Introduction to 2nd-order System Response

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Outline

4th-wall break notes

- Lecture Objective: **why 2nd-order roots of a dynamical system's can result in more interesting responses** (i.e.) the 3 cases as a result from the quadratic equation
- Math background/assumptions:
  - Simple ODEs solutions are covered in prereq and explained again in the intro of this course
  - Specifically, Laplace transform methods and the **inverse-laplace via partial fraction expansion** will be well known to students.
  - In a real course I'd spend time in lecture having students walk me through the derivation of the cases instead of leaving as an exercise/assignment.
- Previous lectures:
  - Discussed 1st order-system response and how time-constant plays into the system impulse and step-response
  - Taught math behind general system response (w/in time and frequency domains)

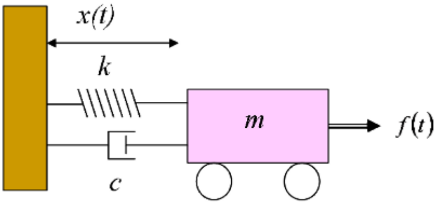
Outline

Reminder: Real-World Dynamical Systems

TODO: add images of dynamical systems

Outline

Review: Spring Mass-Damper System Modeling

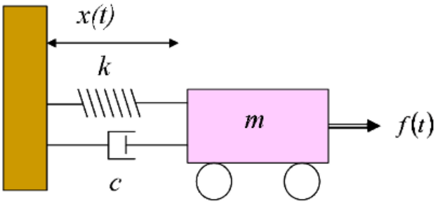


Spring Mass Damper System [1]

Review: Spring Mass-Damper System Modeling

Newton's 2nd Law:

$$F = m\mathbf{a} = m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}\right)$$



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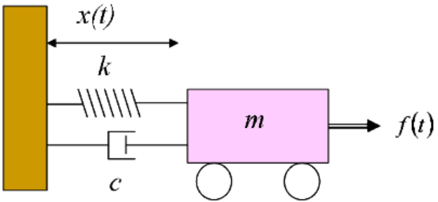
$$m\frac{\mathrm{d}}{\mathrm{d}t^2}x(t) = \sum F = f(t) - b\frac{\mathrm{d}}{\mathrm{d}t}x(t) - kx(t)$$



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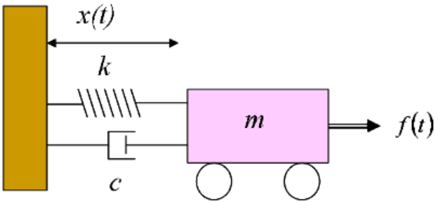
Differential Equation: $(\mathbf{x} = x(t),\ \mathbf{u} = f(t))$

$$m\ddot{\mathbf{x}} + c\dot{\mathbf{x}} + k\mathbf{x} = \mathbf{u}$$



Spring Mass Damper System [1]

Outline

Transfer Function Derivation

Spring-mass-damper Differential Equation:

$$f(t) = m\ddot{x}(t) + b\dot{x}(t) + kx(t) \overset{\mathcal{L}}{\Rightarrow} F(s) = ms^2X(s) + bsX(s) + kX(s)$$

Solve for $X(s)$ and $H(s)$

$$F(s) = (ms^2 + bs + k)X(s)$$

$$X(s) = \frac{1}{ms^2 + bs + k}F(s)$$

$$\begin{aligned} X(s) &= \frac{1}{ms^2 + bs + k}F(s) && = test \\ &= \frac{\frac{k}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}\left(\frac{1}{k}\right)F(s) \end{aligned}$$

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Call the denominator the characteristic polynomial, and demonstrate importance when doing the partial fraction decomposition

Characteristic Polynomial:

$$\Delta(s) = ms^2 + bs + k$$

Factoring

Step Response: $f(t) = u(t) \overset{\mathcal{L}}{\Rightarrow} F(s) = \frac{1}{s}$

$$X(s) = \frac{1}{ms^2 + bs + k}\left(\frac{1}{s}\right) = \frac{1}{s(ms^2 + bs + k)} = \frac{\frac{1}{m}}{s(s^2 + \frac{b}{m}s + \frac{k}{m})}$$

Factoring the characteristic polynomial

In order to do the partial fraction decomposition, it must be in factored form, thus factoring via the quadratic equation: $\Delta(s) = ms^2 + bs + k$

$$s = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = -\frac{b}{2m} \pm \sqrt{\frac{b^2 - 4mk}{4m^2}} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \sqrt{\frac{k}{m}}^2}$$

3 Potential cases:

1. **Damped**: $(\frac{b}{2m})^2 > \frac{k}{m} \Rightarrow (s+a)(s+b)$
2. **Critically Damped**:$(\frac{b}{2m})^2 = \frac{k}{m} \Rightarrow (s+a)^2$
3. **Underdamped**: $(\frac{b}{2m})^2 < \frac{k}{m} \Rightarrow (s + \sigma \pm j\omega)$

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This is equivalent to $\Delta(s) = s^2 + \frac{b}{m}s + \frac{k}{m}$
This motivates the standard characteristic polynomial form:

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 \Rightarrow s = \zeta\omega_0 \pm \sqrt{(\zeta\omega_0)^2 - \omega_0^2} = \omega_0\left(\zeta \pm \sqrt{\zeta - 1}\right)$$

Let $2\zeta\omega_n = \sqrt{\frac{b}{m}}$ and $\omega_0 = \sqrt{\frac{k}{m}}$

$$\Delta(s) = s^2 + \frac{b}{m}s + \left(\sqrt{\frac{k}{m}}\right)^2 \iff \Delta(s) = s^2 + 2\zeta\omega_0 s + \omega_0^2$$

In this instance, the three cases are easily seen based on ζ:

1. Damped: $\zeta > 1$
2. Critically Damped: $\zeta = 1$
3. Underdamped: $\zeta \in [0, 1)$

Case 1 (Damped)

Let $a = \frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \left(\sqrt{\frac{k}{m}}\right)^2}$ and $b = \frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \left(\sqrt{\frac{k}{m}}\right)^2}$

$$X(s) = \frac{1}{ms(s^2 + \frac{b}{m}s + \frac{k}{m})} = \frac{\frac{1}{m}}{s(s+a)(s+b)}$$

Evaluate Coefficients: $C_i = \left.\frac{(s-\lambda_i)}{m(s(s+a)(s+b))}\right|_{s=\lambda_i}$

$$X(s) = \frac{C_1}{s} + \frac{C_2}{s+a} + \frac{C_3}{s+b} \overset{\mathcal{L}}{\Longleftrightarrow} x(t) = \left(C_1 + C_2 e^{-at} + C_3 e^{-bt}\right) u(t)$$

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Evaluate coeficients: $(a)(b) = (\frac{b}{2m})^2 - ((\frac{b}{m})^2 - \frac{k}{m}) = \frac{k}{m}$, $\quad (a-b) = 2\sqrt{((\frac{b}{2m})^2 - \frac{k}{m})}$

$$C_1 = \frac{(s)}{ms(s+a)(s+b)}\Bigg|_{s=0} = \frac{1}{m(a)(b)} \Rightarrow C_1 = \frac{1}{k}\textbf{(Hook's Law @ steady-state)}$$

$$C_2 = \frac{(s+a)}{ms(s+a)(s+b)}\Bigg|_{s=-a} = \frac{1}{m(-a)(-a+b)} = \frac{1}{m(a)(a-b)}$$

$$C_3 = \frac{(s+b)}{ms(s+a)(s+b)}\Bigg|_{s=-b} = \frac{1}{m(-b)(a-b)} = \frac{-1}{m(b)(a-b)}$$

$$C_{2,3} = \frac{\pm 1}{2m\sqrt{((\frac{b}{2m})^2 - \frac{k}{m})}\left(\frac{b}{2m} \pm \sqrt{(\frac{b}{2m})^2 - \left(\sqrt{\frac{k}{m}}\right)^2}\right)}$$

Case 2 (Critically Damped)

Let $a = \frac{b}{2m}$

$$X(s) = \frac{\frac{1}{m}}{s(s+a)^2} = \frac{C_1}{s} + \frac{C_2}{s+a} + \frac{C_3}{(s+a)^2} \overset{\mathcal{L}}{\Rightarrow} x(t) = \left(C_1 + C_2 e^{-at} + C_3 t e^{-at}\right) u(t)$$

Case 3 (Underdamped)

Let $\sigma = \frac{b}{m}$ and $\omega = \sqrt{\sqrt{\frac{k}{m}}^2 - \left(\frac{b}{2m}\right)^2}$

$$X(s) = \frac{\frac{1}{m}}{s(s+\sigma \pm j\omega)} = \frac{C_1}{s} + \frac{C_2}{(s+\sigma+j\omega)} + \frac{C_3}{(s+\sigma-j\omega)}$$

$$\Updownarrow \mathcal{L}$$

$$x(t) = \left(C_1 + C_2 e^{-\sigma t} e^{j\omega t} + C_3 e^{-\sigma t} e^{-j\omega t}\right) u(t)$$

$$= C_1 u(t) + 2e^{-\sigma t} u(t) \frac{C_2 e^{j\omega t} + C_3 e^{-j\omega t}}{2} \Leftarrow \textbf{Convert using Euler's Identity}$$

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Alternative approach

$$X(s) = \frac{\frac{1}{m}}{s(s^2 + \frac{b}{m}s + \frac{k}{m})} = \frac{C_1}{s} + \frac{C_2}{(s + \frac{b}{2m})^2 + \sqrt{\frac{k}{m} - \sqrt{\frac{b}{m}}}^2} \overset{\mathcal{L}}{\Rightarrow}$$

$$\overset{\mathcal{L}}{\Rightarrow} x(t) = \left(C_1 + \frac{C_2}{\sqrt{\frac{k}{m} - \sqrt{\frac{b}{m}}}} \exp\left\{ -\frac{b}{2m} t \right\} \cos\left(\sqrt{\frac{k}{m} - \sqrt{\frac{b}{m}}} \right) t \right) u(t)$$

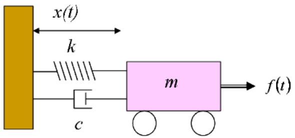
Lecture Overview

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = f(t)$$

$$X(s) = \frac{1}{ms^2 + bs + k} F(s) = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} F(s)$$

$$H(s) = \frac{X(s)}{U(s)} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \zeta = \sqrt{\frac{b^2}{4mk}} \quad U(s) = \frac{1}{k} F(s)$$



Bibliography I

Detroit Mercy University of Michigan, Carnegie Mellon.
Introduction: System modeling.