Title in center of title slide

Author name on title slide
The Erik Jonsson School of Engineering and Computer Science
The University of Texas at Dallas
800 W. Campbell Rd.
Richardson, TX 75080
author.name@utdallas.edu

UTD Beamer Template Hints

Currently the logo in the upper right corner will "follow" the bottom of the top color ribbon i.e. on this slide it looks centered since there is no subtitle, but on the other slides it looks offset because there is a subtitle. I like to not use subtitles so I positioned it this way.

I turned off navigation buttons because they are essentially useless and take up valuable space, but you can turn them on by (un)commenting the relevant commands in `main.tex`. Be sure to add extra space under footnotes to make room for navigation buttons if you do this (there is a command there to do this).

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When you cite a paper [**?**], it will be referenced in the bibliography on the last slides.

Review of Linear Algebra

Vectors

We will next give a brief review of some concepts from **Linear Algebra** that will help in the understanding of **Matrix Representations of Graphs** and also in the discussion of **Dynamical Systems** later in the course.

Definition 1

An n-dimensional vector is a column array of real numbers

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad v_i \in R, \quad i = 1, \ldots, n$$

To save space we often write this as a row vector

$$v = [v_1, v_2, \ldots, v_n]^T$$

Definition 2

An $n \times n$ **matrix** A is an array with n **rows** and m **columns**

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

We write $A = (a_{ij})$ with a_{ij} meaning the element in **row** i and **column** j

$[a_{i1}, a_{i2}, \dots, a_{im}]$ **is the** i**-th row vector**

$[a_{1j}, a_{2j}, \dots, a_{nj}]^T = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$ **is the** j**-th column vector**

Example 1

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

is a 3×4 *matrix.*

$[5, 6, 7, 8]$ *is the second row vector or* **row two**

$[1, 5, 9]^T = \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}$ *is* **column one**.

The **transpose matrix** A^T *is the* 4×3 *matrix*

$$A^T = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$$

Definition 3

Given vectors $x = [x_1, \ldots, x_n]^T$ and $y = [y_1, \ldots, y_n]^T$, the **inner product** or **dot product**, or **scalar product** denoted by

$$< x, y > \text{ or } \; x \cdot y \text{ or } \; x^T y$$

is a **scalar (number)**

$$x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = [x_1, x_2, \ldots x_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Example 2

Let $x = [1, 2, 3]^T$ *and* $y = [4, 5, 6]^T$ *be two vectors in* R^3. *Then*

Definition 4

The **norm** or **length** of a vector $x = [x_1, x_2, \ldots, x_n]^T$ is given by

$$||x|| = \sqrt{x^T x} = (x_1^2 + x_2^2 + \ldots x_n^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$$

The norm, as defined above, is the n-dimensional version of the **Pythagorian Theorem**.

Some facts you may recall from basic vector calculus

- $x \cdot y = ||x|| \cdot ||y|| \cos(\theta)$ where θ is the angle between the vectors x and y.
- Consequently, $|x^T y| \leq ||x|| \cdot ||y||$ **Cauchy-Schwartz Inequality**
- Also, $x \cdot y = 0$ if and only if x and y are **mutually perpendicular (orthogonal)**.

If A is an $n \times m$ matrix and B is a $p \times q$ matrix, then the **Matrix Product AB** exists provided $m = p$.

The result is an $n \times q$ matrix $C = (c_{ij})$ where $c_{ij} = A_i^T B_j$ where A_i is the i-th row of A and B_j is the j-th column of B.

Example 3

Suppose

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ -2 & 1 & 4 \end{bmatrix}; \qquad B = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 2 \\ -2 & 2 & -1 \end{bmatrix}$$

then

$$C = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 7 & 1 \\ -14 & 5 & -4 \end{bmatrix}$$

Some additional definitions and properties of matrix algebra:

- A matrix A is **Symmetric** if $a_{ij} = a_{ji}$. In other words, $A^T = A$, i.e, the i-th row and j-th column of A are the same.

Example 4

Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Then it is easy to see that $A^T = A$, so A is a symmetric matrix.

Some additional properties of matrices are:

- If A and B have the same dimensions $n \times m$, then $C = A + B$ is defined by $c_{ij} = a_{ij} + b_{ij}$.
- $(AB)C = A(BC)$ provided the matrix products are defined.
- $A(B + C) = AB + AC$ provided the matrix products are defined.
- $(A + B)C = AC + BC$ provided the matrix products are defined.
- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

If A and B are $n \times n$ **(square matrices)**, then in general AB is not equal to BA.

If $AB = BA$ then A and B are said to **commute**.

Definition 5

The matrix

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

is the $n \times n$ **Identity Matrix**

Definition 6

The **Inverse** of an $n \times n$ matrix A is an $n \times n$ matrix B satisfying

$$AB = BA = I$$

where I is the $n \times n$ identity matrix.
We denote the inverse B of A as A^{-1}.

Matrices are used to represent **Systems of Linear Equations**

Example 5

The system of m equations in n unknowns

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_n & = & b_2 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_n \end{array}$$

can be written as

$$Ax = b$$

with

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} \quad ; \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \quad ; \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Consider the homogeneous linear equations

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

Eliminating x and y from these equations gives

$$ad - bc = 0$$

This quantity is called the **Determinant** of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

We denote the determinant of a matrix A by $det(A)$ or $|A|$.

The determinant of a 3×3 matrix can be computed as

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - fh) - b(di - fg) + c(dh - ge) \\ &= aei + bfg + cdh - ceg - bdi - afh \end{aligned}$$

The 2×2 determinants $\begin{vmatrix} e & f \\ h & i \end{vmatrix}, -\begin{vmatrix} d & f \\ g & i \end{vmatrix}, \begin{vmatrix} d & e \\ g & h \end{vmatrix}$ are called **Cofactors** of the elements a, b, c, respectively.

Definition 7

The **Cofactor** c_{ij} of an element a_{ij} in an $n \times n$ matrix A is ± 1 times the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting the i-th rown and j-th column of A. The **sign** in front of each cofactor alternates according to the pattern

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \quad \text{sign pattern for the } \; 4 \times 4 \text{ case.}$$

The **Determinant** of any $n \times n$ matrix A can then be calculated by taking **any row** or **column** and multiplying each element of the row or column by its respective cofactor. The determinant is then the **sum** of these **products**.

Example 6

Back to the previous 3×3 *matrix*

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

we can take any row or column, for example, column two, and compute the determinant as

$$\begin{aligned} &= -b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + e \begin{vmatrix} a & c \\ g & i \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ &= -b(di - fg) + e(ai - gc) - h(af - dc) \\ &= -bdi + bfg + eai - egc - haf + hdc \end{aligned}$$

One can check that this is the same expression as computed previously.

Review of Linear Algebra

Determinant and Inverse

The determinant is a scalar function defined for square matrices and satisfies the following properties.

- $|AB| = |A| \cdot |B|$
- $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$
- $|A^T| = |A|$

Note that it is generally **not true** that $|A + B| = |A| + |B|$.

Definition 8

A matrix A is **Singular** if $det(A) = 0$. Otherwise, A is said to be **Nonsingular** or **Invertible**.

Theorem 9

The inverse of an $n \times n$ matrix exists if and only if the **Determinant**, $det(A)$, *is not equal to zero.*

If A and B are invertible, then the product AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

If the $n \times n$ matrix A is nonsingular, the linear system

$$Ax = b$$

has the unique solution

$$x = A^{-1}b$$

Otherwise, there may be no solution or infinitely many solutions.

Example 7

Consider the linear system

$$\begin{aligned} 3x - y &= 4 \\ 6x - 2y &= 8 \end{aligned}$$

The coefficient matrix A is singular in this case and any point on the line $y = 3x - 4$ is a solution.

Example 8

On the other hand, the linear system

$$\begin{aligned} 3x - y &= 4 \\ 6x - 2y &= 2 \end{aligned}$$

has no solution.

Example 9

The linear system

$$\begin{aligned} 3x - y &= 4 \\ 6x + 2y &= 2 \end{aligned}$$

has the unique solution $x = 5/6$, $y = -3/2$.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Then a direct calculation shows

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

A similar calculation shows that $BA = AB$.

Therefore, it follows that

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{|A|} A^{+}$$

which is well defined provided $|A| \neq 0$.

The matrix $A^{+} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is called the **Adjoint of A**.

The inverse of a nonsingular $n \times n$ matrix A is $A^{-1} = \frac{1}{|A|} A^{+}$.

The **Adjoint** of a general $n \times n$ matrix A is given as $A^{+} = C^{T}$ where C is the **Cofactor Matrix**, consisting of elements that are cofactors of the elements of A as defined previously.

Example 10

Find the adjoint of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

First, compute the cofactor of each element of A

Example 11

The cofactors of the given matrix are

$$c_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \quad c_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \quad c_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$c_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \quad c_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \quad c_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$c_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \quad c_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \quad c_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

Example 12

Therefore, the **Cofactor Matrix** *is*

$$C = \begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$$

Finally, the **Adjoint of A** *is the transpose of the Cofactor Matrix*

$$A^{+} = \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$$

Since the determinant of A is $|A| = 22$, it follows that

$$A^{-1} = \frac{1}{} \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \end{bmatrix}$$

Definition 10

Given an $n \times n$ matrix A, the **Trace** of the matrix is

$$Tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{1}^{n} a_{ii}$$

In other words, the Trace is the **sum of the diagonal entries of the matrix**.

The Trace operation satisfies the following:

- $Tr(A + B) = Tr(A) + Tr(B)$
- $Tr(cA) = cTr(A)$ where c is a constant.
- $Tr(AB) = Tr(BA)$
- $Tr(A^T) = Tr(A)$

Definition 11

Suppose we find a scalar λ and a vector x satisfying the equation

$$Ax = \lambda x$$

Then λ is called an **Eigenvalue** of the matrix A and x is called an **Eigenvector** for λ.

Example 13

$$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore $\lambda = 4$ *is an eigenvalue for the matrix* $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ *and* $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *is an eigenvector for* λ.

Note that the equation $Ax = \lambda x$ can be written as

$$(A - \lambda I)x = 0$$

where I is the **identity matrix** defined previously.

Therefore, there will be a nonzero vector x satisfying this equation provided that

$$det(A - \lambda I) = 0$$

Computing $det(A - \lambda I)$ results in a polynomial of degree n for an $n \times n$ matrix, called the **Characteristic Polynomial of A**.

The n roots of the characteristic polynomial, which may be **real** or **complex**, are therefore the n eigenvalues of A.

Theorem 12

If A is a real, symmetric matrix then the eigenvalues of A are real.

Example 14

Back to the previous matrix

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

Then

$$det(A - \lambda I) = det(\begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix}) = \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2)$$

Therefore the eigenvalues are $\lambda = 4$ and $\lambda = 2$.
To find eigenvectors for each λ we need to solve the equation

$$\begin{bmatrix} 3-\lambda & -1 \\ \end{bmatrix} \begin{bmatrix} x_1 \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \end{bmatrix}$$

Example 15

For $\lambda = 4$, *the system of equations become*

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $x_2 = -x_1$ *and so* $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *is an eigenvector for* $\lambda = 4$.

For $\lambda = 2$, *the system of equations become*

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $x_2 = +x_1$ *and so* $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *is an eigenvector for* $\lambda = 2$.

Definition 13

Vectors v_1 and v_2 are said to be **Linearly Dependent** if and only if there are constants α_1 and α_2 such that

$$\alpha_1 v_1 + \alpha_2 v_2 = 0$$

Otherwise, v_1 and v_2 are **Linearly Independent**.

If v_1 and v_2 are linearly independent, then the matrix $T = [v_1 \; v_2]$ is invertible, where v_1 and v_2 are the column vectors of T.

Suppose v_1 and v_2 are two linearly independent eigenvectors for eigenvalues λ_1 and λ_2, respectively.

Then, since $Av_i = \lambda_i v_i$ for $i = 1, 2$ we can write

$$A[v_1 \;\; v_2] = [\lambda_1 v_1 \;\; \lambda_2 v_2]$$

which can be written

$$AT = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} T$$

and so

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \bar{A}$$

a diagonal matrix with the eigenvalues on the diagonal.

The above transformation $T^{-1}AT$ is called a **Similarity Transformation** and the matrices A and $\bar{A}$ are said to be **Similar**.

Example 16

In the previous example, with

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}; \qquad \text{and } \; T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

a straightforward calculation shows that

$$T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and

$$T^{-1}AT = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

Remark 1

1) The **determinant** *of a square matrix A is the* **product** *of the eigenvalues, i.e.,*

$$|A| = \lambda_1 \lambda_2 \cdots \lambda_n = \Pi_{i=1}^{n} \lambda_i$$

2) The **trace** *of a square matrix A is the* **sum** *of the eigenvalues, i.e.,*

$$Tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n = \sum_{i=1}^{n} \lambda_i$$

These properties are **invariant** under **similarity transformation**.

This is because the determinant and trace satisfy

$$\begin{aligned} |T^{-1}AT| &= |A| \\ Tr(T^{-1}AT) &= Tr(A) \end{aligned}$$

We will have occasion to consider so-called **Quadratic Forms**.

Definition 14

A **Quadratic Form** V is a function from $R^n \to R$ of the form

$$V(x) = \sum_{i=i}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j = p_{11} x_2^2 + p_{12} x_1 x_2 + \cdots + p_{nn} x_n^2$$

We assume $p_{ij} = p_{ji}$.

Such a quadratic form can be represented as

$$V(x) = x^T P x$$

where $P = (p_{ij})$ is a symmetric $n \times n$ matrix and $x^T = [x_1, \ldots, x_n]$.

Example 17

The quadratic form

$$V(x) = p_{11}x_1^2 + p_{12}x_1x_2 + p_{21}x_2x_1 + p_{22}x_2^2$$

can be written as

$$V(x) = x^T P x$$

where

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

The simplest example of such a quadratic form is the **norm**

$$V(x) = x_1^2 + x_2^2 = x^T I x$$

where I is the identity matrix.

Note that the equation

$$V(x) = x_1^2 + x_2^2 = r^2$$

defines a **circle of radius r**.

In general, the equation

$$V(x) = x^T P x = c$$

defines an **ellipse** provided the matrix P is **Positive Definite**.

Definition 15

An $n \times n$ matrix $P = (p_{ij})$ is **Positive Definite** if

$$x^T P x > 0 \quad \text{for all} \quad x \neq 0$$

P is **Positive Semi-Definite** or **Nonnegative Definite** if

$$x^T P x \geq 0 \quad \text{for all} \quad x \neq 0$$

Theorem 16

A matrix P is **Positive Definite** *if and only if all eigenvalues of P are positive.*
A matrix P is **Positive Semi-Definite** *if and only if all eigenvalues of P are non-negative.*

Another characterization of a positive definite matrix is

Theorem 17

A matrix P is **Positive Definite** *if and only if all* **Principal Minors** *or* **Principal Minor Determinants** *of P are positive.*

Definition 18

Let

$$P = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{bmatrix}$$

be an $n \times n$ matrix.
The **Principal Minors of P** are

$$M_1 = p_{11} \;; \quad M_2 = \begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix} \;; \quad M_3 = \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{23} & P_{33} \end{vmatrix}$$

$$\dots \;; \; M_n = |P|$$

Remark 2

If the matrix P can be written as $P = C^T C$ for some matrix C, then P is **Positive Semi-Definite**.

To see this, note that a simple calculation shows that

$$x^T P x = x^T C^T C x = y^T y \geq 0 \qquad \text{with} \quad y = Cx$$

y is not strictly positive since there will generally be **nonzero vectors** x such that $Cx = 0$.

Bibliography I