

nonlinear systems

linearization

limit cycle

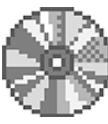
Stability

bifurcations

SYSM 6302

CLASS 25

A nonlinear dynamical system



$$\dot{x} = f(x)$$

may have multiple equilibria (i.e., $\dot{x} = f(x) = 0$), each with a possibly different phase plane portrait. (THM: nearby an equilibrium is isomorphic to a linear system)

← Hartman-Grobman Thm for hyperbolic equilibria

→ Obtain a global (qualitative) picture of the dynamics by "stitching together" the local behavior around each equilibrium.

→ A **linearization** around a point x_0 is given by the system

$$\dot{\delta x} = A \delta x = J_f(x_0) \delta x$$

↑
 $\delta x = x - x_0$

⇒ Evaluate at { node
focus
saddle }



Jacobian

Given a vector field $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$, the Jacobian matrix is the $n \times n$ matrix of partial derivatives:

$$J_f(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

where $f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$

EXAMPLE: Pendulum



$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0$$

$$\dot{x} = \begin{pmatrix} x_2 \\ -\frac{g}{l} \sin x_1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{g}{l} \sin x_1 \end{pmatrix} \Rightarrow \text{Equilibria: } (n\pi, 0)$$

integer

$$J_f(x) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} \cos x_1 & 0 \end{pmatrix}$$

$$\rightarrow J_f(0, n\pi) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{pmatrix}, J_f(0, n\pi) = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{pmatrix}$$

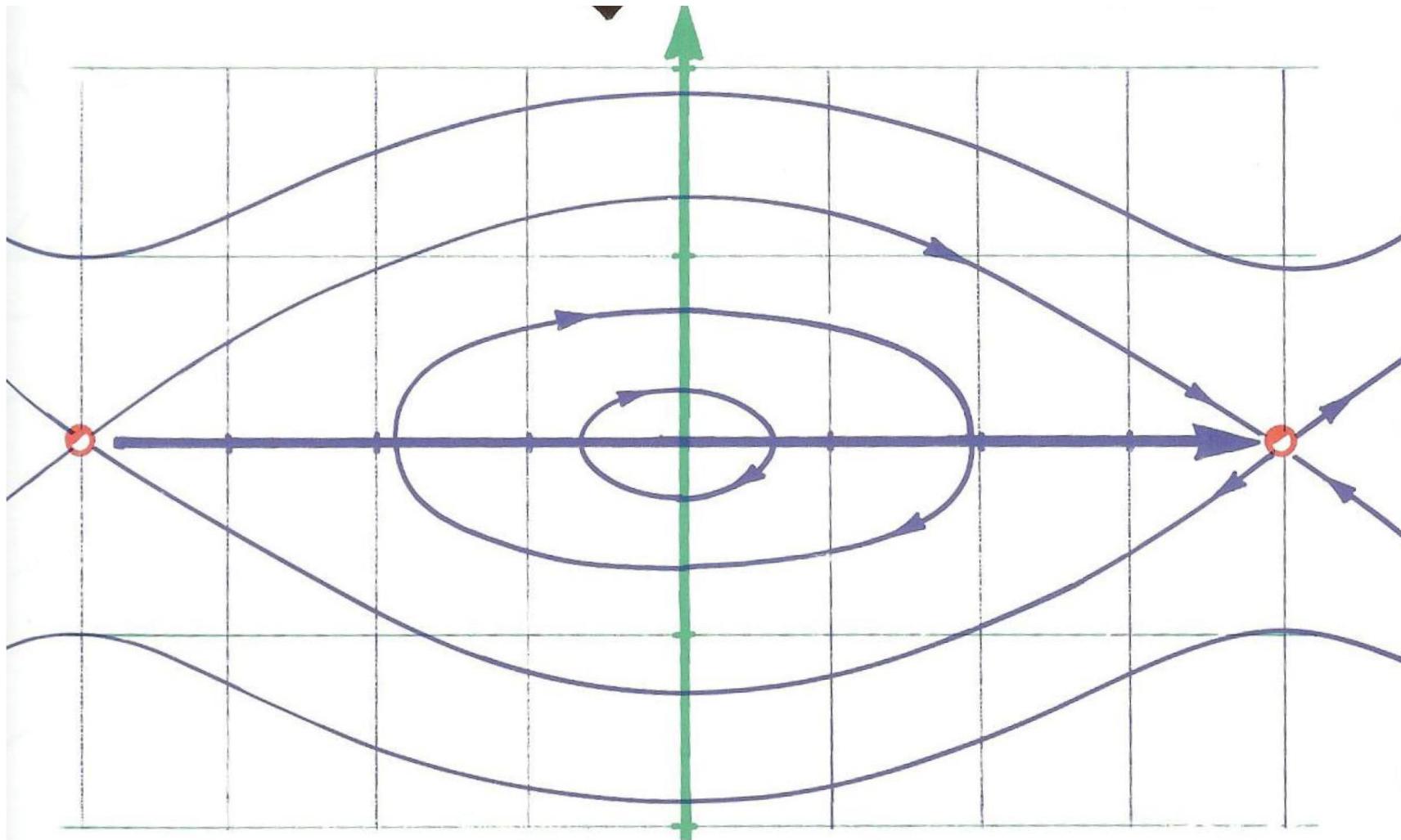
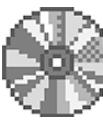
even *odd*

$$\lambda_{1,2} = \pm i \sqrt{\frac{g}{l}}$$

Center

$$\lambda_{1,2} = \pm \sqrt{\frac{g}{l}}$$

Saddle

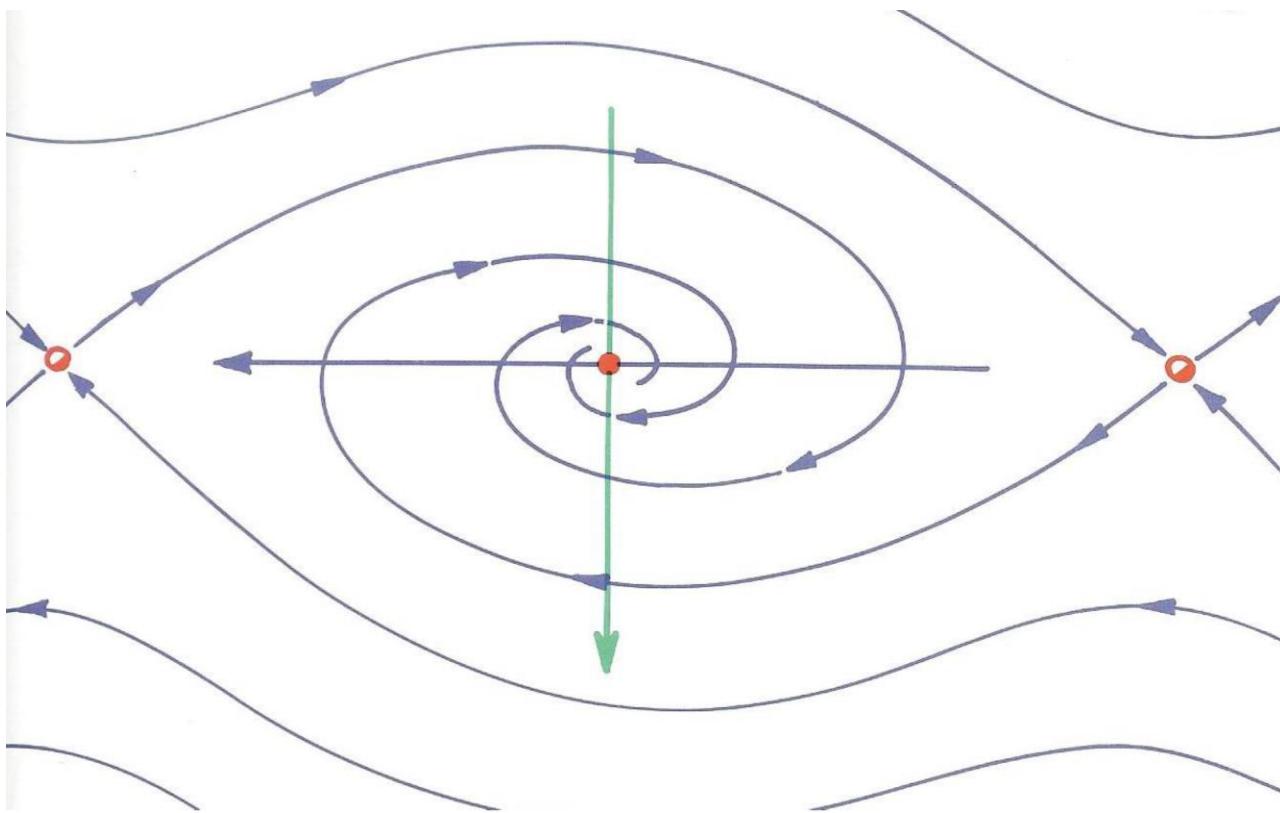


EXAMPLE: Pendulum with Damping

$$\ddot{\theta} + c\dot{\theta} + \frac{g}{l} \sin\theta = 0$$



Equilibria: $(0, n\pi)$



$\overset{\text{even}}{\downarrow}$

$$J_f(0, n\pi) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -c \end{pmatrix}$$

FOCUS

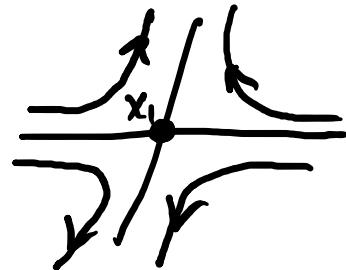
$\overset{\text{odd}}{\downarrow}$

$$J_f(0, (n+1)\pi) = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & -c \end{pmatrix}$$

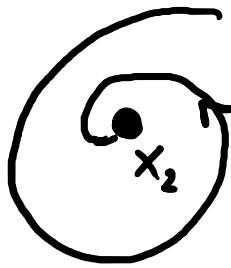
SADDLE

Local to Global

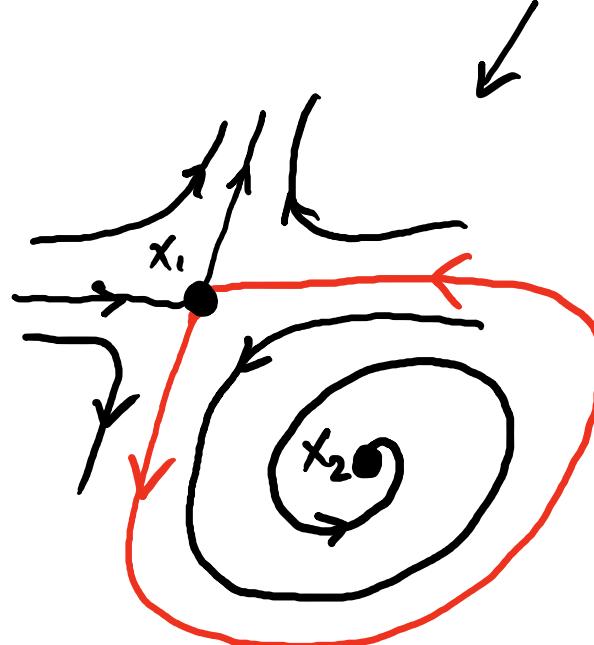
→ Combining local behavior to guess at global behavior can be ambiguous
(especially for centers)



LOCAL

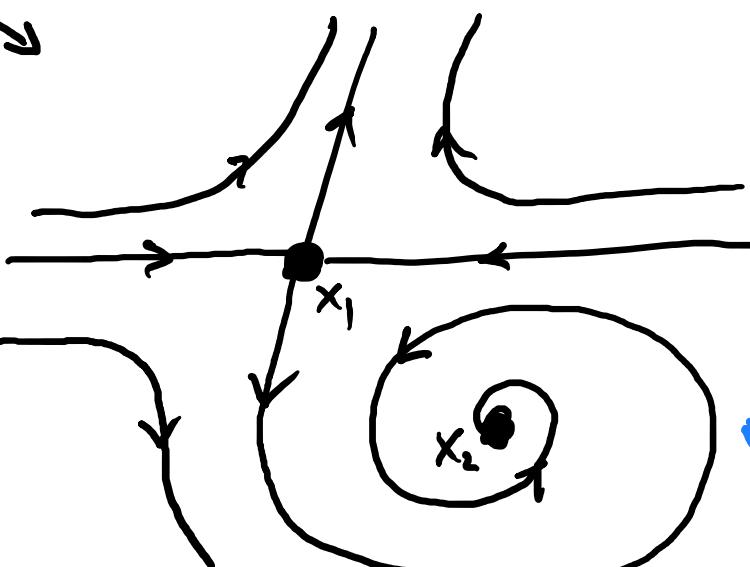


→ Why can't solution lines cross?



GLOBAL

OR

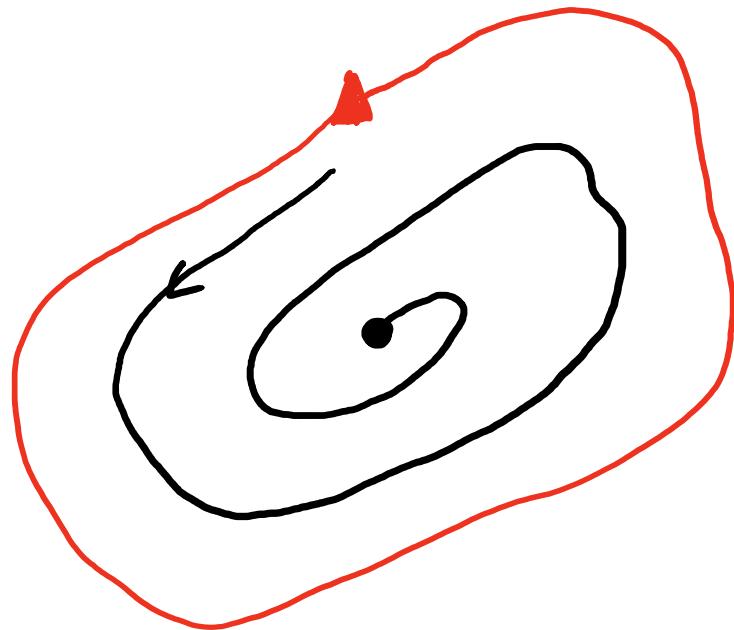


Trajectories that leave x_1 along its unstable eigenvector do not necessarily "blow up"

Limit Cycle - an isolated periodic solution. $x(t) = x(t+T)$, $T > 0$



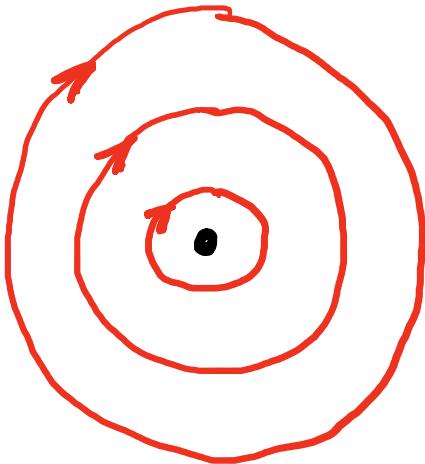
↑ Solutions starting near the limit cycle are, in general, not periodic



For linear systems, the only periodic solutions that are possible are for centers:

⇒ Limit cycles are a purely nonlinear phenomenon

Not isolated!



Van der Pol Equation

$$\frac{d^2x}{dt^2} - \mu(1-x^2) \frac{dx}{dt} + x = 0$$

originally identified in oscillations
of vacuum tubes; used now widely
(e.g., neuroscience, seismology, etc)

→ if $\mu=0$ $\Rightarrow \ddot{x} + x = 0$ ← simple harmonic oscillator without damping

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \mu x_2 (1-x_1^2) \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow \dot{\delta x} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix} \delta x$$

$$\hookrightarrow \dot{x}=0 \Leftrightarrow x=0$$

↑ equilibrium

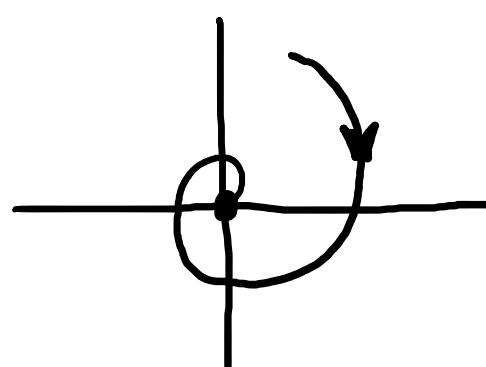
$$\hookrightarrow \lambda^2 - \mu\lambda + 1 = 0$$

$$\lambda = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2} = \frac{\mu}{2} \pm \sqrt{\left(\frac{\mu}{2}\right)^2 - 1}$$

$$\lambda = \frac{\mu}{2} \pm \sqrt{\left(\frac{\mu}{2}\right)^2 - 1}$$

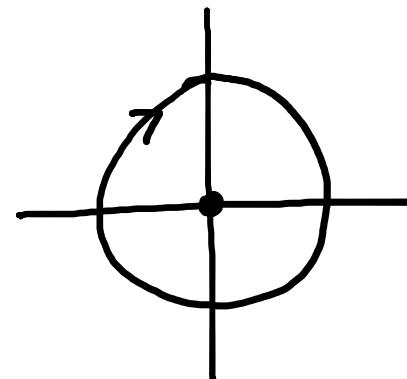
$$\underline{\mu \in [-1, 0)}$$

Stable Focus



$$\underline{\mu = 0}$$

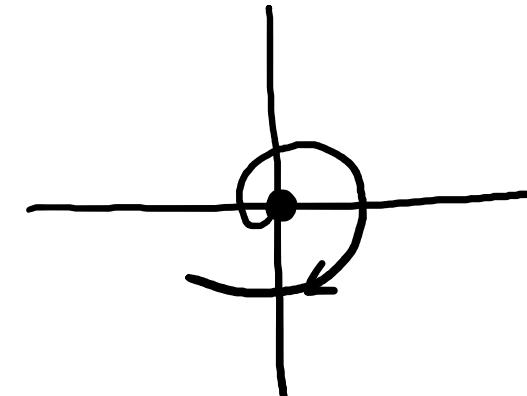
Center



$$\underline{\mu \in (0, 1]}$$

i.e. with damping

Unstable Focus

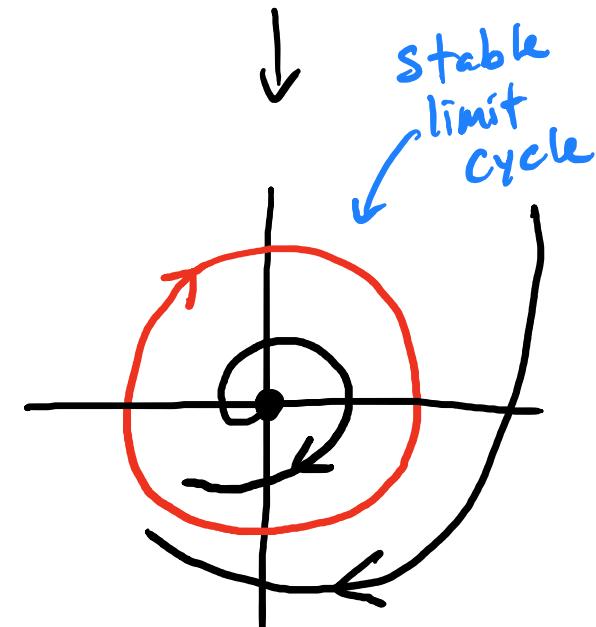
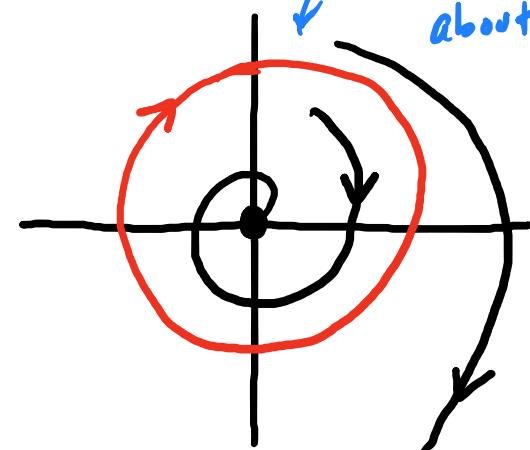


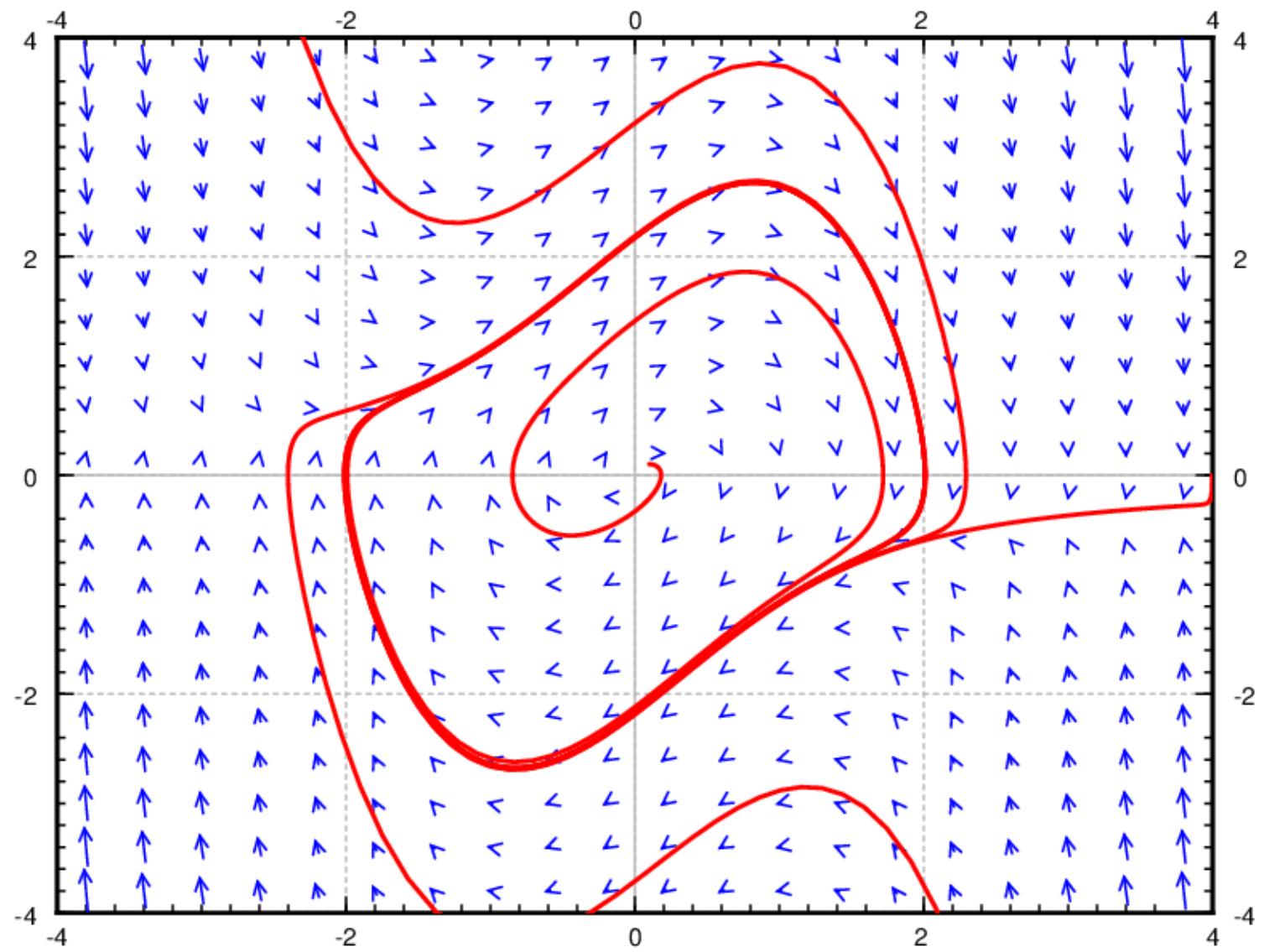
Hopf Bifurcation:

If a linearization passes from sink to source as a parameter passes through μ \Rightarrow for $\mu > \mu_0$ there must be a periodic solution (limit cycle)

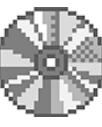
Other bifurcation patterns are possible, but we won't cover them

Hopf's result does not say anything about $\mu < \mu_0$; for the Van der Pol eqn, there is an unstable limit cycle

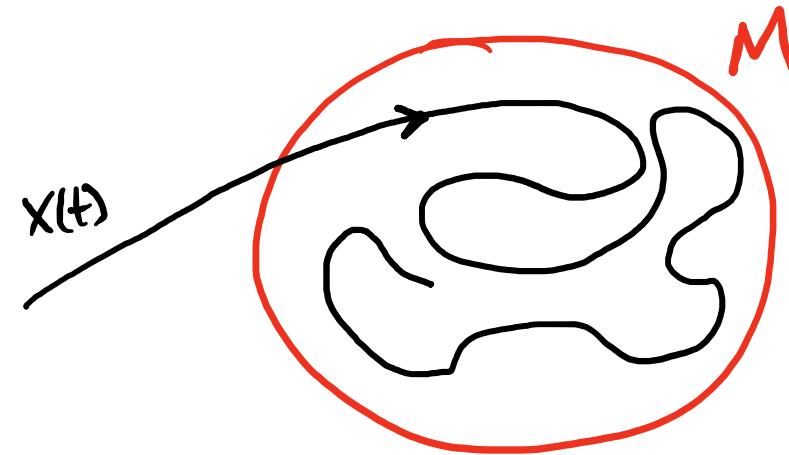




Poincaré - Bendixson Theorem



A set M is **positively invariant** for a nonlinear system $\dot{x} = f(x)$ if, for some time T , $x(T) \in M$, then $x(t) \in M$ for all $t > T$.



A positively invariant set in \mathbb{R}^2 for an autonomous second-order nonlinear system must either contain a periodic orbit or an equilibrium point.

Example: $\dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2)$ $\dot{x}_2 = -x_1 + x_2 - x_2(x_1^2 + x_2^2)$

Eq. pt.
(0,0) \Rightarrow

$$\dot{\delta x} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \delta x, \quad \lambda = 1 \pm i$$

unstable focus 

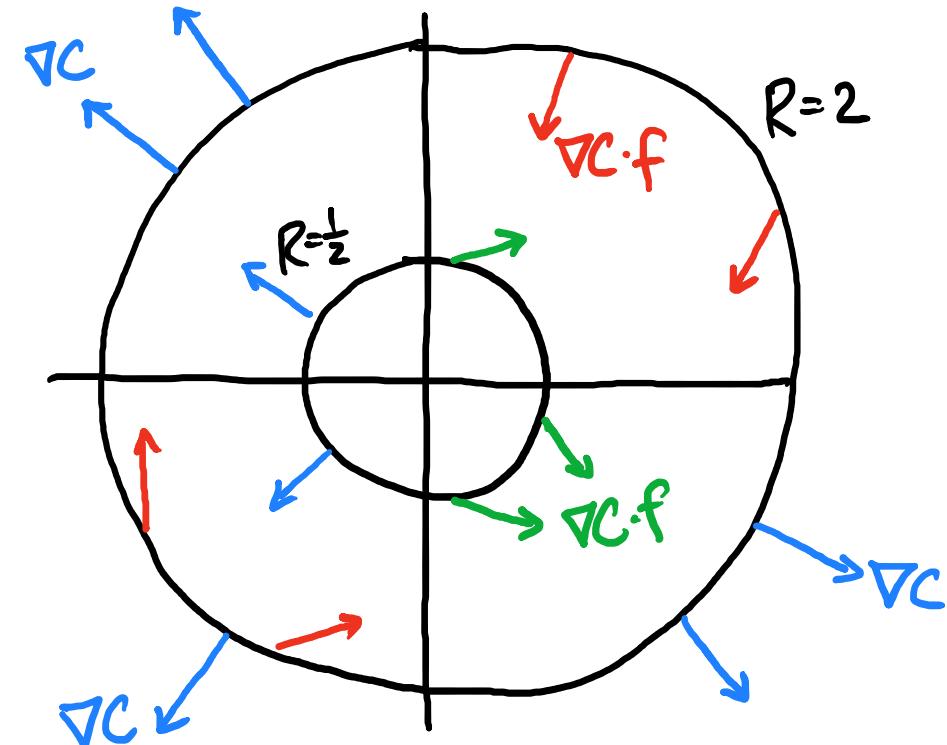
Consider $C(x_1, x_2) = x_1^2 + x_2^2 \rightarrow C=R^2$ defines a circle of radius R

$\hookrightarrow \nabla C = \left(\frac{\partial C}{\partial x_1}, \frac{\partial C}{\partial x_2} \right)$ points radially outward

(direction of greatest rate of change)

$$\begin{aligned} \nabla C \cdot f &= 2x_1 f_1(x_1, x_2) + 2x_2 f_2(x_1, x_2) \\ &= 2(x_1^2 + x_2^2) [1 - (x_1^2 + x_2^2)] \\ &= \begin{cases} < 0 & \text{if } x_1^2 + x_2^2 > 1 \\ > 0 & \text{if } x_1^2 + x_2^2 < 1 \end{cases} \end{aligned}$$

\Rightarrow All trajectories stay inside donut \Rightarrow Limit Cycle
(since no eq pt)



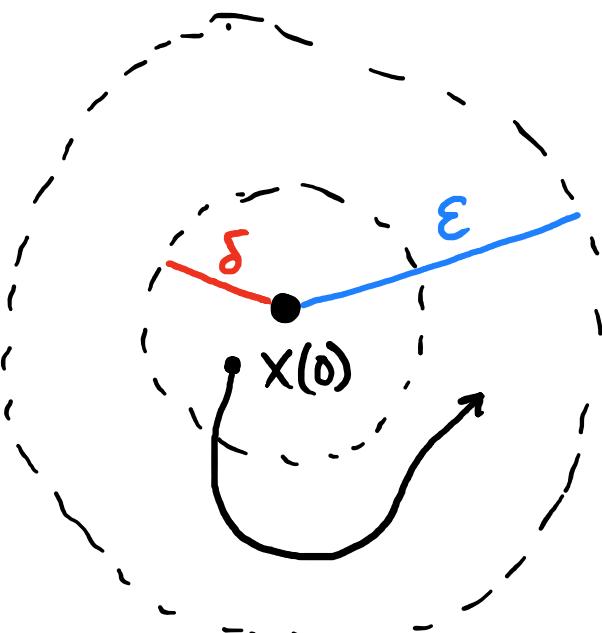


Equilibrium is

STABLE

for any $\epsilon > 0$ there exists
a $\delta > 0$ such that:

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \quad \forall t \geq 0$$



**ASYMPTOTICALLY
STABLE**

$$\lim_{t \rightarrow \infty} x(t) = 0$$

**GLOBALLY ASYMPTOTICALLY
STABLE**

asymptotically stable for
all initial conditions

$$x(0) = x_0 \in \mathbb{R}^n$$

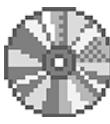
**EXPONENTIALLY
STABLE**

$$\|x(t)\| \leq k e^{-\lambda t} \|x_0\|$$

for some $k > 0, \lambda > 0$

- Linear systems have one equilibrium \Rightarrow Local = Global
- Linear systems: asymptotic stability \Rightarrow exponential stability

Nonlinear Stability (Theorem)



$\dot{x} = f(x)$ with $x^* = 0$ an equilibrium; $\dot{\delta x} = A\delta x$ its linearization at x^*

- ① if all eigenvalues of A have negative real part $\Rightarrow x^*$ is exponentially stable ^{locally}
- ② if any eigenvalue of A has positive real part $\Rightarrow x^*$ is unstable
- ③ if any eigenvalue of A has zero real part $\Rightarrow x^*$ may be

stable	asymptotically stable
unstable	

 $\Rightarrow x^*$ cannot be exponentially stable