MECH 6v29.002 – Model Predictive Control

L22 – Explicit MPC

Project Information

- Tuesday 11/28 and Thursday 11/30 – In-class presentations

Tuesday, 11/28	
8:30 – 8:45	Alex and Harsh
8:45 – 9:00	Michael
9:00 – 9:15	Yuxiang
9:15 – 9:30	Siddharth
9:30 – 9:45	

Thursday, 11/30	
8:30 – 8:45	Diyako and Tiffany
8:45 – 9:00	Shilin
9:00 – 9:15	Jonas
9:15 – 9:30	Sai
9:30 – 9:45	David and Juned

  - Let me know ASAP if this day/time does not work for you
- Friday, 12/08 – Final project report due by 5pm
- Rubric for presentation and final report
  - 20% - Background and motivation
  - 30% - Problem formulation, key features, solution approach
  - 30% - Numerical results and discussion
  - 20% - Overall communication effectiveness

Outline

- Implicit vs Explicit MPC
  - Pros and cons
- Multi-Parametric Programming
  - Simple Examples
- Explicit MPC
  - Main Ideas
  - KKT Conditions
  - General Procedure
  - Example
  - Scalability

Implicit vs Explicit MPC

- Most of what we have done to date would be considered implicit MPC
  - We know that MPC solves an optimal control problem based on feedback (state measurements)
  - Therefore we can think about MPC as implementing the feedback control law $u_k = \kappa\left(x_k\right)$
  - But we have rarely tried to figure out what $\kappa(\cdot)$ is
    - Except for the unconstrained LQR-type case where we could analytically compute the optimal feedback control law using the batch or recursive approaches.
  - Hence, this is an implicit feedback control approach, since we never try to determine the feedback control law and simply solve the optimization problem at every time step instead
- Alternatively, explicit MPC solves a multi-parametric programming problem to determine $\kappa(\cdot)$
  - Find the optimal solution for any value of x_k and store it in a "look-up table"

Pros and Cons

- Explicit MPC is not a replacement for implicit MPC
- Explicit MPC
  - Best for small problems (<= 6 variables, 15 constraints, 8 states)
  - Pros:
    - Can be extremely fast in real-time (very small computation time)
    - Can be very simple to implement/code (look-up table)
  - Cons:
    - Can be memory intensive
    - Offline computation can be very slow
    - Does not scale to large problems
- Implicit MPC
  - Best for medium and large problems
    - depends on the type of solver
  - Pros and cons are basically the opposite of explicit MPC

(Multi-)Parametric Programming

- Conventional optimization problem:
  - Example (a, b, c are known)

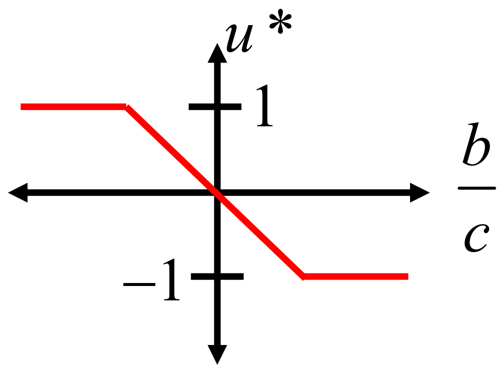
$$\min_u J = a + bu + \frac{1}{2}cu^2 \quad s.t.\ u \in [-1, 1]$$

$$\frac{dJ}{du} = b + cu$$

$$\frac{dJ}{du} = 0$$

$$b + cu^* = 0$$

$$u^* = -\mathrm{sat}\left(\frac{b}{c}\right)$$



- Multi-parametric programming

$$\min_u J = x_1 + x_2 u + \frac{1}{2}x_3 u^2 \quad s.t.\ u \in [-1, 1]$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$u^* = -\mathrm{sat}\left(\frac{x_2}{x_3}\right)$$

Optimal control input is no longer just a point, it is a function of x

(Multi-)Parametric Programming (cont.)

- Since the minimizer of a multi-parametric program is a function, it is reasonable to expect that this only works for some optimization problems
- Turns out that this work for most of the MPC problems that we have covered in this class
- Multi-parametric problems can be solved if (not only if)
  - Cost function is linear or quadratic (in x and u)

$$J = a + b^T x + c^T u \qquad \text{or} \qquad J = \frac{1}{2} x^T Q x + x^T S u + \frac{1}{2} u^T R u$$

  - Constraints are linear

$$\mathcal{U}(x) = \{u \mid Mu \le Nx + p\}$$

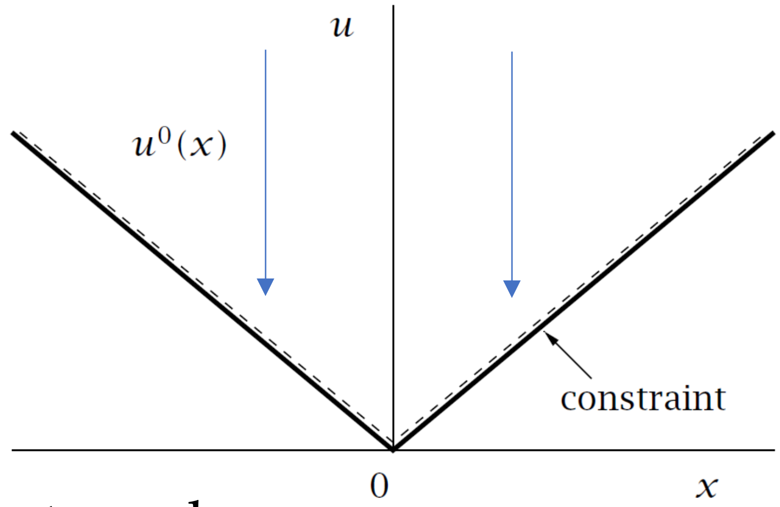
Simple Examples (LP)

- Linear Program

$$\min_u J = x + u$$

$$s.t.\ \ u + x \geq 0$$

$$u - x \geq 0$$



$$\frac{dJ}{du} = 1$$

Tells us to make u as small as possible

$$u^* = |x|$$

- Optimal solution could be implemented as a look-up table

$$u^*(x) = \begin{cases} -x & if\ \ x \leq 0 \\ x & if\ \ x \geq 0 \end{cases}$$

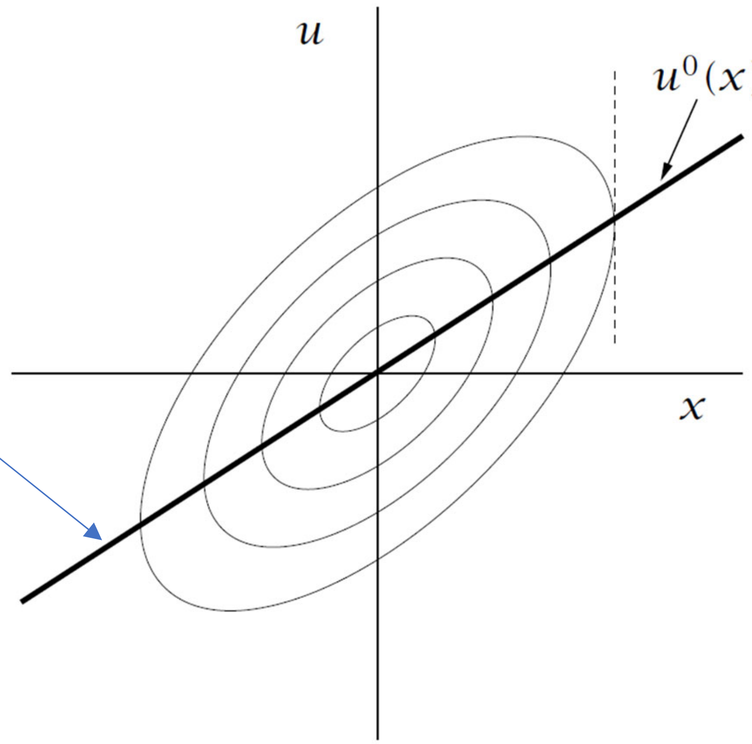
Continuous piecewise-linear function

Simple Examples (Unconstrained QP)

- Unconstrained Quadratic Program

$$\min_u J = \frac{1}{2}(x-u)^2 + \frac{1}{2}u^2$$

$$\frac{dJ}{du} = -(x-u) + u = -x + 2u \quad \Rightarrow \quad u^* = \frac{1}{2}x$$



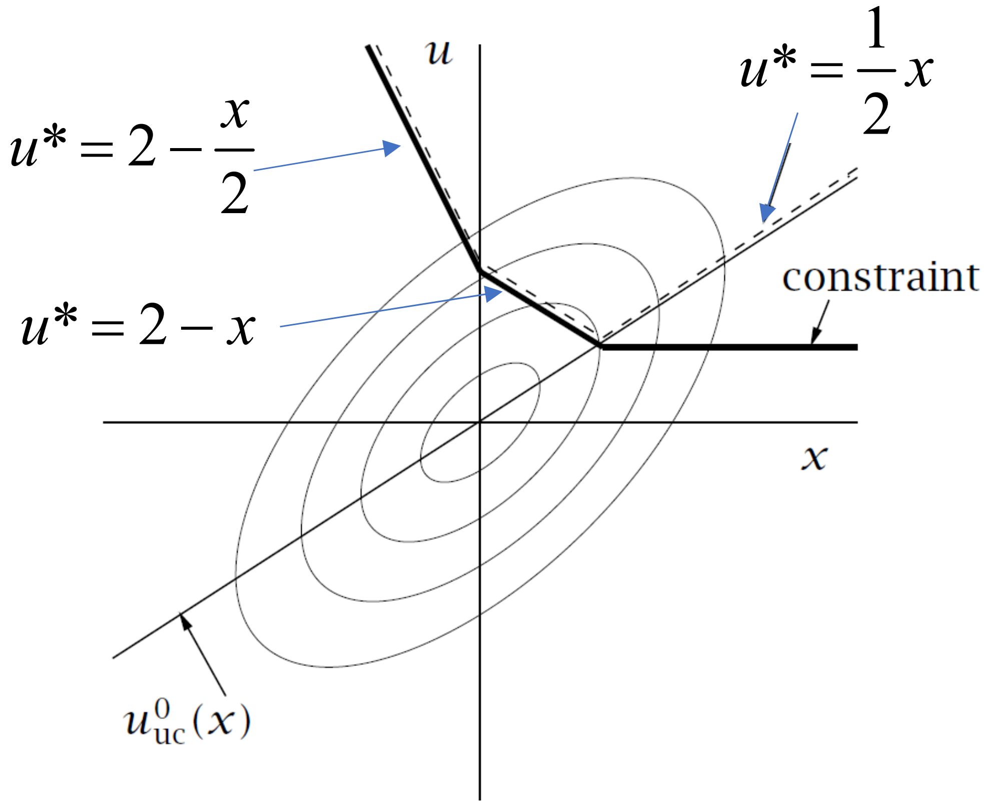
- Optimal input is linear in x
- Optimal cost is quadratic in x $\quad J^*(x) = \frac{1}{4}x^2$

Simple Examples (Constrained QP)

- Constrained Quadratic Program

$$\min_u \; J = \frac{1}{2}(x-u)^2 + \frac{1}{2}u^2$$

$$\begin{aligned} s.t. \quad & u \geq 1 \\ & u + \frac{x}{2} \geq 2 \\ & u + x \geq 2 \end{aligned}$$



- Optimal input is continuous piecewise-linear in x

$$u^*(x) = \begin{cases} 2 - \frac{x}{2} & \text{if} \quad x \leq 0 \\ 2 - x & \text{if} \quad 0 \leq x \leq 2 \\ \frac{1}{2}x & \text{if} \quad 2 \leq x \end{cases}$$

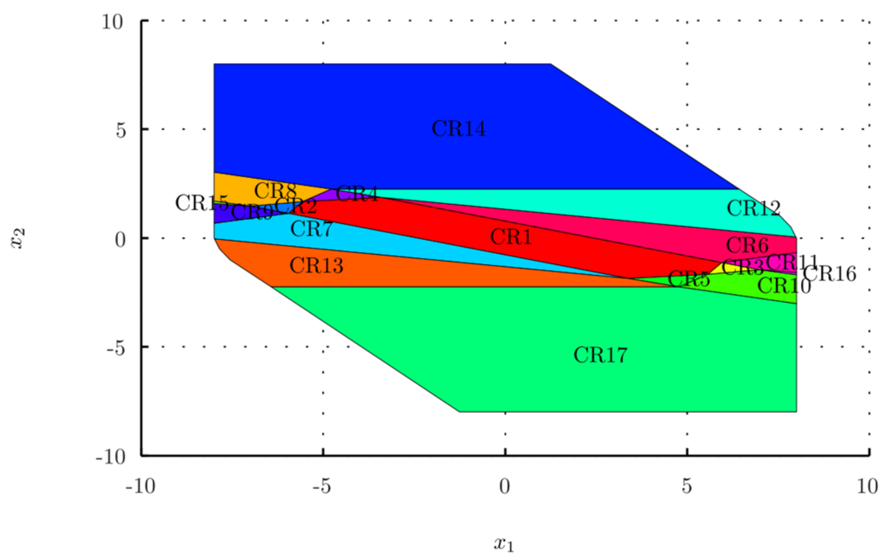
- Optimal cost is piecewise-quadratic in x

Main Ideas[1]

- For LPs and QPs, the explicit feedback control law will be piecewise-linear
  - Under specific conditions, the control law can be continuous piecewise-linear
- The domain of x will be partitioned into a finite set of polyhedral regions
  - In each region, the optimal solution linearly depends on x and the optimal cost is linear or quadratic in x
- Each region corresponds to a different combination of active constraints (inequality constraint is satisfied in equality)
- Control law is then implemented as a look-up table

Critical Region	Active Constraints
CR1	{}
CR2	{1}
CR3	{2}
CR4	{11}
CR5	{12}
CR6	{17}
CR7	{18}
CR8	{1,11}
CR9	{1,18}
CR10	{2,12}
CR11	{2,17}
CR12	{11,17}
CR13	{12,18}
CR14	{1,11,17}
CR15	{1,11,18}
CR16	{2,12,17}
CR17	{2,12,18}

$$u_0^*(x) = \begin{cases} F_1 x + g_1 & \text{if} \quad H_1 x \le K_1 \\ \vdots & \quad\;\; \vdots \\ F_M x + g_M & \text{if} \quad H_M x \le K_M \end{cases}$$



[1] Borrelli, F; Bemporad, A; Morari, M. Predictive Control for linear and hybrid systems, Cambridge University Press, 2017.

KKT Conditions Example

- Karush-Kuhn-Tucker (KKT) Conditions
  - Provide necessary (and sufficient) conditions for optimal solution of (convex) optimization problems
  - Certain technical conditions must hold
    - Convexity, differentiability, strong duality, etc
- Example

$$z^*(x) = \arg\min_z \ f(z,x) = \frac{1}{2}z^2 + 2xz$$

$$s.t. \qquad z - x - 1 \le 0$$

- Lagrangian: $\mathcal{L}(z,x,\lambda) = f(z,x) + \lambda(z - x - 1)$
- KKT Conditions:

Optimality $\quad \frac{\partial}{\partial z}\mathcal{L}(z^*, x^*, \lambda^*) = 0 \quad \Rightarrow \quad z^* + 2x^* + \lambda = 0$

Complementary slackness $\quad \lambda^*(z^* - x^* - 1) = 0$

Dual feasibility $\quad \lambda^* \ge 0$

Primal feasibility $\quad z^* - x^* - 1 \le 0$

KKT Conditions Example (cont.)

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$$s.t. \quad z - x - 1 \le 0$$

KKT Conditions:

$$z^* + 2x^* + \lambda = 0$$

$$\lambda^*(z^* - x^* - 1) = 0$$

$$\lambda^* \ge 0$$

$$z^* - x^* - 1 \le 0$$

- Complementary slackness condition creates two case

$$\lambda^* > 0 \qquad z^* - x^* - 1 = 0$$

$$\lambda^* = 0 \qquad z^* - x^* - 1 < 0$$

- Main idea: Each case corresponds to if the constraint is active or not

General Procedure

- Can always cast linear MPC into the following formulation

$$\min_z \frac{1}{2} z^T Q z + x^T F^T z + \frac{1}{2} x^T Y x$$
$$s.t. \qquad Gz \le W + Sx$$

Think about as:

$$z = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} \qquad x = x(0)$$

$W \in \mathbb{R}^q \rightarrow q$ constraints

- From LQR-like MPC formulation, we can typically assume

$$\begin{bmatrix} Q & F \\ F^T & Y \end{bmatrix} \ge 0, \quad Q = Q^T > 0$$

- Lagrangian and KKT Conditions $\mathcal{L}(z,x,\lambda) = \frac{1}{2} z^T Q z + x^T F^T z + \lambda^T (Gz - W - Sx)$

Optimality $\quad Qz + Fx + G^T \lambda = 0$

Complementary slackness $\quad \lambda_i (G_i z - W_i - S_i x) = 0, \ \forall i \in \{1, \ldots, q\}$

Dual feasibility $\quad \lambda \ge 0$

Primal feasibility $\quad Gz - W - Sx \le 0$

General Procedure (cont.)

- Solving for the explicit MPC control law is an iterative process
- Start at an initial feasible point x_0
- Solve the QP and find

$$z^*(x_0) \text{ and } \lambda^*(x_0)$$

- Use optimal Lagrange multipliers to determine set of active constraints
  - Let the index set of active constraints be

$$I(x_0) \subseteq \{1,\ldots,q\}$$

$$I(x_0) = \{i \in \{1,\ldots,q\} \mid \lambda_i^* > 0\}$$

$$\Leftrightarrow G_i z - W_i - S_i x = 0$$

Active constraint

  - Now we have

$$G_i z^*(x_0) - W_i - S_i x(0) = 0,\ \forall i \in I(x_0)$$

$$G_i z^*(x_0) - W_i - S_i x(0) < 0,\ \forall i \notin I(x_0)$$

$$\min_z \frac{1}{2} z^T Q z + x^T F^T z + \frac{1}{2} x^T Y x$$

$$s.t. \quad Gz \le W + Sx$$

KKT:

$$Qz + Fx + G^T \lambda = 0$$

$$\lambda_i (G_i z - W_i - S_i x) = 0$$

$$\lambda \ge 0$$

$$Gz - W - Sx \le 0$$

General Procedure (cont.)

- Collect all of this as

$$G_i z^*(x_0) - W_i - S_i x(0) = 0, \ \forall i \in I(x_0)$$

$$G_i z^*(x_0) - W_i - S_i x(0) < 0, \ \forall i \notin I(x_0)$$

$$\tilde{G}z - \tilde{W} - \tilde{S}x = 0, \quad \tilde{\lambda} > 0$$

$$\hat{G}z - \hat{W} - \hat{S}x < 0, \quad \hat{\lambda} = 0$$

$$\min_z \frac{1}{2} z^T Q z + x^T F^T z + \frac{1}{2} x^T Y x$$

$$s.t. \quad Gz \le W + Sx$$

KKT:

$$Qz + Fx + G^T \lambda = 0$$

$$\lambda_i (G_i z - W_i - S_i x) = 0$$

$$\lambda \ge 0$$

$$Gz - W - Sx \le 0$$

- From optimality condition, we get

$$z = -Q^{-1}\left(Fx + G^T \lambda\right) = -Q^{-1}\left(Fx + \tilde{G}^T \tilde{\lambda}\right)$$

- Substitute into active constraint equation $\tilde{G}z - \tilde{W} - \tilde{S}x = 0$ and solve for

$$\tilde{\lambda}(x) = -\left(\tilde{G}Q^{-1}\tilde{G}^T\right)^{-1}\left(\tilde{W} + \left(\tilde{S} + \tilde{G}Q^{-1}F\right)x\right)$$

- Plug back into the optimality condition

$$z(x) = Q^{-1}\left[\tilde{G}^T \left(\tilde{G}Q^{-1}\tilde{G}^T\right)^{-1}\left(\tilde{W} + \left(\tilde{S} + \tilde{G}Q^{-1}F\right)x\right) - Fx\right]$$

General Procedure (cont.)

- In a neighborhood of x_0, we have now defined explicit equations for $z(x), \tilde{\lambda}(x)$

$$\tilde{\lambda}(x) = -\left(\tilde{G}Q^{-1}\tilde{G}^T\right)^{-1}\left(\tilde{W} + \left(\tilde{S} + \tilde{G}Q^{-1}F\right)x\right)$$

$$z(x) = Q^{-1}\left[\tilde{G}^T\left(\tilde{G}Q^{-1}\tilde{G}^T\right)^{-1}\left(\tilde{W} + \left(\tilde{S} + \tilde{G}Q^{-1}F\right)x\right) - Fx\right]$$

$$\min_z \frac{1}{2}z^T Qz + x^T F^T z + \frac{1}{2}x^T Yx$$

$$s.t. \quad Gz \le W + Sx$$

KKT:

$$Qz + Fx + G^T\lambda = 0$$

$$\lambda_i\left(G_i z - W_i - S_i x\right) = 0$$

$$\left\{\begin{array}{l}\lambda \ge 0 \\ Gz - W - Sx \le 0\end{array}\right.$$

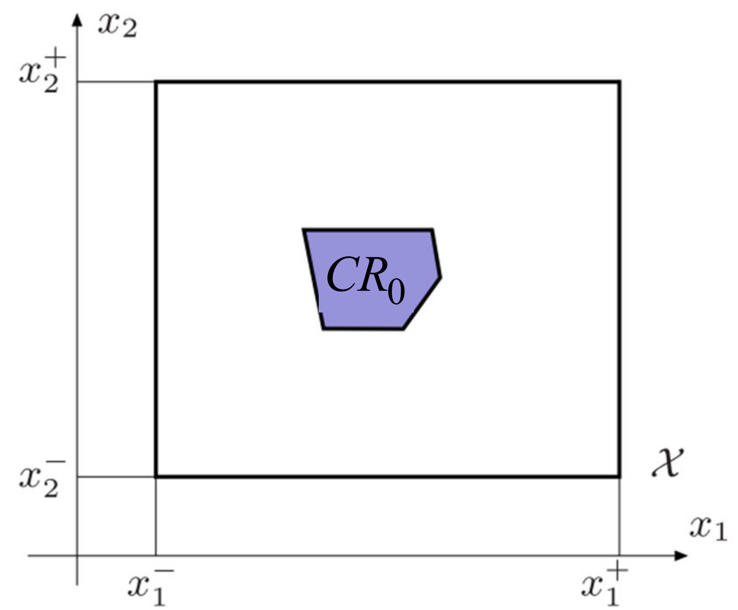
- Combine these equations with primal and dual feasibility constraints, we can define a critical region in terms of x

$$CR_0 = \left\{x \in \mathcal{X} \mid \hat{G}z(x) - \hat{W} - \hat{S}x \le 0,\ \tilde{\lambda}(x) \ge 0\right\}$$

$$= \left\{x \in \mathcal{X} \mid A_0 x \le b_0\right\}$$

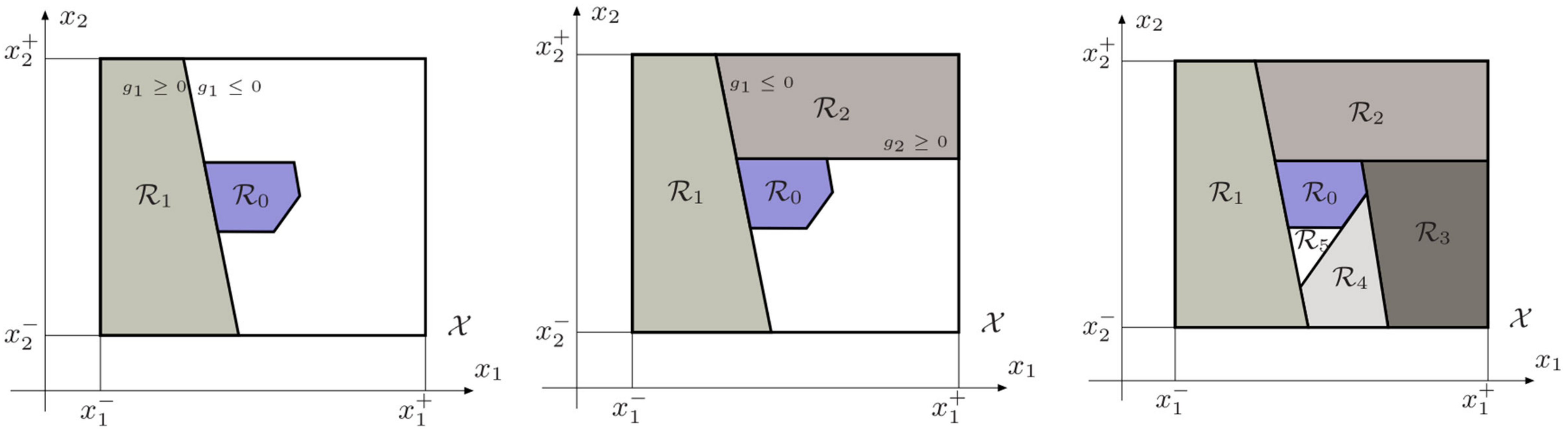
- For all x in this region, the optimal solution (satisfying the KKT conditions) is

$$z(x) = K_0 x + c_0$$



General Procedure (cont.)

- There are multiple methods for finding the remaining regions
- One approach is to split the remaining region



- Repeat the process of picking a point in each region, solving the QP, and defining the new region where the optimal solution is characterized by the same set of active constraints
- This procedure is finite because there are at most 2^q combinations of active constraints
- Majority of computation time is removing redundant inequalities

Double Integrator Example[1]

- Example from lecture notes: "Model Predictive Control Quadratic programming and explicit MPC" by Alberto Bemporad
  - http://cse.lab.imtlucca.it/~bemporad/teaching/mpc/imt/3-qp_explicit.pdf

- Model and constraints

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \qquad -1 \le u_k \le 1$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k$$

- Objective function

$$\min \sum_{k=0}^{\infty} y_k^2 + \frac{1}{100} u_k^2$$

- MPC Design

$$\min_{u_0, u_1} \sum_{k=0}^{1} y_k^2 + \frac{1}{100} u_k^2 + x_2^T P x_2$$

$$\begin{aligned} s.t. \quad & x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \\ & y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \\ & -1 \le u_k \le 1 \end{aligned}$$

Corresponds to infinite horizon discrete-time LQR solution

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- With prediction horizon $N = 2$

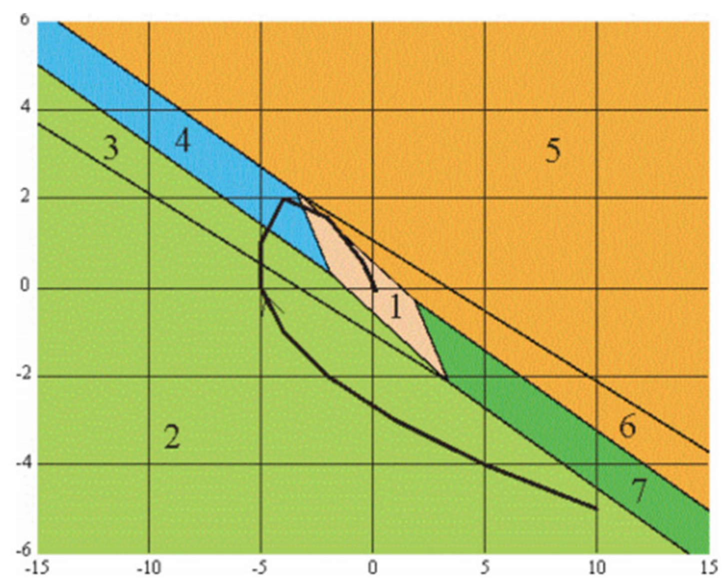
$$\min_{u_0, u_1} \sum_{k=0}^{1} y_k^2 + \frac{1}{100} u_k^2 + x_2^T P x_2$$

$$\text{s.t.} \quad x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k$$

$$-1 \le u_k \le 1$$

$$u(x) = \begin{cases}
\begin{bmatrix} -0.8166 & -1.75 \end{bmatrix} x & \text{if } \begin{bmatrix} -0.8166 & -1.75 \\ 0.8166 & 1.75 \\ 0.6124 & 0.4957 \\ -0.6124 & -0.4957 \end{bmatrix} x \le \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} & \text{(Region \#1)} \\
1 & \text{if } \begin{bmatrix} 0.3864 & 1.074 \\ 0.297 & 0.9333 \end{bmatrix} x \le \begin{bmatrix} -1 \\ -1 \end{bmatrix} & \text{(Region \#2)} \\
1 & \text{if } \begin{bmatrix} -0.297 & -0.9333 \\ 0.8166 & 1.75 \\ 0.9712 & 2.699 \end{bmatrix} x \le \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} & \text{(Region \#3)} \\
\begin{bmatrix} -0.5528 & -1.536 \end{bmatrix} x + 0.4308 & \text{if } \begin{bmatrix} -0.9712 & -2.699 \\ 0.3864 & 1.074 \\ 0.6124 & 0.4957 \end{bmatrix} x \le \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} & \text{(Region \#4)} \\
-1 & \text{if } \begin{bmatrix} -0.3864 & -1.074 \\ -0.297 & -0.9333 \end{bmatrix} x \le \begin{bmatrix} -1 \\ -1 \end{bmatrix} & \text{(Region \#5)} \\
-1 & \text{if } \begin{bmatrix} 0.297 & 0.9333 \\ -0.8166 & -1.75 \\ -0.9712 & -2.699 \end{bmatrix} x \le \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} & \text{(Region \#6)} \\
\begin{bmatrix} -0.5528 & -1.536 \end{bmatrix} x - 0.4308 & \text{if } \begin{bmatrix} -0.3864 & -1.074 \\ 0.9712 & 2.699 \\ -0.6124 & -0.4957 \end{bmatrix} x \le \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} & \text{(Region \#7)}
\end{cases}$$



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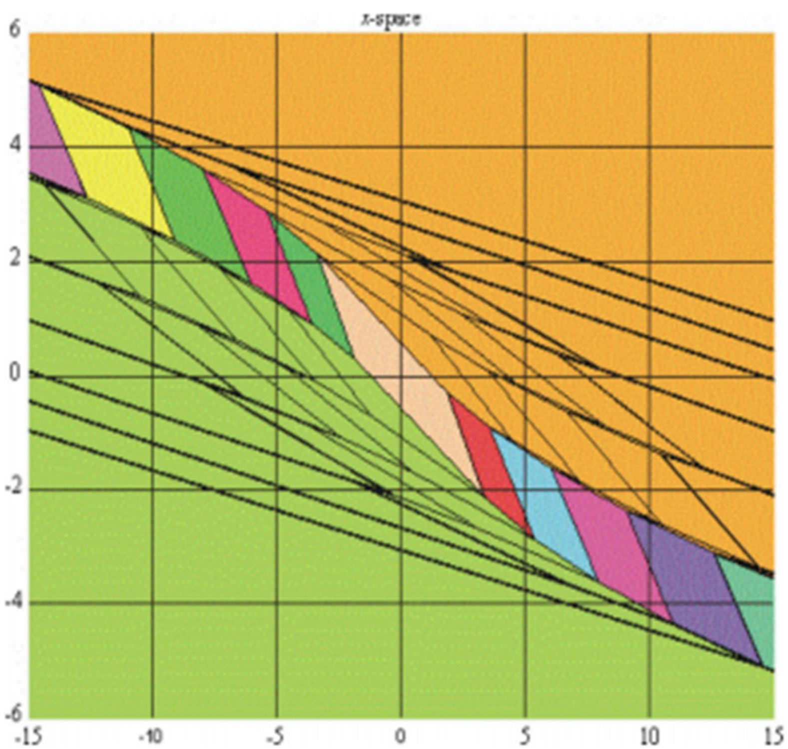
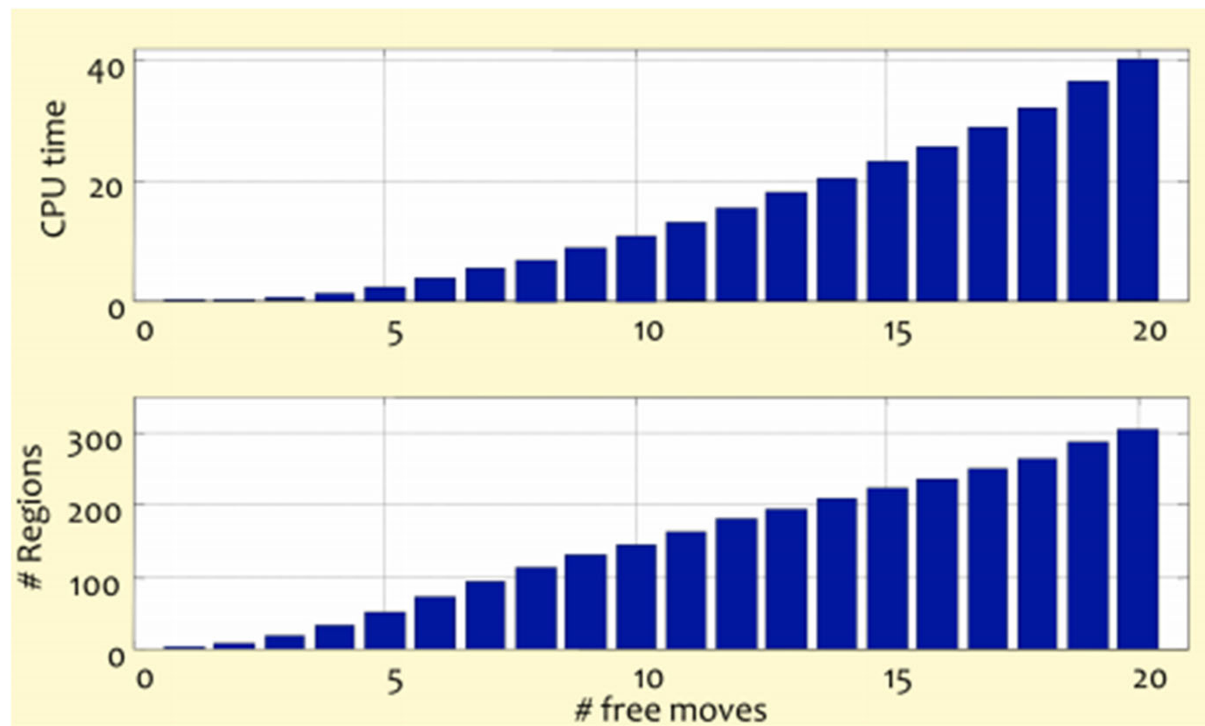
- With prediction horizon $N = 6$





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Scalability Trends

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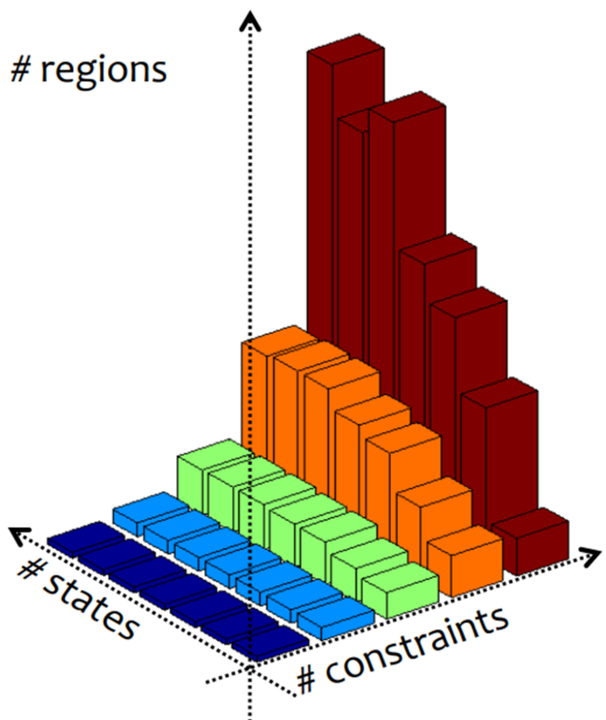
- Maximum number of regions is 2^q
  - Corresponds to the number of active constraint combinations
  - However, most combinations will not be optimal

Mild growth with the # of states



Rapid growth with the # of constraints