MECH 6V29: Multiagent Robotic Systems- HW 2

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Contents

Problem 1

State a summary of lectures ***lectures 6-10***, preferably by creating a concept map diagram (flow diagram). The whole purpose is to make sure that we are clear about the bigger picture, and reiterate why are we doing and discussing the specific topics in the class. Do not merely write the topics, instead create connections between topics to clarify the flow of information.

a) Big Picture Chart

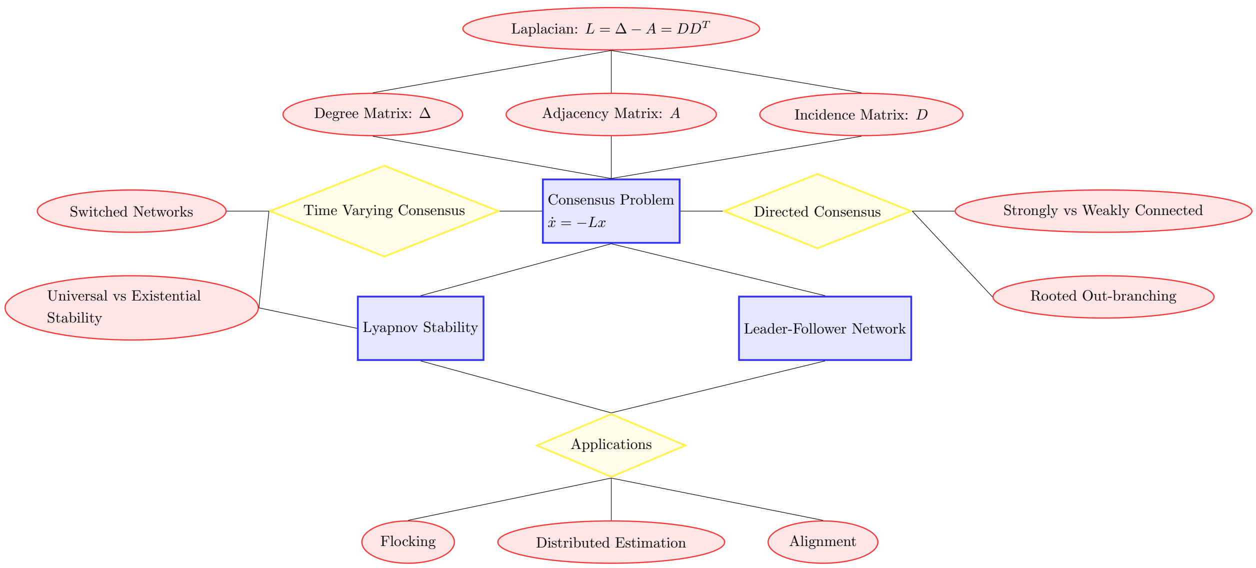


Fig. 1: Diagram of Course Topics (created w/ TikZ)

Problem 2

Recall that if x_i is scalar, with its derivative given by the consensus equation

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i), \; i = 1, 2, \cdots, N$$

this can be written as

$$\dot{x} = -Lx$$

where L is the Laplacian of the undirected graph, and $x = \begin{bmatrix} x_1 x_2 \cdots x_N \end{bmatrix}^T$.

a)

Problem:

If instead

$$\dot{x} = -L^2 x$$

what are the corresponding node level dynamics, that is, find

$$\dot{x}_i = ???$$

Preliminaries

Definition 1. *Let graph $G(V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $E \subseteq V \times V$.*

a. The <u>degree matrix</u> $\Delta \in \mathbb{R}^{n \times n}$ *is a diagonal matrix defined as*

$$\Delta := \begin{bmatrix} \deg(v_1) & & & \\ & \deg(v_2) & & \\ & & \ddots & \\ & & & \deg(v_n) \end{bmatrix}$$

b. The <u>adjacency matrix</u> $A \in \mathbb{R}^{n \times n}$ *is a symmetric matrix ($A = A^T$) defined s.t.*

$$A = [a_{ij}] \;:\; a_{ij} \begin{cases} 1 & (v_i, v_j) \in V \\ 0 & (v_i, v_j) \notin V \end{cases}$$

c. The <u>incidence matrix</u> $D \in \mathbb{R}^{n \times m}$ *is defined as*

$$D = [d_{ij}] \;:\; d_{ij} \begin{cases} 1 & (v_i, -) \in e_j \\ -1 & (-, v_i) \in e_j \\ 0 & \textit{otherwise} \end{cases}$$

d. The <u>Laplacian matrix</u> $L \in \mathbb{R}^{n \times n}$ *is a symmetric ($L = L^T$) and strictly semi-positive definite ($L \succeq 0$) is defined as*

$$L := \Delta - A = DD^T$$

and

$$L = \begin{bmatrix} \deg(v_1) & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & \deg(v_2) & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & \deg(v_n) \end{bmatrix}$$

e. *For a weighted graph $G(V, E, W)$, the diagonal weighted matrix $W \in \mathbb{R}^{m\times m}$ is defined as*

$$W = [w_{ij}]\forall_{ij\in E}$$

were w_{ij} are the corresponding weights for $e_{ij} = (v_i, v_j)$.

Definition 2. *Let undirected and unweighted graph $G(V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $E \subseteq V \times V$. The* consensus dynamics *of network $G(V, E)$ is defined by*

$$\forall_{i=1,\ldots,n}\ \dot{x}_i = \sum_{j\in\mathcal{N}_i}(x_j - x_i) \iff \dot{x} = -Lx$$

For the case with weighted graph $G(V, E, W)$ with diagonal weight matrix $W = [w_{ij}]$, weighted consensus dynamics *are given as*

$$\dot{x}_i = \sum_{j\in\mathcal{N}_i} w_{ij}(x_j - x_i) \implies \dot{x} = -L_w x$$

where weighted Laplacian matrix L_w is defined as

$$L_w = DWD^T$$

Solution:

From the definition of the Laplacian Matrix (Definition 1), we have

$$L = \Delta - A$$

and therefore

$$\begin{aligned} L^2 &= L * L = (\Delta - A)(\Delta - A) \\ &= \Delta^2 - \Delta A - A\Delta + A^2 \\ &= \begin{bmatrix} \deg(v_1)^2 & 0 & 0 & \cdots & 0 \\ 0 & \deg(v_2)^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \deg(v_n)^2 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & -a_{12}\deg(v_1) & -a_{13}\deg(v_1) & \cdots & -a_{1n}\deg(v_1) \\ -a_{21}\deg(v_2) & 0 & -a_{23}\deg(v_2) & \cdots & -a_{2n}\deg(v_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1}\deg(v_n) & -a_{n2}\deg(v_n) & -a_{n3}\deg(v_n) & \cdots & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & -a_{12}\deg(v_1) & -a_{13}\deg(v_2) & \cdots & -a_{1n}\deg(v_3) \\ -a_{21}\deg(v_1) & 0 & -a_{23}\deg(v_2) & \cdots & -a_{2n}\deg(v_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1}\deg(v_1) & -a_{n2}\deg(v_2) & -a_{n3}\deg(v_3) & \cdots & 0 \end{bmatrix} \end{aligned}$$

$$
+\begin{bmatrix}
\sum_{i\neq 1}^{n} a_{1,i}a_{i,1} & \sum_{i\neq 1,2} a_{1,i}a_{i,2} & \cdots & \sum_{i\neq 1,n} a_{1,i}a_{i,2} \\
\sum_{i\neq 2,1} a_{2,i}a_{i,1} & \sum_{i\neq 2} a_{2,i}a_{i,2} & \cdots & \sum_{i\neq 1,n} a_{1,i}a_{i,n} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{i\neq n,1} a_{n,i}a_{i,1} & \sum_{i\neq n,2} a_{n,i}a_{i,2} & \cdots & \sum_{i\neq n} a_{n,i}a_{i,n}
\end{bmatrix}
$$

which results in

$$
\boxed{\dot{x}^{(i)} = -\left(\deg(v_1)^2 + \sum_{j\neq i} a_{i,j}a_{j,i}\right)x^{(i)} - \sum_{j\neq i}\left(-a_{i,j}(\deg(v_i)+\deg(v_j)) + \sum_{k\neq j} a_{i,k}a_{k,j}\right)x^{(j)}}
$$

b)

Problem:

Can you give a graph-theoretic interpretation to your answer in (A)?

Solution:

A potential graph-theoretical interpretation could be would be a network with nonlinear consensus dynamics dependent on more then just a single dynamic link. This is particularly interesting when looking at representations of digital logic gates or neuron type "circuitry".

Problem 3

Consider a leader-follower network with two leaders and two followers, as shown in Fig.2. Assume that leaders and followers are on the real line, and the underlying network graph is a path graph with the end nodes being the static leaders. Moreover, let the dynamics be given by the following:

$$\dot{x}_1 = \alpha_1((x_3 - x_1) + (x_2 - x_1))$$
$$\dot{x}_2 = \alpha_2((x_1 - x_2) + (x_4 - x_2))$$
$$\dot{x}_3 = \dot{x}_4 = 0$$

Where to x_1 and x_2 end up as $t \to \infty$ if $x_3 = \beta$ and $x_4 = \gamma$?

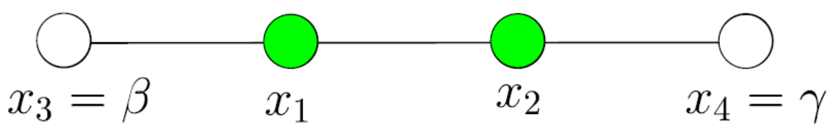


Fig. 2: Leader-follower Network

Solution:

The dynamics of the network overall are described by the dynamics $\dot{x} = A_c l s x$ with matrix

$$A_{cls} = \begin{bmatrix} -2\alpha_1 & \alpha_1 & \alpha_1 & 0 \\ \alpha_2 & -2\alpha_2 & 0 & \alpha_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This can further be redefined into the dynamical system $\dot{x} = Ax + Bu$ with matrices

$$A = \begin{bmatrix} -2\alpha_1 & \alpha_1 \\ \alpha_2 & -2\alpha_2 \end{bmatrix}$$

$$B = \begin{bmatrix} \alpha_1 & \\ & \alpha_2 \end{bmatrix}$$

where u is the static "input" of

$$u = \begin{bmatrix} \beta \\ \gamma \end{bmatrix}$$

Assuming that $\alpha_1, \alpha_2 > 0$, A is stable and therefore the system wll decay to the reference signal u. *Note:* If u changes then the response would be governed by transfer function $(sI - A)^{-1}B$. The final state will be the solution to

$$\dot{x} = 0 = A_{cls}x = Ax + Bu$$

which is calculated to be

$$\boxed{x_\infty = \begin{bmatrix} \dfrac{2\beta + \gamma}{3} \\ \dfrac{\beta + 2\gamma}{3} \end{bmatrix}}$$

Problem 4

Problem:

Given an undirected network containing a total of N nodes. There is a single anchor node that is connected to every one of the follower nodes, that is, anchor has a degree of $N-1$. Find the simple expression for the following quantities.

a. l

b. $L_f\mathbf{1}$

where L_f is the matrix obtained from the partition of the Laplacian matrix as we discussed in class.

Also, relate your answers to where the followers end up as $t \to \infty$.

Preliminaries:

Definition 3. *Let undirected and unweighted graph $G(V,E)$ with $V = \{v_1, \dots, v_n\}$ and $E \subseteq V \times V$. Vertices in V are classified as either* leaders *($v_i \in V_l$) or* followers *($v_i \in V_f$). The* <u>leader-follower</u> *dynamics of the states within network $G(V,E)$ are defined by*

$$\begin{cases} \dot{x}_i = -\sum_{j \neq N_i}(x_i - x_j) & \forall_{i\ :\ v_i \in V_f} \\ \dot{x}_i = 0 & \forall_{i\ :\ v_i \in V_l} \end{cases}$$

or equivalently when $V_l = \{v_n\}$ (single leader node)

$$\dot{x} = \begin{bmatrix} -L_f & -l \\ 0 & 0 \end{bmatrix}$$

Solution:

Assuming that the network is unweighted and the single leader is associated with v_n,

$$\dot{x} = \begin{bmatrix} -L_f & -l \\ 0 & 0 \end{bmatrix}$$

i) l

Since $\deg(v_n) = N-1$,

$$\sum_{i=1}^{N-1} l_i = N-1$$

therefore all elements must be 1 and

$$l = \mathbf{1}_{N-1}$$

ii) $L_f\mathbf{1}$

$$L_f\mathbf{1} = \begin{bmatrix} \deg(v_1) & -a_{1,2} & \cdots & -a_{1,n} \\ -a_{2,1} & \deg(v_2) & \cdots & -a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n,1} & -a_{n,2} & \cdots & \deg(v_n) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \deg(v_1) - \sum_{i\neq 1} a_{1,i} \\ \deg(v_2) - \sum_{i\neq 2} a_{2,i} \\ \vdots \\ \deg(v_n) - \sum_{i\neq n} a_{n,i} \end{bmatrix}$$

Since by definition $\deg(v_i) = \sum j \neq i a_{i,j}$,

$$\boxed{L_f \mathbf{1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}_{N-1}}$$

Problem 5

Problem:

Consider the (undirected) network in Fig.3. At any instance of time, exactly one of the nodes x and y would be included in the network. So, basically, we will get a switched system. Assuming all nodes impliment the consensus dynamics, will they converge at one point (centroid of initial states) asymptotically? Explain.

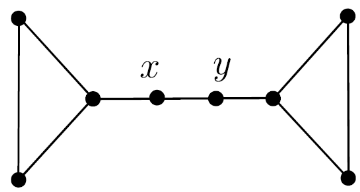


Fig. 3: Network for problem 5

Solution:

Within this switched system shown in Fig.3, the exclusion of nodes x and y during any singe time-step causes the system to be disconnected into 2 connected subgraphs. It is known that consensus will occur within a time-varying (switched) system if the union of multiple time steps cause the network to be connected; however, this does not immediately prove this network will converge since the problem statement does not state either x or y will be included.

Problem 6

Problem:

Assume we are running the (directed) consensus protocol over the three static graphs below. In which of these cases does the protocol drive the state to span$\mathbf{1}$? When does it drive the state to

$$\frac{1}{N}\mathbf{1}\mathbf{1}^T x(0)$$

i.e. to the initial centroid? (Explain the reasons for your answers - don't just state the answer.)

Preliminaries:

Definition 4. *Let directed network (di-graph) $G(V,E)$ with $V = \{v_1, \ldots, v_n\}$ and $E \subseteq V \times V$.*

a. *A directed path ($P : v_i \to v_j$) is a sequence of directed edges constructing a path from v_i to v_j.*

b.

Definition 5. ***Connectivity of Directed Graphs:*** *Let directed network (di-graph) $G(V,E)$ with $V = \{v_1, \ldots, v_n\}$ and $E \subseteq V \times V$.*

a. *Di-graph $G(V,E)$ is* strongly connected *there exists a directed path from any node to every other node.*

b. *Di-graph $G(V,E)$ is* weakly connected *the corresponding undirected graph is connected.*

c. *Di-graph $G(V,E)$ is contains a* rooted out-branching *if*

   (a) *$G(V,E)$ does not contain a directed cycle*

   (b) *$\exists_{v_r \in V}$ such that $\forall_{v_i \neq v_r \in V}$ there is a directed path from v_r to v_i.*

d. *Di-graph $G(V,E)$ is considered* balanced *if $\deg^{in}(v_k) = \deg^{out}(v_k) \forall_{v_k \in V}$.*

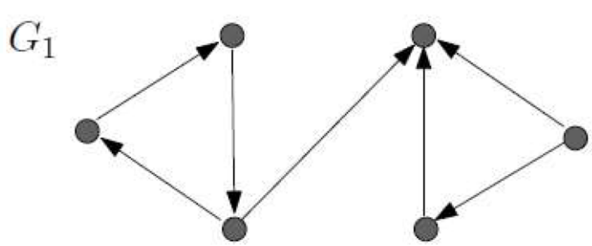
Theorem 1. ***Consensus Requirements:*** *Let directed network (di-graph) $G(V,E)$ with $V = \{v_1, \ldots, v_n\}$ and $E \subseteq V \times V$*

a.

b. *Di-graph $G(V,E)$ drives to x to $\frac{1}{N}\mathbf{1}\mathbf{1}'x(0)$ iff L is balanced and contains a rooted out-branching.*

Solution:

a) G_1



$$D = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

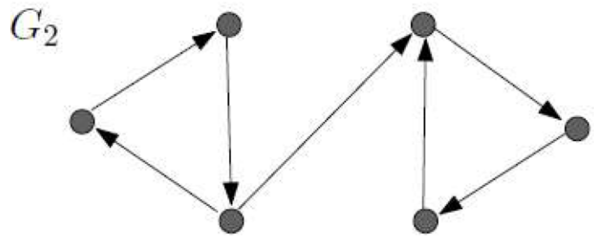
$$Delta^{\text{in}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0` \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

$$\text{rank}(L) = 5 = N - 1 \implies \text{null}(L) = \text{span}(\mathbf{1})$$

This means G_1 has rooted out-branching and consensus will occur to $\text{span}(\mathbf{1})$. However, L is not balenced as $\deg^{in}(v_k) \neq \deg^{out}(v_k) \forall_{v_k \in V}$

b) G_2



$$D = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

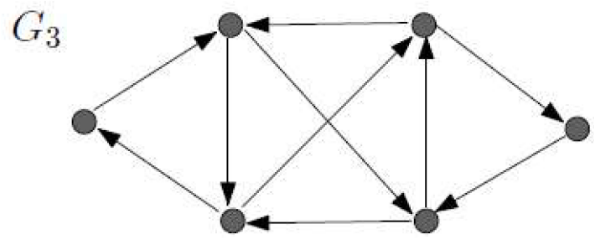
$$Delta^{\text{in}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

$$\text{rank}(L) = 5 = N - 1 \implies \text{null}(L) = \text{span}(\mathbf{1})$$

This means G_1 has rooted out-branching and consensus will occur to span($\mathbf{1}$). However, L is not balenced as $\deg^{in}(v_k) \neq \deg^{out}(v_k) \forall_{v_k \in V}$

c) G_3



$$D = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$Delta^{\text{in}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & -1 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{bmatrix}$$

$$\text{rank}(L) = 5 = N - 1 \implies \text{null}(L) = \text{span}(\mathbf{1})$$

This means G_1 has rooted out-branching and consensus will occur to span($\mathbf{1}$). Additionally, L is balenced as $\deg^{in}(v_k) = \deg^{out}(v_k) \forall_{v_k \in V}$

Problem 7

Problem:

Show that a graph that is weakly connected and balanced is also strongly connected. (I do not expect a formal proof. Just discuss your main argument.)

Solution:

Theorem 2. *Let graph $G(V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $E \subseteq V \times V$.*

If $G(V, E)$ is weakly connected and balanced, then $G(V, E)$ is strongly connected. (i.e)

$$G(V, E)\,Weakly\ Connected \wedge G(V, E)Balanced \implies G(V, E)Strongly\ Connected$$

Proof. Not a formal proof... Essentially the related undirected network being fully connected, as implied by weakly connected, is then connected to the fact that the network is balanced implying in and out degrees are the same with can be used to prove directed paths exist in addition to just undirected connectivity. □

A MATLAB Code:

All code I write in this course can be found on my GitHub repository:
https://github.com/jonaswagner2826/MECH6V29_MultiagentRoboticSystems