Homework 2 – Solution

(Multiagent Robot Systems)

Problem 2:

a

Let

$$p = Lx \;\Rightarrow\; p_i = \sum_{j\in N_i}(x_i - x_j).$$

Moreover, let

$$q = Lp \;\Rightarrow\; q_i = \sum_{j\in N_i}(p_i - p_j) = \sum_{j\in N_i}\left(\sum_{k\in N_i}(x_i - x_k) - \sum_{\ell\in N_j}(x_j - x_\ell)\right).$$

But, as $\dot{x} = -q$ we have that

$$\dot{x}_i = -\sum_{j\in N_i}\left(\sum_{k\in N_i}(x_i - x_k) - \sum_{\ell\in N_j}(x_j - x_\ell)\right).$$

b

One conclusion one can draw from **a** is that information from nodes two hops away are included as well, i.e., what we get is in essence a new information exchange network where two edges are adjacent if and only if they share a neighbor or are adjacent in the original graph.

Problem 3:

We have a leader-follower network and we know that this will drive the followers to an equilibrium point where $\dot{x}_1 = \dot{x}_2 = 0$. We get

$$\begin{aligned} 0 &= \alpha_1((\beta - x_1) + (x_2 - x_1)) \\ 0 &= \alpha_2((x_1 - x_2) + (\gamma - x_2)), \end{aligned}$$

i.e.,

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \beta \\ \gamma \end{bmatrix} \;\Rightarrow\; \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2\beta + \gamma \\ \beta + 2\gamma \end{bmatrix}.$$

Problem 4:

We have that

$$\ell = -\mathbf{1}$$

since every follower is connected to the leader.

Moreover,

$$L_f\mathbf{1} = \mathbf{1}$$

since each element is the degree minus all adjacencies except the leader, i.e. degree -(degree-1)=1.

Finally, as $t \to \infty$, we have (since $L_f \succ 0$)

$$\dot{x}_f(\infty) = 0 = -L_f x_f(\infty) - \ell x_N = -L_f x_f(\infty) + \mathbf{1}x_N.$$

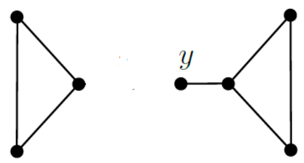
And, since $L_f\mathbf{1} = \mathbf{1}$ we can try $x_f(\infty) = \mathbf{1}\alpha$ for some $\alpha \in \mathbb{R}$, with the result that

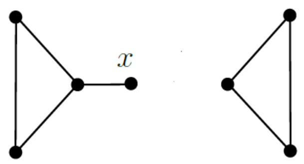
$$0 = -\mathbf{1}\alpha + \mathbf{1}x_N$$

and hence (due to the uniqueness of solutions), we must have $\alpha = x_N$, i.e. all followers end up at x_N.

Problem 5:

This is a switched network and we know that in such a network, all agents converge at the centroid of initial positions if and only if the graph is jointly connected (refer to Notes 7). Under the conditions mentioned in the problem statement, the graph cannot be jointly connected. The reason is that there will never be an edge between nodes x and y as only one of them will exist in the network at any instant of time (as shown in below figures). So, agents in this switching network will not converge to the centroid of initial points.





Problem 6:

We see that only G_2 and G_3 contain a rooted, out-branching tree. As such, the agreement protocol will not converge to span$\{\mathbf{1}\}$ under graph G_1, while it will under graphs G_2 and G_3. However, since only G_3 is balanced, it will converge to the initial, static centroid under graph G_3 but not under graph G_2.

Problem 7:

1. First, we show that if the graph is balanced, then for any vertex v, there exists a directed cycle that contains v.
2. Second, we observe that since the graph is weakly connected. As a result, these cycles overlap, i.e., cycles have common vertices.
3. Since cycles have common vertices, so we can go from a node in one cycle to any node in any other cycle. Hence, there is a path from every node to every other node, that is, the graph is strongly connected.

Now, to show (1), we can proceed as follows (sketch):

Arbitrarily choose a vertex v. Then select an arbitrary out-going edge of v, say (v,w). Since the graph is balanced, i.e., *in-deg*(w) = *out-deg*(w), node w must have an out-going edge. Select any out-going edge of w, and continue visiting out-neighbors of nodes. The edges traversed during this process are excluded from further consideration. Note that every time we visit a node (that is, there is an edge incident on the node), it must have an out-going edge that remains unvisited due to balanced condition. The only node for which there may not be an unvisited outgoing edge is v because we started the cycle by visiting one of v's outgoing edges. Since there's always an out-going edge we can visit for any node other than v, eventually the cycle must return to v. Thus, there is a directed cycle containing v.