Appendix A.

Vehicle Model Equations

Referring to figure A.1, the equations of motion of the seven degrees of freedom vehicle model read (for the meaning of the symbols see the Notation section):

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \left[(F_{x1} - F_{x2}) \cdot \frac{t_f}{2} + (F_{x3} - F_{x4}) \cdot \frac{t_r}{2} + (F_{y1} + F_{y2}) \cdot l_f - (F_{y3} + F_{y4}) \cdot l_r \right] \frac{1}{I_{zz}}$$

$$\dot{x}_3 = [F_{x1} + F_{x2} + F_{x3} + F_{x4} - F_{ax}] \frac{1}{M} + x_2 x_4$$

$$\dot{x}_4 = [F_{y1} + F_{y2} + F_{y3} + F_{y4}] \frac{1}{M} - x_2 x_3$$

$$\dot{x}_5 = x_3 \cdot \cos(x_1) - x_4 \cdot \sin(x_1)$$

$$\dot{x}_6 = x_3 \cdot \sin(x_1) + x_4 \cdot \cos(x_1)$$

$$\dot{x}_7 = x_8$$

$$\dot{x}_8 = (T_1 - F_{xw1} \cdot R_f) \frac{1}{J_{wf}}$$

$$\dot{x}_9 = x_{10}$$

$$\dot{x}_{10} = (T_2 - F_{xw2} \cdot R_f) \frac{1}{J_{wf}}$$

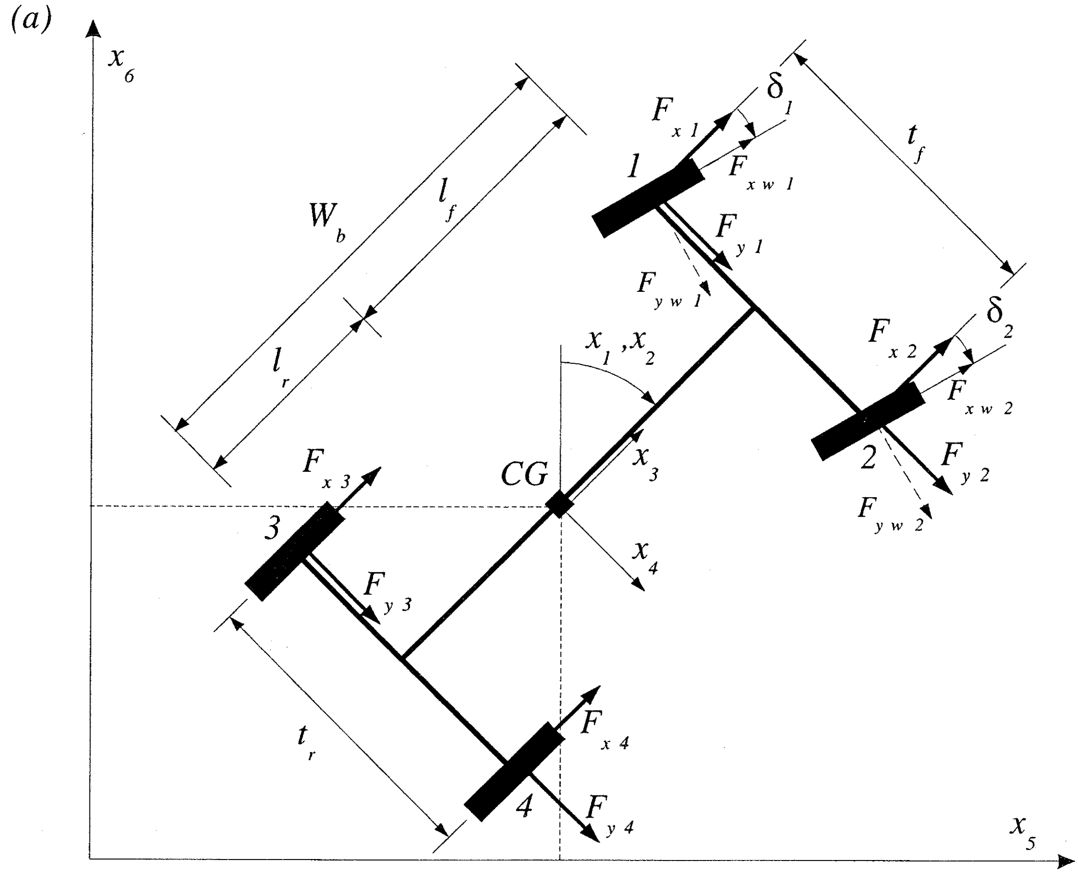
$$\dot{x}_{11} = x_{12}$$

$$\dot{x}_{12} = \frac{(T_3 - F_{x3} \cdot R_r) \cdot \left(J_{wr} + J_m \cdot \left(\frac{G_R}{2} \right)^2 \right) - (T_4 - F_{x4} \cdot R_r) \cdot J_m \cdot \left(\frac{G_R}{2} \right)^2}{J_{wr}^2 + 2 \cdot J_{wr} \cdot J_m \cdot \left(\frac{G_R}{2} \right)^2}$$

Eq. A.1

$$\dot{x}_{13} = x_{14}$$

$$\dot{x}_{14} = \frac{(T_4 - F_{x4} \cdot R_r) \cdot \left(J_{wr} + J_m \cdot \left(\frac{G_R}{2} \right)^2 \right) - (T_3 - F_{x3} \cdot R_r) \cdot J_m \cdot \left(\frac{G_R}{2} \right)^2}{J_{wr}^2 + 2 \cdot J_{wr} \cdot J_m \cdot \left(\frac{G_R}{2} \right)^2}$$



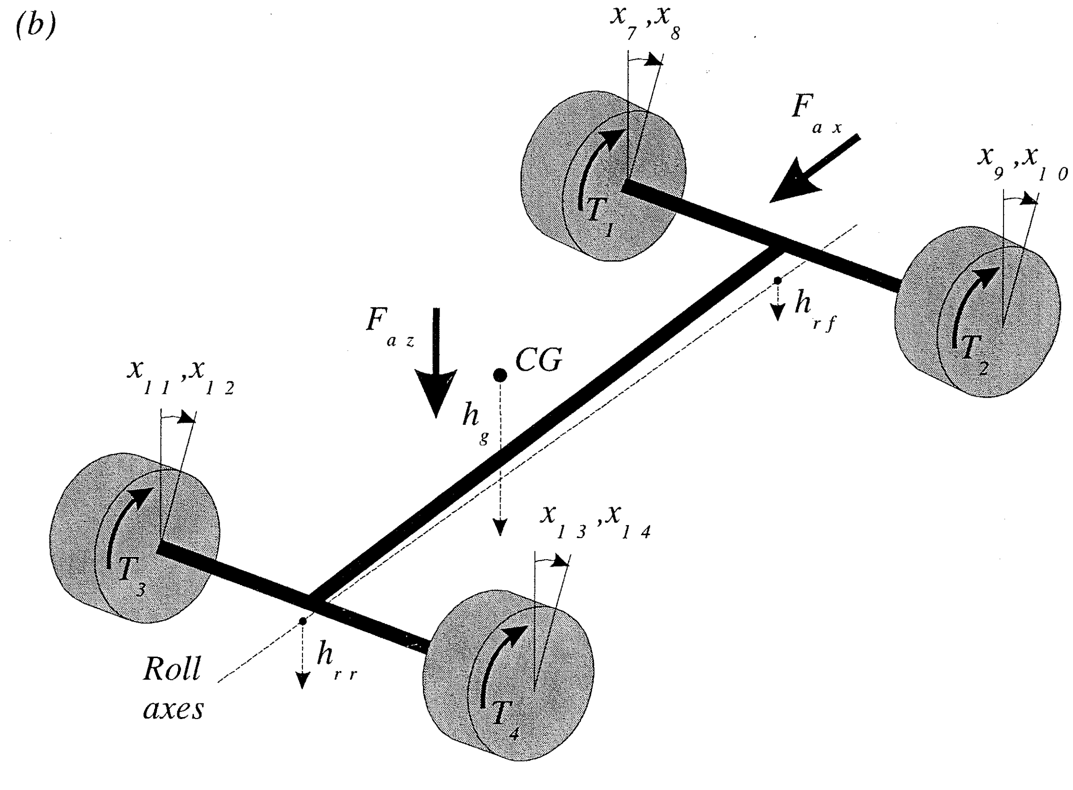


Figure A.1 Scheme of the vehicle model.

For the front wheels, the tyre forces are firstly evaluated into the wheel's reference axis system (F_{xw1}, F_{yw1}, F_{xw2} and F_{yw2} in figure A.1 (a)) and then projected onto the longitudinal and lateral directions:

$$\begin{cases} F_{x1} = F_{xw1} \cdot \cos(\delta_1) - F_{yw1} \cdot \sin(\delta_1) \\ F_{y1} = F_{xw1} \cdot \sin(\delta_1) + F_{yw1} \cdot \cos(\delta_1) \\ F_{x2} = F_{xw2} \cdot \cos(\delta_2) - F_{yw2} \cdot \sin(\delta_2) \\ F_{y2} = F_{xw2} \cdot \sin(\delta_2) + F_{yw2} \cdot s\cos(\delta_2) \end{cases} \quad \text{Eq. A.2}$$

The individual steer angles for the front wheels are assumed to be equal:

$$\delta_1 = \delta_2 = \delta \quad \text{Eq. A.3}$$

The model of the powertrain includes:

- Engine output torque described using an experimental engine map;
- Global transmission ratios;
- Torque sensitive, limited slip differential.

The experimental engine map allows to evaluate the engine output torque as a function of two parameters, the throttle aperture and the engine rotational velocity. At any time in the simulation, the engine rotational velocity may be evaluated from the rear wheel velocities by applying the differential rule:

$$\omega_E = \frac{(x_{12} + x_{14})}{2} \cdot G_R \cdot \frac{30}{\pi} [rpm] \quad \text{Eq. A.4}$$

Here, the gear ratio G_R is selected upon the vehicle forward velocity x_3. Using the current values of the longitudinal control variable u_{tb} and the engine rotational velocity, the engine output torque T_E is estimated from the engine map $\mathbf{E_{trq}}$ by means of bi-linear interpolation:

$$T_E = \text{bi\_linear\_interp}\left(\mathbf{E_{rpm}}, \mathbf{E_{th}}, \mathbf{E_{trq}}, \omega_E, u_{tb}\right) \quad \text{Eq. A.5}$$

The torque applied to each driven wheel may then be computed as follows:

$$\begin{cases} T_3 = \dfrac{T_E \cdot G_R}{2}_{\ r} + \Delta_T \\ T_4 = \dfrac{T_E \cdot G_R}{2} - \Delta_T \end{cases} \quad \text{Eq. A.6}$$

Δ_T is the amount of torque transfer due to the limited slip differential. The simple model used to compute the torque transfer is based on the Salisbury type differential (Milliken and Milliken, 1995). In this differential clutch packs are used to progressively lock the two output shafts together. The axles of the differential pinions rest against wedges which are spread by the input torque, so that they apply a pressure to the clutch packs which is proportional to the input torque. Furthermore, the wedges may have different angles to give different characteristics in driving and in overrun. The amount of torque transfer may then be evaluated as a constant portion of the input torque. Two different torque gains G_{D2_D} and G_{D2_O} are used to represent the different characteristics of the differential in driving and in overrun respectively. Also a constant off-set G_{D1} is included to account for the natural friction in the differential. According to the SAE sign conventions, the driving torque is negative and the braking torque is positive. Hence, if T_D is the input torque to the differential cage, the torque transfer reads:

$$\begin{aligned} T_T &= -G_{D1} + G_{D2\_D} \cdot T_D \qquad \mathit{if} \quad T_D < 0 \quad (\mathit{driving}) \\ T_T &= -G_{D1} - G_{D2\_O} \cdot T_D \qquad \mathit{if} \quad T_D > 0 \quad (\mathit{overrun}) \end{aligned} \qquad \text{Eq. A.7}$$

Eq. A.7 always returns a negative value for the torque transfer. The direction of the torque transfer which is actually applied to the rear wheels depends on the sign of their differential velocity:

$$\Delta T = T_T \cdot \operatorname{sign}(\omega_{RR} - \omega_{LR}) \qquad \text{Eq. A.8}$$

When the equations of motion for the rear wheels are derived, it is necessary to take into account that the torque values T_3 and T_4 do not only accelerate the rotating masses associated with the wheels. The inertia due to the rotating mass of the engine has to be taken into account, since it depends on the gear ratio and its influence is significant in lower gears. Let J_m be the inertia of the rotating mass of the engine. The input torque to the differential cage T_D may be written as follows:

$$T_D = T_E \cdot G_R - J_m \cdot \dot{\omega}_E \cdot G_R = T_E \cdot G_R - J_m \cdot \frac{(\dot{x}_{12} + \dot{x}_{14})}{2} \cdot G_R^2 \qquad \text{Eq. A.9}$$

Neglecting the inertia of the differential, we may write that the torque T_D is divided between the two half shafts accounting for the torque transfer Δ_T as follows:

$$\begin{cases} T_3' = \dfrac{T_D}{2} + \Delta_T \\ T_4' = \dfrac{T_D}{2} - \Delta_T \end{cases} \qquad \text{Eq. A.10}$$

Since T_3' and T_4' are the actual torque values applied to the rear wheels, we may write the following equations for the rear wheel spin dynamics:

$$\begin{cases} T_3' - J_{wr} \cdot \dot{x}_{12} - F_{x3} \cdot R_r = 0 \\ T_4' - J_{wr} \cdot \dot{x}_{14} - F_{x4} \cdot R_r = 0 \end{cases} \qquad \text{Eq. A.11}$$

Eq. A.9 may be modified using Eq. A.10 as follows:

$$T_3' + T_4' = T_E \cdot G_R - J_m \cdot \frac{(\dot{x}_{12} + \dot{x}_{14})}{2} \cdot G_R^2 \qquad \text{Eq. A.12}$$

From Eq. A.10 we may also derive:

$$T_3' = T_4' + 2 \cdot \Delta_T \qquad \text{Eq. A.13}$$

Hence, by substituting Eq. A.13 into Eq. A.12 we obtain:

$$T_4' = \left(\frac{T_E \cdot G_R}{2} - \Delta_T \right) - J_m \cdot \frac{(\dot{x}_{12} + \dot{x}_{14})}{4} \cdot G_R^2 = T_4 - J_m \cdot \frac{(\dot{x}_{12} + \dot{x}_{14})}{4} \cdot G_R^2 \qquad \text{Eq. A.14}$$

In the same way we obtain for the rear left wheel:

$$T_3' = T_3 - J_m \cdot \frac{(\dot{x}_{12} + \dot{x}_{14})}{4} \cdot G_R^2 \qquad \text{Eq. A.15}$$

Substituting Eqs. A.14 and A.15 into Eq. A.11, and solving for $\dot{x}_{12}$ and $\dot{x}_{14}$, allows to obtain the equations of motion for the rear wheels as functions of the torque values T_3 and T_4 computed by using Eq. A.6.

When braking, the amount of negative torque applied to the four wheels is evaluated by multiplying the maximum braking torque available by the value of the longitudinal control variable:

$$T_B = u_{tb} \cdot T_{B\max} \qquad u_{tb} < 0 \qquad \text{Eq. A.16}$$

T_B is shared between the front and rear axles according to the constant coefficients B_f and B_r. Then, the braking torque is divided evenly between the wheels on the same axle. The braking effect of the engine is accounted for by evaluating Eq. A.5 with u_{tb} equal to 0. Also the torque transfer due to the differential is included. Hence, the expressions of the braking torque for each wheel read:

$$\begin{cases} T_1 = \dfrac{T_B \cdot B_f}{2} \\ T_2 = \dfrac{T_B \cdot B_f}{2} \\ T_3 = \dfrac{T_B \cdot B_r}{2} + \dfrac{T_E \cdot G_R}{2} + \Delta_T \\ T_4 = \dfrac{T_B \cdot B_r}{2} + \dfrac{T_E \cdot G_R}{2} - \Delta_T \end{cases} \qquad \text{Eq. A.17}$$

The aerodynamic drag and down-force in Eq. A.1 are evaluated as follows:

$$F_{ax} = \frac{1}{2}\rho \cdot S_f \cdot C_x \cdot x_3^2$$
$$F_{az} = \frac{1}{2}\rho \cdot S_f \cdot C_z \cdot x_3^2 \qquad \text{Eq. A.18}$$

The vertical load acting on each wheel is evaluated including a steady state approximation of the lateral and longitudinal load transfers. Firstly, the static distribution of the vehicle weight and the aerodynamic down-force on the front and rear axles is calculated as follows:

$$F_{fz\_static} = \left(M \cdot g \cdot l_r + F_{az} \cdot D_f\right) / W_b$$
$$F_{rz\_static} = \left(M \cdot g \cdot l_f + F_{az} \cdot D_r\right) / W_b \qquad \text{Eq. A.19}$$

Then, the longitudinal load transfer, which occurs in driving or braking conditions, is evaluated by considering the equilibrium of the vehicle about its centre of gravity. Assuming the aerodynamic drag acts at the height of the CG, the expression for the longitudinal load transfer reads:

$$\Delta F_{z\_long} = \left(F_{x1} + F_{x2} + F_{x3} + F_{x4}\right)\frac{h_g}{W_b} \cong \left(\frac{T_1 + T_2}{R_f} + \frac{T_3 + T_4}{R_r}\right)\frac{h_g}{W_b} \qquad \text{Eq. A.20}$$

Since the longitudinal tyre forces are functions of the tyre vertical loads and therefore unknown, their values are approximated by substituting the torque applied to each wheel divided by the wheel radius. In doing this, the contribution of the inertia of the rotating masses is neglected and the longitudinal load transfer will be slightly overestimated.

The lateral load transfer is estimated by evaluating the steady state cornering equilibrium of the vehicle. It is assumed that the roll axis has a fixed position and that the roll stiffness distribution is constant. The vehicle chassis is considered infinitely stiff, hence front and rear roll angles are the same. Furthermore, the contribution of the unsprung masses is neglected. Referring to figure A.2, which represents the front end of the vehicle, the roll equilibrium of the body about the roll axis reads:

$$T_{rf} - M_f \cdot \frac{x_3^2}{r_t} \cdot \left(h_g - h_{rf}\right) \cdot \cos(\phi) + M_f \cdot g \cdot \left(h_g - h_{rf}\right) \cdot \sin(\phi) - C_{f\phi} \cdot \phi = 0 \qquad \text{Eq. A.21}$$

Analogously, the roll equilibrium of the whole vehicle front end about a ground origin located in the middle of the vehicle reads:

$$\left(\frac{M_f \cdot g}{2} + \Delta F_{fz}\right)\cdot\frac{t_f}{2} - \left(\frac{M_f \cdot g}{2} - \Delta F_{fz}\right)\cdot\frac{t_f}{2} + \\ - M_f \cdot \frac{x_3^2}{r_t}\cdot\left[h_{rf} + \left(h_g - h_{rf}\right)\cdot\cos(\phi)\right] + M_f \cdot g \cdot \left(h_g - h_{rf}\right)\cdot\sin(\phi) + T_{rf} = 0 \qquad \text{Eq. A.22}$$

Eq. A.22 may be simplified by assuming small roll angles, i.e. $\cos(\phi) \cong 1$ and $\sin(\phi) \cong \phi$. Furthermore, solving Eq. A.21 for T_{rf} and substituting in Eq. A.22 yields:

$$\Delta F_{fz} \cdot t_f - M_f \cdot \frac{x_3^2}{r_t} \cdot h_{fr} + C_{f\phi} \cdot \phi = 0 \qquad \text{Eq. A.23}$$

From Eq. A.23, the front lateral load transfer may be derived:

$$\Delta F_{fz} = \frac{M_f \cdot \frac{x_3^2}{r_t} \cdot h_{fr} - C_{f\phi} \cdot \phi}{t_f} \qquad \text{Eq. A.24}$$

If we now consider the roll equilibrium of the whole vehicle body, the roll angle resulting from a constant lateral acceleration of the centre of mass may be evaluated as follows:

$$\phi = -\frac{E \cdot \frac{x_3^2}{r_t}}{C_{f\phi} + C_{r\phi} - E \cdot g} \qquad \text{Eq. A.25}$$

where the term E reads:

$$E = M_f \cdot \left(h_g - h_{rf}\right) + M_r \cdot \left(h_g - h_{rr}\right) \qquad \text{Eq. A.26}$$

Eq. A.26 may be simplified by using the longitudinal position of the centre of mass to compute the front and rear load distribution:

$$E = M \cdot \left[\frac{l_r}{W_b}\cdot\left(h_g - h_{rf}\right) + \frac{l_f}{W_b}\cdot\left(h_g - h_{rr}\right)\right] = \\ = M \cdot \left(h_g - \frac{l_r}{W_b}\cdot h_{rf} - \frac{l_f}{W_b}\cdot h_{rr}\right) = \\ = M \cdot \left(h_g - h_{rc}\right) \qquad \text{Eq. A.27}$$

Here, h_{rc} is the roll centre height at the position of the vehicle centre of gravity. Next, Eq. A.27 may be substituted in Eq. A.25. The resulting expression may be simplified further by observing that in general $C_{f\phi} + C_{r\phi} >> M \cdot g \cdot \left(h_g - h_{rc}\right)$:

$$\phi = -\frac{M \cdot (h_g - h_{rc}) \cdot \frac{x_3^2}{r_t}}{C_{f\phi} + C_{r\phi} - M \cdot g \cdot (h_g - h_{rc})} \cong -\frac{M \cdot (h_g - h_{rc}) \cdot \frac{x_3^2}{r_t}}{C_{f\phi} + C_{r\phi}} \qquad \text{Eq. A.28}$$

Substituting Eq. A.28 into Eq. A.24 yields the expression for the lateral load transfer for the front axle as function of the vehicle states and parameters. However, since it is not normally possible to compute the actual turning radius r_t, it is more convenient to use the following expression to compute the vehicle lateral acceleration:

$$\frac{x_3^2}{r_t} = x_2 \cdot x_3 \qquad \text{Eq. A.29}$$

We are now able to write the expressions of the lateral load transfer for the front and rear axles read respectively:

$$\Delta F_{fz\_lat} = \frac{x_2 \cdot x_3 \cdot M}{t_f}\left[(l_r \cdot h_{rf}) / W_b + R_{sf}(h_g - h_{rc})\right]$$
$$\Delta F_{rz\_lat} = \frac{x_2 \cdot x_3 \cdot M}{t_r}\left[(l_f \cdot h_{rr}) / W_b + R_{sr}(h_g - h_{rc})\right] \qquad \text{Eq. A.30}$$

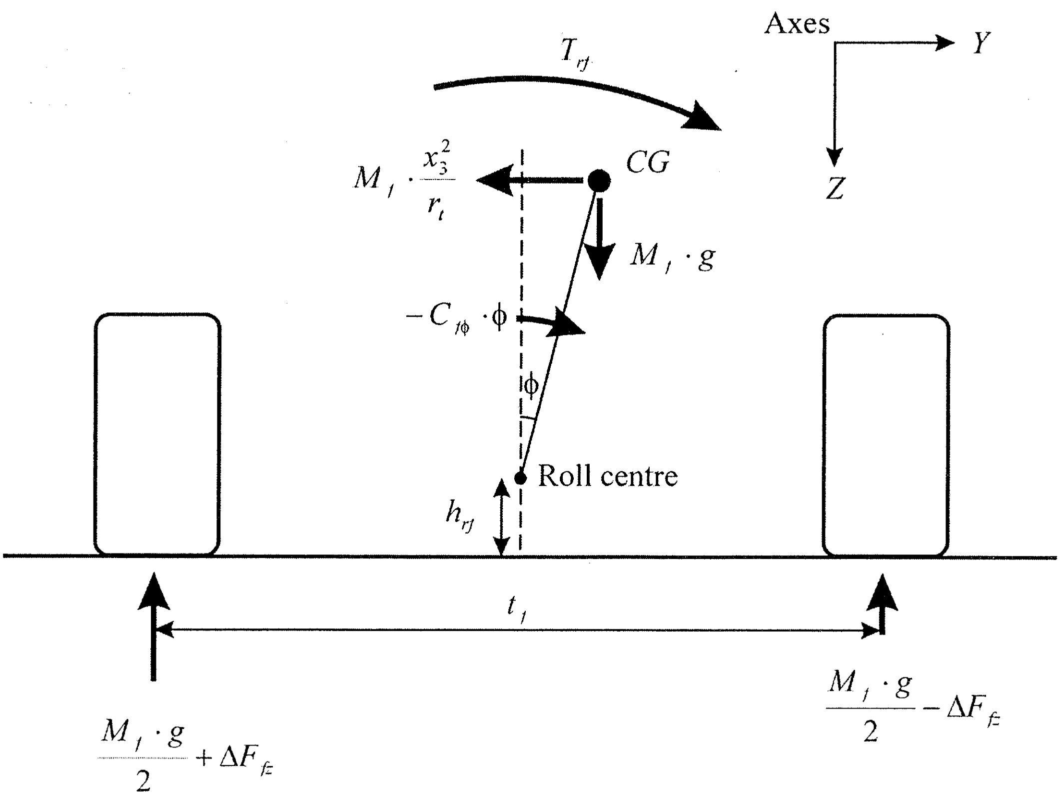


Figure A.2 Schematic rear view of the front suspension.

Finally, the above expressions may be combined to evaluate the vertical load on each wheel. According to the SAE conventions, which implies negative tyre loads, we may write:

$$
\begin{aligned}
F_{z1} &= -\left[\frac{1}{2}F_{fz\_static} - \frac{1}{2}\Delta F_{z\_long} + \Delta F_{fz\_lat}\right] \\
F_{z2} &= -\left[\frac{1}{2}F_{fz\_static} - \frac{1}{2}\Delta F_{z\_long} - \Delta F_{fz\_lat}\right] \\
F_{z3} &= -\left[\frac{1}{2}F_{rz\_static} + \frac{1}{2}\Delta F_{z\_long} + \Delta F_{rz\_lat}\right] \\
F_{z4} &= -\left[\frac{1}{2}F_{rz\_static} + \frac{1}{2}\Delta F_{z\_long} - \Delta F_{rz\_lat}\right]
\end{aligned}
\qquad \text{Eq. A.31}
$$

The tyre forces are evaluated by using the Magic Formula Tyre Model which features the use of weighting functions to represent the tyre behaviour at combined slip conditions (Pacejka and Besselink, 1997). We may formally write:

$$
\begin{cases}
F_{ywi\,(combined)} = F_{ywi\,(free\ rolling)}\left(\alpha_i, \gamma_i, F_{zi}\right)\cdot G_y\left(\alpha_i, k_i, \gamma_i, F_{zi}\right) \\
F_{xwi\,(combined)} = F_{xwi\,(pure\ slip)}\left(k_i, F_{zi}\right)\cdot G_x\left(\alpha_i, k_i\right)
\end{cases}
\quad i = 1,2,3,4 \qquad \text{Eq. A.32}
$$

The tyre forces are referred to the wheel reference axes system. For the steered wheels, the longitudinal and lateral forces are projected onto the vehicle reference axes system using equations A.2. A value for the static camber γ_i is assigned to each wheel. The tyre slip quantities are evaluated with the assumption of small angles and they are referred to the centre of the contact area of each wheel. The expressions for the slip angles in degrees and the longitudinal slip in percent read respectively:

$$
\begin{cases}
\alpha_1 = -\delta_1 + \dfrac{x_4 + l_f x_2}{x_3 + x_2\cdot\dfrac{t_f}{2}}\dfrac{180}{\pi}; \quad \alpha_2 = -\delta_2 + \dfrac{x_4 + l_f x_2}{x_3 - x_2\cdot\dfrac{t_f}{2}}\dfrac{180}{\pi} \\
\alpha_3 = \dfrac{x_4 - l_r x_2}{x_3 + x_2\cdot\dfrac{t_r}{2}}\dfrac{180}{\pi}; \quad \alpha_4 = \dfrac{x_4 - l_r x_2}{x_3 - x_2\cdot\dfrac{t_r}{2}}\dfrac{180}{\pi}
\end{cases}
\qquad \text{Eq. A.33}
$$

$$
\begin{cases}
k_1 = -\left(1 - \dfrac{x_8}{x_3 + x_2\cdot\dfrac{t_f}{2}}R_f\right)\times 100; \quad k_2 = -\left(1 - \dfrac{x_{10}}{x_3 - x_2\cdot\dfrac{t_f}{2}}R_f\right)\times 100 \\
k_3 = -\left(1 - \dfrac{x_{12}}{x_3 + x_2\cdot\dfrac{t_r}{2}}R_r\right)\times 100; \quad k_4 = -\left(1 - \dfrac{x_{14}}{x_3 - x_2\cdot\dfrac{t_f}{2}}R_r\right)\times 100
\end{cases}
\qquad \text{Eq. A.34}
$$

Two sets of tyre data were employed. One is representative of the typical tyre forces for a Formula One car on dry tarmac, the other of the typical tyre forces for a rally car on a low friction surface, e.g. gravel or mud. Figures A.3 and A.4 show an example of such forces at pure slip for a range of loads representative of the vehicle working conditions.

When presenting results we often referred to the tyre saturation level as a measure for evaluating whether the vehicle is working more or less close to the limit of its performance envelope. We define the saturation level of a tyre either in longitudinal or lateral direction as follows. Let us consider a tyre generating the lateral force F_{yi} and the longitudinal force F_{xi} at the actual working conditions a_i, k_i, F_{zi} and γ_i. The lateral saturation is defined as the ratio between F_{yi} and the maximum lateral force obtainable while keeping k_i constant:

$$F_{y\_MAX} = \max_{\alpha}\left(F_{y(combined)}\right) \quad \text{constant longitudinal slip } k_i, \alpha \text{ varying}$$

$$Lat\_sat = \frac{F_{yi}}{F_{y\_MAX}} \times 100 \quad [\%] \qquad \text{Eq. A.35}$$

Similarly, for the longitudinal saturation we may write:

$$F_{x\_MAX} = \max_{k}\left(F_{x(combined)}\right) \quad \text{constant slip angle } \alpha_i, k \text{ varying}$$

$$Long\_sat = \frac{F_{xi}}{F_{x\_MAX}} \times 100 \quad [\%] \qquad \text{Eq. A.36}$$

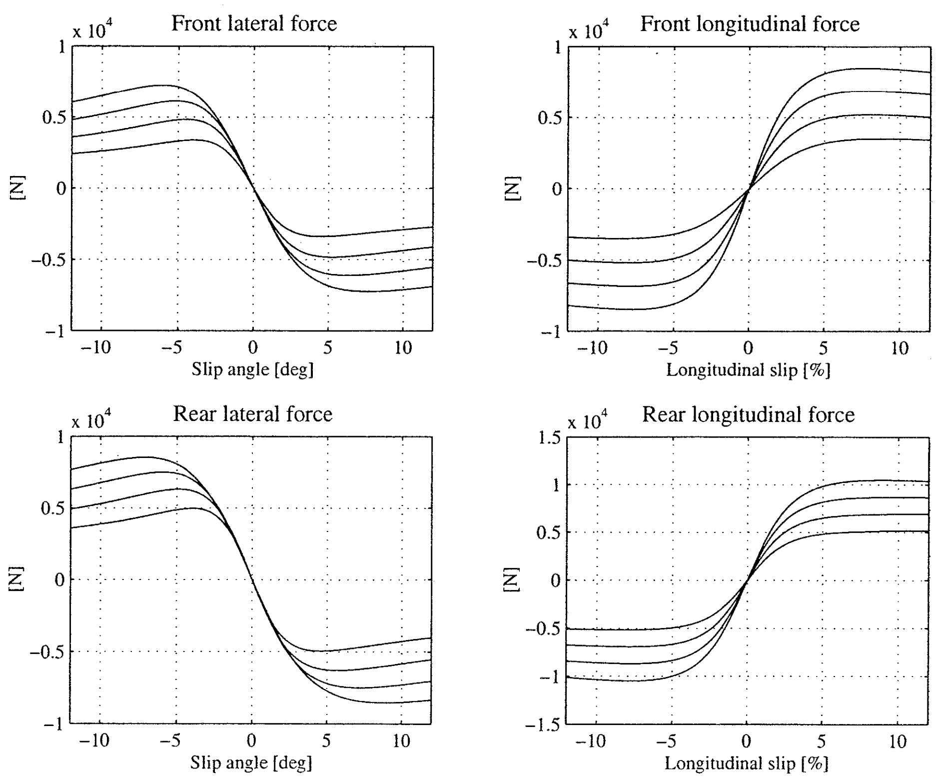


Figure A.3 Tyre force characteristics, F1 car, front loads equal to 2, 3, 4 and 5 kN, rear loads equal to 3, 4, 5 and 6 kN.

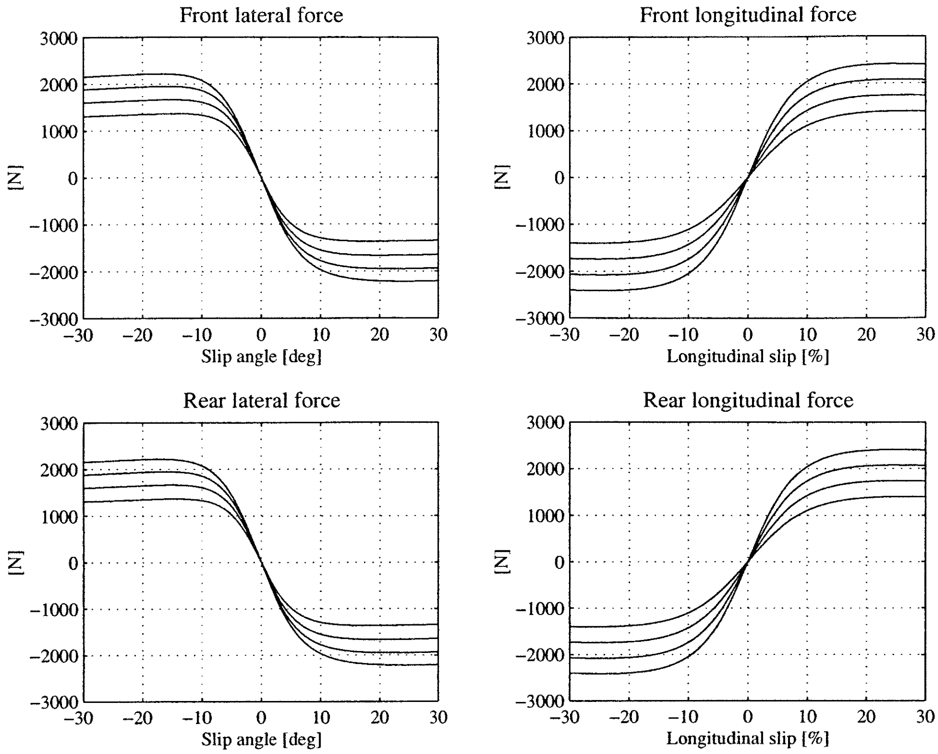


Figure A.4 Tyre force characteristics, rally car, front and rear loads equal to 2, 2.5, 3 and 3.5 kN.