Introduction to 2nd-order System Response

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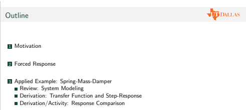
Outline

2nd-order System Dynamics

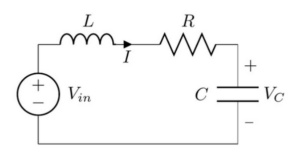
└Outline

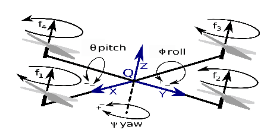


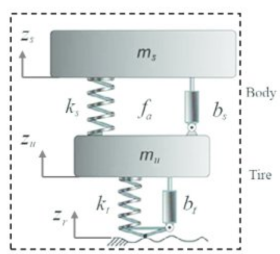
4th-wall break notes

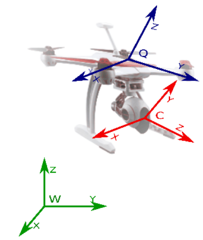
- Lecture Objective: **why 2nd-order roots of a dynamical system's can result in more interesting transient dynamics** (i.e.) the 3 cases as a result from the quadratic equation
- Math background/assumptions:
  - Simple ODEs solutions are covered in prereq and explained ealier in this course
  - Laplace transform methods and the **inverse-laplace via partial fraction expansion** will be well known to students
  - In a real course I'd spend time in lecture having students walk me through the various derivations instead of walking through them or leaving as an exercise/assignment.
- Previous lectures:
  - 1st order-system response and how time-constant plays into the system impulse and step-response
  - Solutions to ODEs (w/in both time and frequency domains)

Real-World Dynamical Systems









Step Response - 1st vs 2nd order

Step Input: $\mathbf{u}(t) \overset{\mathcal{L}}{\Rightarrow} U(s) = \frac{1}{s}$

1$^{\text{st}}$-order:

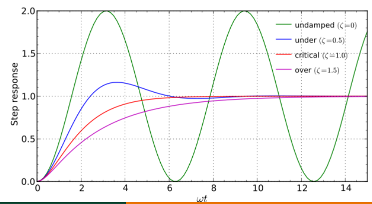
$$Y(s) = \frac{K}{\tau s + 1}\left(\frac{1}{s}\right) \quad \overset{\mathcal{L}^{-1}}{\Rightarrow} \quad y(t) = K(1 - e^{-t/\tau})\mathbf{u}(t)$$

2$^{\text{nd}}$-order:

$$Y(s) = \frac{K}{(s+p_1)(s+p_2)}\left(\frac{1}{s}\right) \quad \overset{\mathcal{L}^{-1}}{\Rightarrow} \quad y(t) = \left(C_1 + C_2 e^{-p_1 t} + C_3 e^{-p_2 t}\right)\mathbf{u}(t)$$

3 distinct cases:

- Damped: $p_1 \neq p_2$
- Critically Damped: $p_1 = p_2$ Special Case: $C_2 e^{-p_{1,2}t} + C_3 t e^{-p_{1,2}t}$
- Underdamped: $p_{1,2} = \sigma \pm j\omega$



Spring Mass-Damper System Modeling

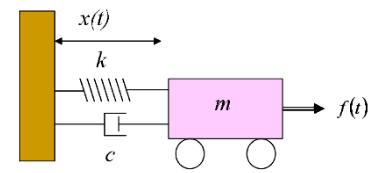
Newton's 2$^{\text{nd}}$ Law:

$$F = m\mathbf{a} = m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}\right)$$

$$m\frac{\mathrm{d}}{\mathrm{d}t^2}x(t) = \sum F = f(t) - b\frac{\mathrm{d}}{\mathrm{d}t}x(t) - kx(t)$$

Differential Equation: $(\mathbf{x} = x(t),\ \mathbf{u} = f(t))$

$$m\ddot{\mathbf{x}} + c\dot{\mathbf{x}} + k\mathbf{x} = \mathbf{u}$$



Spring Mass Damper System [1]

Activity: `https://www.sccs.swarthmore.edu/users/12/abiele1/Linear/examples/simple.html`

Transfer Function Derivation

Convert Differential Equation to Laplace: $(x(t) = \dot{x}(t) = 0)$

$$f(t) = m\ddot{x}(t) + b\dot{x}(t) + kx(t) \quad \overset{\mathcal{L}}{\Rightarrow} \quad F(s) = ms^2X(s) + bsX(s) + kX(s)$$

Solve for $X(s)$ **in terms of** $F(s)$

$$X(s) = \frac{1}{ms^2 + bs + k}F(s) = \frac{1}{m(s^2 + \frac{b}{m}s + \frac{k}{m})}\left(\frac{k}{k}\right)F(s) = \left(\frac{1}{k}\right)\frac{\frac{k}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}F(s)$$

Transfer Function:

$$H(s) = \frac{X(s)}{F(s)} = \left(\frac{1}{k}\right)\frac{\frac{k}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} \Leftarrow \underset{\text{(Hook's Law)}}{F = k\Delta x}$$

Factoring the characteristic polynomial

Apply the quadratic formula to find the roots of the characteristic polynomial:

$$\Delta(s) = ms^2 + bs + k \quad \Rightarrow \quad s = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

3 Potential cases:

1. **Damped**: $b^2 > 4mk \Rightarrow p_1 \neq p_2 \Rightarrow (s+p_1)(s+p_2)$
2. **Critically Damped**: $b^2 = 4mk \Rightarrow p_1 = p_2 \Rightarrow (s+p_{1,2})^2$
3. **Underdamped**: $b^2 < 4mk \Rightarrow p_{1,2} = \sigma \pm j\omega \Rightarrow (s + \sigma \pm j\omega) = ((s+\sigma)^2 + \omega^2)$

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2nd-order System Dynamics
└Applied Example: Spring-Mass-Damper
  └Derivation: Transfer Function and Step-Response
    └Factoring the characteristic polynomial

This motivates the standard characteristic polynomial form:

$$s^2+2\zeta\omega_0 s+\omega_0^2 \Rightarrow s=\zeta\omega_0 \pm\sqrt{(\zeta\omega_0)^2-\omega_0^2}=\omega_0\left(\zeta\pm\sqrt{\zeta-1}\right)$$

Let $2\zeta\omega_n=\sqrt{\frac{b}{m}}$ and $\omega_0=\sqrt{\frac{k}{m}}$

$$\Delta(s)=s^2+\frac{b}{m}s+\left(\sqrt{\frac{k}{m}}\right)^2 \iff \Delta(s)=s^2+2\zeta\omega_0 s+\omega_0^2$$

In this instance, the three cases are easily seen based on ζ:

1. Damped: $\zeta>1$
2. Critically Damped: $\zeta=1$
3. Underdamped: $\zeta\in[0,1)$

Case 1 (Damped)

Distinct real roots: $p_1\neq p_2 \Rightarrow \Delta(s)=(s+p_1)(s+p_2)$

$$X(s)=\left(\frac{1}{k}\right)\frac{\frac{k}{m}}{s(s^2+\frac{b}{m}s+\frac{k}{m})}=\frac{K}{s(s+p_1)(s+p_2)}$$

Partial Fraction Expansion:

$$X(s)=\frac{C_1}{s}+\frac{C_2}{s+p_1}+\frac{C_3}{s+p_2}$$

Inverse Laplace:

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} x(t)=\left(C_1+C_2e^{-p_1t}+C_3e^{-p_2t}\right)u(t)$$

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2nd-order System Dynamics
└Applied Example: Spring-Mass-Damper
  └Derivation/Activity: Response Comparison
    └Case 1 (Damped)

Let $a=\frac{b}{2m}+\sqrt{\left(\frac{b}{2m}\right)^2-\left(\sqrt{\frac{k}{m}}\right)^2}$ and $b=\frac{b}{2m}-\sqrt{\left(\frac{b}{2m}\right)^2-\left(\sqrt{\frac{k}{m}}\right)^2}$ Evaluate coeficients:

$(a)(b)=(\frac{b}{2m})^2-((\frac{b}{m})^2-\frac{k}{m})=\frac{k}{m}$, $\quad (a-b)=2\sqrt{((\frac{b}{2m})^2-\frac{k}{m})}$

$$C_1=\frac{(s)}{ms(s+a)(s+b)}\bigg|_{s=0}=\frac{1}{m(a)(b)}\Rightarrow C_1=\frac{1}{k}\textbf{(Hook's Law @ steady-state)}$$

$$C_2=\frac{(s+a)}{ms(s+a)(s+b)}\bigg|_{s=-a}=\frac{1}{m(-a)(-a+b)}=\frac{1}{m(a)(a-b)}$$

$$C_3=\frac{(s+b)}{ms(s+a)(s+b)}\bigg|_{s=-b}=\frac{1}{m(-b)(a-b)}=\frac{-1}{m(b)(a-b)}$$

$$C_{2,3}=\frac{\pm 1}{2m\sqrt{((\frac{b}{2m})^2-\frac{k}{m})}\left(\frac{b}{2m}\pm\sqrt{\left(\frac{b}{2m}\right)^2-\left(\sqrt{\frac{k}{m}}\right)^2}\right)}$$

Case 2 (Critically Damped)

Repeated Roots: $b^2=4mk\Rightarrow p_1=p_2\Rightarrow\Delta(s)=(s+p)^2$

$$X(s)=\left(\frac{1}{k}\right)\frac{\frac{k}{m}}{s(s^2+\frac{b}{m}s+\frac{k}{m})}=\frac{K}{s(s+p)^2}$$

Partial Fraction Expansion:

$$X(s)=\frac{C_1}{s}+\frac{C_2}{s+p}+\frac{C_3}{(s+p)^2}$$

Inverse Laplace:

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} x(t)=\left(C_1+C_2e^{-pt}+C_3te^{-pt}\right)u(t)$$

Case 3 (Underdamped)

Complex Roots: $b^2 < 4mk \Rightarrow p_{1,2} = \sigma \pm j\omega \Rightarrow \Delta(s) = (s+\sigma\pm j\omega)^2 = ((s+\sigma)^2+\omega^2)$

$$X(s) = \left(\frac{1}{k}\right)\frac{\frac{k}{m}}{s(s^2+\frac{b}{m}s+\frac{k}{m})} = \frac{K}{s(s+\sigma\pm j\omega)}$$

Partial Fraction Expansion:

$$X(s) = \frac{C_1}{s} + \frac{C_2}{(s+\sigma+j\omega)} + \frac{C_3}{(s+\sigma-j\omega)}$$

Inverse Laplace:

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} x(t) = \left(C_1 + C_2 e^{-\sigma t}e^{j\omega t} + C_3 e^{-\sigma t}e^{-j\omega t}\right)u(t)$$
$$= C_1 u(t) + 2e^{-\sigma t}\left(\frac{C_2 e^{j\omega t}+C_3 e^{-j\omega t}}{2}\right)u(t) \Leftarrow \textbf{Convert using Euler's Identity}$$

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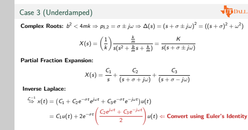
2nd-order System Dynamics
└Applied Example: Spring-Mass-Damper
 └Derivation/Activity: Response Comparison
  └Case 3 (Underdamped)



Alternative approach

$$X(s) = \frac{K}{s(s+\sigma\pm j\omega)} = \frac{K}{s((s+\sigma)^2+\omega^2)}$$
$$= \frac{C_1}{s} + \frac{C_2}{(s+\sigma)^2+\omega^2} \overset{\mathcal{L}^{-1}}{\Rightarrow}$$
$$\overset{\mathcal{L}^{-1}}{\Rightarrow} x(t) = \left(C_1 + \frac{C_2}{\sigma}e^{-\sigma t}\cos(\omega t)\right)$$

Specific case:

$$X(s) = \frac{\frac{1}{m}}{s(s^2+\frac{b}{m}s+\frac{k}{m})} = \frac{C_1}{s} + \frac{C_2}{(s+\frac{b}{2m})^2+\sqrt{\frac{k}{m}-\sqrt{\frac{b}{m}}}^{\,2}} \overset{\mathcal{L}^{-1}}{\Rightarrow}$$

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} x(t) = \left(C_1 + \frac{C_2}{\sqrt{\frac{k}{m}-\sqrt{\frac{b}{m}}}}\exp\left\{-\frac{b}{2m}t\right\}\cos\left(\left(\sqrt{\frac{k}{m}-\sqrt{\frac{b}{m}}}\right)t\right)\right)u(t)$$

Activity: Response Comparison

TODO:

1. Experiment with different m,k, and b parameters to gain an intuitive understanding of how each parameter effects the response
2. Select one of each case and derive the functional form (i.e. solve for poles and coefficients)

Online Tool: `https://www.sccs.swarthmore.edu/users/12/abiele1/Linear/examples/simple.html`

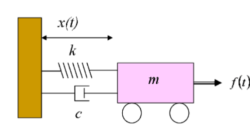
Lecture Overview

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = f(t)$$

$$X(s) = \frac{1}{ms^2+bs+k}F(s) = \frac{\frac{k}{m}}{s^2+\frac{b}{m}s+\frac{k}{m}}\frac{1}{k}F(s)$$

$$H(s) = \frac{X(s)}{F(s)} = (K)\frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$$

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \zeta = \sqrt{\frac{b^2}{4mk}} \quad K = \frac{1}{k}$$



Bibliography I

- Detroit Mercy University of Michigan, Carnegie Mellon.
  Introduction: System modeling.
- Engineer on a Disk.
  ebook: Dynamic system modeling and control.

TLDR: Second-Order System Dynamics

Transfer Function

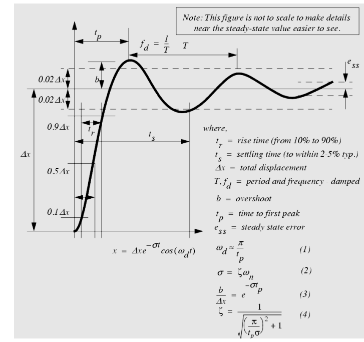
$$H(s) = \frac{Y(s)}{U(s)} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

System Poles

$$s = -\zeta\omega_0 \pm \omega_0\sqrt{1-\zeta^2}$$

Spring Mass Damper System Parameters

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \zeta = \sqrt{\frac{c^2}{4mk}}$$



2nd Order System Response [2]