MECH 6V29: Multiagent Robotic Systems- HW 1

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Contents

Problem 1

State a summary of first five lectures, preferably by creating a concept map diagram (flow diagram). The whole purpose is to make sure that we are clear about the bigger picture, and reiterate why are we doing and discussing the specific topics in the class. Do not merely write the topics, instead create connections between topics to clarify the flow of information.

a) Big Picture Chart

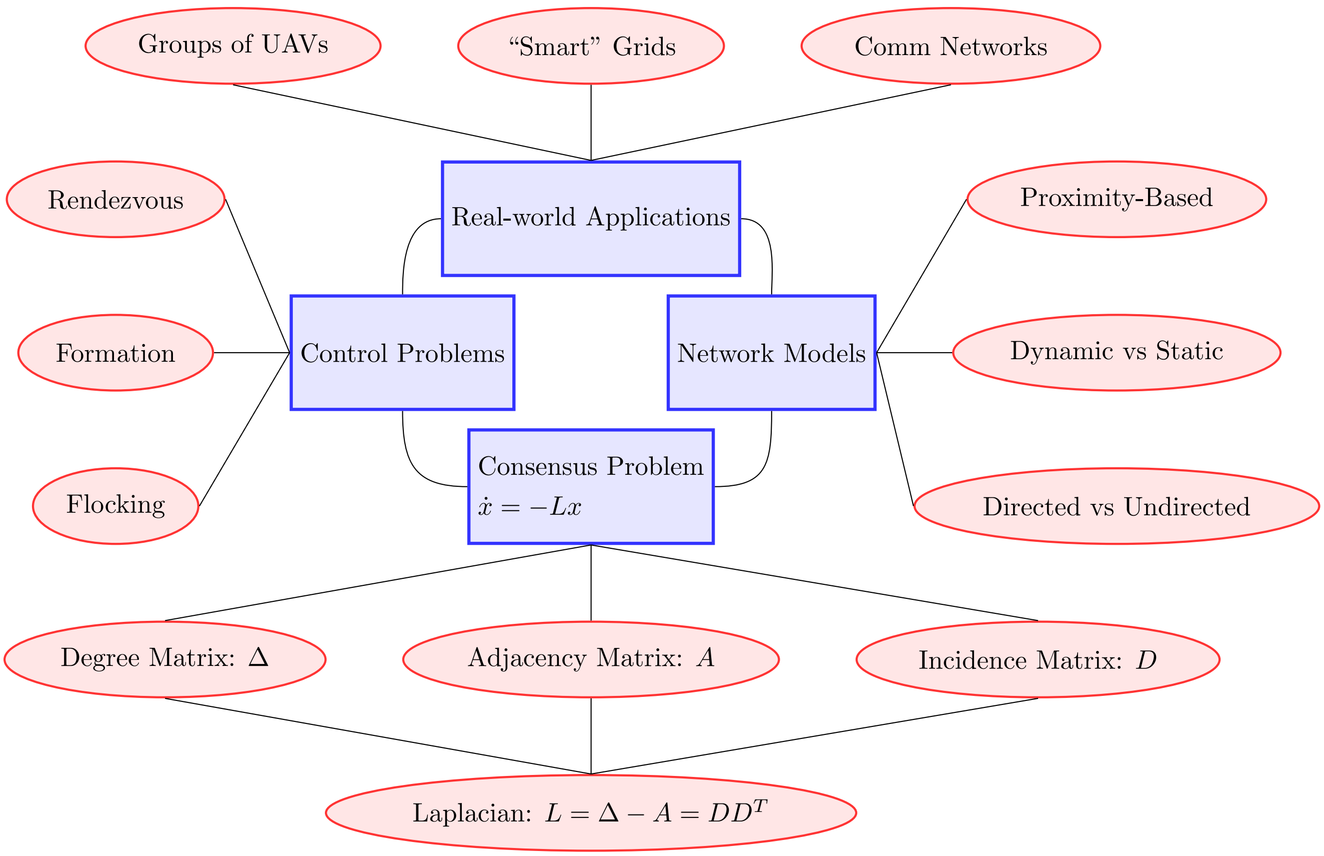

Fig. 1: Diagram of Course Topics (created w/ TikZ)

Problem 2

Go to the following youtube link, and watch a TED talk by Dr. Magnus Egerstedt titled "Swarm robotics – From local rules to global behaviors". https://www.youtube.com/watch?v=ULKyXnQ9xWA Then, write down a one paragraph summary, just highlighting the main point or the 'take-away' message of the talk. Be brief and to the point.

a) Video Summary

The TED talk by Dr. Magnus Egerstedt title "Swarm Robotics – From local rules to global behaviors" was an introduction to the control of swarm robotic systems and how the field has evolved in the last 10 years. Swarm robotics algorithms must consist of simple, local, and scalable rules that are safe and reactive for individual agents and result in a predictable emergence. Much of this is done based on how animals operate within nature and are then particular rules are proven using math. Currently the consensus problem is fundamental to swarm robotic algorithms and control of the swarm is being done by weighted terms on distances on distances to local agents. Fundamentally, Swarm Robotics aims to control multi-agent systems with local rules that result in emergence affecting the global behavior of the entire swarm.

Problem 3

In class we saw that an undirected graph G is connected if and only if its Laplacian's second smallest eigenvalue, λ_2 is non-zero. Using a similar argument as the one in class, show that the number of connected components (i.e. connected subgraphs that are disconnected from each other) is equal to the number of zero eigenvalues of the Laplacian.

a) Fundamental Graph Theory Definitions

Definition 1. *Graph $G(V, E)$ is constructed with vertex set*

$$V = \{v_1, v_2, \dots, v_n\}$$

of n discrete vertices and edge set

$$E = \{e_1, \dots, e_m\} \subseteq V \times V$$

consisting of m edges $e_{k=(i,j)} = (v_i, v_j)\forall_{k=1,\dots,m}$ connecting vertices v_i and v_j.

Definition 2. *Let graph $G(V, E)$ with $V = \{v_1, \dots, v_n\}$ and $E \subseteq V \times V$.*

a. *$G(V, E)$ is considered Undirected if*

$$(v_i, v_j) \in E \iff (v_j, v_i) \in E$$

otherwise, $G(V, E)$ is considered directed.

b. *An undirected graph $G(V, E)$ is considered connected if there exists a path between any two vertices.*

c. *A directed graph $G(V, E)$ is considered strongly connected if there exists a directed path between any two vertices.*

d. *A directed graph $G(V, E)$ is considered weakly connected if the corresponding undirected graph is connected.*

Definition 3. *Let graph $G(V, E)$ with $V = \{v_1, \dots, v_n\}$ and $E \subseteq V \times V$.*

a. *The degree matrix $\Delta \in \mathbb{R}^{n\times n}$ is a diagonal matrix defined as*

$$\Delta := \begin{bmatrix} \deg(v_1) & & & \\ & \deg(v_2) & & \\ & & \ddots & \\ & & & \deg(v_n) \end{bmatrix}$$

b. *The adjacency matrix $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix ($A = A^T$) defined s.t.*

$$A = [a_{ij}] \ : \ a_{ij} \begin{cases} 1 & (v_i, v_j) \in V \\ 0 & (v_i, v_j) \notin V \end{cases}$$

c. *The incidence matrix $D \in \mathbb{R}^{n\times m}$ is defined as*

$$D = [d_{ij}] \ : \ d_{ij} \begin{cases} 1 & (v_i, -) \in e_j \\ -1 & (-, v_i) \in e_j \\ 0 & \textit{otherwise} \end{cases}$$

d. *The* Laplacian matrix $L \in \mathbb{R}^{n\times n}$ *is a symmetric* ($L = L^T$) *and strictly semi-positive definite* ($L \succeq 0$) *is defined as*

$$L := \Delta - A = DD^T$$

and

$$L = \begin{bmatrix} \deg(v_1) & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & \deg(v_2) & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & \deg(v_n) \end{bmatrix}$$

e. *For a weighted graph* $G(V, E, W)$*, the diagonal weighted matrix* $W \in \mathbb{R}^{m\times m}$ *is defined as*

$$W = [w_{ij}]\forall_{ij\in E}$$

were w_{ij} *are the corresponding weights for* $e_{ij} = (v_i, v_j)$.

Definition 4. *Let undirected and unweighted graph* $G(V, E)$ *with* $V = \{v_1, \dots, v_n\}$ *and* $E \subseteq V \times V$*. The* consensus *dynamics of network* $G(V, E)$ *is defined by*

$$\forall_{i=1,\dots,n}\ \dot{x}_i = \sum_{j\in\mathcal{N}_i}(x_j - x_i) \iff \dot{x} = -Lx$$

For the case with weighted graph $G(V, E, W)$ *with diagonal weight matrix* $W = [w_{ij}]$*,* weighted consensus dynamics *are given as*

$$\dot{x}_i = \sum_{j\in\mathcal{N}_i} w_{ij}(x_j - x_i) \implies \dot{x} = -L_w x$$

where weighted Laplacian matrix L_w *is defined as*

$$L_w = DWD^T$$

b) Preliminary Theorems

Theorem 1. *Let* $L = L^T \succeq 0 \in R^{n\times n}$ *be the Laplacian matrix of undirected graph* $G(V, E)$ *with eigenvalues*

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$$

Theorem 2. *Let* $G(V, E)$ *be an undirected graph. The Laplacian matrix* $L \in \mathbb{R}^{n\times n}$ *of* $G(V, E)$*, is symmetric* ($L = L^T$) *and positive semi-definite* ($L \succeq 0$)*. Let the eigenvalues of* L *be*

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$$

In this undirected case, L *will always have eigenvalue* $\lambda_1 = 0$ *with the corresponding eigenvector of* $v_1 = \mathbf{1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$.

Proof. $\lambda_1 = 0$ and $v_1 = \mathbf{1}$ implies

$$L\mathbf{1} = \lambda_1\mathbf{1} = 0$$

Thus

$$L\mathbf{1} = \begin{bmatrix} \deg(v_1) & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & \deg(v_2) & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & \deg(v_n) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \deg(v_1) - \sum_{i \neq 1} a_{1i} \\ \deg(v_2) - \sum_{i \neq 2} a_{2i} \\ \vdots \\ \deg(v_n) - \sum_{i \neq n} a_{ni} \end{bmatrix}$$

$$= \begin{bmatrix} \deg(v_1) - \sum_{i \in \mathcal{N}_1} a_{1i} \\ \deg(v_2) - \sum_{i \in \mathcal{N}_2} a_{2i} \\ \vdots \\ \deg(v_n) - \sum_{i \in \mathcal{N}_n} a_{ni} \end{bmatrix} = \begin{bmatrix} \deg(v_1) - \deg(v_1) \\ \deg(v_2) - \deg(v_2) \\ \vdots \\ \deg(v_n) - \deg(v_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ddots \\ 0 \end{bmatrix} = \mathbf{0}$$

This implies that $\text{span}\{1\} \subseteq (L)$. □

c) Solution

Theorem 3. *Let $G(V, E)$ be an undirected graph. The Laplacian matrix $L \in \mathbb{R}^{n \times n}$ of $G(V, E)$, is symmetric ($L = L^T$) and positive semi-definite ($L \succeq 0$). Let the eigenvalues of L be*

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

Undirected graph $G(V, E)$ will have k connected subgraphs iff L has k zero-valued eigenvalues. (i.e)

$$G(V, E) \textit{ contains } k \textit{ connected subgraphs} \iff m_1(\Lambda(L), \lambda_1 = 0) = k$$

Proof. The number of block diagonal matrices in the Jordan form of A is equivalent to the number of connected subgraphs, k. Similarly, if A can be in k blocks then L will also have k blocks. In other words,

$$G(V, E) \text{ contains } k \text{ connected subgraphs} \iff A \equiv \text{diag}[A_1, A_2, \ldots, A_k] \iff L \equiv \text{diag}[L_1, L_2, \ldots, L_k]$$

From Theorem 2,

$$\forall_{i=1,\ldots,k}\ \Lambda(L_i) \ni \lambda_{i,1} = 0$$

It is also known that the eigenvalues for a block diagonal matrix is composed of all the eigenvalues from the block matrices. (i.e.)

$$\Lambda(L) = \bigcup_{i=1,\ldots,k} \Lambda(L_i)$$

with multiplicities of each similar eigenvalue being summed. Since each block, representing a subgraph, has a single zero eigenvalue the multiplicity of $\lambda_1 = 0 \in \Lambda(L)$ is equal to $\sum_{i=1}^{k} 1 = k$. Therefore,

$$G(V, E) = [G_{i=1,\ldots,k}], G_i \text{ connected} \iff m_1(\Lambda(L), \lambda_1 = 0) = k$$

□

Problem 4

Let the subspace S be

$$S = \text{span}\{\mathbf{1}\}^{\perp}$$

, i.e.,

$$x \in S \iff x^T\mathbf{1} = 0$$

Show that S is L-invariant, i.e. $LS \subseteq S$ (i.e. $Lx \in S, \forall_{x\in S}$), where L is the Laplacian of an undirected, connected graph.

Problem 5

$K_{1,6}$ is a star graph with one central node and six leaf nodes as shown in Fig.2

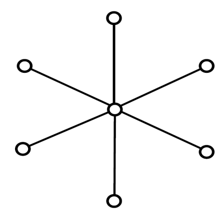

Fig. 2: Star graph, $K_{1,6}$.

Your task is to show that $K_{1,6}$ can never be an induced subgraph of a Δ-disk proximity graph.

Problem 6

If $l_{i,j}$ is the shortest path distance (number of edges one needs to follow) between vertices v_i and v_j, the diameter of the graph is defined as

$$\text{diam}(G) = \max_{v_i, v_j \in V} l_{i,j}$$

Similarly, if we let l_i^* (known as the eccentricity of vertex v_i) be the longest distance to any vertex from the vertex v_i, i.e.,

$$l_i^* = \max_{v_j \in V} l_{i,j}$$

then the radius of a graph is defined as

$$\text{radius}(G) = \min_{v_i \in V} l_i^*$$

Find the radius and diameter of the following graphs.

Problem 7

Following are some undirected networks on four nodes with the same initial positions. In which of these networks, nodes converge fastest under the distributed consensus dynamics? Explain your answer.

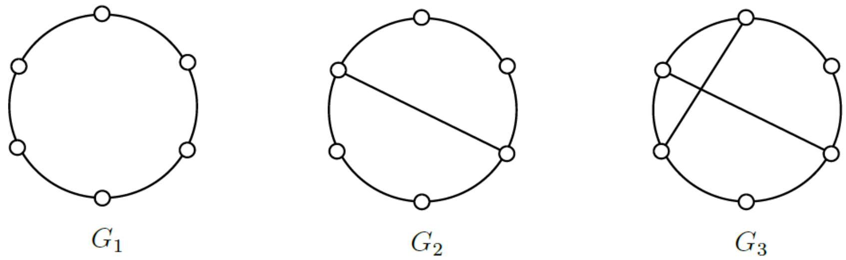

Fig. 3: Graphs G_1, G_2, and G_3.

Problem 8

What is the necessary and sufficient condition for the consensus to happen in the case of static directed networks? Derive this condition.

References

[1] A. Marsden, "Eigenvalues of the laplacian and their relationship to the connectedness of a graph," *University of Chicago, REU*, 2013.

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