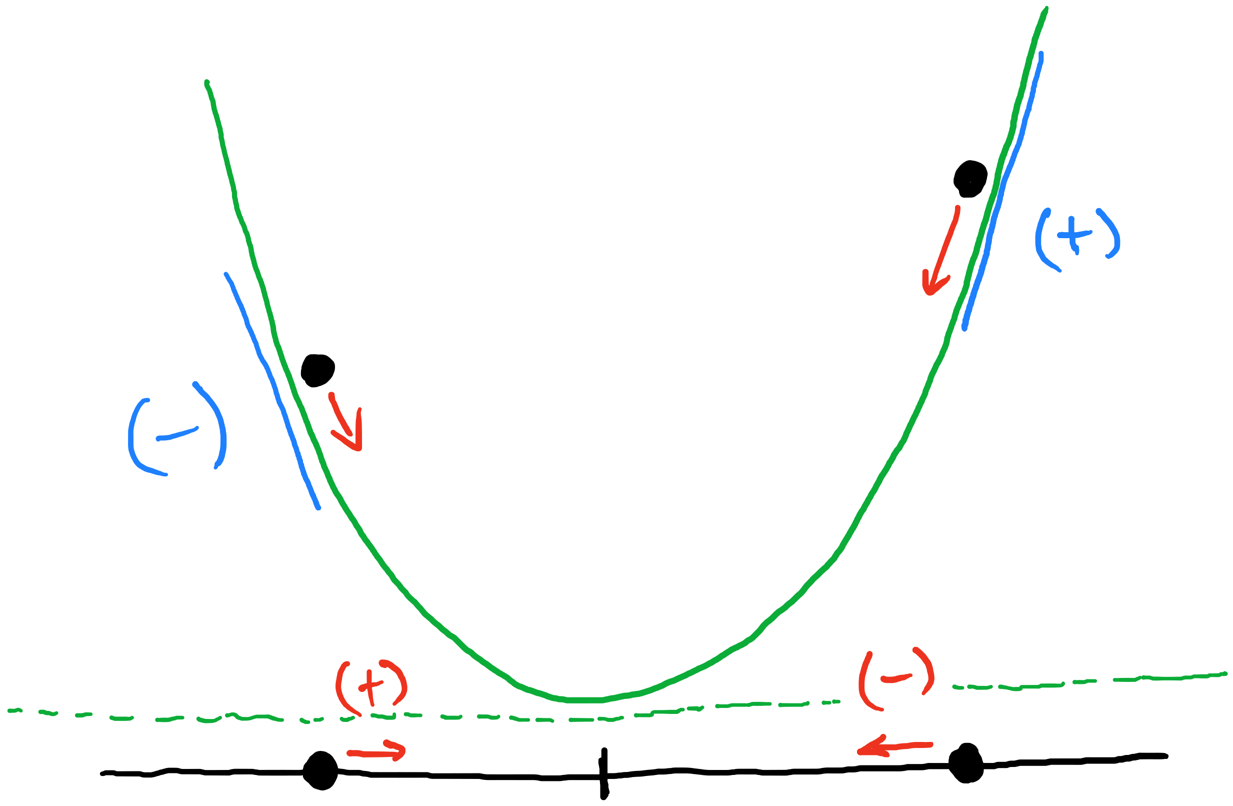
Lyapunov stability

SYSM 6302

CLASS 26

Intuition



Pick a (nonnegative) function such that the value decreases along the trajectories of the system

motion × slope < 0

Consider a mass (m) acted on by a dissipative force (e.g., drag, friction)

$$m\ddot{x} = -\overset{>0}{b}\dot{x} \quad \longrightarrow \quad m\dot{v} = -bv$$

$\leftarrow$ $x=0,\ v=0$ is an equilibrium

Kinetic Energy: $E(t) = \frac{1}{2}mv^2$

$$\dot{E}(t) = mv\dot{v} = -bv^2 < 0$$

Since $E(t) \geq 0$ and $\dot{E}(t) < 0$, ($E(t)$ is positive (or zero) and decreasing)

$$E(t) \to 0 \quad \text{as} \quad t \to \infty$$

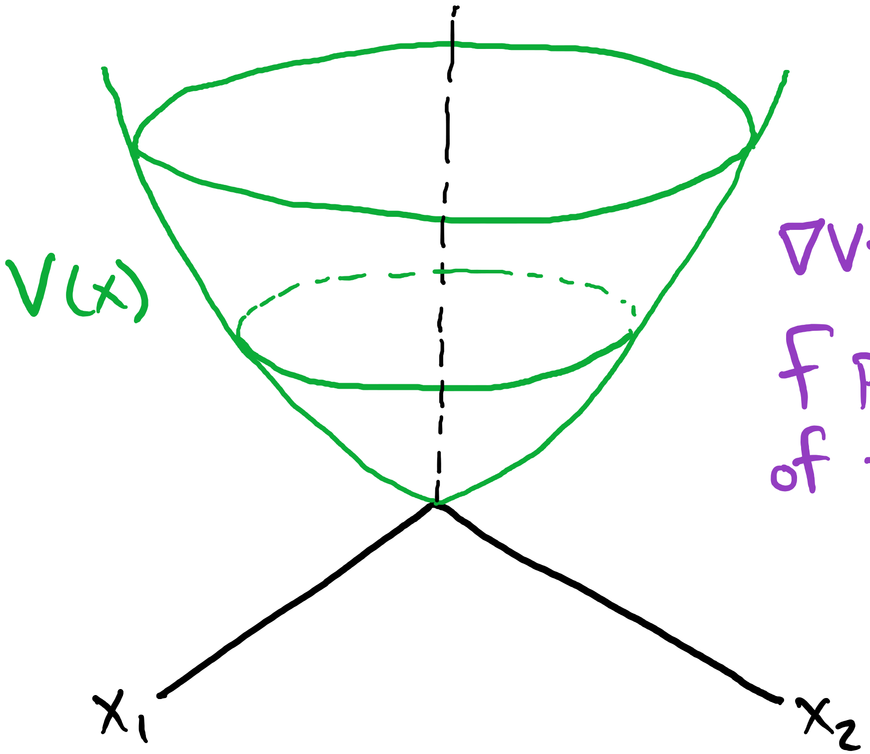
$$\Rightarrow v(t) \to 0 \quad \text{as} \quad t \to \infty$$

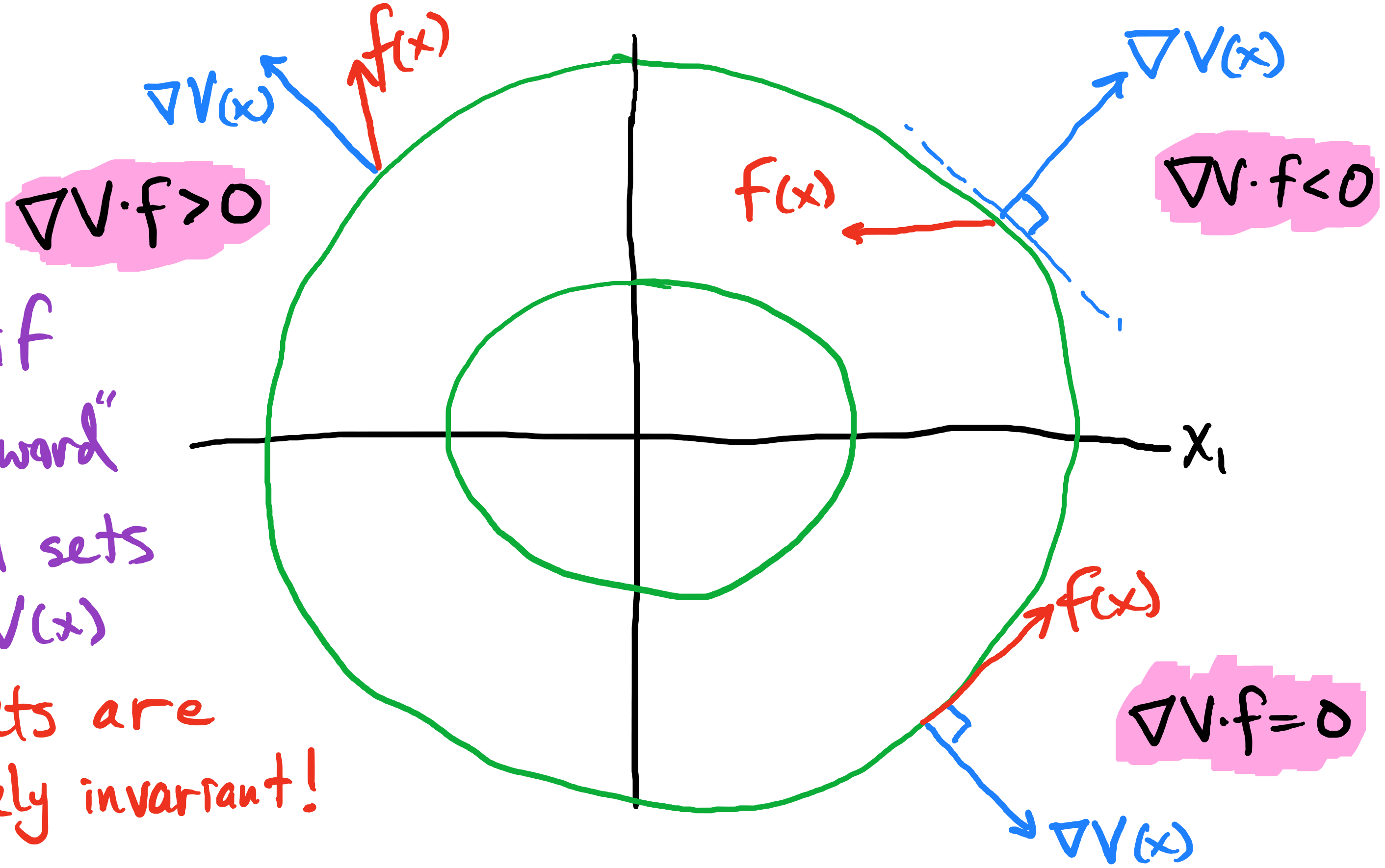
Determined stability without eigenvalues!

Pick a nonnegative scalar function that decreases along the trajectories of $\dot{x} = f(x)$

$V(x)$

$$\dot{V}(x) = \sum_{i=1} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \nabla V(x) \cdot f(x)$$





$\nabla V \cdot f < 0$ if f points "inward" of the level sets of $V(x)$

Level sets are positively invariant!

Stability in the sense of Lyapunov

Let → $x^* = 0$ be an equilibrium point of $\dot{x} = f(x)$

→ $V(x)$ be a continuously differentiable function

If (1) $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$

(2) $\dot{V}(x) \leq 0$

Then x^* is stable.

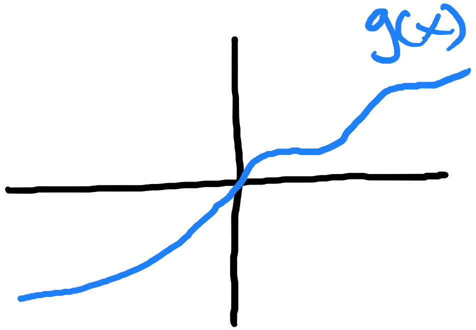
If (2) $\dot{V}(x) < 0$, then x^* is asymptotically stable

If (2) $\dot{V}(x) \leq 0$ and $\dot{V}(x) = 0$ only at $x = x^*$ $\Rightarrow$ x^* is asymptotically stable

Example: $\dot{x} = -x$ and pick $V(x) = x^2 > 0$

$\dot{V} = 2x\dot{x} = -2x^2 < 0 \quad \Rightarrow \quad x^* = 0$ is asymptotically stable

Example: $\dot{x} = -g(x)$ with $g(0) = 0$ & $x\,g(x) > 0$:

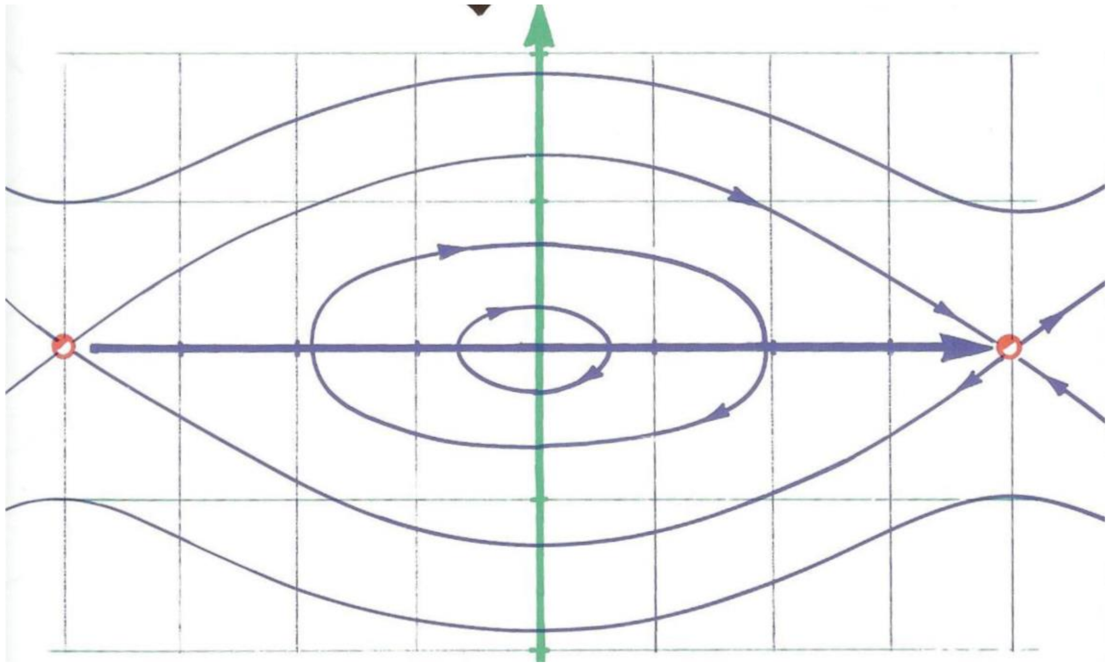


Try $V(x) = \frac{1}{2}x^2 > 0$

$\dot{V} = x\dot{x} = -x\,g(x) < 0$

Pendulum

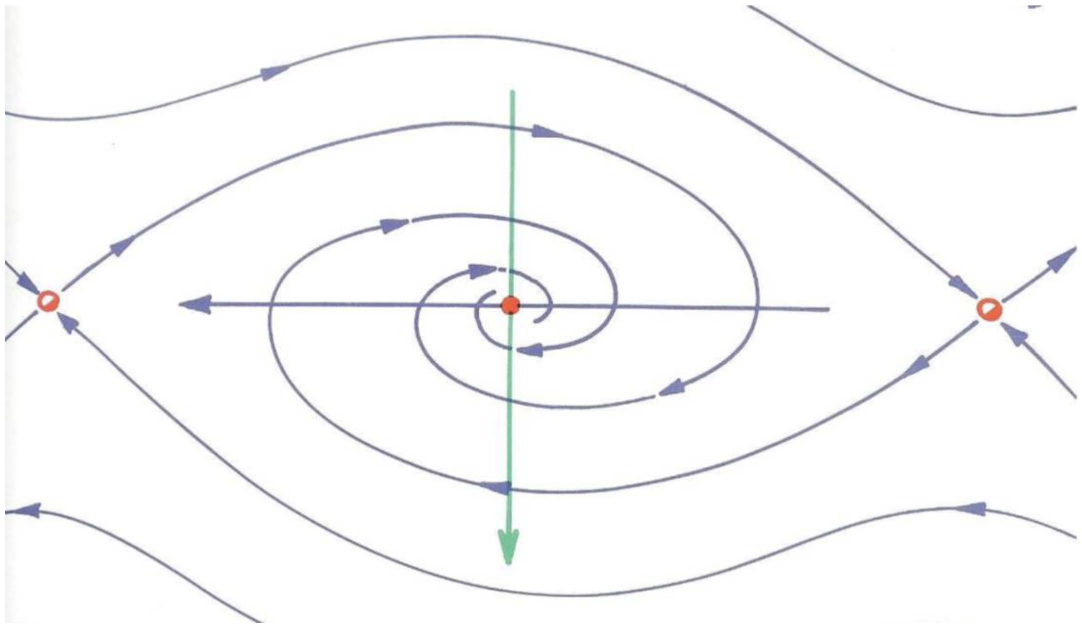
→ What we could tell from linearization and eigenvalues



Without Damping

Center ⇒ stability is unknown

↖ not hyperbolic equilibrium



With Damping

Stable Focus ⇒ Asymptotic Stability

Pendulum $\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$; with friction $\ddot{\theta} + B\dot{\theta} + \frac{g}{\ell}\sin\theta = 0$

$$E(t) = \underbrace{\tfrac{1}{2} m\ell^2\dot{\theta}^2}_{\text{kinetic}} + \underbrace{mg\ell(1-\cos\theta)}_{\text{potential}}$$

$$\dot{E} = m\ell^2\dot{\theta}\ddot{\theta} + mg\ell\sin\theta\cdot\dot{\theta} = -mg\ell\sin\theta\cdot\dot{\theta} + mg\ell\sin\theta\cdot\dot{\theta} = 0$$

$\Rightarrow$ E is a Lyapunov function and system is stable near $\theta = x_1 = 0$, $\dot{\theta} = x_2 = 0$.

the linearized system is a center, so we were not able to claim stability until now!

$\dot{E} \equiv 0$ comes from the fact that $\nabla E \cdot f = 0$

↳ f points tangent to the level sets of E

$\Rightarrow$ With friction $\dot{E} = -m\ell^2 B\dot{\theta}^2 \leq 0$ ← this implies the origin is stable, but not asymptotically stable

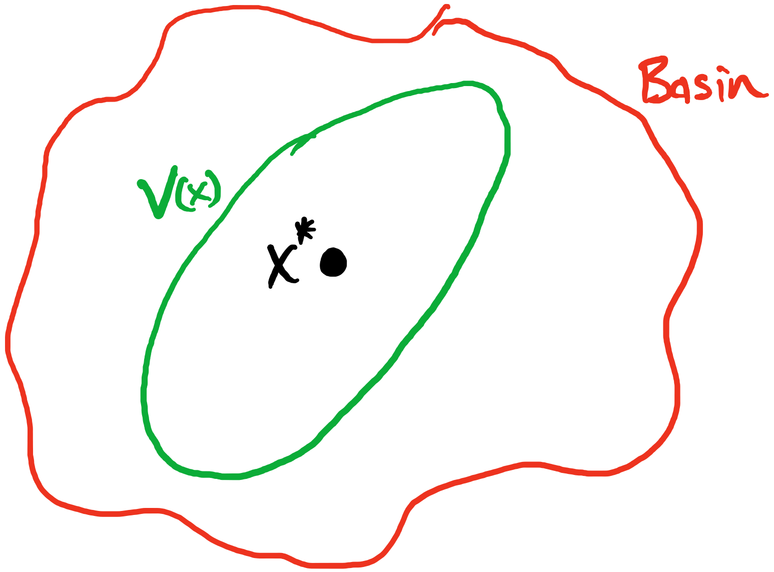
↳ Lyapunov provides sufficient conditions & depends on choice of $V(x)$!

Basin / Domain / Region of Attraction

$$B = \{ y \in \mathbb{R}^n \mid \lim_{t\to\infty} x(t) = x^*,\ x(0) = y \}$$

→ The set of initial states that converge to the equilibrium x^*

→ Lyapunov functions can provide estimates →



→ If $B = \mathbb{R}^n \Rightarrow x^*$ is globally asymptotically stable

↳ Why can there only be one equilibrium point?

If (1) $V(0) = 0$, $V(x) > 0 \quad \forall x \neq 0$

(2) $V(x) \to \infty$ as $\|x\| \to \infty$ (Radially unbounded)

(3) $\dot{V}(x) < 0 \quad \forall x \neq 0$

Then $x^* = 0$ is globally asymptotically stable

Back to Consensus!

Assuming $A = A^T$

Recall: $\dot{x}_i = \sum_{j \in N_i} (x_j - x_i) \longrightarrow \dot{x} = -Lx$

Signed edge adjacency matrix $M \in \mathbb{R}^{m \times n}$:

$$M_{ev} = \begin{cases} 1 & v = i \\ -1 & v = j \\ 0 & \text{otherwise} \end{cases}$$

edge (i,j)

$i < j$

$$L = M^T M$$

Since each edge connects two nodes,

$$y = Mx = \left.\begin{pmatrix} \vdots \\ x_i - x_j \\ \vdots \end{pmatrix}\right\} \text{\# of edges}$$

Try $V(x) = \frac{1}{2}x^T x$

showed how to do this previously

$$\dot{V}(x) = x^T \dot{x} = -x^T L x = -x^T M^T M x = -y^T y \leq 0$$

equality only at $y=0$ (the equilibrium)

$\Rightarrow$ $y=0$ is globally asymptotically stable

↳ $x_i - x_j \to 0$ for all $(i,j) \in G$

↳ if G is connected, then $\exists$ an edge adjacent to every vertex