Chapter 4

Normal Forms

Introduction

Note.

(i) An important problem is now to know how to bring a fast-slow system into a **normal form**.

(ii) There is no complete general theory for what a "normal form" for a fast-slow system should be.

(iii) Some transformations can be used for specific classes of systems to bring them into fast-slow form.

4.1 The Normally Hyperbolic Case

4.2 Fold Points

Having demonstrated how to bring a normally hyperbolic system into fast-slow form, we now ask whether there are "normal forms" for singularities of the critical manifold. Here, we consider the case when the critical manifold contains a **fold point**, i.e., a point where normal hyperbolicity is lost.

4.3 Fold Curves

Next, we ask whether our previous normal form approach can be extended to folds in systems with more than one slow variable, that is, **fold curves**.

4.4 Systems of First Approximation

Remark. The next results truly separate the theory of *fast-slow systems* from *classical bifurcation theory*.

4.5 A Note On Linear Systems

We have applied various techniques to simplify fast-slow systems given in the **standard form**

$$\begin{aligned} \varepsilon\frac{\mathrm{d}x}{\mathrm{d}\tau} &= \dot{x} = f(x,y,\varepsilon), \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} &= \dot{y} = g(x,y,\varepsilon), \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $0 < \varepsilon \ll 1$. However, not all fast-slow systems can be transformed into this useful form, so we must consider techniques to analyze more general systems. To begin, consider a more general system of the form

$$\varepsilon\frac{\mathrm{d}z}{\mathrm{d}\tau} = \varepsilon\dot{z} = F(z,\varepsilon), \quad z\in\mathbb{R}^N, \quad F(\varepsilon)\in M_{N\times N}(\mathbb{R}). \tag{4.2}$$

We can make the decomposition $F(z) = F_0 + \varepsilon F_1(\varepsilon)$, where $F_0 = F(0)$. Our goal is to find a suitable transformation that brings this system into fast-slow form, conveniently written as

$$\begin{aligned}\varepsilon\dot{x} &= A_{11}x + \varepsilon A_{12}y,\\ \dot{y} &= A_{21}x + A_{22}y,\end{aligned} \tag{4.3}$$

where the matrix A given by

$$A = \begin{pmatrix} A_{11}(\varepsilon) & A_{12}(\varepsilon)\\ A_{21}(\varepsilon) & A_{22}(\varepsilon)\end{pmatrix} \in \mathrm{GL}(N;\mathbb{R}). \tag{4.4}$$

Example (Calculation of normal form). A concrete 2×2 example is given by

$$\frac{dz}{dt} = \underbrace{\begin{pmatrix} -1.2 & 1.6\\ 0.6 & -0.8\end{pmatrix}}_{=:F_0} z + \varepsilon\underbrace{\begin{pmatrix} 0.2 & -0.6\\ 0.4 & -1.2\end{pmatrix}}_{=:F_1} z, \tag{4.5}$$

(with $t = \tau/\varepsilon$). Although (??) is not in standard form, its phase portrait and time series show a clear fast transient followed by slow evolution.

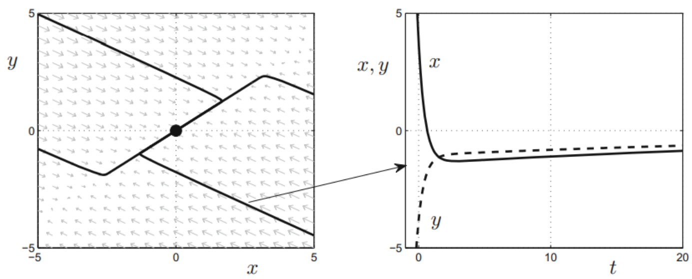

Figure 4.1: Phase portrait and time series for

Note (Splitting assumption). Let $\dim\big(\ker(F_0)\big) = n$ and set $N = m+n$. Assume that there exists the direct-sum decomposition

$$\mathrm{im}(F_0)\oplus\ker(F_0) = \mathbb{R}^{m+n}. \tag{4.6}$$

This is the linear analogue of the Fenichel-type splitting for normally hyperbolic fast-slow systems: every vector $z\in\mathbb{R}^{m+n}$ can be uniquely decomposed as

$$z = z_{\text{fast}} + z_{\text{slow}}, \quad z_{\text{fast}}\in\ker(F_0), \quad z_{\text{slow}}\in\mathrm{im}(F_0).$$

it separates the *fast directions* coming from $\mathrm{im}(F_0)$ from the *slow directions* coming from $\ker(F_0)$.

How can we construct fast and slow coordinates?

- Choose m linearly independent row vectors orthogonal to $\ker(F_0)$ and use them as rows of a matrix $Q\in\mathbb{R}^{m\times(m+n)}$. Then $\ker(F_0) = \ker(Q)$, and thus

$$\nu\in\ker(F_0) \Leftrightarrow F_0\nu = 0 \Leftrightarrow Q\nu = 0,$$

so $x := Qz$ is a natural candidate for the fast variable(s). If z lies purely in the slow directions, then $x = Qz = 0$. For this, see the following example.

- Choose $P\in\mathbb{R}^{n\times(m+n)}$ from the *left nullspace* of F_0 so that

$$PF_0 = 0.$$

Then $y := Pz$ is the natural candidate for the slow variable(s). This is because if the linear fast dynamics is $\dot{x} = F_0 z$, then

$$\dot{y} = \frac{\mathrm{d}}{\mathrm{d}t}(Pz) = P\dot{z} = PF_0 z = 0.$$

Therefore, y is constant under the fast flow, and thus it is a slow variable.

- Stack the two maps into an invertible matrix (since the rows of Q and P span $\mathbb{R}^{m+n}$):

$$T := \begin{pmatrix} Q \\ P \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}, \qquad \begin{pmatrix} x \\ y \end{pmatrix} = Tz.$$

T is the linear analogue of *Fenichel coordinates.*

Example. For the same F_0, one computes

$$\ker(F_0) = \operatorname{span}\left(\begin{pmatrix} -0.8 \\ -0.6 \end{pmatrix}\right).$$

A vector orthogonal to this nullspace is $(0.6, -0.8)^\top$, hence one can take

$$Q = (0.6,\ -0.8) : \mathbb{R}^2 \to \mathbb{R}, \qquad x := Qz,$$

so that $x = 0$ on the line corresponding to the critical manifold candidate.

Theorem 4.5.1 (Kuehn 4.5.3 normal form for linear systems)**.** Suppose $F(0) = F_0$ satisfies the decomposition

$$\operatorname{im}(F_0) \oplus \ker(F_0) = \mathbb{R}^{m+n} = \mathbb{R}^N.$$

as mentioned above. Then the coordinate change

$$x = Qv, \quad y = Pv, \quad T := \begin{pmatrix} Q \\ P \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}$$

transforms the system (**??**) into the normal form

$$\begin{aligned} \varepsilon\dot{x} &= A_{11}(\varepsilon)x + \varepsilon A_{12}(\varepsilon)y, \\ \dot{y} &= A_{21}(\varepsilon)x + A_{22}(\varepsilon)y, \end{aligned} \tag{4.7}$$

where $A_{11}(0) \in \mathbb{R}^{m\times m}$ is nonsingular, $A_{12}(\varepsilon) \in \mathbb{R}^{m\times n}$, $A_{21}(\varepsilon) \in \mathbb{R}^{n\times m}$, and $A_{22}(\varepsilon) \in \mathbb{R}^{n\times n}$.

Proof. One convenient way to express the blocks is as follows: let the columns of V and W be bases of $\operatorname{im}(F_0)$ and $\ker(F_0)$, respectively, so that $T^{-1} = (V\ W)$. Then the blocks can be chosen as

$$\begin{aligned} A_{11}(\varepsilon) &:= QF_0V + \varepsilon\, QF_1(\varepsilon)V, \\ A_{12}(\varepsilon) &:= QF_1(\varepsilon)W, \\ A_{21}(\varepsilon) &:= PF_1(\varepsilon)V, \\ A_{22}(\varepsilon) &:= PF_1(\varepsilon)W. \end{aligned}$$

By construction $A_{11}(0) = QF_0V$ is invertible, and it is the only nonzero block of TF_0T^{-1}.

Explanation:

(1) Assume the splitting $\operatorname{Im}(F_0) \oplus \ker(F_0) = \mathbb{R}^{m+n}$, or equivalently, $\operatorname{Im}(F_0) \cap \ker(F_0) = \{0\}$. Notice that the restriction $F_0|_{\operatorname{Im}(F_0)} : \operatorname{Im}(F_0) \to \operatorname{Im}(F_0)$ is injective, for if $w \in \operatorname{Im}(F_0)$ and $F_0 w = 0$, then $w \in \operatorname{Im}(F_0) \cap \ker(F_0) = \{0\}$, hence $w = 0$. Since $\dim(\operatorname{Im}(F_0)) = m$ is finite, $F_0|_{\operatorname{Im}(F_0)}$ is a bijection, and hence an isomorphism.

Next, choose $Q \in \mathbb{R}^{m\times(m+n)}$ such that $\ker(Q) = \ker(F_0)$. Then the restriction $Q|_{\operatorname{Im}(F_0)}$:

$\mathrm{Im}(F_0) \to \mathbb{R}^m$ is also injective. Hence, $Q|_{\mathrm{Im}(F_0)}$ is an isomorphism onto $\mathbb{R}^m$. Consequently,

$$\mathrm{Im}(F_0) \xrightarrow{F_0} \mathrm{Im}(F_0) \xrightarrow{Q} \mathbb{R}^m$$

is an isomorphism. Writing this map in the basis V of $\mathrm{Im}(F_0)$ yields the matrix

$$A_{11}(0) = QF_0V \in \mathbb{R}^{m\times m},$$

which must therefore be invertible.

(2) Let

$$T := \begin{pmatrix} Q \\ P \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}, \qquad T^{-1} = \begin{pmatrix} V & W \end{pmatrix},$$

where the columns of V form a basis of $\mathrm{Im}(F_0)$ and the columns of W form a basis of $\ker(F_0)$. Then

$$TF_0T^{-1} = \begin{pmatrix} Q \\ P \end{pmatrix} F_0 \begin{pmatrix} V & W \end{pmatrix} = \begin{pmatrix} QF_0V & QF_0W \\ PF_0V & PF_0W \end{pmatrix}. \tag{4.8}$$

Since $W \subset \ker(F_0)$, we have $F_0W = 0$, hence $QF_0W = 0$. Moreover, by construction P lies in the left nullspace of F_0, so $PF_0 = 0$, which implies $PF_0V = PF_0W = 0$. Therefore,

$$TF_0T^{-1} = \begin{pmatrix} QF_0V & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.9}$$

■

Note (summary). How do we know whether a (possibly nonlinear) system has a multiple time scale structure? Theorem 4.5.3 provides a partial answer for linear systems of the form (**??**). We should:

(i) Consider a linearized system.

(ii) Identify a small parameter.

(iii) Consider the eigenvalue structure of the singular limit system.

It often helps to identify the fast and slow variables and rewrite the system in fast-slow normal form.

Exercise (4.5.4). We start from Example 4.5.1:

$$\frac{dz}{dt} = F_0z + \varepsilon F_1 z, \qquad F_0 = \begin{pmatrix} -1.2 & 1.6 \\ 0.6 & -0.8 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0.2 & -0.6 \\ 0.4 & -1.2 \end{pmatrix}, \tag{4.10}$$

with $t = \tau/\varepsilon$. Hence

$$\dot{z} = \frac{dz}{d\tau} = \frac{1}{\varepsilon}F_0z + F_1z, = \left(\frac{1}{\varepsilon}F_0 + F_1\right)z. \tag{4.11}$$

Then, we bring this system into fast-slow form by following the steps in the proof of Theorem 4.5.3:

(1) Compute $\ker(F_0)$ and pick Q: A direct computation (Example 4.5.2) gives

$$\ker(F_0) = \mathrm{span}\left(\begin{pmatrix} -0.8 \\ -0.6 \end{pmatrix}\right). \tag{4.12}$$

A vector orthogonal to $(-0.8, -0.6)^\top$ is $(0.6, -0.8)^\top$, so we take

$$Q = \begin{pmatrix} 0.6 & -0.8 \end{pmatrix}, \qquad x := Qz. \tag{E4}$$

(2) Pick P from the left nullspace and define y: Choose P so that $PF_0 = 0$ (a basis for the left nullspace of F_0). Let $P = (a \;\; b)$. Then

$$(a \;\; b)\begin{pmatrix} -1.2 & 1.6 \\ 0.6 & -0.8 \end{pmatrix} = (0 \;\; 0) \quad \Longrightarrow \quad -1.2a + 0.6b = 0 \Longrightarrow b = 2a.$$

We may take $a = 1$, hence

$$P = \begin{pmatrix} 1 & 2 \end{pmatrix}, \qquad y := Pz. \tag{4.13}$$

(3) Form T and its inverse: Define

$$T := \begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} 0.6 & -0.8 \\ 1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} x \\ y \end{pmatrix} = Tz. \tag{4.14}$$

One checks $\det(T) = 2 \neq 0$, so T is invertible and

$$T^{-1} = \begin{pmatrix} 1 & 0.4 \\ -0.5 & 0.3 \end{pmatrix} = \begin{pmatrix} V & W \end{pmatrix}, \qquad V = \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}, \quad W = \begin{pmatrix} 0.4 \\ 0.3 \end{pmatrix}. \tag{4.15}$$

(Indeed, V spans $\operatorname{im}(F_0)$ and W spans $\ker(F_0)$.)

(4) Compute the normal-form blocks: From the proof of Theorem 4.5.3, in (x, y)-coordinates

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = T\left(\frac{1}{\varepsilon}F_0 + F_1\right)T^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\varepsilon}QF_0V + QF_1V & QF_1W \\ PF_1V & PF_1W \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.16}$$

Therefore the standard fast-slow form

$$\varepsilon\dot{x} = A_{11}(\varepsilon)x + \varepsilon A_{12}(\varepsilon)y, \qquad \dot{y} = A_{21}(\varepsilon)x + A_{22}(\varepsilon)y \tag{4.17}$$

is obtained with

$$A_{11}(\varepsilon) = QF_0V + \varepsilon QF_1V, \quad A_{12}(\varepsilon) = QF_1W, \quad A_{21}(\varepsilon) = PF_1V, \quad A_{22}(\varepsilon) = PF_1W. \tag{4.18}$$

Now compute each term (here $F_1(\varepsilon) = F_1$ is constant):

$$QF_0V = -2, \qquad QF_1V = -\frac{1}{2}, \qquad QF_1W = \frac{1}{10}, \tag{4.19}$$

$$PF_1V = \frac{5}{2}, \qquad PF_1W = -\frac{1}{2}. \tag{4.20}$$

Hence

$$A_{11}(\varepsilon) = -2 - \frac{\varepsilon}{2}, \qquad A_{12}(\varepsilon) = \frac{1}{10}, \qquad A_{21}(\varepsilon) = \frac{5}{2}, \qquad A_{22}(\varepsilon) = -\frac{1}{2}. \tag{4.21}$$

The final answer is given as:

$$\boxed{\varepsilon\dot{x} = \left(-2 - \frac{\varepsilon}{2}\right)x + \varepsilon\left(\frac{1}{10}\right)y, \qquad \dot{y} = \left(\frac{5}{2}\right)x - \left(\frac{1}{2}\right)y.} \tag{4.22}$$