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Ellipsoidal state estimation for dynamical systems☆

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Abstract

The ellipsoidal estimation of state for dynamical systems is an efficient technique for the set-membership modelling of uncertain dynamical systems. In the paper, an overview of recent results on the method of ellipsoids for the approximation of reachable sets of dynamical systems is presented. Optimal ellipsoidal estimates of reachable sets are considered for various optimality criteria, including the new one equal to the projection of an ellipsoidal onto a given direction. Nonlinear differential equations governing the evolution of ellipsoids are analyzed and simplified. The method of ellipsoids is extended to the case where the parameters of the system are uncertain and/or subjected to unknown but bounded perturbations. Generalizations of the method of ellipsoids and its applications to control problems for dynamical systems are discussed.

Keywords: Uncertain systems; Control; Estimation

1. Introduction

Modelling of systems in the presence of unknown but bounded perturbations attracts permanent attention of researchers in the fields of estimation and control. The set-membership approach to such systems is a natural counterpart to the well-known stochastic, or probabilistic, approach.

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In the framework of the set-membership approach, the ellipsoidal estimation seems to be the most efficient technique. Among its advantages are the explicit form of approximation, smoothness of the boundary, invariance with respect to linear transformations, possibility of optimization, etc. The ellipsoidal estimation originates at the end of 1960s [1,19,22]. The earlier results have been summarized in books [14,20]. The concept of optimality for the approximating ellipsoids have been first introduced in [2]. These results have been generalized, extended and summarized in books [3,4,6]. Various aspects of ellipsoidal estimation have been developed and summarized in books [15,17], survey papers [11,12], and special issues of journals [18,21].

In this paper, basic concepts and results of the method of ellipsoids are outlined. Recent developments, generalizations, and applications of the method are presented [5,7–10,13].

2. Ellipsoidal estimation

Consider a linear system of ordinary differential equations

$$\dot{x} = A(t)x + B(t)u + f(t), \quad t \geqslant s. \tag{1}$$

Here, $x \in R^n$ is the n-vector of state, $u \in R^m$ is the m-vector of unknown perturbations, the dot denotes differentiation with respect to time t, A is an $n \times n$ matrix, B is an $m \times m$ matrix, and f is an n-vector. The matrices $A(t)$ and $B(t)$ as well as the vector $f(t)$ are given functions of t for $t \geqslant s$, where s is the initial time instant.

Denote by $E(a, Q)$ the following n-dimensional ellipsoid:

$$E(a, Q) = \{x : (Q^{-1}(x - a), (x - a)) \leqslant 1\}, \tag{2}$$

where $a \in R^n$ is its center, Q is a positive definite $n \times n$ matrix, and $(\cdot, \cdot)$ denotes the scalar product of vectors. Suppose the unknown perturbation $u(t)$ is bounded by the ellipsoid as follows:

$$u(t) \in E(0, G(t)), \quad t \geqslant s, \tag{3}$$

where $G(t)$ is an $m \times m$ matrix specified for $t \geqslant s$. The initial data for Eq. (1) are also uncertain and can be described by the inclusion

$$x(s) \in M = E(a_0, Q_0), \tag{4}$$

where M is a given ellipsoid in R^n, a_0 is a given n-vector, and Q_0 is a given positive definite $n \times n$ matrix.

The *reachable* or *attainable* set $D(t, s, M)$ of system (1) for $t \geqslant s$ is defined as the set of all end points $x(t)$ at the instant t of all state trajectories $x(\cdot)$ compatible with Eqs. (1), (3), and (4). The reachable set has the following evolutionary property:

$$D(t, s, M) = D(t, \tau, D(\tau, s, M)), \tag{5}$$

which is true for all $\tau \in [s, t]$. Since the precise determination of reachable sets presents usually a very complicated problem, one can be often satisfied with finding two-sided (inner and outer) ellipsoidal approximations of these sets such that

$$E(a^-(t), Q^-(t)) \subset D(t, s, M) \subset E(a^+(t), Q^+(t)), \quad t \geqslant s. \tag{6}$$

The families of ellipsoids $E^+(t) = E(a^+(t), Q^+(t))$ and $E^-(t) = E(a^-(t), Q^-(t))$ are called *superreachable* and *subreachable*, respectively, if $E^+(t) \supset D(t, \tau, E^+(\tau))$ and $E^-(t) \subset D(t, \tau, E^-(\tau))$ for all $\tau \in [s, t]$. These properties are similar to (5). We shall consider mostly outer estimates of reachable sets in the class of superreachable ellipsoids and omit the superscript $^+$. Later, some remarks about inner ellipsoidal estimates will be made.

3. Optimality

To make the approximating ellipsoids closer to reachable sets, we shall impose certain optimality conditions.

Let us characterize an ellipsoid $E(a, Q)$ by a scalar optimality criterion J which is a given function $L(Q)$ of the matrix Q, i.e. $J(E(a, Q)) = L(Q)$. The function $L(Q)$ is defined for all symmetric positive definite matrices Q, is smooth and monotone. The later property means that $L(Q_1) \geqslant L(Q_2)$, if $Q_1 - Q_2$ is a nonnegative definite matrix. Consider some important particular cases of the general criterion $L(Q)$.

1. The volume of an ellipsoid [2–4,6] is given by $J = c_n(\det Q)^{1/2}$, where c_n is a constant depending on n.
2. The sum of the squared semiaxes of an ellipsoid is equal to $J = \mathrm{Tr}\, Q$.
3. A more general linear criterion [4,6] is $J = \mathrm{Tr}(CQ)$, where C is a symmetric nonnegative definite $n \times n$ matrix.
4. The criterion $J = (Qv, v)$, where v is a given nonzero n-vector, is a particular case of the previous one [7]. Here,

$$C = v * v, \quad C_{ij} = v_i v_j, \quad i, j = 1, \ldots, n, \tag{7}$$

   where $*$ denotes the dyadic product of vectors. This criterion has a clear geometric interpretation: it is related to the projection $\Pi_v(E)$ of the ellipsoid $E(a, Q)$ onto the direction of the vector v as follows: $\Pi_v(E) = 2(Qv, v)^{1/2}/|v|$. Hence, the minimization of $J = (Qv, v)$ is equivalent to the minimization of the projection of an ellipsoid onto the direction of the vector v. Other examples of optimality criteria are given in [4,6].

We consider below locally optimal and globally optimal outer ellipsoids [4].

A smooth family of ellipsoids $E^*(t) = E(a(t), Q(t))$ is called *locally optimal*, if it is superreachable and $\mathrm{d}L(Q(\tau))/\mathrm{d}\tau|_{\tau=t} \to \min$ for all $t \geqslant s$, where the minimum is taken over all smooth families of superreachable ellipsoids $E^+(t)$ such that $E^+(t) = E^*(t)$.

A smooth family of superreachable ellipsoids is called *globally optimal* for a given $t = T$, if the minimum of $L(Q(T))$ over all superreachable families of ellipsoids is attained on this family.

Note that all definitions and results related to optimal ellipsoids are true also for the case, where the criterion also depends on time t so that $L = L(Q, t)$. In particular, the linear criterion can be taken equal to $J = \mathrm{Tr}[C(t)Q]$.

4. Equations of ellipsoids

As shown in [2–4,6], the center $a(t)$ and the matrix $Q(t)$ of locally optimal ellipsoids satisfy the following initial value problems for ordinary differential equations

$$\dot{a} = A(t)a + f(t), \quad a(s) = a_0, \tag{8}$$

$$\dot{Q} = A(t)Q + QA^{\mathrm{T}}(t) + hQ + h^{-1}K(t), \quad K(t) = B(t)G(t)B^{\mathrm{T}}(t), \quad Q(s) = Q_0. \tag{9}$$

Here, $^{\mathrm{T}}$ denotes a transposed matrix, and the following notation is used:

$$h = \left[\mathrm{Tr}\left(\frac{\partial L}{\partial Q} K \right) \Big/ \mathrm{Tr}\left(\frac{\partial L}{\partial Q} Q \right) \right]^{1/2}. \tag{10}$$

Note that Eq. (8) for the vector $a(t)$ is linear and does not depend on the chosen optimality criterion $L(Q)$. In contrast, Eqs. (9) and (10) for the matrix $Q(t)$ are nonlinear and depend on $L(Q)$. For ellipsoids optimal in the sense of volume, expression (10) becomes [2–4,6]

$$h = [n^{-1}\,\mathrm{Tr}(Q^{-1}K)]^{1/2}.$$

For the linear criterion $L(Q) = \mathrm{Tr}(CQ)$, it follows from (10):

$$h = [\mathrm{Tr}(CK)/\mathrm{Tr}(CQ)]^{1/2}. \tag{11}$$

The center $a(t)$ of globally optimal ellipsoids coincides with that of locally optimal ones and satisfies the initial value problem (8). The matrix $Q(t)$ of globally optimal ellipsoids satisfies Eq. (9), where instead of (10), we have [4]

$$h = [\mathrm{Tr}(PK)/\mathrm{Tr}(PQ)]^{1/2}. \tag{12}$$

Here, $P(t)$ is a symmetric positive definite $n \times n$ matrix satisfying the following differential equation and initial condition at $t = T$:

$$\dot{P} = -PA - A^{\mathrm{T}}P, \quad P(T) = [\partial L(Q)/\partial Q]_{t=T}. \tag{13}$$

Hence, we have a two-point boundary value problem for the matrices Q and P described by Eqs. (9), (12), and (13).

For the linear criterion $L(Q) = \mathrm{Tr}(CQ)$, Eqs. (13) are reduced to

$$\dot{P} = -PA - A^{\mathrm{T}}P, \quad P(T) = C(T). \tag{14}$$

Therefore, for ellipsoids globally optimal in the sense of the linear criterion, the boundary value problem for the matrices Q and P becomes decoupled and reduces to two initial value problems: a linear one (14) for $P(t)$ (to be solved from $t = T$ to s) and a nonlinear one for $Q(t)$ defined by Eqs. (9) and (12).

Further simplifications are possible for the criterion $J = (Qv, v)$. Substituting C from (7) into (11), we obtain

$$h = [(Kv, v)/(Qv, v)]^{1/2}$$

for locally optimal ellipsoids. Here, $v = v(t)$ is a given nonzero vector function.

For globally optimal ellipsoids, we choose $C(T) = v_T * v_T$, where v_T is a given constant n-vector. Let us introduce the adjoint n-vector satisfying the initial value problem:

$$\dot{\psi} = -A^{\mathrm{T}}\psi, \quad \psi(T) = v_T. \tag{15}$$

Then we obtain [9,10]

$$P(t) = \psi(t) * \psi(t).$$

Thus, in order to find the matrix $Q(t)$ of globally optimal ellipsoids in case of the criterion $J = (Qv_T, v_T)$, one is to solve first the linear n-dimensional initial value problem (15) for $\psi(t)$ (instead of $n(n+1)/2$-dimensional problem for P) and then a nonlinear initial value problem for Q given by Eq. (9), where

$$h = [(K\psi, \psi)/(Q\psi, \psi)]^{1/2}.$$

Thus, the optimality criterion $J = \mathrm{Tr}(CQ)$ and its particular case $J = (Qv, v)$ lead to considerable simplifications of equations for optimal outer ellipsoids. Also, these criteria can give quite satisfactory outer approximations of reachable sets (sometimes, better than the approximations optimal in the sense of volume [10]). However, only ellipsoids optimal in the sense of volume are invariant with respect to linear transformations [4,6].

Equations for inner approximating ellipsoids $E(a^-(t), Q^-(t))$ locally optimal in the sense of volume have been obtained in [2–4,6]. The center of these ellipsoids coincides with that of outer ones, so that $a^-(t) \equiv a(t)$, where $a(t)$ is defined by (8). The matrix $Q^-(t)$ satisfies the following nonlinear initial value problem:

$$\dot{Q}^- = AQ^- + Q^-A^{\mathrm{T}} + 2K^{1/2}(K^{-1/2}Q^-K^{-1/2})^{1/2}K^{1/2}, \quad Q^-(s) = Q_0.$$

Here, $K(t)$ is defined in (9).

While the outer ellipsoids are useful for evaluating the influence of uncertain perturbations $u(t)$, the inner ellipsoids can be used for control problems. If $u(t)$ is a control applied to system (1), the inner ellipsoids determine the states which are certainly reachable under control constraint (3). Moreover, the control $u(t)$, $t \in [s, T]$, bringing the system to any prescribed state $x(T)$ at the given instant $T \geqslant s$ can be readily obtained [4], if $x(T) \in E(a^-(T), Q^-(T))$.

5. Properties of optimal ellipsoids

Outer approximating ellipsoids $E(a(t), Q(t))$ optimal in the sense of the criterion $J = (Qv, v)$ have the following properties [9,10]:

1. Globally optimal ellipsoids touch the reachable sets $D(t, s, M)$ for all $t \in [s, T]$ at points $x(t)$, where the normal to the boundary of these sets is parallel to the vector $\psi(t)$ defined by (15). In other words, these ellipsoids are tight in the sense of [16].
2. Globally optimal ellipsoids are also locally optimal for the vector $v(t) = \psi(t)$, where $\psi(t)$ is defined by (15).

3. Locally optimal ellipsoids for the vector $v(t)$ defined by

$$v(t) = \psi(t), \quad \dot{\psi} = -A^{\mathrm{T}}\psi, \quad \psi(s) = v^0, \tag{16}$$

where v^0 is an arbitrary vector, are also globally optimal for any time instant $T \geqslant s$ and for the optimality criterion $J = (Qv(T), v(T))$.

To construct these locally (and also globally) optimal ellipsoids, one is to solve the linear initial value problem (8) for $a(t)$ and also the initial value problem consisting of Eq. (9) for $Q(t)$ and (16) for $v(t) = \psi(t)$. Here, the initial vector v^0 can be chosen arbitrarily, and different vectors v^0 correspond to different approximating ellipsoids touching reachable sets at different points.

Various properties of nonlinear equations (9) governing the evolution of the matrix $Q(t)$ of locally optimal ellipsoids have been studied in [4,6,7,10] for different optimality criteria.

As a rule, the nonlinear differential equations for the approximating ellipsoids are to be integrated numerically. However, certain explicit analytical solutions have been obtained [2–4,6,10].

Special transformations have been proposed that simplify these nonlinear equations and reduce them to the forms in which either $A = 0$ or $K = I$ in (9), where I is a unit matrix [4,6,7,10].

Asymptotic behavior of the solution of Eq. (9) have been analyzed in the vicinity of the initial point $t = s$, if $Q_0 = Q$ in [4]. This important case corresponds to the situation, where the initial set M is a point $x(s) = a_0$. In this case, Eq. (9) has a singularity (see Eqs. (10) and (11)), and the obtained asymptotic solution can be used for starting the numerical integration of Eq. (9) near the initial point $t = s$ for the case where $Q_0 = 0$.

Asymptotic behavior for solutions of Eq. (9) at infinity ($t \to \infty$) also have been investigated [4,10].

6. Generalizations and applications

The method of ellipsoids has been extended to the case, where the parameters of the linear system (1) are uncertain and/or subjected to unknown but bounded perturbations [5]. Consider the following system:

$$\dot{x} = [A_0(t) + A_1(t)]x + f(t). \tag{17}$$

Here, $x \in R^n$ is the state, the matrix $A_0(t)$ and the n-vector $f(t)$ are given functions of time, whereas the matrix $A_1(t)$ is unknown, and its elements $a_{ij}(t)$ are bounded:

$$|a_{ij}(t)| \leqslant b_{ij}, \quad i, j = 1, \ldots, n, \quad t \geqslant s. \tag{18}$$

Here, b_{ij} are given nonnegative numbers. The system described by Eqs. (17) and (18) includes the case of unknown but bounded perturbations (e.g., the case of parametric excitation) as well as the case of fixed but unknown parameters of the system.

Outer ellipsoidal estimates $E(a(t), Q(t))$ on the reachable set of the system described by Eqs. (17) and (18) have been obtained [5]. The equation for the center $a(t)$ is still the same as (8). Nonlinear matrix equation governing the evolution of the matrix $Q(t)$ has been derived in [5]. In contrast to Eq. (9), here, the right-hand side of the equation for Q depends on the vector a.

The method of ellipsoids has been generalized for nonlinear systems subjected to perturbations. The main idea is to construct a linear comparison system described by Eqs. (1), (3), (4) and such that the reachable sets of the nonlinear system at all time instants $t \geqslant s$ lie within the reachable sets of the comparison system. For same classes of nonlinear systems it is possible. Then the outer ellipsoidal estimates for the linear comparison system can serve also as the outer estimates for the original nonlinear system [4]. Similarly, inner ellipsoidal estimates for reachable sets of nonlinear systems can be obtained [4].

The class of approximating sets can be extended: besides ellipsoids, also intersections and unions of several ellipsoids can significantly improve the approximation of reachable sets [4].

Outer and inner ellipsoidal approximations can be used for the solution and approximation of various problems in control and estimation. Here, we shall only mention some of them; see [4,6] for details.

1. Two-sided approximations for optimal control problems.
2. Construction of suboptimal control bringing the system to the prescribed terminal state.
3. Analysis of practical stability under uncertain perturbations for a bounded time interval.
4. Two-sided estimates in differential games.
5. Evaluation of the influence of parameter uncertainty.
6. Evaluation of parametric excitation.
7. State estimation in the presence of observations corrupted by unknown but bounded errors. Here, set-membership estimation algorithms have been developed, see [4,6] and recent developments in [8,13].
8. Problems of control in the presence of uncertain perturbations have been analyzed and solved [7,10].

7. Conclusion

Method of ellipsoids is an efficient technique for the analysis of dynamical systems subjected to perturbations. By means of this approach, exact and approximate solutions as well as two-sided bounds for the basic problems in control and estimation can be obtained.

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