UWaterloo 2026 Winter PMATH352 - Complex Analysis

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January 14, 2026

Contents

1 Complex Differentiation

Date: 12/1/2026

Theorem 1.1 (Cauchy-Riemann). *Let f be a differentiable function from an open set $U \subseteq \mathbb{C}$ to $\mathbb{C}$. Write $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, $u, v : U \to \mathbb{R}$. Then the partial derivatives of u and v satisfy the* Cauchy-Riemann equations *given by*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1}$$

This is a necessary condition for complex differentiability. To see why it is not sufficient, we can consider the following example.

Example 1.1 (Non-differentiability). Let $f(z)$ be defined by $z^5/|z|^4$ when $z \neq 0$

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Theorem 1.2. *If $f(z) = \sum a_n z^n \in \mathbb{C}[[z]]$ has radius of convergence $R > 0$ and if $g(z) = \sum +n n a_n z^{n-1}$. Then $g(z)$ converges in $B(0,R)$ and $f'(z) = g(z)$ for all $z \in B(0,R)$.*

Proof. We know from previous result that $g(z)$ has radius of convergence R. Now fix $z_0 \in B(0,R)$. There exists $\delta > 0$ such that $B(z_0,\delta) \subseteq B(0,R)$, so for all h such that $|h| < \delta$, $z_0 + h \in B(0,R)$. Let $h \in \mathbb{C}$, $0 < |h| < \delta$, consider

$$\begin{aligned}
\left|\frac{f(z_0+h)-f(z_0)}{h} - g(z_0)\right| &= \frac{\sum_{n=0}^{\infty} a_n(z_0+h)^n - \sum_{n=0}^{\infty} a_n z_0^n}{h} - \sum_{n=1}^{\infty} n a_n z_0^{n-1} \\
&= \left|\sum_{n=1}^{\infty} a_n \frac{(z_0+h)^n - z_0^n}{h} - \sum_{n=1}^{\infty} n a_n z_0^{n-1}\right| \\
&= \left|\sum_{n=1}^{\infty} a_n \left(\frac{(z_0+h)^n - z_0^n}{h} - n z_0^{n-1}\right)\right| \\
&= \left|\sum_{n=0}^{\infty} a_n \left[\sum_{j=2}^{n} \binom{n}{j} z_0^{n-j} h^{j-1}\right]\right| \\
&\le \sum_{n=0}^{\infty} |a_n| \sum_{j=2}^{n} \binom{n}{j} |z_0|^{n-j} |h|^{j-1} \\
&\le |h| \sum_{n=0}^{\infty} |a_n| \sum_{l=0}^{n-2} \binom{n}{2}\binom{n-2}{l} |z_0|^{n-2-l} |h|^l, \quad l = j-2 \\
&= |h| \sum_{n=2}^{\infty} |a_n| \binom{n}{2} (|z_0| + |h|)^{n-2} \\
&\le \frac{|h|}{2} \sum_{n=2}^{\infty} |a_n| n(n-1)\rho^{n-2} = C|h| \to 0, \quad \text{as } h \to 0.
\end{aligned}$$

In the above derivation, we used the fact that the series $\sum_{n=0}^{\infty}$ is absolutely convergent, $|z_0| + |h| < |z_0| + \delta \le \rho < R$. Therefore, the limit

$$\lim_{h\to 0}\left|\frac{f(z_0+h)-f(z_0)}{h}\right|$$

existss and is equal to $g(z_0)$. Hence, for all $z \in B(0,R)$, we have

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

□

Corollary 1.3. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is convergent in $B(0,R)$, then $f^{(j)}(z)$ exists and*

$$\frac{f^{(j)}}{j!} = \sum_{n=j}^{\infty} \binom{n}{j} a_n z^{n-j}, \quad \textit{for all } z \in B(0,R).$$

Definition 1.1. *We define*

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \textit{for all } z \in \mathbb{C}. \tag{2}$$

This power series has $R = \infty$.

Theorem 1.4. *The power series* $\exp z$ *has the following properties:*

(i) $\exp x = e^x > 0$ *for all* $x \in \mathbb{R}$.

(ii) $\exp \overline{z} = \overline{\exp z}$ *for all* $z \in \mathbb{C}$.

(iii) $\exp(z+w) = \exp z \cdot \exp w$ *for all* $z, w \in \mathbb{C}$.

(iv) $\frac{d}{dz} \exp z = \exp z$ *for all* $z \in \mathbb{C}$.

(v) $|\exp z| = e^{\operatorname{Re}(z)}$ *for all* $z \in \mathbb{C}$.

Proof.

(i) For $x \in \mathbb{R}$, we have $\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x > 0$, by definition.

(ii) If $S_N = \sum_{n=0}^{N} a_n \to S$, then $\overline{S_N} = \overline{\sum_{n=0}^{N} a_n} = \sum_{n=0}^{N} \overline{a_n} \to \overline{S}$. Therefore, $\exp \overline{z} = \overline{\exp z}$.

(iii) Let $c \in \mathbb{C}$ and $f(z) = \exp(c-z)\exp z$. Assuming (iV) to be true, we have

$$\frac{\mathrm{d}}{\mathrm{d}z} f(z) = \exp(c-z)\exp z - \exp(c-z)\exp z = 0.$$

Therefore, $f(z)$ is constant. Evaluating at $z=0$, we have $f(z) = f(0) = \exp c$. Let $c = z+w$, we get the desired result.

(iv) By the corollary above, we have

$$\frac{\mathrm{d}}{\mathrm{d}z} \exp z = \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp z.$$

(v) we have

$$|\exp(z)|^2 = \exp z \overline{\exp z} = \exp z \exp \overline{z} = \exp(z + \overline{z}) = \exp(2\operatorname{Re}(z)) = e^{2\operatorname{Re}(z)}.$$

Therefore, $|\exp z| = e^{\operatorname{Re}(z)}$.

□

Definition 1.2 (Trigonometric functions)**.** *We also define*

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad \textit{for all } z \in \mathbb{C}. \tag{3}$$

These power series have $R = \infty$.

Corollary 1.5. *For all* $z \in \mathbb{C}$, *we have*

$$\frac{\mathrm{d}}{\mathrm{d}z} \sin z = \cos z, \quad \frac{\mathrm{d}}{\mathrm{d}z} \cos z = -\sin z. \tag{4}$$

Corollary 1.6 (Euler's identity)**.** *For all* $z \in \mathbb{C}$, *we have*

$$\exp(iz) = \cos z + i \sin z. \tag{5}$$

Proof. Notice that $\exp(iz) = e^{iz} = 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots$ absolutely converges, so we may rearrange the terms as we wish. □