# Relations and Their Properties

Slides by A. Bloomfield

# What is a relation

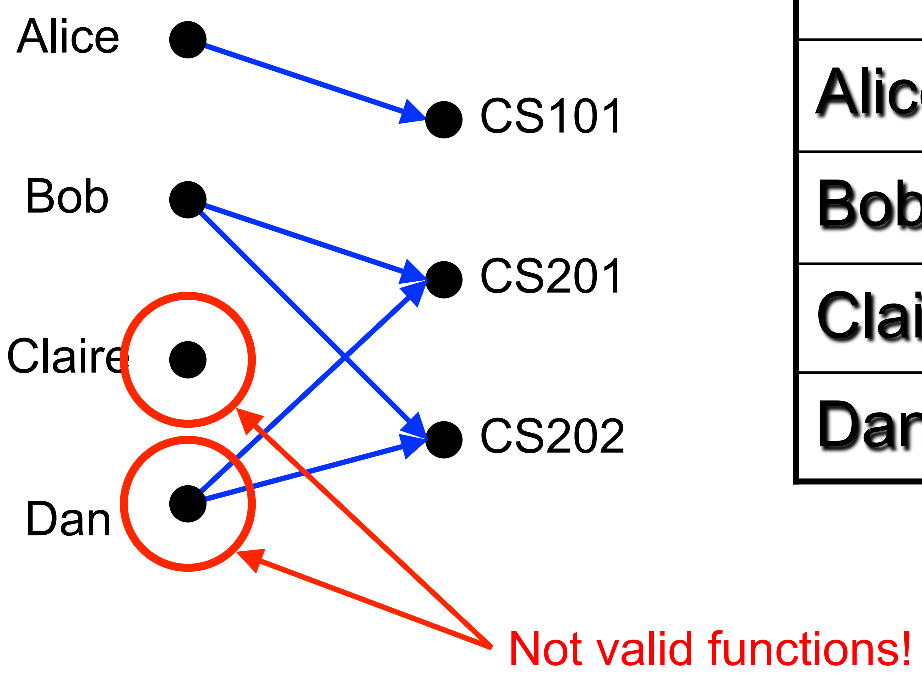
- Let $A$ and $B$ be sets. A binary relation $R$ is a subset of $A \times B$
- Example
  - Let $A$ be the students in a the CS major
    - $A$ = {Alice, Bob, Claire, Dan}
  - Let $B$ be the courses the department offers
    - $B$ = {CS101, CS201, CS202}
  - We specify relation $R = A \times B$ as the set that lists all students $a \in A$ enrolled in class $b \in B$
  - R = { (Alice, CS101), (Bob, CS201), (Bob, CS202),
    (Dan, CS201), (Dan, CS202) }

# More relation examples

- Another relation example:
  - Let *A* be the cities in the US
  - Let *B* be the states in the US
  - We define *R* to mean *a* is a city in state *b*
  - Thus, the following are in our relation:
    - (C' ville, VA)
    - (Philadelphia, PA)
    - (Portland, MA)
    - (Portland, OR)
    - etc…
- Most relations we will see deal with ordered pairs of integers

# Representing relations

We can represent relations graphically:

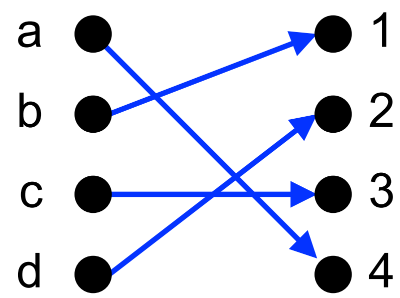



We can represent relations in a table:

| | CS101 | CS201 | CS202 |
|---|---|---|---|
| Alice | X | | |
| Bob | | X | X |
| Claire | | | |
| Dan | | X | X |

# Relations vs. functions

- Not all relations are functions
- But consider the following function:




- All functions are relations!

# When to use which?

- A function is used when you need to obtain a SINGLE result for any element in the domain
  - Example: sin, cos, tan

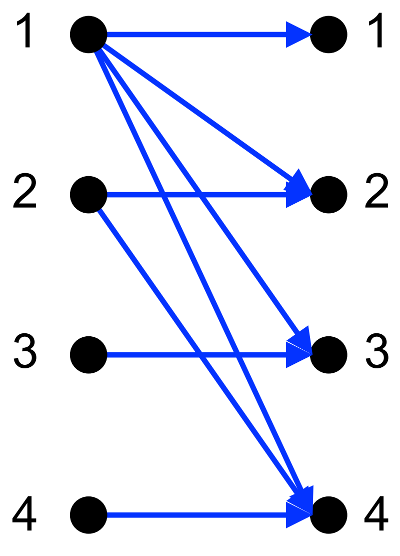
- A relation is when there are multiple mappings between the domain and the co-domain
  - Example: students enrolled in multiple courses

# Relations on a set

- A relation on the set $A$ is a relation from $A$ to $A$
  - In other words, the domain and co-domain are the same set
  - We will generally be studying relations of this type

# Relations on a set

- Let $A$ be the set { 1, 2, 3, 4 }
- Which ordered pairs are in the relation $R$ = { ($a$,$b$) | $a$ divides $b$ }
- R = { (1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4) }




| *R* | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | X | X | X | X |
| 2 | | X | | X |
| 3 | | | X | |
| 4 | | | | X |

# More examples

- Consider some relations on the set **Z**
- Are the following ordered pairs in the relation?

| | (1,1) | (1,2) | (2,1) | (1,-1) | (2,2) |
|---|---|---|---|---|---|
| • $R_1 = \{ (a,b) \mid a \leq b \}$ | X | X | | | X |
| • $R_2 = \{ (a,b) \mid a > b \}$ | | | X | X | |
| • $R_3 = \{ (a,b) \mid a = \|b\| \}$ | X | | | X | X |
| • $R_4 = \{ (a,b) \mid a = b \}$ | X | | | | X |
| • $R_5 = \{ (a,b) \mid a = b+1 \}$ | | | X | | |
| • $R_6 = \{ (a,b) \mid a+b \leq 3 \}$ | X | X | X | X | |

# Relation properties

- Six properties of relations we will study:
  - Reflexive
  - Irreflexive
  - Symmetric
  - Asymmetric
  - Antisymmetric
  - Transitive

# Reflexivity

- A relation is reflexive if every element is related to itself
  - Or, $(a,a) \in R$
- Examples of reflexive relations:
  - $=$, $\leq$, $\geq$
- Examples of relations that are not reflexive:
  - $<$, $>$

# Irreflexivity

- A relation is irreflexive if every element is *not* related to itself
  - Or, $(a,a) \notin R$
  - Irreflexivity is the opposite of reflexivity
- Examples of irreflexive relations:
  - $<$, $>$
- Examples of relations that are not irreflexive:
  - $=$, $\leq$, $\geq$

# Reflexivity vs. Irreflexivity

- A relation can be neither reflexive nor irreflexive
  - Some elements are related to themselves, others are not
- We will see an example of this later on

# Symmetry

- A relation is symmetric if, for every $(a,b) \in R$, then $(b,a) \in R$

- Examples of symmetric relations:
  - =, isTwinOf()

- Examples of relations that are not symmetric:
  - $<$, $>$, $\leq$, $\geq$

# Asymmetry

- A relation is asymmetric if, for every $(a,b) \in R$, then $(b,a) \notin R$
  - Asymmetry is the opposite of symmetry

- Examples of asymmetric relations:
  - $<$, $>$

- Examples of relations that are not asymmetric:
  - =, isTwinOf(), ≤, ≥

# Antisymmetry

- A relation is antisymmetric if, for every $(a,b) \in R$, then $(b,a) \in R$ is true only when $a=b$
  - Antisymmetry is *not* the opposite of symmetry

- Examples of antisymmetric relations:
  - $=$, $\leq$, $\geq$

- Examples of relations that are not antisymmetric:
  - $<$, $>$, isTwinOf()

# Notes on *symmetric relations

- A relation can be neither symmetric or asymmetric
  - R = { (a,b) | a=|b| }
  - This is not symmetric
    - -4 is not related to itself
  - This is not asymmetric
    - 4 is related to itself
  - Note that it is antisymmetric

# Transitivity

- A relation is transitive if, for every $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$

- If $a < b$ and $b < c$, then $a < c$
  - Thus, $<$ is transitive

- If $a = b$ and $b = c$, then $a = c$
  - Thus, $=$ is transitive

# Transitivity examples

- Consider isAncestorOf()
  - Let Alice be Bob's parent, and Bob be Claire's parent
  - Thus, Alice is an ancestor of Bob, and Bob is an ancestor of Claire
  - Thus, Alice is an ancestor of Claire
  - Thus, isAncestorOf() is a transitive relation

- Consider isParentOf()
  - Let Alice be Bob's parent, and Bob be Claire's parent
  - Thus, Alice is a parent of Bob, and Bob is a parent of Claire
  - However, Alice is *not* a parent of Claire
  - Thus, isParentOf() is *not* a transitive relation

# Relations of relations summary

| | = | < | > | ≤ | ≥ |
|---|---|---|---|---|---|
| Reflexive | X | | | X | X |
| Irreflexive | | X | X | | |
| Symmetric | X | | | | |
| Asymmetric | | X | X | | |
| Antisymmetric | X | | | X | X |
| Transitive | X | X | X | X | X |

# Combining relations

- There are two ways to combine relations $R_1$ and $R_2$
  - Via Boolean operators
  - Via relation "composition"

# Combining relations via Boolean operators

- Consider two relations $R_{\geq}$ and $R_{\leq}$
- We can combine them as follows:
  - $R_{\geq} \cup R_{\leq}$ = all numbers $\geq$ OR $\leq$
    - That's all the numbers
  - $R_{\geq} \cap R_{\leq}$ = all numbers $\geq$ AND $\leq$
    - That's all numbers equal to
  - $R_{\geq} \oplus R_{\leq}$ = all numbers $\geq$ or $\leq$, but not both
    - That's all numbers not equal to
  - $R_{\geq} - R_{\leq}$ = all numbers $\geq$ that are not also $\leq$
    - That's all numbers strictly greater than
  - $R_{\leq} - R_{\geq}$ = all numbers $\leq$ that are not also $\geq$
    - That's all numbers strictly less than
- Note that it's possible the result is the empty set

# Combining relations via relational composition

- Let $R$ be a relation from $A$ to $B$, and $S$ be a relation from $B$ to $C$
  - Let $a \in A$, $b \in B$, and $c \in C$
  - Let $(a,b) \in R$, and $(b,c) \in S$
  - Then the composite of R and S consists of the ordered pairs $(a,c)$
    - We denote the relation by $S \circ R$
    - Note that $S$ comes first when writing the composition!

# Combining relations via relational composition

- Let $M$ be the relation "is mother of"
- Let $F$ be the relation "is father of"
- What is $M \circ F$?
  - If $(a,b) \in F$, then $a$ is the father of $b$
  - If $(b,c) \in M$, then $b$ is the mother of $c$
  - Thus, $M \circ F$ denotes the relation "maternal grandfather"
- What is $F \circ M$?
  - If $(a,b) \in M$, then $a$ is the mother of $b$
  - If $(b,c) \in F$, then $b$ is the father of $c$
  - Thus, $F \circ M$ denotes the relation "paternal grandmother"
- What is $M \circ M$?
  - If $(a,b) \in M$, then $a$ is the mother of $b$
  - If $(b,c) \in M$, then $b$ is the mother of $c$
  - Thus, $M \circ M$ denotes the relation "maternal grandmother"
- Note that $M$ and $F$ are not transitive relations!!!

# Combining relations via relational composition

- Given relation $R$
  - $R \circ R$ can be denoted by $R^2$
  - $R^2 \circ R = (R \circ R) \circ R = R^3$
  - Example: $M^3$ is your mother's mother's mother

# Representing Relations

Slides by A. Bloomfield

# In this slide set…

- Matrix review
- Two ways to represent relations
  - Via matrices
  - Via directed graphs

# Matrix review

- This is from Rosen, page 201 and 202
- We will only be dealing with zero-one matrices
  - Each element in the matrix is either a 0 or a 1

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

- These matrices will be used for Boolean operations
  - 1 is true, 0 is false

# Matrix transposition

- Given a matrix **M**, the transposition of **M**, denoted **M**$^t$, is the matrix obtained by switching the columns and rows of **M**

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\mathbf{M}^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

$$\mathbf{M}^t = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

- In a "square" matrix, the main diagonal stays unchanged

# Matrix join

- A *join* of two matrices performs a Boolean OR on each relative entry of the matrices
  - Matrices must be the same size
  - Denoted by the or symbol: $\vee$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

# Matrix meet

- A *meet* of two matrices performs a Boolean AND on each relative entry of the matrices
  - Matrices must be the same size
  - Denoted by the or symbol: ∧

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

# Matrix Boolean product

- A *Boolean product* of two matrices is similar to matrix multiplication

$$c_{1,1} = a_{1,1} * b_{1,1} + a_{1,2} * b_{2,1} + a_{1,3} * b_{3,1} + a_{1,4} * b_{4,1}$$

  - Instead of the sum of the products, it's the conjunction (and) of the disjunctions (ors)

$$c_{1,1} = a_{1,1} \wedge b_{1,1} \vee a_{1,2} \wedge b_{2,1} \vee a_{1,3} \wedge b_{3,1} \vee a_{1,4} \wedge b_{4,1}$$

  - Denoted by the or symbol: $\odot$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

# Relations using matrices

- List the elements of sets *A* and *B* in a particular order
  - Order doesn't matter, but we'll generally use ascending order
- Create a matrix $\mathbf{M}_R = [m_{ij}]$

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

# Relations using matrices

- Consider the relation of who is enrolled in which class
  - Let A = { Alice, Bob, Claire, Dan }
  - Let B = { CS101, CS201, CS202 }
  - R = { $(a,b)$ | person $a$ is enrolled in course $b$ }

| | CS101 | CS201 | CS202 |
|---|---|---|---|
| Alice | X | | |
| Bob | | X | X |
| Claire | | | |
| Dan | | X | X |

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

# Relations using matrices

- What is it good for?
  - It is how computers view relations
    - A 2-dimensional array
  - Very easy to view relationship properties
- We will generally consider relations on a single set
  - In other words, the domain and co-domain are the same set
  - And the matrix is square

# Reflexivity

- Consider a reflexive relation: $\leq$
  - One which every element is related to itself
  - Let A = { 1, 2, 3, 4, 5 }

$$\mathbf{M}_{\leq} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If the center (main) diagonal is all 1's, a relation is reflexive

# Irreflexivity

- Consider a reflexive relation: <
  - One which every element is *not* related to itself
  - Let A = { 1, 2, 3, 4, 5 }

$$\mathbf{M}_{\leq} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If the center (main) diagonal is all 0's, a relation is irreflexive

# Symmetry

- Consider an symmetric relation *R*
  - One which if *a* is related to *b* then *b* is related to *a* for all (*a*,*b*)
  - Let A = { 1, 2, 3, 4, 5 }

$$\mathbf{M}_{\leq} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

- If, for *every* value, it is the equal to the value in its transposed position, then the relation is symmetric

# Asymmetry

- Consider an asymmetric relation: <
  - One which if *a* is related to *b* then *b* is *not* related to *a* for all (*a*,*b*)
  - Let A = { 1, 2, 3, 4, 5 }

$$\mathbf{M}_{\leq} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- If, for every value and the value in its transposed position, if they are not both 1, then the relation is asymmetric
- An asymmetric relation must also be irreflexive
- Thus, the main diagonal must be all 0's

# Antisymmetry

- Consider an antisymmetric relation: ≤
  - One which if *a* is related to *b* then *b* is *not* related to *a* unless *a*=*b* for all (*a*,*b*)
  - Let A = { 1, 2, 3, 4, 5 }

$$\mathbf{M}_{\leq} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- If, for every value and the value in its transposed position, if they are not both 1, then the relation is antisymmetric
- The center diagonal can have both 1's and 0's

# Transitivity

- Consider an transitive relation: ≤
  - One which if *a* is related to *b* and *b* is related to *c* then *a* is related to *c* for all (*a*,*b*), (*b*,*c*) and (*a*,*c*)
  - Let A = { 1, 2, 3, 4, 5 }

$$\mathbf{M}_{\leq} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- If, for every spot (*a*,*b*) and (*b*,*c*) that each have a 1, there is a 1 at (*a*,*c*), then the relation is transitive

- Matrices don't show this property easily

# Combining relations: via Boolean operators

- Example 4 from Rosen, section 7.3
- Let:

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{M}_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- Join:

$$\mathbf{M}_{R \cup S} = \mathbf{M}_R \vee \mathbf{M}_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

- Meet:

$$\mathbf{M}_{R \cap S} = \mathbf{M}_R \wedge \mathbf{M}_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

# Combining relations: via relation composition

- Example 4 from Rosen, section 7.3
- Let:

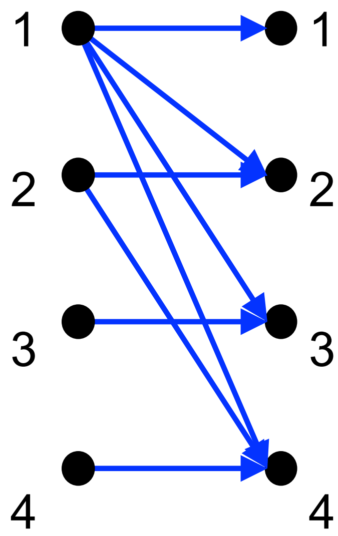
$$\mathbf{M}_R = \begin{array}{c|c} & \begin{matrix} d & e & f \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \qquad \mathbf{M}_S = \begin{array}{c|c} & \begin{matrix} g & h & i \end{matrix} \\ \begin{matrix} d \\ e \\ f \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

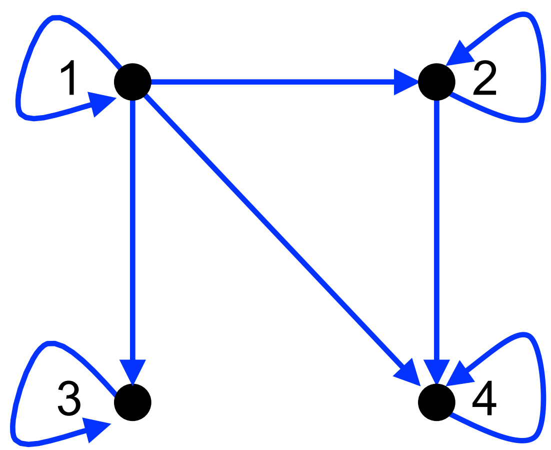
$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{array}{c|c} & \begin{matrix} g & h & i \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

- But why is this the case?

# Representing relations using directed graphs

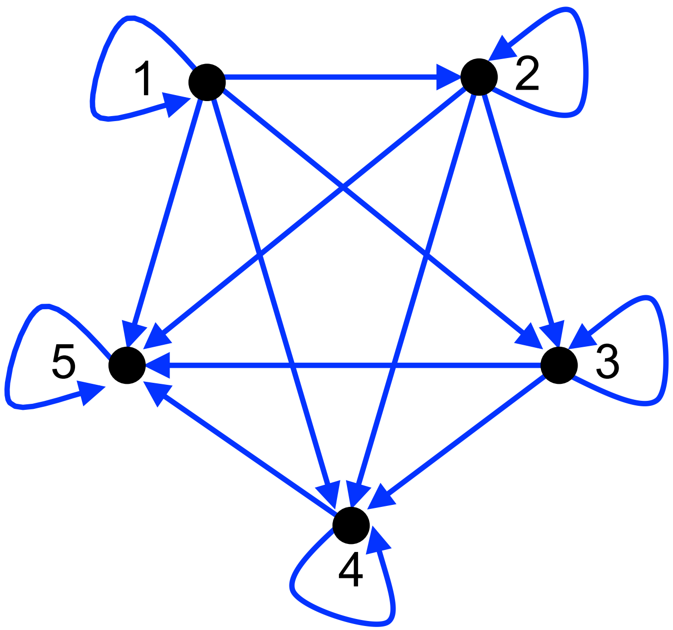
- A directed graph consists of:
  - A set *V* of vertices (or nodes)
  - A set *E* of edges (or arcs)
  - If (*a*, *b*) is in the relation, then there is an arrow from *a* to *b*
- Will generally use relations on a single set
- Consider our relation *R* = { (*a*,*b*) | *a* divides *b* }
- Old way:



# Reflexivity

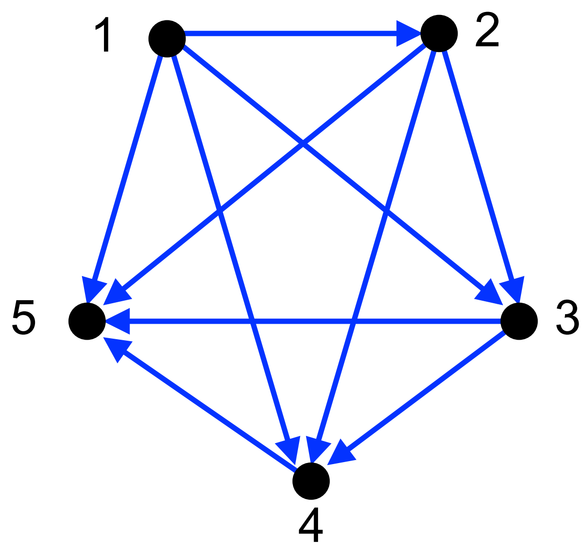
- Consider a reflexive relation: ≤
  - One which every element is related to itself
  - Let A = { 1, 2, 3, 4, 5 }




If every node has a loop, a relation is reflexive

# Irreflexivity

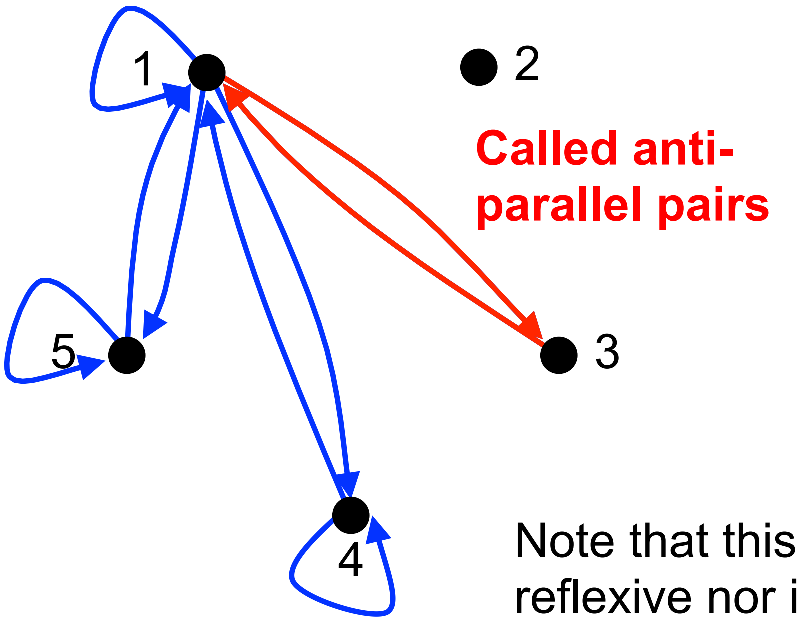
- Consider a reflexive relation: <
  - One which every element is *not* related to itself
  - Let A = { 1, 2, 3, 4, 5 }




If every node does *not* have a loop, a relation is irreflexive

# Symmetry

- Consider an symmetric relation *R*
  - One which if *a* is related to *b* then *b* is related to *a* for all (*a*,*b*)
  - Let A = { 1, 2, 3, 4, 5 }

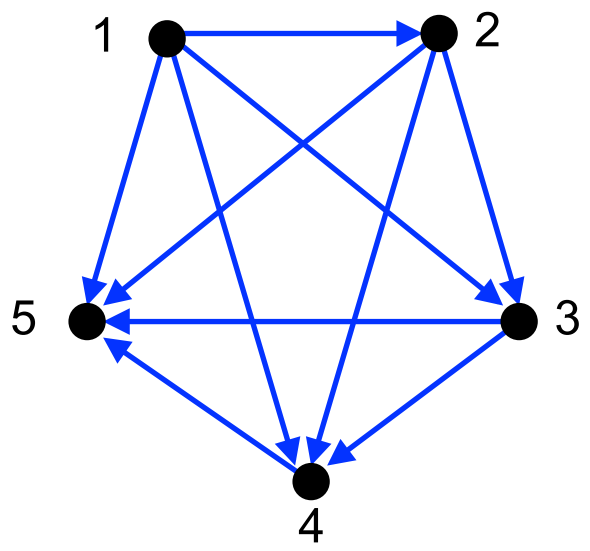



- If, for every edge, there is an edge in the other direction, then the relation is symmetric
- Loops are allowed, and do not need edges in the "other" direction

Note that this relation is neither reflexive nor irreflexive!

# Asymmetry

- Consider an asymmetric relation: <
  - One which if *a* is related to *b* then *b* is *not* related to *a* for all (*a*,*b*)
  - Let A = { 1, 2, 3, 4, 5 }

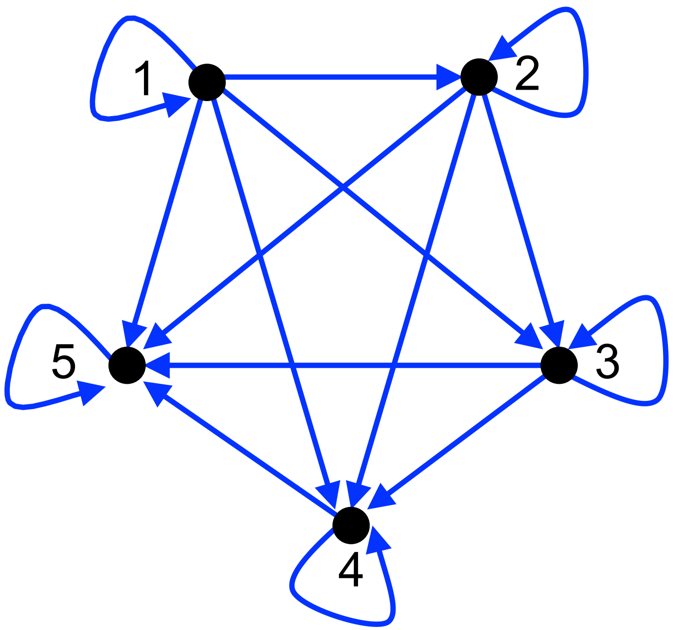



- A digraph is asymmetric if:

1. If, for every edge, there is *not* an edge in the other direction, then the relation is asymmetric
2. Loops are *not* allowed in an asymmetric digraph (recall it must be irreflexive)

# Antisymmetry

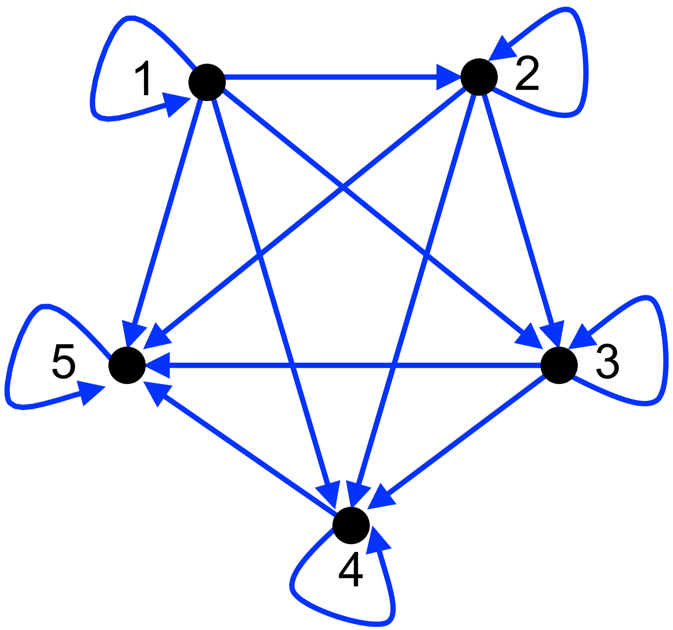
- Consider an antisymmetric relation: ≤
  - One which if *a* is related to *b* then *b* is *not* related to *a* unless *a*=*b* for all (*a*,*b*)
  - Let A = { 1, 2, 3, 4, 5 }




- If, for every edge, there is *not* an edge in the other direction, then the relation is antisymmetric
- Loops are allowed in the digraph

# Transitivity

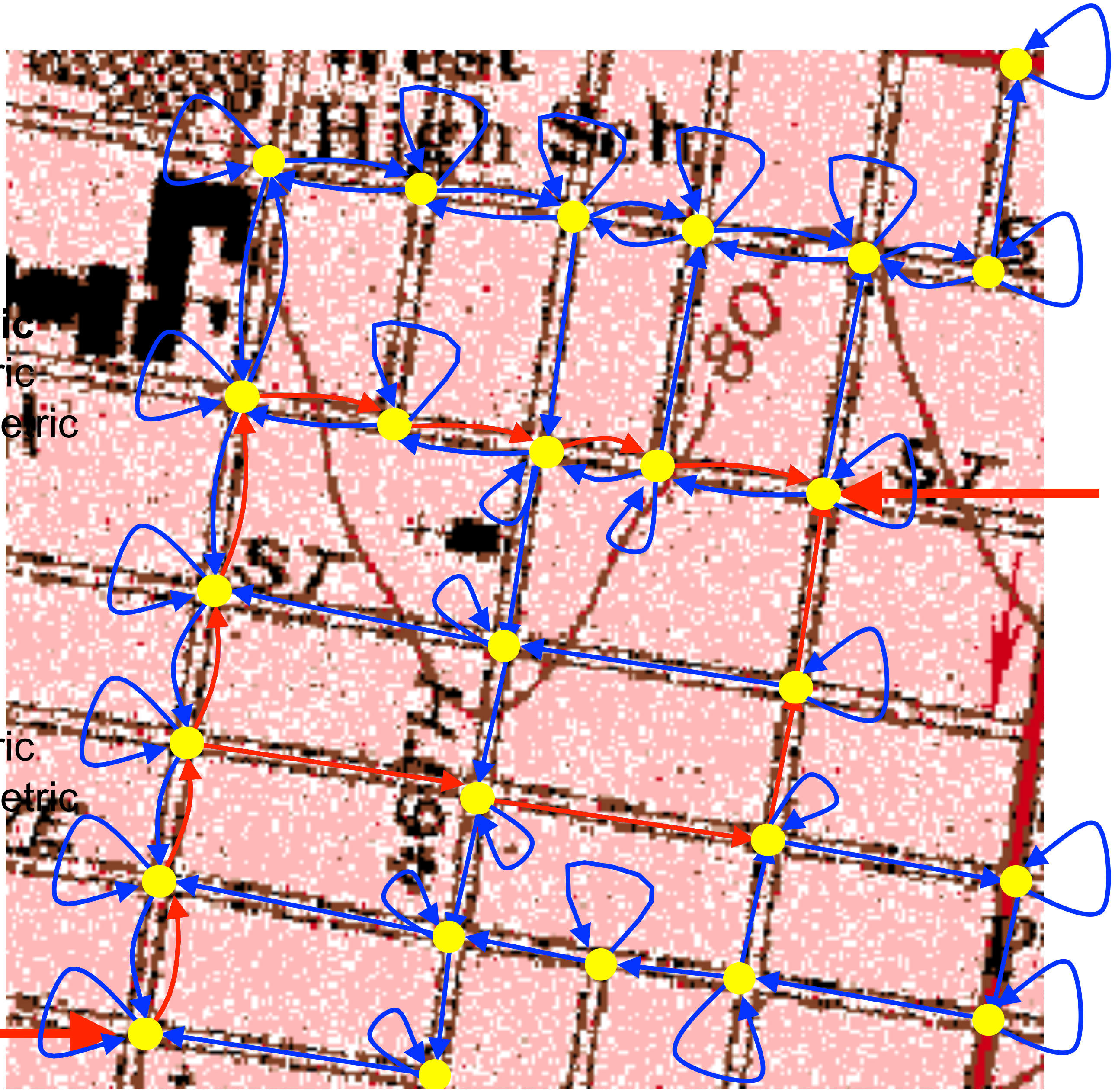
- Consider an transitive relation: ≤
  - One which if *a* is related to *b* and *b* is related to *c* then *a* is related to *c* for all (*a*,*b*), (*b*,*c*) and (*a*,*c*)
  - Let A = { 1, 2, 3, 4, 5 }




- A digraph is transitive if, for there is a edge from *a* to *c* when there is a edge from *a* to *b* and from *b* to *c*

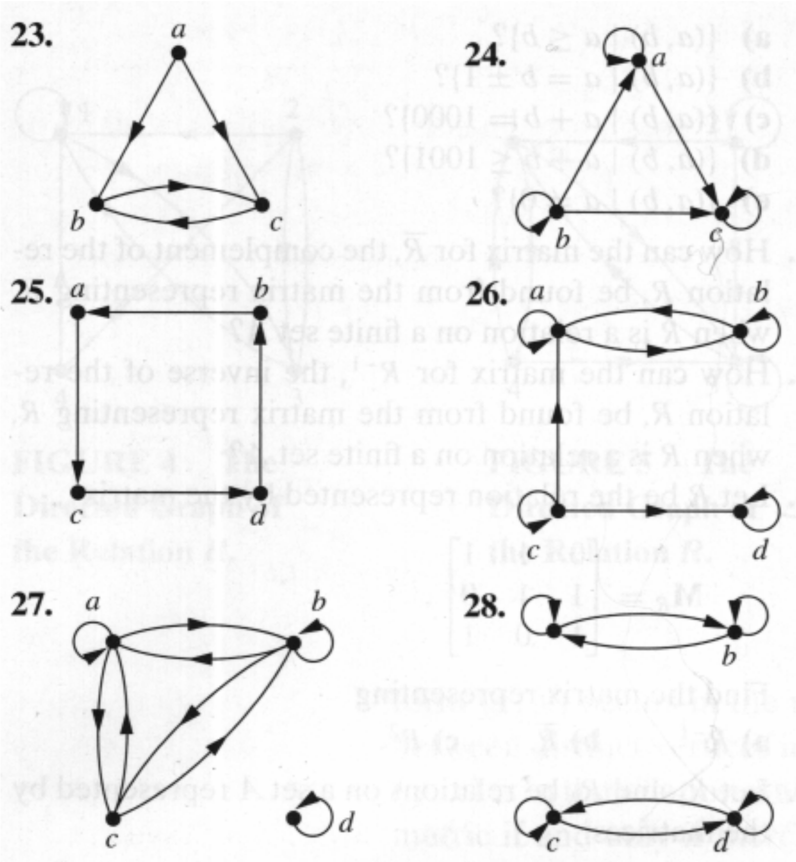
# Applications of digraphs: MapQuest

- Not reflexive
- Is irreflexive
- **Not symmetric**
- Not asymmetric
- Not antisymmetric
- Not transitive




- Not reflexive
- Is irreflexive
- **Is symmetric**
- Not asymmetric
- Not antisymmetric
- Not transitive

# Rosen, questions 31 & 32, section 7.3




Which of the graphs are reflexive, irreflexive, symmetric, asymmetric, antisymmetric, or transitive

| | 23 | 24 | 25 | 26 | 27 | 28 |
|---|---|---|---|---|---|---|
| Reflexive | | Y | | Y | | Y |
| Irreflexive | Y | | Y | | | |
| Symmetric | | | | | Y | Y |
| Asymmetric | | | Y | | | |
| Anti-symmetric | | Y | Y | | | |
| Transitive | | | | | | Y |

# Rosen, section 7.1 (sic) question 45 (a)

- How many symmetric relations are there on a set with *n* elements?

- Solution guide explanation is pretty poorly worded

- So instead we'll use matrices

# Rosen, section 7.1 (sic) question 45 (a)

- Consider the matrix representing symmetric relation R on a set with *n* elements:

$$\begin{bmatrix} 1 & & \cdots & 0 & 1 \\ & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 0 & \\ 1 & 0 & \cdots & & 1 \end{bmatrix}$$

- The center diagonal can have any values
- Once the "upper" triangle is determined, the "lower" triangle must be the transposed version of the "upper" one
- How many ways are there to fill in the center diagonal and the upper triangle?
- There are $n^2$ elements in the matrix
- There are *n* elements in the center diagonal
  - Thus, there are $2^n$ ways to fill in 0's and 1's in the diagonal
- Thus, there are $(n^2-n)/2$ elements in each triangle
  - Thus, there are $2^{(n^2-n)/2}$ ways to fill in 0's and 1's in the triangle
- Answer: there are $2^n * 2^{(n^2-n)/2} = 2^{(n^2+n)/2}$ possible symmetric relations on a set with *n* elements

# Closures of Relations

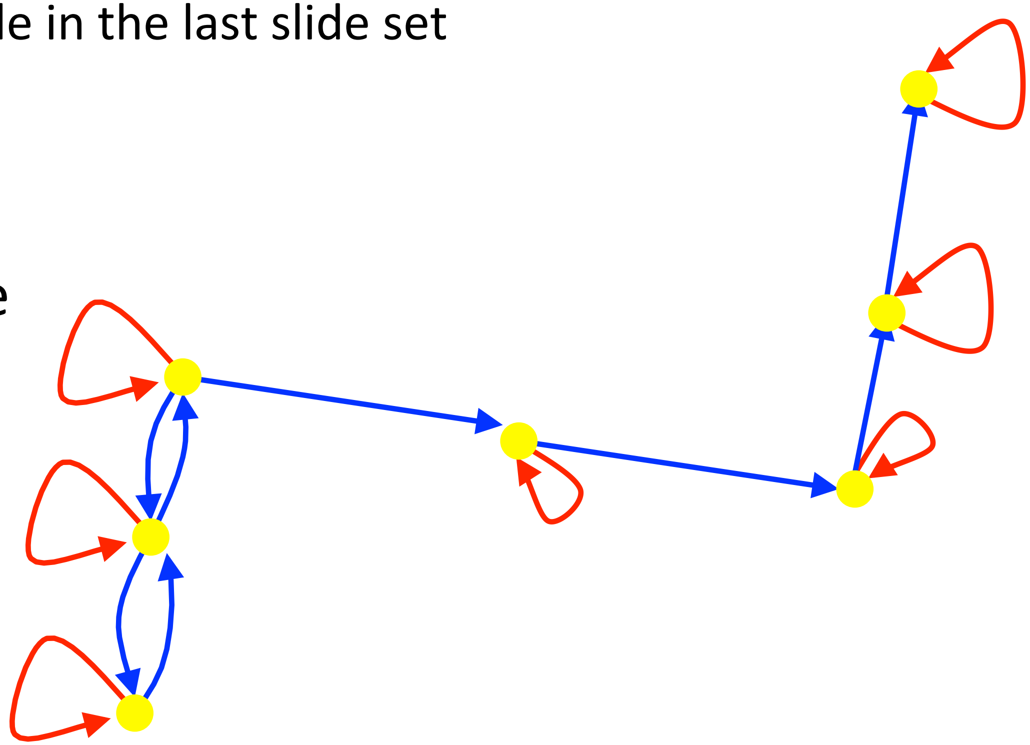
Slides by A. Bloomfield

# Relational closures

- Three types we will study
  - Reflexive
    - Easy
  - Symmetric
    - Easy
  - Transitive
    - Hard

# Reflexive closure

- Consider a relation *R*:
  - From our MapQuest example in the last slide set
  - Note that it is not reflexive
- We want to add edges to make the relation reflexive
- By adding those edges, we have made a non-reflexive relation *R* into a reflexive relation



- This new relation is called the **reflexive closure** of *R*

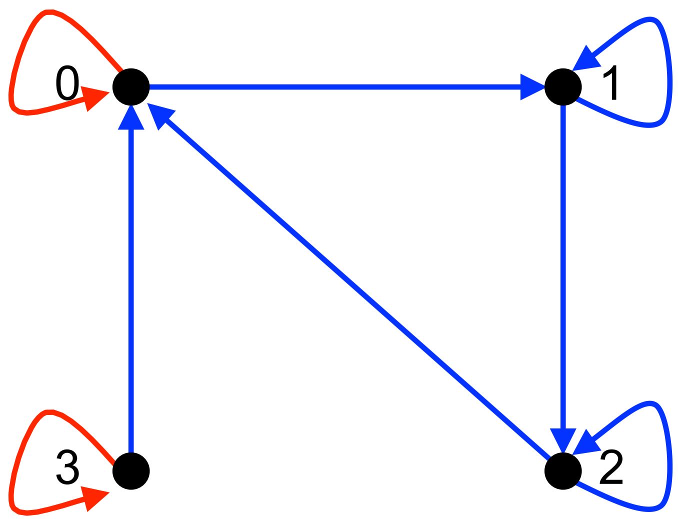
# Reflexive closure

- In order to find the reflexive closure of a relation $R$, we add a loop at each node that does not have one

- The reflexive closure of $R$ is $R \cup \Delta$
  - Where $\Delta = \{ (a,a) \mid a \in R \}$
    - Called the "diagonal relation"
  - With matrices, we set the diagonal to all 1's

# Rosen, section 7.4, question 1(a)

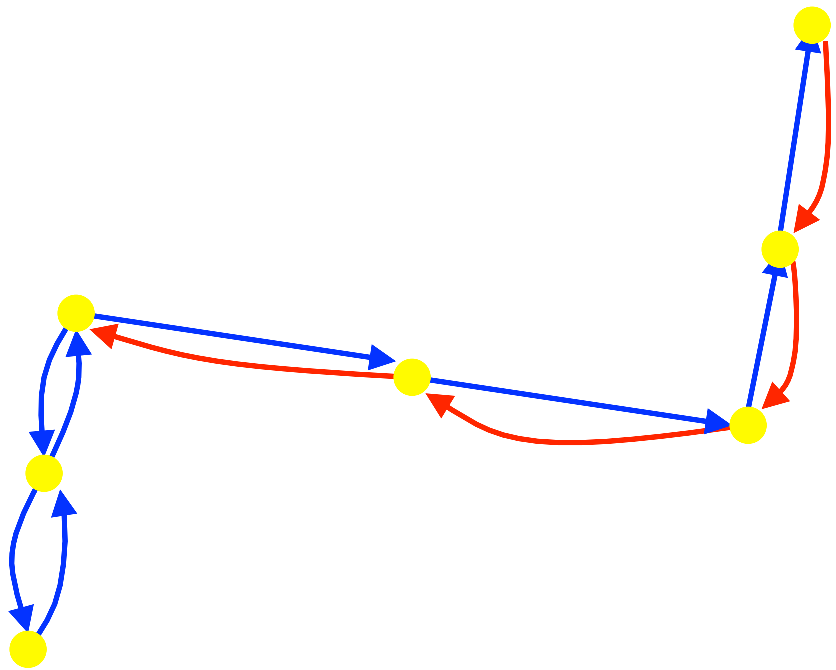
- Let *R* be a relation on the set { 0, 1, 2, 3 } containing the ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), and (3,0)
- What is the reflexive closure of *R*?
- We add all pairs of edges (*a*,*a*) that do not already exist




We add edges:
(0,0), (3,3)

# Symmetric closure

- Consider a relation *R*:
  - From our MapQuest example in the last slide set
  - Note that it is not symmetric
- We want to add edges to make the relation symmetric
- By adding those edges, we have made a non-symmetric relation *R* into a symmetric relation



- This new relation is called the **symmetric closure** of *R*

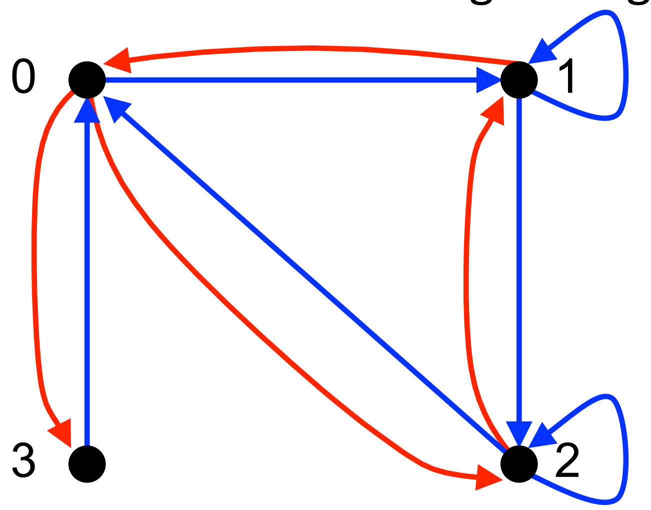
# Symmetric closure

- In order to find the symmetric closure of a relation $R$, we add an edge from $a$ to $b$, where there is already an edge from $b$ to $a$

- The symmetric closure of $R$ is $R \cup R^{-1}$
  - If $R = \{ (a,b) \mid \ldots \}$
  - Then $R^{-1} = \{ (b,a) \mid \ldots \}$

# Rosen, section 7.4, question 1(b)

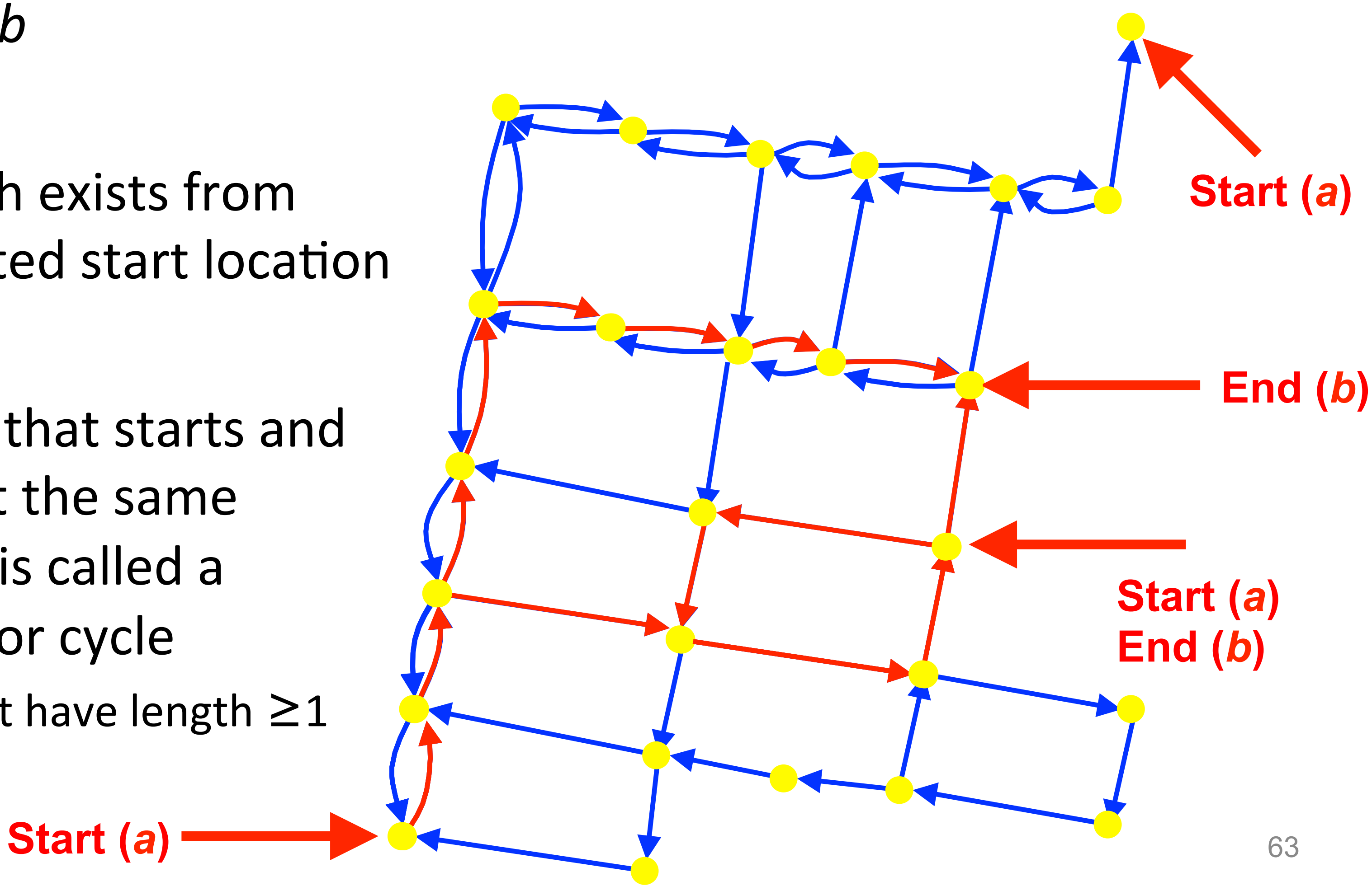
- Let *R* be a relation on the set { 0, 1, 2, 3 } containing the ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), and (3,0)
- What is the symmetric closure of *R*?
- We add all pairs of edges (*a*,*b*) where (*b*,*a*) exists
  - We make all "single" edges into anti-parallel pairs




We add edges:
(0,2), (0,3)
(1,0), (2,1)

# Paths in directed graphs

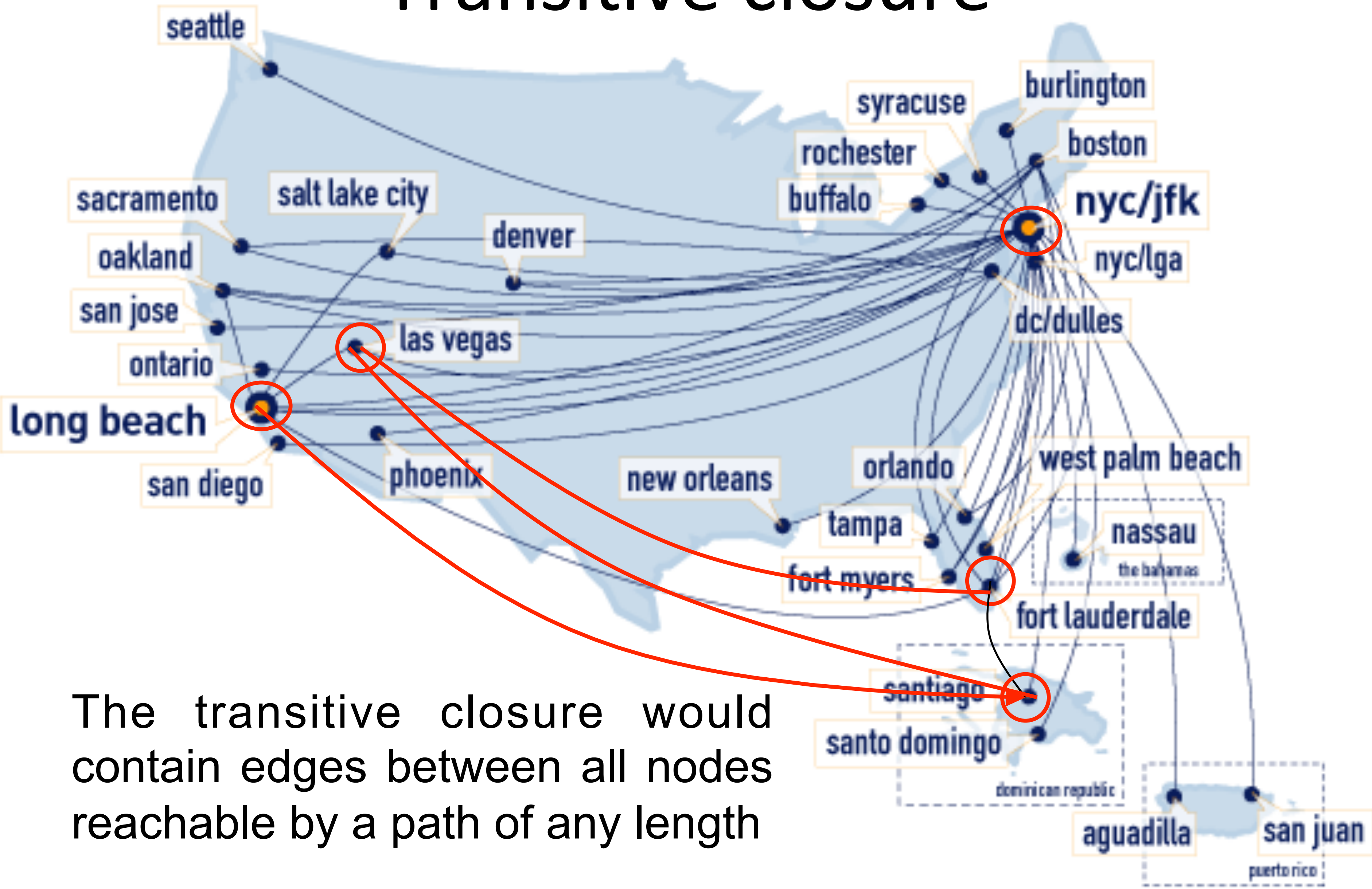
- A *path* is a sequences of connected edges from vertex *a* to vertex *b*
- No path exists from the noted start location
- A path that starts and ends at the same vertex is called a circuit or cycle
  - Must have length ≥1



# More on paths…

- The length of a path is the number of **edges** in the path, not the number of nodes

# Transitive closure

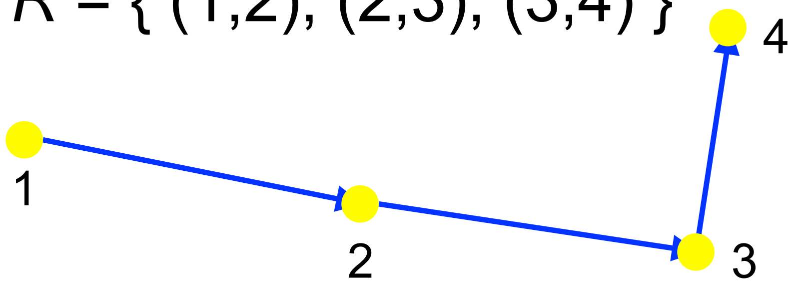



The transitive closure would contain edges between all nodes reachable by a path of any length

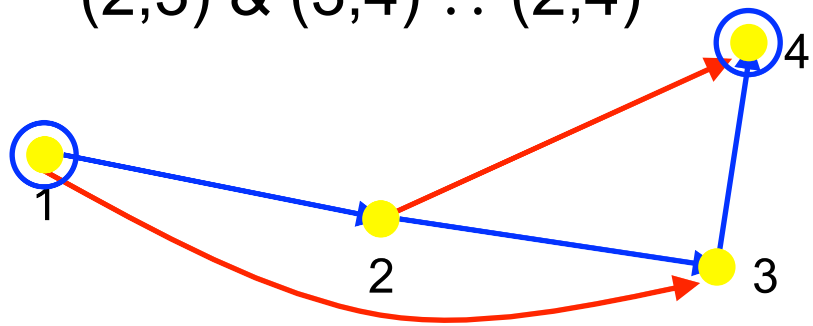
# Transitive closure

- Informal definition: If there is a path from *a* to *b*, then there should be an edge from *a* to *b* in the transitive closure
- First take of a definition:
  - In order to find the transitive closure of a relation *R*, we add an edge from *a* to *c*, when there are edges from a to *b* and *b* to *c*
- But there is a path from 1 to 4 with no edge!

$R = \{ (1,2), (2,3), (3,4) \}$




$(1,2)$ & $(2,3)$ $\therefore$ $(1,3)$
$(2,3)$ & $(3,4)$ $\therefore$ $(2,4)$

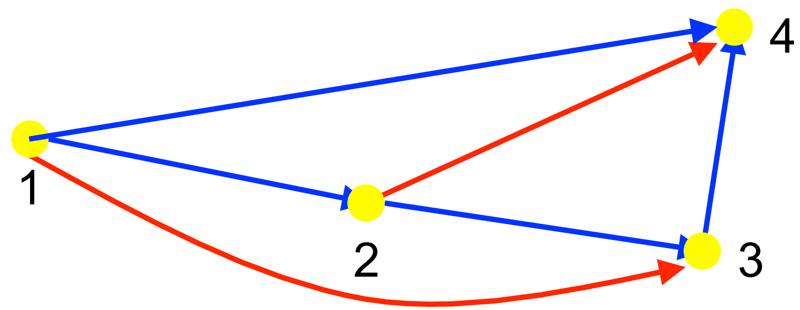


# Transitive closure

- Informal definition: If there is a path from *a* to *b*, then there should be an edge from *a* to *b* in the transitive closure

- Second take of a definition:
  - In order to find the transitive closure of a relation *R*, we add an edge from *a* to *c*, when there are edges from a to *b* and *b* to *c*
  - Repeat this step until no new edges are added to the relation
- We will study different algorithms for determining the transitive closure

- red means added on the first repeat
- teal means added on the second repeat



# 6 degrees of separation

- The idea that everybody in the world is connected by six degrees of separation
  - Where 1 degree of separation means you know (or have met) somebody else
- Let $R$ be a relation on the set of all people in the world
  - $(a,b) \in R$ if person $a$ has met person $b$
- So six degrees of separation for *any* two people $a$ and $g$ means:
  - $(a,b)$, $(b,c)$, $(c,d)$, $(d,e)$, $(e,f)$, $(f,g)$ are all in $R$
- Or, $(a,g) \in R^6$

# Connectivity relation

- $R$ contains edges between all the nodes reachable via 1 edge
- $R \circ R = R^2$ contains edges between nodes that are reachable via 2 edges in $R$
- $R^2 \circ R = R^3$ contains edges between nodes that are reachable via 3 edges in $R$
- $R^n$ = contains edges between nodes that are reachable via $n$ edges in $R$

- $R^*$ contains edges between nodes that are reachable via any number of edges (i.e. via any path) in $R$
  – Rephrased: $R^*$ contains all the edges between nodes $a$ and $b$ when is a path of length at least 1 between $a$ and $b$ in $R$

- $R^*$ is the transitive closure of $R$
  – The definition of a transitive closure is that there are edges between any nodes $(a,b)$ that contain a path between them

# How long are the paths in a transitive closure?

- Let *R* be a relation on set *A*, and let *A* be a set with *n* elements
  - Rephrased: consider a graph *G* with *n* nodes and some number of edges
- Lemma 1: If there is a path (of length at least 1) from *a* to *b* in *R*, then there is a path between *a* and *b* of length not exceeding *n*
- Proof preparation:
  - Suppose there is a path from *a* to *b* in *R*
  - Let the length of that path be *m*
  - Let the path be edges $(x_0, x_1)$, $(x_1, x_2)$, …, $(x_{m-1}, x_m)$
  - That's nodes $x_0, x_1, x_2, \ldots, x_{m-1}, x_m$
  - If a node exists twice in our path, then it's not a shortest path
    - As we made no progress in our path between the two occurrences of the repeated node
  - Thus, each node may exist at most once in the path

# How long are the paths in a transitive closure?

- Proof by contradiction:
  - Assume there are more than $n$ nodes in the path
    - Thus, $m > n$
    - Let $m = n+1$
  - By the pigeonhole principle, there are $n+1$ nodes in the path (pigeons) and they have to fit into the $n$ nodes in the graph (pigeonholes)
  - Thus, there must be at least one pigeonhole that has at least two pigeons
  - Rephrased: there must be at least one node in the graph that has two occurrences in the nodes of the path
    - Not possible, as the path would not be the shortest path
  - Thus, it cannot be the case that $m > n$

- If there exists a path from *a* to *b*, then there is a path from *a* to *b* of at most length *n*

# Finding the transitive closure

- Let $\mathbf{M}_R$ be the zero-one matrix of the relation $R$ on a set with $n$ elements. Then the zero-one matrix of the transitive closure $R^*$ is:

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}$$

Nodes reachable with one application of the relation

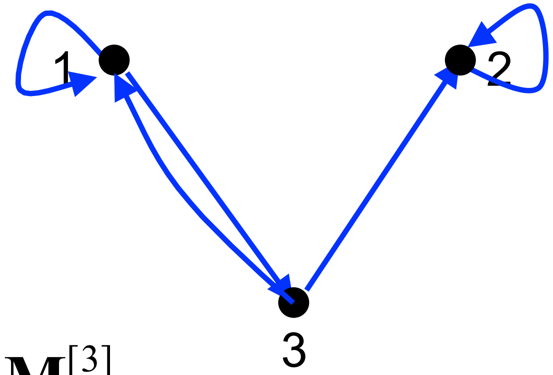
Nodes reachable with two applications of the relation

Nodes reachable with *n* applications of the relation

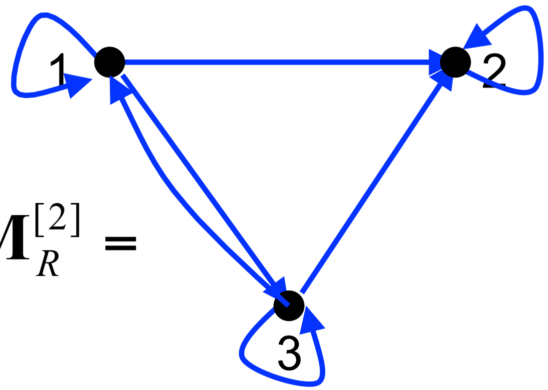
# Rosen, section 7.4, example 7

- Find the zero-one matrix of the transitive closure of the relation R given by:

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$




$$\mathbf{M}_R^{[2]} =$$

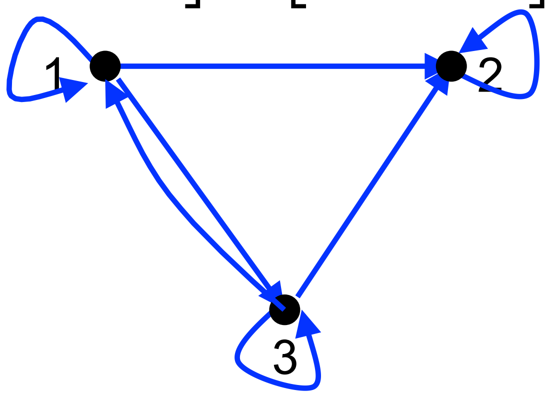



$$\mathbf{M}_{R*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]}$$

$$\mathbf{M}_R^{[2]} = \mathbf{M}_R \odot \mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

# Rosen, section 7.4, example 7

$$\mathbf{M}_R^{[3]} = \mathbf{M}_R^{[2]} \odot \mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$




$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

# Transitive closure algorithm

- What we did (or rather, could have done):
  - Compute the next matrix $\mathbf{M}_R^{[i]}$ where $1 \le i \le n$
  - Do a Boolean join with the previously computed matrix
- For our example:
  - Compute $\mathbf{M}_R^{[2]} = \mathbf{M}_R \circ \mathbf{M}_R$
  - Join that with $\mathbf{M}_R$ to yield $\mathbf{M}_R \vee \mathbf{M}_R^{[2]}$
  - Compute $\mathbf{M}_R^{[3]} = \mathbf{M}_R^{[2]} \circ \mathbf{M}_R$
  - Join that with $\mathbf{M}_R \vee \mathbf{M}_R^{[2]}$ from above

# Transitive closure algorithm

**procedure** *transitive_closure* ($\mathbf{M}_R$: zero-one $n \times n$ matrix)

  $\mathbf{A} := \mathbf{M}_R$

  $\mathbf{B} := \mathbf{A}$

  **for** $i := 2$ **to** $n$

  **begin**

    $\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$

    $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$

  **end** { $\mathbf{B}$ is the zero-one matrix for $R^*$ }

# More transitive closure algorithms

- More efficient algorithms exist, such as Warshall's algorithm
  - We won't be studying it in this class
  - Thus, the material on pages 503-506 won't be on the test

# Equivalence Relations

Slides by A. Bloomfield

# Introduction

- Certain combinations of relation properties are very useful
  - We won't have a chance to see many applications in this course
- In this set we will study equivalence relations
  - A relation that is reflexive, symmetric and transitive
- Next slide set we will study partial orderings
  - A relation that is reflexive, antisymmetric, and transitive
- The difference is whether the relation is symmetric or antisymmetric

# Outline

- What is an equivalence relation
- Equivalence relation examples
- Related items
  - Equivalence class
  - Partitions

# Equivalence relations

- A relation on a set $A$ is called an *equivalence relation* if it is reflexive, symmetric, and transitive
  - This is definition 1 in the textbook

- Consider relation $R = \{ (a,b) \mid len(a) = len(b) \}$
  - Where $len(a)$ means the length of string $a$
  - It is reflexive: $len(a) = len(a)$
  - It is symmetric: if $len(a) = len(\mathrm{b})$, then $len(b) = len(a)$
  - It is transitive: if $len(a) = len(\mathrm{b})$ and $len(\mathrm{b}) = len(\mathrm{c})$, then $len(\mathrm{a}) = len(\mathrm{c})$
  - Thus, $R$ is a equivalence relation

# Equivalence relation example

- Consider the relation $R = \{ (a,b) \mid a \equiv b \pmod{m} \}$
  - Remember that this means that $m \mid a-b$
  - Called "congruence modulo $m$"
- Is it reflexive: $(a,a) \in R$ means that $m \mid a-a$
  - $a-a = 0$, which is divisible by $m$
- Is it symmetric: if $(a,b) \in R$ then $(b,a) \in R$
  - $(a,b)$ means that $m \mid a-b$
  - Or that $km = a-b$. Negating that, we get $b-a = -km$
  - Thus, $m \mid b-a$, so $(b,a) \in R$
- Is it transitive: if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$
  - $(a,b)$ means that $m \mid a-b$, or that $km = a-b$
  - $(b,c)$ means that $m \mid b-c$, or that $lm = b-c$
  - $(a,c)$ means that $m \mid a-c$, or that $nm = a-c$
  - Adding these two, we get $km+lm = (a-b) + (b-c)$
  - Or $(k+l)m = a-c$
  - Thus, $m$ divides $a-c$, where $n = k+l$
- Thus, congruence modulo $m$ is an equivalence relation

# Rosen, section 7.5, question 1

- Which of these relations on {0, 1, 2, 3} are equivalence relations? Determine the properties of an equivalence relation that the others lack

a) { (0,0), (1,1), (2,2), (3,3) }
  - Has all the properties, thus, is an equivalence relation

b) { (0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3) }
  - Not reflexive: (1,1) is missing
  - Not transitive: (0,2) and (2,3) are in the relation, but not (0,3)

c) { (0,0), (1,1), (1,2), (2,1), (2,2), (3,3) }
  - Has all the properties, thus, is an equivalence relation

d) { (0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2) (3,3) }
  - Not transitive: (1,3) and (3,2) are in the relation, but not (1,2)

e) { (0,0), (0,1) (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3) }
  - Not symmetric: (1,2) is present, but not (2,1)
  - Not transitive: (2,0) and (0,1) are in the relation, but not (2,1)

# Rosen, section 7.5, question 5

- Suppose that $A$ is a non-empty set, and $f$ is a function that has $A$ as its domain. Let $R$ be the relation on $A$ consisting of all ordered pairs $(x,y)$ where $f(x) = f(y)$
  - Meaning that $x$ and $y$ are related if and only if $f(x) = f(y)$
- Show that $R$ is an equivalence relation on $A$

- Reflexivity: $f(x) = f(x)$
  - True, as given the same input, a function always produces the same output
- Symmetry: if $f(x) = f(y)$ then $f(y) = f(x)$
  - True, by the definition of equality
- Transitivity: if $f(x) = f(y)$ and $f(y) = f(z)$ then $f(x) = f(z)$
  - True, by the definition of equality

# Rosen, section 7.5, question 8

- Show that the relation $R$, consisting of all pairs $(x,y)$ where $x$ and $y$ are bit strings of length three or more that agree except perhaps in their first three bits, is an equivalence relation on the set of all bit strings

- Let $f(x)$ = the bit string formed by the last $n$-3 bits of the bit string $x$ (where $n$ is the length of the string)

- Thus, we want to show: let $R$ be the relation on $A$ consisting of all ordered pairs $(x,y)$ where $f(x) = f(y)$

- This has been shown in question 5 on the previous slide

# Equivalence classes

- Let $R$ be an equivalence relation on a set $A$. The set of all elements that are related to an element $a$ of $A$ is called the *equivalence class* of $a$.
- The equivalence class of $a$ with respect to $R$ is denoted by $[a]_R$
- When only one relation is under consideration, the subscript is often deleted, and $[a]$ is used to denote the equivalence class
- Note that these classes are disjoint!
  - As the equivalence relation is symmetric
- This is definition 2 in the textbook

# More on equivalence classes

- Consider the relation $R$ = { ($a,b$) | a mod 2 = b mod 2 }
  - Thus, all the even numbers are related to each other
  - As are the odd numbers
- The even numbers form an equivalence class
  - As do the odd numbers
- The equivalence class for the even numbers is denoted by [2] (or [4], or [784], etc.)
  - [2] = { …, -4, -2, 0, 2, 4, … }
  - 2 is a *representative* of it's equivalence class
- There are only 2 equivalence classes formed by this equivalence relation

# More on equivalence classes

- Consider the relation $R = \{ (a,b) \mid a = b$ or $a = -b \}$
  - Thus, every number is related to additive inverse
- The equivalence class for an integer $a$:
  - $[7] = \{ 7, -7 \}$
  - $[0] = \{ 0 \}$
  - $[a] = \{ a, -a \}$
- There are an infinite number of equivalence classes formed by this equivalence relation

# Partitions

- Consider the relation $R$ = { ($a,b$) | a mod 2 = b mod 2 }
- This splits the integers into two equivalence classes: even numbers and odd numbers
- Those two sets together form a partition of the integers
- Formally, a <u>partition of a set *S*</u> is a collection of non-empty disjoint subsets of S whose union is *S*
- In this example, the partition is { [0], [1] }
  - Or { {…, -3, -1, 1, 3, …}, {…, -4, -2, 0, 2, 4, …} }

# Rosen, section 7.5, question 32

- Which of the following are partitions of the set of integers?

a) The set of even integers and the set of odd integers
   - Yes, it's a valid partition

b) The set of positive integers and the set of negative integers
   - No: 0 is in neither set

c) The set of integers divisible by 3, the set of integers leaving a remainder of 1 when divided by 3, and the set of integers leaving a remaineder of 2 when divided by 3
   - Yes, it's a valid partition

d) The set of integers less than -100, the set of integers with absolute value not exceeding 100, and the set of integers greater than 100
   - Yes, it's a valid partition

e) The set of integers not divisible by 3, the set of even integers, and the set of integers that leave a remainder of 3 when divided by 6
   - The first two sets are not disjoint (2 is in both), so it's not a valid partition

# Partial Orderings

Slides by A. Bloomfield

# Introduction

- An equivalence relation is a relation that is reflexive, symmetric, and transitive
- A partial ordering (or partial order) is a relation that is reflexive, *anti*symmetric, and transitive
  - Recall that antisymmetric means that if $(a,b) \in R$, then $(b,a) \notin R$ unless $b = a$
  - Thus, $(a,a)$ is allowed to be in $R$
  - But since it's reflexive, all possible $(a,a)$ must be in $R$
- A set $S$ with a partial ordering $R$ is called a *partially ordered set*, or *poset*
  - Denoted by $(S,R)$

# Partial ordering examples

- Show that ≥ is a partial order on the set of integers
  - It is reflexive: $a \geq a$ for all $a \in \mathbf{Z}$
  - It is antisymmetric: if $a \geq b$ then the only way that $b \geq a$ is when $b = a$
  - It is transitive: if $a \geq b$ and $b \geq c$, then $a \geq c$

- Note that ≥ is the partial ordering on the set of integers
- $(\mathbf{Z}, \geq)$ is the partially ordered set, or poset

# Symbol usage

- The symbol $\preccurlyeq$ is used to represent *any* relation when discussing partial orders
  - Not just the less than or equals to relation
  - Can represent $\leq$, $\geq$, $\subseteq$, etc
  - Thus, $a \preccurlyeq b$ denotes that $(a,b) \in R$
  - The poset is $(S, \preccurlyeq)$

# Comparability

- The elements $a$ and $b$ of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$.
  - Meaning if $(a,b) \in R$ or $(b,a) \in R$
  - It can't be both because $\preccurlyeq$ is antisymmetric
    - Unless $a = b$, of course
  - If neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then $a$ and $b$ are *incomparable*
    - Meaning they are not related to each other
  - This is definition 2 in the text

- If all elements in $S$ are comparable, the relation is a *total ordering*

# Comparability examples

- Let $\preccurlyeq$ be the "divides" operator |
- In the poset ($\mathbf{Z}^+$,|), are the integers 3 and 9 comparable?
  - Yes, as 3 | 9
- Are 7 and 5 comparable?
  - No, as $7 \nmid 5$ and $5 \nmid 7$

- Thus, as there are pairs of elements in $\mathbf{Z}^+$ that are not comparable, the poset ($\mathbf{Z}^+$,|) is a partial order

# Comparability examples

- Let $\preccurlyeq$ be the less than or equals operator $\leq$
- In the poset $(\mathbf{Z}^+, \leq)$, are the integers 3 and 9 comparable?
  - Yes, as $3 \leq 9$
- Are 7 and 5 comparable?
  - Yes, as $5 \leq 7$

- **As all pairs of elements** in $\mathbf{Z}^+$ are comparable, the poset $(\mathbf{Z}^+, \leq)$ is a **total order**
  - a.k.a. totally ordered poset, linear order, chain, etc.

# Well-ordered sets

- ($S$,$\preccurlyeq$) is a well-ordered set if:
  - ($S$, $\preccurlyeq$) is a totally ordered poset
  - Every non-empty subset of $S$ has a least element

- Example: (**Z**,≤)
  - Is a total ordered poset (every element is comparable to every other element)
  - It has no least element
  - Thus, it is not a well-ordered set

- Example: ($S$,≤) where $S$ = { 1, 2, 3, 4, 5 }
  - Is a total ordered poset (every element is comparable to every other element)
  - Has a least element (1)
  - Thus, it is a well-ordered set

# Lexicographic ordering

- Consider two posets: $(S, \preccurlyeq_1)$ and $(T, \preccurlyeq_2)$
- We can order Cartesian products of these two posets via lexicographic ordering
  - Let $s_1 \in S$ and $s_2 \in S$
  - Let $t_1 \in T$ and $t_2 \in T$
  - $(s_1,t_1) \preccurlyeq (s_2,t_2)$ if either:
    - $S_1 \preccurlyeq_1 s_2$
    - $s_1 = s_2$ and $t_1 \preccurlyeq_2 t_2$

- Lexicographic ordering is used to order dictionaries

# Lexicographic ordering

- Let *S* be the set of word strings (i.e. no spaces)
- Let *T* bet the set of strings with spaces
- Both the relations are alphabetic sorting
  - We will formalize alphabetic sorting later
- Thus, our posets are: $(S, \preccurlyeq)$ and $(T, \preccurlyeq)$

- Order ("run", "noun: to…") and ("set", "verb: to…")
  - As "run" $\preccurlyeq$ "set", the "run" Cartesian product comes before the "set" one
- Order ("run", "noun: to…") and ("run", "verb: to…")
  - Both the first part of the Cartesian products are equal
  - "noun" is first (alphabetically) than "verb", so it is ordered first

# Lexicographic ordering

- We can do this on more than 2-tuples
- $(1,2,3,5) \preccurlyeq (1,2,4,3)$
  - When $\preccurlyeq$ is $\leq$

# Lexicographic ordering

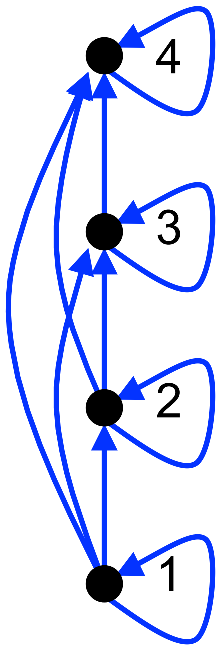
- Consider the two strings $a_1a_2a_3…a_m$, and $b_1b_2b_3…b_n$
- Here follows the formal definition for lexicographic ordering of strings

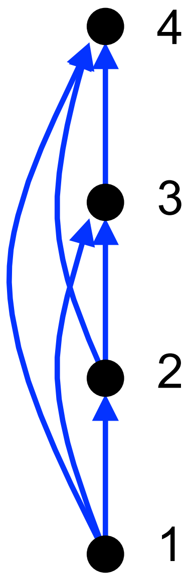
- If $m = n$ (i.e. the strings are equal in length)
  - $(a_1, a_2, a_3, …, a_m) \preccurlyeq (b_1, b_2, b_3, …, b_n)$ using the comparisons just discussed
  - Example: "run" $\preccurlyeq$ "set"

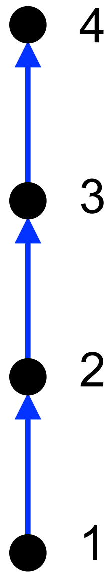
- If $m \neq n$, then let $t$ be the minimum of $m$ and $n$
  - Then $a_1a_2a_3…a_m$, is less than $b_1b_2b_3…b_n$ if and only if either of the following are true:
  - $(a_1, a_2, a_3, …, a_t) \preccurlyeq (b_1, b_2, b_3, …, b_t)$
    - Example: "run" $\preccurlyeq$ "sets" ($t = 3$)
  - $(a_1, a_2, a_3, …, a_t) = (b_1, b_2, b_3, …, b_t)$ and $m < n$
    - Example: "run" $\preccurlyeq$ "running"

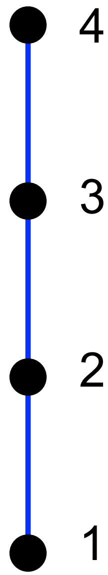
# Hasse Diagrams

- Consider the graph for a finite poset ({1,2,3,4}, ≤)
- When we KNOW it's a poset, we can simplify the graph



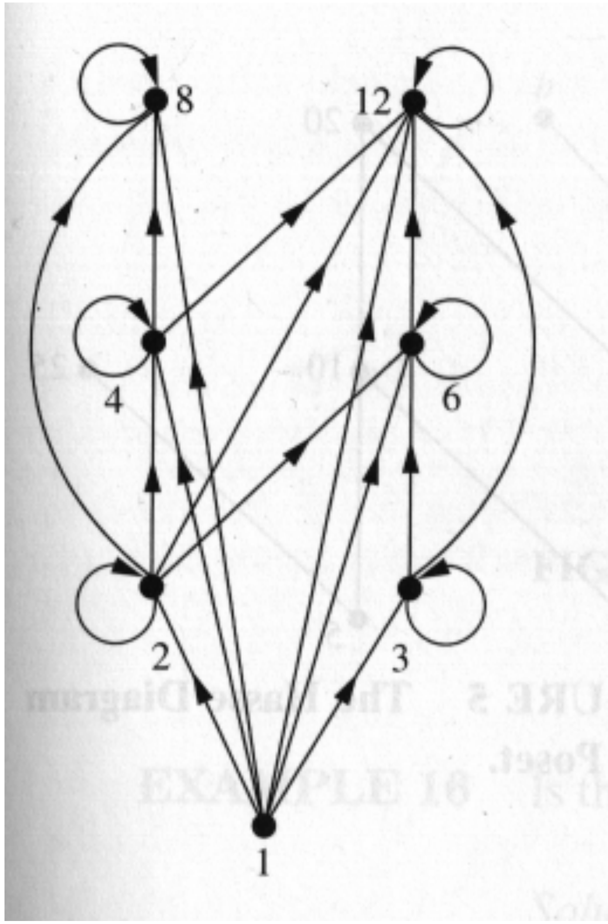



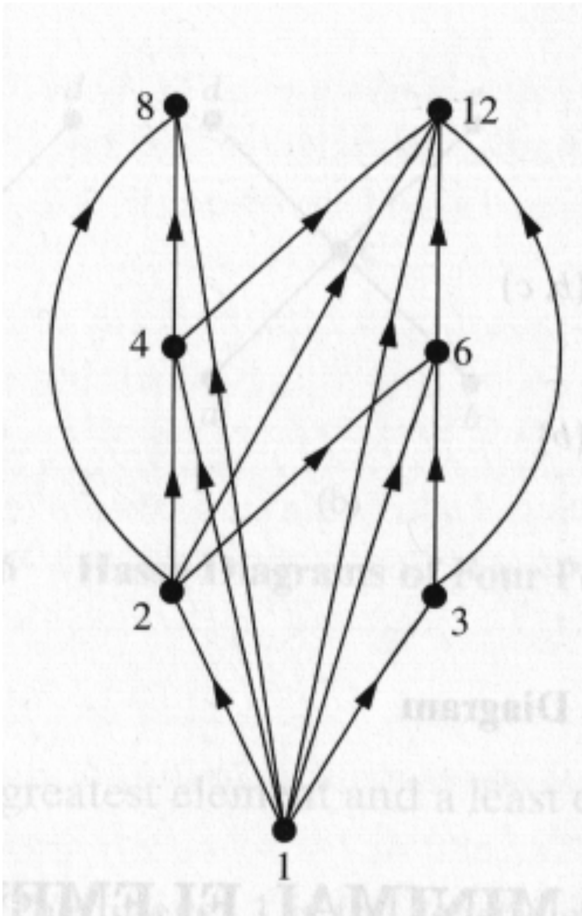



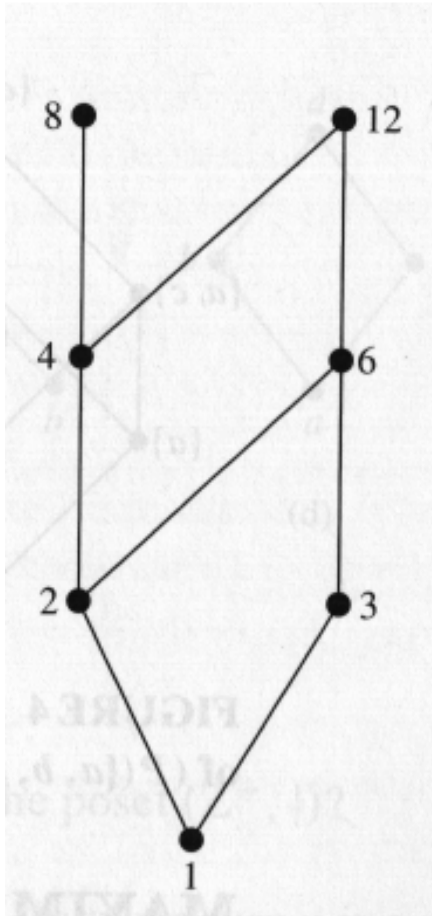

Called the Hasse diagram

# Hasse Diagram

- For the poset ({1,2,3,4,6,8,12}, |)



# Maximal and Minimal elements in Posets

- Let $(A, \mathcal{R})$ be a poset.
- $x \in A$ is a **maximal element** of $A$ if $(x, a) \notin \mathcal{R}$ for all $a \in A$ and $a \neq x$.[a]
  - The poset $(\{1, 2, 3, 4, 5, 6, 7, 8\}, |)$ on p. 332 has 4 maximal elements: 5, 6, 7, 8 as none of them divides a number in $\{1, 2, 3, 4, 5, 6, 7, 8\}$.
- $y \in A$ is a **minimal element** of $A$ if $(b, y) \notin \mathcal{R}$ for all $b \in A$ and $b \neq y$.
  - The poset $(\mathbb{Z}^+, \leq)$ has minimal element 1 but no maximal elements.

# Maximum and Minimum elements

## Least and Greatest Elements of Posets

- Let $(A, \mathcal{R})$ be a poset.
- $x \in A$ is a **least element** if $(x, a) \in \mathcal{R}$ for all $a \in A$.
- $y \in A$ is a **greatest element** if $(b, y) \in \mathcal{R}$ for all $b \in A$.
- Least element and greatest element, if they exist, are unique.
  - For example, suppose $x, y$ are both greatest elements.
  - Then $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$, which imply $x = y$ because of antisymmetry (p. 303).

# Not being covered

- The remainder of 7.6 is not being covered due to lack of time
  - Lattices
  - Topological sorting