differential equations

linear systems

phase plane portraits

stability

SYSM 6302

CLASS 24

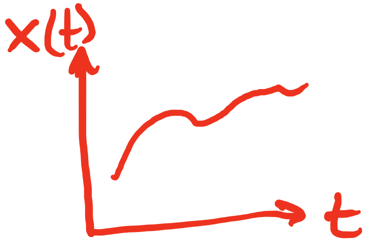
First Order Ordinary Differential Equation

state

vector field

dependent variable

independent variable



$$\frac{d}{dt}x(t) = f(t, x(t)) \xrightarrow{\text{Solve}} x(t) = \cdots$$

ordinary derivative
(not partial derivative)

First order because
the highest derivative is 1

Autonomous Differential Equation

$$\frac{d}{dx}x(t) = f(x(t))$$

no explicit dependence
on independent variable

↳ when independent variable is time: autonomous = time-invariant

System of Autonomous Differential Equations

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

LINEAR System:

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2$$

$$\Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \dot{x} = Ax$$

→ Many physical systems are modeled by linear systems

↳ Newton's laws: $F = ma$

Example of converting a higher order linear differential equation to a linear system:

$$\ddot{y} + \alpha \dot{y} + \beta y = 0 \longrightarrow \text{let } \begin{array}{l} x_1 = y \\ x_2 = \dot{y} \end{array} \longrightarrow \begin{array}{l} \dot{x}_1 = \dot{y} = x_2 \\ \dot{x}_2 = \ddot{y} = -\alpha \dot{y} - \beta y \\ \qquad = -\alpha x_2 - \beta x_1 \end{array}$$

$$\Rightarrow \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\alpha x_2 - \beta x_1 \end{array}$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\beta & -\alpha \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\Rightarrow$ Eigenvalues: $\lambda = \frac{1}{2} p \pm \frac{1}{2}\sqrt{p^2 - 4q}$

$p = \underbrace{a_{11} + a_{22}}_{tr(A)}$, $q = \underbrace{a_{11}a_{22} - a_{12}a_{21}}_{det(A)}$

For now, we assume A has no eigenvalues with zero real part.

↳ only equilibrium $\dot{x} = 0$ is at $x = 0$

Eigenvalues can be:

$p^2 < 4q$: ① complex conjugates: $\lambda = \alpha \pm i\beta$

$p^2 > 4q$: real:

$$\left.\begin{array}{l} \text{② both positive} \\ \text{③ both negative} \\ \text{④ opposite sign} \end{array}\right\} \lambda_1, \lambda_2$$

Note that $x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is always a solution of a linear system and that at $x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\dot{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, thus this point is an equilibrium.

Linear System solutions (Linear, constant coefficients)

Guess a non-trivial solution: $x(t) = e^{\lambda t} v$

$\dot{x} = Ax \implies \lambda e^{\lambda t} v = A e^{\lambda t} v \rightarrow \lambda v = A v$

must hold for all t

eigenvalue! eigenvector!

Solutions are of the form: $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \dots + c_n e^{\lambda_n t} v_n$

$$= \sum_{i=1}^{n} c_i e^{\lambda_i t} v_i$$

things can get a little more complicated, but this is true for all "non-tricky" cases

this should look vaguely familiar to our discussion on diffusion

Phase Portrait: SADDLE ($\lambda_1 < 0$, $\lambda_2 > 0$; both real)

$$X(t) = C_1 e^{\lambda_1 t} V_1 + C_2 e^{\lambda_2 t} V_2$$

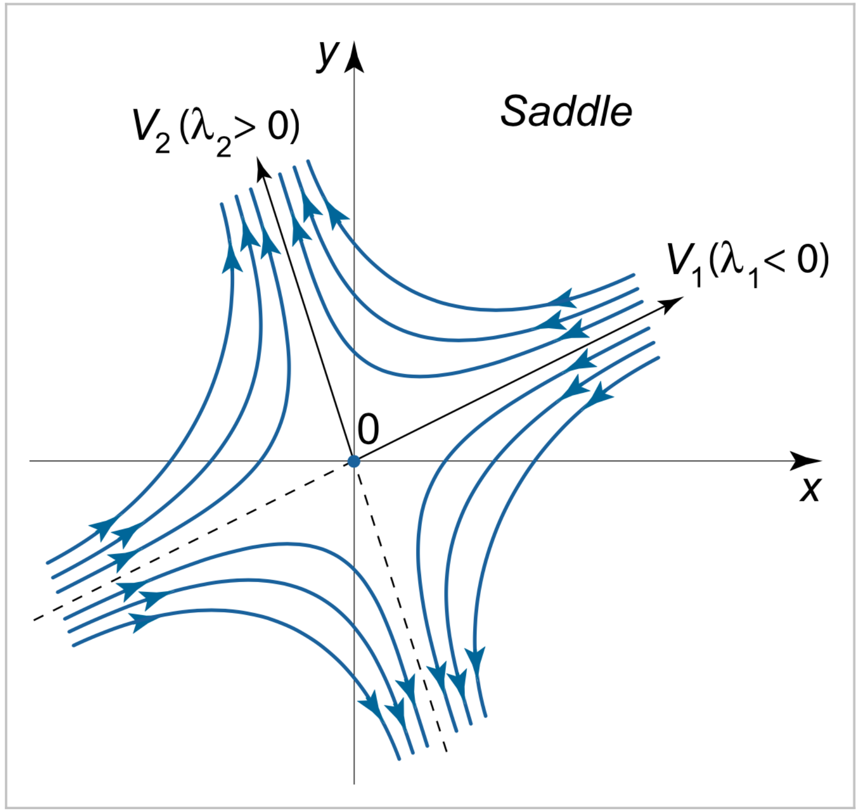
→ Linearly separating the trajectory into the part along V_1 & the part along V_2

→ A trajectory that starts on an eigenvector will stay on that eigenvector

As $t \to \infty$, only initial vectors along V_1 go to $\begin{pmatrix}0\\0\end{pmatrix}$.
All others go to ∞.

↳ the equilibrium is stable only along V_1.

How to plot a phase portait



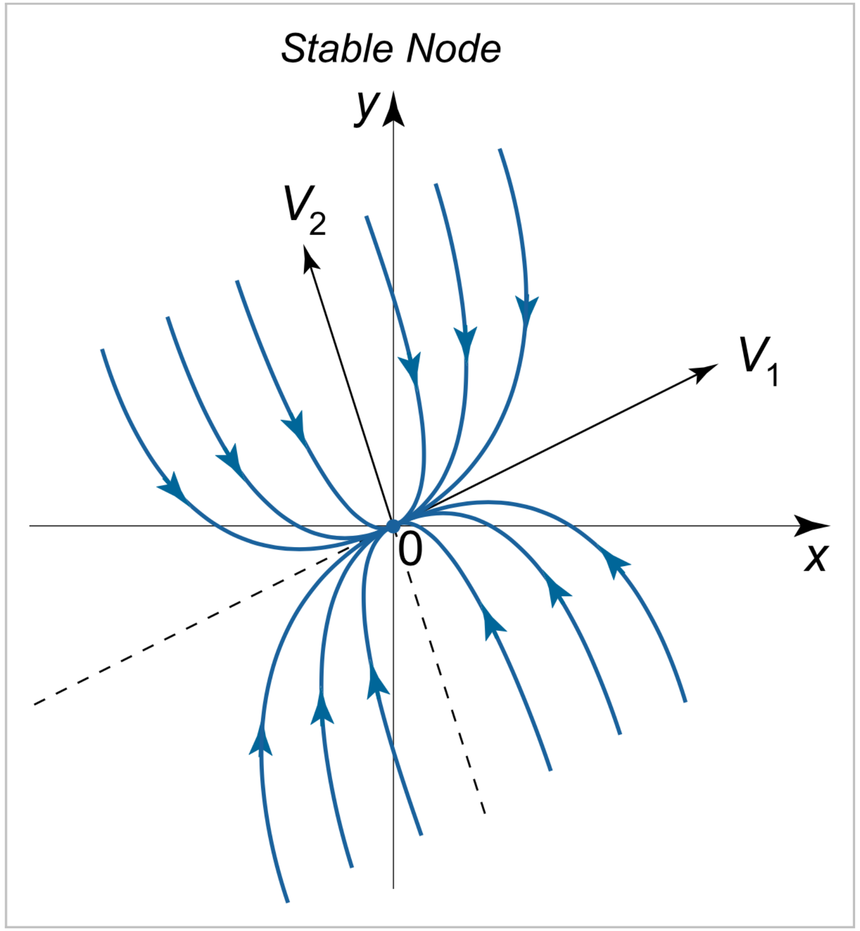
https://www.math24.net/linear-autonomous-systems-equilibrium-points-page-2/#example2

Note
$\vec{x} = \begin{pmatrix}x\\y\end{pmatrix}$ in my borrowed figures

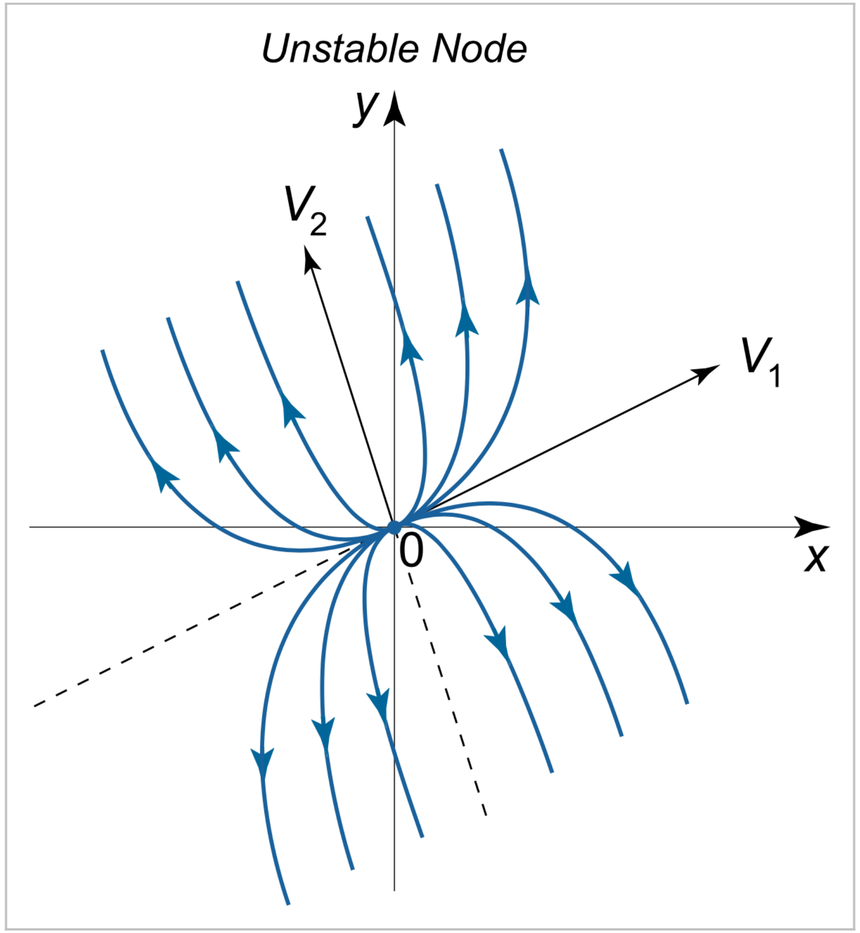
Phase Portrait: NODE ($\lambda_1 \neq \lambda_2$, both real)

$$X(t) = C_1 e^{\lambda_1 t} V_1 + C_2 e^{\lambda_2 t} V_2$$

$\lambda_1, \lambda_2 < 0$, $X(t) \to 0$ as $t \to \infty$



$\lambda_1, \lambda_2 > 0$, $X(t) \to \infty$ as $t \to \infty$



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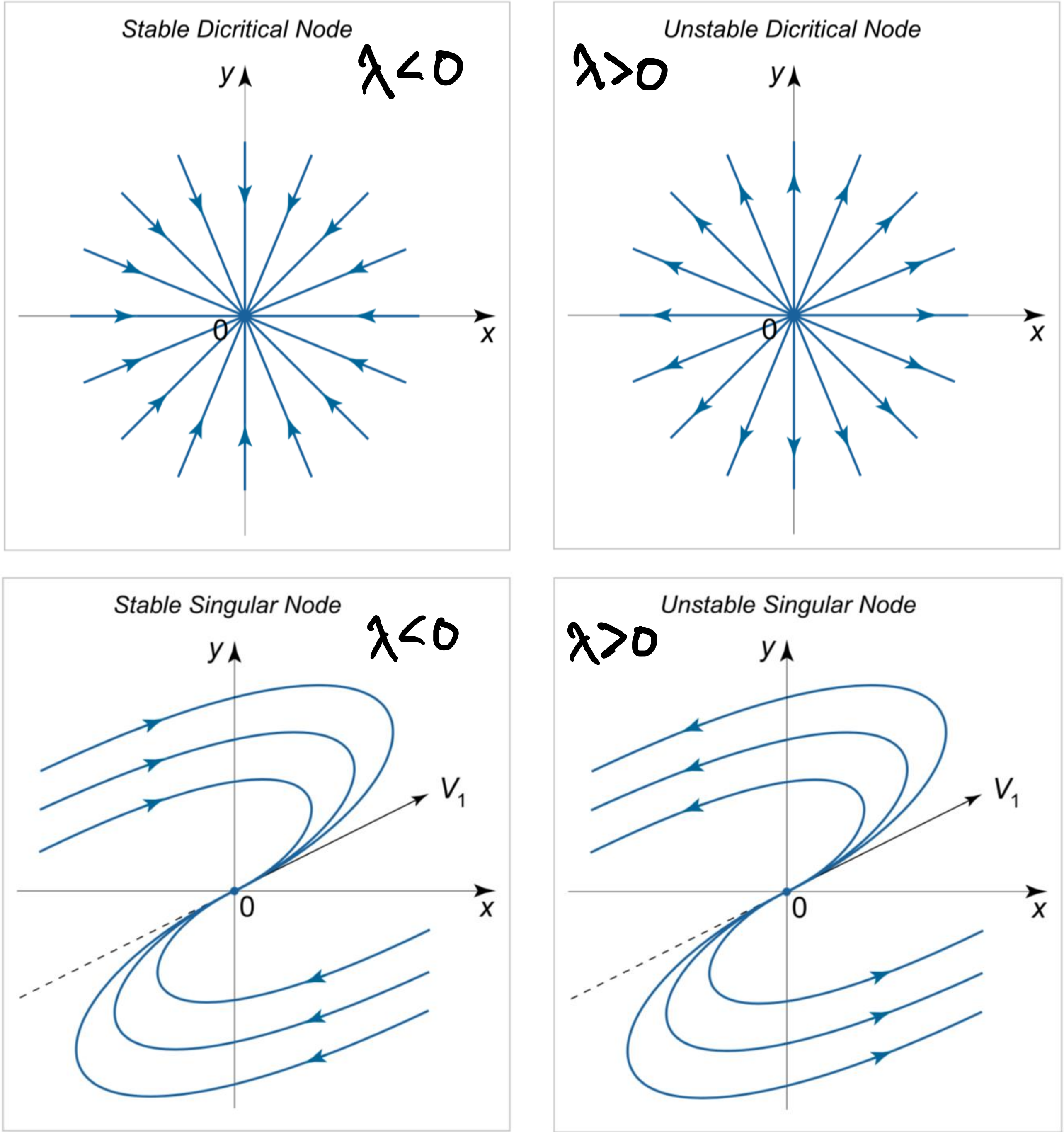
These diagram the case when $|\lambda_2| > |\lambda_1|$ because of the slope of the trajectories

Degenerate Node Phase Portraits

$$\lambda_1 = \lambda_2 = \lambda \neq 0$$



Two types of behavior - WHY?

2

Geometric Multiplicity!

1

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \dots + c_n e^{\lambda_n t} v_n$$

THM: When $\{\lambda_i\}$ are distinct $\Rightarrow$ $\{v_i\}$ are linearly independent

↳ if $\lambda_i = \lambda_j$ ("repeated eigenvalue"), TWO CASES:

← # of repeats: algebraic multiplicity

① λ_i has two linearly independent eigenvectors

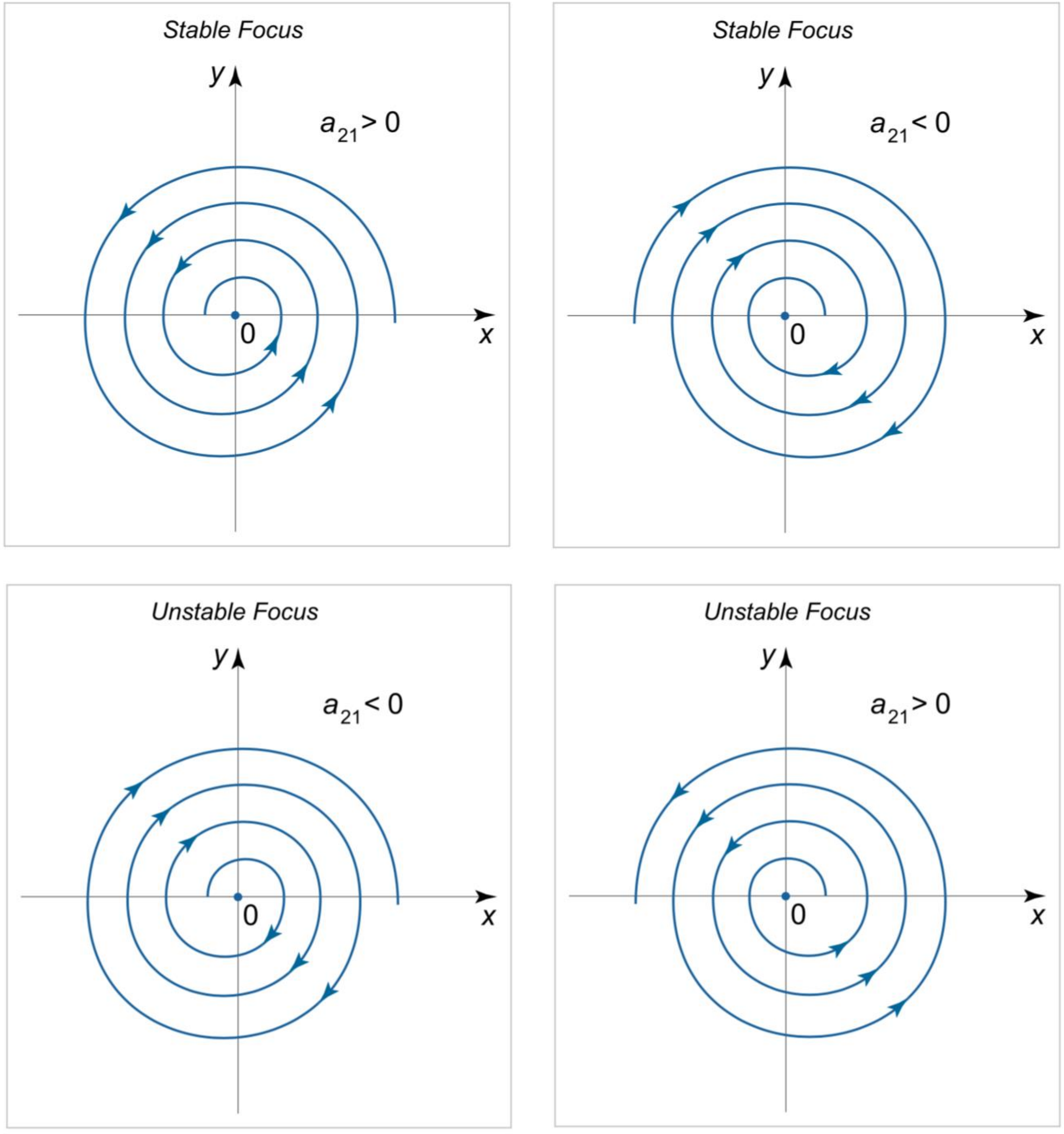
← # of lin. indep. eigenvectors: geometric multiplicity

② λ_i has one linearly independent eigenvector

$\Rightarrow$ When geometric mult < algebraic mult., we "lose" one of the solutions & get solutions of the form:

$$c_1 e^{\lambda_i t} v_1 + c_2 e^{\lambda_i t}(t v_1 + v_2)$$

generalized eigenvector



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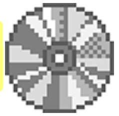
Phase Portrait: Spiral / Focus



→ λ_1, λ_2 complex conjugates

↳ $\lambda_{1,2} = \alpha \pm i\beta$

in general complex

$$x(t) = C\underbrace{e^{\lambda_1 t}}_{\text{Euler Formula}} v_1$$

⋮

$$= e^{\alpha t}(\cos\beta t + i\sin\beta t)(v_a + i v_b)$$

⇒ THM: $Re[x(t)]$ & $Im[x(t)]$ are independent solutions

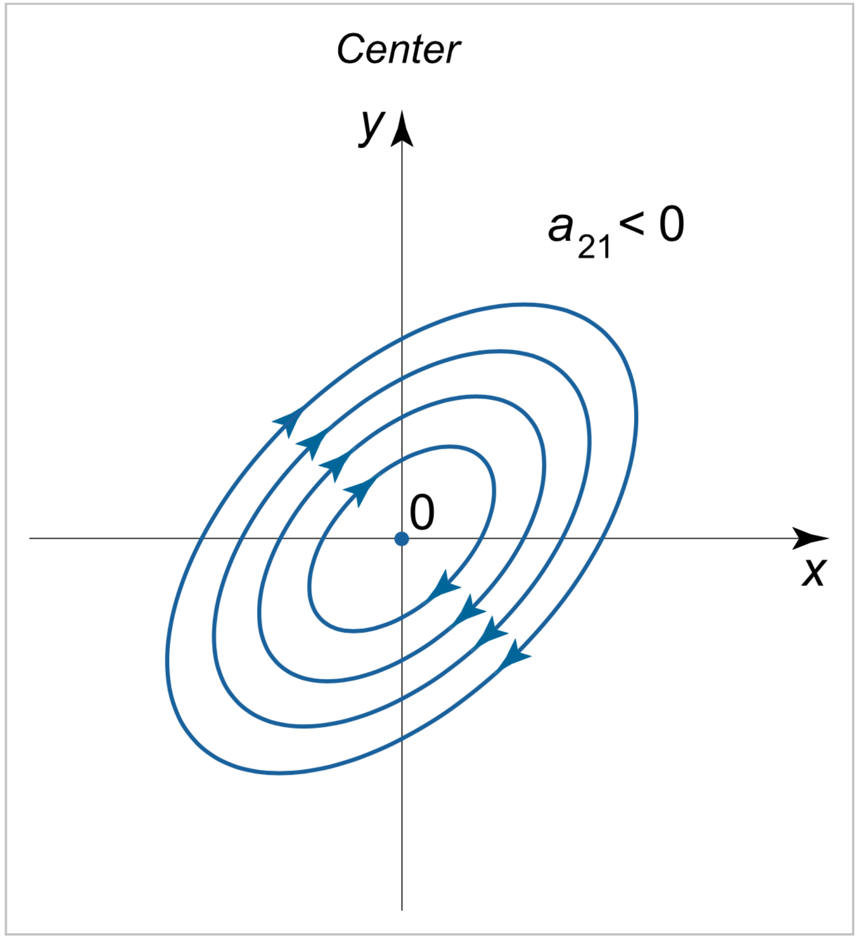
→ $\alpha > 0 \Rightarrow$ unstable

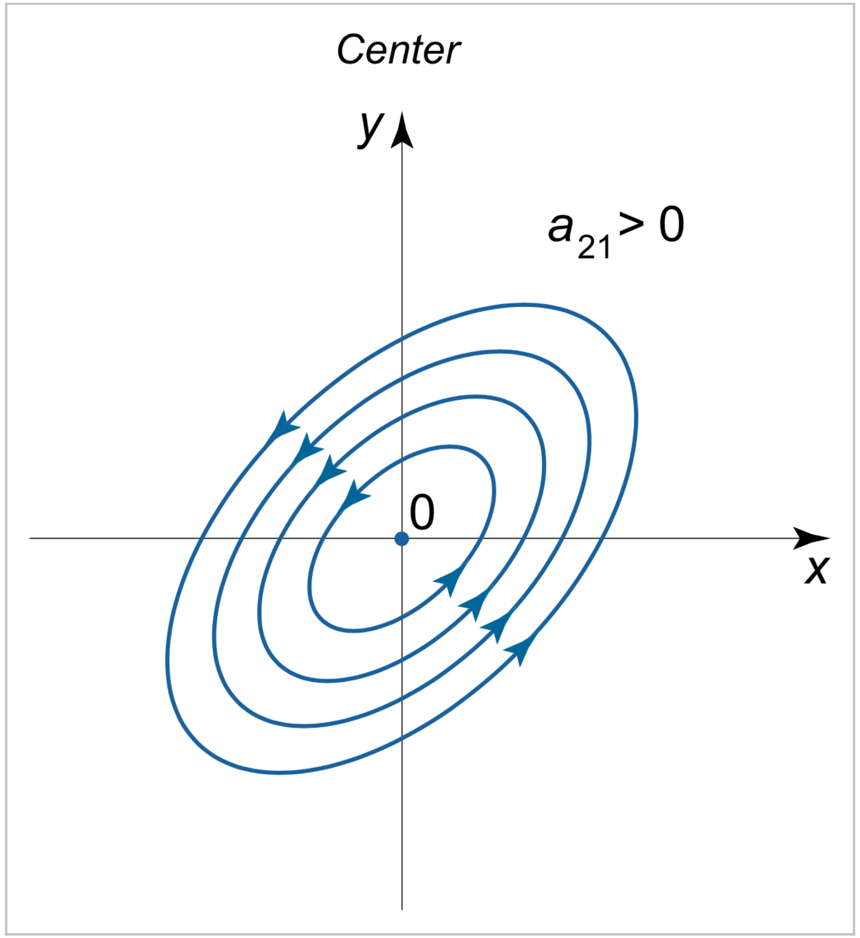
$\alpha < 0 \Rightarrow$ stable

Phase Portrait: Center (complex; $\alpha = 0$)

→ Special case of focus with marginal stability

$$x(t) = (\cos\beta t + i\sin\beta t)(v_a + i v_b)$$





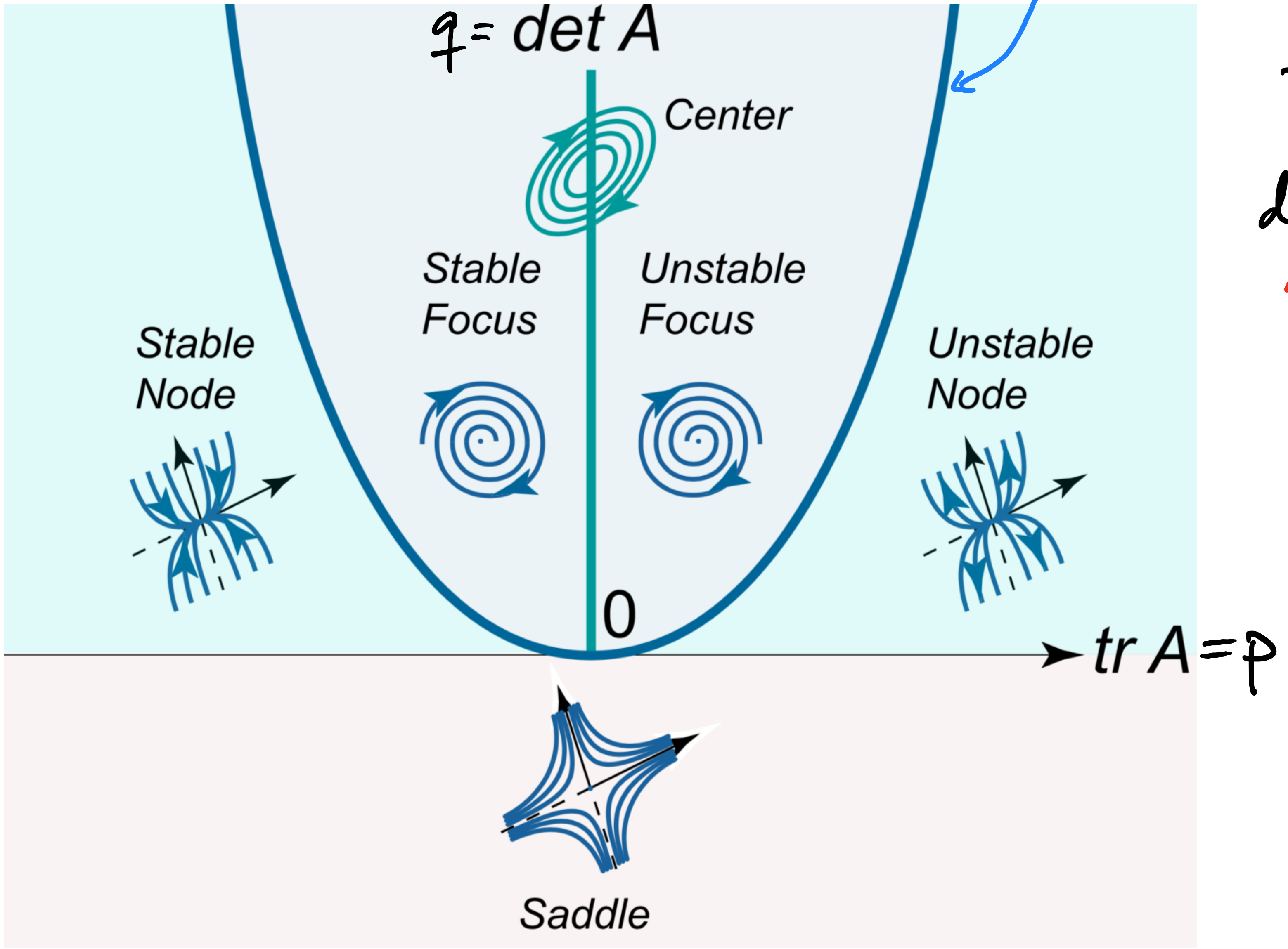
Because the origin is not the only invariant set (i.e., points x such that Ax does not leave the set), center equilibria are handled differently

→ it is not a "hyperbolic equilibria"

Phase Plane Portraits

$p^2 = 4q$

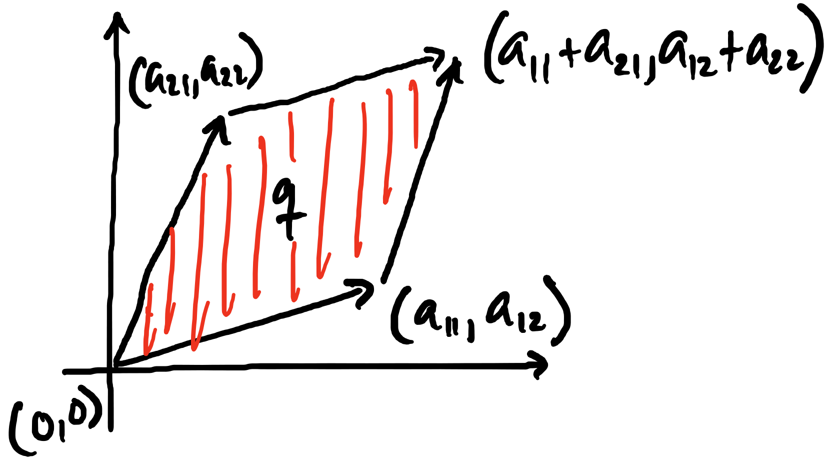
$$\lambda = \frac{1}{2}p \pm \frac{1}{2}\sqrt{p^2 - 4q}$$



$$\mathrm{tr}\,A = p = \lambda_1 + \lambda_2$$ SUM!

$$\det A = q = \lambda_1 \lambda_2$$ PRODUCT!

determinant = (signed) area



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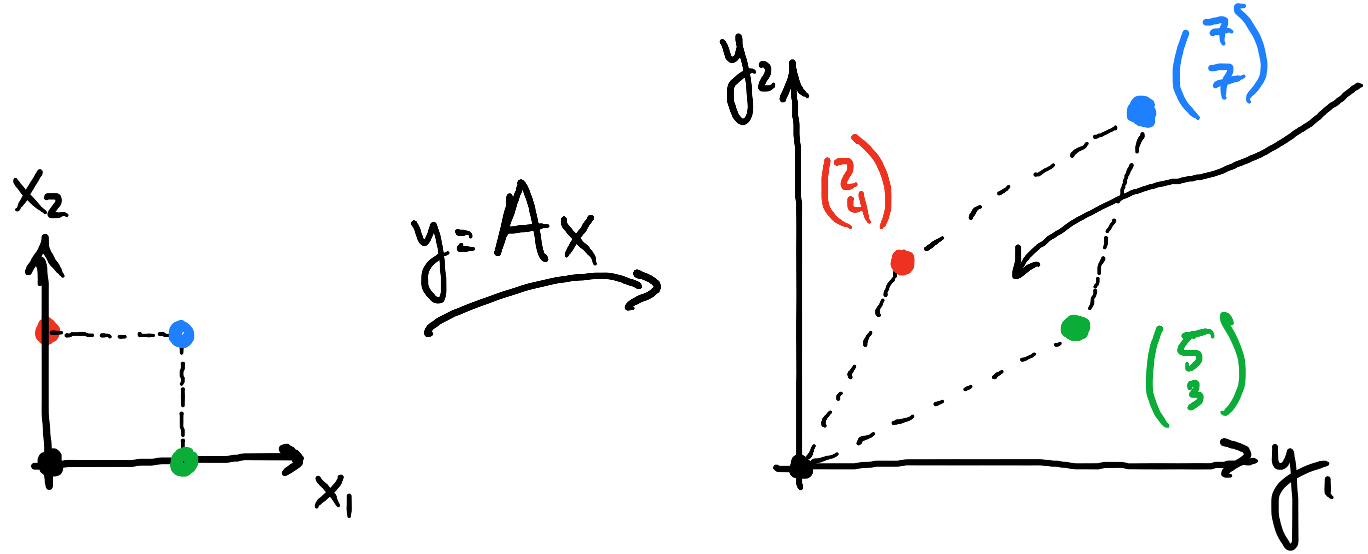
$$A = \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix} \longrightarrow \underbrace{\lambda_1 = 2,\ \lambda_2 = 7}_{\text{Amount of scaling}}\ ;\ \underbrace{v_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\text{Directions of scaling}}$$

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Ax$$

$$q = \lambda_1 \cdot \lambda_2 = 14$$



$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \underset{v_1}{\begin{pmatrix} 2 \\ -3 \end{pmatrix}} + 1 \underset{v_2}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \longrightarrow \begin{pmatrix} 7 \\ 7 \end{pmatrix} = \underset{\lambda_1}{2} \cdot 0 \cdot \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \underset{\lambda_2}{7} \cdot 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{5} \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \frac{2}{5} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2 \cdot -\frac{1}{5} \begin{pmatrix} 2 \\ -3 \end{pmatrix} + 7 \cdot \frac{2}{5} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$