Assignment 1

(SYSM 6v80.001 / MECH 6v29.001 – Multiagent Robotic Systems)

Problem 3

In class we saw that an undirected graph G is connected if and only if its Laplacian's second smallest eigen value, λ_2 is non-zero. Using a similar argument as the one in class, show that the number of connected components (i.e. connected subgraphs that are disconnected from each other) is equal to the number of zero eigen values of the Laplacian.

Solution:

Approach 1:

We know that for L, the number of zero eigen values is equal to the dimension of the null space of L. Thus, we need to find out the dimension of the null space of L, and show that it is equal to the number of connected components in G.

Since, $L = DD^T$, and null(DD^T) =null(D^T). Therefore, we need to characterize z such that $D^T z = 0$, or equivalently $z^T D = 0$. The structure of D (incidence matrix) directly gives us

$$z^T D = 0 \;\Rightarrow\; z_i = z_j,\ \forall\ (v_i, v_j) \in E$$

If we assume that the indexing of the vertices is such that the first indices correspond to component 1, and so forth, we get that

$$\begin{gathered} z_1 = z_2 = \cdots = z_\alpha, \\ z_{\alpha+1} = z_{\alpha+2} = \cdots = z_{\alpha+\beta}, \\ \cdots, \end{gathered}$$

where α is the number of vertices in the first connected component, β is the number of nodes in the second connected component and so on. In other words,

$$\text{null}(L) = \text{span}\left(\begin{bmatrix} \mathbf{1}_\alpha \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \mathbf{1}_\beta \\ \mathbf{0} \end{bmatrix}, \cdots, \begin{bmatrix} \mathbf{0} \\ \mathbf{1}_\gamma \end{bmatrix} \right)$$

where, $\mathbf{1}_\alpha$ is a column vector containing α number of 1's. Thus the number of zero eigenvalues is equal to the number of connected components in the graph.

Approach 2:

We can write the Laplacian matrix of G consisting of multiple connected components as a block diagonal matrix (matrix whose main diagonal entries are square matrices themselves and off diagonal blocks are all zeros). If we assume that the indexing of the vertices is such that the first indices correspond to component 1, and so forth, the Laplacian of given G can be written as,

$$L = \begin{bmatrix} L_1 & & & \\ & L_2 & & \\ & & \ddots & \\ & & & L_k \end{bmatrix}$$

where L_i is the Laplacian corresponding to the i^{th} connected component.

Now, it is know that eigen values of a block diagonal matrix is the union of the eigen values of the matrices that are the diagonal blocks, i.e.,

$$\text{eig}(L) = \text{eig}(L_1) \ \cup \ \text{eig}(L_2) \ \cup \cdots \cup \ \text{eig}(L_k)$$

Since L_i is a connected graph by it self, and therefore, has only one zero eigen value. Thus, the number of zero eigen values in L will be equal to the number of connected components in G.

Problem 4

Let the subspace S be

$$S \ = \ \text{span}\{\mathbf{1}\}^{\perp},$$

i.e.,

$$x \in S \ \Leftrightarrow \ x^T\mathbf{1} \ = \ 0$$

Show that S is $L-$invariant, i.e., $LS \subseteq S$ (i.e., $Lx \in S, \ \forall \ x \in S$), where L is the Laplacian of an undirected, connected graph.

Solution:

To show that S is $L-$invariant is same as showing that $Lx \in S, \forall \ x \in S$, which in turn is same as showing that Lx is orthogonal to the elements in $\text{span}\{\mathbf{1}\}$ (as $S = \text{span}\{\mathbf{1}\}^{\perp}$).

Thus, we take a dot product of $\mathbf{1}$ with Lx, where $x \in S$.

$$\mathbf{1}^T Lx = (\mathbf{1}^T L)x = (L\mathbf{1})^T x = 0x = 0$$

and hence Lx is orthogonal to $\text{span}\{\mathbf{1}\}$ for all $x \in S$.

Problem 5

$K_{1,6}$ is a star graph with one central node and six leaf nodes as shown in Fig. 2.

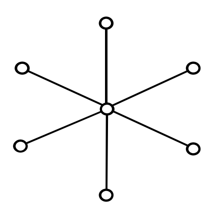

Figure 1: Star graph, $K_{1,6}$.

Your task is to show that $K_{1,6}$ can never be an induced subgraph of a Δ-disk proximity graph.

Solution:

$K_{1,6}$ is a star graph with one central node and six leaf nodes. Let v_i and v_j be two nodes that are adjacent to the node v, and $\angle v_i v v_j$ be the angle between v_i and v_j through v. Then it is easy to show that

$$(v_i, v_j) \notin E \iff \angle v_i v v_j > 60^o$$

Now assume that we have six neighbors of the node v, denoted by $\{v_1, v_2, \cdots, v_6\}$ as shown in Fig. 2. Further, we assume that v_i and v_j are pair-wise non adjacent, i.e.,

$$(v_i, v_j) \notin E, \ \forall \, i, j \ \in \ \{1, 2, \cdots, 6\}$$

Thus,

$$\angle v_1 v v_2 + \angle v_2 v v_3 + \angle v_3 v v_4 + \angle v_4 v v_5 + \angle v_5 v v_6 + \angle v_6 v v_1 \ > \ 6(60^o) \ = \ 360^o$$

which is not possible. So, $\angle v_i v v_j \leq 60$ for some v_i, v_j, where $i, j \in \{1, 2, \cdots, 6\}$ and $i \neq j$. This directly implies that $k_{1,6}$ cannot be an induced subgraph of a Δ-disk proximity graph.

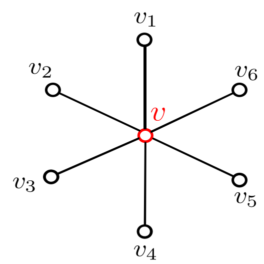

Figure 2: Star graph, $K_{1,6}$.

Problem 6

If $l_{i,j}$ is the shortest path distance (number of edges one needs to follow) between vertices v_i and v_j, the diameter of the graph is defined as

$$\text{diam}(G) \ = \ \max_{v_i, v_j \in V} l_{i,j}$$

Similarly, if we let l_i^* (known as the eccentricity of vertex v_i) be the longest distance to any vertex from the vertex v_i, i.e.,

$$l_i^* \ = \ \max_{v_j \in V} l_{i,j}$$

then the radius of a graph is defined as

$$\text{radius}(G) \ = \ \min_{v_i \in V} l_i^*$$

Find the radius and diameter of thegraphs in Figure 3.

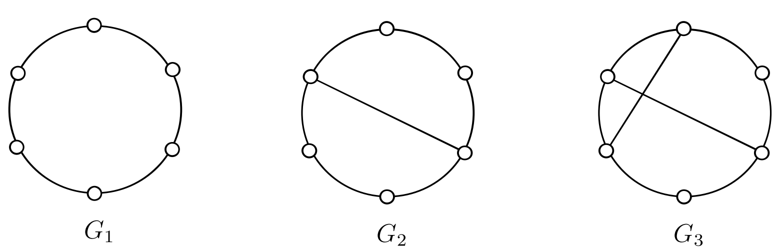

Figure 3:

Solution:

We get

$$\begin{array}{ccc} \text{diam}(G_1) = 3 & \text{diam}(G_2) = 3 & \text{diam}(G_3) = 2 \\ \text{rad}(G_1) = 3 & \text{rad}(G_2) = 2 & \text{rad}(G_3) = 2 \end{array}$$

Problem 7

Following are some undirected networks on four nodes with the same initial positions. In which of these networks, nodes converge fastest under the distributed consensus dynamics? Explain your answer.

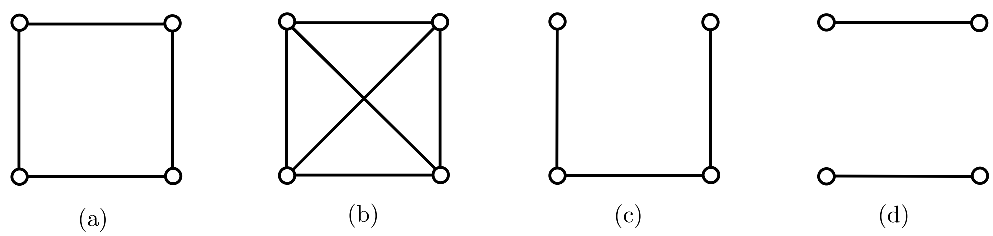

Figure 4:

Solution:

Nodes in the network (b) will converge fastest, as λ_2 of the Laplacian has the highest value in the case of network (b). We know that the convergence of consensus dynamics are directly related to λ_2. See **Notes 5 (Section 3)** for details.

Furthermore, note that consensus won't happen in the case of network (d) as the graph is not connected and hence do not satisfy the necessary condition for consensus in undirected networks.

Problem 8

What is the *necessary and sufficient* condition for the consensus to happen in the case of static directed networks? Derive this condition.

Solution:

Please see **Notes 6 - Supplementary** for the details.