MECH 6V29: Multiagent Robotic Systems- HW 5

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Contents

Preliminary Notes

a) Definitions

Definition 1. *Graph $G(V,E)$ is constructed with vertex set*

$$V = \{v_1, v_2, \ldots, v_n\}$$

of n discrete vertices and edge set

$$E = \{e_1, \ldots, e_m\} \subseteq V \times V$$

consisting of m edges $e_{k=(i,j)} = (v_i, v_j) \forall_{k=1,\ldots,m}$ connecting vertices v_i and v_j.

Definition 2. *Let graph $G(V,E)$ with $V = \{v_1, \ldots, v_n\}$ and $E \subseteq V \times V$.*

a. *$G(V,E)$ is considered undirected if*

$$(v_i, v_j) \in E \iff (v_j, v_i) \in E$$

otherwise, $G(V,E)$ is considered directed.

b. *An undirected graph $G(V,E)$ is considered connected if there exists a path between any two vertices.*

c. *An undirected graph $G(V,E)$ is considered planer if the graph can be drawn on a plane without any edges crossing.*

d. *A directed graph $G(V,E)$ is considered strongly connected if there exists a directed path between any two vertices.*

e. *A directed graph $G(V,E)$ is considered weakly connected if the corresponding undirected graph is connected.*

Definition 3. *Let $G(V,E)$ be a undirected graph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\} \subseteq V \times V$ with $e_k = e_{ij} = (v_i, v_j)$.*

a. *A path between two vertices v_i and v_j is a sequence of edges $[e_{i,*}, \ldots, e_{*,j}]$ that joins a sequence of vertices $[v_i, \ldots, v_j]$.*

b. *A path length is the number of edges in the path.*

c. *The shortest path length $l_{i,j}$ is the minimum length of all paths between vertices v_i and v_j. This quantity is also known as the distance between v_i and v_j, $dist(v_i, v_j)$.*

d. *The diameter of graph $G(V,E)$ is the maximum distance between any two vertices in the graph. (i.e.)*

$$\operatorname{diam}(G(V,E)) := \max_{v_i, v_j \in V} l_{i,j}$$

e. *The eccentricity of vertex v_i, l_i^*, is the largest distance from v_i to any other vertex in the graph. (i.e)*

$$l_i^* := \max_{v_j \in V} l_{i,j}$$

f. *The radius of graph $G(V,E)$ is the minimum eccentricity of the vertices of the graph. (i.e)*

$$\operatorname{radius}(G(V,E)) := \min_{v_i \in V} l_i^* = \min_{v_i \in V} \max_{v_j \in V} l_{i,j}$$

Definition 4. *Let undirected and unweighted graph $G(V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $E \subseteq V \times V$. Vertices in V are classified as either leaders ($v_i \in V_l$) or followers ($v_i \in V_f$). The leader-follower dynamics of the states within network $G(V, E)$ are defined by*

$$\begin{cases} \dot{x}_i = -\sum_{j \neq N_i}(x_i - x_j) & \forall_{i\,:\,v_i \in V_f} \\ \dot{x}_i = 0 & \forall_{i\,:\,v_i \in V_l} \end{cases}$$

or equivalently when $V_l = \{v_n\}$ (single leader node)

$$\dot{x} = \begin{bmatrix} -L_f & -l \\ 0 & 0 \end{bmatrix}$$

Problem 1

State a summary of **Notes 16.1-17.4**, (which Include the topics of Lloyd's algorithm, network controllability (upper and lower bounds)) preferably by creating a concept map diagram (flow diagram). The whole purpose is to make sure that we are clear about the bigger picture, and reiterate why are we doing and discussing the specific topics in the class. Do not merely write the topics, instead create connections between topics to clarify the flow of information.

a) Big Picture Chart

TO DO: Draw graph

Problem 2

Preliminaries

Definition 5. *The zero forcing method is used to find a graph-theoretic lower bound on the controllability of a network undergoing consensus (i.e. the Laplacian dynamics)*

Let $G = (V, E)$ with each $v \in V$ is either colored black or white.

a. *The Zero Forcing (ZF Process) updates the colors of vertices from an initial V' according to the following rule:*

   If $v \in V$ is black and has exactly one white neighbor u, then change u to black.

b. *The Derived Set, $D(G, V')$, is the set of black vertices after undergoing the ZF Process.*

c. *For $G = (V, E)$, $V' \subseteq V$ is a Zero Forcing Set (ZFS), $Z(G)$, if and only if $D(G, V') = V$. The size of the minimum ZFS is known as the zero forcing number, $\zeta(G)$.*

Definition 6. *Let $G(V, E)$ with n agents, where $V = \{v_1, v_2, \ldots, v_n\}$ with m agents denoted as leaders with $V_L = \{\updownarrow_1, \updownarrow_2, \ldots, \updownarrow_m\} \subseteq V$. We say G is undergoing the controlled Laplacian (consensus) dynamics of*

$$\dot{x} = -Lx + Bu$$

with L the Laplacian matrix and input matrix B defined as

$$[B]_{ij} = \begin{cases} 1 & if v_i = l_j \\ 0 & otherwise \end{cases}$$

Theorem 1. *If a leader set V_L is ZFS then the system is completely controllable with V_L.*

Define the following set of matrices

$$Q_{ss}(G) = \left\{ X \in \mathbb{R}^{|V| \times |V|} \ : \ X = X^T \ \wedge \ \forall_{i \neq j}(X_{ij} \neq 0 \iff (i,j) \in E) \wedge \ \forall_{i \neq j}(X_{ij} \geq 0 \vee X_{ij} \leq 0) \right\}$$

Theorem 2. *The leader set V_L completely controls the dynamics of $\dot{x} = Mx + Bu$, where $M \in Q_{ss}(G)$, if and only if V_L is a ZFS.*

Theorem 3. *The rank of the controllability matrix, $\Gamma(G, V_L)$, is at least the size of the derived set for V_L, $D(G, V_L)$. (i.e.)*

$$|\Gamma(G, V_L)| \geq |D(G, V_L)|$$

Problem

Consider the graph in Figure 1, and the following statement:

"Three leader nodes are sufficient to make the graph completely controllable."

Using the zero forcing method (Notes 17.3), discuss if the statement is true or false? If it is true, find a set of three leader nodes making the network completely controllable and justify your choice. If the statement is false, discuss why?

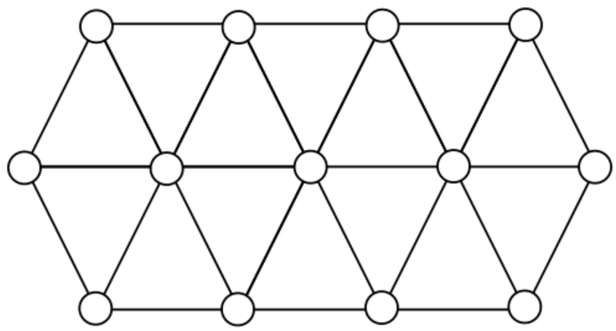


Fig. 1: Caption

Solution

It is possible since if you select the 3 on one end it would be able to progress throug the entire 'chain' and therefore V' of the three on either end would be ZFS which implies they can completely control the system.

Problem 3

(Each part has 5 pints.)

a)

Problem:

Consider the graph in Figure 2. Let $V_L = \{v_4, v_5\}$ be a leader set. Using zero forcing method, what is the minimum rank of the controllability matrix?

Solution:

We know that $|\Gamma(G, V_L)| \geq |D(G, V_L)|$, so following the ZF process we get

$$D(G, V_L) = \{v_1, v_4, v_5, v_6, v_8\}$$

hence

$$|D(G, V_L)| = 5$$

therefore the minimum rank of the controllability matrix is 5.

b)

Problem:

Consider a graph with unity edge weights and $V_l \subset V$ be a zero forcing set of the graph. If we change the edge weights from unity to arbitrary positive values, what would be the rank of the controllability matrix with the same set of leaders V_L? Please explain your answer.

Solution:

Well, since changing the dynamics does not change the topology of the graph, $D(G, V_L)$ will not change and the same conclusion can be drawn:

$$|\Gamma(G, V_L) \geq 5|$$

c)

Problem:

Can you find a set of three leaders through which the graph in Figure 2 becomes completely controllable? If yes, which three nodes can be leaders?

Solution:

Well, I haven't proven it by checking all ${}_9C_3 = 84$ selections, but upon guessing a few intuitively selected ones I was unable to find 3 nodes to achieve complete controllability.

d)

Problem:

This part is about finding the worst leaders from the perspective of zero forcing method. Can you identify two different sets of leader nodes, each of which containing four nodes, such that the zero forcing based bound on the rank of the controllability matrix is four with each of those leader sets.

Solution:

$$V_L = \{v_3, v_5, v_8, v_9\} \rightarrow D(G, V_L) = V_L$$

$$V_L = \{v_1, v_4, v_6, v_9\} \rightarrow D(G, V_L) = V_L$$

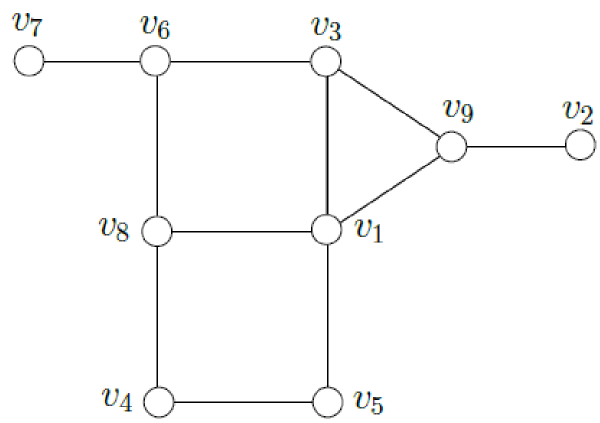


Fig. 2: Graph for Problem 3

Problem 4

Preliminaries

Recall the definition of distance between v_i and v_j, $\text{dist}(v_i, v_j)$, as seen in Definition 3, which will be renotated as $d(u, v) = \text{dist}(v_i = u, v_j = v)$.

Definition 7. *Let $G = (V, E)$ be a leader-follower network with leader set $V_L \subseteq V$ containing m nodes:*

$$V_L = \{\updownarrow_1, \updownarrow_2, \ldots, \updownarrow_m\}$$

a. We define the distance-to-leader vector, $D_i \in \mathbb{R}^m$, as the distance of node v_i to each of the leader nodes. (i.e.)

$$D_i = \begin{bmatrix} d(v_i, \updownarrow_1) \\ d(v_i, \updownarrow_2) \\ \vdots \\ d(v_i, \updownarrow_m) \end{bmatrix}$$

b. The Sweeping Bar Game is the game that you play to maximize the final score. ***To Do:*** *Explain the principle of the Sweeping Bar Game*

c.

Theorem 4. *A lower bound of the controllability matrix can be found from solving the Sweeping Bar Game.*

For $m = 1$, the rank of $\Gamma(G, V_L = \{l\}) \geq \max_{v \in V} d(v, l)$

Problem

In Problem 3(A), you computed the zero forcing based controllability bound for $V_L = \{v_4, v_5\}$. Now compute the distance-based bound on the rank of the controllability matrix (as we discussed in the class and also explained in Notes 17.4). Which one is a better bound here?

Solution

$D = \begin{bmatrix} 2 & 4 & 3 & 0 & 1 & 2 & 3 & 1 & 3 \\ 1 & 3 & 2 & 1 & 0 & 3 & 4 & 2 & 2 \end{bmatrix}$ which will end up with a bound of 3 which is not as great of a bound as the ZPK method.

Problem 5

Preliminaries

Definition 8. ***To Do:*** *Definitions for the partitions and things.*

a)

Problem

(Each part has 5 points.)

Consider the graph in Figure 3, in which **only black** node is a leader node.

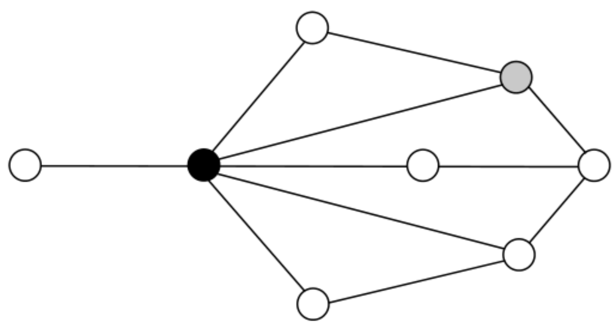


Fig. 3: A Leader-follower network in Problem 2

How many external equitable partitions (EEP) does the graph have? Which of these EEPs is the maximal leader-invariant (LIEEP), and based on that comment if the system is completely controllable or not.

Solution

I'm seeing 3 EEP's. The 2 trivial ones and then splitting groupings of 3 vertically:

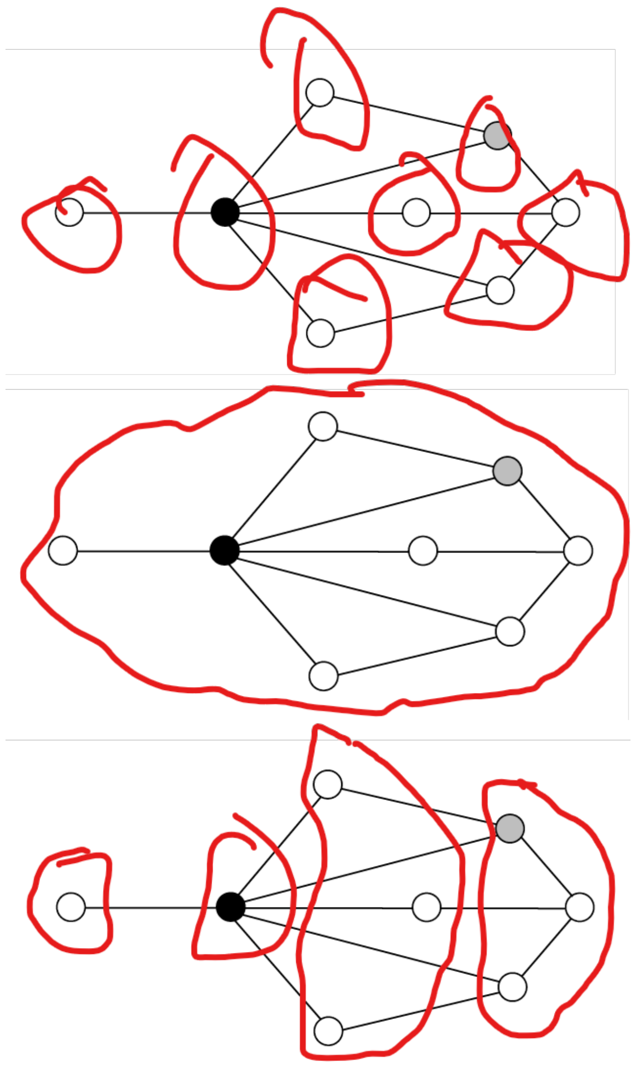
The LIEEP is then the non-trivial one with a total of 4 partitions. The leads to the having the number of cells not including the leaders to be 3, so the rank of Γ is upper bounded by 3. (i.e.)

$$\text{rank}(\Gamma) \leq 3$$

From this we know that it is actually not completely controllable as you would only ever be able to control the centroids of the cells but not each node individually.

Part - B Now consider that **both black and gray nodes** are leaders, and repeat the same as in Part A.

Well since the only one of the possible EEP's found in the last part that had the leaders as singleton cells was the trivial one, then that is the only LIEEP. This then correlates to it being completely controllable (or that $\alpha = 6$ which implies it can be completely controlled by just those two leaders).



Problem 6

Problem

Let G be a leader-follower network with a single leader $\updownarrow_G$. Similarly, H is another leader follower network with a single leader $\updownarrow_H$ and n follower nodes. Now we obtain a new graph M by connecting H and n copies of G as follows: Connect a copy of G to H by replacing a follower node i in H by the leader node $\updownarrow_G$ in G. See Figure 4 for the example.

Either prove or disprove (by giving a counter example) the following statement:

"If G and H has trivial maximal LIEEPs, then M also has a trivial maximal $LIEEP$"

(recall that trivial maximal LIEEP means that each node in the graph is in a singleton cell)

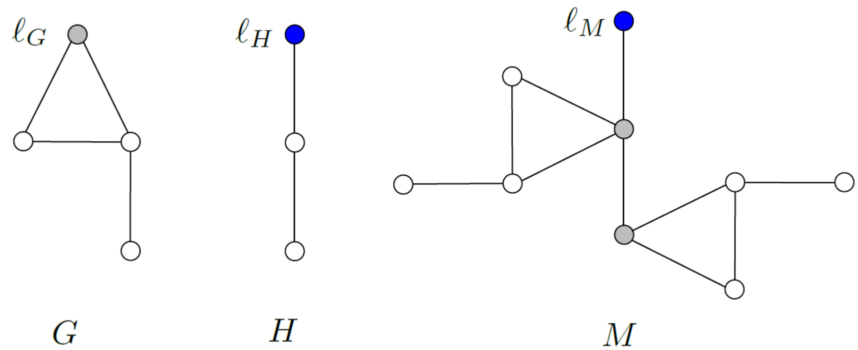


Fig. 4: A Leader-follower network in Problem 3

Solution

Note: Assuming that the statement of "trival maximal LIEEPs" means that the maximal LIEEP is the trival maximum and that no others exist.

Well this statement is true. One way I would prove this would be through a transitive property type of step. (i.e.) Use the fact that M is created by essentially convolving the two graphs transfer functions from one node to another (if you look at it that way, I got into that habit sometimes of seeing the linking from a node to another as being able to be described by some TF...) and then applying a transitive property.