Introduction to 2^{nd}-order System Response

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The University of Texas at Dallas

Outline

1. Motivation
2. Forced Response
3. Applied Example: Spring-Mass-Damper
   - Review: System Modeling
   - Derivation: Transfer Function and Step-Response
   - Derivation/Activity: Response Comparison

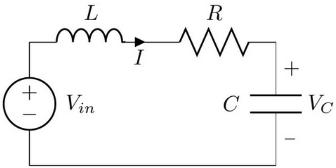
Outline

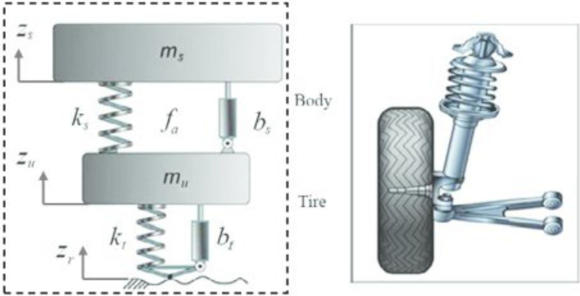
4th-wall break notes

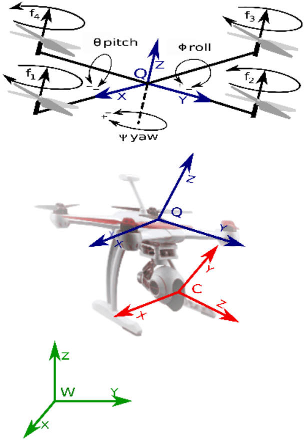
- Lecture Objective: **why 2nd-order roots of a dynamical system's can result in more interesting transient dynamics** (i.e.) the 3 cases as a result from the quadratic equation
- Math background/assumptions:
  - Simple ODEs solutions are covered in prereq and explained ealier in this course
  - Laplace transform methods and the **inverse-laplace via partial fraction expansion** will be well known to students
  - In a real course I'd spend time in lecture having students walk me through the various derivations instead of walking through them or leaving as an exercise/assignment.
- Previous lectures:
  - 1st order-system response and how time-constant plays into the system impulse and step-response
  - Solutions to ODEs (w/in both time and frequency domains)

Outline

Real-World Dynamical Systems







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Step Response - 1st vs 2nd order

Step Input: $\mathbf{u}(t) \overset{\mathcal{L}}{\Rightarrow} U(s) = \frac{1}{s}$

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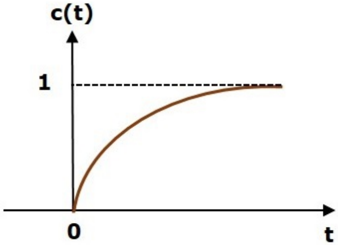

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$\Delta(s)$ dictates transient dynamics

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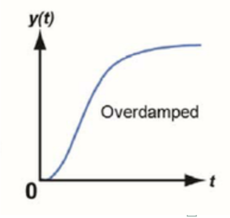

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  Special
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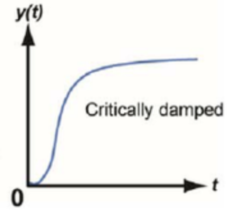

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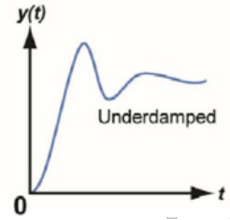

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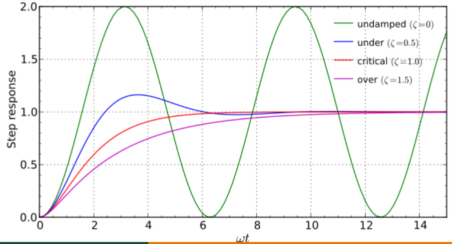

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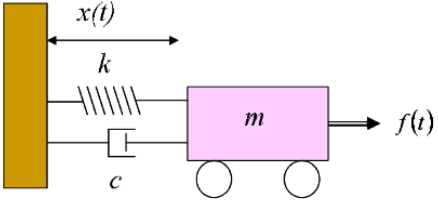
Spring Mass-Damper System Modeling

Newton's 2nd Law:

$$F = m\mathbf{a} = m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}\right)$$



Spring Mass Damper System [1]

Activity: `https://www.sccs.swarthmore.edu/users/12/abiele1/Linear/examples/simple.html`

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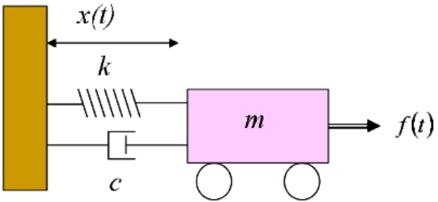

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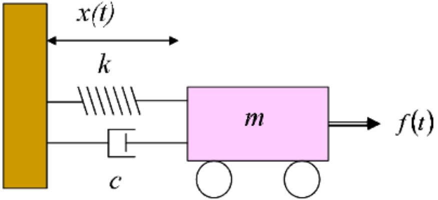

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Differential Equation: $(\mathbf{x} = x(t), \mathbf{u} = f(t))$

$$m\ddot{\mathbf{x}} + c\dot{\mathbf{x}} + k\mathbf{x} = \mathbf{u}$$

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Transfer Function Derivation

Convert Differential Equation to Laplace: $(x(t) = \dot{x}(t) = 0)$

$$f(t) = m\ddot{x}(t) + b\dot{x}(t) + kx(t) \quad \overset{\mathcal{L}}{\Rightarrow} \quad F(s) = ms^2X(s) + bsX(s) + kX(s)$$

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Solve for $X(s)$ in terms of $F(s)$

$$F(s) = (ms^2 + bs + k)X(s)$$

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$$X(s) = \frac{1}{ms^2 + bs + k}F(s) = \frac{1}{m(s^2 + \frac{b}{m}s + \frac{k}{m})}\left(\frac{k}{k}\right)F(s)$$

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Transfer Function:

$$H(s) = \frac{X(s)}{F(s)} = \left(\frac{1}{k}\right)\frac{\frac{k}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$

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(Hook's Law)

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(Standard Form)

Factoring the characteristic polynomial

Apply the quadratic formula to find the roots of the characteristic polynomial:

$$\Delta(s) = ms^2 + bs + k \quad \Rightarrow \quad s = \frac{-\,b \pm \sqrt{b^2 - 4mk}}{2m}$$

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This motivates the standard characteristic polynomial form:

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 \Rightarrow s = \zeta\omega_0 \pm \sqrt{(\zeta\omega_0)^2 - \omega_0^2} = \omega_0\left(\zeta \pm \sqrt{\zeta - 1}\right)$$

Let $2\zeta\omega_n = \sqrt{\frac{b}{m}}$ and $\omega_0 = \sqrt{\frac{k}{m}}$

$$\Delta(s) = s^2 + \frac{b}{m}s + \left(\sqrt{\frac{k}{m}}\right)^2 \iff \Delta(s) = s^2 + 2\zeta\omega_0 s + \omega_0^2$$

In this instance, the three cases are easily seen based on ζ:

1. Damped: $\zeta > 1$
2. Critically Damped: $\zeta = 1$
3. Underdamped: $\zeta \in [0, 1)$

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Distinct real roots: $p_1 \neq p_2 \Rightarrow \Delta(s) = (s + p_1)(s + p_2)$

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Let $a = \frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \left(\sqrt{\frac{k}{m}}\right)^2}$ and $b = \frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \left(\sqrt{\frac{k}{m}}\right)^2}$ Evaluate coeficients:

$(a)(b) = (\frac{b}{2m})^2 - ((\frac{b}{m})^2 - \frac{k}{m}) = \frac{k}{m}, \quad (a-b) = 2\sqrt{((\frac{b}{2m})^2 - \frac{k}{m})}$

$$C_1 = \left.\frac{(s)}{ms(s+a)(s+b)}\right|_{s=0} = \frac{1}{m(a)(b)} \Rightarrow C_1 = \frac{1}{k}\textbf{(Hook's Law @ steady-state)}$$

$$C_2 = \left.\frac{(s+a)}{ms(s+a)(s+b)}\right|_{s=-a} = \frac{1}{m(-a)(-a+b)} = \frac{1}{m(a)(a-b)}$$

$$C_3 = \left.\frac{(s+b)}{ms(s+a)(s+b)}\right|_{s=-b} = \frac{1}{m(-b)(a-b)} = \frac{-1}{m(b)(a-b)}$$

$$C_{2,3} = \frac{\pm 1}{2m\sqrt{((\frac{b}{2m})^2 - \frac{k}{m})}\left(\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \left(\sqrt{\frac{k}{m}}\right)^2}\right)}$$

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Inverse Laplace:

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} x(t) = \left(C_1 + C_2 e^{-p_1 t} + C_3 e^{-p_2 t}\right)u(t)$$

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$$X(s) = \frac{C_1}{s} + \frac{C_2}{s+p} + \frac{C_3}{(s+p)^2}$$

Inverse Laplace:

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} x(t) = \left(C_1 + C_2 e^{-pt} + C_3 t e^{-pt}\right) u(t)$$

Case 3 (Underdamped)

Complex Roots: $b^2 < 4mk \Rightarrow p_{1,2} = \sigma \pm j\omega \Rightarrow \Delta(s) = (s + \sigma \pm j\omega)^2 = ((s + \sigma)^2 + \omega^2)$

Case 3 (Underdamped)

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Partial Fraction Expansion:

$$X(s) = \frac{C_1}{s} + \frac{C_2}{(s + \sigma + j\omega)} + \frac{C_3}{(s + \sigma - j\omega)}$$

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Inverse Laplace:

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} x(t) = \left(C_1 + C_2 e^{-\sigma t} e^{j\omega t} + C_3 e^{-\sigma t} e^{-j\omega t}\right) u(t)$$

$$= C_1 u(t) + 2e^{-\sigma t} \left(\frac{C_2 e^{j\omega t} + C_3 e^{-j\omega t}}{2}\right) u(t)$$

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Complex Roots: $b^2 < 4mk \Rightarrow p_{1,2} = \sigma \pm j\omega \Rightarrow \Delta(s) = (s + \sigma \pm j\omega)^2$

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$$= C_1 u(t) + 2e^{-\sigma t} \left(\frac{C_2 e^{j\omega t} + C_3 e^{-j\omega t}}{2}\right) u(t) \Leftarrow \textbf{Convert using Euler's Identity}$$

Case 3 (Underdamped)

Complex Roots: $b^2 < 4mk \Rightarrow p_{1,2} = \sigma \pm j\omega \Rightarrow \Delta(s) = (s+\sigma \pm j\omega)^2$

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Alternative approach

$$X(s) = \frac{K}{s(s+\sigma \pm j\omega)} = \frac{K}{s((s+\sigma)^2+\omega^2)}$$
$$= \frac{C_1}{s} + \frac{C_2}{(s+\sigma)^2+\omega^2} \overset{\mathcal{L}^{-1}}{\Rightarrow}$$
$$\overset{\mathcal{L}^{-1}}{\Rightarrow} x(t) = \left(C_1 + \frac{C_2}{\sigma}e^{-\sigma t}\cos(\omega t)\right)$$

Specific case:

$$X(s) = \frac{\frac{1}{m}}{s(s^2+\frac{b}{m}s+\frac{k}{m})} = \frac{C_1}{s} + \frac{C_2}{(s+\frac{b}{2m})^2 + \sqrt{\frac{k}{m}-\sqrt{\frac{b}{m}}}^{\,2}} \overset{\mathcal{L}^{-1}}{\Rightarrow}$$

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} x(t) = \left(C_1 + \frac{C_2}{\ } \exp\left\{-\frac{b}{\ }t\right\}\cos\left(\left(\sqrt{\frac{k}{\ }-\sqrt{\frac{b}{\ }}}\right)t\right)\right)u(t)$$

Activity: Response Comparison

TODO:

1. Experiment with different m, k, and b parameters to gain an intuitive understanding of how each parameter effects the response
2. Select one of each case and derive the functional form (i.e. solve for poles and coefficients)

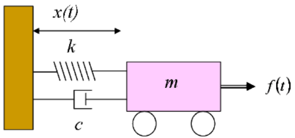
Online Tool: `https://www.sccs.swarthmore.edu/users/12/abiele1/Linear/examples/simple.html`

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = f(t)$$

$$X(s) = \frac{1}{ms^2 + bs + k}F(s) = \frac{\frac{k}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}\frac{1}{k}F(s)$$

$$H(s) = \frac{X(s)}{F(s)} = (K)\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \zeta = \sqrt{\frac{b^2}{4mk}} \quad K = \frac{1}{k}$$



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Transfer Function

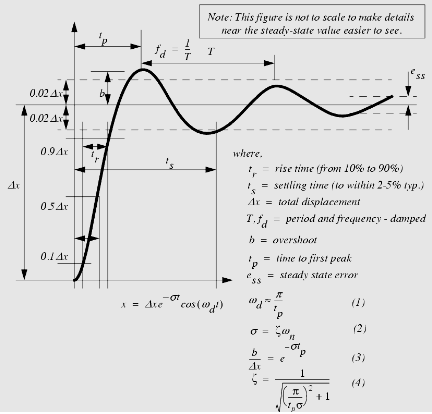
$$H(s) = \frac{Y(s)}{U(s)} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

System Poles

$$s = -\zeta\omega_0 \pm \omega_0\sqrt{1-\zeta^2}$$

Spring Mass Damper System Parameters

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \zeta = \sqrt{\frac{c^2}{4mk}}$$



2nd Order System Response [2]