Introduction to 2nd-order System Response

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Background

Outline

2 Forced Response Solution

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2nd-order System Dynamics

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Backgroun

Motivation: Real-World Dynamical System



TODO: add examples of dynamical systems

Background

Review: Spring Mass-Damper System Model



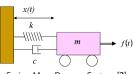
Newton's 2nd Law:

$$F = m\mathbf{a} = m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}\right)$$

$$m\frac{\mathrm{d}}{\mathrm{d}t^2}x(t) = \sum F = f(t) - b\frac{\mathrm{d}}{\mathrm{d}t}x(t) - kx(t)$$

Differential Equation: $(\mathbf{x} = x(t), \mathbf{u} = f(t))$

$$m\ddot{\mathbf{x}} + c\dot{\mathbf{x}} + k\mathbf{x} = \mathbf{u}$$



Spring Mass Damper System [?]

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 ${\sf Spring-mass-damper\ Differential\ System:}$

$$m\ddot{\mathbf{x}}(t) + b\dot{\mathbf{x}}(t) + k\mathbf{x}(t) = f(t) \quad \Rightarrow \quad \omega_0 = \sqrt{\frac{c}{m}}, \\ \zeta = \sqrt{\frac{c^2}{4mk}}, \\ u(t) = \frac{1}{m}f(t)$$

2nd-Order Dynamical System:

$$\ddot{x}(t) + 2\zeta\omega_0\dot{x}(t) + \omega_0^2x(t) = u(t)$$

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2nd-order System Dynamics

2nd-order System Dynamics
—Forced Response Solution
Derivation: Laplace

Derivation: Laplace method to derive Step-Response

$$\begin{split} m\ddot{x}(t) + b\dot{x}(t) + kx(t) &= f(t) \\ & \quad \Downarrow \mathcal{L} \\ ms^2X(s) + bsX(s) + kX(s) &= F(s) \\ (ms^2 + bs + k)X(s) &= F(s) \\ X(s) &= \frac{1}{ms^2 + bs + k}F(s) \end{split}$$

Charectoristic Polynomial :

$$\Delta(s) = ms^2 + bs + k \Rightarrow s^2 + \frac{b}{m}s + \frac{k}{m} = s^2 + 2\zeta\omega_0s + \omega_0^2$$

2nd-order System Dynamics Forced Response Solution

Laplace method to derive Step-Response



In order to do the partial fraction decomposition, it must be in factored form... thus the quadratic equation:

$$\Delta(s) = ms^2 + bs + k = 0 \Rightarrow s = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = \frac{b}{2m} \pm \sqrt{\frac{b^2 - 4mk}{4m^2}} = \frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

3 Potential cases: (may have learned in differential equations)

1. Damped: $(\frac{b}{2m})^2 > \frac{k}{m} \Rightarrow (s+a)(s+b)$

2. Critically Damped: $(\frac{b}{2m})^2 = \frac{k}{m} \Rightarrow (s+a)^2$

3. Underdamped: $(\frac{b}{2m})^2 < \frac{k}{m} \Rightarrow (s + \sigma \pm j\omega_d)$

We will continue under the assumption that the system is underdamped and then return for a more general case.

2nd-order System Dynamics Forced Response Solution

 $\cup {\sf Derivation: Laplace method to derive Step-Response}$

Detection: Lipitace method to derive Step-Bargoness $\frac{1}{\sqrt{1+|x|^2}} \text{Detection}$ Spring was described bytem: $m(t) + \log(t + \log(t + \log t)) = m_t - \sqrt{\frac{t}{m_t}} < \sqrt{\frac{t}{2+m_t}} - \log(t) - \frac{1}{m_t} \log t$ $2^{m_t} \text{ Cash in Exposure Symmetry}$ $(t) + 2 \lambda_{t+1} \log(t) + \frac{1}{m_t} \log(t) + \frac{1}{m_t} \log(t)$

Step Response:
$$f(t) = u(t) \stackrel{\mathcal{L}}{\Rightarrow} F(s) = \frac{1}{s}$$

$$X(s) = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} \left(\frac{1}{s}\right) = \frac{\frac{1}{m}}{s(s^2 + \frac{b}{m}s + \frac{k}{m})}$$

Factored Form (quadratic formula):
$$as^2+bs+c=0 \Rightarrow s=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

TODO: factorit...3cases...

Partial Fraction Expansion

$$X(s) =$$

2nd-order System Dynamics —Forced Response Solution

Laplace method to derive Step-Response



Side Note: (skip unless time allows/go back to this) Looking at the original differential equation we can see how the standard form $s^2+2\zeta\omega_0s+(\omega_0)^2$ would fit in as

$$\zeta\omega_0\pm\sqrt{(\zeta\omega_0)^2-\omega_0^2}=\zeta\omega_0\pm\sqrt{\omega_0^2(\zeta^2-1)}=\zeta\omega_0\pm\omega_0\sqrt{\zeta^2-1}$$

TLDR: Second-Order System Dynamics



Transfer Function

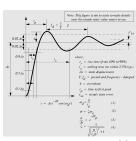
$$H(s) = \frac{Y(s)}{U(s)} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

System Poles

$$s = -\zeta \omega_0 \pm \omega_0 \sqrt{1 - \zeta^2}$$

Spring Mass Damper System Parameters

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \zeta = \sqrt{\frac{4}{4}}$$



2nd Order System Response [?]

Lecture Overview



$$H(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$
$$\omega_0 = \sqrt{\frac{k}{m}} \quad \zeta = \sqrt{\frac{c^2}{4mk}}$$

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \zeta = \sqrt{\frac{c^2}{4m}}$$

$$\begin{array}{c} x(t) \\ k \\ c \end{array} \longrightarrow f(t)$$