isomorphism
permutation matrix
similarity

SYSM 6302

CLASS 16

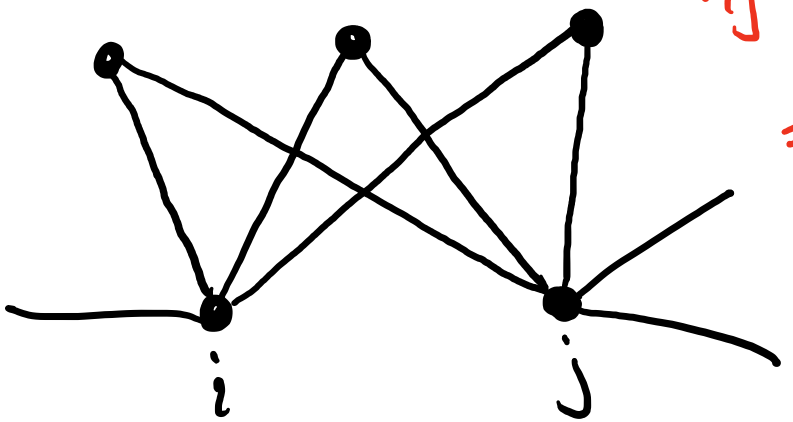
Similarity

Two nodes can be similar because they

Connect to the same nodes

$$n_{ij} = \sum_{k=1}^{n} A_{ik} A_{jk} = \sum_{k=1}^{n} A_{ik} A_{kj} = [A^2]_{ij}$$

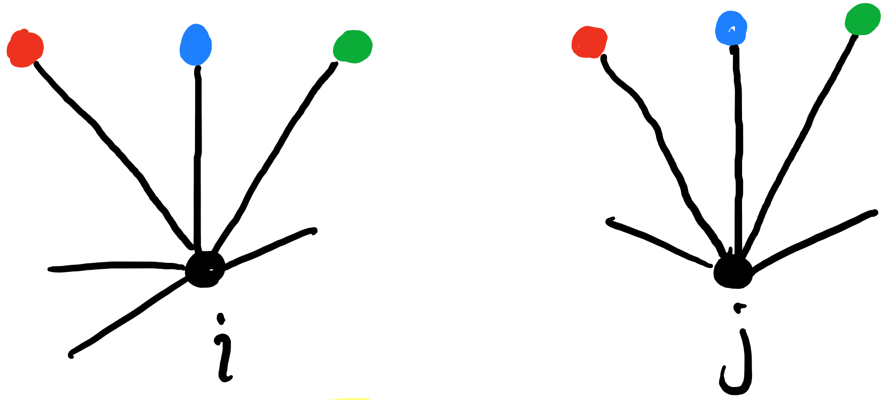
undirected graph



Structural Equivalence

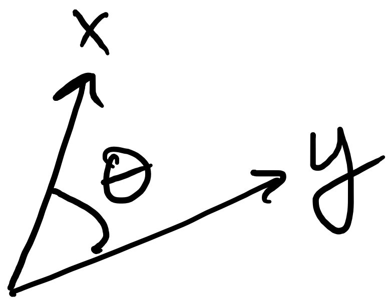
(aka Revenge of Cocitation)

Connect to different nodes, but that are similar



Regular Equivalence

Cosine Similarity



Similar vectors point in the same direction

$$x \cdot y = |x||y| \cos\theta$$

$$\cos\theta = \frac{x \cdot y}{|x||y|}$$

Let $x =$ i^{th} row of A

$y =$ j^{th} row of A

Connections to node i

Connections to node j

$$A_{ij} \in \{0,1\} \Rightarrow A_{ij} = (A_{ij})^2$$

$$0 \leq \sigma_{ij} \leq 1$$

$$\sigma_{ij} = \cos\theta = \frac{x \cdot y}{|x||y|} = \frac{\sum_{k=1}^{n} A_{ik} A_{kj}}{\sqrt{\sum_k A_{ik}^2}\sqrt{\sum_k A_{jk}^2}} = \frac{n_{ij}}{\sqrt{\sum_k A_{ik}}\sqrt{\sum_k A_{jk}}} = \frac{n_{ij}}{\sqrt{k_i k_j}}$$

Pearson Correlation Coefficient for Similarity

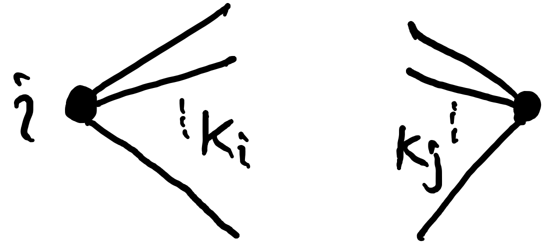
Compare n_{ij} to the number of common neighbors expected at random

$$\sigma_{ij} = n_{ij} - \frac{k_i k_j}{n} = \sum_{l=1}^{n} A_{il}A_{jl} - n \underbrace{\frac{k_i}{n}}_{=\langle A_i \rangle} \underbrace{\frac{k_j}{n}}_{=\langle A_j \rangle} = \sum_{l=1}^{n} \left(A_{il}A_{jl} - \langle A_i \rangle \langle A_j \rangle \right)$$

average of elements in row i

$$= \sum_{l=1}^{n} \left(A_{il} - \langle A_i \rangle \right)\left(A_{jl} - \langle A_j \rangle \right)$$

$$= \mathrm{cov}(A_i, A_j)$$



$\frac{k_i}{n-1} \approx \frac{k_i}{n}$: probability of connecting node i to any other node at random

k_j : number of "tries" to connect i to j

$$r_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}}, \quad -1 \le r_{ij} \le 1$$

Regular Equivalence

similarity of nbrs of i and j

matrix!

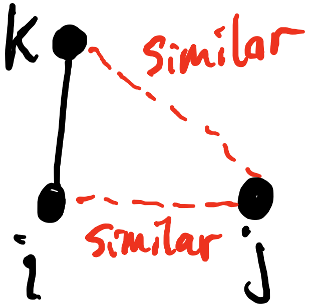
$$\sigma_{ij} = \alpha \sum_{k=1}^{n} A_{ik} \sum_{l=1}^{n} A_{jl}\, \sigma_{kl} \quad \Rightarrow \quad \sigma = \alpha A \sigma A$$

sum over nbrs of i

sum over nbrs of j

nodes are not similar to themselves!

$$\sigma_{ij} = \alpha \sum_{k=1}^{n} A_{ik} \sum_{l=1}^{n} A_{jl}\, \sigma_{kl} + \delta_{ij} \quad \Rightarrow \quad \sigma = \alpha A \sigma A + I$$



$$\sigma_{ij} = \alpha \sum_{k=1}^{n} A_{ik}\, \sigma_{kj} + \delta_{ij} \quad \Rightarrow \quad \sigma = \alpha A \sigma + I$$

$$\Rightarrow \sigma = (I - \alpha A)^{-1}$$

Alternate definition

Isomorphism - a one-to-one mapping between graphs.

Graphs G_1 & G_2, $\phi: V(G_1) \to V(G_2)$ is an isomorphism

verticies

if vertices $u, v \in G_1$ are adjacent $\Rightarrow \phi(u), \phi(v) \in G_2$ are adjacent

→ Two graphs are isomorphic if an isomorphism exists

→ ϕ^{-1} is the inverse isomorphism $\phi^{-1}: V(G_2) \to V(G_1)$

→ Necessary conditions: same # nodes, # edges, degree sequence/distribution

→ Isomorphism of G to itself: automorphism (a strict notion of similarity)

Permutation Matrix – square matrix formed by swapping rows of the identity matrix.

Element in first spot goes to 3rd spot

$$\pi(1,2,3,4) = (2,4,1,3) \implies P = \begin{array}{c} \\ 1^{st} \\ 2^{nd} \\ 3^{rd} \\ 4^{th} \end{array} \begin{array}{c} \begin{array}{cccc} 1^{st}\text{ element} & 2^{nd}\text{ element} & 3^{rd} & 4^{th} \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

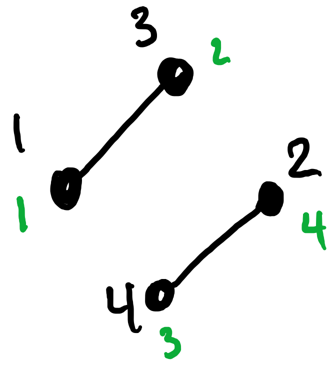
element in 2^{nd} spot goes to 1^{st}

→ P is orthogonal, i.e., $P^{-1} = P^T$

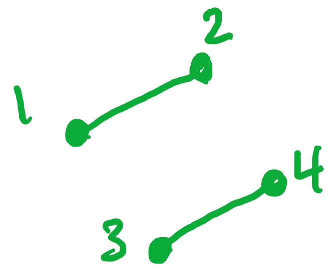
→ A permutation matrix defines an isomorphism in matrix form (on the adjacency)

↳ G_1 & G_2 are isomorphic $\iff$ there exists a permutation matrix, P, such that $PA_1P^T = A_2$

$$\pi(1,2,3,4) = (1,3,4,2) \qquad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

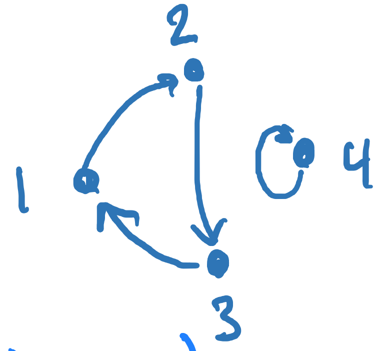


$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$



$$A_2 = PA_1P^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_1 P^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



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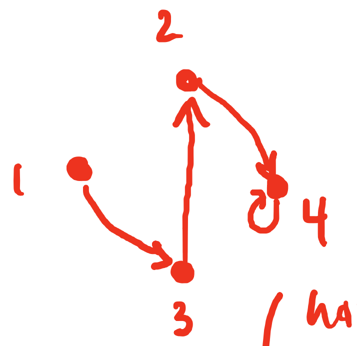

(has no meaning)