# Linear Algebra Review

Here, we will review some of the most relevant matrix identities and operations that we will use in this course. Given square matrices $A$, $B$, $C$ of size $n \times n$ and vectors $v_i$ of size $n$:

- $A_{ij}$ is the entry in the $i^{\text{th}}$ row and $j^{\text{th}}$ column.
- A set of vectors is linearly dependent if one can be written as the linear combination of the others: $v_1 = \alpha_2 v_2 + \alpha_3 v_3 + \cdot + \alpha_n v_n$, $\alpha_k$ is a number and not all are zero. Otherwise, they are linearly independent.
- Matrix multiplication is defined $C = AB$, $C_{ij} = \sum_{k=0}^{n} A_{ik} B_{kj}$, which multiplies the $i^{\text{th}}$ row of $A$ and the $j^{\text{th}}$ column of $B$ element-by-element.
- In general, $AB \neq BA$.
- $A^T$ transforms the matrix $A$ by swapping row and columns - visually this is done by flipping it over its diagonal.
- A matrix is called symmetric if $A = A^T$.
- A matrix is called diagonal if it only has non-zero elements on the diagonal.
- A matrix is called upper [lower] triangular if it only has non-zero elements on or above [below] the diagonal.
- The rank of a matrix is the number of linearly independent columns or linearly independent rows.
- A matrix has an inverse ($A^{-1}$ such that $AA^{-1} = I$) if and only if it has rank equal to $n$.
- Eigenvalues $\lambda_i$ are real numbers such that $Av_i = \lambda_i v_i$, for some non-zero $v_i$.
- Eigenvectors are the vectors $v_i$ corresponding to eigenvalues $\lambda_i$ such that $Av_i = \lambda_i v_i$.
- If a matrix has $n$ distinct eigenvalues, then it has $n$ linearly independent eigenvectors.
- A matrix is diagonalizable ($A = P\Lambda P^{-1}$) if and only if it has $n$ linearly independent eigenvectors. $\Lambda$ is the diagonal matrix of eigenvalues and $P$ is the matrix whose columns are the eigenvectors.
- A symmetric matrix has real (not imaginary) eigenvalues and we can always find eigenvectors that are orthogonal (i.e., $v_i \cdot v_j = 0$, for $i \neq j$).
- An orthogonal matrix $U = [v_1 \ v_2 \ \ldots \ v_n]$ has columns $v_i$ that are orthogonal to each other (i.e., $v_i \cdot v_j = 0$, for $i \neq j$) and of unit length (i.e., $\|v_i\| = v_i \cdot v_i = 1$). The transpose of an orthogonal matrix is its inverse: $U^T = U^{-1}$.
- A symmetric matrix is always diagonalizable, where $P$, the matrix of eigenvectors, is orthogonal. Therefore, $A = U\Lambda U^T$.
- Left eigenvalues ($\sigma_i$) and eigenvectors ($u_i$) obey a similar relation: $u_i A = \sigma_i u_i$.