2025 Fall Introduction to ODE

Homework 8 (Due November 10 12:00, 2025)

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Exercise 1. Suppose a, b, c are nonnegative continuous functions on $[0, \infty)$, u is a nonnegative bounded continuous solution of the inequality

$$u(t) \leq a(t) + \int_0^t b(t-s)u(s)\,ds + \int_0^\infty c(s)u(t+s)\,ds, \quad t \geq 0,$$

and $a(t) \to 0$, $b(t) \to 0$ as $t \to \infty$, $\int_0^\infty b(s)\,ds < \infty$, $\int_0^\infty c(s)\,ds < \infty$. Prove that $u(t) \to 0$ as $t \to \infty$ if

$$\int_0^\infty b(s)\,ds + \int_0^\infty c(s)\,ds < 1.$$

Solution 1.

Steps:

1. Choose a sequence that approaches the limsup of $u(t)$ as $t \to \infty$.
2. Split the integral appropriately and estimate the terms.
3. Combine everything and show that the limsup must be zero.

Method:

1. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence such that $t_n \to \infty$ as $n \to \infty$ and $u(t_n) \to L = \limsup_{t\to\infty} u(t)$. By a change of variables, we have

$$u(t) \leq a(t) + \int_0^t \mathrm{d}r\, b(r)u(t-r) + \int_0^\infty \mathrm{d}s\, c(s)u(t+s).$$

2. Since $b(t) \geq 0$ and $\int_0^\infty \mathrm{d}s\, b(s) < \infty$, we have $\lim_{R\to\infty} \int_R^\infty \mathrm{d}s\, b(s) = 0$. This holds similarly for $c(t)$, so there exist $R_b, R_c > 0$ such that

$$\int_{R_b}^\infty \mathrm{d}s\, b(s) < \varepsilon, \quad \int_{R_c}^\infty \mathrm{d}s\, c(s) < \varepsilon.$$

By the definition of L, there exists $T_\varepsilon > 0$ such that $u(t) \leq L + \varepsilon$ for all $t \geq T_\varepsilon$. Let's take $T = T_\varepsilon + \max\{R_b, R_c\}$, and n sufficiently large such that $t_n \geq T$.

3. Now we will estimate each term at $t = t_n$: First, since $a(t) \to 0$ as $t \to \infty$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $a(t_n) < \varepsilon$. Next, we split the first integral:

$$\int_0^{t_n} \mathrm{d}r\, b(r)u(t_n - r) = \int_0^{R_b} \mathrm{d}r\, b(r)u(t_n - r) + \int_{R_b}^{t_n} \mathrm{d}r\, b(r)u(t_n - r).$$

The first term is bounded by $(L+\varepsilon)\int_0^{R_b} \mathrm{d}r\, b(r)$, and the second term is bounded by $M\int_{R_b}^\infty \mathrm{d}r\, b(r) < M\varepsilon$, where $M = \sup_{t\geq 0} u(t)$. Similarly, we split the second integral:

$$\int_0^\infty \mathrm{d}s\, c(s)u(t_n + s) = \int_0^{R_c} \mathrm{d}s\, c(s)u(t_n + s) + \int_{R_c}^\infty \mathrm{d}s\, c(s)u(t_n + s).$$

The first term is bounded by $(L+\varepsilon)\int_0^{R_c} \mathrm{d}s\, c(s)$, and the second term is bounded by $M\int_{R_c}^{\infty} \mathrm{d}s\, c(s) < M\varepsilon$. Combining everything, we have for all n sufficiently large,

$$
\begin{aligned}
u(t_n) &\le a(t_n) + \int_0^{t_n} \mathrm{d}r\, b(r)u(t_n - r) + \int_0^{\infty} \mathrm{d}s\, c(s)u(t_n + s) \\
&= \varepsilon + (L+\varepsilon)\int_0^{R_b} \mathrm{d}r\, b(r) + M\varepsilon + (L+\varepsilon)\int_0^{R_c} \mathrm{d}s\, c(s) + M\varepsilon \\
&= (L+\varepsilon)\left(\int_0^{R_b} \mathrm{d}r\, b(r) + \int_0^{R_c} \mathrm{d}s\, c(s)\right) + \varepsilon(1+2M).
\end{aligned}
$$

Rearranging gives

$$
L \le \frac{1+B+C+2M}{1-(B+C)}\varepsilon,
$$

where $B = \int_0^{\infty} \mathrm{d}r\, b(r)$ and $C = \int_0^{\infty} \mathrm{d}s\, c(s)$. Since $\varepsilon > 0$ is arbitrary, we conclude that $L \le \limsup_{t\to\infty} u(t) = 0$. Since $u(t) \ge 0$, we have $\lim_{t\to\infty} u(t) = 0$.

Exercise 2. For any real matrix D, $\det D \neq 0$, show there is a real matrix B such that $e^B = D^2$. If C is a real matrix in Lemma 7.1 and there is a real matrix B such that $e^B = C$, must there exist a real matrix D such that $C = D^2$?

Solution 2.

Steps:

1. Write down the real Jordan form of D.
2. Prove the existence of logarithm of D^2, and hence the existence of B such that $e^B = D^2$.
3. Discuss whether there must exist a real matrix D such that $C = D^2$.

Method:

1. Let $D \in M_n(\mathbb{R})$ be a real invertible matrix. Let λ be an eigenvalue of D with eigenvector v, so $Dv = \lambda v$. Then, we have

$$
D^2 v = D(\lambda v) = \lambda D v = \lambda^2 v.
$$

Therefore, the eigenvalues of D^2 are $\lambda_i^2 > 0$, where λ_i are the eigenvalues of D and are nonzero since $\det D \neq 0$. By the **Real Jordan Form Theorem**, there is a matrix $P \in \mathrm{GL}_n(\mathbb{R})$ such that

$$
D = PJP^{-1},
$$

where J is block-diagonal with blocks of two types:

(1) Real eigenvalue λ: real Jordan blocks of the form

$$
J = \begin{pmatrix}
\lambda & 1 & & & 0 \\
 & \lambda & 1 & & \\
 & & \ddots & \ddots & \\
 & & & \lambda & 1 \\
0 & & & & \lambda
\end{pmatrix}.
$$

(2) Complex eigenvalue pair $\alpha \pm i\beta$, $\beta > 0$: a block $K \in M_{2m\times 2m}(\mathbb{R})$ of the form

$$
K = \begin{pmatrix}
C & I_2 & & & 0 \\
 & C & I_2 & & 0 \\
 & & \ddots & \ddots & \\
 & & & C & I_2 \\
0 & & & & C
\end{pmatrix},
$$

where

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in M_{2\times 2}(\mathbb{R}).$$

For the real eigenvalues, replace each Jordan block $J(\lambda_k)$ corresponding to eigenvalue λ_k^2 by a Jordan block $\tilde{J}(\lambda_k)$ with eigenvalue $\log(\lambda_k^2)$, which is real since $\lambda_k^2 > 0$. For complex eigenvalue pairs, let's define the standard representation of complex matrices: let $M = X + iY \in M_n(\mathbb{C})$ be a complex matrix with $X, Y \in M_n(\mathbb{R})$, then its standard representation is given by

$$\Phi(M) = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in M_{2n}(\mathbb{R}).$$

Then $\Phi : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$ is an injective homomorphism, and $\Phi(e^M) = e^{\Phi(M)}$. Then, notice that $K_k = \Phi(J(\mu_k))$, where $J(\mu_k)$ is the Jordan block corresponding to the complex eigenvalue $\mu_k = \alpha_k + i\beta_k$ of D^2. Construct the logarithm of $J(\mu_k) \in M_m(\mathbb{C})$ as

$$L_c(\mu_k) = \log(\mu_k^2)I + \log(I + N), \quad N = \frac{J(\mu_k^2) - \mu_k^2 I}{\mu_k^2} \text{ is nilpotent},$$

where

$$\log(I + N) = \sum_{j=1}^{m-1} \frac{(-1)^{j+1}}{j} N^j.$$

Let $\tilde{J}(\mu_k) = \Phi(L_c(\mu_k)) \in M_{2m}(\mathbb{R})$ be the real logarithm, where $\exp\left(\tilde{J}(\mu)\right) = e^{\Phi(L_c)} = \Phi(e^{L_c}) = \Phi(J(\mu_k)) = K_k$. Assemble all the $\tilde{J}_k = \tilde{J}(\nu_k)$, where $\nu = \lambda$ or μ, into a real block-diagonal matrix J_B, then $J^2 = e^{J_B}$, and we have

$$D^2 = PJ^2P^{-1} = Pe^{J_B}P^{-1} = e^{PJ_BP^{-1}} \equiv e^B.$$

Thus, such a real matrix B exists.

2. First we begin with a lemma from the textbook:

   Lemma 1 (Hale, Lemma 7.1)**.** *If C is an $n \times n$ matrix with $\det C \neq 0$, then there is a matrix B such that $e^B = C$.*

   That is, any invertible matrix has a logarithm. We will show that any such matrix has a square root as well. Since there is a matrix B such that $e^B = C$, consider the matrix $D = e^{B/2}$, then
   $$D^2 = e^{B/2}e^{B/2} = e^B = C$$
   since $\frac{B}{2}$ commutes with itself. Thus, such a real matrix D exists.

Exercise 3. Let $X = X(t) \in \mathbb{R}^{n\times n}$ be the fundamental matrix solution of $E\dot{\vec{x}} = H(t)\vec{x}$, where E is given by

$$E = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$$

and $H = H(t)$ is symmetric and periodic with period T. Suppose H is a nonconstant matrix, and let $\mathbb{H} = \{X(t) : t \in \mathbb{R}\}$. Must $\mathbb{H}$ be a subgroup of the symplectic group $G := \{M \in \mathbb{R}^{2k\times 2k} : M'EM = E\}$? Justify your answer. Here, M' denotes the transpose of the matrix M.

Solution 3.

Steps:

1. Show that $X(t)$ is symplectic for all $t \in \mathbb{R}$, and hence $\mathbb{H} \subseteq G$.
2. Show that $\mathbb{H}$ is not closed under matrix multiplication with a counterexample. Hence, $\mathbb{H}$ is not a subgroup of G.

Method:

1. Let $X(t)$ be the fundamental matrix solution of the system $E\dot{\vec{x}} = H(t)\vec{x}$. We will show that $X(t)$ is symplectic for all $t \in \mathbb{R}$. Consider the matrix $Y(t) = X(t)'EX(t)$. Differentiating $Y(t)$ with respect to t, we have

$$\frac{dY}{dt} = \dot{X}(t)'EX(t) + X(t)'E\dot{X}(t).$$

Taking transpose of $\dot{X}(t) = E^{-1}H(t)X(t)$, we have $(E\dot{X})' = (H(t)X)'$. UUsing $E' = -E$, $H' = H$, we get $\dot{X}'E = -X'H$. Substituting back gives $\dot{Y} = 0$. At $t = 0$, $X(t) = I$, so $X'(t)EX(t) = Y(t) = Y(0) = I$. Therefore, $X(t)$ is symplectic for all $t \in \mathbb{R}$, and $\mathbb{H} \subseteq G$.

2. To show that $\mathbb{H}$ is not a subgroup of G, we will show that matrix multiplication is not closed in $\mathbb{H}$ by constructing a counterexample with $k = 1$. Let

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $H(t) = \sin t I_2$, where I_2 is the 2×2 identity matrix. Then $H(t)$ is non-constant in time, symmetric, and periodic with period 2π. The system $E\dot{x} = H(t)x$ can be written as

$$\begin{cases} \dot{x_2} &= \sin t x_1, \\ \dot{x_1} &= -\sin t x_2, \end{cases}$$

which is easily solved by the change of variable $z = x_1 + ix_2$. The solution is

$$z(t) = z(0)\exp\left(\int_0^t \sin s \, ds\right) = z(0)e^{i(1-\cos t)},$$

and the fundamental matrix solution is a rotational matrix given by

$$X(t) = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix}, \text{where } \theta(t) = 1 - \cos t.$$

As can be seen, $X(t)$ is symplectic, and since $\cos t \in [-1, 1]$, $\theta \in [0, 2]$, our set $\mathbb{H}$ is given by

$$\mathbb{H} = \{X(t) \mid t \in \mathbb{R}\} = \left\{ \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \mid \phi \in [0, 2] \right\}.$$

Now, pick t_1, t_2 such that $\phi_1 = \theta(t_1)$, $\phi_2 = \theta(t_2)$ are equal to $\frac{3}{2}$. Then, by property of rotation matrices, we have

$$X(t_1)X(t_2) = \begin{pmatrix} \cos 3 & -\sin 3 \\ \sin 3 & \cos 3 \end{pmatrix},$$

where none of $3 + 2n\pi$ lies inside $[0, 2]$. Therefore, $X(t_1)X(t_2) \notin \mathbb{H}$, and $\mathbb{H}$ is not closed under matrix multiplication. Thus, $\mathbb{H}$ is not a subgroup of G.