Polytopic Observers for LPV Discrete-Time Systems

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Chapter 5
Polytopic Observers for LPV Discrete-Time Systems

Meriem Halimi, Gilles Millerioux, and Jamal Daafouz

Abstract. The main goal of this work is to give a general treatment on observer synthesis for LPV systems in the framework of Linear Matrix Inequalities. A special Parameter Dependent Lyapunov Function, called poly-quadratic Lyapunov function, is considered. It incorporates the parameter variations for LPV systems with polytopic parameter dependence and allows to guarantee a so-called poly-quadratic stability which is sufficient to ensure Global Asymptotic Stability. The concept of polytopic observers is introduced. A LMI-based method for the synthesis of this type of observers is proposed. The case when LPV systems are subjected to disturbances or when the parameter is known with a bounded level of uncertainty is further addressed. Conditions to guarantee performances like Input-to-State Stability (ISS), bounded peak-to-peak gain and L2 gain are given. The design of polytopic Unknown Input Observers both in the deterministic and in the noisy or uncertain cases is also presented. Finally, two illustrative examples dealing with polytopic observers for chaos synchronization and air path management of a turbocharged Spark Ignition engine are detailed.

5.1 Introduction

Linear Parameter Varying (LPV) systems are linear models whose state representation depends on a parameter vector which can vary in time. Since several years these systems give rise to more and more attention, both in control [2] [32] [24] [38] [23] [16] and in observation and filtering [4] [20] [40] [37]. Contrarily to systems with parametric uncertainties, in this case the current values of the parameters are assumed to be known. The variation of the parameters within a bounded set might be arbitrarily fast or restricted by a certain rate of variation. This LPV modeling techniques have gained a lot of interest as they provide a systematic procedure to design

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gain-scheduled controllers, especially those related to aerospace control [36]. The main goal of this work is to give a general treatment on observer synthesis for LPV systems in the framework of Linear Matrix Inequalities [22].

A key stage for the synthesis of observer is the search for an adequate Lyapunov function. A usual approach is to call for a single quadratic Lyapunov function [5][22]. This approach suffers from conservatism since it does not take into account the parameter variations of the LPV system. In some cases, it may cause the problem to be infeasible, meaning that quadratic stabilization cannot be achieved. A significant improvement can be obtained by considering Parameter Dependent Lyapunov Functions (PDLF) which incorporate the parameter variations. A special parameter dependence is the affine one [18][17] and can be extended to a polynomial one [6]. Unfortunately, affine parameter dependent Lyapunov functions lead to an infinite number of constraints because all the values of the parameters which continuously vary in some prescribed range have to be considered. Thus, one must discretize the range of all admissible values in order to obtain a finite set of constraints. Another usual dependence is the polytopic one which allows to overpass the discretization and to turn the problem into the resolution of a finite set of constraints by only considering the vertices of the polytope. This is precisely the option which is chosen in the present treatment. A suitable Parameter Dependent Lyapunov function associated to the polytopic description is the so-called poly-quadratic Lyapunov function [11]. It allows to guarantee the so-called poly-quadratic stability which is sufficient to ensure Global Asymptotic Stability.

The layout of the paper is the following. In Section 5.2, some basic definitions are recalled including the notions of Global Asymptotical Stability and Input-to-State Stability (ISS). Section 5.3 is devoted to LPV models with special emphasis on the polytopic one. In Section 5.4 is introduced the concept of polytopic observers. A LMI-based method for the synthesis of this type of observers is proposed. It is based on the notion of poly-quadratic stability. Section 5.5 addresses the case when the LPV system is subjected to disturbances or when the parameter is known with a bounded level of uncertainty. Conditions to guarantee performances like ISS, bounded peak-to-peak gain and $\mathscr{L}_2$ gain are given. Section 5.6 deals with the design of polytopic Unknown Input Observers both in the deterministic and in the noisy or uncertain cases. Finally, illustrative examples are detailed in Section 5.7.

Notation
$\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$: the field of real numbers, the set of non-negative real numbers and the set of non-negative integers, respectively.

$z^{(i)}$: the ith component of a real vector z. z^T: the transpose for the vector z. $\|z\| = \sqrt{z^T z}$: the Euclidean norm of z. $\|z\|_\infty$: the infinity norm of z given by $\max_i |z^{(i)}|$. $\{z\}$: a sequence of samples $z_k, z_{k+1}, \ldots$ without explicit initial and final discrete-time $k \in \mathbb{N}$. $\{z\}_{k_1}^{k_2}$: a sequence of samples $z_{k_1}, \ldots, z_{k_2}$. $\|z\|_2 = \sqrt{\sum_{k=0}^{\infty} z_k^T z_k}$: the Euclidean norm for a sequence $\{z\}$. $\|z\|_\infty$: the supremum norm given by $\|z\|_\infty = sup_{k\in\mathbb{N}} \|z_k\|$ for a sequence $\{z\}$.

1: the identity matrix of appropriate dimension. **0**: the zero matrix of appropriate dimension. X^T: the transpose for the matrix X. $X > 0 \ \ (X < 0)$: a positive definite (negative definite) matrix X. $X \geq 0$ $(X \leq 0)$: a semi-positive definite (semi-negative definite) matrix X. $\|X\| = \sqrt{\lambda_{max}(X^T X)}$: the spectral norm of the matrix X, where λ_{max} is the largest eigenvalues of $X^T X$. $X^\dagger$: the generalized inverse (Moore-Penrose) of X satisfying $X^\dagger X$ symmetric, $XX^\dagger$ symmetric, $XX^\dagger X = X$ and $X^\dagger XX^\dagger = X^\dagger$. If X is nonsingular then $X^\dagger = X^{-1}$. $(\bullet)$: the blocks of a matrix induced by symmetry.

5.2 Preliminaries

Definition 5.1. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathscr{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$, and to class $\mathscr{L}_\infty$ if additionally $\varphi(s) \to \infty$ as $s \to \infty$

Definition 5.2. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathscr{K}\mathscr{L}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(.,k) \in \mathscr{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s,.)$ is decreasing and $lim_{k\to\infty}\beta(s,k) = 0$.

Consider the discrete-time nonlinear systems

$$x_{k+1} = f(x_k) \tag{5.1}$$
$$x_{k+1} = f_w(x_k, w_k) \tag{5.2}$$

with $x_k \in \mathbb{R}^n$ is the state vector, $w_k \in \mathbb{R}^{d_w}$ is an unknown disturbance input.

Definition 5.3. The system (5.1) is called Globally Asymptotically Stable (GAS) if there exists a $\mathscr{K}\mathscr{L}$-function β such that, for each $x_0 \in \mathbb{R}^n$, it holds that the corresponding state trajectory satisfies for all $k \in \mathbb{N}$

$$\|x_k\| \leq \beta(\|x_0\|, k)$$

Definition 5.4. The system (5.2) is said to be Input-to-State Stable (ISS) with respect to w_k if there exist a $\mathscr{K}\mathscr{L}$ function β and a $\mathscr{K}$ function γ such that, for all input sequences $\{w\}$, for each $x_0 \in \mathbb{R}^n$, it holds that the corresponding state trajectory satisfies for all $k \in \mathbb{N}$

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|w\|_\infty) \tag{5.3}$$

If β can be taken of the form $\beta(s,k) = ds\zeta^k$ for some $d \geq 0$ and $0 < \zeta < 1$, ζ is the decay factor for (5.1) and the function γ is an ISS gain for (5.2).

5.3 LPV Models

We investigate LPV systems given by the following form:

$$\begin{cases} x_{k+1} = A(\rho_k)x_k + Bu_k \\ y_k = Cx_k + Du_k \end{cases} \tag{5.4}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input, $y_k \in \mathbb{R}^p$ is the output vector, $A \in \mathbb{R}^{n\times n}$ is the dynamical matrix depending on the possibly time varying parameter vector $\rho_k = \left[\rho_k^{(1)}, \rho_k^{(2)}, ..., \rho_k^{(L)}\right] \in \mathbb{R}^L$, $C \in \mathbb{R}^{p\times n}$ is the output matrix, $B \in \mathbb{R}^{n\times m}$ is the input matrix.

For obvious practical considerations and as usual in the framework of LPV systems, we assume that each component $\rho^{(i)}$ $(i = 1,\ldots,L)$ of ρ_k lies in a bounded range $[\rho_{min}^{(i)}, \rho_{max}^{(i)}]$. As a result, ρ_k lies in a bounded set $\Omega_\rho \subset \mathbb{R}^L$. The dependence of $A(\rho_k)$ with respect to ρ_k can take many forms. However, some of them are of special importance when it comes to analysis and synthesis. We focus here on two specific decompositions, namely, affine and polytopic.

The affine decomposition refers to an affine dependency of $A(\rho_k)$ with respect to ρ_k. $A(\rho_k)$ is thereby of class C^1 with respect to ρ_k and so, can be rewritten as

$$A(\rho_k) = \bar{A_0} + \sum_{j=1}^{L} \rho_k^{(j)} \bar{A_j} \tag{5.5}$$

where $\bar{A_0}$ and $\bar{A_j}$ are constant matrices obtained by separating constant terms and terms depending on $\rho_k^{(j)}$.

The polytopic decomposition refers to a dependence on ρ_k of $A(\rho_k)$ which reads

$$A(\rho_k) = \sum_{i=1}^{N} \xi_k^{(i)}(\rho_k) A_i \tag{5.6}$$

where ξ_k belongs to the compact set S

$$S = \left\{ \mu_k \in \mathbb{R}^N, \mu_k = \left[\mu_k^{(1)}, \ldots, \mu_k^{(N)}\right], \mu_k^{(i)} \geq 0 \; \forall\, i \text{ and } \textstyle\sum_{i=1}^N \mu_k^{(i)} = 1 \right\}$$

Owing to the convexity of S, the set of matrices $\{A_1, \ldots, A_N\}$ defines a polytope denoted D_A and the matrices A_i correspond to the vertices of D_A. Hereafter, for the sake of simplicity and whenever possible, the parameter dependency on ρ_k of $\xi_k^{(i)}$ will be omitted, that is the notation $\xi_k^{(i)}$ will be used instead of $\xi_k^{(i)}(\rho_k)$.

It is noteworthy to mention that the affine decomposition (5.5) can be rewritten in the polytopic form (5.6). Indeed, since ρ_k belongs to a bounded set Ω_ρ, it can be embedded in a polytope D_ρ with vertices $\theta_i, \ldots, \theta_N \in \mathbb{R}^L$, such that

$$\rho_k = \sum_{i=1}^{N} \xi_k^{(i)} \theta_i, \; \xi_k \in S \tag{5.7}$$

Substituting (5.7) into (5.5) yields:

$$A(\rho_k) = \bar{A_0} + \sum_{j=1}^{L} (\sum_{i=1}^{N} \xi_k^{(i)} \theta_i^{(j)}) \bar{A_j}, \; \xi_k \in S \tag{5.8}$$

Since $\sum_{i=1}^{N} \xi_k^{(i)} = 1$ and A_0 is a constant matrix, it follows that $A_0 = \sum_{i=1}^{N} \xi_k^{(i)} A_0$ and therefore (5.8) turns into:

$$A(\rho_k) = \sum_{i=1}^{N} \xi_k^{(i)} (\bar{A_0} + \sum_{j=1}^{L} \theta_i^{(j)} \bar{A}_j), \ \xi_k \in S \tag{5.9}$$

Identifying (5.6) and (5.9) yields

$$A_i = \bar{A_0} + \sum_{j=1}^{L} \theta_i^{(j)} \bar{A}_j \tag{5.10}$$

The constraint "$\xi_k \in S$" is equivalent to "$\rho_k \in D_\rho$". Since $\Omega_\rho \subseteq D_\rho$, it should be pointed out that the polytopic description (5.6) may describe a broader class of systems than the original one, leading thereby to some conservatism. However, when the components $\rho_k^{(j)}$ ($j = 1, \ldots, L$) of (5.5) are independent, Ω_ρ turns into a specific polytope D_ρ called hypercube with $N = 2^L$ vertices and in this case $\Omega_\rho = D_\rho$.

5.3.1 Minimal Polytope

It may happen that obtaining analytically the polytope D_ρ is either a hard task or even is not possible. Moreover, we should be concerned, for the sake of conservatism, to get a minimal polytope. Let us assume that we can get, by simulation or experimentally, a sufficient number of vectors ρ_k, collected in a finite set Γ_ρ of cardinality N_ρ, to describe the set Ω_ρ with proper accuracy. The minimal polytope D_ρ^* wherein Ω_ρ is embedded can thereby be considered as the convex hull of the set of points Γ_ρ. We recall that an element of a finite set of points is an extreme point if it is not a convex combination of other points in this set. Hence, finding out D_ρ^* amounts to finding out the extreme points of Γ_ρ. It turns out that the computation can be performed by standard methods. They are briefly recalled below while a detailed review is provided in [25].

The computation of the convex hull for the dimension $L = 2$ has been extensively studied and several efficient algorithms are available. The most popular is the Graham Scan [19]. It is based on the consideration that the angle between two consecutive faces (formed by three consecutive vertices) of the convex hull is lower than π. The complexity of the algorithm is $O(N_\rho log N_\rho)$. This algorithm has a main drawback in that it cannot be extended to dimensions greater than 2. Another efficient algorithm called Quick hull is based on the "divide and conquer" approach. It has been introduced in slightly different forms by [15] and [35]. Such an algorithm uses the property which stipulates that given a triangle of three points of the original set, the points strictly inside this triangle do not belong to the convex hull. Hence, they can be discarded. The complexity of this algorithm is also $O(N_\rho log N_\rho)$. Moreover it can be easily extended to any dimension (see [1] for the dimension $L = 3$).

However, the complexity grows up rapidly and becomes redhibitory for large dimensions. The algorithm Quick hull is incorporated into the built-in function *convhull* of the software Matlab. The algorithm Random Sampling presented in [9] is based on iterative projections on hyperplanes randomly chosen. Finally, a linear program approach is proposed in [33] which calls for solving an optimization problem.

5.3.2 On Line Polytopic Decomposition

In this subsection, we are concerned with a way of working out on-line the vector $\xi_k = \left[\xi_k^{(1)} \cdots \xi_k^{(N)}\right]^T$ involved in the polytopic decomposition (5.7) of ρ_k and (5.6) of $A(\rho_k)$. The vector ξ_k is solution of

$$\begin{aligned} W_k &= Z \ \xi_k \\ s.t\ \xi_k^{(i)} &\geq 0, \ \ i = 1, \ldots, N \end{aligned} \tag{5.11}$$

where

$$W_k = [\rho_k^{(1)} \cdots \rho_k^{(L)} \ 1]^T \quad \text{and} \quad Z = \begin{bmatrix} \theta_1^{(1)} & \cdots & \theta_N^{(1)} \\ \vdots & \cdots & \vdots \\ \theta_1^{(L)} & & \theta_N^{(L)} \\ 1 & \ldots & 1 \end{bmatrix}$$

and where the entries θ_i are given, the matrix Z of dimension $(L+1) \times N$ being thereby constant and known. Indeed, it is assumed that ρ_k is on-line available, as usual in the framework of LPV systems. The matrix Z may be of large dimension and is likely to be not amenable for an efficient on-line computation of ξ_k. A method has been proposed in [25] to circumvent this problem.

5.3.3 LPV Models for the Description of Nonlinear Systems

LPV systems can model nonlinear systems under certain conditions. Standard procedures call for interpolation of linearized systems but the resulting LPV model is only an approximation of the actual nonlinear system. As a result, concluding on stability and performances of the nonlinear system based on the LPV approximation may be misleading [24]. Hence, we should rather be interested in an exact description. Such a purpose has been investigated in the works reported in [7] or in the paper [25].

Consider the nonlinear system

$$x_{k+1} = g(x_k, u_k) \tag{5.12}$$

where $x_k \in X \subseteq \mathbb{R}^n$ is the state vector and $u_k \in \mathbb{R}^m$ is the control input.

Proposition 1. *If the following conditions are fulfilled*

- *there exists a function* $\rho : \mathbb{R}^n \to \mathbb{R}^L$ *such that* $A(\rho(x_k))x_k + Bu_k = g(x_k, u_k)$
- $\rho(x_k)$ *depends only on measured signals*
- $\rho(x_k)$ *is bounded when* x_k *lies in the admissible set* $X \subseteq \mathbb{R}^n$

then the nonlinear system (5.12) *admits an exact LPV description in the form of the first equation of* (5.4) *with* $\rho_k = \rho(x_k)$.

It is worth pointing out that, most often, the LPV description is not unique and multiple functions ρ can be candidates. Furthermore, the resulting LPV model describes a larger class of systems than the original nonlinear one. Indeed, a trajectory of the nonlinear system is also a trajectory of the LPV model, among an infinite number of possibilities, but the converse is not true. More formally, an LPV system is a linear differential inclusion parameterized in the vector ρ_k. And yet, there is no unique linear differential inclusion of a nonlinear system. The choice can be guided by the objective of reducing the conservatism of the stability conditions or enhancing their tractability both for analysis of synthesis issues. It can also be interesting to select an appropriate function ρ so as the domains of attraction of both models are as most coincident as possible.

In the rest of this work, it will be assumed that the matrix $A(\rho_k)$ in (5.4) is rewritten in the polytopic form (5.6).

The extension to LPV systems with time varying matrices B, C and D is possible. A first option is to merely consider an augmented vector $\bar{\rho}_k$ which involves all the parameter vectors associated to the respective matrices A, B, C and D and to get a polytopic description of the matrix

$$\begin{bmatrix} A(\bar{\rho}_k) & B(\bar{\rho}_k) \\ C(\bar{\rho}_k) & D(\bar{\rho}_k) \end{bmatrix} = \sum_{i=1}^{N} \xi_k^{(i)} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, \quad \xi_k \in S$$

Another alternative would follow the same line of reasoning as the one provided in Theorem 1 of [31].

5.4 Polytopic Observers in a Noise-Free Context

5.4.1 Observability and Detectability

5.4.1.1 Observability

As far as the observability of LPV systems is concerned, the following theorem, borrowed from [40] holds.

Theorem 5.1. *System (5.4) is completely observable if rank*$(\mathscr{O}_n(\rho_{k:k+n-1})) = n$ *for all* $k \in \mathbb{Z}$.

where $\mathscr{O}_n(\rho_{k:k+n-1})$ is the so-called parameter varying state-observability matrix of (5.4) defined, for $n > 1$, as

$$\mathscr{O}_n(\rho_{k:k+n-1}) = \begin{bmatrix} C \\ CA(\rho_k) \\ \vdots \\ C\prod_{l=0}^{n-2} A(\rho_{k+n-2-l}) \end{bmatrix} \tag{5.13}$$

and $\rho_{k:k+n-1} = [\rho_k, \cdots, \rho_{k+n-1}]$. For $n = 1$, $\mathscr{O}_n(\rho_{k:k+n-1})$ reduces to $\mathscr{O}_1(\rho_{k:k}) = C$.

In other words, the concept of observability is defined similarly to the linear case when considering all possible trajectories of the parameter $\rho_k \in \Omega_\rho$. Actually, Theorem 5.1 is a straightforward extension of the condition of observability stated in [34] which deals with linear time-varying systems. Hence, in Theorem 5.1, the constraint "for all $k \in \mathbb{Z}$" can be reinterpreted in the case of LPV systems as "for all $\rho_k \in \Omega_\rho$".

The problem lies in that the conditions are much less tractable for LPV systems than for linear systems since, in the general case, the number of trajectories of $\rho_k \in \Omega_\rho$, and so the number of vectors $\rho_{k:k+n-1}$ is infinite. And yet, unfortunately, the observability of the pairs (C, A_i) assigned to the vertices of the polytope D_A does not necessarily induce the observability for all the pairs $(C, A(\rho_k))$. As an example, let us consider the system obeying the form (5.4) with

$$A(\rho_k) = \begin{bmatrix} 0.6 + \rho_k & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0.5 \end{bmatrix}$$

The parameter ρ_k belongs to the range $[0\ 1]$.

The observability matrix is given by

$$\mathscr{O}_2([\rho_k, \rho_{k+1}]) = \begin{bmatrix} 1 & 0.5 \\ \rho_k + 1.1 & 1 \end{bmatrix}$$

The observability matrix for the respective pairs (C, A_1) and (C, A_2), with $A_1 = A(0)$ and $A_2 = A(1)$ numerically reads

$$\mathscr{O}_2([0, *]) = \begin{bmatrix} 1 & 0.5 \\ 1.1 & 1 \end{bmatrix} \quad \text{and} \quad \mathscr{O}_2([1, *]) = \begin{bmatrix} 1 & 0.5 \\ 2.1 & 1 \end{bmatrix}$$

where * stands for an arbitrary value of ρ_{k+1}, the observability matrix depending exclusively on ρ_k.

It is clear that $rank(\mathscr{O}_2([0, *])) = rank(\mathscr{O}_2([1, *])) = 2$. However, for $\rho_k = 0.9$, the observability matrix numerically reads

$$\mathscr{O}_2([0.9, *]) = \begin{bmatrix} 1 & 0.5 \\ 2 & 1 \end{bmatrix}$$

and so $rank(\mathscr{O}_2([0.9,*])) = 1$. As a result, the two pairs (C,A_1) and (C,A_2) are observable whereas the observability is not satisfied inside the polytope D_A when $\rho_k = 0.9$.

A reduction of the computational cost for testing the observability rank condition of Theorem 5.1 is most often either a hard task or merely infeasible.

5.4.1.2 Detectability

The notion of detectability relies on the notion of stability of the unobservable subspace. And yet, similarly to general nonlinear systems, stability of LPV systems can match different definitions. Hence, despite the resulting conservatism, we must resort to specific ones. For instance in [41], detectability is defined analogously to quadratic stability, that is

Theorem 5.2. *The LPV system* (5.4) *is quadratically detectable, if there exists a matrix $P = P^T > 0$ and a matrix function $L(\rho_k)$ such that*

$$(A(\rho_k) + L(\rho_k)C)^T P + P(A(\rho_k) + L(\rho_k)C) < 0 \;\; \forall \rho_k \in \Omega_\rho$$

It turns out that checking for the conditions of Theorem 5.2 is equally computationally demanding as the actual observer synthesis. Let us also notice that the computation of related invariant subspaces associated to the notion of detectability is not trivial (see however a special treatment in [3] for example).

As a conclusion of this section, the practical use of observability and detectability is often of limited interest and these notions do not deserve in general extensive investigation.

5.4.2 Synthesis

Let us recall that it is assumed that the matrix $A(\rho_k)$ in (5.4) is rewritten in the polytopic form (5.6).

A polytopic observer for (5.4) obeys the following state space description

$$\begin{cases} \hat{x}_{k+1} = A(\rho_k)\hat{x}_k + Bu_k + L(\rho_k)(y_k - \hat{y}_k) \\ \hat{y}_k = C\hat{x}_k + Du_k \end{cases} \tag{5.14}$$

where L is a time varying gain matrix depending on ρ_k which reads

$$L(\rho_k) = \sum_{i=1}^{N} \xi_k^{(i)}(\rho_k) L_i, \;\; \xi_k \in S \tag{5.15}$$

and where the $\xi_k^{(i)}(\rho_k)$ in (5.15) coincide, for every discrete time k, with the ones involved in the polytopic decomposition (5.6) of $A(\rho_k)$.

It's a simple matter to see that, from (5.4) and (5.14), the reconstruction error $e_k = x_k - \hat{x}_k$ is governed by the dynamics

$$e_{k+1} = (A(\rho_k) - L(\rho_k)C)\, e_k \tag{5.16}$$

The dynamics of the state reconstruction is nonlinear since A and L depend on ρ_k. However, (5.16) can be viewed as an autonomous LPV polytopic system with state vector $e_k \in \mathbb{R}^n$. Indeed, from (5.6) and (5.15), and taking into account the coincidence between the $\xi_k^{(i)}$s involved in (5.15) and (5.6), we get that

$$e_{k+1} = \sum_{i=1}^{N} \xi_k^{(i)} (A_i - L_i C) e_k, \;\; \xi_k \in S \tag{5.17}$$

Global Asymptotical Stability around the equilibrium point $e^* = 0$ can be ensured by a suitable choice of the gains L_i ($i = 1, \ldots, N$) involved in (5.15). To this end, the following theorem is central.

Theorem 5.3. *If there exist symmetric matrices P_i, matrices G_i and matrices F_i fulfilling, $\forall (i, j) \in \{1...N\} \times \{1...N\}$, the Linear Matrix Inequalities*

$$\begin{bmatrix} P_i & (\bullet)^T \\ G_i A_i - F_i C & G_i^T + G_i - P_j \end{bmatrix} > 0 \tag{5.18}$$

then the polytopic observer (5.14) *with gain $L(\rho_k) = \sum_{i=1}^{N} \xi_k^{(i)}(\rho_k) L_i$ and $L_i = G_i^{-1} F_i$ ensures that the system* (5.16) *is GAS.*

Proof 1. *The detailed proof is given in [12]. It is shown that (5.18) ensures the existence of a Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^L \to \mathbb{R}_+$ defined by $V(e_k, \rho_k) = e_k^T P(\rho_k) e_k$ with $P(\rho_k) = \sum_{i=1}^{N} \xi_k^{(i)}(\rho_k) P_i$ and $\xi_k \in S$, called poly-quadratic Lyapunov function, fulfilling for all $e_k \in \mathbb{R}^n$, for all $\xi_k \in S$*

$$V(e_{k+1}, \rho_{k+1}) - V(e_k, \rho_k) < 0 \tag{5.19}$$

Such a function ensures the poly-quadratic stability of (5.16) *which is sufficient for Global Asymptotical Stability.*

5.4.3 Decay Rate

We should be concerned with monitoring the rate of convergence towards $e^* = 0$. In this respect, the decay rate is well suited. The global asymptotical convergence of (5.16) towards $e^* = 0$ with decay rate $\alpha > 1$ is formalized as follows:

$$\forall e_0 \in \mathbb{R}^n, \quad \lim_{k \to \infty} \alpha^k \|e_k\| = 0 \tag{5.20}$$

In other words, $\|e_k\|$ decreases faster than α^{-k}. A sufficient condition for the global convergence of (5.16) towards $e^* = 0$ with decay rate α is given by the following theorem.

Theorem 5.4. *If there exist symmetric matrices P_i, matrices F_i and G_i fulfilling, for a prescribed scalar κ, $\forall (i,j) \in \{1,\cdots,N\} \times \{1,\cdots,N\}$, the Linear Matrix Inequalities*

$$\begin{bmatrix} \kappa P_i & (\bullet)^T \\ G_i A_i - F_i C & G_i^T + G_i - P_j \end{bmatrix} > 0 \tag{5.21}$$

then the polytopic observer (5.14) *with gain $L(\rho_k) = \sum_{i=1}^{N} \xi_k^{(i)}(\rho_k) L_i$ and $L_i = G_i^{-1} F_i$ ensures the global convergence of* (5.16) *with decay rate α no less than $\kappa^{-\frac{1}{2}}$* $(0 < \kappa < 1)$.

Proof 2. *The proof is detailled in [26]. It is shown that (5.21) ensures the existence of a Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^L \to \mathbb{R}_+$, defined by $V(e_k, \rho_k) = e_k^T P(\rho_k) e_k$ with $P(\rho_k) = \sum_{i=1}^{N} \xi_k^{(i)}(\rho_k) P_i$ and $\xi_k \in S$, called poly-quadratic Lyapunov function, fulfilling for all $e_k \in \mathbb{R}^n$, for all $\xi_k \in S$*

$$V(e_{k+1}, \rho_{k+1}) - \kappa V(e_k, \rho_k) < 0 \tag{5.22}$$

which is sufficient to obtain (5.20) *with $\alpha \geq \kappa^{-\frac{1}{2}}$.*

5.5 Polytopic Observers in a Noisy or Uncertain Context

In this section, we are concerned with the situation when the system (5.4) is subjected to disturbances and obeys

$$\begin{cases} x_{k+1} = A(\rho_k) x_k + B u_k + E w_k^d \\ y_k = C x_k + D u_k + H w_k^o \end{cases} \tag{5.23}$$

where $w_k^d \in \mathbb{R}^{d_{w^d}}$ is the disturbance acting on the dynamics through E while $w_k^o \in \mathbb{R}^{d_{w^o}}$ is the disturbance acting on the output through H.

In such a case, Equation (5.16) of the state reconstruction error $e_k = x_k - \hat{x}_k$ turns into

$$e_{k+1} = (A(\rho_k) - L(\rho_k) C) e_k + v_k \tag{5.24}$$

with $v_k = E w_k^d - L(\rho_k) H w_k^o$.

Besides, we can also be concerned with the case when ρ_k is not available but only an estimated parameter $\hat{\rho}_k \in \Omega_{\hat{\rho}}$ is available. The uncertainty level Δ satisfies $\|\rho_k - \hat{\rho}_k\|_\infty < \Delta$. Then, the polytopic observer (5.14) can take the form

$$\begin{cases} \hat{x}_{k+1} = A(\hat{\rho}_k) \hat{x}_k + B u_k + L(\hat{\rho}_k)(y_k - \hat{y}_k) \\ \hat{y}_k = C \hat{x}_k + D u_k \end{cases} \tag{5.25}$$

with

$$L(\hat{\rho}_k) = \sum_{i=1}^{N} \hat{\xi}_k^{(i)}(\hat{\rho}_k) L_i \tag{5.26}$$

In such a case, (5.24) still holds provided that ρ_k is replaced by $\hat{\rho}_k$ and that $v_k = \Delta A(\rho_k, \hat{\rho}_k) x_k$ with $\Delta A(\rho_k, \hat{\rho}_k) = A(\rho_k) - A(\hat{\rho}_k)$.

5.5.1 Input-to-State-Stability (ISS)

Many approaches to derive sufficient conditions to guarantee the ISS are based on the notion of ISS Lyapunov functions.

Definition 5.5. Let $d_1, d_2 \in \mathbb{R}_+$, let $a, b, c, l \in \mathbb{R}_+$ with $a \leq b$ and let $\alpha_1(s) = as^l, \alpha_2(s) = bs^l, \alpha_3(s) = cs^l$ and $\tau \in \mathscr{K}$. A function $V : \mathbb{R}^n \times \mathbb{R}^L \to \mathbb{R}_+$ which satisfies

$$\alpha_1(\|e_k\|) \leq V(e_k, \rho_k) \leq \alpha_2(\|e_k\|) \tag{5.27}$$

$$V(e_{k+1}, \rho_{k+1}) - V(e_k, \rho_k) \leq -\alpha_3(\|e_k\|) + \tau(\|v_k\|) \tag{5.28}$$

for all $e_k \in \mathbb{R}^n$, all $v_k \in \mathbb{R}^n$ and all $\rho_k \in \Omega_\rho$ is called an ISS Lyapunov Function for (5.24).

Theorem 5.5. *If the system* (5.24) *admits an ISS Lyapunov function, then* (5.24) *is ISS with respect to v_k, that is there exist a $\mathscr{KL}$ function β and a $\mathscr{K}$ function γ such that, for all sequences $\{v\}$, for each $e_0 \in \mathbb{R}^n$, it holds that, for all $k \in \mathbb{N}$*

$$\|e_k\| \leq \beta(\|e_0\|, k) + \gamma(\|v\|_\infty) \tag{5.29}$$

5.5.1.1 Link between Poly-Quadratic Stability and ISS

Theorem 5.6. *If the LMIs* (5.18) *are feasible, then the system* (5.24) *is ISS with respect to v_k and*

$$\|e_k\| \leq \sqrt{\frac{c_2}{c_1}} \left(1 - \frac{c_3 - \delta}{c_2}\right)^{k/2} \|e_0\| + \sqrt{\frac{c_2 + \delta^{-1} c_4^2}{c_1} \cdot \frac{c_2}{c_3 - \delta}} \cdot \|v\|_\infty \tag{5.30}$$

c_1, c_2, c_3, c_4 and δ are some scalars depending on the eigenvalues of the matrices derived from the solution of (5.18). *The quantity* $(1 - \frac{c_3 - \delta}{c_2})^{1/2}$ *is called the decay factor.*

In other words, the polytopic observer (5.14) with gain $L(\rho_k)$ given by (5.15) and derived from the solution of (5.18), built from the matrices of the noise-free system (5.4) and from the assumption that ρ_k is perfectly known, ensures the poly-quadratic stability of (5.16), also guarantees the ISS of (5.24), that is of the state reconstruction error derived from the system (5.23) which describes the system (5.4) subjected to disturbances or/and bounded uncertainties on ρ_k.

Proof 3. *The proof is detailed in [30]. It is shown that* (5.18) *ensures the existence of an ISS Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^L \to \mathbb{R}_+$ with $P(\rho_k) = \sum_{i=1}^N \xi_k^{(i)}(\rho_k) P_i$ and $\xi_k \in S$, which is sufficient to derive* (5.30).

5.5.1.2 Minimization of the ISS Gain: A LMI Formulation

The point is that both the decay factor and the ISS gain in (5.30) cannot be prescribed beforehand. The following theorem is an attempt to handle this problem.

Theorem 5.7. *If there exist symmetric matrices P_i, matrices G_i, matrices F_i, fulfilling, for a prescribed scalar $\sigma_{ev} \geq 1$, $\forall (i,j) \in (1,\ldots,N) \times (1,\ldots,N)$, the Linear Matrix Inequalities*

$$\begin{bmatrix} G_i^T + G_i - P_j & \mathbf{0} & G_iA_i - F_iC & G_i \\ (\bullet)^T & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ (\bullet)^T & (\bullet)^T & P_i & \mathbf{0} \\ (\bullet)^T & (\bullet)^T & (\bullet)^T & \sigma_{ev}\mathbf{1} \end{bmatrix} > 0 \tag{5.31}$$

then the polytopic observer (5.14) *with gain $L(\rho_k) = \sum_{i=1}^N \xi_k^{(i)}(\rho_k)L_i$ and $L_i = G_i^{-1}F_i$, ensures that the system* (5.24) *is ISS with respect to v_k and*

$$\|e_k\| \leq \sqrt{\sigma_{ev}}(1 - \frac{1}{\sigma_{ev}})^{k/2}\|e_0\| + \sigma_{ev}\|v\|_\infty \tag{5.32}$$

Proof 4. *The detailed proof is provided in [13] [20]. It is shown that* (5.31) *ensures the existence of an ISS Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^L \to \mathbb{R}_+$ with $P(\rho_k) = \sum_{i=1}^N \xi_k^{(i)}(\rho_k)P_i$ and $\xi_k \in S$ which verifies for all $e_k \in \mathbb{R}^n$, all $v_k \in \mathbb{R}^n$ and all $\xi_k \in S$ of* (5.24)*, the following conditions*

$$\|e_k\|^2 \leq V(e_k,\rho_k) \leq \sigma_{ev}\|e_k\|^2 \tag{5.33}$$

$$V(e_{k+1},\rho_{k+1}) - V(e_k,\rho_k) \leq -\|e_k\|^2 + \sigma_{ev}\|v_k\|^2 \tag{5.34}$$

And yet, the existence of V is sufficient to obtain (5.32).

Moreover, we should be interested in optimizing the ISS gain by minimizing σ_{ev} in (5.32). Insofar as σ_{ev} appears in a linear way in the Matrix Inequalities (5.31), the problem

$$\begin{array}{ll} min & \sigma_{ev} \\ s.t\ (5.31) & \end{array} \tag{5.35}$$

is still a convex problem.

5.5.1.3 Decoupling of the Decay Rate and the ISS Gain

It is worth pointing out that, in the previous formulation and in particular when considering (5.32), the quantity σ_{ev} is both involved in the decay factor and in the ISS gain. We should be interested in monitoring independently both of them. This is the purpose of the following theorem.

Theorem 5.8. *If there exist symmetric matrices P_i, matrices G_i, matrices F_i and two real numbers $\mu > 0$ and ν fulfilling, for a prescribed $\lambda \in]0\ 1[$, $\forall (i,j) \in \{1\cdots N\} \times \{1\cdots N\}$, the Linear Matrix Inequalities*

$$\begin{bmatrix} (1-\lambda)P_i & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & \mu\mathbf{1} & (\bullet)^T \\ G_iA_i - F_iC & G_i & G_i^T + G_i - P_j \end{bmatrix} > 0 \tag{5.36}$$

and

$$\begin{bmatrix} \lambda P_i & (\bullet)^T & (\bullet)^T \\ \mathbf{0} & (\nu-\mu)\mathbf{1} & (\bullet)^T \\ \mathbf{1} & \mathbf{0} & \nu\mathbf{1} \end{bmatrix} > 0 \tag{5.37}$$

then the polytopic observer (5.14) *with gain* $L(\rho_k) = \sum_{i=1}^{N} \xi_k^{(i)}(\rho_k)L_i$ *and* $L_i = G_i^{-1}F_i$*, ensures that the system* (5.24) *is ISS with respect to* v_k *and*

$$\|e_k\| \leq \sqrt{\nu\lambda\mu}(1-\lambda)^{k/2}\|e_0\| + \nu\|v\|_\infty \tag{5.38}$$

Proof 5. *The proof follows the same lines of reasoning that the ones provided in [28]. It is shown that* (5.36)-(5.37) *ensures the existence of an ISS Lyapunov function* $V : \mathbb{R}^n \times \mathbb{R}^L \to \mathbb{R}_+$ *with* $P(\rho_k) = \sum_{i=1}^{N} \xi_k^{(i)}(\rho_k)P_i$ *and* $\xi_k \in S$ *which verifies for all* $e_k \in \mathbb{R}^n$*, all* $v_k \in \mathbb{R}^n$ *and all* $\xi_k \in S$*, the following conditions*

$$\frac{1}{\nu\lambda}\|e_k\|^2 \leq V(e_k,\rho_k) \leq \mu\|e_k\|^2 \tag{5.39}$$

$$V(e_{k+1},\rho_{k+1}) - V(e_k,\rho_k) \leq -\frac{1}{\nu}\|e_k\|^2 + \mu\|v_k\|^2 \tag{5.40}$$

And yet, the existence of V *is sufficient to obtain* (5.38).

Remark 1. *It can be shown that the conditions (5.36)-(5.37) are less conservative than* (5.31) *and that* (5.31) *is a special case of (5.36)-(5.37) when* $\nu = \sigma_{ev}, \lambda = \frac{1}{\sigma_{ev}}$ *and* $\mu = \sigma_{ev}$ *as well.*

Remark 2.*The Matrix Inequalities (5.36)-(5.37) are not linear because of the products* λP_i *in (5.36). Actually, they turn into LMIs if* λ *is fixed. Hence, they can be easily solved due to the fact that* λ *is a scalar and that the range of* λ *is bounded since* $\lambda \in]0,1[$*. As a result, a simple line search can be performed and* $\lambda = \frac{1}{\sigma_{ev}}$*, that is the solution of* (5.31)*, may be used as an admissible initial starting value.*

Moreover, we should be interested in optimizing the ISS gain by minimizing ν in (5.38). Insofar as ν appears in a linear way in the Matrix Inequalities (5.37), for a prescribed $\lambda \in]0\ 1[$, the problem

$$\begin{array}{c} \min \quad \nu \\ s.t\ (5.36)-(5.37) \end{array} \tag{5.41}$$

is still a convex problem.

5.5.2 Peak-to-Peak Gain

The peak-to-peak gain of the state error equation (5.24) is defined as the ratio

$$\sup_{0<\|v\|_\infty<\infty,\ \rho_k\in\Omega_\rho} \frac{\|e\|_\infty}{\|v\|_\infty} \tag{5.42}$$

The peak-to-peak gain is defined in the same way as in the linear case, except that, in addition, all possible trajectories $\rho_k \in \Omega_\rho$ have to be considered.

The following theorem holds.

Theorem 5.9. *If the Linear Matrix Inequalities* (5.31) *(resp. (5.36)-(5.37)) are fulfilled, then the polytopic observer* (5.14) *with gain* $L(\rho_k) = \sum_{i=1}^{N} \xi_k^{(i)}(\rho_k)L_i$ *and* $L_i = G_i^{-1}F_i$*, ensures that the error* e_k *of* (5.24) *admits a peak-to-peak gain smaller than* σ_{ev} *(resp. smaller than* ν*). One gets respectively*

$$\sup_{0<\|v\|_\infty<\infty,\ \rho_k\in D_\rho} \frac{\|e\|_\infty}{\|v\|_\infty} < \sigma_{ev} \tag{5.43}$$

$$\sup_{0<\|v\|_\infty<\infty,\ \rho_k\in D_\rho} \frac{\|e\|_\infty}{\|v\|_\infty} < \nu \tag{5.44}$$

Proof 6. *The result can be directly inferred from the inequality* (5.32) *(resp.* (5.38)*) by taking the limit of k to infinity and assuming that* $\|e_0\| = 0$.

Let us notice that (5.31) or (5.36)-(5.37) guarantees that the peak-to-peak gain is bounded for all possible trajectories ρ_k in $D_\rho \supseteq \Omega_\rho$ and not in Ω_ρ.

The minimization of the peak-to-peak gain can be performed through (5.35) or (5.41).

5.5.3 $\mathscr{L}_2$ Gain

Let $z_k = \tilde{H}e_k$ be a linear combination of the state reconstruction error e_k obeying the dynamics (5.24).

Definition 5.6. The $\mathscr{L}_2$ gain of the state error equation (5.24) is defined as

$$\sup_{\|v\|_2\neq 0,\ \rho_k\in\Omega_\rho} \frac{\|z\|_2}{\|v\|_2} \tag{5.45}$$

Similarly to the peak-to-peak gain, the $\mathscr{L}_2$ gain is defined in the same way as in the linear case, except that, in addition, all possible trajectories $\rho_k \in \Omega_\rho$ have to be considered.

Theorem 5.10. *If there exist symmetric matrices* P_i*, matrices* G_i *and matrices* F_i*, fulfilling, for a prescribed real number* σ_2*,* $\forall (i,j) \in \{1\cdots N\} \times \{1\cdots N\}$*, the Linear Matrix Inequalities*

$$\begin{bmatrix} P_i & (\bullet)^T & (\bullet)^T & (\bullet)^T \\ \boldsymbol{0} & \sigma_2 \boldsymbol{1} & (\bullet)^T & (\bullet)^T \\ G_i A_i - F_i C & G_i E - F_i H & G_i^T + G_i - P_j & (\bullet)^T \\ \tilde{H} & \boldsymbol{0} & \boldsymbol{0} & \sigma_2 \boldsymbol{1} \end{bmatrix} > 0 \tag{5.46}$$

then the polytopic observer (5.14) *with gain* $L(\rho_k) = \sum_{i=1}^{N} \xi_k^{(i)}(\rho_k) L_i$ *and* $L_i = G_i^{-1} F_i$*, ensures that the error* e_k *of* (5.24) *admits a* $\mathscr{L}_2$ *gain smaller than* σ_2.

Proof 7. *The proof follows the same lines of reasoning than the ones provided in [27]. It is shown that* (5.46) *ensures the existence of a Lyapunov function* $V : \mathbb{R}^n \times \mathbb{R}^L \to \mathbb{R}_+$ *defined by* $V(e_k, \rho_k) = e_k^T P(\rho_k) e_k$ *with* $P(\rho_k) = \sum_{i=1}^{N} \xi_k^{(i)}(\rho_k) P_i$ *and* $\xi_k \in S$*, fulfilling for all* $e_k \in \mathbb{R}^n$*, all* $\xi_k \in S$

$$V(e_{k+1}, \rho_{k+1}) - V(e_k, \rho_k) + \sigma_2^{-1} (\tilde{C} e_k)^T (\tilde{C} e_k) - \sigma_2 v_k^T v_k < 0$$

which is sufficient to obtain

$$\sup_{\|v\|_2 \neq 0,\ \rho_k \in D_\rho} \frac{\|z\|_2}{\|v\|_2} < \sigma_2$$

Since σ_2 appears in a linear way in (5.46), the minimization problem

$$\begin{array}{l} \min \quad \sigma_2 \\ s.t \ (5.46) \end{array} \tag{5.47}$$

is still a convex problem.

Let us notice that (5.46) guarantees that the $\mathscr{L}_2$ gain is bounded for all possible trajectories ρ_k in $D_\rho \supseteq \Omega_\rho$ and not in Ω_ρ.

5.6 Unknown Input Observers

This section is devoted to the design of polytopic Unknown Input Observers for LPV systems. State reconstruction error dynamics and its analysis are investigated both in the deterministic case (consideration of Equation 5.4)) and in the case when the system (5.4) is subjected to disturbances (consideration of Equation (5.23)).

5.6.1 Notation and Definitions

In the deterministic case (consideration of Equation (5.4)), when the system (5.4) is driven by an input sequence $\{u\}_0^\infty$, the output y_{k+i} of (5.4) $(i = 0, \cdots, \infty)$ reads

$$y_{k+i} = C(\rho_{k+i}) A_{\rho_k}^{\rho_{k+i-1}} x_k + \sum_{j=0}^{i} \mathscr{T}_{i,j}(\rho_k) u_{k+j} \tag{5.48}$$

with

$$\mathscr{T}_{i,j}(\rho_k) = C(\rho_{k+i})A^{\rho_{k+i-1}}_{\rho_{k+j+1}}B(\rho_{k+j}) \text{ if } j \leq i-1, \ \ \mathscr{T}_{i,i}(\rho_k) = D(\rho_{k+i})$$

Stacking up the outputs (5.48) yields

$$\underline{y}^i_k = \mathscr{O}^i(\rho_k)x_k + M^i(\rho_k)\underline{u}^i_k \tag{5.49}$$

with

$$\mathscr{O}^i(\rho_k) = \begin{bmatrix} C(\rho_k) \\ C(\rho_{k+1})A(\rho_k) \\ \vdots \\ C(\rho_{k+i})A^{\rho_{k+i-1}}_{\rho_k} \end{bmatrix} \tag{5.50}$$

$$\underline{u}^i_k = \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+i} \end{bmatrix} \tag{5.51}$$

$$\underline{y}^i_k = \begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+i} \end{bmatrix} \tag{5.52}$$

and the Toeplitz-like matrix $M^i(\rho_k)$ defined as the following.

For $i < 0$ $M^i(\rho_k) = \mathbf{0}$, for $i = 0$ $M^0(\rho_k) = D(\rho_k)$ and for $i > 0$,

$$M^i(\rho_k) =$$

$$\begin{bmatrix} D(\rho_k) & \mathbf{0}_{p\times m} & \cdots & \cdots & \cdots \\ C(\rho_{k+1})B(\rho_k) & D(\rho_{k+1}) & \mathbf{0}_{p\times m} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ C(\rho_{k+i})A^{\rho_{k+i-1}}_{\rho_{k+1}}B(\rho_k) & C(\rho_{k+i})A^{\rho_{k+i-1}}_{\rho_{k+2}}B(\rho_{k+1}) & \ldots & C(\rho_{k+i})B(\rho_{k+i-1}) & D(\rho_{k+i}) \end{bmatrix} \tag{5.53}$$

where

$$\begin{aligned} A^{\rho_{k_1}}_{\rho_{k_0}} &= A(\rho_{k_1})A(\rho_{k_1-1})\ldots A(\rho_{k_0}) \text{ if } \ k_1 \geq k_0 \\ &= \mathbf{1}_n \ \text{ if } \ k_1 < k_0 \end{aligned}$$

is the transition matrix.

In the case when the system (5.4) is subjected to disturbances (consideration of Equation (5.23), when the system (5.23) is driven by an input sequence $\{u\}^{\infty}_0$, the output y_{k+i} of (5.23) $(i = 0, \cdots, \infty)$ reads

$$y_{k+i} = C(\rho_{k+i})A_{\rho_k}^{\rho_{k+i-1}}x_k + \sum_{j=0}^{i} \mathscr{T}_{i,j}(\rho_k)u_{k+j} + \sum_{j=0}^{i} \mathscr{S}_{i,j}(\rho_k)w_{k+j}^d + Hw_{k+i}^o \quad (5.54)$$

with

$$\mathscr{T}_{i,j}(\rho_k) = C(\rho_{k+i})A_{\rho_{k+j+1}}^{\rho_{k+i-1}}B(\rho_{k+j}) \text{ if } j \leq i-1, \quad \mathscr{T}_{i,i}(\rho_k) = D(\rho_{k+i})$$

$$\mathscr{S}_{i,j}(\rho_k) = C(\rho_{k+i})A_{\rho_{k+j+1}}^{\rho_{k+i-1}}E \text{ if } j \leq i-1, \quad \mathscr{S}_{i,i}(\rho_k) = \mathbf{0}$$

Stacking up the outputs (5.54) yields

$$\underline{y}_k^i = \mathscr{O}^i(\rho_k)x_k + M^i(\rho_k)\underline{u}_k^i + F^i(\rho_k)\underline{w}_k^{di} + N^i \, \underline{w}_k^{oi} \quad (5.55)$$

with

$$\underline{w}_k^{di} = \begin{bmatrix} w_k^d \\ w_{k+1}^d \\ \vdots \\ w_{k+i}^d \end{bmatrix} \quad (5.56)$$

$$\underline{w}_k^{oi} = \begin{bmatrix} w_k^o \\ w_{k+1}^o \\ \vdots \\ w_{k+i}^o \end{bmatrix} \quad (5.57)$$

The matrix N^i is defined as the following.

$$N^0 = \mathbf{0}, \quad N^1 = \begin{bmatrix} H & \mathbf{0}_{p \times d_{w^o}} \\ \mathbf{0}_{p \times d_{w^o}} & H \end{bmatrix} \quad (5.58)$$

and for $i > 1$, N^i is recursively defined as

$$N^{i+1} = \begin{bmatrix} N^i & \mathbf{0}_{(i+1) \cdot p \times d_{w^o}} \\ \mathbf{0}_{p \times d_{w^o} \cdot (i+1)} & H \end{bmatrix} \quad (5.59)$$

The matrix $F^i(\rho_k)$ is defined as the following.
For $i \leq 0$ $F^i(\rho_k) = \mathbf{0}$ and for $i > 0$,

$$F^i(\rho_k) = \begin{bmatrix} \mathbf{0}_{p \times d_{w^d}} & \mathbf{0}_{p \times d_{w^d}} & \cdots & \cdots & \cdots \\ C(\rho_{k+1})E & \mathbf{0}_{p \times d_{w^d}} & \mathbf{0}_{p \times d_{w^d}} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ C(\rho_{k+i})A_{\rho_{k+1}}^{\rho_{k+i-1}}E & C(\rho_{k+i})A_{\rho_{k+2}}^{\rho_{k+i-1}}E & \cdots & C(\rho_{k+i})E & \mathbf{0}_{p \times d_{w^d}} \end{bmatrix} \quad (5.60)$$

Central notions for the design of UIO are left invertibility and inherent delay.

Definition 5.7. The LPV system (5.4) is *left invertible* if it is possible to recover the input u_0 from a finite number of $r+1$ measurements y_i $(i=0,\dots,r)$, the state vector x_0 at time $k=0$ and the sequence $\{\rho\}_0^r$ of the parameter ρ_k being known. The least integer r for which (5.4) is left invertible is called the *left inherent delay*.

Definition 5.7 is an extension of the notion introduced in [39] for linear MIMO systems. Let us point out that the inherent delay generalizes the notion of relative degree which only holds for SISO systems.

Theorem 5.11. *The LPV system* (5.4) *is* left invertible *if there exists a nonnegative integer $r<\infty$ such that for all $\rho_k \in \Omega_\rho$,*

$$rank\, M^r(\rho_k) - rank\, M^{r-1}(\rho_{k+1}) = m \tag{5.61}$$

Proof 8. *The proof follows the same lines of reasoning than the ones provided in [29].*

5.6.2 Deterministic Case

A polytopic unknown input observer for (5.4) obeys the following equations

$$\begin{cases} \hat{x}_{k+r+1} &= \bar{P}^r(\rho_k)\hat{x}_{k+r} + \bar{Q}^r(\rho_k)\underline{y}_k^r + L(\rho_k)(y_k - \hat{y}_{k+r}) \\ \hat{y}_{k+r} &= C(\rho_k)\hat{x}_{k+r} \end{cases} \tag{5.62}$$

with $\bar{Q}^r(\rho_k)$ obeying

$$\bar{Q}^r(\rho_k) = B(\rho_k)\bar{I}_m M^{r^\dagger}(\rho_k) + Y(\rho_k)(\mathbf{1}_{p(r+1)} - M^r(\rho_k)M^{r^\dagger}(\rho_k)) \tag{5.63}$$

with

$$\bar{I}_m = (\mathbf{1}_m\ \mathbf{0}_{m\times(m\cdot r)}) \tag{5.64}$$

and

$$\bar{P}^r(\rho_k) = A(\rho_k) - \bar{Q}^r(\rho_k)\mathscr{O}^r(\rho_k) \tag{5.65}$$

Let $e_k = x_k - \hat{x}_{k+r}$ be the state error reconstruction. Assuming that Theorem 5.11 is fulfilled, it can be shown, from (5.4) and (5.62)-(5.65) that

$$\begin{aligned} e_{k+1} = &\ (A(\rho_k) - B(\rho_k)\bar{I}_m M^{r^\dagger}(\rho_k)\mathscr{O}^r(\rho_k) \\ &- Y(\rho_k)(\mathbf{1}_{p(r+1)} - M^r(\rho_k)M^{r^\dagger}(\rho_k))\mathscr{O}^r(\rho_k) \\ &- L(\rho_k)C(\rho_k))e_k \end{aligned} \tag{5.66}$$

The matrix $Y(\rho_k)$ is an arbitrary matrix which plays the role of a parameterization. However, it is worth stressing that in some special cases, an arbitrary choice of $Y(\rho_k)$ may not be suitable for state reconstruction purposes (see [14] in the linear case). To overcome this problem, the computation of $Y(\rho_k)$ should be included in the design procedure. From this perspective, we proceed to the change

of variable $\tilde{A}(\rho_k) = A(\rho_k) - B(\rho_k)\bar{I}_m M^{r^\dagger}(\rho_k)\mathscr{O}^r(\rho_k)$, $\tilde{L}(\rho_k) = [Y(\rho_k) \quad L(\rho_k)]$ and $\tilde{C}(\rho_k) = \begin{bmatrix} (\mathbf{1}_{p(r+1)} - M^r(\rho_k)M^{r^\dagger}(\rho_k))\mathscr{O}^r(\rho_k) \\ C(\rho_k) \end{bmatrix}$.
Consequently, (5.66) turns into:

$$e_{k+1} = (\tilde{A}(\rho_k) - \tilde{L}(\rho_k)\tilde{C}(\rho_k))e_k \tag{5.67}$$

The problem of guaranteeing the Global Asymptotical Stability of (5.67) around $e^* = 0$ can be tackled in a similar way than in the previous sections devoted to polytopic observers, that is resorting to poly-quadratic stability.

5.6.3 Noisy Case

Likewise in Section 5.5 when the inputs were supposed to be known, we are concerned with the situation when the system (5.4) is subjected to disturbances and obeys (5.23).

Assuming that Theorem 5.11 is fulfilled, it can be shown from (5.4) and the polytopic UIO (5.62) that, after some heavy but quite basic manipulations, the reconstruction error $e_k = x_k - \hat{x}_{k+r}$ obeys

$$\begin{aligned} e_{k+1} = &(\bar{P}^r(\rho_k) - L(\rho_k)C(\rho_k))e_k + (Ew_k^d - L(\rho_k)Hw_k^o) \\ &-\bar{Q}^r(\rho_k)(F^r(\rho_k)\underline{w}_k^{dr} + N^r\underline{w}_k^{or}) \end{aligned} \tag{5.68}$$

Replacing the expression (5.63) of $\bar{Q}^r(\rho_k)$ into $\bar{P}^r(\rho_k)$ yields

$$\begin{aligned} e_{k+1} = &(A(\rho_k) - B(\rho_k)\bar{I}_m M^{r^\dagger}(\rho_k)\mathscr{O}^r(\rho_k) \\ &-Y(\rho_k)(\mathbf{1}_{p(r+1)} - M^r(\rho_k)M^{r^\dagger}(\rho_k))\mathscr{O}^r(\rho_k) \\ &-L(\rho_k)C(\rho_k))e_k + Ew_k^d - L(\rho_k)Hw_k^o \\ &-(B(\rho_k)\bar{I}_m M^{r^\dagger}(\rho_k) + Y(\rho_k)(\mathbf{1}_{p(r+1)} - M^r(\rho_k)M^{r^\dagger}(\rho_k))) \\ &(F^r(\rho_k)\underline{w}_k^{dr} + N^r\underline{w}_k^{or}) \end{aligned} \tag{5.69}$$

Similarly to the previous case, $Y(\rho_k)$ is an arbitrary matrix which plays the role of a parameterization and we should proceed to the change of variable $\tilde{A}(\rho_k) = A(\rho_k) - B(\rho_k)\bar{I}_m M^{r^\dagger}(\rho_k)\mathscr{O}^r(\rho_k)$, $\tilde{L}(\rho_k) = [Y(\rho_k) \;\; L(\rho_k)]$,
$\tilde{C}(\rho_k) = \begin{bmatrix} (\mathbf{1}_{p(r+1)} - M^r(\rho_k)M^{r^\dagger}(\rho_k))\mathscr{O}^r(\rho_k) \\ C(\rho_k) \end{bmatrix}$.
Consequently, (5.69) turns into:

$$e_{k+1} = (\tilde{A}(\rho_k) - \tilde{L}(\rho_k)\tilde{C}(\rho_k))e_k + v_k \tag{5.70}$$

with:

$$\begin{aligned} v_k = &Ew_k^d - L(\rho_k)Hw_k^o \\ &-(B(\rho_k)\bar{I}_m M^{r^\dagger}(\rho_k) + Y(\rho_k)(\mathbf{1}_{p(r+1)} - M^r(\rho_k)M^{r^\dagger}(\rho_k))) \\ &(F^r(\rho_k)\underline{w}_k^{dr} + N^r\underline{w}_k^{or}) \end{aligned}$$

Again, the problem of polytopic unknown input observer design for guaranteeing performances of the convergence behavior around $e^* = 0$ of (5.70) can be tackled in a similar way than in the previous sections devoted to polytopic observers, that is resorting to poly-quadratic stability.

5.7 Illustrative Examples

5.7.1 Example 1

The purpose of this example is to illustrate both an LPV polytopic description of a nonlinear system and the search for the minimal polytope D_ρ^* wherein the set Ω_ρ is embedded. Let us consider the map, with state vector $x_k = [x_k^{(1)} \;\; x_k^{(2)} \;\; x_k^{(3)} \;\; x_k^{(4)}]^T$, given by

$$\begin{cases} x_{k+1}^{(1)} &= (x_k^{(1)})^2 - (x_k^{(2)})^2 + ax_k^{(1)} + bx_k^{(2)} \\ x_{k+1}^{(2)} &= 2x_k^{(1)}x_k^{(2)} + cx_k^{(1)} + dx_k^{(2)} \\ x_{k+1}^{(3)} &= 0.1bx_k^{(2)} - 0.1(x_k^{(2)})^2 + 0.1x_k^{(3)} \\ x_{k+1}^{(4)} &= 0.5x_k^{(1)} + 0.1x_k^{(2)} + 0.3x_k^{(4)} \\ y_k^{(1)} &= x_k^{(1)} \\ y_k^{(2)} &= x_k^{(2)} \end{cases} \tag{5.71}$$

with $a = 0.9$, $b = -0.6013$, $c = 2$, and $d = 0.5$. For this typical parameters setting, the system exhibits a chaotic motion. A projection of the corresponding chaotic attractor in the 3-dimensional space $(x_k^{(1)}, x_k^{(2)}, x_k^{(3)})$ is depicted in Figure 5.1. We aim at rewriting (5.71) into the LPV form (5.4). To this end, let us choose ρ_k as a parameter vector obeying

$$\begin{aligned} \rho_k^{(1)} &= a + x_k^{(1)} \\ \rho_k^{(2)} &= b - x_k^{(2)} \end{aligned} \tag{5.72}$$

Then, (5.71) can be written as an LPV system of the form (5.4) with

$$A(\rho_k) = \begin{bmatrix} \rho_k^{(1)} & \rho_k^{(2)} & 0 & 0 \\ c & d + 2(\rho_k^{(1)} - a) & 0 & 0 \\ 0 & 0.1\rho_k^{(2)} & 0.1 & 0 \\ 0.5 & 0.1 & 0 & 0.3 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

while B and D are zero since the system (5.71) is autonomous.

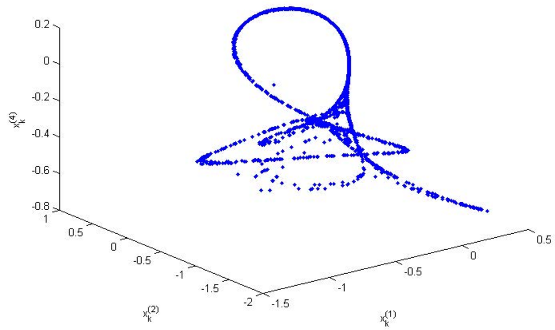


Fig. 5.1 Chaotic attractor in the 3-dimensional space $(x_k^{(1)}, x_k^{(2)}, x_k^{(3)})$

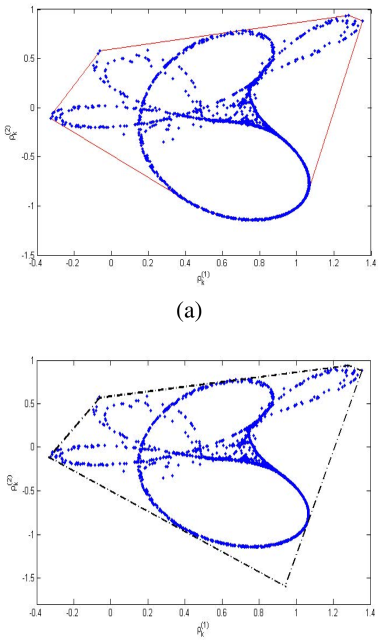
Let us point out that such a choice for ρ_k matches the conditions of the Proposition 1 provided in Subsection 5.3.3. In particular, ρ_k is accessible from the output y_k. Indeed, $\rho_k^{(1)} = a + y_k^{(1)}$ and $\rho_k^{(2)} = b - y_k^{(2)}$.

Now, by simulating (5.71) from an initial condition $x_0 = [-0.72\ \ -0.64\ \ 0.1\ \ 0]^T$ which belongs to the chaotic attractor, we collect 2000 vectors ρ_k in order to build up the set Γ_ρ. Then, the Quick hull approach, implemented in the built-in function *convhull* of the software Matlab, is performed to find out the minimal polytope D_ρ^*. It turns out that 108 vertices θ_i have been found. Both the set Ω_ρ and the minimal polytope D_ρ^* are depicted in Figure 2(a). It should be noted that if the number of the vertices of the polytope is very large, it can be more interesting to minimize the number of vertices in order to enhance the tractability of the LMIs. This is what has precisely been done, as shown in Figure 2(b). The number of the vertices has been reduced to 5 vertices. Let us point out however that the LMIs become more conservative than the ones derived from the minimal polytope.

5.7.2 Example 2

This section illustrates the synthesis of a polytopic observer both in a noise-free as well as in a noisy context. The system under consideration is borrowed from [10] and is called "turbocharged SI engines". Actually, only a part of the system is investigated here. It is briefly described and motivated.

From the perspective of reducing fuel consumption and pollutant emissions of Spark Ignition (SI) engines, new air path management systems have to be proposed. Hence, efficient control of the air actuators is required. For any set point of the torque intended to move the engine, the ratio between the air mass m_{air} and the fuel quantity trapped in the cylinder must be kept constant in order to minimize the pollutant emissions. To this end, the air mass m_{air} trapped in the cylinder must be



(a)

(b)

Fig. 5.2 Set Ω_ρ and polytopes D^*_ρ (a) and D_ρ (b)

known with optimal accuracy in order to predict the quantity of fuel to be injected. It turns out that m_{air} is directly linked to the air flow Q_{cyl} captured in the cylinder.

Actually, a compressor produces a flow from the ambient air. The resulting air flow Q_{th} enters a manifold of which volume is V_{man} and is characterized by a pressure p_{man} and temperature T_{man}. Then, two flows leave the manifold: the first flow which is captured in the cylinder Q_{cyl} and the flow Q_{sc} scavenged from the intake to the exhaust. Unlike Q_{sc}, Q_{th} and p_{man} which are accessible quantities, Q_{cyl} must be estimated from an open loop model of the in-cylinder air mass which delivers $\hat{Q}_{cyl}$. As a result, $Q_{cyl} = \hat{Q}_{cyl} + \Delta Q_{cyl}$ and the error ΔQ_{cyl} can be assumed to be constant because slowly time-varying regarding the overall dynamics. The flow balance in the manifold reads

$$\dot{p}_{man} = \frac{rT_{man}}{V_{man}}(Q_{th} - \hat{Q}_{cyl} - \Delta Q_{cyl} - Q_{sc})$$

As a conclusion, we must estimate ΔQ_{cyl}. However, a direct estimation from the flow balance equation cannot be directly done since it requires the derivative $\dot{p}_{man}$ of p_{man}. Thus, the flow balance equation is written in a state space form where the state vector is composed of p_{man} and ΔQ_{cyl}, the input are the accessible variables, namely, Q_{th}, $\hat{Q}_{cyl}$ and Q_{sc} and finally, the output is p_{man}. An observer is designed to reconstruct both p_{man} and ΔQ_{cyl}. For implementation reasons, we must discretize the equations. After discretization, one obtains

$$\begin{cases} x_{k+1} = A_d(\rho_k)x_k + B_d(\rho_k)u_k \\ y_k = C_d x_k + D_d u_k \end{cases} \tag{5.73}$$

with:

$$x_k = \begin{bmatrix} p_{man}(k) \\ \Delta Q_{cyl}(k) \end{bmatrix}, \quad u_k = \begin{bmatrix} Q_{th}(k) \\ Q_{cyl}(k) \\ Q_{sc}(k) \end{bmatrix}, \quad y_k = p_{man}(k)$$

$$A_d(\rho_k) = \begin{bmatrix} 1 & \rho_k \\ 0 & 1 \end{bmatrix}, \; B_d = \begin{bmatrix} -\rho_k & \rho_k & \rho_k \\ 0 & 0 & 0 \end{bmatrix}, \; C_d = [1 \quad 0], \; D_d = 0$$

and

$$\rho_k = -r \frac{T_{man}(k)}{V_{man}} t_{tdc}(k)$$

The scalar r is the ideal gas constant and t_{tdc} is the sampling period which is actually time-varying because it depends on the engine speed. After normalization, ρ_k lies in the range $[\rho_{min} \;\; \rho_{max}] = [-\,3.3453 \;\; -\,0.0174]$.

We get typically an LPV system which admits a simple polytopic description since D_ρ has clearly only two vertices $\theta_1 = \rho_{min} = -3.3453$ and $\theta_2 = \rho_{max} = -0.0174$. The corresponding matrices A_1 and A_2 are computed according to (5.10) with $L = 2$:

$$A_1 = \begin{bmatrix} 1 & \rho_{min} \\ 0 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 1 & \rho_{max} \\ 0 & 1 \end{bmatrix}$$

In our present case, the matrix B is also parameter dependent. Since it also depends on ρ_k in a similar way, it admits the same polytopic decomposition with vertices

$$B_1 = \begin{bmatrix} -\rho_{min} & \rho_{min} & \rho_{min} \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} -\rho_{max} & \rho_{max} & \rho_{max} \\ 0 & 0 & 0 \end{bmatrix}$$

For the state reconstruction of x_k, we resort to a polytopic observer of the form (5.14). The toolbow *Yalmip* of Matlab is used to solve the LMIs required to derive the gain $L(\rho_k)$ of the observer.

Results

Poly-Quadratic stability

It turns out that the LMIs (5.18) are feasible. The resulting gains are respectively $L_1 = [1.9623 \;\; -0.2877]^T$, $L_2 = [1.0056 \;\; -0.2977]^T$ and ensure the Global Asymptotical convergence of the observer.

Performances in a noisy context: ISS

We solve the problem (5.35) involving the LMIs (5.31) in order to minimize σ_{ev} in the ISS gain of (5.32). The optimal solution is given by $\sigma^*_{ev} = 1.3401\ 10^4$ with observer gains $L_1 = [2.8856 \;\; -0.5637]^T$ and $L_2 = [1.0101 \;\; -0.5779]^T$. Next, in order to minimize ν in the ISS gain of (5.38), we solve the problem (5.41) involving the LMIs (5.36)-(5.37) for different values of λ within the admissible range

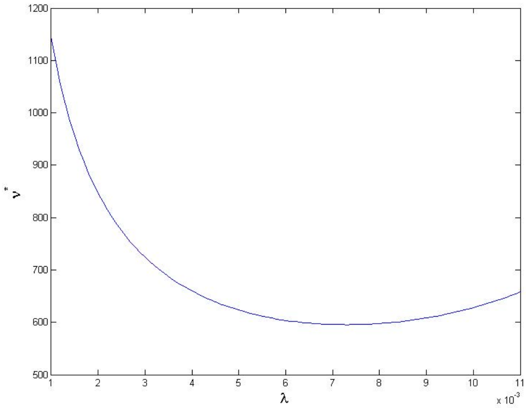


Fig. 5.3 variation of ν^* with respect to λ

$[0.0001 \ -0.9999]$. The variation of the optimal solution ν^* with respect to λ is plotted in Figure 5.3. As we can see, this variation is convex. The best ISS gain corresponds to $\nu^* = 596.2197$, $\lambda^* = 7.2\ 10^{-3}$ and $\mu^* = 595.9576$. The gains are given by $L_1 = [2.3882 \ -0.4150]^T$ and $L_2 = [1.0073 \ -0.4195]^T$. As expected, one has $\sigma_{ev}^* > \nu^*$ since the LMIs (5.31) are more conservative than (5.36)-(5.37).

It can also be interesting to compare the decay factor obtained respectively from (5.31) and (5.36)-(5.37) for the same ISS gain. Let us check the conditions for $\nu = \sigma_{ev}^* = 13401$, that is, for the optimal value of σ_{ev} when considering (5.31). On one hand, the decay factor is given by: $\sqrt{1 - \frac{1}{\sigma_{ev}^*}} = 0.99996$. On the other hand, the solution of (5.36)-(5.37) for $\nu = \sigma_{ev}^* = 13401$ gives $\mu^* = 12207.0353$ and $\lambda^* = 0.0202$ and so a decay factor $\sqrt{1-\lambda^*} = 0.98985$. The corresponding gains are $L_1 = [2.7256 - 0.5158]^T$ and $L_2 = [1.0103 - 0.5891]^T$. As expected by the consideration on the conservatism, for a same ISS gain, we can also get a better decay rate when considering the LMIs (5.36)-(5.37) instead of (5.31).

5.8 Conclusion

Observer design for LPV systems has been discussed. It has been shown that a polytopic approach can be used to take into account not only the classical situation where the parameters can be perfectly measured but also the more realistic situation when the measured parameters are affected by a bounded uncertainty and/or when the system is subjected to disturbances. Both stability of the observation error and performance (ISS, bounded peak-to-peak gain and $\mathscr{L}_2$ gain) have been considered. The theoretical developements end up with the design of polytopic unknown input observers both in the deterministic and in the noisy or uncertain cases. These results can be extended to take into account constraints related to the parameter rate of variation. Indeed, motivated by practical issues, taking into account the bounds on the rate of variation of the parameters has already been discussed for control design (see [8, 21] and references therein). And yet, it turns out that such a consideration makes sense for observer design as well. Finally, polytopic LPV observers discussed here can be used for both control and diagnosis. As an example, a solution based on polytopic observers for output feedback observer based controllers for LPV systems with inexact parameter measurement is proposed in [20].

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