Assouad dimension and fractal geometry: updates on open problems

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Chapter 17 of [5] contained several open questions. I will maintain this document with up to date information on the status of these questions to the best of my knowledge. I will only state that a problem has been solved (or partially solved) if the solution can be found in a published paper or on the arXiv.

I include the section labels and questions as stated in [5] and with the same numbering. Additional text will be included below a given question in the case there is an update on the status. All other text from [5, Chapter 17] has been removed. For context and other information surrounding the questions, see [5]. Occasionally I will pose further questions in this document. These will be labelled **Bonus question** and numbered separately.

17 Future directions

17.1 Finite stability of modified lower dimension

Question 17.1.1. Is it true that, for arbitrary $E, F \subseteq \mathbb{R}^d$,

$$\dim_{\mathrm{ML}} E \cup F \;=\; \max\{\dim_{\mathrm{ML}} E,\ \dim_{\mathrm{ML}} F\}?$$

SOLVED! Question 17.1.1 was answered in the affirmative by Balka, Elekes and Kiss [1, Theorem 2.2].

17.2 Dimensions of measures

Question 17.2.1. Does there exist a compact set $F \subseteq \mathbb{R}^d$ such that $\dim_{\mathrm{L}} \mu < \dim_{\mathrm{L}} F$ for all doubling measures fully supported on F?

17.3 Weak tangents

Question 17.3.1. Is it true that for all closed sets $F \subseteq \mathbb{R}^d$ there exists a weak tangent E of F such that $\dim_{\mathrm{ML}} E = \dim_{\mathrm{A}} F$?

Question 17.3.2. Given a closed set $F \subseteq \mathbb{R}^d$, is the set $\Delta(F)$ always analytic or Borel and, if so, does it always belong to a particular finite Borel class?

SOLVED! Question 17.3.2 was answered by Balka, Elekes and Kiss [2, Theorem 3.12]. It turns out that $\Delta(F)$ is always analytic and, moreover, any analytic set $\Delta \subseteq [0, d]$ containing its supremum and infimum can be realised as $\Delta(F)$ for some compact $F \subseteq \mathbb{R}^d$.

17.4 Further questions of measurability

Question 17.4.1. Is the function $\dim_{\mathrm{ML}} : \mathcal{K}(\mathbb{R}^d) \to \mathbb{R}$ Borel measurable? If so, does it belong to a particular Baire class?

SOLVED! Question 17.4.1 was answered in the affirmative by Balka, Elekes and Kiss and [1, Theorem 4.3]. It turns out that $\dim_{\mathrm{ML}}$ is Baire 2. (It is easy to see that it is not Baire 1.)

17.5 IFS attractors

Question 17.5.1. Let $F \subseteq \mathbb{R}^2$ be a self-affine set satisfying the SSC and which is not self-similar. Does there always exist $\pi \in G(1,2)$ such that

$$\dim_{\mathrm{A}} F = \dim_{\mathrm{A}} \pi F + \max_{x \in \pi F} \dim_{\mathrm{A}} (\pi^{\perp} + x) \cap F?$$

Configuration	carpet	self-affine	self-similar	Fuchsian	Kleinian
L = H = B = A	✓	✓	✓	✓	✓
L = H = B < A	×	✓	✓	✓	✓
L = H < B = A	×	?	×	×	×
L < H = B = A	×	✓	×	×	✓
L = H < B < A	×	?	×	×	×
L < H = B < A	×	?	×	×	✓
L < H < B = A	×	?	×	×	×
L < H < B < A	✓	✓	×	×	×

Table 1: This table summarises which configurations of dimension are possible in certain families of fractal sets. *Carpet* refers to Bedford-McMullen carpets, *self-affine* refers to general self-affine sets, *self-similar* refers to general self-similar sets, *Fuchsian* refers to limit sets of geometrically finite Fuchsian groups, and *Kleinian* refers to limit sets of geometrically finite Kleinian groups. The configurations refer to the relationships between lower, Hausdorff, box and Assouad dimensions, with the obvious labelling. The symbol ✓ means the configuration is known to be possible within the given class, × means the configuration is known to be impossible within the given class, and ? means we do not know if the configuration is possible within the given class.

Question 17.5.2. Are any of the configurations marked with a question mark in Table 1 possible in the relevant class of sets?

PARTIAL PROGRESS! Bárány, Käenmäki and Yu provided an example of a planar self-affine set F satisfying:

$$\dim_{\mathrm{L}} F < \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F < \dim_{\mathrm{A}} F,$$

see [3, Example 3.3].

Question 17.5.3. Is it true that, for all self-similar sets $F \subseteq \mathbb{R}^d$,

$$\dim_{\mathrm{qA}} F = \dim_{\mathrm{B}} F?$$

Question 17.5.4. Does there exist a self-affine set $F \subseteq \mathbb{R}^d$ which satisfies the SSC and is such that $\dim_{\mathrm{qA}} F < \dim_{\mathrm{A}} F$?

SOLVED! Fraser and Rutar [6] constructed explicit examples of rectangular dominated self-affine sets in the plane satisfying the SSC for which the Assouad dimension is 1 and the quasi-Assouad dimension can be made arbitrarily small, answering this questions in the affirmative.

Question 17.5.5. Let $F \subseteq \mathbb{R}^d$ be the attractor of an IFS consisting of bi-Lipschitz contractions and assume F is not a singleton. Is it true that $\dim_{\mathrm{L}} F > 0$?

17.6 Random sets

Question 17.6.1. Suppose

$$\frac{\log(R/\phi(R))}{\log|\log R|}$$

neither converges to 0 nor diverges. Then what is the almost sure value of $\dim_{\mathrm{A}}^{\phi} M$?

Question 17.6.2. Is it true that for all $t \in [s, d]$, there exists ϕ such that almost surely $\dim_{\mathrm{A}}^{\phi} M = t$?

17.7 General behaviour of the Assouad spectrum

Question 17.7.1. Is it true that for any set $F \subseteq \mathbb{R}^d$, there exists $\theta_F \in (0, 1)$ such that $\dim_{\mathrm{A}}^{\theta} F$ is non-decreasing on the interval $(\theta_F, 1)$?

SOLVED! Rutar [7] has provided a precise characterisation of functions attainable as the Assouad spectrum of a compact set in $\mathbb{R}^d$. It is remarkably flexible and in fact the bounds from [5, Theorem 3.3.1] are necessary and sufficient for such a classification. Using this, Rutar established many new results about possible forms of the Assouad spectrum including answering the above question in the negative.

17.8 Projections

Question 17.8.1. Is it true that, for any non-empty set $F \subseteq \mathbb{R}^d$ and $1 \leqslant k < d$,

$$\dim_{\mathrm{H}}\{\pi \in G(k, d) : \dim_{\mathrm{A}} \pi F < \min\{k, \dim_{\mathrm{A}} F\}\} \leqslant k(d - 1 - k)?$$

Question 17.8.2. Is it true that, for any non-empty set $F \subseteq \mathbb{R}^2$,

$$\dim_{\mathrm{P}}\{\pi \in G(1, 2) : \dim_{\mathrm{A}} \pi F < \min\{1, \dim_{\mathrm{A}} F\}\} = 0?$$

Question 17.8.3. Given a compact set $F \subseteq \mathbb{R}^2$, is the function $\pi \mapsto \dim_{\mathrm{A}} \pi F$ Borel measurable? If so, does it lie in a finite Borel class?

SOLVED! Well, actually this was not a good question. The answer is immediately seen to be yes, and even that $\pi \mapsto \dim_{\mathrm{A}} \pi F$ is Baire 2. This follows since $\pi \mapsto \pi F$ is continuous (easy) and $E \mapsto \dim_{\mathrm{A}} E$ is Baire 2 ([4, Theorem 2.6]). I am grateful to Richárd Balka for pointing this out. A better question (and what I claim I really meant to ask) is:

Bonus question 1. Which functions can be realised as $\pi \mapsto \dim_{\mathrm{A}} \pi F$ for a fixed compact set $F \subseteq \mathbb{R}^2$? For example, is it true that if $\phi : S^1 \to [0,1]$ is Baire 2, then there exists a compact set $F \subseteq \mathbb{R}^2$ such that $\dim_{\mathrm{A}} \pi F = \phi(\pi)$ for all $\pi \in S^1$?

17.9 Distance sets

Question 17.9.1. Is it true that if $F \subseteq \mathbb{R}^d$ with $d \geqslant 3$, then

$$\dim_{\mathrm{A}} D(F) \geqslant \min\left\{\frac{2}{d}\dim_{\mathrm{A}} F, 1\right\}?$$

PARTIAL PROGRESS! Shmerkin and Wang proved that if $F \subseteq \mathbb{R}^d$ with $\dim_{\mathrm{A}} F \geqslant d/2$, then $\dim_{\mathrm{A}} D(F) = 1$, see [8, Corollary 8.1]. This follows from [8, Theorem 1.4] which proves that if $F \subseteq \mathbb{R}^d$ is a Borel set with $\dim_{\mathrm{H}} F = \dim_{\mathrm{P}} F \geqslant d/2$, then $\dim_{\mathrm{H}} D(F) = 1$.

Question 17.9.2. For sets $F \subseteq \mathbb{R}^d$ with $d \geqslant 3$, can we move the Assouad dimension version of the distance set problem 'ahead' of the Hausdorff or box dimension version? For example, can we prove that $\dim_{\mathrm{A}} F \geqslant s$ guarantees $\dim_{\mathrm{A}} D(F) = 1$ for some s for which we do not yet have the corresponding Hausdorff or box dimension result?

SOLVED! As mentioned above, Shmerkin and Wang proved that if $F \subseteq \mathbb{R}^d$ with $\dim_{\mathrm{A}} F \geqslant d/2$, then $\dim_{\mathrm{A}} D(F) = 1$, see [8, Corollary 8.1]. The analogous Hausdorff or box dimension result is not known.

Question 17.9.3. Is it true that if $F \subseteq \mathbb{R}^2$ is a set with $\dim_{\mathrm{A}} F > 1$, then there exists $x \in F$ such that $\dim_{\mathrm{A}} D_x(F) = 1$?

Question 17.9.4. Let dim denote one of the Hausdorff, box, packing or Assouad dimensions. Given $s \in (0, d]$, what is

$$\inf\{\dim D(F) : \dim F = s\}?$$

Is the function

$$s \mapsto \inf\{\dim D(F) : \dim F = s\}$$

independent of the specific choice of dim?

17.10 The Hölder mapping problem and dimension

No questions explicitly asked

17.11 Dimensions of graphs

Question 17.11.1. What are the Assouad dimensions, quasi-Assouad dimensions, and Assouad spectra of the graphs of the Weierstrass and Takagi functions?

References

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[3] B. Bárány, A. Käenmäki and H. Yu, Finer geometry of planar self-affine sets, preprint, arXiv:2107.00983

[4] J. M. Fraser, Assouad type dimensions and homogeneity of fractals, *Trans. Amer. Math. Soc.*, **366**, (2014), 6687–6733.

[5] J. M. Fraser, *Assouad dimension and fractal geometry*, CUP, Tracts in Mathematics Series, **222**, (2020).

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