Observer-Based Control of Discrete-Time LPV Systems with Uncertain Parameters

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Abstract

In this paper we consider output-based stabilization of discrete-time LPV systems in the situation where the parameters are not exactly known, but only available with a finite accuracy. The controllers are obtained using a separate design of an observer and a state feedback and the interconnection is proven to stabilize the LPV system despite the mismatch between the true and available parameters. The approach allows to make tradeoffs between the admissible level of mismatch and the performance in terms of decay factors. All the design conditions will be formulated in term of LMIs, which can be solved efficiently, as is also illustrated by a numerical example.

Index Terms

LPV systems, output feedback and observers, robust control, LMIs, separation principle

I. Introduction

Linear Parameter-Varying (LPV) systems have received considerable attention from the control community in recent years due to their applicability in many practical situations (see [16], [3], [2], [14], [15], [4] and references therein). Controllers that are designed on the basis of LPV system models have to satisfy two important properties, when they are implemented in practice:

- First of all, the controller needs to be output-based, as in practice it is rarely the case that the full state variable is available for feedback.
- Secondly, the controller must be robust with respect to some degree of mismatch between the available and the true parameters as the real parameters are not always known exactly, although this is often assumed in the LPV literature.

The goal of this paper is to design controllers for LPV systems that satisfy these two properties, which means that we would like to solve the output-based controller design problem for discrete-time LPV systems with not exactly known parameters. The problem that the scheduling parameters measurements are only known up to a given precision was mentioned for the first time in [2] in the case of continuous-time LPV systems. Unfortunately, the synthesis of robust dynamic output feedback controllers in [2] has to be performed by the solution of bilinear matrix inequalities. In [11], one considered dynamic output feedback control of continuous-time LPV systems, where only some of the parameters are measured and available for feedback. The derived conditions for the construction of the controllers, which depend only on the measured parameters, are expressed in terms of linear matrix inequalities (LMIs) and an additional coupling constraint, which destroys the convexity of the conditions. Recently, a solution is given in [8] to the robust dynamic output feedback design for continuous-time LPV systems when the measured varying parameters do not exactly fit the real ones using convex programming.

Output feedback control design for discrete-time LPV systems for which the measured parameters do not exactly fit the real ones is an open problem. Convexity is only obtained in case of stability analysis [13]. In [13] it is also shown that an observer that is asymptotically recovering the state when the parameters are exactly measured, is input-to-state stable (ISS) [17], [9] with respect to mismatch between the true and the available parameters. However, [13] does not study the observer synthesis nor the output-based stabilization problem. In addition, it does not allow for minimizing the ISS gain (as a measure for the influence of the mismatch of the parameters on the estimation error). These two important features will be considered in this paper.

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Closely related to LPV systems are switched linear (SL) systems and piecewise affine (PWA) systems, which can be perceived as a subclass of LPV systems in which the parameters only take a *finite* number of values. Observer-based control design for SL systems has been considered in [5] under the assumption of having *exact* knowledge of the parameter values. In case of unknown parameters, [1] proposes design conditions for observers that include an estimation procedure for the parameters. In [10], [18] observers and observer-based controllers were designed for PWA systems based on LMIs. In this case the parameters are also unknown as they depend on the state variable that has to be estimated. However, as for SL and PWA systems the number of parameter values is finite, these results are not applicable to general LPV systems.

This paper contributes to this open problem. In particular, the main contributions are LMI-based conditions for the separate design of state observers and input-to-state stabilizing state feedbacks for discrete-time LPV systems. Next we prove that the resulting closed-loop system is globally exponentially stable for some level of mismatch between the true parameters and the available ones. Interestingly, the flexibility in our framework allows to make tradeoffs between this level of mismatch and the performance of the closed-loop in terms of the decay factor. All the design conditions will be formulated in terms of LMIs, which can be solved efficiently [6].

II. Notation and basic definitions

$\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{N}$ are the field of real numbers, the set of non-negative reals and the set of non-negative integers, respectively. The i-th entry of a real vector x is denoted by x^i (subscripts are used for denoting discrete-time dependence). We denote by $\|x\| = \sqrt{x^T x}$ the Euclidean norm of x in $\mathbb{R}^n$, where M^T denotes the transpose for a vector or matrix M, and by $\|x\|_\infty$ its infinity norm given by $\max_i |x^i|$. For a sequence $\{v_k\}_{k\in\mathbb{N}}$ with $v_k \in \mathbb{R}^n$ we denote its supremum norm $\sup_{k\in\mathbb{N}} \|v_k\|$ by $\|v\|_\infty$. For a matrix $M \in \mathbb{R}^{n\times m}$ we denote its spectral norm $\sqrt{\lambda_{max}(M^T M)}$ by $\|M\|$, where $\lambda_{max}(M^T M)$ denotes the largest eigenvalue of $M^T M$. When a matrix P is positive definite (including symmetry), we write $P \succ 0$. If it is positive semi-definite, we use $P \succeq 0$. Similarly, for (semi-)negative definiteness we write $\prec$ and $\preceq$. By $\mathbf{0}$ and $\mathbf{1}$ we denote the zero and the identity matrix of appropriate dimensions. Below we will use the following result.

Lemma 1: For all $\varepsilon > 0$ and all $a, b \in \mathbb{R}$ it holds that $ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$.

Proof: $0 \leq (\sqrt{\varepsilon}a - \frac{1}{\sqrt{\varepsilon}}b)^2 = \varepsilon a^2 - 2ab + \frac{1}{\varepsilon}b^2$. ∎

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$ and to class $\mathcal{K}_\infty$ if additionally $\varphi(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is decreasing and $\lim_{k\to\infty} \beta(s, k) = 0$. Consider now the discrete-time nonlinear systems

$$x_{k+1} = G(x_k, \omega_k), \qquad \text{and} \tag{1}$$

$$x_{k+1} = G_v(x_k, v_k, \omega_k), \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the state, $v_k \in \mathbb{R}^{d_v}$ is an unknown disturbance input and $\omega_k \in \mathbb{R}^{d_\omega}$ is an uncertainty parameter at discrete time $k \in \mathbb{N}$. $G : \mathbb{R}^n \times \mathbb{R}^{d_\omega} \to \mathbb{R}^n$ and $G_v : \mathbb{R}^n \times \mathbb{R}^{d_v} \times \mathbb{R}^{d_\omega} \to \mathbb{R}^n$ are arbitrary nonlinear functions. We assume that $\omega_k \in \Omega$, $k \in \mathbb{N}$ for some set $\Omega \subset \mathbb{R}^{d_\omega}$.

Definition 2: [17], [9] The system (1) with uncertainty set Ω is called globally asymptotically stable (GAS), if there exists a $\mathcal{KL}$-function β such that, for each $x_0 \in \mathbb{R}^n$ and all $\{\omega_k\}_{k\in\mathbb{N}}$ with $\omega_k \in \Omega$, $k \in \mathbb{N}$, it holds that the corresponding state trajectory satisfies $\|x_k\| \leq \beta(\|x_0\|, k)$ for all $k \in \mathbb{N}$. If β can be taken of the form $\beta(s, k) = ds\lambda^k$ for some $d \geq 0$ and $0 \leq \lambda < 1$ the system (1) with uncertainty set Ω is called globally exponentially stable (GES). The system (2) with uncertainty set Ω is said to be input-to-state stable (ISS) with respect to v if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that, for each $x_0 \in \mathbb{R}^n$, all $\{v_k\}_{k\in\mathbb{N}}$ and all $\{\omega_k\}_{k\in\mathbb{N}}$ with $\omega_k \in \Omega$, $k \in \mathbb{N}$, it holds for all $k \in \mathbb{N}$ that $\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|v\|_\infty)$.

We call λ a *decay factor* for (1) and the function γ an *ISS gain* of (2). Next we state sufficient conditions for ISS using so-called *ISS Lyapunov functions*. The proofs are omitted for shortness, but can be based on [9], [12] by adopting parameter-dependent Lyapunov functions.

Theorem *3:* Let $d_1, d_2 \in \mathbb{R}_{\geq 0}$, let $a, b, c, \mu \in \mathbb{R}_{>0}$ with $c \leq b$ and let $\alpha_1(s) := as^\mu$, $\alpha_2(s) := bs^\mu$, $\alpha_3(s) := cs^\mu$ and $\sigma \in \mathcal{K}$. Furthermore, let $V : \mathbb{R}^n \times \mathbb{R}^{d_\omega} \to \mathbb{R}_{\geq 0}$ be a function such that

$$\alpha_1(\|x\|) \leq V(x, \omega) \leq \alpha_2(\|x\|) \tag{3a}$$

$$V(G_v(x, v, \omega_1), \omega_2) - V(x, \omega_1) \leq -\alpha_3(\|x\|) + \sigma(\|v\|) \tag{3b}$$

for all $x \in \mathbb{R}^n$, all $v \in \mathbb{R}^{d_v}$ and ω, ω_1, $\omega_2 \in \Omega$. Then system (2) with uncertainty set Ω is ISS with respect to v. In case (3a) and $V(G(x,\omega_1),\omega_2) - V(x,\omega_1) \leq -\alpha_3(\|x\|)$ hold for all x and ω, ω_1, $\omega_2 \in \Omega$, then system (1) with uncertainty set Ω is GES with decay factor $1 - \frac{c}{b} \in [0,1)$.
A function V that satisfies (3) is called an *ISS Lyapunov function.*

III. Problem statement

We consider discrete-time linear parameter-varying (LPV) systems given by

$$
\begin{aligned}
x_{k+1} &= A(\rho_k)x_k + Bu_k && \text{(4a)}\\
y_k &= Cx_k + Du_k && \text{(4b)}
\end{aligned}
$$

with $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^m$ and $u_k \in \mathbb{R}^r$ the state, output and control input at discrete time $k \in \mathbb{N}$ and $\rho_k \in \mathbb{R}^L$ is a time-varying parameter. The matrices $A(\rho) \in \mathbb{R}^{n\times n}$ for each ρ, $B \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times r}$ have appropriate dimensions. The parameter ρ lies in some set $\Theta \subset \mathbb{R}^L$ and we assume that $A : \Theta \to \mathbb{R}^{n\times n}$ can be written in the polytopic form $A(\rho) = \sum_{i=1}^N \xi^i(\rho)A_i$ for certain continuous functions $\xi^i : \Theta \to \mathbb{R}$ and matrices $A_i \in \mathbb{R}^{n\times n}$, $i = 1,\ldots,N$. In addition we assume that the mapping $\xi : \Theta \to \mathbb{R}^N$ given by $\xi := (\xi^1,\ldots,\xi^N)^\top$ is such that $\xi(\Theta) \subset \mathcal{S}$ with $\mathcal{S} = \{\mu \in \mathbb{R}^N \mid \mu^i \geq 0,\ i = 1,\ldots,N \text{ and } \sum_{i=1}^N \mu^i = 1\}$. Hence, $A(\rho)$ lies for each $\rho \in \Theta$ in the convex hull $\mathbf{Co}\{A_1,\ldots,A_N\}$.

In this paper, we focus on the situation where the true (time-varying) parameter ρ_k is actually not available, but only an estimated parameter $\hat\rho_k \in \Theta$ fulfilling $\|\rho_k - \hat\rho_k\|_\infty \leq \Delta$, where Δ is some nonnegative constant indicating the uncertainty level.

Problem 4: Design an observer-based controller of the form

$$
\begin{aligned}
\hat x_{k+1} &= A(\hat\rho_k)\hat x_k + Bu_k + L(\hat\rho_k)(y_k - \hat y_k) && \text{(5a)}\\
\hat y_k &= C\hat x_k + Du_k && \text{(5b)}\\
u_k &= K(\hat\rho_k)\hat x_k && \text{(5c)}
\end{aligned}
$$

with $L(\hat\rho_k) = \sum_{i=1}^N \xi^i(\hat\rho_k)L_i$ and $K(\hat\rho_k) = \sum_{i=1}^N \xi^i(\hat\rho_k)K_i$ by appropriately choosing the gains L_i and K_i, $i = 1,\ldots,N$ such that the closed-loop system (4)-(5) is GAS when the uncertainty satisfies $\|\rho_k - \hat\rho_k\|_\infty \leq \Delta$ and $\hat\rho_k \in \Theta$ for all $k \in \mathbb{N}$.

As a second goal we aim at designing an observer-based controller as in the above problem formulation that guarantees GAS of the closed-loop system for the largest uncertainty level Δ.

IV. Observer Design

We first focus on the estimation of the state x_k using a polytopic observer of the form

$$
\left\{\begin{array}{lllllll}
\hat x_{k+1} & = & A(\hat\rho_k)\hat x_k & + & Bu_k & + & L(\hat\rho_k)(y_k - \hat y_k)\\
\hat y_k & = & C\hat x_k & + & Du_k, & &
\end{array}\right. \tag{6}
$$

where $\hat\rho_k \in \Theta$ and possibly $\rho_k \neq \hat\rho_k$. The estimation error $e_k{:=}x_k - \hat x_k$ is governed by

$$
e_{k+1} = \mathcal{A}_e(\hat\rho_k)e_k + v_k \tag{7}
$$

with $\mathcal{A}_e(\rho_k) := \sum_{i=1}^N \xi^i(\rho_k)\tilde A_i$, where $\tilde A_i = A_i - L_iC$ and

$$
v_k = \underbrace{(A(\rho_k) - A(\hat\rho_k))}_{=:\Delta A(\rho_k,\hat\rho_k)}x_k \tag{8}
$$

The next theorem provides polytopic observers (6) that render (7) ISS with respect to v.

Theorem *5:* Assume that there exist symmetric matrices $P_i \in \mathbb{R}^{n\times n}$, matrices $G_i \in \mathbb{R}^{n\times n}$, $F_i \in \mathbb{R}^{n\times m}$, $i = 1,\ldots,N$ and a scalar $\sigma_{ev} \geq 1$ satisfying for all $i,j = 1,\ldots,N$ the following LMIs

$$
\begin{bmatrix}
G_i^T + G_i - P_j & \mathbf{0} & G_iA_i - F_iC & G_i\\
\mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0}\\
A_i^TG_i^T - C^TF_i^T & \mathbf{1} & P_i & \mathbf{0}\\
G_i^T & \mathbf{0} & \mathbf{0} & \sigma_{ev}\mathbf{1}
\end{bmatrix} \succ \mathbf{0}, \tag{9}
$$

then the error dynamics (7) with uncertainty set Θ for $\hat{\rho}$ and[1] $L_i = G_i^{-1}F_i$ is ISS with respect to v and $V_e(e_k,\hat{\xi}_k) = e_k^T(\sum_{i=1}^N \hat{\xi}_k^i P_i)e_k$ is an ISS Lyapunov function that satisfies

$$V_e(e_{k+1},\hat{\xi}_{k+1}) - V_e(e_k,\hat{\xi}_k) \leq -\|e_k\|^2 + \sigma_{ev}\|v_k\|^2 \tag{10a}$$

$$\|e_k\|^2 \leq V_e(e_k,\hat{\xi}_k) \leq \sigma_{ev}\|e_k\|^2 \tag{10b}$$

for all $\hat{\xi}_k,\ \hat{\xi}_{k+1} \in \mathcal{S}$, $e_k \in \mathbb{R}^n$, $v_k \in \mathbb{R}^n$. The ISS gain γ can be taken linear as $\gamma(s) = \sigma_{ev}s$.

Proof: Since $(P_j^{-\frac{1}{2}}G_i^T - P_j^{\frac{1}{2}})^T(P_j^{-\frac{1}{2}}G_i^T - P_j^{\frac{1}{2}}) \succeq 0$ implies

$$G_iP_j^{-1}G_i^T \succeq G_i + G_i^T - P_j, \tag{11}$$

feasibility of the LMIs (9) gives for all i,j the LMI

$$\begin{bmatrix} G_iP_j^{-1}G_i^T & \mathbf{0} & G_iA_i - F_iC & G_i \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ A_i^TG_i^T - C^TF_i^T & \mathbf{1} & P_i & \mathbf{0} \\ G_i^T & \mathbf{0} & \mathbf{0} & \sigma_{ev}\mathbf{1} \end{bmatrix} \succ \mathbf{0}. \tag{12}$$

This is equivalent for all i,j to

$$N_{ij}\Psi_{ij}N_{ij}^T \succ \mathbf{0} \text{ with } N_{ij} = \begin{bmatrix} G_iP_j^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}, \quad \Psi_{ij} = \begin{bmatrix} P_j & \mathbf{0} & P_j(A_i - L_iC) & P_j \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ (A_i - L_iC)^TP_j & \mathbf{1} & P_i & \mathbf{0} \\ P_j & \mathbf{0} & \mathbf{0} & \sigma_{ev}\mathbf{1} \end{bmatrix}. \tag{13}$$

Hence, we have that for all i,j

$$\Psi_{ij} \succ \mathbf{0}. \tag{14}$$

For shortness we write $\hat{\xi}_k^i = \xi^i(\hat{\rho}_k)$ and $\xi_k^i = \xi^i(\rho_k)$. Multiplying (14) by $\hat{\xi}_k^i$ and summing, multiplying by $\hat{\xi}_{k+1}^j$ and summing, and using the Schur lemma yield

$$\begin{bmatrix} \mathcal{P}_k & \mathbf{0} \\ \mathbf{0} & \sigma_{ev}\mathbf{1} \end{bmatrix} - \begin{bmatrix} \mathcal{A}_e(\hat{\rho}_k)^T & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathcal{P}_{k+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathcal{A}_e(\hat{\rho}_k) & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \succ \mathbf{0}$$

with $\mathcal{P}_k := P(\hat{\xi}_k) := \sum_{i=1}^N \hat{\xi}_k^i P_i$ and $\mathcal{P}_{k+1} := P(\hat{\xi}_{k+1}) = \sum_{j=1}^N \hat{\xi}_{k+1}^j P_j$. Note that we used $\hat{\xi}_k,\ \hat{\xi}_{k+1} \in \mathcal{S}$ (due to $\hat{\rho}_k,\ \hat{\rho}_{k+1} \in \Theta$). Hence, for all $e_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^n$ $\begin{pmatrix} e_k^T & v_k^T \end{pmatrix} M \begin{pmatrix} e_k \\ v_k \end{pmatrix} \leq 0$ with

$$M = \begin{bmatrix} \mathbf{1} - \mathcal{P}_k + \mathcal{A}_e(\hat{\rho}_k)^T\mathcal{P}_{k+1}\mathcal{A}_e(\hat{\rho}_k) & \mathcal{A}_e(\hat{\rho}_k)^T\mathcal{P}_{k+1} \\ \mathcal{P}_{k+1}\mathcal{A}_e(\hat{\rho}_k) & \mathcal{P}_{k+1} - \sigma_{ev}\mathbf{1} \end{bmatrix}.$$

This implies for all e_k and all v_k that

$$(\mathcal{A}_e(\hat{\rho}_k)e_k + v_k)^T\mathcal{P}_{k+1}(\mathcal{A}_e(\hat{\rho}_k)e_k + v_k) - e_k^T\mathcal{P}_ke_k \leq -e_k^Te_k + \sigma_{ev}v_k^Tv_k.$$

This can be rewritten as

$$V_e(e_{k+1},\hat{\xi}_{k+1}) - V_e(e_k,\hat{\xi}_k) \leq -e_k^Te_k + \sigma_{ev}v_k^Tv_k \tag{15}$$

Moreover, the feasibility of the LMIs (9) for all $i,j = 1,\ldots,N$ implies that

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & P_i \end{bmatrix} \succ \mathbf{0} \text{ and } \begin{bmatrix} G_i^T + G_i - P_j & G_i \\ G_i^T & \sigma_{ev}\mathbf{1} \end{bmatrix} \succ \mathbf{0} \tag{16}$$

are satisfied for all $i,j = 1,\ldots,N$. From (16), it follows that $P_i \succ \mathbf{1}$ and $G_i^T + G_i - P_j \succ \sigma_{ev}^{-1}G_iG_i^T \succeq 0$ for all $i,j = 1,\ldots,N$. This with (11) leads to $G_iP_j^{-1}G_i^T \succeq \sigma_{ev}^{-1}G_iG_i^T$ and thus $P_j \preceq \sigma_{ev}\mathbf{1}$, $j = 1,\ldots,N$, because G_i is invertible. Invertibility of G_i follows from $G_i^T + G_i \succ P_j$ as it would imply for $G_ix = 0$ that $x^TP_jx \leq 0$ and thus $x = 0$. As such, we have (10b) for all $e_k \in \mathbb{R}^n$ and all $\hat{\xi}_k \in \mathcal{S}$. We could base ourselves on Theorem 3 to obtain ISS, but we proceed here to explicitly compute the ISS gain. From (15) and (10b), one has

$$V_e(e_{k+1},\hat{\xi}_{k+1}) \leq (1 - \tfrac{1}{\sigma_{ev}})V_e(e_k,\hat{\xi}_k) + \sigma_{ev}\|v_k\|^2. \tag{17}$$

[1]The LMIs (9) imply that G_i is invertible for each $i = 1,\ldots,N$ as is shown in the proof.

Applying (17) repetitively leads to

$$\begin{array}{rcl} V_e(e_k,\hat{\xi}_k) & \leq & (1-\frac{1}{\sigma_{ev}})^k V_e(e_0,\hat{\xi}_0)+\sigma_{ev}\sum_{l=0}^{k-1}(1-\frac{1}{\sigma_{ev}})^{k-l-1}\|v_l\|^2 \\ & \leq & (1-\frac{1}{\sigma_{ev}})^k V_e(e_0,\hat{\xi}_0)+\sigma_{ev}^2\|v\|_\infty^2. \end{array}$$

Finally, by using again (10b), taking the square root, we obtain the inequality

$$\|e_k\| \leq \sqrt{\sigma_{ev}}(1-\frac{1}{\sigma_{ev}})^{k/2}\|e_0\|+\sigma_{ev}\|v\|_\infty. \tag{18}$$

This inequality shows ISS with respect to v with linear ISS gain $\gamma(s)=\sigma_{ev}s,\ s\in\mathbb{R}_{\geq 0}$. ■

In case the conditions of Theorem 5 hold, the polytopic observer (6) guarantees GES of the error dynamics (7) in the *nominal* case where $\rho_k=\hat{\rho}_k$ for all $k\in\mathbb{N}$ (as then $v_k=0$, $k\in\mathbb{N}$). In case $\rho_k\neq\hat{\rho}_k$, ISS (see (18)) guarantees only a steady state estimation error e that is smaller than $\delta\sigma_{ev}\sup_{k\in\mathbb{N}}\|x_k\|$ with $\delta:=\sup\{\Delta A(\rho,\hat{\rho})\mid\|\rho-\hat{\rho}\|_\infty\leq\Delta\}$. Hence, a kind of *steady state relative error* can be obtained in the sense that $\frac{\limsup_{k\to\infty}\|e_k\|}{\limsup_{k\to\infty}\|x_k\|}\leq\delta\sigma_{ev}$, which implies that $e_k\to 0$ $(k\to\infty)$, if $(\rho_k-\hat{\rho}_k)\to 0$ $(k\to\infty)$. In [10] also the concept of steady state relative error was used in the context of observer design for discontinuous PWA systems in which the mode of the plant can be different than the mode of the observer. Here this mismatch between observer and plant model is caused by $\rho_k\neq\hat{\rho}_k$.

The smallest ISS gain σ_{ev} based on the above design procedure can be obtained by selecting among all possible solutions σ_{ev}, P_i, G_i, and F_i of the LMIs (9) for $i,j=1,\ldots,N$ the ones leading to the smallest value for σ_{ev}, which amounts to solving the convex optimization problem

$$\min\{\sigma_{ev}\mid P_i,F_i,G_i,\sigma_{ev}\text{ satisfying (9) for } i,j=1,\ldots,N\} \tag{19}$$

Remark 6: Note that the normalization of certain constants in (10) to 1 is without loss of generality as any ISS Lyapunov function V_e for (7) can be multiplied by a sufficiently large positive constant to satisfy (10). See also the proof of Theorem 7 below.

As mentioned, if the hypotheses of Theorem 5 are satisfied, the polytopic observer (6) guarantees GES of the error dynamics in the *nominal* case ($\rho_k=\hat{\rho}_k$ for all $k\in\mathbb{N}$). Actually, the observer satisfies the matrix inequalities

$$(A_i-L_iC)^T\tilde{P}_j(A_i-L_iC)-\tilde{P}_i\prec 0,\ \ i,j=1,\ldots,N \text{ and } \tilde{P}_i\succ 0,\ \ i=1,\ldots,N \tag{20}$$

In [13] it is proven that (20) is sufficient for the observer (6) with $\hat{\rho}_k=\rho_k$ (a nominal observer) to recover the state of (4) asymptotically. Actually, the conditions (20) are both necessary and sufficient for the existence of a parameter-dependent quadratic Lyapunov function proving GES of the estimation error dynamics in the *nominal* case ($\hat{\rho}_k=\rho_k$) [7]. Interestingly, the conditions in (20) also guarantee that the hypotheses of Theorem 5 are satisfied (as will be shown in Theorem 7 below). This shows the *non-conservatism* of the LMIs (9) as the existence of a *nominal* observer for the *exact* LPV system, with a parameter-dependent quadratic Lyapunov function proving GES of the error dynamics, is sufficient for (9) to hold. This also shows that any GES observer for the exact LPV system has some degree of robustness.

Theorem *7:* If there exist $\tilde{P}_i$ and L_i, $i=1,\ldots,N$ such that (20) holds, then there are symmetric matrices $\tilde{P}_i$ and matrices F_i, G_i, $i=1,\ldots,N$ and a scalar $\sigma_{ev}\geq 1$ satisfying for all $i,j=1,\ldots,N$ the LMIs (9).

Proof: Due to the strictness in (20) there exist positive constants r, c_1 and c_2 such that for all $e\in\mathbb{R}^n$ and all $v\in\mathbb{R}^n$ it holds that

$$\begin{array}{rcl} ((A_i-L_iC)e+v)^T\tilde{P}_j((A_i-L_iC)e+v)-e^T\tilde{P}_ie & \leq & -r\|e\|^2+c_1\|v\|\|e\|+c_2\|v\|^2 \\ & \leq & -\frac{r}{2}\|e\|^2+(c_2+\frac{c_1^2}{2r})\|v\|^2, \end{array} \tag{21}$$

where in the second inequality we applied Lemma 1 with $a=\|v\|$, $b=\|e\|$ and $\varepsilon=\frac{r}{c_1}$. By multiplying this inequality by a suitable factor $M\geq\frac{2+2\tau}{r}$ and defining $P_i=M\tilde{P}_i$ we obtain for a small $\tau>0$ and some positive constant c_3

$$((A_i-L_iC)e+v)^TP_j((A_i-L_iC)e+v)-e^TP_ie \leq -\|e\|^2+c_3\|v\|^2-\tau(\|e\|^2+\|v\|^2) \tag{22}$$

for all $i,j=1,\ldots,N$ and $P_i\succeq\mathbf{1}$, $i=1,\ldots,N$. The inequality (22) implies that $(\ e^T\ \ v^T\ )[M_{ij}+\tau\mathbf{1}](\ e^T\ \ v^T\ )^T\leq 0$ with $M_{ij}=\begin{bmatrix}\mathbf{1}-P_i+(A_i-L_iC)^TP_j(A_i-L_iC) & (A_i-L_iC)^TP_j \\ P_j(A_i-L_iC) & P_j-\sigma_{ev}\mathbf{1}\end{bmatrix}$ and $\sigma_{ev}=c_3$. Hence, this gives $M_{ij}\prec 0$ and thus (14), $i,j=1,\ldots,N$ by using the Schur lemma. The proof of Theorem 5 can now be traced back following (13) and (12) leading to (9) for any set of invertible matrices G_i, $i=1,\ldots,N$ and matrices $F_i=G_iL_i$. ■

V. STATE FEEDBACK DESIGN

We now focus on the design of a state feedback for (4a) using an estimated state given by

$$u_k = K(\hat{\rho}_k)\hat{x}_k = K(\hat{\rho}_k)(x_k - e_k) \tag{23}$$

with $K(\hat{\rho}_k) = \sum_{i=1}^N \xi^i(\hat{\rho}_k)K_i$ and e_k the estimation error. This results in the closed loop

$$x_{k+1} = \mathcal{A}_x(\hat{\rho}_k)x_k + v_k - BK(\hat{\rho}_k)e_k \tag{24}$$

with, as before, v_k is given by (8) and $\mathcal{A}_x(\hat{\rho}_k) = \sum_{i=1}^N \xi^i(\hat{\rho}_k)\underbrace{(A_i + BK_i)}_{A_{BK\,i}}$. Again, we sometimes write $\hat{\xi}_k^i = \xi^i(\hat{\rho}_k)$ and $\xi_k^i = \xi^i(\rho_k)$. We now study ISS of (24).

Theorem *8:* Assume that there exist symmetric matrices $Y_i \in \mathbb{R}^{n\times n}$ and matrices $Z_i \in \mathbb{R}^{m\times n}$, $i = 1,\dots,N$ and scalars $\sigma_{xv}, \sigma_{xe}, \mu$ with $\mu > 0$ and $\sigma_{xv} \geq 1$ satisfying for $i,j = 1,\dots,N$ the LMI conditions

$$\begin{bmatrix} Y_i & \mathbf{0} & \mathbf{0} & Y_iA_i^T + Z_i^TB^T & Y_i \\ \mathbf{0} & \sigma_{xv}\mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \sigma_{xe}\mathbf{1} & -Z_i^TB & \mathbf{0} \\ A_iY_i + BZ_i & \mathbf{1} & -BZ_i & Y_j & \mathbf{0} \\ Y_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \succ \mathbf{0} \text{ and } Y_i \succeq \mu\mathbf{1}, \tag{25}$$

then the closed-loop system (24) with uncertainty set Θ for $\hat{\rho}$ and $K_i = Z_iY_i^{-1}$, $i = 1,\dots,N$ is ISS with respect to e and v and $V_x(x_k,\hat{\xi}_k) = x_k^T\sum_{i=1}^N \hat{\xi}_k^i S_i x_k$ is an ISS Lyapunov function that satisfies for all $\hat{\xi}_k$, $\hat{\xi}_{k+1} \in \mathcal{S}$, all $x_k \in \mathbb{R}^n$, all $e_k \in \mathbb{R}^n$ and all $v_k \in \mathbb{R}^n$

$$V_x(x_{k+1},\hat{\xi}_{k+1}) - V_x(x_k,\hat{\xi}_k) \leq -\|x_k\|^2 + \sigma_{xv}\|v_k\|^2 + \mu^{-2}\sigma_{xe}\|e_k\|^2 \tag{26a}$$

$$\|x_k\|^2 \leq V_x(x_k,\hat{\xi}_k) \leq \sigma_{xv}\|x_k\|^2. \tag{26b}$$

Proof: Assume that the LMIs in (25) are feasible and define $S_i := Y_i^{-1}$. Premultiply the first LMIs in (25) by $\mathcal{T} = diag(S_i, \mathbf{1}, S_i, \mathbf{1}, \mathbf{1})$, postmultiply it by $\mathcal{T}^T$ and apply the Schur lemma to arrive for $i,j = 1,\dots,N$ at

$$\begin{bmatrix} S_i - \mathbf{1} & \mathbf{0} & \mathbf{0} & (A_i + BK_i)^T \\ \mathbf{0} & \sigma_{xv}\mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \sigma_{xe}S_i^2 & -K_i^TB^T \\ A_i + BK_i & \mathbf{1} & -BK_i & S_j^{-1} \end{bmatrix} \succ \mathbf{0}. \tag{27}$$

Multiply (27) by $\hat{\xi}_k^i$, sum for $i = 1,\dots,N$ (note that $\sum_{i=1}^N \hat{\xi}_k^i = 1$, since $\hat{\rho}_k \in \Theta$) and use the Schur lemma pivoting again around the south east block to obtain for $j = 1,\dots,N$

$$\begin{bmatrix} (\sum_{i=1}^N \hat{\xi}_k^i A_{BK\,i})^T S_j (\sum_{i=1}^N \hat{\xi}_k^i A_{BK\,i}) - \sum_{i=1}^N \hat{\xi}_k^i S_i + \mathbf{1} & (\bullet)^T & (\bullet)^T \\ S_j\sum_{i=1}^N \hat{\xi}_k^i A_{BK\,i} & S_j - \sigma_{xv}\mathbf{1} & (\bullet)^T \\ -(\sum_{i=1}^N \hat{\xi}_k^i BK_i)^T S_j (\sum_{i=1}^N \hat{\xi}_k^i A_{BK\,i}) & -(\sum_{i=1}^N \hat{\xi}_k^i BK_i)^T S_j & T_{33} \end{bmatrix} \prec \mathbf{0}, \tag{28}$$

where the term T_{33} corresponds to $(\sum_{i=1}^N \hat{\xi}_k^i BK_i)^T S_j (\sum_{i=1}^N \hat{\xi}_k^i BK_i) - \sigma_{xe}\sum_{i=1}^N \hat{\xi}_k^i S_i^2$. Using that $S_i^2 \preceq \mu^{-2}\mathbf{1}$, $i = 1,\dots,N$ due to the second LMIs in (25), multiplying (28) by $\hat{\xi}_{k+1}^j$ and summing for $j = 1,\dots,N$ (note that $\sum_{j=1}^N \hat{\xi}_{k+1}^j = 1$, since $\hat{\rho}_{k+1} \in \Theta$) leads to

$$V_x(x_{k+1},\hat{\xi}_{k+1}) - V_x(x_k,\hat{\xi}_k) \leq -\|x_k\|^2 + \sigma_{xv}\|v_k\|^2 + \mu^{-2}\sigma_{xe}\|e_k\|^2 \tag{29}$$

As (27) implies $S_i \succ \mathbf{1}$ and $S_i \preceq \sigma_{xv}\mathbf{1}$, we have for all $x_k \in \mathbb{R}^n$ and $\xi_k \in \mathcal{S}$ that (26b) holds. From Theorem 3 it follows now that the closed-loop system is ISS with respect to v and e. ■

The following corollary applies when the full state x_k is known (i.e. $e_k = 0$ for all $k \in \mathbb{N}$).

Corollary 9: Let the hypotheses of Theorem 8 be satisfied. Then the LPV system consisting of (4a) and the state feedback $u_k = K(\hat{\rho}_k)x_k$ with uncertainty set Θ for ρ and $K_i = Z_iY_i^{-1}$, $i = 1,\dots,N$ is GES for all uncertainties satisfying $\|\Delta A(\rho,\hat{\rho})\| \leq \delta$, when $\delta < \frac{1}{\sigma_{xv}}$.

Proof: From (26a) with $e = 0$ and $v_k = \Delta A(\rho_k,\hat{\rho}_k)x_k$ it follows that

$$V_x(x_{k+1},\hat{\xi}_{k+1}) - V_x(x_k,\hat{\xi}_k) \leq -(1 - \sigma_{xv}\delta)\|x_k\|^2. \tag{30}$$

Together with (26b) this proves GES on the basis of Theorem 3. ■

An analogous result to Theorem 7 can also be shown for the state feedback. In particular, a nominal state feedback $u_k = K(\rho_k)x_k$ with $K(\rho_k) = \sum_{i=1}^{N} \xi^i(\rho_k)K_i$ (i.e. with estimation error $e_k = 0$, $k \in \mathbb{N}$ and exact knowledge of parameters, $\rho_k = \hat{\rho}_k$, $k \in \mathbb{N}$) coupled to the LPV system (4a) is GAS if there are K_i, $\tilde{S}_i$, $i = 1, \dots, N$ such that

$$(A_i + BK_i)^T \tilde{S}_j (A_i + BK_i) - \tilde{S}_i \prec \mathbf{0},\ i,j = 1,\dots,N \text{ and } \tilde{S}_i \succ \mathbf{0},\ i = 1,\dots,N. \tag{31}$$

Clearly, a state feedback (23) that renders (24) ISS (proved by parameter-dependent quadratic ISS Lyapunov functions) certainly satisfies (31). Interestingly, the converse also holds in the sense that a *nominally* stabilizing state feedback for (4a) has some robustness properties in the sense that (26) holds for some V_x and even stronger, the LMIs in (25) are feasible. This clearly indicates the non-conservatism of the derived LMIs in Theorem 8. However, note that (31) does not allow any minimization of the ISS gains, while the results of Theorem 8 do.

Theorem *10:* Suppose that there exist K_i, $\tilde{S}_i$, $i = 1, \dots, N$ such that (31) is satisfied. Then there are symmetric matrices Y_i and matrices Z_i, $i = 1, \dots, N$ and scalars $\sigma_{xv}, \sigma_{xe}, \mu$ with $\mu > 0$ satisfying the LMIs (25) for $i, j = 1, \dots, N$.

Proof: Due to (31) there exist positive constants r and $c_l > 0$, $l = 1, \dots, 7$ such that for all $i, j = 1, \dots, N$ and all $x \in \mathbb{R}^n$, $v \in \mathbb{R}^n$, $e \in \mathbb{R}^n$ it holds that

$$\begin{aligned}(A_{BK_i}x + v - BK_ie)^T \tilde{S}_j (A_{BK_i}x + v - BK_ie) - x^T \tilde{S}_i x \le -r\|x\|^2 + c_1\|x\|\|v\| + \\ + c_2\|x\|\|e\| + c_3\|e\|\|v\| + c_4\|v\|^2 + c_5\|e\|^2 \le -\frac{r}{2}\|x\|^2 + c_6\|v\|^2 + c_7\|e\|^2,\end{aligned} \tag{32}$$

where we applied Lemma 1 three times with $a = \|x\|$, $b = \|v\|$, $\varepsilon = \frac{r}{2c_1}$ to obtain $c_1\|x\|\|v\| \le \frac{r}{4}\|x\|^2 + \frac{c_1^2}{r}\|v\|^2$, with $a = \|x\|$, $b = \|e\|$, $\varepsilon = \frac{r}{2c_2}$ to obtain $c_2\|x\|\|e\| \le \frac{r}{4}\|x\|^2 + \frac{c_2^2}{r}\|e\|^2$ and with $a = \|v\|$, $b = \|e\|$ and some arbitrary $\varepsilon > 0$, respectively. Multiplying (32) by a sufficiently large constant $M \ge \frac{2+2\tau}{r}$ and setting $S_j = M\tilde{S}_j$ for all $j = 1, \dots, N$ yield

$$\begin{aligned}(A_{BK_i}x + v - BK_ie)^T S_j (A_{BK_i}x + v - BK_ie) - x^T S_i x \\ \le -\|x\|^2 + c_8\|v\|^2 + c_9\|e\|^2 - \tau(\|x\|^2 + \|v\|^2 + \|e\|^2),\end{aligned} \tag{33}$$

with $c_8 = Mc_6 + \tau$ and $c_9 = Mc_7 + \tau$ and some $\tau > 0$, and, $S_j \succ \mathbf{1}$, $j = 1, \dots, N$. With $\sigma_{xv} = c_8$ and $\sigma_{xe} = \max_{j=1,\dots,N} \frac{c_9}{\lambda_{min}(S_j^2)}$, (33) gives

$$\begin{aligned}(A_{BK_i}x + v - BK_ie)^T S_j (A_{BK_i}x + v - BK_ie) - x^T S_i x \\ \le -\|x\|^2 + \sigma_{xv}\|v\|^2 + \sigma_{xe}e^T S_i^2 e - \tau(\|x\|^2 + \|v\|^2 + \|e\|^2).\end{aligned} \tag{34}$$

This yields for all $i, j = 1, \dots, N$

$$\begin{bmatrix} A_{BK}{}_i^T S_j A_{BKi} - S_i + \mathbf{1} & (\bullet)^T & (\bullet)^T \\ S_j A_{BKi} & S_j - \sigma_{xv}\mathbf{1} & (\bullet)^T \\ -(BK_i)^T S_j A_{BKi} & -(BK_i)^T S_j & (BK_i)^T S_j (BK_i) - \sigma_{xe} S_i^2 \end{bmatrix} \prec \mathbf{0}, \tag{35}$$

which is equivalent to (27) (using Schur lemma and S_j is positive definite). These are also equivalent to the first LMI in (25) for all i, j with $Y_j := S_j$, which can be observed by tracing back the proof of Theorem 8. In addition, since Y_j is positive definite for all $j = 1, \dots, N$ there is a positive μ such that the second LMI in (25) holds for all $i = 1, \dots, N$. ■

VI. OBSERVER-BASED CONTROL DESIGN

Next we will show that the separate design of the observer as in section IV and a state feedback as in section V leads to a stabilizing output-based controller for some nontrivial level of uncertainty $\delta := \sup\{\|\Delta A(\rho, \hat{\rho})\| \mid \|\rho - \hat{\rho}\|_\infty \le \Delta\}$. The closed-loop system is given by

$$\begin{pmatrix} x_{k+1} \\ e_{k+1} \end{pmatrix} = \begin{bmatrix} A(\rho_k) + BK(\hat{\rho}_k) & -BK(\hat{\rho}_k) \\ A(\rho_k) - A(\hat{\rho}_k) & A(\hat{\rho}_k) - L(\hat{\rho}_k)C \end{bmatrix} \begin{pmatrix} x_k \\ e_k \end{pmatrix} \tag{36}$$

Theorem *11:* Let an observer (6) that satisfies the hypotheses of Theorem 5 and a state feedback law that satisfies the hypotheses of Theorem 8 be given[2]. Then for any $\max\{1-\frac{1}{\sigma_{ev}}, 1-\frac{1}{\sigma_{xv}}\} \leq \varepsilon < 1$ and any $0 < \beta \leq \frac{1-(1-\varepsilon)\sigma_{ev}}{\mu^{-2}\sigma_{xe}}$ the closed-loop system (36) is GES with decay factor equal to $\sqrt{\varepsilon}$ for all uncertainties satisfying $\|\Delta A(\rho,\hat{\rho})\| \leq \delta := \sqrt{\frac{\beta(1-(1-\varepsilon)\sigma_{xv})}{\sigma_{ev}+\beta\sigma_{xv}}}$.

Proof: Consider the candidate Lyapunov function $V_\beta(x_k, e_k, \hat{\xi}_k) := \beta V_x(x_k, \hat{\xi}_k) + V_e(e_k, \hat{\xi}_k)$ for the closed-loop system (36) with $\beta > 0$. From (10) and (26) and noting that $v_k = \Delta A(\rho_k, \hat{\rho}_k)x_k$ with $\delta = \sup\{\|\Delta A(\rho,\hat{\rho})\| \mid \|\rho - \hat{\rho}\|_\infty \leq \Delta\}$, we have that

$$\Delta V_\beta(x_k, e_k, \hat{\xi}_k, \hat{\xi}_{k+1}) \leq (-\beta + \beta\sigma_{xv}\delta^2 + \sigma_{ev}\delta^2)\|x_k\|^2 - (1-\beta\mu^{-2}\sigma_{xe})\|e_k\|^2, \tag{37}$$

where $\Delta V_\beta(x_k, e_k, \hat{\xi}_k, \hat{\xi}_{k+1}) := V_\beta(x_{k+1}, e_{k+1}, \hat{\xi}_{k+1}) - V_\beta(x_k, e_k, \hat{\xi}_k)$ with $(x_{k+1}^T, e_{k+1}^T)^T$ as in (36). To obtain GES with decay factor $\sqrt{\varepsilon}$ it suffices to guarantee $V_\beta(x_{k+1}, e_{k+1}, \hat{\xi}_{k+1}) \leq \varepsilon V_\beta(x_k, e_k, \hat{\xi}_k)$ as V_β can be bounded by quadratic $\mathcal{K}$ functions $\alpha_1(s) = as^2$ and $\alpha_2(s) = bs^2$ as in (3a) in the norm $\|(x_k^T, e_k^T)^T\|$. To obtain this inequality it is sufficient to have

$$\Delta V_\beta(x_k, e_k, \hat{\xi}_k, \hat{\xi}_{k+1}) \leq -(1-\varepsilon)(\beta\sigma_{xv}\|x_k\|^2 + \sigma_{ev}\|e_k\|^2), \tag{38}$$

because $V_\beta(x_k, e_k, \hat{\xi}_k) \leq \beta\sigma_{xv}\|x_k\|^2 + \sigma_{ev}\|e_k\|^2$. Due to (37), the inequality (38) holds when (i) $\beta - \beta\sigma_{xv}\delta^2 - \sigma_{ev}\delta^2 \geq (1-\varepsilon)\beta\sigma_{xv}$ and (ii) $1 - \beta\mu^{-2}\sigma_{xe} \geq (1-\varepsilon)\sigma_{ev}$. Obviously, under the hypotheses of the theorem these conditions are true, which completes the proof. ■

It is of interest to find the Lyapunov function V_β that provides the largest robustness in terms of δ. To maximize the value for δ^2 (for a fixed value of the decay factor $\sqrt{\varepsilon}$) it is clear that we have to maximize $f(\beta) := \frac{\beta}{\sigma_{ev}+\beta\sigma_{xv}}$. Since $\frac{df(\beta)}{d\beta} = \frac{\sigma_{ev}}{(\sigma_{ev}+\beta\sigma_{xv})^2} \geq 0$, the maximum is obtained for the largest allowable value of β, which is $\frac{1-(1-\varepsilon)\sigma_{ev}}{\mu^{-2}\sigma_{xe}}$ and thus the maximum of δ is

$$\delta(\varepsilon) = \sqrt{\frac{(1-[1-\varepsilon]\sigma_{ev})(1-[1-\varepsilon]\sigma_{xv})}{\mu^{-2}\sigma_{xe}\sigma_{ev} + (1-[1-\varepsilon]\sigma_{ev})\sigma_{xv}}}. \tag{39}$$

Suppose we now would like to find the value of ε such that the admissible uncertainty level $\delta(\varepsilon)$ is maximal. Since it can be inspected that $\frac{d\delta^2(\varepsilon)}{d\varepsilon} > 0$ for any $\max\{1-\frac{1}{\sigma_{ev}}, 1-\frac{1}{\sigma_{xv}}\} \leq \varepsilon < 1$, maximizing robustness requires maximizing (actually taking supremum of) ε and thus taking it close to 1. This yields that the maximal value of δ can become arbitrarily close to

$$\delta(1) = \sqrt{\frac{1}{\mu^{-2}\sigma_{xe}\sigma_{ev} + \sigma_{xv}}}, \tag{40}$$

while still guaranteeing stability. Hence, for maximizing robustness in terms of maximizing $\delta(\varepsilon)$, we should maximize ε meaning that the performance in terms of the decay factor $\sqrt{\varepsilon}$ is worst. As such, we encountered a "classical" tradeoff between robustness and performance.

The reasoning above maximizes robustness for *fixed* values of σ_{xv}, σ_{ev} and σ_{xe}. Since we have determined this maximum given these σ's, we can now optimize robustness by appropriately selecting the gains L_i and K_i, $i = 1, \ldots, N$. From (40) it is clear that we have to minimize $\mu^{-2}\sigma_{xe}\sigma_{ev} + \sigma_{xv}$ to get the maximal value for $\delta(1) = \sqrt{\frac{1}{\mu^{-2}\sigma_{xe}\sigma_{ev}+\sigma_{xv}}}$ (for decay factor equal to 1). This gives rise to the following procedure to get maximal robustness in the mismatch between the scheduling parameter $\hat{\rho}_k$ and the actual one ρ_k as reflected in δ.

Design procedure

Step 1 : Minimize σ_{ev} subject to (9) for $i, j = 1, \ldots, N$. This gives the minimum σ_{ev}^* and the corresponding observer gains L_i, $i = 1, \ldots, N$.

Step 2 : Given σ_{ev}^* as in Step 1. Fix $\mu > 0$ and minimize the expression $\mu^{-2}\sigma_{xe}\sigma_{ev}^* + \sigma_{xv}$ subject to the LMIs given in (25). This results in the feedback gains K_i, $i = 1, \ldots, N$.

The optimization problems in Step 1 and 2 are convex problems as we are minimizing linear costs subject to LMI constraints. Step 2 might even be extended by performing a line search in μ and applying the above procedure repetitively. Once, the minimal value $\mu^{*-2}\sigma_{xe}^*\sigma_{ev}^* + \sigma_{xv}^*$ is found, one can on the basis of Theorem 11 and (39)

[2]Note that the hypotheses guarantee that $\sigma_{ev} > 1$ and $\sigma_{xv} > 1$.

still make tradeoffs between transient performance in terms of the decay factor $\sqrt{\varepsilon}$ and robustness in terms of $\delta(\varepsilon)$. Letting ε increase from $\max\{1-\frac{1}{\sigma_{ev}}, 1-\frac{1}{\sigma_{xv}}\}$ (maximal performance, minimal robustness) to 1 (minimal performance, maximal robustness), tradeoff curves between performance and robustness are obtained.

VII. Illustrative Example

Consider the LPV system (4) with $A(\rho_k) = \begin{bmatrix} 0.25 & 1 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.6+\rho_k \end{bmatrix}$, $B = [1 \ 0 \ 1]^T$, $C = [1 \ 0 \ 2]$, $D = 0$ and $\rho_k \in [0\ ,\ 0.5]$, $k \in \mathbb{N}$. In this case we can take the functions $\xi^1(\rho) = \frac{0.5-\rho}{0.5}$ and $\xi^2(\rho) = \frac{\rho}{0.5}$ with $A_1 = A(0)$ and $A_2 = A(0.5)$. The observer is designed using Theorem 5 along with the optimization problem (19) (Step 1). The optimal solution is given by $\sigma^*_{ev} = 5.8277$ with observer gains $L_1 = \begin{bmatrix} -0.0835 & -0.0011 & 0.3870 \end{bmatrix}^T$ and $L_2 = \begin{bmatrix} -0.0835 & -0.0011 & 0.7094 \end{bmatrix}^T$. With this optimal observer and the associated slope of the linear ISS gain σ^*_{ev}, a line search involving $\mu > 0$ is performed in order to minimize the cost $J = \mu^{-2}\sigma_{xe}\sigma^*_{ev} + \sigma_{xv}$ subject to the LMIs given in (25) for all i, j (Step 2). Fig. 1 shows the minimum of J for each fixed μ, which is the smallest for $\mu^* = 0.2986$ yielding $\sigma^*_{xe} = 0.2663$ and $\sigma^*_{xv} = 13.9284$ and corresponds to the controller gains $K_1 = [-0.0327 \ -0.1241 \ -0.2387]$, $K_2 = [0.0005 \ -0.0010 \ -0.6148]$.

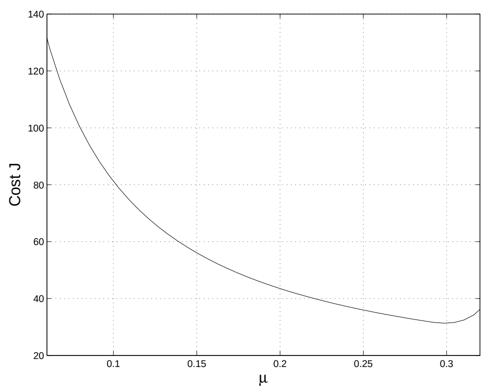

Fig. 1. Line search μ

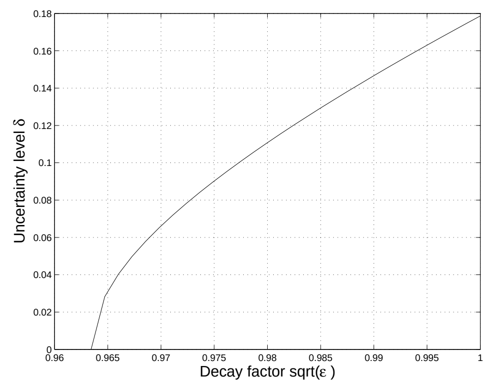

Fig. 2. Tradeoff performance/robustness

As a consequence, the maximum level of uncertainty is $\delta^*_{max} = \sqrt{\frac{1}{\mu^{*-2}\sigma^*_{xe}\sigma^*_{ev}+\sigma^*_{xv}}} = 0.1786$. Hence, for $\Delta A(\rho_k, \hat{\rho}_k) = |\rho_k - \hat{\rho}_k| \leq \delta < 0.1786$ GES of the closed-loop system (36) is guaranteed (with a decay factor close to 1). Letting ε increase from $\max\{1-\frac{1}{\sigma^*_{ev}}, 1-\frac{1}{\sigma^*_{xv}}\}$ to 1 leads to the tradeoff curves between performance in terms of the decay factor $\sqrt{\varepsilon}$ and robustness to uncertainty $\Delta A(\rho_k, \hat{\rho}_k)$ in terms of δ as depicted in Fig. 2.

VIII. Conclusions

In this paper the design of robustly stabilizing output-based feedback controllers is considered for discrete-time LPV systems in which the scheduling parameters are only known up to a given precision. The output-based controllers are obtained using separate design of the observer and the state feedback and we showed that the interconnection of the LPV plant, observer and state feedback leads to a stable closed-loop system for certain levels of mismatch between estimated and true parameters. The non-conservatism of our approach is demonstrated by showing that well known conditions for nominally stabilizing observers and feedbacks (i.e. without mismatch between true and available parameters) imply our LMI-based conditions. The flexibility in the framework allows to construct the controller that guarantees stability for the largest level of parameter uncertainty and to make tradeoffs between performance in terms of decay factor and robustness with respect to parameter uncertainty.

References

[1] A. Alessandri, M. Baglietto, and G. Battistelli. Luenberger observers for switching discrete-time linear systems. *Int. J. Control*, 80(12):1931–1943, 2007.

[2] P. Apkarian and P. Gahinet. A convex characterization of gain-scheduled Hinfinity controllers. *IEEE Trans. on Automatic Control*, 40:853–864, 1995.

[3] G. Becker and A. Packard. Robust performance of linear parametrically varying systems using parametrically-dependent linear feedback. *Syst. Contr. Lett.*, 23:205–215, 1994.

[4] F. Blanchini and S. Miani. Stabilization of LPV systems: state feedback, state estimation, and duality. *SIAM J. Control Optim.*, 42:76–97, 2003.

[5] F. Blanchini, S. Miani, and F. Mesquine. A separation principle for linear switching systems and parametrization of all stabilizing controllers. *Proc. IEEE Conference on Decision and Control*, pages 953–958, 2008.

[6] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15. SIAM Studies in Applied Mathematics, 1994.

[7] J. Daafouz and J. Bernussou. Parameter dependent Lyapunov functions for discrete time systems with time varying parametric uncertainties. *Systems and Control Letters*, 43:355–359, 2001.

[8] J. Daafouz, J. Bernussou, and J.C. Geromel. On inexact LPV control design of continuous−time polytopic systems. *IEEE Trans. Automatic Control*, 53:1674–1678, 2008.

[9] Z.-P. Jiang and Y. Wang. Input-to-state stability for discrete-time nonlinear systems. *Automatica*, 37:857–869, 2001.

[10] A.Lj. Juloski, W.P.M.H. Heemels, and S. Weiland. Observer design for a class of piecewise linear systems. *Intern. J. Robust and Nonlinear Control*, 17(15):1387–1404, 2007.

[11] I.E. Kose and F. Jabbari. Control of LPV systems with partly-measured parameters. *Proceedings of the 36th IEEE Conference on Decision and Control*, 1997.

[12] M. Lazar, D. Munoz de la Pena, W.P.M.H. Heemels, and T. Alamo. On the stability of min-max nonlinear model predictive control. *Syst. Contr. Lett.*, 57(1):39–48, 2008.

[13] G. Millerioux, L. Rosier, G. Bloch, and J. Daafouz. Bounded state reconstruction error for LPV systems with estimated parameters. *IEEE Trans. Automatic Control*, pages 1385–1389, 2004.

[14] W.J. Rugh and J.S. Shamma. A survey of research on gain-scheduling. *Automatica*, 36:1401–1425, 2000.

[15] C. Scherer. LPV control and full block multipliers. *Automatica*, 37:361–375, 2001.

[16] J. Shamma and M. Athans. Guaranteed properties of gain scheduled control of linear parameter-varying plants. *Automatica*, 27:559–564, 1991.

[17] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automatic Control*, 34:435–443, 1989.

[18] N. van de Wouw and A. Pavlov. Tracking and synchronisation for a class of PWA systems. *Automatica*, 44(11):2909–2915, 2008.