蔣正偉 Quantum Mechanics I 105-1

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Instruction

1. This is an open-book, 120-minute exam. You are only allowed to use the main textbook (by Sakurai and Napolitano), your own notes and homework assighments. You are allowed to refer to derived results in the textbook only. Please indicate both equation and page numbers when you do so. For example, Eq. (3.1) on p.90 of Sakurai and Napolitano.
2. Each sub-problem has 10 points. The total points is 100. Please arrange your time and do not waste too much time on a single problem.
3. To avoid any misunderstanding, ask if you have any question about the problems or notations.

Source: https://www.ptt.cc/bbs/NTUcourse/M.1485016916.A.0DD.html

1 Problems

Problem 1. Remember the operator identity for operators A and B and a real parameter λ:

$$e^{\lambda A}Be^{-\lambda A} = B + \lambda[A,B] + \frac{\lambda^2}{2!}[A,[A,B]] + \frac{\lambda^3}{3!}[A,[A,[A,B]]] + \cdots . \tag{1}$$

(a) Suppose $[A,[A,B]] = \beta B$ for some constant β, show that

$$e^{\lambda A}Be^{-\lambda A} = B\cosh\lambda\sqrt{\beta} + \frac{[A,B]}{\sqrt{\beta}}\sinh\lambda\sqrt{B}. \tag{2}$$

(b) If $A = A(t)$, show that

$$\frac{\mathrm{d}}{\mathrm{d}t} = e^A\left\{A' - \frac{1}{2!}[A,A'] + \frac{1}{3!}[A,[A,A]] - \cdots\right\}. \tag{3}$$

Problem 2. The wave function $\Psi(\mathbf{r},t)$ expanded in terms of plane waves with definite momenta is

$$\Psi(\mathbf{r},t) = \frac{1}{(2\pi\hbar)^{3/2}}\int \mathrm{d}^3p\,\Phi(\mathbf{p},t)\exp\left[\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r})\right]. \tag{4}$$

(a) Derive the Schrödinger equation for $\Phi(\mathbf{p},t)$, state any assumption used in the derivation.

(b) Simplify the above result by further assuming that the potential $V(\mathbf{r})$ is an analytic function of $\mathbf{r}$.

Problem 3. Suppose the Hamiltonian H for a particular quantum system is a well-behaved (differentiable) function of some parameter λ. Denote E_n and $|\psi_n\rangle$ as the eigenvalues and the corresponding orthonormal eigenkets of $H(\lambda)$. The variable n may be a discrete or continuous set of indices. Assume either that E_n is nondegenerate, or that the eigenkets are the "good" combinations.

(a) Prove the Feynman-Hellmann theorem:

$$\frac{\mathrm{d}E_n}{\mathrm{d}\lambda} = \left\langle \psi_n \left| \frac{\mathrm{d}H}{\mathrm{d}\lambda} \right| \psi_n \right\rangle. \tag{5}$$

(b) Apply the theorem to the one-dimensional simple harmonic oscillator, using $\lambda = m$, and explain what physics you find. Here you can directly make use the energy spectrum $E_n = (n + \frac{1}{2})\hbar\omega$.

Problem 4. Consider a one-dimensional simple harmonic oscillator of mass m and charge q. Suppose the system is placed in a static electric field of strength E. Therefore, the Hamiltonian of this oscillator is given by

$$\frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 - qEx. \tag{6}$$

(a) Suppose the electric field is a constant, i.e., $E = E_0$. Derive the energy level for all states.

(b) Write down the wave function $\psi_0(x)$ for the ground state in Part (a). Determine the most likely position of the oscillator in this ground state and give the physical interpretation for your result.

Problem 5. Work in Heisenberg picture.

(a) Derive the quantum mechanic version of the Lorentz force

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = Qe\left[\mathbf{E} + \frac{1}{2c}\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{B} - \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{B}\right)\right]. \tag{7}$$

(b) Explain whether the terms in the round parentheses are symmetric in the two operators?