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A quantitative investigation into Parkinson’s Law detailing the observed inefficiency of decision-making bodies

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I investigate the relationship between dissensus, that is, the degree to which a decision making body (referred to as a cabinet or committee throughout this study) reaches differing views after a deliberation period. By showing that some key parameters defining a cabinet are quantifiable as variables, I justify the use of evolving a node network to model the behaviour of real deliberating committees. Running the simulation over a range of committee sizes enables the plotting of dissensus as a function of the number of committee members. In analysing these results, the observations on bureaucratic inefficiency made by C.N. Parkinson reveal themselves. I then vary the parameters within their realistic constraints and extend the model to accommodate for increasing cabinet complexity. I draw conclusions throughout the discussion by comparing the results derived from different models with each other and with Parkinson’s own conclusions.

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1. Background

Parkinson's Law

'The importance of Parkinson's Law lies in the fact that it is a law of growth based upon an analysis of the factors by which that growth is controlled.'[1] C.N. Parkinson determined that decision-making bodies are inherently inefficient, because they are limited by their size. Purportedly, as their size increases, a given member bears less influence, so new members are inducted more easily. According to his arguments, we should see a committee reach disagreement more and more as their size increases, because large committees are more likely to dissociate into smaller sub-groups. This is but one of a cohort of criticisms made by Parkinson on the subject of bureaucratic inefficiency and it is our subject of investigation.

Graphs

I use a graph data structure to represent a collection of individuals able to interact with other individuals. An individual is a node and their links to other nodes form the edges of our graph. Graphs can be traversed and manipulated using Python array handling protocols that can enact some basic operations from graph theory, a branch of mathematics. A visual representation of such a data structure is shown in Figure 1.[2]

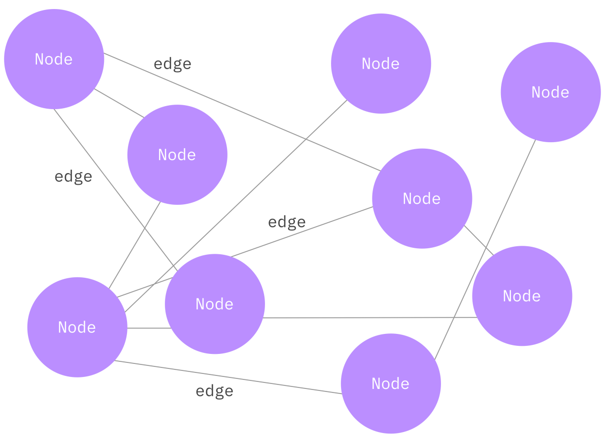

Figure 1: Graph data structure

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[1] The Economist - From the archive: Parkinson's Law (19th November 1955)

[2] medium.com - Creating Graphs With JavaScript

2. Introduction

A decision-making body, given a binary argument to debate, can develop in a wide variety of ways, despite the fact that in many cases their function is to reach a common a consensus. They might, for example, form clusters in which an opinion is shared so that coalitions emerge; they could reach a perfect disagreement, dividing the cabinet in half, or they could, of course, all end up sharing the same opinion on a matter, thus arriving at the elusive consensus.

At times in this paper, I take the Cabinet of the United Kingdom as a contextual example of a decision-making body with which to draw reference. The committee is composed of the Prime Minister and 21 cabinet ministers[3] (the most senior of the government ministers). Interestingly, 21 is the number above which Parkinson claimed a committee is plagued by an innate inefficiency. Independent of his Law of Triviality, which claims that committees spend more time discussing trivialities than pressing concerns, this figure is taken as a consequence of the tendency for opinion formation to occur in small groups not large ones. The result of this is the aforementioned clustering.

This paper employs fundamental bayesian statistics to model the most likely evolution of a decision-making body debating a binary argument. I vary the range of parameters to see how they influence the relationship between dissensus and committee size before extending the model to incorporate more complex committee structures.

3. Method

Data Structures

For the base implementation, I generally use Numpy arrays to store information about the committee. To store the opinion states of each node, I use a one-dimensional array of length N, where N is the number of committee members. The array contains N binary states, representing the distribution of opinions (i.e. ones and zeros representing 'yes's and 'no's).

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[3] members.parliament.uk - Her Majesty's Government: The Cabinet

I store the links between nodes as an adjacency matrix, but I do not label the connections themselves, thus defining each edge of my graph only by the two nodes it connects. I demand that the graph is always connected, that is, the formation of completely independent sub-graphs is not permitted, as this would reflect the genesis of a completely new committee with no ties to the parent body and is therefore beyond the scope of this study. As such, it is necessary to traverse the graph and perform a Depth First Search (DFS)[4], which can identify whether or not there is a traversable path between any two nodes. To implement this algorithm, I convert the two-dimensional Numpy array representation of the aforementioned adjacency matrix into a Python dictionary of the same format.

Re-Wiring

I define the number of connections held by each node as k and ensure that $N > k + 1$, such that the graph is not wholly connected (i.e. each node is not connected to every single other node). I then define e to be the probability that a given connection is randomly re-wired, that is to say a node replaces its connection to one node with a connection to another node. Given the semantic context, this could be likened to ending a conversation with one cabinet minister and hearing what someone else has to say on the matter. Insisting that $0 < e < 1$, k and e become two of our variable parameters that can be changed in order to gauge their influence on the evolving system of opinion states.

Re-Evaluating

Given that the opinion states are essentially stored as a list, it is sensible to traverse the array random-sequentially, so as not to introduce any directional bias to a system in which spatial dimensions are arbitrary. At each node, I evaluate the average opinion of its connected nodes and determine whether or not this exceeds a pre-determined threshold h in favour of either opinion. I define h such that $0.5 < h < 1$ and it becomes the final of our three focal variables. The condition under which a node changes its opinion is described by Equation (1).

[4] brilliant.org - Depth-First Search (DFS)

$$| < opinion > | \geqslant h \tag{1}$$

Where I have used the absolute notation to indicate that we are dealing with the degree to which the average opinion favours either '1' or '0'.

Simulation

Once k, e and h are defined within their constraints, I randomly establish the initial edges of the graph such that each node has k connections and assign a particular distribution of initial opinion states. I then iterate the opinion re-evaluation procedure, followed by the re-wiring procedure a number of times (three for the primary experiment) and output the resulting distribution of opinions after each iteration as a row in a block diagram. From this, it is possible to view the evolution of the committee's array of opinions.

The finite limit on the number of iterations is justified contextually by relating it to a restricted discussion period that may even be quantised by, say, a fixed number of ballots.

Plotting Dissensus

To plot dissensus as a function of N, I iterate the simulation process over all possible combinations of initial opinion states and do this for a range of values for $N = 0,1,2,...,N$. Therefore, we take S_i to be the number of nodes initially holding the opinion '1', which of course means that there are $N - S_i$ nodes initially holding the opinion '0'. S_f is the final number of nodes holding the opinion '1' when the simulation for a given N and S_i is complete. The dissensus function is then given by Equation (2).

$$D(N) = \left\langle \Theta\left(1 - \frac{max(S_f, N - S_f)}{N}\right)\right\rangle_{S_i} \tag{2}$$

Where $\Theta(x)$ is the Heaviside step function (0 for $x < 0$, 1 for $x > 0$) and $\langle\ .\ \rangle_{S_i}$ denotes the average over all possible initial conditions. It is therefore clear to see that for a given

N, $D(N)$ gives the expectation value of a final state that does not resemble consensus[5]. In other words, the dissensus function measures a group of this size’s disposition to ending up in dispute. To plot dissensus, I simply iterate the calculation of $D(N)$ over the range of N values and create a scatter plot.

4. Results

Simulations

I use block diagrams to illustrate the opinion states of committee members at each iterative stage in the simulation. An example of this is demonstrated by Figure 2.

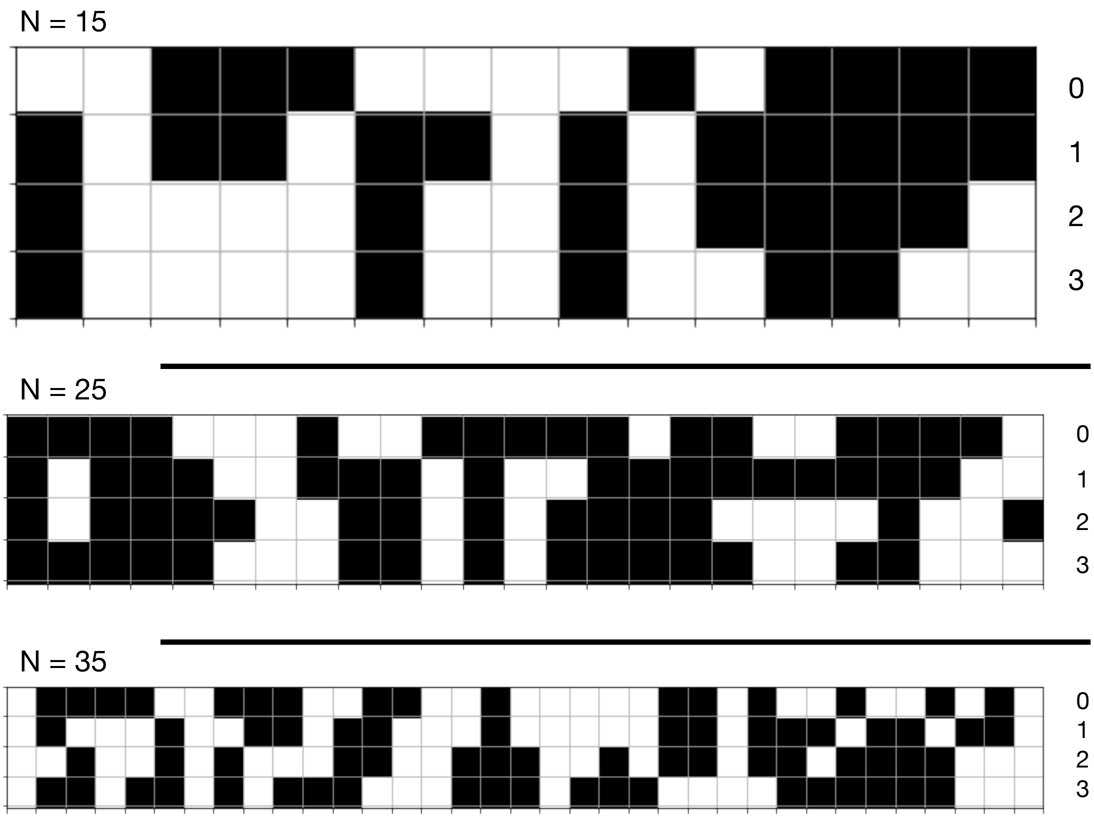

Figure 2: Dynamical evolution of opinion states demonstrated in groups of three different sizes (N = 15, 25, 35). Variables are set to: k = 8, e = 0.1 and h = 0.6. An individual is represented by a tile and their opinion is indicated by their colour (black=‘1’ or white=‘0’). For each value of N, I run the simulation three times with the same values assigned to our key variables. For this simulation (row ‘0’), the initial opinion states are assigned randomly and the following lines are obtained by running the iterative re-wiring and re-evaluating processes.

[5] Parkinson’s Law Quantified: Three Investigations on Bureaucratic Inefficiency, Klimek et al (12th August 2008)

For the values of N used in Figure 2, the final state of the committee tends to exhibit dissensus, with the disordered clustering of opinion states proposed by Parkinson being visually evident in the block diagrams.

Interestingly though, with k set to a lower value, such as 5 (as in Figure 3), so that the $N > k + 1$ restriction allows for N to be as low as 7; running the simulation as above we see that consensus is nearly always reached for $N < 10$.

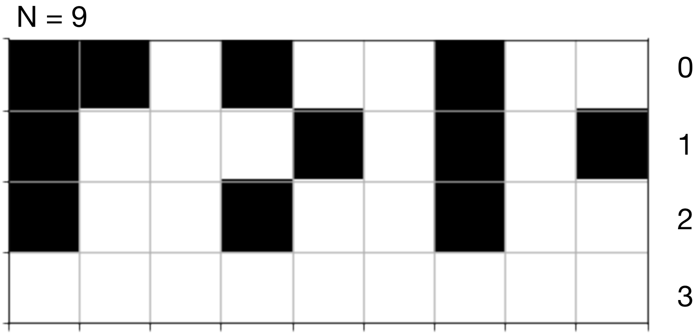

Figure 3: Example evolution of opinion states for N < 10; demonstrating the arrival at consensus by the end of the third iterative update.

Plots of Dissensus

I plot dissensus as a function of N, $D(N)$, for the range $k + 1 < N \leqslant 40$ and do so for several different values of k, e and h. Consistently and most predominantly we see dissensus increasing with the number of committee members. This relationship does depend significantly on the three focal variables, which is made apparent by Figure 4, where I have chosen scatter plots, because the stochastic nature of the simulations means that the point distribution is largely too broad to fit an appropriate line.

5. Discussion

To contextualise the variables in use, I argue that k could be considered the number of people that an individual can converse with at once. The finite number of iterations is like a finite time frame imposed upon a committee meeting; or, more closely relating to the quantised nature of the iterative process, a number of polls prior to a final

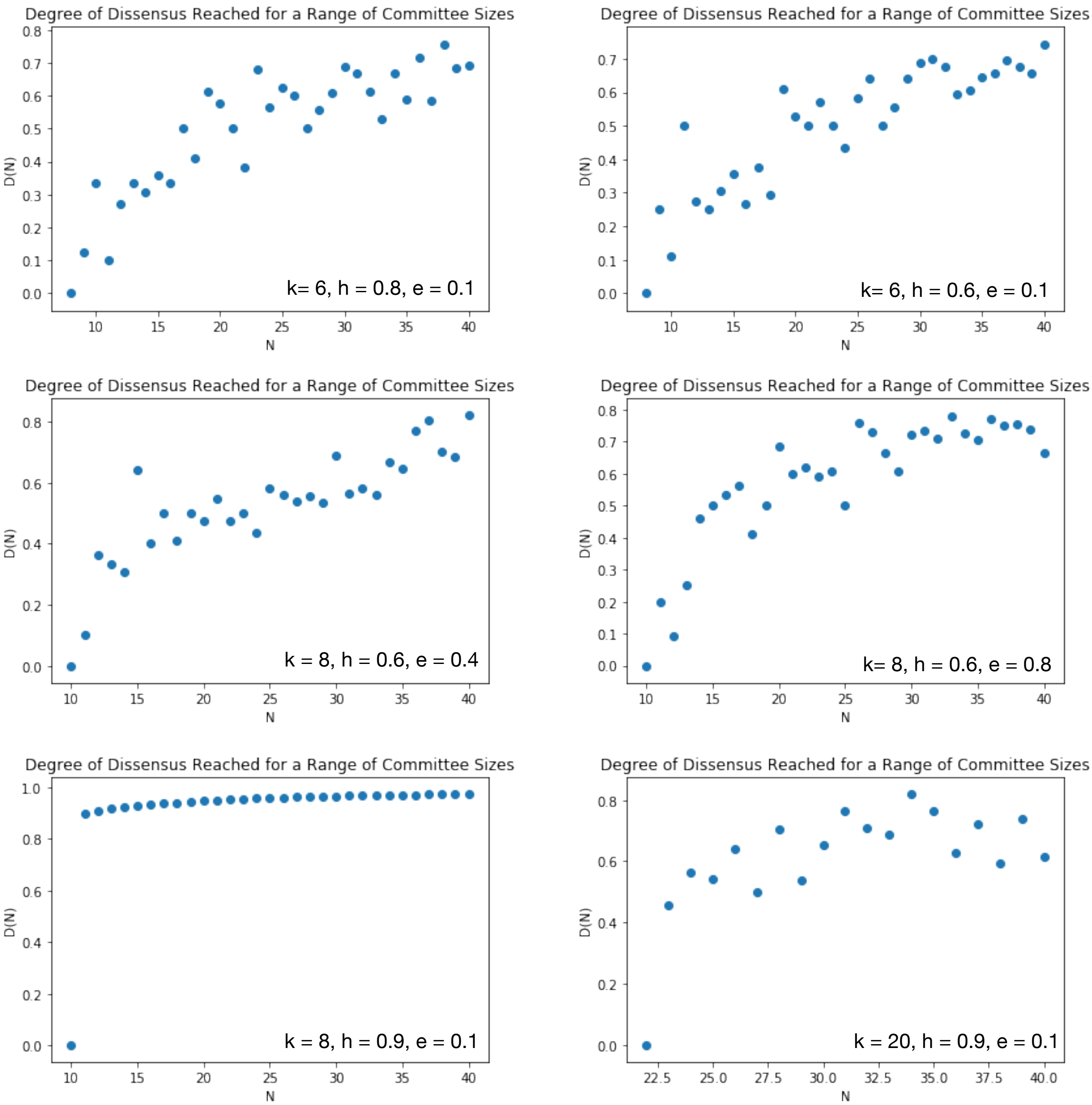

Figure 4: Six scatter plots of D(N) showing how dissensus increases with committee size for a selection of variable combinations. N, the number of committee members and k, the number of connections held by each member are varied within the constraint N > k + 1. In every instance, the maximum value of N is set to 40. The probability that a given connection is rewired during an iterative update, e, is varied such that 0 < e < 1. The threshold value, h, that the fraction of connected nodes sharing one opinion must exceed for a given node to change its opinion to that of the majority is varied such that 0.5 < h < 1.

vote. Edges of the graph represent a communication medium such that e simply relates to the tendency individuals have to end communications with one person and compare their opinion with a different committee member. The variable h is essentially the regard with which individuals hold their current opinion - how likely are they to be swayed by their peers?

In the series of updates, as N increases beyond values around ten, we see increased clustering. That is to say the final line in each block diagram transitions from an often homogenous state (i.e. all blocks the same colour) to an orderless array of black and white blocks. The disorder arises as a consequence of the arbitrary positioning of each block. By this I mean that there is no association between the adjacency of blocks in the diagrams and whether or not they are connected by an edge of the graph.

To test whether the nodes were not just forming dissensus, but doing so in clusters i.e. smaller sub-groups in which opinions are shared, I calculate the average percentage of connected nodes that share an opinion. Running repeated simulations with the same fixed variables and outputting this average percentage each time reveals that it varies between 50 - 100%, thus implying that by the end of the simulation there are never circumstances in which, for a given node, the fraction of nodes connected to it that share its opinion is less than half. Those values particularly close to 100% are of course not to be taken as particularly indicative of this trend, as they are due to the less likely scenarios in which the committee arrives at consensus. This indicates that connected nodes tend to share the same opinion by the end of the third iteration, as is illustrated by Figure 5.

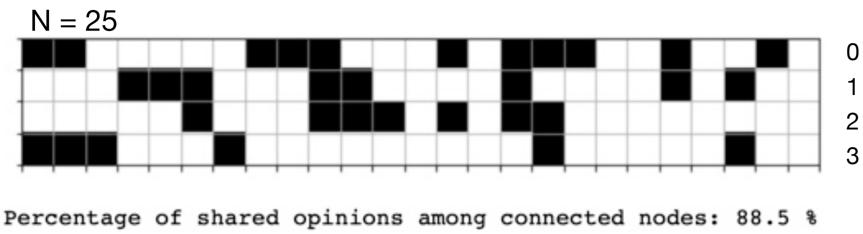

Figure 5: Example evolution of opinion states with variables set to: k = 8, e = 0.1, h = 0.6. The average percentage of nodes connected to one given node that share the same opinion is outputted below the block diagram. Repeated trials show that the result is indicative of the general trend even for greater D(N).

The writings of C.N. Parkinson on this matter are the foundation of this paper. They postulate that large committees, in particular those above a proposed 'Threshold of Inefficiency' (purportedly about 20), are prone to this clustering behaviour, which prevents the committee from arriving at a consensus even though this is arguably a committee's sole function. By showing that, by the end of a third iteration, a node's opinion is always that of the majority of its connected nodes, I have illustrated a potential reason for the correlation between dissensus and committee size exhibited in Figure 3. That is, small committees largely prohibit clustering because they are essentially one sub-group and are therefore likely to arrive at consensus, following the very argument that connected nodes eventually share opinions. Arguably then, it is the difference between k and N that matters here, but generally, smaller N values mean less difference between k and N, as k will have some upper bound in real social contexts.[6]

In Figure 4, I only include one case in which k is particularly large (bottom right graph). Comparing the two bottom-most graphs in Figure 4, it becomes clear that increasing k to twenty lowers the very high, converging $D(N)$ values show in the bottom left graph. This consistent, almost maximal dissensus is a consequence of the high opinion state re-evaluation threshold used when plotting the bottom two graphs ($h = 0.9$). This implies that restricting the likelihood that a node will change its opinion during a given iterative update causes dissensus to increase - stubbornness emerges as a counterproductive diplomatic quality.

Comparing the middle two graphs in Figure 4, we see that $D(N)$ increases more quickly with N when the likelihood that a connection is rewired, e, is higher. This faster growth is highlighted by the fact that, excluding the bottom left graph, this is the only graph for which $D(N) \geqslant 0.7$ by $N = 20$ and it is the only graph plotted for $e > 0.5$. From this, I conclude that maintaining connections more often than making new ones is a beneficial attribute for a decision-making body seeking consensus, which is somewhat antithetical to the intuition that interacting with more committee members would be a productive behaviour. This is in part because real decision-making involves novel perspectives and subtleties beyond binary arguments, which this simplified model cannot incorporate. It is also because the rewiring process is random-sequential, meaning it introduces stochastic uncertainty into the updating of connections, whereas in real systems, connections are made as a consequence of logical factors that an individual considers.

Finally, the two top-most graphs in Figure 4 display steady growth of $D(N)$ with N, thus implying that the associated variable values are non-extremal. Of the two, the

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[6] One can only talk to so many people at once!

graph with higher h simply has a slightly steeper gradient, which is in line with previous discussion.

6. Extending the Model

Hierarchies

Many committees are hierarchical. That is to say, they have a non-negligible power dynamic. Often, pyramid structures are associated with hierarchies, as there are increasingly fewer individuals at each incremental step-up in power. This is illustrated in Figure 6.[7]

Figure 6: Diagrammatic representation of a pyramid hierarchy. There are less people associated with each rung in the pyramid, with a higher rung being attributed to greater power.

Descending from a single node at the top of the pyramid (i.e. the most powerful node), a simple sequence, with each step representing a rung in the pyramid, goes as 1, 3, 5, 7, … which is an arithmetic series of the form given by Equation (3).

$$\Sigma_1^H (2n - 1) = N \tag{3}$$

Where H is the number of rungs in the pyramid. Equation (3) can be rearranged, using the common formula for summations of arithmetic series in terms of n, to find Equation (4).

$$H = \sqrt{N} \tag{4}$$

From this, it is clear that the hierarchical implementation requires that N be a square number, as H must be an integer.

To create a hierarchy, I apply a weight to each node's opinion state by extending the one-dimensional opinion state array to a two-dimensional array that stores each node's opinion state and weighting. The distribution of weights is constructed to

[7] rabbisacks.org - Pyramid Hierarchy Homepage

resemble the common social pyramid structure. I achieve this by creating a variable p, such that the most powerful node has an authority equivalent to a fraction, $\frac{N}{p}$, of the total committee size; let $p = 6$ for our uses here. The other weights are assigned such that the bottom rung is associated with an opinion state weighting of 1 and that the difference in weighting between rungs is evenly distributed, as given by Equation (5).

$$diff. = \frac{\frac{N}{p} - 1}{H} \tag{5}$$

To determine whether or not a node will change its opinion during an iterative update, I sum the weights in favour of zero and those in favour of one for the k connected nodes and divide each sum by a normalising factor, which is the sum of the N total weights multiplied by the fraction $\frac{k}{N}$. If the result is greater than the usual threshold variable, h, then the node changes its opinion in favour of the majority.

Running the new simulation four times produces the block diagrams shown in Figure 7.

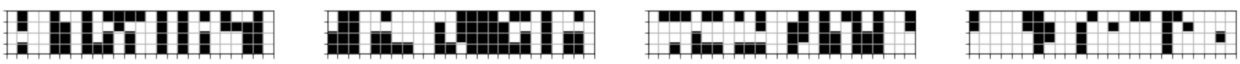

Figure 7: Block diagrams showing the dynamical evolution of opinion states in the same manner as before, only they now show data for the hierarchical implementation. The top row of nodes have their opinion states randomly designated and each consecutive row below is the consequence of the iterative updating process described above. In this instance, N = 25, k = 8, e = 0.1 and h = 0.6, just as in Figure 2 and Figure 5.

Comparing these block diagrams to the block diagram in Figure 2 for which $N = 25$, we see that they exhibit similar behaviour, with clustering and a tendency to maintain dissensus being attributes exhibited in both circumstances.

Plotting dissensus for the hierarchical case, as in Figure 8, reveals that $D(N)$ is consistently higher than in the standard implementation, as it tends to grow particularly quickly with N initially and then more slowly, if at all, where $D(N) > 0.5$. This is likely

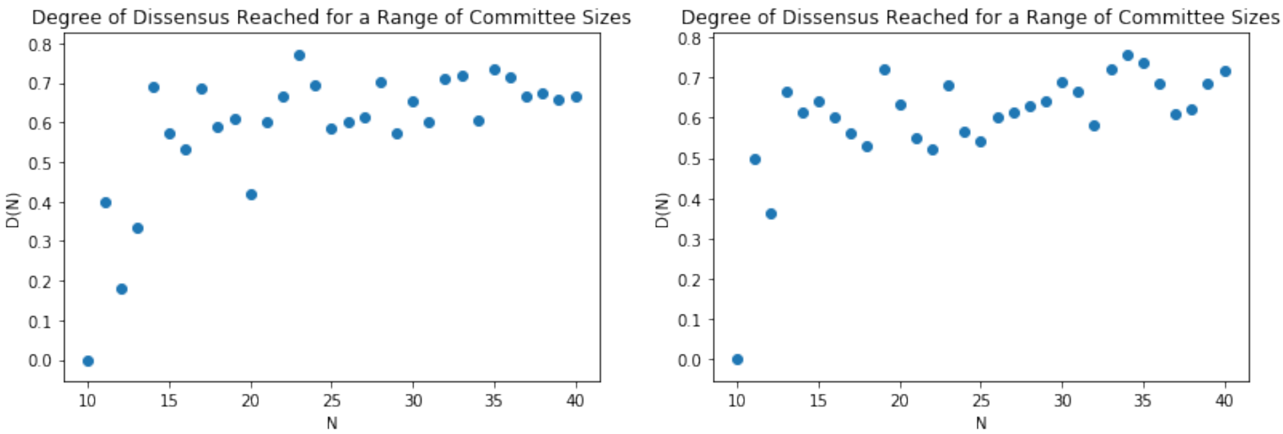

Figure 8: Two scatter plots of dissensus as a function of committee size for nodes with opinions weighted in a manner corresponding to a pyramid hierarchy.

due to the finer dependencies on N for the hierarchical model. For example, H, the number of weighting levels, scales with $\sqrt{N}$ while the range of weightings is equivalent to $\frac{N}{p} - 1$. Therefore, greater values of N correspond to both an increased number of weighting levels and a greater range of weightings, i.e. increased disparity between the top and bottom rung of nodes, increased disparity between two adjacent rungs and an increased number of rungs.

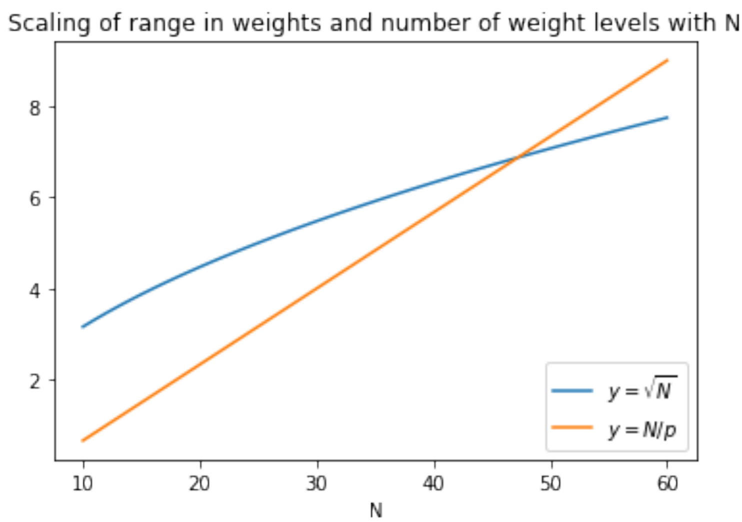

Figure 9: Plot showing the dependencies of the range of weights (orange line) and the number of weight levels (blue line) on N.

These two contributing dependencies do however scale differently with N, can be seen in Figure 9.

We see that the range grows more quickly than the number of levels, but that it is always lower for $N \leqslant 40$, which are the values of N that I use for plots of dissensus. Given that the disparity between the two functions of N shown in Figure 9 is highest at $N = 10$ and decreases as N approaches 40, it seems as though this is the difference responsible for the changing gradients of $G(N)$ evident in Figure 8. One could subsequently conclude that a hierarchical committee is less likely to reach consensus for smaller committee sizes, but that this impact is reduced as N increases and the number of hierarchical layers approaches the total range in opinion state weightings. Given that there are no other factors discerning the hierarchical case from the standard implementation, this is the only analytical explanation to the difference in trend between the two cases.

Perhaps if one were to adjust the value p, or reshape the hierarchical structure itself, there might be more of a case in favour of hierarchies. I investigate this for a loosely quantified contextual case study below.

The Cabinet of the United Kingdom

On October 17th 2019, Boris Johnson and EU leaders agreed on a new Brexit deal. By October 31st, the proposed deadline, a parliamentary agreement was not reached. I explore a simplified example of this case study by reducing the decision-making body comprising the 650 members of Parliament within the House of Commons to the UK cabinet, which is comprised of just the Prime Minister and 21 Cabinet ministers.

Henceforth, in this example we have $N = 22$ and I set our variables to values that gave non-extremal results in the standard implementation ($k = 8$, $e = 0.1$, $h = 0.6$). I alter the hierarchical committee structure, such that the 21 Cabinet ministers have the same opinion state weighting of 1, but the Prime Minister's opinion has a weighting of 2. I create this disparity, because it is the Prime Minister's role to defend their agenda after having put it forward, so their opinion, being the one to be agreed or disagreed with, should hold minimally more weight given that it is the only independent, autonomously formed opinion. The use of a weighting equivalent to two Cabinet ministers is simply what I deme reasonable conjecture here.

Again due to the fact that the Prime Minister is solely acting to defend their agenda (e.g. Mr Johnson was not going to change his mind that his deal should be passed), I have made any connection with the Prime Minister non-reciprocal, meaning they can influence other nodes, but will not be influenced by other nodes.

I quantise the timeframe such that each day between 17th-31st October is equivalent to one iteration.

Two sequential outputs from the resulting simulation are given in Figure 10.

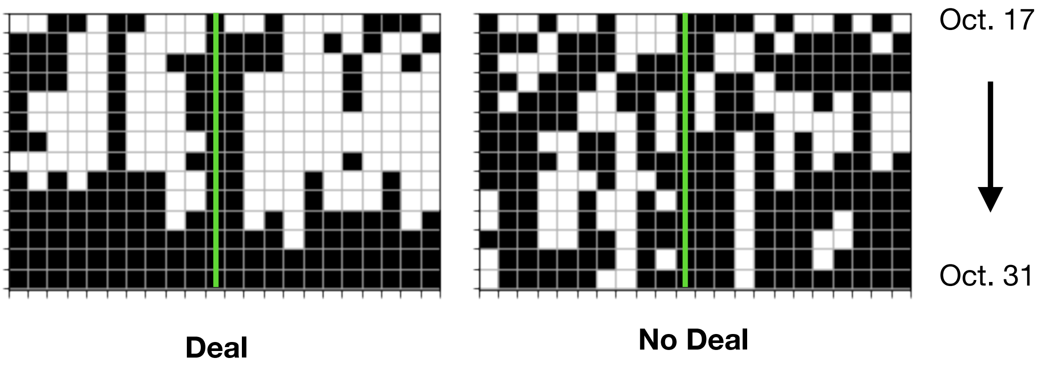

Figure 10: Two block diagrams showing the dynamical evolution of opinion states for a hierarchical model designed to resemble the Cabinet of the United Kingdom. There are 14 rows to represent the 14 days within which the UK Government had to decide whether or not to leave the European Union under a deal negotiated by Prime Minister Boris Johnson. The block representing the Prime Minister is highlighted by a green line and its opinion state (colour) is fixed after the stochastic assignment of initial opinions seen in the first row. In the left diagram, the Cabinet reaches consensus with two days left to spare. The right diagram shows a final state of dissensus, thus resembling the real outcome in which the members are divided and the deal is not accepted.

We observe two outcomes under the same parameters that would have very different real-world consequences, which highlights the limitations of a stochastic model that relies on a great number of assumptions and simplifications.

To see which outcome is more likely within this model, I run the simulation 100 times and each time calculate the proportion of nodes that are in consensus with reference to the majority opinion, using Equation (6).

$$C(N) = \frac{|\ \Sigma(Zeros) - \Sigma(Ones)\ |}{N} \tag{6}$$

Where the proportion in consensus is denoted by $C(N)$. I then calculate the average of this value over the 100 repetitions. This returned a value of $C(N) = 0.85$, but this is not a fixed figure as we are dealing with stochastic processes, hence the need to calculate an average over a large number of trials; here I use 100. Nevertheless, we have a result that indicates consensus is likely given that 85% of nodes share their final opinion on average. Though of course the model neglects contextual factors, such as the actual contents of the proposed deal and the biased sentiments held by individuals.

Further Work

I propose that the following improvements and developments can be made to the investigation of this subject:

- Rows of block diagrams could be ordered so as to illustrate the clustering more effectively, by making connected nodes adjacent to one another.

- Given the opportunity to import further Python modules, one could create a three-dimensional visualisation of the evolving graph structure.

- The basic implementation could include more key variables to more closely resemble the complex social structures it attempts to quantify. For example, k could become flexible so that each iteration there is a probability that it increases or decreases by one.

- The variable h could be made inconsistent among the nodes, so that the subsequent committee resembles a collection of individuals with a range of inclinations to change their opinion.

- The model could be extended further still to accommodate for different committee structures to those that I have explored; a committee of army generals, for example.

- One could explore dissensus for much larger committees, say of the order of hundreds or thousands.