Zonotope Notes for State Estimation & Detection

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Contents

1 Basic Definitions

1.1 General Sets / Nomenclature

Basically a ***set*** is the collection of things.
A set is ***Bounded*** if ...

$$\exists_M \ : \ \forall_{x \in X} x \leq M$$

A set is ***closed*** if (operations on members of a class results in another member of the class)

$$X \text{closed under} f(\cdot, \cdot, \dots) \iff \forall_{x,y} \in X \implies f(x, y, \dots) \in X$$

The space $\mathbb{R}$ is ... The space R^n is ...
All the other definitions...

1.1.1 Set Operations

Definition 1. *Let $Z, W \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^k$, and $\mathbf{R} \subset R^{k \times n}$.*

1. *A **Linear Mapping** of Z defined as*

$$\mathbf{R}Z \equiv \{\mathbf{R}\mathbf{z} \ : \ \mathbf{z} \in Z\} \tag{1}$$

2. *A **Minkowski Sum** of Z and W is defined as*

$$Z + W \equiv \{\mathbf{z} + \mathbf{w} \ : \ \mathbf{z} \in Z, \mathbf{w} \in W\} \tag{2}$$

3. *A **Generalized Intersection** of Z and Y is defined as*

$$Z \cap_{\mathbf{R}} Y \equiv \{\mathbf{z} \in Z \ : \ \mathbf{R}\mathbf{z} \in Y\} \tag{3}$$

   which is a standard intersection $\cap$ for $k = n$ and $\mathbf{R} = \mathbf{I}_{n \times n}$.

1.2 Specific Set Definitions

1.2.1 Convex Polytope

Definition 2. *$P \subset \mathbb{R}^n$ is a* **Convex Polytope** *if it is Bounded and*

$$\exists(\mathbf{H}, \mathbf{k}) \in \mathbb{R}^{n_h \times n} \times \mathbb{R}^{n_h} \ : \ P = \{\mathbf{z} \in \mathbb{R}^n \ : \ \mathbf{H}\mathbf{z} \leq \mathbf{k}\} \tag{4}$$

Notes:

- (4) is known as a *halfspace-representation* (H-rep) of P. [1]
- A polytope can also be represented as the convex hull of the vertices (V-rep).

1.2.2 Zonotope

Definition 3. *$Z \subset \mathbb{R}^n$ is* **Zonotope** *if*

$$\exists(\mathbf{G}, \mathbf{c}) \in \mathbb{R}^{n \times n_g} \times \mathbb{R}^n \ : \ Z = \{\mathbf{G}\xi + \mathbf{c} \ : \ \|\xi\|_\infty \leq 1\} \tag{5}$$

Notes:

- Z defined by (5) can be denoted by $Z = \{\mathbf{G}, \mathbf{c}\}$.
- (5) is known as the *generator-representation* (G-rep) where $\mathbf{c}$ is called the *center* and the columns of $\mathbf{G}$ are the *generators*.

[1] I've also known this as an Affine version of a polytope as opposed to the standard convex hull definition.

- The ***order*** of a Zonotope is n_g/n.

Special Zonotopes:

1. Z is a ***parallelotope*** if Z is a zonotope with $n_g = n$.
2. Z is an ***interval*** if $\mathbf{G} = \mathbf{I}_{n\times n}$.

Properties:

1. Zonotopes are ***centrally symentric*** (i.e. every chord through $\mathbf{c}$ is bisected by $\mathbf{c}$).

$$\text{A convex polytope is a zonotope} \iff \text{every 2-face is centrally symmetric}$$

2. All Zonotopes are affine image of the ∞-norm unit ball.

Operations: Let $Z = \{\mathbf{G}_z, \mathbf{c}_z\}$ and $W = \{\mathbf{G}_w, \mathbf{c}_w\}$.

$$\mathbf{R}Z = \{\mathbf{R}\mathbf{G}_z, \mathbf{R}\mathbf{c}_z\} \tag{6}$$

$$Z + W = \{\begin{bmatrix}\mathbf{G}_z & \mathbf{G}_w\end{bmatrix}, \mathbf{c}_z + \mathbf{c}_w\} \tag{7}$$

1.2.3 Ellipsoid

Definition 4. *$E \subset \mathbb{R}^n$ is an* **Ellipsoid** *if*

$$\exists(\mathbf{Q}, \mathbf{c}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \ : \ E = \{\mathbf{Q}\xi + \mathbf{c} \ : \ \|\xi\|_2 \leq 1\} \tag{8}$$

Notes:

- (4) represents the degenerate ellipsoid when $\mathbf{Q}$ is singular.
- If $\mathbf{Q}$ is invertable, then (4) is equivelent to

$$E = \left\{\mathbf{z} \ : \ (\mathbf{z}-\mathbf{c})^T\left(\mathbf{Q}\mathbf{Q}^T\right)^{-1}(\mathbf{z}-\mathbf{c}) \leq 1\right\}$$

- We denote shorthand for the ellipsoid E defined in (4) as $E = \{\mathbf{Q}, \mathbf{c}\}$ where Q is known as the covariance matrix and c is the center.

Properties:

1. All Ellipsoids are affine image of the 2-norm unit ball.

2 Constrained Zonotopes

2.1 Basic Definition

Definition 5. $Z \subset \mathbb{R}^n$ *is* **Constrained Zonotope** *if*

$$\exists(\mathbf{G}, \mathbf{c}, \mathbf{A}, \mathbf{b}) \in \mathbb{R}^{n \times n_g} \times \mathbb{R}^n \times \mathbb{R}^{n_c \times n_g} \times \mathbb{R}^{n_c} \ : \ Z = \{\mathbf{G}\xi + \mathbf{c} \ : \ \|\xi\|_\infty \leq 1 \wedge \mathbf{A}\xi = \mathbf{b}\} \tag{9}$$

Notes:

- Z defined by (9) can be denoted by $Z = \{\mathbf{G}, \mathbf{c}, \mathbf{A}, \mathbf{b}\}$.
- $Z = \{\mathbf{G}, \mathbf{c}, \mathbf{A}, \mathbf{b}\}$ is known as a ***constrained generator representation*** (CG-rep).

Properties:

1. Constrained Zonotopes are affine image of a constrained unit hypercube $B_\infty(\mathbf{A}, \mathbf{b}) \equiv \{\xi \in B_\infty \ : \ \mathbf{A}\xi \ : \ \mathbf{A}\xi = \mathbf{b}\}$.

2.2 Basic Operations:

Let $Z = \{\mathbf{G}_z, \mathbf{c}_z, \mathbf{A}_z, \mathbf{b}_z\} \subset \mathbb{R}^n$, $W = \{\mathbf{G}_w, \mathbf{c}_w, \mathbf{A}_w, \mathbf{b}_q\} \subset \mathbb{R}^n$, $Y = \{\mathbf{G}_y, \mathbf{c}_y, \mathbf{A}_y, \mathbf{b}_y\} \subset \mathbb{R}^k$, and $\mathbf{R} \in \mathbb{R}^{k \times n}$.
Linear Mapping:

$$\mathbf{R}Z = \{\mathbf{R}\mathbf{G}_z, \mathbf{R}\mathbf{c}_z, \mathbf{A}_z, \mathbf{b}_z\} \tag{10}$$

Minkowski Sum:

$$Z + W = \left\{ \begin{bmatrix} \mathbf{G}_z & \mathbf{G}_w \end{bmatrix}, \mathbf{c}_z + \mathbf{c}_w, \begin{bmatrix} \mathbf{A}_z & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix}, \begin{bmatrix} \mathbf{b}_z \\ \mathbf{b}_w \end{bmatrix} \right\} \tag{11}$$

Generalized Intersection:

$$Z \cap_{\mathbf{R}} Y = \left\{ \begin{bmatrix} \mathbf{G}_z & \mathbf{0} \end{bmatrix}, \mathbf{c}_z, \begin{bmatrix} \mathbf{A}_z & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_y \\ \mathbf{R}\mathbf{G}_z & -\mathbf{G}_y \end{bmatrix}, \begin{bmatrix} \mathbf{b}_z \\ \mathbf{b}_y \\ \mathbf{c}_y - \mathbf{R}\mathbf{c}_z \end{bmatrix} \right\} \tag{12}$$

2.3 Relationship to Convex Polytope

$Z \subset \mathbb{R}^n$ is a constrained zonotope iff it is a convex polytope. (i.e.) $\forall_{Z \subset \mathbb{R}^n}$

$$Z \text{ constrained polytope } \iff Z \text{ convex polytope}$$

or equivalently $\forall_{Z = P \subset \mathbb{R}^n}$

$$\exists(\mathbf{G}, \mathbf{c}, \mathbf{A}, \mathbf{b}) \ : \ Z = \{\mathbf{G}, \mathbf{c}, \mathbf{A}, \mathbf{b}\} \iff \exists(\mathbf{H}, \mathbf{k}) \ : \ P = \{\mathbf{H}, \mathbf{k}\}$$

2.4 Complexity Reduction

2.4.1 Rescaling Constrained Zonotopes

Let $Z = \{\mathbf{G}, \mathbf{c}, \mathbf{A}, \mathbf{b}\}$. For $\xi^L, \xi^U \in \mathbb{R}^n \ : \ B_\infty(\mathbf{A}, \mathbf{b}) \subset [\xi^L, \xi^U] \subset [-\mathbf{1}, \mathbf{1}]$, there is an equivelent CG-rep as

$$Z = \{\mathbf{G}\text{diag}(\xi_r), \mathbf{c} + \mathbf{G}\xi_m, \mathbf{A}\text{diag}(\xi_r), \mathbf{b} - \mathbf{A}\xi_m\} \tag{13}$$

with $\xi_m = \frac{1}{2}(\xi^U + \xi^L)$ and $\xi_r = \frac{1}{2}(\xi^U - \xi^L)$.
Notes:

- Best interval is just solving for an LP (min or max satisfying the equivelency and ∞-norm ball ineqality) but it's cheaper to do a different method...

2.4.2 Constraint Reduction

Let $Z = \{\mathbf{G}, \mathbf{c}, \mathbf{A}, \mathbf{b}\}$. A reduced constraint set $\tilde{Z}$ with $Z \subset \tilde{Z}$ exists $\forall \Lambda_G \in \mathbb{R}^{n \times n_c}, \Lambda_A \in \mathbb{R}^{n_c \times n_c}$ defined by

$$\tilde{Z} = \{\mathbf{G} - \Lambda_G \mathbf{A}, \mathbf{c} + \Lambda_G \mathbf{b}, \mathbf{A} - \Lambda_A \mathbf{A}, \mathbf{b} - \Lambda_A \mathbf{b}\} \tag{14}$$

Eliminate single constraints A single constraint equation is given by

$$\xi_j = a_{1j}^{-1} b_j - a_{1j}^{-1} \sum_{k \neq j} a_{1k} \xi_k \tag{15}$$

with the entire form as

$$\mathbf{z} = \mathbf{G}\xi + \mathbf{c}$$

To eliminate the j-th constraint we construct Λ_G and λ_A as

$$\Lambda_G \equiv \mathbf{G}\mathbf{E}_{j1} a_{1j}^{-1}, \quad \Lambda_A \equiv \mathbf{A}\mathbf{E}_{j1} a_{1j}^{-1} \tag{16}$$

using $\mathbf{E}_{j1} \in \mathbb{R}^{n_g \times n_c}$ with all zero except for a 1 in $(j, 1)$.
This results in

$$\tilde{Z} = \{\mathbf{G} - \Lambda_G \mathbf{A}, \mathbf{c} + \Lambda_G \mathbf{b}, \mathbf{A} - \Lambda_A \mathbf{A}, \mathbf{b} - \Lambda_A \mathbf{b}\} = \left\{\tilde{\mathbf{G}}, \tilde{\mathbf{c}}, \tilde{\mathbf{A}}, \tilde{\mathbf{b}}\right\}$$

We can eliminate the