Introduction to 2nd-order System Response

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The University of Texas at Dallas

Outline

1. Motivation
2. Forced Response
3. Applied Example: Spring-Mass-Damper
   - Review: System Modeling
   - Derivation: Transfer Function and Step-Response
   - Activity: Response Comparison

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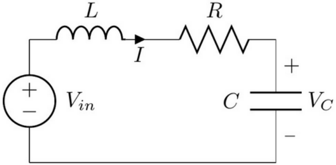
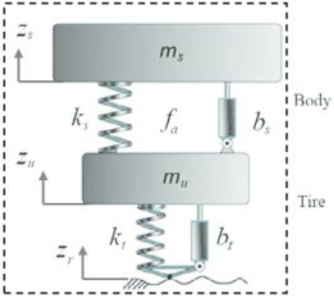
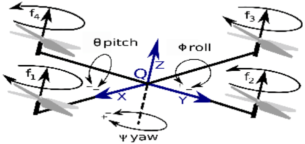
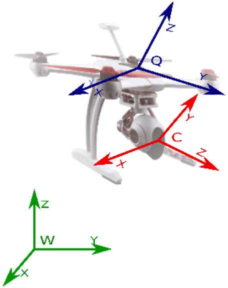
Real-World Dynamical Systems









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Step Response - 1st vs 2nd order

Step Input: $\mathbf{u}(t) \overset{\mathcal{L}}{\Rightarrow} U(s) = \frac{1}{s}$

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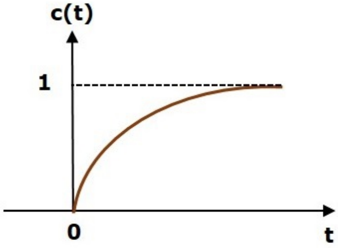

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$\Delta(s)$ dictates transient dynamics

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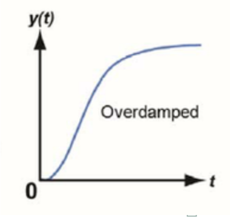

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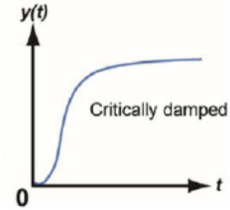
  Special Case: $C_2 e^{-p_{1,2} t} + C_3 t e^{-p_{1,2} t}$



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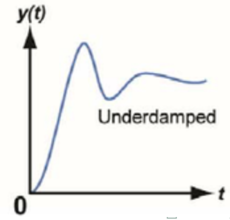

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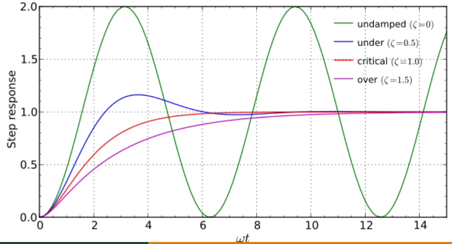

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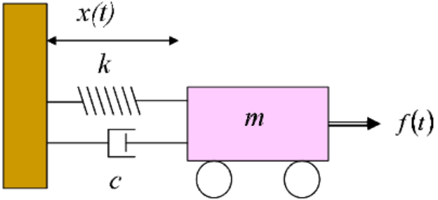
Spring Mass-Damper System Modeling

Newton's 2nd Law:

$$F = m\mathbf{a} = m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}\right)$$



Spring Mass Damper System [1]

Activity: `https://www.sccs.swarthmore.edu/users/12/abiele1/Linear/examples/simple.html`

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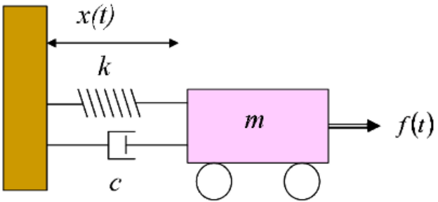

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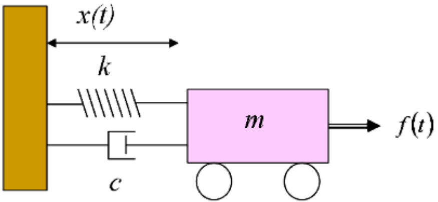

Spring Mass Damper System [1]

Differential Equation: $(\mathbf{x} = x(t), \mathbf{u} = f(t))$

$$m\ddot{\mathbf{x}} + c\dot{\mathbf{x}} + k\mathbf{x} = \mathbf{u}$$

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Transfer Function Derivation

Convert Differential Equation to Laplace: $(x(t) = \dot{x}(t) = 0)$

$$f(t) = m\ddot{x}(t) + b\dot{x}(t) + kx(t) \quad \overset{\mathcal{L}}{\Rightarrow} \quad F(s) = ms^2X(s) + bsX(s) + kX(s)$$

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Solve for $X(s)$ in terms of $F(s)$

$$F(s) = (ms^2 + bs + k)X(s)$$

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Transfer Function:

$$H(s) = \frac{X(s)}{F(s)} = \left(\frac{1}{k}\right)\frac{\frac{k}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$

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(Hook's Law)

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(Standard Form)

Factoring the characteristic polynomial

Apply the quadratic formula to find the roots of the characteristic polynomial:

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$$\overset{\mathcal{L}^{-1}}{\Rightarrow} x(t) = \left(C_1 + C_2 e^{-p_1 t} + C_3 e^{-p_2 t}\right) u(t)$$

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$$\begin{aligned} \overset{\mathcal{L}^{-1}}{\Rightarrow} x(t) &= \left(C_1 + C_2 e^{-\sigma t} e^{j\omega t} + C_3 e^{-\sigma t} e^{-j\omega t}\right) u(t) \\ &= C_1 u(t) + 2e^{-\sigma t} \left(\frac{C_2 e^{j\omega t} + C_3 e^{-j\omega t}}{2}\right) u(t) \end{aligned}$$

Case 3 (Underdamped)

Complex Roots: $b^2 < 4mk \Rightarrow p_{1,2} = \sigma \pm j\omega \Rightarrow \Delta(s) = (s + \sigma \pm j\omega)^2$

$$X(s) = \left(\frac{1}{k}\right) \frac{\frac{k}{m}}{s(s^2 + \frac{b}{m}s + \frac{k}{m})} = \frac{K}{s(s + \sigma \pm j\omega)}$$

Partial Fraction Expansion:

$$X(s) = \frac{C_1}{s} + \frac{C_2}{(s + \sigma + j\omega)} + \frac{C_3}{(s + \sigma - j\omega)}$$

Inverse Laplace:

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} x(t) = \left(C_1 + C_2 e^{-\sigma t} e^{j\omega t} + C_3 e^{-\sigma t} e^{-j\omega t}\right) u(t)$$

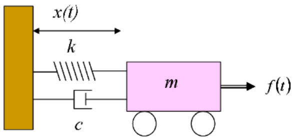
$$= C_1 u(t) + 2e^{-\sigma t} \left(\frac{C_2 e^{j\omega t} + C_3 e^{-j\omega t}}{2}\right) u(t) \Leftarrow \textbf{Convert using Euler's Identity}$$

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = f(t)$$

$$X(s) = \frac{1}{ms^2 + bs + k} F(s) = \frac{\frac{k}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} \frac{1}{k} F(s)$$

$$H(s) = \frac{X(s)}{F(s)} = (K)\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \zeta = \sqrt{\frac{b^2}{4mk}} \quad K = \frac{1}{k}$$



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Transfer Function

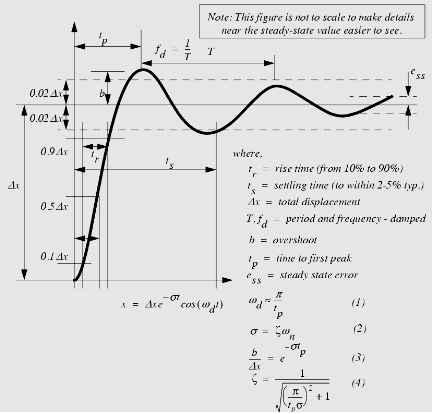
$$H(s) = \frac{Y(s)}{U(s)} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

System Poles

$$s = -\zeta\omega_0 \pm \omega_0\sqrt{1-\zeta^2}$$

Spring Mass Damper System Parameters

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \zeta = \sqrt{\frac{c^2}{4mk}}$$



2nd Order System Response [2]