Time Series Analysis

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Estimation

Some different approaches to estimate the unknown parameters:

$$\hat{\boldsymbol{\theta}}_{LS} = \arg\min_{\boldsymbol{\theta}} \left\| \boldsymbol{\Pi}_{\mathbf{X}}^{\perp} \mathbf{y} \right\|_2^2 = \left(\mathbf{X}^* \mathbf{X} \right)^{-1} \mathbf{X}^* \mathbf{y}$$

$$\hat{\boldsymbol{\theta}}_{PEM} = \arg\min_{\boldsymbol{\theta}} \sum \left| \epsilon_{t+1|t}(\boldsymbol{\Theta}) \right|^2$$

$$\hat{\boldsymbol{\theta}}_{ML} = \arg\max_{\boldsymbol{\theta}} f_{\mathbf{e}}(\mathbf{y}) = \arg\max_{\boldsymbol{\theta}} \ln f_{\mathbf{e}}(\mathbf{y})$$

where

$$\epsilon_{t+1|t}(\boldsymbol{\Theta}) = y_{t+1} - \hat{y}_{t+1|t}(\boldsymbol{\Theta})$$

Estimation

The quality of the estimates will be bounded by the CRLB, i.e.,

$$V\{\hat{\boldsymbol{\theta}}\} \geq \mathbf{I}_{\boldsymbol{\theta}}^{-1} \geq \mathbf{0}$$

where the FIM is given as

$$\begin{aligned} \left[\mathbf{I}_{\boldsymbol{\theta}}\right]_{k,l} &= -E\left\{ \frac{\partial^2 \ln f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_k \partial \boldsymbol{\theta}_l} \right\} \\ &= E\left\{ \frac{\partial \ln f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_k} \frac{\partial \ln f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_l} \right\} \end{aligned}$$

A statistically efficient estimator can be found iff

$$\frac{\partial \ln f_{\mathbf{x}}(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathbf{I}_{\boldsymbol{\theta}} \Big(\mathbf{g}(\mathbf{x}) - \boldsymbol{\theta} \Big)$$

If the data is Gaussian, the FIM simplifies to Slepian-Bangs formula

$$\left[\mathbf{I}_{\boldsymbol{\theta}}^{-1}\right]_{ij} = \left[\frac{\partial \mathbf{m}_{\boldsymbol{\theta}}}{\partial \boldsymbol{\theta}_i}\right]^T \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1} \left[\frac{\partial \mathbf{m}_{\boldsymbol{\theta}}}{\partial \boldsymbol{\theta}_j}\right] + \frac{1}{2}\left[\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1} \frac{\partial \boldsymbol{\Sigma}_{\boldsymbol{\theta}}}{\partial \boldsymbol{\theta}_i} \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1} \frac{\partial \boldsymbol{\Sigma}_{\boldsymbol{\theta}}}{\partial \boldsymbol{\theta}_j}\right]$$

Estimation

Under reasonable conditions, the ML estimate is (asymptotically)

$$\hat{\boldsymbol{\theta}} \in \mathcal{N}\left(\boldsymbol{\theta}, \mathbf{I}_{\boldsymbol{\theta}}^{-1}\right)$$

and is thus statistically efficient. In the particular case of a Gaussian linear system

$$\mathbf{x} = \mathbf{A}\boldsymbol{\theta} + \mathbf{e}$$

the ML estimate coincides with the WLS estimate

$$\hat{\boldsymbol{\theta}} = \left(\mathbf{A}^T \mathbf{C}^{-1} \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{C}^{-1} \mathbf{x}$$

with

$$\hat{\boldsymbol{\theta}} \in \mathcal{N}\left(\boldsymbol{\theta}, \left(\mathbf{A}^T \mathbf{C}^{-1} \mathbf{A}\right)^{-1}\right)$$

Estimation

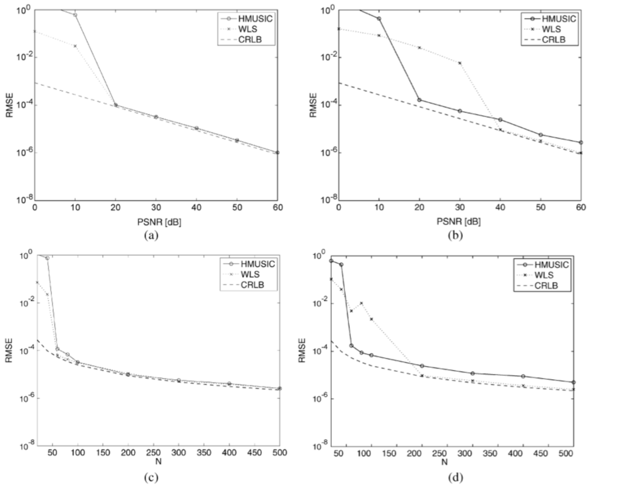

Fig. 7. (a) RMSE as a function of PSNR for $N = 200$ with constant amplitudes. (b) RMSE as a function of PSNR for $N = 200$ with randomized amplitudes. (c) RMSE as a function of N for PSNR $= 40$ dB with constant amplitudes. (d) RMSE as a function of N for PSNR $= 40$ dB with randomized amplitudes.