Counterexamples in Algebra

Introduction to Algebra, Fall 2024

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August 17, 2025

1 Linear Algebra

2 Abstract Algebra

2.1 Groups

Example 1 (Noncyclic group whose proper subgroups are all cyclic)**.**

The Klein-four group $V_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The result follows from the fact that it is the smallest noncyclic group.

Example 2 (Nonisomorphic groups with the same lattice)**.**

Example 3 (The converse of Lagrange's Theorem is not true)**.**

A_4 does not have a subgroup of index 2.

2.2 Rings

The term **ring** is reserved for a ring with identity, while a ring without identity shall be called a **rng** in the following article.

Example 4 (Are associates always unit multiples? Part I)**.**

Following the notation of Anderson, D.D et al., we adopt the following definitions and notations. Let R be a commutative ring, and let $a, b \in R$. We say a and b are *associates*, denoted $a \sim b$, if $(a) = (b)$. On the other hand, if $a = ub$ for some $u \in U(R)$ the unit group of R, we say a and b are *strong associates* and write $a \approx b$.

Now take the ring R as the set of continuous functions on the closed interval $[0, 3]$, i.e. $R = \{f \in \mathcal{C}([0, 3])\}$. It is straightforward to show that R is a ring with respect to the usual addition and multiplication in $\mathbb{R}$.

Now consider the piecewise functions

$$f(x) = \begin{cases} 1 - x, & 0 \le x \le 1, \\ 0, & 1 \le x \le 2, \\ x - 2, & 2 \le x \le 3. \end{cases}$$

$$g(x) = \begin{cases} 1 - x, & 0 \le x \le 1, \\ 0, & 1 \le x \le 2, \\ -(x - 2), & 2 \le x \le 3. \end{cases}$$

Claim. $f(x) \mid g(x)$ and $g(x) \mid f(x)$. Or, equivalently, f and g are associates: $f \sim g$.

Proof. Let

$$h(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 3 - 2x, & 1 \le x \le 2, \\ -1, & 2 \le x \le 3. \end{cases}$$

Then we can check that $h(x)$ is continuous, so $h \in R$, and $f(x) = h(x)g(x)$, $g(x) = h(x)f(x)$. Therefore $f \in (g)$ and $g \in (f)$, which implies $f \sim g$. $\square$

Now suppose $h(x)$ is a unit in R, then there exists $k \in R$ such that $h(x)k(x) = 1$. But such a $k(x)$ cannot be continuous, since it has to satisfy $(3 - 2x)k(x) = 1$ on $[1, 2]$, contradiction. Therefore $h \notin U(R)$ is not a unit.

Example 5 (Are associates always unit multiples? Part II)**.**

Here we give another counterexample of the above claim. Consider the ring

Example 6 (E.D.s, P.I.D.s, and U.F.D.s)**.**

1. A *P.I.D.* that is not a *E.D.*:
2. A *U.F.D.* that is not a *P.I.D.*:
3. Rings that are not a *U.F.D.*s:

Example 7 (Ring which is isomorphic to its own square)**.**

Let $R = \prod_{i=1}^{\infty} \mathbb{Z}$. Then $R \cong R \times R$ by the isomorphism

$$\phi : R \to R \times R,\ \phi((x_1, x_2, \dots)) = ((x_1, x_3, \dots), (x_2, x_4, \dots)).$$

2.3 Fields