones(1,N+1));

    constraints = [];
    objective = 0;
    for k = 1:N
        objective = objective + x_{k}'*Q*x_{k} + u_{k}'*R*u_{k} +
 s_{k}; %norm(Q*x_{k},2) + norm(R*u_{k},2);
        constraints = [constraints, s_{k} >= 0];
        constraints = [constraints, x_{k+1} == A*x_{k} + B*u_{k}];
        constraints = [constraints, cons(x_{k+1},u_{k},s_{k})];
    end
    constraints = [constraints,cons_f(x_{k+1},u_{k},s_{k})];
    objective = objective + x_{k+1}'*P*x_{k+1};

    opts = sdpsettings;
    controller = optimizer(constraints, objective,opts,x_{1},u_{1});
end
```

```
function [X,U,diagnostics_] = run_sim(A,B,controller,x0, tf)

    X_{tf+1} = []; U_{tf} = []; diagnostics_{tf} = [];
    X_{1} = x0;
    for k = 1:tf
        [U_{k},diagnostics_{k}] = controller{X_{k}};
        X_{k+1} = A*X_{k} + B*U_{k};
    end
    X = [X_{:}]; U = [U_{:}];
end

function fig = plot_trajectory(X, U)
    fig = figure(...
        WindowStyle="normal",...
        Position=[0 0 750 500]);
    hold on; grid on;
    subplot(2,1,1);
    stairs(X')
    title('State Trajectory')
    legend({'x_1','x_2'})
    subplot(2,1,2);
    stairs(U');
    title('Input Trajectory')
    legend({'u_1'})
end

function [S_x, S_u] = construct_Sx_Su(A, B, N, augmentRow)
    arguments
        A
        B
        N
        augmentRow = true
    end
    nx = size(A,1);
    nu = size(B,2);
    % Construct S_x,and S_u
    for i = 1:N
        S_x_{i} = A^(i);

        for j = 1:N
            if j <= i
                S_u__{j} = A^(i-1+j-1)*B;
            else
                S_u__{j} = zeros(nx,nu);
            end
        end
        S_u_{i} = horzcat(S_u__{:});
    end
    S_x = vertcat(S_x_{:});
    S_u = vertcat(S_u_{:});

    % augment w/ extra row
    if augmentRow
```

```
        S_x = vertcat(eye(nx), S_x);
        S_u = vertcat(zeros(nx,N*nu), S_u);
    end
end
```

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