diffusion

Laplacian

consensus

SYSM 6302

CLASS 20

Processes on Networks (Dynamics & Control)

→ A state that evolves over time

(everything we need to know to describe the system

follows a rule to find the state at the next moment in time

differential or difference equation

→ The state typically describes the nodes of the network, but sometimes represents edges.

→ The topology of the network (the location of edges between nodes), indicates which nodes/states have an effect on other nodes/states

→ In simple models, nodes pass their state to their neighbors. More sophisticated models have nodes pass some function of their state, or have a way to preserve their own state

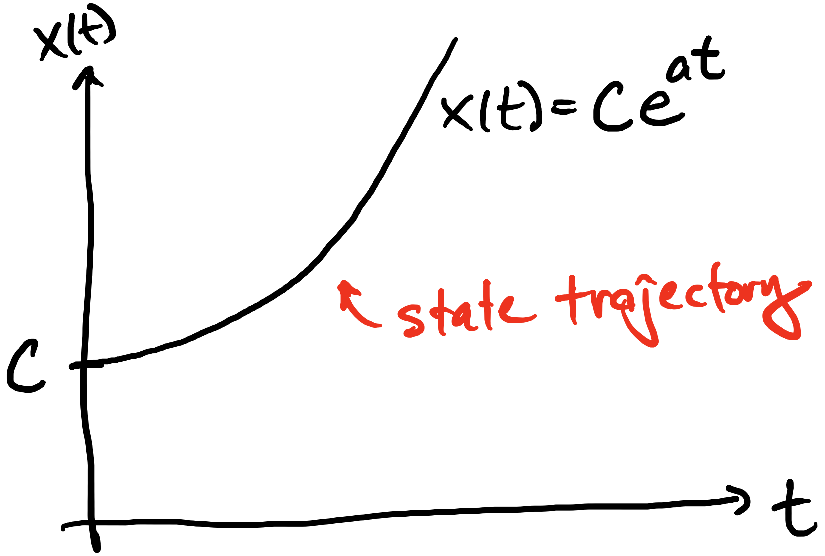
Solutions to differential Equations

$$\dot{x}(t) = a\,x(t) \Rightarrow \frac{dx}{dt} = a\,x \Rightarrow \int \frac{dx}{x} = \int a\,dt \Rightarrow \ln(x) = at + \tilde{c}$$

$x(t) \in \mathbb{R}$ $\quad a \in \mathbb{R}$

$$\Rightarrow x(t) = e^{at+\tilde{c}} = C e^{at}$$

$$x(0) = Ce^{a \cdot 0} = C$$



General Network Dynamics

Note: we are assuming an undirected graph

$$\dot{x}_i = \underbrace{f_i(x_i)}_{\text{intrinsic dynamics of the node itself}} + \sum_{j=1}^{n} \underbrace{A_{ij}}_{\text{A_{ij} term only includes contribution from node j if there is an edge between i \& j}} \overbrace{g_{ij}(x_i, x_j)}^{\text{due to connectivity}}$$

$\dot{x}_i$: the time rate of change of the state of node i

→ often we assume the same form for all nodes:

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{n} A_{ij}\, g(x_i, x_j)$$

Diffusion

→ Physical diffusion follows a pressure differential

→ Rate of exchange between nodes $\sim$ $x_j - x_i$

if $x_i = x_j$ nothing flows

if $x_i > x_j$ flow is out, so negative

no intrinsic dynamics

$$\dot{x}_i = \sum_{j=1}^{n} A_{ij}\, C(x_j - x_i)$$

← simple passing of state ← conservation of "state"

$$= C \sum_j A_{ij} x_j - C x_i \underbrace{\sum_j A_{ij}}_{\text{sum of row } i \,=\, k_i}$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases}$$

$$= C \sum_j A_{ij} x_j - C x_i k_i = C \sum_j \left(A_{ij} - \delta_{ij} k_i\right) x_j$$

$$\dot{x}_i = c \sum_j \left(A_{ij} - \delta_{ij} k_i\right) x_j$$

$$\dot{x} = c \underbrace{(A - D)}_{= -L} x \quad , \quad D = \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_n \end{pmatrix}$$

← the Laplacian

$$\frac{dx}{dt} + cLx = 0$$

← Analogous to diffusion of a gas

time derivative

spatial derivative

$$\frac{dx}{dt} + c\nabla^2 x = 0$$

Properties of the Laplacian

recall $a_i(t) \in \mathbb{R}$

$v_i \in \mathbb{R}^n$

① Let $x(t) = \sum_{i=1}^{n} a_i(t)\, v_i$ ← eigenvectors of L

$$\frac{dx}{dt} + cLx = 0 \;\longrightarrow\; \sum_i \frac{da_i}{dt} v_i + cL\sum a_i v_i = 0$$

$$\sum_i \left[\frac{da_i}{dt} + cLa_i\right] v_i = 0$$

$$\sum_i \left[\frac{da_i}{dt} + c\lambda_i a_i\right] v_i = 0$$

$$\Big\langle \sum_i \left[\frac{da_i}{dt} + c\lambda_i a_i\right] v_i ,\, v_j \Big\rangle = \langle 0, v_j \rangle$$

since L is symmetric

$$\langle v_i, v_j \rangle = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$a_i(t) = a_i(0)\, e^{-c\lambda_i t} \;\longleftarrow\; \frac{da_i}{dt} + c\lambda_i a_i = 0 \quad \text{for } i = 1, 2, \dots, n$$

② It is possible to show that L is positive semi-definite

$\Rightarrow \lambda_i \geq 0$ (eigenvalues of L are non negative)

①+②: $a_i(t) = a_i(0)\, e^{-c\lambda_i t}$, $\lambda_i \geq 0$ $\rightarrow$ all $a_i(t)$ converge

$\Rightarrow X(t) = \sum_i a_i(t) v_i \longrightarrow$ equilibrium

let $x^* = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ $\Rightarrow$ $L x^* = (D-A)x^* = \left[\begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_n \end{pmatrix} - \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \right] \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\rightarrow \sum_j (D_{ij} - A_{ij}) x_j^* = k_i - A_{i1} - A_{i2} - \dots - A_{in} = 0 \quad \forall i$

$\Rightarrow x^*$ is the eigenvector of L with $\lambda = 0$

Thus $\quad x(t) = \sum_{i=1}^{n} a_i(t)\, v_i \longrightarrow x^* \quad$ as $t \to \infty$

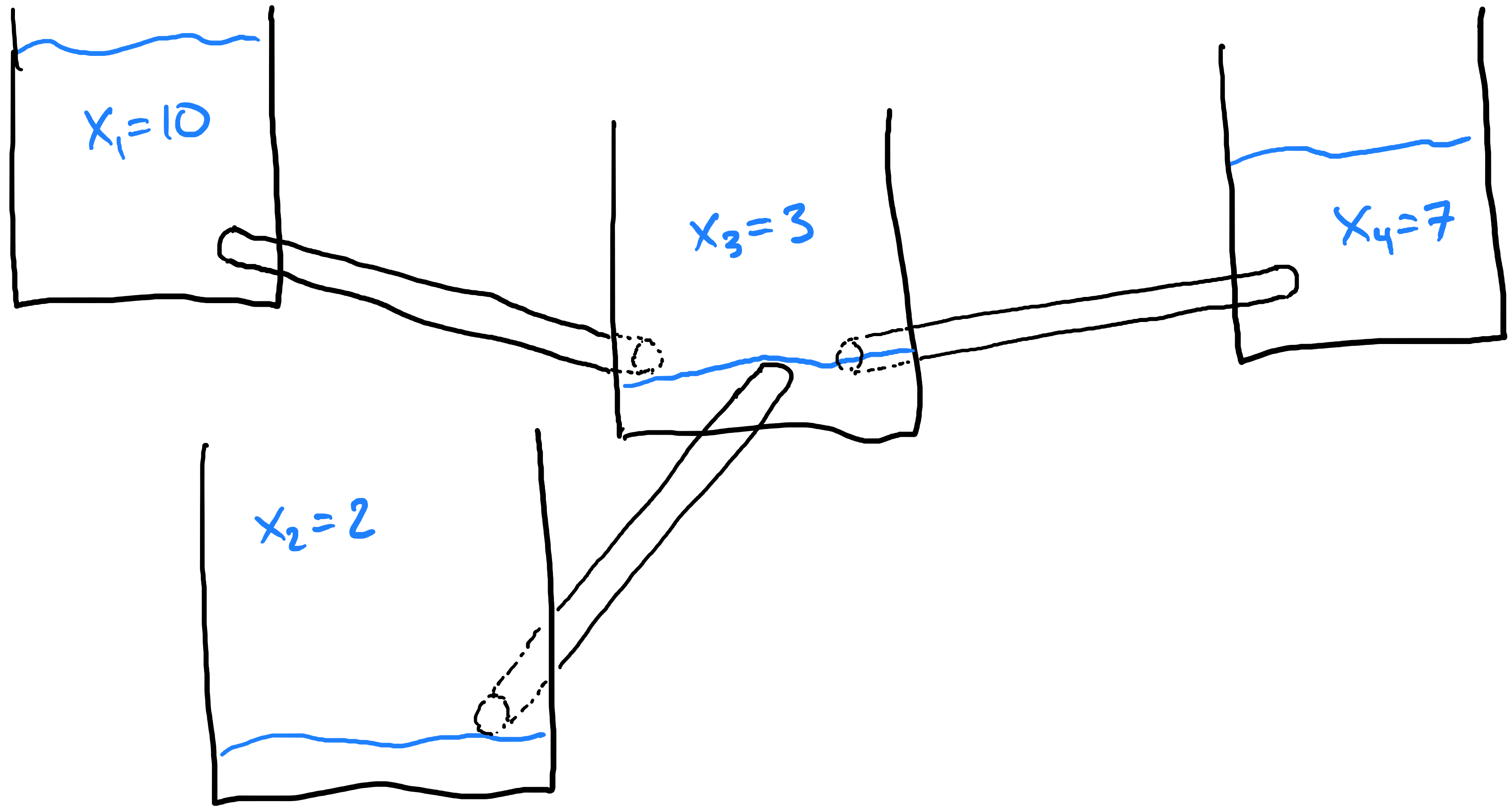
$$a_i(t) \longrightarrow 0 \qquad \forall \;\; \lambda_i > 0$$

$$a_i(t) \longrightarrow a_i(0) \qquad \forall \;\; \lambda_i = 0$$

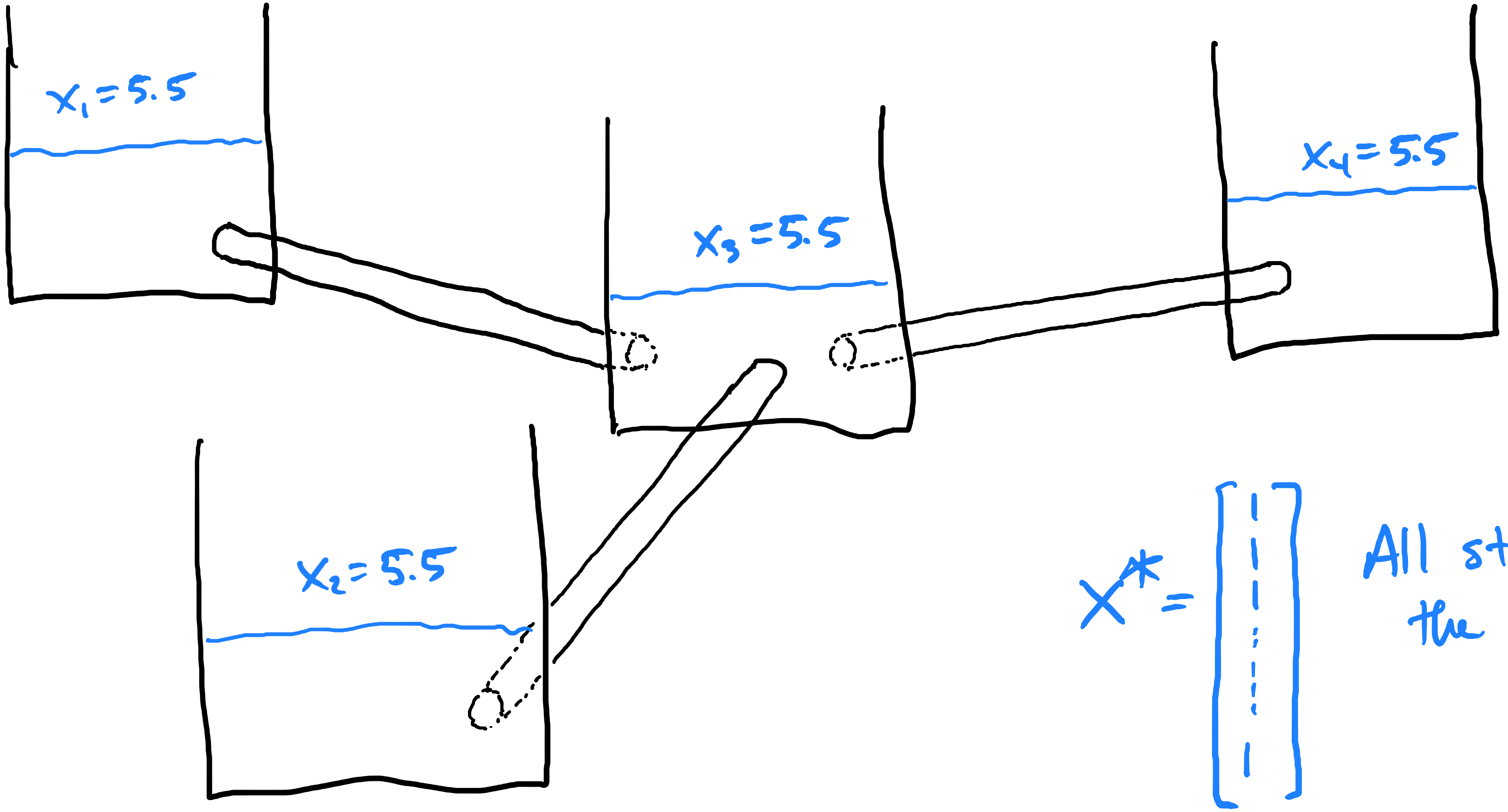
③ The multiplicity of $\lambda = 0$ is equal to the number of components of A.

$$L x_1^* = (D - A) x_1^* = \left[\left(\begin{array}{cccc:cccc} k_1 & & & & & & & \\ & \ddots & & & & & & \\ & & & k_l & & & & \\ \hdashline & & & & k_{l+1} & & & \\ & & & & & \ddots & & \\ & & & & & & & k_n \end{array} \right) - \left(\begin{array}{c:c} A_1 & 0 \\ \hdashline 0 & A_2 \end{array} \right) \right] \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \hdashline 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \hdashline 0 \\ \vdots \\ 0 \end{bmatrix}$$

we can find a permutation to order the nodes by component



$x_1=10$
$x_3=3$
$x_4=7$
$x_2=2$

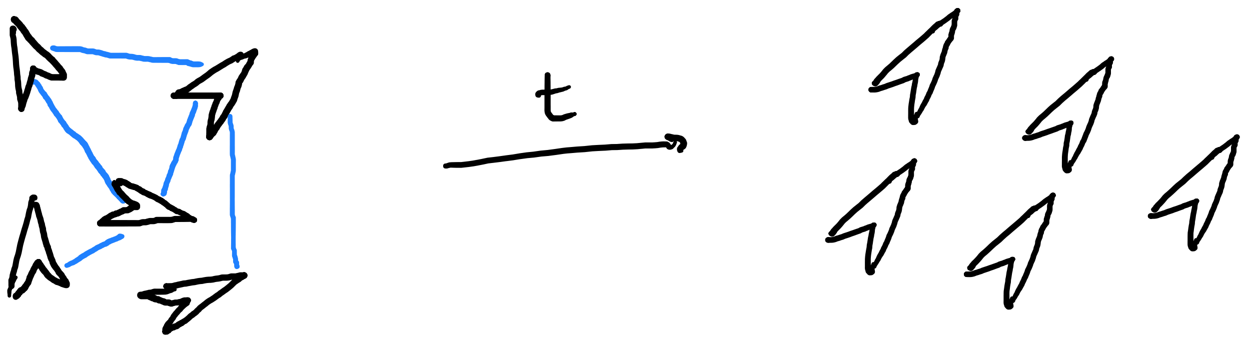


$$x^* = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

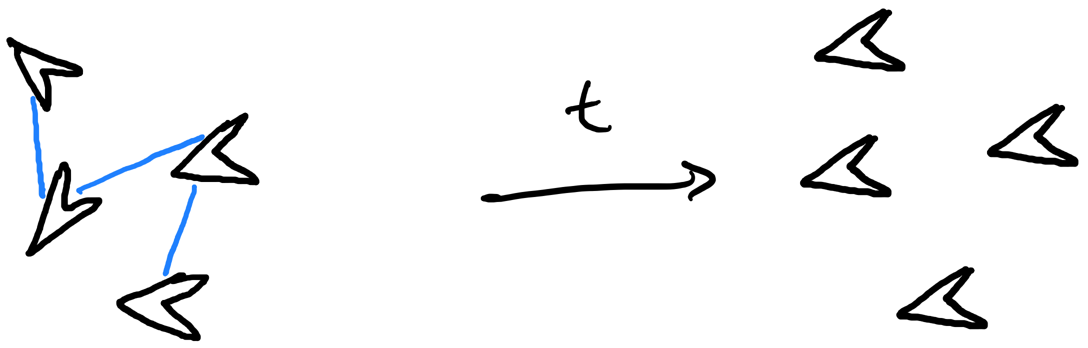
All states are the same!

Consensus

$$\dot{x}_i = \sum_{j=1}^{n} A_{ij}(x_j - x_i) \Rightarrow \dot{x} = -Lx$$



Directed graphs require having a directed spanning tree to reach consensus.



For switching topologies, the union of graphs must be jointly connected (undirected) or have a minimum spanning tree (directed) in some finite time interval.

Designation of leader / follower