Introduction to Frequency Response Analysis

Mechanical System Forced Response, Transfer Functions, and Input-Output Characterization

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Outline

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Review: Mechanical System Modeling

$$F = m\mathbf{a} = m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}\right)$$

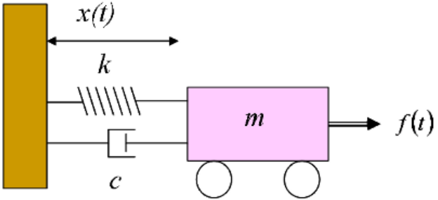
$$m\frac{\mathrm{d}}{\mathrm{d}t^2}x(t) = \sum F = f(t) - b\frac{\mathrm{d}}{\mathrm{d}t}x(t) - kx(t)$$

Let $\mathbf{x} = x(t)$ and $\mathbf{u} = f(t)$

$$m\ddot{\mathbf{x}} + c\dot{\mathbf{x}} + k\vec{\mathbf{x}} = \mathbf{u}$$

$$\ddot{\mathbf{x}} = \left(\frac{-c}{m}\right)\dot{\mathbf{x}} + \left(\frac{-k}{m}\right)\mathbf{x} + \left(\frac{1}{m}\right)\mathbf{u}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\dot{\mathbf{x}}\\ \mathbf{x}\end{bmatrix} = \begin{bmatrix}1 & 0\\ -\frac{c}{m} & -\frac{k}{m}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{x}}\\ \mathbf{x}\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}[\mathbf{u}]$$



Spring Mass Damper System [1]

Review: Second-Order System Dynamics

Transfer Function

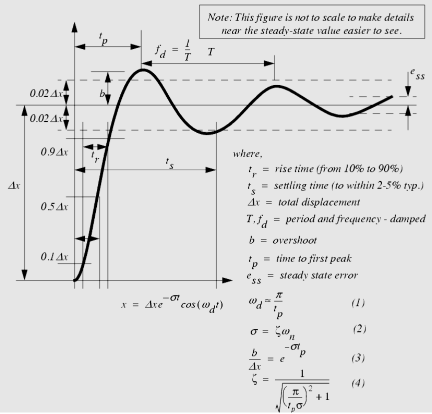
$$H(s) = \frac{Y(s)}{U(s)} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

System Poles

$$s = -\zeta\omega_0 \pm \omega_0\sqrt{1-\zeta^2}$$

Spring Mass Damper System Parameters

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \zeta = \sqrt{\frac{c^2}{4mk}}$$



2nd Order System Response [2]

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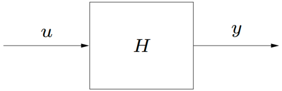
Review: Steady-state Input System Response



Convolution:

$$y(t) = \int_0^t h(\tau)u(t-\tau)\mathrm{d}\tau \quad Y(s) = H(s)U(s)$$

Sinusoidal Input:

$$u(t) = cos(\omega t) = (e^{j\omega t} + e^{-j\omega t})/2 = 1e^{j0}$$

Steady-State Output:

$$y(t) = \int_0^\infty h(\tau)cos(\omega(t-\tau))\mathrm{d}\tau = Ae^{j\phi} = A\cos(\omega t + \phi)$$

where $A = |H(j\omega)|$ and $\phi = \angle(H(j\omega))$.

Steady-state System Response

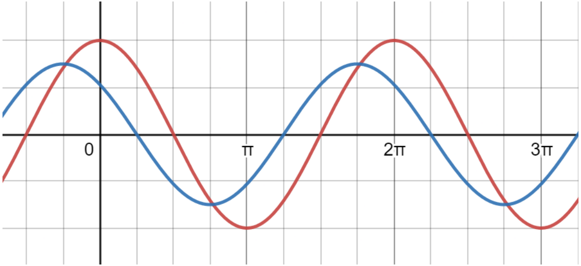
$$\color{red}{u(t) = U_0 \cos(\omega t)} \quad \color{blue}{y(t) = Y_0 \cos(\omega t + \phi)}$$

Magnitude Gain:

$$|H(j\omega)| = \frac{Y_0}{U_0} = \frac{1.5}{2} = 0.75$$

Phase Shift:

$$\angle H(j\omega) = \phi = \frac{\pi}{4}$$



Steady-state System Response with $\omega = 1$.

Homework 6: Sketch Bode Plots for given Transfer Functions See posted review lectures for examples

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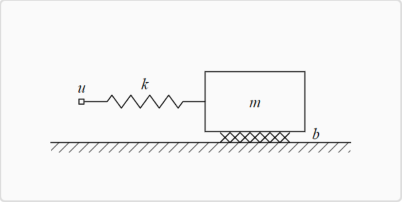
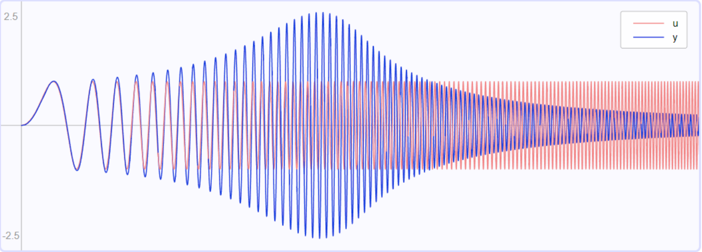
Activity: Spring Mass Damper Frequency Response [3]





https://www.sccs.swarthmore.edu/users/12/abiele1/Linear/examples/freq.html

Bode Plot

Transfer Function:

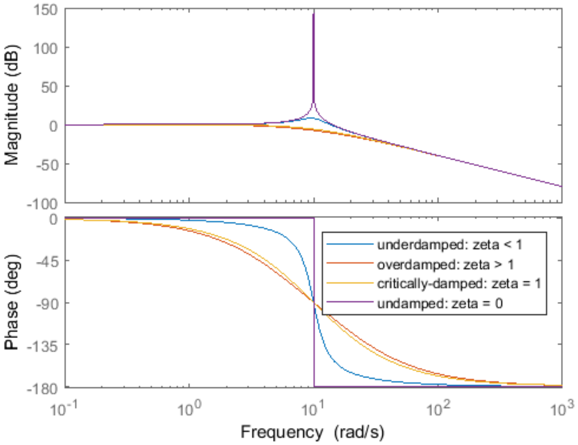
$$H(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Magnitude Gain:

$$|H(j\omega)|_{db} = 20\log_{10}\left|\frac{Y_0}{U_0}\right|$$

Phase Shift:

$$\angle H(j\omega) = \phi$$



Bode Diagram varying dampening factor, ζ.

Activity: Transfer Function Frequency Response

Investigate the system response by varying the input frequency, amplitude, and phase. [4]

Interactive Demo

Choose a transfer function.

○ $H(s) = \frac{1}{1+2s}$ $\qquad H(j\omega) = \frac{1}{1+j2\omega}$

◉ $H(s) = \frac{1.6}{s^2+0.5s+1.6}$ $\qquad H(j\omega) = \frac{1.6}{(1.6-\omega^2)+j(0.5\cdot\omega)}$

Set input parameters, $V_{in}(t) = A\cdot\cos(\omega\cdot t+\varphi)$.

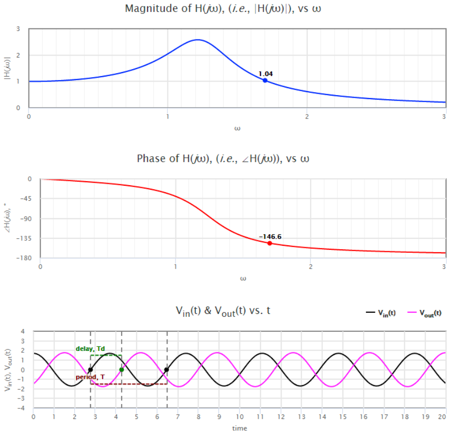
Set ω: 1.500 ω 0 — 3

Set A: 1.7 A 0.2 — 2

Set φ: 0 φ -180 — 180

At ω =1.5, $H(j\omega) = 1.6/(0.31 + j0.75) = 1.61\angle{-130.9°} = M\angle\theta$.
Since the input can be represented as $1.7\angle 0°$,
The output is $M\cdot A\angle(\theta+\varphi) = 2.74\angle{-130.9°}$.

	Magnitude	Phase	Time Domain
H(jω)	1.61	-130.9°	1.61·cos(1.5·t + -130.9°)
Input	1.7	0°	1.7·cos(1.5·t + 0°)
Output	2.74	-130.9°	2.74·cos(1.5·t + -130.9°)



https://lpsa.swarthmore.edu/Bode/BodeWhat.html

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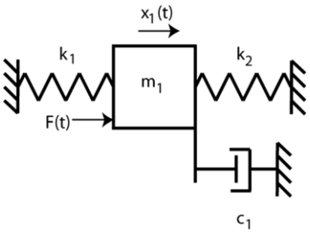
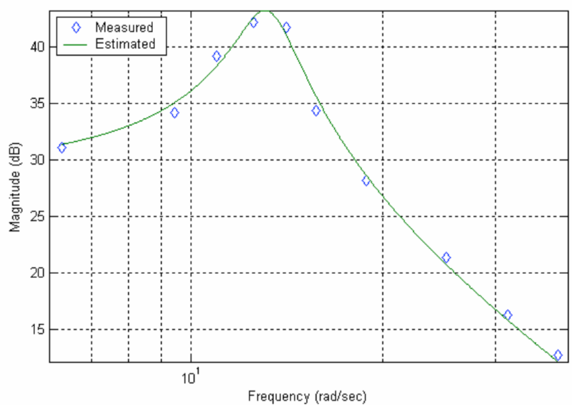
Real-World Applications: System Characterization



$$\frac{X_1(s)}{F(s)} = \frac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

Parameter	Initial Estimate	Final Estimate
K	30.6	28.9
ω_n	13.2	13.2
ζ	0.10	0.10

System Characterization Paper Results [5]



Lab 3: Experiment with selected system to obtain frequency response and characterize dynamics with an appropriate transfer function. See lab instructions for more details

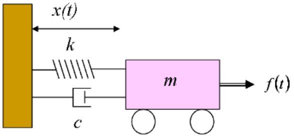
Lecture Overview

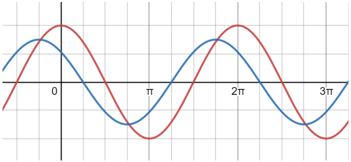
$$H(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

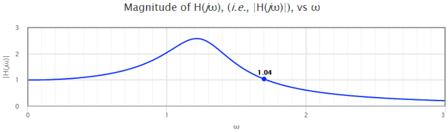
$$\omega_0 = \sqrt{\frac{k}{m}} \quad \zeta = \sqrt{\frac{c^2}{4mk}}$$

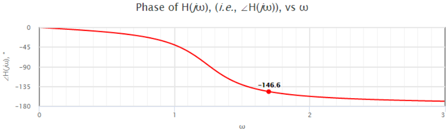
$$u(t) = U_0 \cos(\omega t)$$

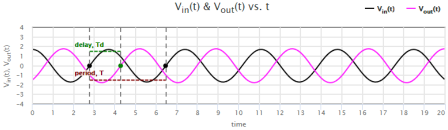
$$y(t) = Y_0 \cos(\omega t + \phi)$$











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