2025 Fall Introduction to ODE

Notes for Midterm

物理三 黃紹凱 B12202004

October 16, 2025

1 Chapter 1

1.1 Reduction of Order

Given a second-order linear ODE

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = 0, \tag{1}$$

with one solution $y(x) = u_1(x)$ known, assume the second solution is of the form $u_2(x) = u_1(x)U(x)$. Find $U(x)$:

$$U(x) = \int^x \mathrm{d}t\, \frac{1}{u_1^2(t)} \exp\left(-\int^t \mathrm{d}s\, p(s)\right). \tag{2}$$

1.2 Variation of Parameters

For an initial value problem, the solution has the form

$$y(x) = c_1 u_1(x) + c_2 u_2(x) + g(x),$$

where $u_1(x), u_2(x)$ are the complementary functions, and $g(x)$ is the particular integral. Find the particular integral given the complementary functions:

$$y(x) = \int^x \mathrm{d}s\, f(s) \frac{\begin{vmatrix} u_1(s) & u_2(s) \\ u_1(x) & u_2(x) \end{vmatrix}}{\begin{vmatrix} u_1(s) & u_2(s) \\ u_1'(s) & u_2'(s) \end{vmatrix}} = \int^x \mathrm{d}s\, f(s) \frac{u_1(s)u_2(x) - u_2(s)u_1(x)}{W(s)}, \tag{3}$$

where $W(s)$ is the **Wronskian**.

Find Wronskian without constructing the solutions: **Abel's formula**.

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^x \mathrm{d}t\, p(t)\right). \tag{4}$$

1.3 Method of Frobenius

Indicial equation:

1. Roots differ by integer and one of the coefficients is indeterminate when $c = \alpha$: Obtain both solutions with $c = \alpha$.
2. Roots differ by non-integer: Obtain two solutions with $c = \alpha$ and $c = \beta$.
3. Double root: Obtain one solution with $c = \alpha$, and obtain the second with $(\partial y/\partial c)|_{c=\alpha}$.

2 Chapter 2

2.1 Legendre's Differential Equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \quad n = 0, 1, 2, \ldots. \tag{5}$$

The general solution is

$$y(x) = AP_n(x) + BQ_n(x), \tag{6}$$

where $P_n(x)$ is a degree n polynomial, and $Q_n(x)$ is an infinite series converging for $|x| < 1$.

2.2 Generating Function

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad |t| < 1. \tag{7}$$

Then we have **Bonnet's formula** and a recurrence relation for derivatives:

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0, \tag{8}$$

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x). \tag{9}$$

2.3 Rodrigue's Formula

Rodrigue's formula gives an explicit expression for $P_n(x)$:

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n. \tag{10}$$

From this we have **Schläfli's representation**:

$$P_n(z) = \frac{1}{2^{n+1}\pi i}\oint_{\mathcal{C}} \mathrm{d}\xi\, \frac{(\xi^2-1)^n}{(\xi-z)^{n+1}}, \tag{11}$$

where $\mathcal{C}$ is a simple closed contour enclosing z. Then we have **Laplace's representation**:

$$P_n(z) = \frac{1}{2\pi}\int_0^{\pi} \mathrm{d}\theta\, \left(z + \sqrt{z^2-1}\cos\theta\right)^n. \tag{12}$$

2.4 Orthogonality

Legendre polynomials are orthogonal on the interval $[-1, 1]$:

$$\int_{-1}^{1} \mathrm{d}x\, P_n(x)P_m(x) = \frac{2}{2n+1}\delta_{nm}. \tag{13}$$

They form a complete basis for piecewise continuous functions on $[-1, 1]$, so we have the **Fourier-Legendre series**:

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \quad a_n = \frac{2n+1}{2}\int_{-1}^{1} \mathrm{d}t\, f(t)P_n(t). \tag{14}$$

2.5 Associated Legendre Equation

The associated Legendre equation is

$$(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0, \quad n = 0, 1, 2, \ldots, \quad m = 0, 1, 2, \ldots, n. \tag{15}$$

The general solution is

$$y(x) = AP_n^m(x) + BQ_n^m(x), \tag{16}$$

where $P_n^m(x), Q_n^m(x)$ are the associated Legendre functions, $P_n^m(x) = 0$ when $m > n$, and $Q_n^m(x)$ is singular at $x = \pm 1$. We have

$$P_n^m(x) = (1 - x^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_n(x). \tag{17}$$

The **spherical harmonics** that solve the angular part of the spherical Laplace equation are defined as

$$Y_n^m(\theta, \phi) = (-1)^m \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta) e^{im\phi}, \quad n = 0, 1, 2, \ldots, \quad m = -n, -n+1, \ldots, n. \tag{18}$$

3 Chapter 3

3.1 The Bessel Equation

The Bessel equation is

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0. \tag{19}$$

When ν is not a half-integer, the general solution is

$$y(x) = AJ_\nu(x) + BJ_{-\nu}(x), \tag{20}$$

where

$$J_{\pm\nu} = \frac{x^{\pm\nu}}{2^{\pm\nu}\Gamma(1\pm\nu)} \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n}}{n!(n+1\pm\nu)_n} \tag{21}$$

and $(\alpha)_r = \alpha(\alpha+1)\cdots(\alpha+r-1)$ is the **Pochhammer symbol**.

When $\nu = 0$, the other solution is **Weber's Bessel function** of order zero:

$$Y_0(x) = J_0(x)\log x - \sum_{n=0}^{\infty} (-1)^n \frac{H_n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}, \tag{22}$$

where H_n is the n-th **harmonic number**. When ν is not a positive integer, the other solution is

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}. \tag{23}$$

3.2 Generating Function

For integer n, the Bessel function is

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n}}{2^{2k+n}k!(k+n)!}. \tag{24}$$

The generating function is

$$e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n. \tag{25}$$

We have the alternative integral representation

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta\, \cos\left(n\theta - x\sin\theta\right). \tag{26}$$

3.3 Recurrence Relations

The Bessel functions and their derivatives satisfy the recurrence relations

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x), \tag{27}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^\nu N_\nu(x)\right) = x^\nu N_{\nu-1}(x), \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-\nu} N_\nu(x)\right) = -x^{-\nu} N_{\nu+1}(x). \tag{28}$$

where N_n denotes either J_n or Y_n.

3.4 Orthogonality

The Bessel functions are orthogonal on the interval $[0,1]$ with respect to the weight x:

$$\int_0^a \mathrm{d}x\, x J_\nu(\lambda_m x) J_\nu(\lambda_n x) = \frac{a^2\delta_{mn}}{2}[J_{\nu+1}(\lambda_m)]^2, \tag{29}$$

so we have the **Fourier-Bessel series**:

$$f(x) = \sum_{n=1}^{\infty} a_n J_\nu(\lambda_n x), \quad a_n = \frac{2}{a^2[J_{\nu+1}(\lambda_n)]^2}\int_0^a \mathrm{d}t\, t f(t) J_\nu(\lambda_n t). \tag{30}$$

3.5 Solutions Expressible as Bessel Functions

The modified Bessel equation is

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0, \tag{31}$$

with solutions

$$I_\nu(x) = i^{-\nu}J_\nu(ix), \quad K_\nu(x) = \frac{\pi}{2}i^{\nu+1}H_\nu^{(1)}(ix), \tag{32}$$

where $H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$ is the **Hankel function of the first kind**.

The transformation

4 Chapter 6

4.1 Integral Transforms

Integral transforms are of the form

$$F(s) = \int_0^\infty \mathrm{d}t\, K(s,t) f(t), \tag{33}$$

where $K(s,t)$ is the kernel of the transform. The Fourier transform, Laplace transform, Mellin transform, and Hankel transform are all examples of integral transforms.

4.2 Laplace Transforms

Definition of a Laplace transform:

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty \mathrm{d}t\, e^{-st} f(t). \tag{34}$$

Theorem 1 (Lerch)**.** A function $f(t)$ is said to be of exponential order on $[0,\infty)$ if there exist constants $M, c, T > 0$ such that

$$|f(t)| \leq Me^{ct}, \quad t > T.$$

If $f(t)$ is piecewise continuous on every finite interval in $[0,\infty)$ and of exponential order, then the Laplace transform $F(s) = \mathcal{L}[f(t)]$ exists for $s > c$.

Theorem 2 (Existence of Laplace Transform)**.** If f is of exponential order on $[0,\infty)$, then $\mathcal{L}[f] \to 0$ as $s \to \infty$.

A useful Laplace transform identity for calculating integrals is

$$\int_0^\infty \mathrm{d}x\, f(x)g(x) = \int_0^\infty \mathrm{d}x\, (\mathcal{L}f)(x)(\mathcal{L}^{-1}g)(x) = \int_0^\infty \mathrm{d}x\, (\mathcal{L}^{-1}f)(x)(\mathcal{L}g)(x) \tag{35}$$

4.3 Convolution Theorem

4.4 Inverse Laplace Transform

The inverse Laplace transform can be found using the **Bromwich integral**:

$$g(t) = \mathcal{L}^{-1}[G(s)] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathrm{d}s\, e^{st} G(s), \tag{36}$$

where γ is the smallest real number such that $e^{-\gamma t} g(t)$ is bounded as $t \to \infty$.