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IFAC PapersOnLine 50-1 (2017) 9778–9783

# Joint State and Parameter Estimation for Discrete-Time Polytopic Linear Parameter-Varying Systems ⋆

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**Abstract:** Linear parameter-varying systems are very suitable for modelling nonlinear systems, since well-established methods from the linear-systems domain can be applied. Knowledge about the scheduling parameter is an important condition in this modelling framework. In case this parameter is not known, joint state and parameter-estimation methods can be employed, e.g., using interacting multiple-model estimation methods, or using an extended Kalman filter. However, these methods cannot be directly used in case the parameters lie in a polytopic set. Furthermore, these existing methods require tuning in order to have convergence and stability. In this paper, we propose to solve the joint-estimation problem in a two-step, Dual Estimation approach, where we first solve the parameter-estimation problem by solving a constrained optimisation problem in a recursive manner and secondly, employ a robust polytopic observer design for state estimation. Simulations show that our novel method outperforms the existing joint-estimation methods and is a promising first step for further research.



*Keywords:* State estimation, parameter estimation, observer design, polytopic systems, LPV

## 1. INTRODUCTION

Today, the field of Linear Parameter-Varying (LPV) modelling receives a great deal of attention and it can be seen as a bridge between nonlinear and linear systems. An LPV system can be interpreted as a combination of Linear Time-Invariant (LTI) systems, combined through a scheduling parameter. Therefore, well-established results in the linear-systems domain are applicable to these systems, which has created significant interest from many fields of research for obtaining accurate models and performing controller and observer synthesis. For example, the advantages of the LPV framework are used in aerospace engineering (Marcos and Balas, 2004) and battery management for automotive applications (Hu and Yurkovich, 2012), amongst numerous other fields. In particular, results for robust controller and observer synthesis are an advantage in the sense that robustness, stability and possibly optimal performance can be guaranteed. An important assumption in the LPV framework is that the scheduling parameter is known, e.g., through measurements. Even for uncertain measurements of the scheduling parameter, it has been shown that it is still possible to synthesise a robustly stabilising observer (Zhang et al., 2016), up to large levels of uncertainty (Heemels et al., 2010).

However, when the scheduling parameter is unknown, an important assumption in the LPV framework is not satisfied. In order to deal with nonlinear systems with the idea of the LPV framework in mind, in combination with performing joint state and parameter estimation, multiple-model estimation methods have been proposed (Bar-Shalom et al., 2001). These methods perform very well in applications such as aircraft tracking, where a finite number of models is sufficient. However, when the scheduling parameter varies in a polytopic set, these methods can only lead to approximate results by considering a finite number of models in the polytopic set. An alternative has been proposed in the form of an interpolation-based multiple-model method or even a polytopic Extended Kalman Filter (EKF) approach, where the scheduling parameter is estimated (Hallouzi et al., 2009). The problem of the latter approach is that, provided that a set of assumptions is met, stability and convergence can only be locally guaranteed with the correct tuning of the filter (Ljung, 1979; Reif et al., 1999; Boutayeb et al., 1997). The above challenges motivate our research to revisit the problem of joint estimation for polytopic systems.

In this paper, we propose a two-step Dual Estimation (DE) approach, which guarantees robust stability and convergence, for solving the joint-estimation problem for polytopic systems. The first step (i.e., state estimation) consists of synthesising a robustly stabilising parameter-varying observer for all possible systems in the polytopic set. In the second step, the parameter-estimation step, the scheduling parameter for the polytopic parameter-varying system is estimated in two stages. Firstly, a Recursive Least Squares (RLS) algorithm with forgetting is proposed for finding the Transfer-Function (TF) coefficients of the system and secondly, the TF coefficients are mapped to the scheduling-parameter values by solving a small-sized constrained optimisation problem. Guarantees on convergence can be given in the sense of convergence of

⋆ This work has received financial support from the H2020 programme of the European Commission under the grant 3CCar (grant no.662192).


10.1016/j.ifacol.2017.08.880

state and scheduling parameter to the true values, while robust stability of the state-estimation error is guaranteed by the robust parameter-varying observer.

The organisation of the paper is as follows. The problem statement is given in Section 2 and the proposed approach is introduced in Section 3. Subsequently, an existing joint state and parameter-estimation method, namely, Extended Kalman filtering, will be presented and adapted for polytopic parameter-varying systems in Section 4 and a numerical example with a comparison of the presented methods is given in Section 5. Finally, the conclusions are drawn in Section 6.

*Nomenclature* The following notational conventions will be used. The transpose of a matrix $A$ will be denoted by $A^\top$, and its determinant by $\det(A)$. The adjoint or adjugate matrix of $A$ is given by $\mathrm{adj}(A) = C^\top$, where $C$ is the matrix containing the cofactors of $A$. If $\det(A) \neq 0$, then $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$. Finally, $\{u_\ell\}_{\ell=\bar{k}}^{k}$ denotes the sequence $\{u_{\bar{k}}, \dots, u_k\}$ for some $\bar{k} \le k$ and $\alpha^i$ denotes the $i$-th element of a vector $\alpha \in \mathbb{R}^n$.

## 2. PROBLEM STATEMENT

Let us consider a discrete-time polytopic parameter-varying system given by

$$\begin{cases} x_{k+1} = \sum_{i=1}^{N} \alpha^i (A_i x_k + B_i u_k) \\ y_k \quad\;\; = C x_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}$ and $y_k \in \mathbb{R}$ denote the state, the input and the output, respectively, of the system at discrete time $k \in \mathbb{N}$. Moreover, $N \in \mathbb{N}$ is the number of vertices of the polytopic system and the scheduling parameter $\alpha \in \mathcal{A}$ is unknown, where

$$\mathcal{A} = \Big\{ \alpha \in \mathbb{R}^N \Big| \sum_{i=1}^{N} \alpha^i = 1, \alpha^i \geqslant 0 \text{ for } i \in \{1, 2, \dots, N\} \Big\}. \tag{2}$$

In this paper, we focus on the simultaneous estimation of both the state $x_k$ as well as the scheduling parameter $\alpha \in \mathcal{A}$. In particular, we would like to develop a (recursive) update rule for the estimates of $x_k$ and $\alpha$ of (1) at time $k \in \mathbb{N}$, denoted by $\hat{x}_k$ and $\hat{\alpha}_k$, which we will now formalise.

*Problem 1.* (Joint-Estimation Problem). Find functions $f_x$ and $f_\alpha$ so that the recursive update rules of the form

$$\begin{cases} \hat{x}_{k+1} = f_x(\hat{x}_k, \hat{\alpha}_k, \{u_\ell, y_\ell\}_{\ell=\bar{k}_x}^{k}) \\ \hat{\alpha}_{k+1} = f_\alpha(\hat{x}_k, \hat{\alpha}_k, \{u_\ell, y_\ell\}_{\ell=\bar{k}_\alpha}^{k}) \end{cases} \tag{3}$$

for some $\bar{k}_x, \bar{k}_\alpha < k$ yield that $\|x_k - \hat{x}_k\| \to 0$ and $\|\alpha - \hat{\alpha}_k\| \to 0$ as $k \to \infty$.

For simplicity, we assume in this paper that $\alpha \in \mathcal{A}$ is constant, or at most slowly time varying and that the matrix $C$ in (1) does not depend on the scheduling parameter $\alpha$. Moreover, we assume that the pair $\left(\sum_{i=1}^{N} \hat{\alpha}^i A_i, C\right)$ is observable for all $\hat{\alpha} \in \mathcal{A}$. Finally, to guarantee a unique solution to the joint-estimation problem, we assume that $C(qI - \sum_{i=1}^{N} \alpha^i A_i)^{-1} \sum_{i=1}^{N} \alpha^i B_i = C(qI - \sum_{i=1}^{N} \hat{\alpha}^i A_i)^{-1} \sum_{i=1}^{N} \hat{\alpha}^i B_i$ for almost all $q \in \mathbb{C}$ implies that $\alpha = \hat{\alpha}$.

## 3. JOINT ESTIMATION FOR POLYTOPIC SYSTEMS

In this section, we propose a solution to the joint-estimation problem posed in the previous section. The solution strategy that we will present is based on separating the state and the parameter-estimation problem. In particular, we propose to use a polytopic Luenberger-type of observer given by

$$\hat{x}_{k+1} = \sum_{i=1}^{N} \hat{\alpha}_k^i (A_i \hat{x}_k + B_i u_k + L_i(C\hat{x}_k - y_k)) \tag{4}$$

for some $\hat{\alpha}_k \in \mathcal{A}$ to estimate the state $x_k$, for well-designed matrices $L_i$, $i \in \{1, \dots, N\}$. As we will show with the observer design, the matrices $L_i$, $i \in \{1, \dots, N\}$ can be designed such that the prediction error of the observer (4) yields $\|x_k - \hat{x}_k\| \to 0$ when $k \to \infty$ if $\|\alpha - \hat{\alpha}_k\| \to 0$. For the parameter-estimation problem, we can develop an update rule for $\hat{\alpha}_k$ that uses past inputs and outputs $\{u_\ell, y_\ell\}_{\ell=\bar{k}}^{k}$ only, for some $\bar{k} < k$, that achieves $\|\alpha - \hat{\alpha}_k\| \to 0$ as $k \to \infty$.

### 3.1 Robust Observer Design

In the first step of the design procedure, we will design the observer gains $L_i$ so that the estimation error $e_k = x_k - \hat{x}_k$, given by

$$e_{k+1} = \sum_{i=1}^{N} \Big( \hat{\alpha}_k^i (A_i + L_i C) e_k + (\alpha^i - \hat{\alpha}_k^i)(A_i x_k + B_i u_k) \Big) \tag{5}$$

is stable in some appropriate sense. It should be noted that, in this paper, we investigate systems for which the matrix $C$ does not depend on the scheduling parameter. Considering systems with $C_i$, in which case cross products between $C_i$ and $L_i$ would appear in the prediction error, see (5), can be seen as an extension to this work.

The stability concept employed in the observer design in this paper is Input-to-State-Stability (ISS), see, e.g., (Khalil, 2013). Namely, the estimation error can be written as

$$e_{k+1} = \sum_{i=1}^{N} \hat{\alpha}_k^i (A_i + L_i C) e_k + v_k \tag{6}$$

with

$$v_k = \sum_{i=1}^{N} (\alpha^i - \hat{\alpha}_k^i)(A_i x_k + B_i u_k) \tag{7}$$

and ISS of (6) with respect to $\{v_k\}_{k \in \mathbb{N}}$ for all $\hat{\alpha} \in \mathcal{A}$ implies that the solutions satisfy

$$\|e_k\| \leqslant c\lambda^k \|e_0\| + \sum_{\ell=0}^{k-1} \sigma \lambda^{k-1-\ell} \|v_\ell\| \tag{8}$$

for some $c > 0$, $0 < \lambda < 1$, $\sigma > 0$ and all $e_0, v_k \in \mathbb{R}^{n_x}$, $k \in \mathbb{N}$. Roughly speaking, the property guarantees that, if the disturbance $v$ vanishes, the state error $e$ converges to zero, where it should be noted that the observer is robust against all variations of $\hat{\alpha}_k$. To ensure that $v$ vanishes, proper estimation of $\hat{\alpha}$ needs to be addressed.

Guaranteeing ISS of the system (6) can be done using the notion of an ISS Lyapunov function. An ISS Lyapunov function is a function $V(e, \hat{\alpha}) > 0$ for all e $\neq 0$, $V(0, \hat{\alpha}) = 0$ and

$$V(e_{k+1}, \hat{\alpha}_{k+1}) - V(e_k, \hat{\alpha}_k) < -\|e_k\|^2 + \zeta\|v_k\|^2 \tag{9}$$

for all $k \in \mathbb{N}$. The following Theorem, taken from (Heemels et al., 2010), provides conditions in the form of Linear Matrix Inequalities (LMIs) that can be used for the synthesis of matrices $L_i$ that render (6) ISS.

*Theorem 1.* Assume there exist matrices $P_i$, $F_i$, $G_i$, $i \in \{1, \ldots, N\}$ and a scalar $\zeta$ such that

$$\begin{bmatrix} G_i + G_i^\top - P_j & 0 & G_iA_i + F_iC & G_i \\ 0 & I & I & 0 \\ A_i^\top G_i^\top + C^\top F_i^\top & I & P_i & 0 \\ G_i^\top & 0 & 0 & \zeta I \end{bmatrix} \succ 0 \tag{10}$$

for all $i, j \in \{1, \ldots, N\}$. Then the system (6) with

$$L_i = G_i^{-1} F_i \tag{11}$$

is ISS with respect to $v$.

### 3.2 Parameter Estimation

Since the state estimation error is guaranteed to be bounded for any $\hat{\alpha}_k \in \mathcal{A}$, we will now propose a method which yields that $\|\alpha - \hat{\alpha}_k\| \to 0$. In particular, we propose an on-line recursive parameter-estimation method that uses input-output data, i.e., $\{u_\ell, y_\ell\}_{\ell=0}^k$, only. While the state-estimation problem is coupled to the parameter-estimation method, and is robust with respect to uncertainty in the parameters, the parameter-estimation procedure does not use the state estimates. This will prevent convergence or instability issues that might occur in alternative approaches, see Section 4.

The parameter-estimation problem will be solved by minimising a prediction error, i.e.,

$$\hat{\alpha}_k = \arg\min_{\alpha \in \mathcal{A}} \sum_{\ell=0}^{k} \gamma^{k-\ell} \|y_k - \varphi_k \vartheta(\alpha)\|^2 \tag{12a}$$

where $0 < \gamma \leqslant 1$ is a 'forgetting factor' and

$$\varphi_k = [y_{k-1} \, \ldots \, y_{k-n_x} \; u_{k-1} \, \ldots u_{k-n_x}] \tag{12b}$$

$$\vartheta(\alpha) = [a_1(\alpha) \, \ldots \, a_{n_x}(\alpha) \; b_1(\alpha) \, \ldots b_{n_x}(\alpha)]^\top \tag{12c}$$

in which the variables $a_i$ and $b_i$ are the coefficients from the polynomials

$$\sum_{i=1}^{n_x} a_i(\alpha) q^i = \det\big(qI - \sum_{i=1}^{N} \alpha^i A_i\big) \tag{13a}$$

$$\sum_{i=1}^{n_x} b_i(\alpha) q^i = C \operatorname{adj}\big(qI - \sum_{i=1}^{N} \alpha^i A_i\big) \sum_{i=1}^{N} \alpha^i B_i. \tag{13b}$$

The polynomials (13) are formed from computing the coefficients of the transfer function of system (1), in which $(qI - A)^{-1} = \frac{1}{\det(qI-A)} \operatorname{adj}(qI - A)$. It should be noted that solving (12) is a nonlinear, and possibly nonconvex, optimisation problem.

Instead of solving (12) directly, we propose a two-step approach in which we first determine the coefficients $a_i$ and $b_i$ by solving

$$\hat{\vartheta}_k = \arg\min_{\vartheta} \sum_{\ell=0}^{k} \gamma^{k-\ell} \|y_k - \varphi_k \vartheta\|^2, \tag{14}$$

which is an unconstrained linear least-squares optimisation problem, which can be done in a recursive manner. Subsequently, we will determine the scheduling parameter $\hat{\alpha}_k$ corresponding to the parameter $\hat{\vartheta}_k$. The latter problem is still a constrained nonlinear optimisation problem, but with a less-complex objective function. The solution to the linear least-squares problem in a recursive form is given by

$$P_k = \tfrac{1}{\gamma} P_{k-1} - \tfrac{1}{\gamma} P_{k-1} \varphi_k^\top (\gamma I + \varphi_k P_{k-1} \varphi_k^\top)^{-1} \varphi_k P_{k-1} \tag{15a}$$

$$\hat{\vartheta}_k = \hat{\vartheta}_{k-1} + P_k \varphi_k^\top (y_k - \varphi_k \hat{\vartheta}_{k-1}) \tag{15b}$$

for a given initial estimate $\hat{\vartheta}_0$ and some (positive definite) matrix $P_0$. The second step in the parameter-estimation procedure that we propose is solving

$$\hat{\alpha}_k = \arg\min_{\alpha \in \mathcal{A}} (\hat{\vartheta}_k - \vartheta(\alpha))^\top P_k^{-1} (\hat{\vartheta}_k - \vartheta(\alpha)), \tag{16}$$

which completes the procedure. Although this step requires the use of nonlinear optimisation solvers, the computations can be executed relatively efficiently by supplying the solvers with the required Hessian and Jacobian of the objective function symbolically. Also, $P_k^{-1}$ can be computed in a recursive manner instead of computing the inverse of $P_k$ directly. The proposed two-step method solves (12), as presented in the Theorem below.

*Theorem 2.* The optimal solution to the two-step procedure consisting of (15) and (16) is equal to the optimal solution to (12).

**Proof.** First, we use that the inverse of the left-hand and the right-hand side of (15a) satisfy

$$P_k^{-1} = \gamma P_{k-1}^{-1} + \varphi_k^\top \varphi_k, \tag{17}$$

whose solution satisfies

$$P_k^{-1} = \gamma^k P_0^{-1} + \sum_{\ell=1}^{k} \gamma^{k-\ell} \varphi_\ell^\top \varphi_\ell. \tag{18}$$

Using (18) and the fact that $\sum_{\ell=0}^{k-1} \gamma^{k-\ell} \varphi_\ell^\top \varphi_\ell \vartheta_{k-1} = \sum_{\ell=0}^{k-1} \gamma^{k-\ell} \varphi_\ell^\top y_\ell$, expression (15b) can be rewritten as

$$\hat{\vartheta}_k = \Phi_k^{-1} \sum_{\ell=0}^{k} \gamma^{k-\ell} \varphi_\ell^\top y_\ell, \tag{19}$$

with $\Phi_k = \sum_{\ell=0}^{k} \gamma^{k-\ell} \varphi_\ell^\top \varphi_\ell$, which is the solution to the unconstrained least-squares problem (14). Subsequently, substituting this solution into the optimisation problem

$$\min_{\alpha \in \mathcal{A}} (\hat{\vartheta}_k - \vartheta(\alpha))^\top P_k^{-1} (\hat{\vartheta}_k - \vartheta(\alpha)) + Q_k \tag{20}$$

with

$$Q_k = \sum_{\ell=0}^{k} \big(\gamma^{k-\ell} y_\ell^\top y_\ell\big) - \big(\sum_{\ell=0}^{k} \gamma^{k-\ell} \varphi_\ell^\top y_\ell\big)^\top \Phi_k^{-1} \big(\sum_{\ell=0}^{k} \gamma^{k-\ell} \varphi_\ell^\top y_\ell\big) \tag{21}$$

leads to

$$\begin{aligned} &\min_{\alpha \in \mathcal{A}} (\Phi_k^{-1} \textstyle\sum_{\ell=0}^{k} \gamma^{k-\ell} \varphi_\ell^\top y_\ell - \vartheta(\alpha))^\top \Phi_k \\ &\qquad \times (\Phi_k^{-1} \textstyle\sum_{\ell=0}^{k} \gamma^{k-\ell} \varphi_\ell^\top y_\ell - \vartheta(\alpha)) + Q_k \\ &= \min_{\alpha \in \mathcal{A}} \sum_{\ell=1}^{k} \gamma^{k-\ell} \vartheta(\alpha)^\top \varphi_\ell^\top \varphi_\ell \vartheta(\alpha) \\ &\qquad - (\varphi_\ell^\top y_\ell)^\top \vartheta(\alpha) - \vartheta(\alpha)^\top \varphi_\ell^\top y_\ell + y_\ell^\top y_\ell \\ &= \min_{\alpha \in \mathcal{A}} \sum_{\ell=1}^{k} \gamma^{k-\ell} (y_\ell - \varphi_\ell^\top \vartheta(\alpha))^\top (y_\ell - \varphi_\ell^\top \vartheta(\alpha)), \end{aligned} \tag{22}$$

which is the optimisation problem (12). Now, we can observe that the solution to (20) corresponds to the solution to (16), meaning that the optimal solution to the optimisation problem together with (15) is equivalent to solving (12), which concludes the proof. □

## 4. EXTENDED KALMAN FILTERING

A well-established method, often used for solving the joint state and parameter estimation as described in Section 3, is the Extended Kalman Filter (Ljung, 1979). The main is idea is to augment the state with the parameters, i.e.,

$$\begin{bmatrix} x_{k+1} \\ \alpha_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \sum_{i=1}^{N} \alpha^i (A_i x_k + B_i u_k) \\ \alpha_k \end{bmatrix}}_{:=f(x_k, \alpha_k, u_k)} \tag{23}$$

and to apply a method that is very similar to Kalman filtering. Since (23) is no longer a linear system, due to the products of $\alpha$ and $e$, a Kalman filter cannot be applied directly. Instead, EKF uses a linearised system to update the Kalman filtering equations, see below.

In summary, the extended Kalman filter uses two steps: a prediction step and an update step. The prediction step is given by

$$\hat{x}_{k|k-1} = \textstyle\sum_{i=1}^{N} \hat{\alpha}^i_{k-1|k-1} (A_i \hat{x}_{k-1|k-1} + B_i u_{k-1}) \tag{24a}$$

$$\hat{\alpha}_{k|k-1} = \hat{\alpha}_{k|k} \tag{24b}$$

$$P_{k|k-1} = \hat{A}_{k-1} P_{k-1|k-1} \hat{A}_{k-1}^\top + Q \tag{24c}$$

and the update step is given by

$$\begin{bmatrix} \hat{x}_{k|k} \\ \bar{\alpha}_{k|k} \end{bmatrix} = \begin{bmatrix} \hat{x}_{k|k-1} \\ \hat{\alpha}_{k|k-1} \end{bmatrix} + L_k (y_k - C \hat{x}_{k|k-1}) \tag{25a}$$

$$P_{k|k} = (I - L_k \hat{C}) P_{k|k-1}, \tag{25b}$$

where, to ensure that $\hat{\alpha}_{k|k} \in \mathcal{A}$, we adapt the standard EKF and use the projection

$$\hat{\alpha}_{k|k} = \arg\min_{\alpha \in \mathcal{A}} \|\alpha - \bar{\alpha}_{k|k}\| \tag{25c}$$

and

$$L_k = P_{k|k-1} \hat{C}^\top (R + \hat{C} P_{k|k-1} \hat{C}^\top)^{-1} \tag{26a}$$

$$\begin{aligned} \hat{A}_k &= \begin{bmatrix} \frac{\partial f(\hat{x}_{k|k}, \hat{\alpha}_{k|k}, u_k)}{\partial x} & \frac{\partial f(\hat{x}_{k|k}, \hat{\alpha}_{k|k}, u_k)}{\partial \alpha} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^{N} \hat{\alpha}^i_{k|k} A_i & \left[ (A_1 x_{k|k} + B_1 u_k) \ \ldots \ (A_N x_{k|k} + B_N u_k) \right] \\ 0 & I \end{bmatrix} \end{aligned} \tag{26b}$$

$$\hat{C} = [C \quad 0]. \tag{26c}$$

It should be noted that, even though the positive definite matrices $Q$ and $R$ originate from covariance matrices in the conventional Kalman filter, they are often used as tuning parameters when used in the extended Kalman filter for solving the joint-estimation problem. The stochastic assumptions with the conventional Kalman filter that the process noise and measurement noise (corresponding to $Q$ and $R$, respectively) are Gaussian, are not always valid for the actual system. Moreover, it is known that the EKF converges when the initial state and parameter estimates are close enough, the disturbing noise terms are small enough and the nonlinearities are not discontinuous, even though no quantitative statements on the domain of attraction are known (Reif et al., 1999). Also, in general, EKF estimates are biased and divergent (Ljung, 1979) and only mild converge conditions can be given based on the linearisation of the nonlinear system in the EKF (Boutayeb et al., 1997). Furthermore, the EKF cannot handle constraints $\alpha \in \mathcal{A}$ and in order to ensure that $\hat{\alpha} \in \mathcal{A}$, we modify the EKF in the sense that we project the parameters found using the EKF into the set $\mathcal{A}$ at every time $k \in \mathbb{N}$ as in (25c).

## 5. NUMERICAL EXAMPLE

A simulation study is performed in order to evaluate the performance of the proposed method, as well as the performance of the EKF with projection and the performance of a third method for solving the joint-estimation problem, the Interacting Multiple Model estimation (IMM) method (Bar-Shalom et al., 2001). IMM uses a discrete number of models (or modes) $M$ with $M$ different Kalman filters for each mode $j \in \{1, \ldots, M\}$ and the algorithm computes $\mu^j_k$, the probability of being in a certain mode at time $k$. In order to use IMM in the context of parameter estimation for polytopic systems (where we have infinitely many modes), the parameter space $\mathcal{A}$ is gridded, leading to a finite number of modes $\{\bar{\alpha}_i, \ldots, \bar{\alpha}_M\} = \mathcal{A}_M \subset \mathcal{A}$. This leads to a trade-off between complexity and accuracy of the IMM estimation method, leading to

$$\bar{A}_j = \sum_{i=1}^{N} \bar{\alpha}^i_j A_i \qquad \text{and} \qquad \bar{B}_j = \sum_{i=1}^{N} \bar{\alpha}^i_j B_i. \tag{27}$$

For a detailed description of IMM, the reader is referred to (Bar-Shalom et al., 2001).

Now, let us define a discrete-time polytopic parameter-varying system given by (1) for $N = 4$. The vertices of the polytopic system are selected randomly, under the condition that they are stable, i.e., the eigenvalues of every $A_i$ lie inside the unit disc, and observable. In this example, the polytopic system in (1) is given by $C = [1 \ 0]$ and

$$\begin{aligned} A_1 &= \begin{bmatrix} -0.80 & 0.25 \\ 0.25 & -0.30 \end{bmatrix}, B_1 = \begin{bmatrix} 1.9 \\ 0 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0.30 & 0.70 \\ 0.70 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} -1 \\ 1.50 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} -0.30 & 0.65 \\ 0.55 & 0.10 \end{bmatrix}, B_3 = \begin{bmatrix} 0.30 \\ -2 \end{bmatrix}, \\ A_4 &= \begin{bmatrix} 0.55 & -0.20 \\ -0.40 & -0.30 \end{bmatrix}, B_4 = \begin{bmatrix} -0.60 \\ 0 \end{bmatrix}. \end{aligned} \tag{28}$$

In order to guarantee informative input-output data of the polytopic system, a system input $u$ is chosen so that excitation of the dynamics of the polytopic system is ensured. Therefore, $u$ is taken to be a square wave of period $T = 10$ samples with a random, uniformly distributed duty cycle and an amplitude of one (i.e. $u_k \in [0, 1]$ for all $k \in \mathbb{N}$). The $\alpha$-tracking performance of the methods can be evaluated by setting the scheduling parameter $\alpha$ to

$$\alpha_k = \begin{cases} [0.5, 0.3, 0.2, 0]^\top & k < 500 \\ [0.35, 0.4, 0.1, 0.15]^\top & k \geq 500 \end{cases} \tag{29}$$

and setting the initial conditions $\alpha$ for all of the methods to $\alpha_0 = [0.25, 0.25, 0.25, 0.25]^\top$.

A measure for the performance of the estimation methods can be given by the Euclidean vector norm of the state error and scheduling-parameter error at time $k$, i.e.,

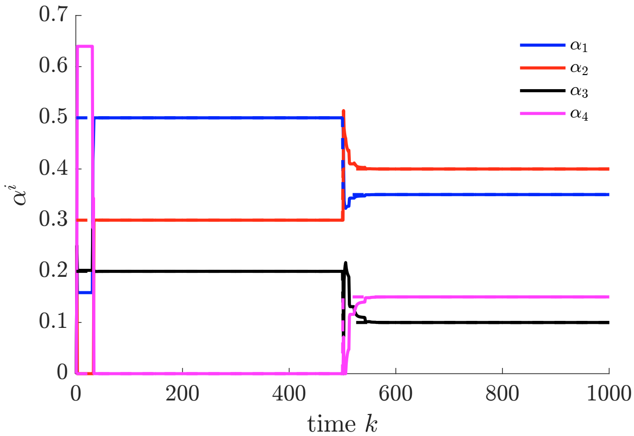


Fig. 1. $\alpha$ (dashed) and $\hat{\alpha}$ (solid) for the DE method.

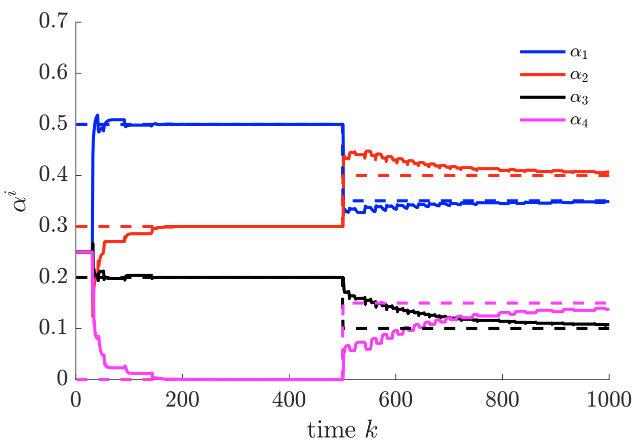


Fig. 2. $\alpha$ (dashed) and $\hat{\alpha}$ (solid) for the EKF method.

$\|x_k - \hat{x}_k\|$ and $\|\alpha_k - \hat{\alpha}_k\|$, respectively. Ideally, these norms should converge to zero as $k \to \infty$. Since the IMM estimation method uses a finite number of modes by means of gridding of the parameter space $\mathcal{A}$, $\|\alpha_k - \hat{\alpha}_k\|$ can be obtained by using $\hat{\alpha}_k = \sum_{j=1}^{M} \mu_k^j \bar{\alpha}_j$. Also, in order to illustrate the trade-off between complexity and accuracy with IMM estimation, we define $\{\alpha_k | k < 500\} \in \mathcal{A}_M$, whilst $\{\alpha_k | k \geq 500\} \notin \mathcal{A}_M$.

The tuning parameters for the estimation methods are chosen such that the methods have good performance with respect to the defined error norms. For DE, $\gamma = 0.9$ is chosen, for IMM, we tune the probabilities $p^{ij}$, which describe the probability of a transition from mode $i$ to mode $j$, to $p^{ii} = 0.99$ (with appropriate choices for $Q$ and $R$ in the $M$ Kalman filters) and for the EKF, $R = 0.01$ and the state-covariance matrix $Q$ has non-zero diagonal entries for the scheduling-parameter covariances with $q_{ii} = 100$.

In Figs. 1, 2 and 3, the estimated scheduling parameter $\hat{\alpha}$ for the estimation methods is depicted. In each figure, the solid lines show the estimated scheduling parameter $\hat{\alpha}$ and the dashed lines depict $\alpha$ from (29). In all figures, it can be seen that $\hat{\alpha}$ tracks $\alpha$ reasonably well for $k < 500$. Both the DE and the EKF, in Fig. 1 and Fig. 2, respectively, converge to the true values of $\alpha$, with DE having faster convergence. It follows from $\sum_{j=1}^{M} \mu_k^j \bar{\alpha}_j$ that $\hat{\alpha}$ in the

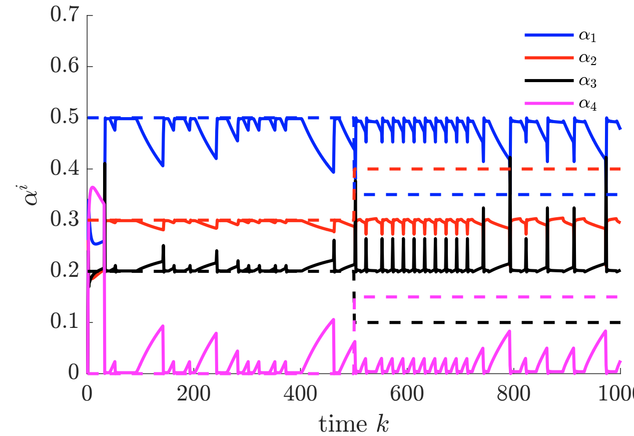


Fig. 3. $\alpha$ (dashed) and $\hat{\alpha}$ (solid) for the IMM method.

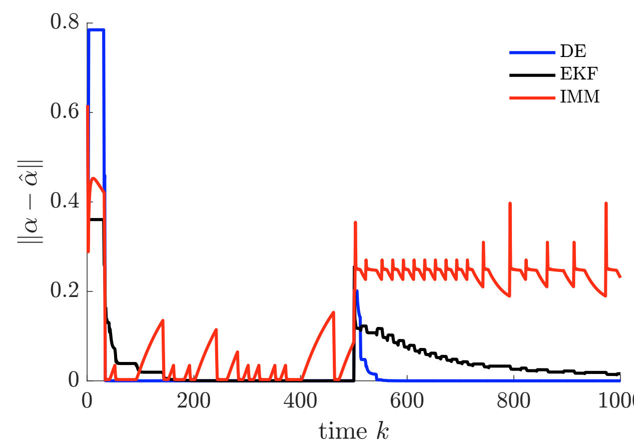


Fig. 4. $\|\alpha_k - \hat{\alpha}_k\|$ for DE, EKF and IMM.

IMM method in Fig. 3 converges to a value around $\alpha$, since the mode probabilities $\mu_k^j$ do not converge to zero by definition, see (Bar-Shalom et al., 2001). For $k \geq 500$, the DE method has fast convergence, whereas the EKF has a slower convergence rate, possibly due to changed system dynamics as a consequence of the value of $\alpha$. Since the mode for $\{\alpha_k | k \geq 500\}$ is not in the set of modes for the IMM, it does not converge to $\alpha$. In Fig. 4, $\|\alpha_k - \hat{\alpha}_k\|$ is shown for the DE, EKF and IMM estimation methods and allows for comparing the results of Figs. 1, 2 and 3 in one figure. The state error, i.e., $\|x_k - \hat{x}_k\|$, is shown in Fig. 5. Since all of the estimation methods have a relatively small estimation error in $\alpha$ for $k < 500$ in Fig. 4, also the state error in Fig. 5 is relatively small after initial convergence to $\alpha$ and $x$. For $k \geq 500$, it can be seen in Fig. 5 that the DE method and the EKF converge to the new value of $\alpha$, but the IMM estimation state error is large as to be expected from the scheduling-parameter error. Also, the inset in Fig. 5 shows that the EKF convergence to zero state error is slower than the corresponding DE convergence.

Table 1. Results of Monte-Carlo simulation study.

| **Method** | $\|x - \hat{x}\|$ | $\|\alpha - \hat{\alpha}\|$ |
|---|---|---|
| DE | $3.0 \cdot 10^{-2}$ | $6.0 \cdot 10^{-2}$ |
| EKF | $8.6 \cdot 10^{-2}$ | $18.8 \cdot 10^{-2}$ |
| IMM1 | $3.8 \cdot 10^{-2}$ | $8.8 \cdot 10^{-2}$ |
| IMM2 | $15.1 \cdot 10^{-2}$ | $20.4 \cdot 10^{-2}$ |

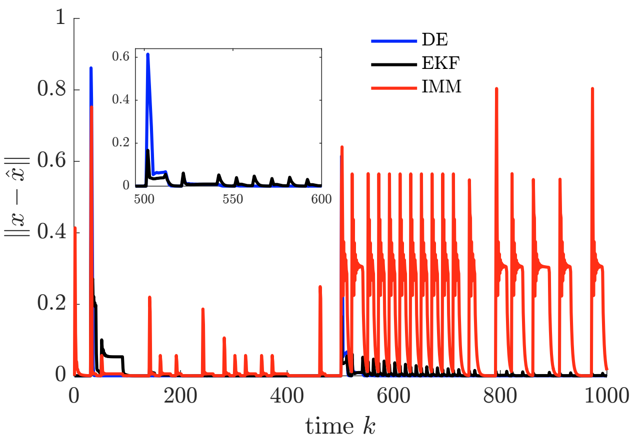


Fig. 5. $\|x_k - \hat{x}_k\|$ for DE, EKF and IMM.

A more extensive evaluation of the estimation methods can be done by performing a Monte-Carlo study of the aforementioned simulation experiment with a number of $N_{mc} \in \mathbb{N}$ Monte-Carlo runs. Instead of the vertices in (28), every run, random (stable and observable) vertices for the polytopic system in (1) are selected. Also, a new square-wave input sequence $u$ is generated every run. Subsequently, the previously defined performance measure for the estimation methods can now be taken as $\sum_{j=1}^{N_{mc}} \|x_k^j - \hat{x}_k^j\|$ and $\sum_{j=1}^{N_{mc}} \|\alpha_k^j - \hat{\alpha}_k^j\|$. Again, the tuning parameters for all methods are chosen such that the methods have good performance with respect to these error norms.

A comparison of all methods can be given by taking the average of the Monte-Carlo error norms over time $k$. Results for this comparison with $N_{mc} = 100$ can be found in Table 1. For the IMM method, two results are obtained, namely, IMM1 where $\alpha_k \in \mathcal{A}_M$ for all $k$ and IMM2 where $\{\alpha_k | k < 500\} \in \mathcal{A}_M$ and $\{\alpha_k | k \geq 500\} \notin \mathcal{A}_M$. The results in Table 1 show that IMM has good performance when $\alpha_k \in \mathcal{A}_M$, as with IMM1. However, in the case of IMM2, where $\alpha \notin \mathcal{A}_M$ for $k \geq 500$, IMM performs poorly. Overall, the EKF performs relatively well, but DE method outperforms the EKF, since the DE method has better convergence properties. For some systems (and corresponding dynamics) the EKF converges slowly or, in the worst case, not at all, whilst the DE method has relatively fast convergence as shown in Fig. 1.

In conclusion, it can be stated that IMM can perform relatively well, however, it requires (very fine) gridding op the parameter space $\mathcal{A}$. The EKF and DE method do not require gridding and DE performs better than EKF as shown in Table 1. The DE method guarantees convergence and ISS whereas the EKF does not, however, at this point in time, the nonlinear optimisation step of DE in (16) is computationally relatively heavy and therefore, this step will be subject to further investigation.

## 6. CONCLUSIONS

Modelling nonlinear systems can be done in a very suitable way through the use of polytopic systems. However, this LPV or polytopic modelling framework is based on the fact that the scheduling parameter is known. In case this parameter is not known, in the current literature, joint state and parameter-estimation methods are employed outside of the polytopic framework. For example, an EKF is used on the nonlinear system directly and alternatively, multiple-model estimation methods such as the IMM method are used. Both of these alternatives have disadvantages. For the EKF, convergence and stability properties are unsatisfactory and for multiple-model estimation, scalability of the method is a drawback. In this paper, we have proposed to solve the joint-estimation problem in a two-step, Dual Estimation (DE) approach, where we first solve the parameter-estimation problem by solving a constrained optimisation problem and secondly, employ a robust polytopic observer design for the state estimation. This way, stability and convergence can be guaranteed with less tuning efforts. Simulations are performed with the DE, EKF and IMM method in order to compare the methods. The results show that the DE method outperforms the alternative joint-estimation methods. Concerning the constrained optimisation step of DE, we see opportunities for improvement in the future. Therefore, these results can be seen as a promising first step for further research.

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