Introduction to 2$^{\text{nd}}$-order System Response

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Outline

1. Background
2. Forced Response Solution

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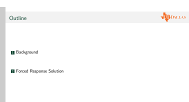
2nd-order System Dynamics

└Outline



TODO: include 4th-wall break notes

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Motivation: Real-World Dynamical System

TODO: add examples of dynamical systems

Review: Spring Mass-Damper System Model

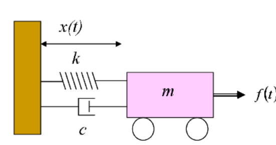
Newton's 2$^{\text{nd}}$ Law:

$$F = m\mathbf{a} = m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}\right)$$

$$m\frac{\mathrm{d}}{\mathrm{d}t^2}x(t) = \sum F = f(t) - b\frac{\mathrm{d}}{\mathrm{d}t}x(t) - kx(t)$$

Differential Equation: $(\mathbf{x} = x(t),\, \mathbf{u} = f(t))$

$$m\ddot{\mathbf{x}} + c\dot{\mathbf{x}} + k\mathbf{x} = \mathbf{u}$$



Spring Mass Damper System [?]

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Derivation: Laplace method to derive Step-Response

Spring-mass-damper Differential System:

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = f(t) \quad \Rightarrow \quad \omega_0 = \sqrt{\frac{k}{m}}, \zeta = \sqrt{\frac{c^2}{4mk}}, u(t) = \frac{1}{m}f(t)$$

2nd-Order Dynamical System:

$$\ddot{x}(t) + 2\zeta\omega_0\dot{x}(t) + \omega_0^2 x(t) = u(t)$$

$$\begin{aligned} m\ddot{x}(t) + b\dot{x}(t) + kx(t) &= f(t) \\ &\Downarrow \mathcal{L} \\ ms^2X(s) + bsX(s) + kX(s) &= F(s) \\ (ms^2 + bs + k)X(s) &= F(s) \\ X(s) &= \frac{1}{ms^2 + bs + k}F(s) \end{aligned}$$

Charectoristic Polynomial :

$$\Delta(s) = ms^2 + bs + k \Rightarrow s^2 + \frac{b}{m}s + \frac{k}{m} = s^2 + 2\zeta\omega_0 s + \omega_0^2$$

In order to do the partial fraction decomposition, it must be in factored form... thus the quadratic equation:

$$\Delta(s) = ms^2 + bs + k = 0 \Rightarrow s = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = \frac{b}{2m} \pm \sqrt{\frac{b^2 - 4mk}{4m^2}} = \frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

3 Potential cases: (may have learned in differential equations)

1. **Damped**: $(\frac{b}{2m})^2 > \frac{k}{m} \Rightarrow (s+a)(s+b)$
2. **Critically Damped**:$(\frac{b}{2m})^2 = \frac{k}{m} \Rightarrow (s+a)^2$
3. **Underdamped**: $(\frac{b}{2m})^2 < \frac{k}{m} \Rightarrow (s + \sigma \pm j\omega_d)$

We will continue under the assumption that the system is underdamped and then return for a more general case.

Step Response: $f(t) = u(t) \overset{\mathcal{L}}{\Rightarrow} F(s) = \frac{1}{s}$

$$X(s) = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}\left(\frac{1}{s}\right) = \frac{\frac{1}{m}}{s(s^2 + \frac{b}{m}s + \frac{k}{m})}$$

Factored Form (quadratic formula): $as^2 + bs + c = 0 \Rightarrow s = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$TODO : factorit...3cases...$$

Partial Fraction Expansion

$$X(s) =$$

Side Note: (skip unless time allows/go back to this) Looking at the original differential equation we can see how the standard form $s^2 + 2\zeta\omega_0 s + (\omega_0)^2$ would fit in as

$$\zeta\omega_0 \pm \sqrt{(\zeta\omega_0)^2 - \omega_0^2} = \zeta\omega_0 \pm \sqrt{\omega_0^2(\zeta^2 - 1)} = \zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

TLDR: Second-Order System Dynamics

Transfer Function

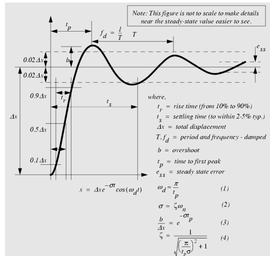
$$H(s) = \frac{Y(s)}{U(s)} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

System Poles

$$s = -\zeta\omega_0 \pm \omega_0\sqrt{1-\zeta^2}$$

Spring Mass Damper System Parameters

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \zeta = \sqrt{\frac{c^2}{4mk}}$$



2nd Order System Response [?]

Lecture Overview

$$H(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \zeta = \sqrt{\frac{c^2}{4mk}}$$