Introduction to 2nd-order System Response

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Outline

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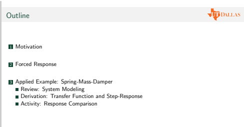
1 Motivation

2 Forced Response

3 Applied Example: Spring-Mass-Damper
- Review: System Modeling
- Derivation: Transfer Function and Step-Response
- Activity: Response Comparison



4th-wall break notes

- Lecture Objective: **why 2nd-order roots of a dynamical system's can result in more interesting responses** (i.e.) the 3 cases as a result from the quadratic equation
- Math background/assumptions:
  - Simple ODEs solutions are covered in prereq and explained again in the intro of this course
  - Specifically, Laplace transform methods and the **inverse-laplace via partial fraction expansion** will be well known to students.
  - In a real course I'd spend time in lecture having students walk me through the derivation of the cases instead of leaving as an exercise/assignment.
- Previous lectures:
  - 1st order-system response and how time-constant plays into the system impulse and step-response
  - Solutions to differential equations (w/in time and frequency domains)

Real-World Dynamical Systems

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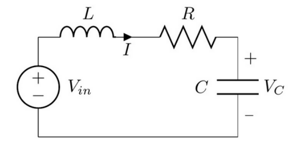
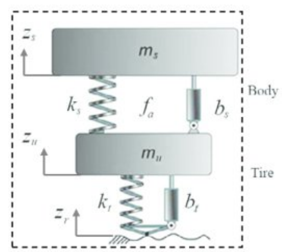
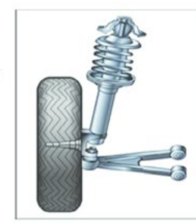
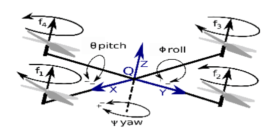
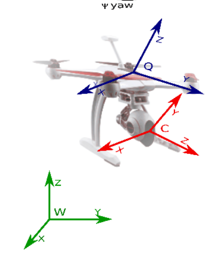










Step Response - 1st vs 2nd order

Step Input: $\mathbf{u}(t) \overset{\mathcal{L}}{\Rightarrow} U(s) = \frac{1}{s}$

1st-order:

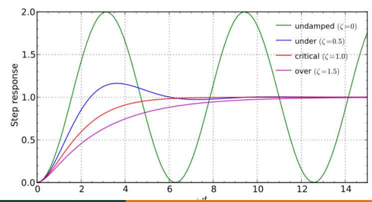
$$Y(s) = \frac{K}{\tau s + 1}\left(\frac{1}{s}\right) \quad \overset{\mathcal{L}^{-1}}{\Rightarrow} \quad y(t) = K(1 - e^{-t/\tau})\mathbf{u}(t)$$

2nd-order:

$$Y(s) = \frac{K}{(s+p_1)(s+p_2)}\left(\frac{1}{s}\right) \quad \overset{\mathcal{L}^{-1}}{\Rightarrow} \quad y(t) = \left(C_1 + C_2 e^{-p_1 t} + C_3 e^{-p_2 t}\right)\mathbf{u}(t)$$

3 distinct cases:

- Damped: $p_1 \neq p_2$
- Critically Damped: $p_1 = p_2$
  Special Case:
  $C_2 e^{-p_1 t} + C_3 e^{-p_2 t} \rightarrow$
  $C_2 e^{-p_{1,2} t} + C_3 t e^{-p_{1,2} t}$
- Underdamped: $p_{1,2} = \sigma \pm j$

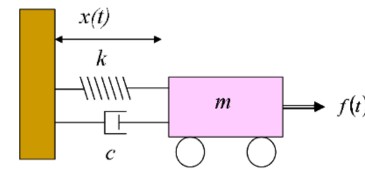


Spring Mass-Damper System Modeling

Newton's 2nd Law:

$$F = m\mathbf{a} = m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}\right)$$

$$m\frac{\mathrm{d}}{\mathrm{d}t^2}x(t) = \sum F = f(t) - b\frac{\mathrm{d}}{\mathrm{d}t}x(t) - kx(t)$$



Spring Mass Damper System [1]

Differential Equation: ($\mathbf{x} = x(t)$, $\mathbf{u} = f(t)$)

$$m\ddot{\mathbf{x}} + c\dot{\mathbf{x}} + k\mathbf{x} = \mathbf{u}$$

Activity: `https://www.sccs.swarthmore.edu/users/12/abiele1/Linear/examples/simple.html`

Transfer Function Derivation

Convert Differential Equation to Laplace: $(x(t) = \dot{x}(t) = 0)$

$$f(t) = m\ddot{x}(t) + b\dot{x}(t) + kx(t) \quad \overset{\mathcal{L}}{\Rightarrow} \quad F(s) = ms^2X(s) + bsX(s) + kX(s)$$

Solve for $X(s)$ in terms of $F(s)$

$$X(s) = \frac{1}{ms^2 + bs + k}F(s) = \frac{1}{m(s^2 + \frac{b}{m}s + \frac{k}{m})}\left(\frac{k}{k}\right)F(s) = \left(\frac{1}{k}\right)\frac{\frac{k}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}F(s)$$

Transfer Function:

$$H(s) = \frac{X(s)}{F(s)} = \left(\frac{1}{k}\right)\frac{\frac{k}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}} \Leftarrow \underset{\textbf{(Hook's Law)}}{F = k\Delta x} \Leftarrow \underset{\textbf{(Standard Form)}}{\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}}$$

Factoring the characteristic polynomial

Apply the quadratic formula to find the roots of the characteristic polynomial:

$$\Delta(s) = ms^2 + bs + k \quad \Rightarrow \quad s = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

3 Potential cases:

1. **Damped**: $b^2 > 4mk \Rightarrow p_1 \neq p_2 \Rightarrow (s+p_1)(s+p_2)$
2. **Critically Damped**: $b^2 = 4mk \Rightarrow p_1 = p_2 \Rightarrow (s+p_{1,2})^2$
3. **Underdamped**: $b^2 < 4mk \Rightarrow p_{1,2} = \sigma \pm j\omega$

2024-02-20

2nd-order System Dynamics
└Applied Example: Spring-Mass-Damper
  └Derivation: Transfer Function and Step-Response
    └Factoring the characteristic polynomial

This motivates the standard characteristic polynomial form:

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 \Rightarrow s = \zeta\omega_0 \pm \sqrt{(\zeta\omega_0)^2 - \omega_0^2} = \omega_0\left(\zeta \pm \sqrt{\zeta - 1}\right)$$

Let $2\zeta\omega_n = \sqrt{\frac{b}{m}}$ and $\omega_0 = \sqrt{\frac{k}{m}}$

$$\Delta(s) = s^2 + \frac{b}{m}s + \left(\sqrt{\frac{k}{m}}\right)^2 \iff \Delta(s) = s^2 + 2\zeta\omega_0 s + \omega_0^2$$

In this instance, the three cases are easily seen based on ζ:

1. Damped: $\zeta > 1$
2. Critically Damped: $\zeta = 1$
3. Underdamped: $\zeta \in [0, 1)$

Case 1 (Damped)

Distinct real roots: $p_1 \neq p_2 \Rightarrow \Delta(s) = s(s+p_1)(s+p_2)$

$$X(s) = \left(\frac{1}{k}\right)\frac{\frac{k}{m}}{s(s^2 + \frac{b}{m}s + \frac{k}{m})} = \frac{K}{s(s+p_1)(s+p_2)}$$

Partial Fraction Expansion:

$$X(s) = \frac{C_1}{s} + \frac{C_2}{s+p_1} + \frac{C_3}{s+p_2}$$

Inverse Laplace:

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} x(t) = \left(C_1 + C_2 e^{-p_1 t} + C_3 e^{-p_2 t}\right)u(t)$$

2024-02-20

2nd-order System Dynamics
└Applied Example: Spring-Mass-Damper
  └Activity: Response Comparison
    └Case 1 (Damped)

Let $a = \frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \left(\sqrt{\frac{k}{m}}\right)^2}$ and $b = \frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \left(\sqrt{\frac{k}{m}}\right)^2}$ Evaluate coeficients:

$(a)(b) = (\frac{b}{2m})^2 - ((\frac{b}{m})^2 - \frac{k}{m}) = \frac{k}{m}$, $\quad (a-b) = 2\sqrt{((\frac{b}{2m})^2 - \frac{k}{m})}$

$$C_1 = \left.\frac{(s)}{ms(s+a)(s+b)}\right|_{s=0} = \frac{1}{m(a)(b)} \Rightarrow C_1 = \frac{1}{k}\textbf{(Hook's Law @ steady-state)}$$

$$C_2 = \left.\frac{(s+a)}{ms(s+a)(s+b)}\right|_{s=-a} = \frac{1}{m(-a)(-a+b)} = \frac{1}{m(a)(a-b)}$$

$$C_3 = \left.\frac{(s+b)}{ms(s+a)(s+b)}\right|_{s=-b} = \frac{1}{m(-b)(a-b)} = \frac{-1}{m(b)(a-b)}$$

$$C_{2,3} = \frac{\pm 1}{2m\sqrt{((\frac{b}{2m})^2 - \frac{k}{m})}\left(\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \left(\sqrt{\frac{k}{m}}\right)^2}\right)}$$

Case 2 (Critically Damped)

Repeated Roots: $b^2 = 4mk \Rightarrow p_1 = p_2 \Rightarrow \Delta(s) = s(s+p)^2$

$$X(s) = \left(\frac{1}{k}\right)\frac{\frac{k}{m}}{s(s^2 + \frac{b}{m}s + \frac{k}{m})} = \frac{K}{s(s+p)^2}$$

Partial Fraction Expansion:

$$X(s) = \frac{C_1}{s} + \frac{C_2}{s+p} + \frac{C_3}{(s+p)^2}$$

Inverse Laplace:

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} x(t) = \left(C_1 + C_2 e^{-pt} + C_3 t e^{-pt}\right)u(t)$$

Case 3 (Underdamped)

Repeated Roots: $b^2 = 4mk \Rightarrow p_1 = p_2 \Rightarrow \Delta(s) = s(s+p)^2$

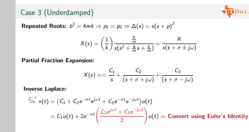
$$X(s) = \left(\frac{1}{k}\right)\frac{\frac{k}{m}}{s(s^2 + \frac{b}{m}s + \frac{k}{m})} = \frac{K}{s(s+\sigma \pm j\omega)}$$

Partial Fraction Expansion:

$$X(s) == \frac{C_1}{s} + \frac{C_2}{(s+\sigma+j\omega)} + \frac{C_3}{(s+\sigma-j\omega)}$$

Inverse Laplace:

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} x(t) = \left(C_1 + C_2 e^{-\sigma t}e^{j\omega t} + C_3 e^{-\sigma t}e^{-j\omega t}\right)u(t)$$
$$= C_1 u(t) + 2e^{-\sigma t}\left(\frac{C_2 e^{j\omega t} + C_3 e^{-j\omega t}}{2}\right)u(t) \Leftarrow \textbf{Convert using Euler's Identity}$$



Alternative approach

$$X(s) = \frac{\frac{1}{m}}{s(s^2 + \frac{b}{m}s + \frac{k}{m})} = \frac{C_1}{s} + \frac{C_2}{(s + \frac{b}{2m})^2 + \sqrt{\frac{k}{m} - \sqrt{\frac{b}{m}}}^2} \overset{\mathcal{L}}{\Rightarrow}$$

$$\overset{\mathcal{L}}{\Rightarrow} x(t) = \left(C_1 + \frac{C_2}{\sqrt{\frac{k}{m} - \sqrt{\frac{b}{m}}}} \exp\left\{-\frac{b}{2m}t\right\} \cos\left(\left(\sqrt{\frac{k}{m} - \sqrt{\frac{b}{m}}}\right)t\right)\right)u(t)$$

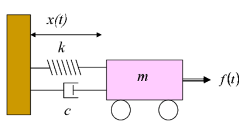
Lecture Overview

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = f(t)$$

$$X(s) = \frac{1}{ms^2 + bs + k}F(s) = \frac{\frac{k}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}\frac{1}{k}F(s)$$

$$H(s) = \frac{X(s)}{F(s)} = (K)\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \zeta = \sqrt{\frac{b^2}{4mk}} \quad K = \frac{1}{k}$$



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TLDR: Second-Order System Dynamics

Transfer Function

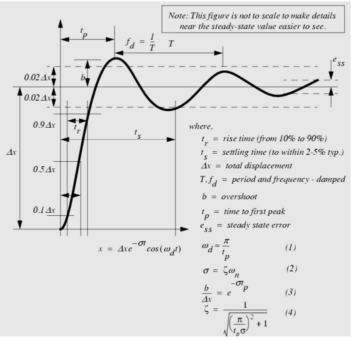
$$H(s) = \frac{Y(s)}{U(s)} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

System Poles

$$s = -\zeta\omega_0 \pm \omega_0\sqrt{1-\zeta^2}$$

Spring Mass Damper System Parameters

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \zeta = \sqrt{\frac{c^2}{4mk}}$$



2nd Order System Response [2]