2025 Fall Nonlinear Optics

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Homework #1 (Due Sep 18, 2025)
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Problem 1. Consider a two-dimensional system with square symmetry (D_4 symmetry). There is a set of 8 symmetry transformations.

E	identity
R^+	rotation by $\pi/2$
R^-	rotation by $-\pi/2$
R	rotation by π
M_x	mirror symmetry about the x axis
M_y	mirror symmetry about the y axis
D_1	mirror symmetry about diagonal D_1 ($y = x$)
D_2	mirror symmetry about diagonal D_2 ($y = -x$)

The product of any two symmetyr transformaations T_2T_1 may be defined as the transformation which results from performing T_1 followed by T_2. For example, $M_xR^+ = D_2$.

(a) Construct the multiplication table for the symmetry operations.

(b) Now consider the permittivity tensor ϵ. For a general system in two dimensions with no symmetry, ϵ is of the form:

$$\epsilon = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_{yy} \end{pmatrix}. \tag{1}$$

Show that for a system with square symmetry (D_4 symmetry), ϵ must be **isotropic** :

$$\epsilon = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}. \tag{2}$$

Solution 1.

(a) The multiplication table is given as follows:

	E	R_+	R_-	R	M_x	M_y	D_1	D_2
E	E	R_+	R_-	R	M_x	M_y	D_1	D_2
R_+	R_+	R	E	R_-	D_2	D_1	M_y	M_x
R_-	R_-	E	R	R_+	D_2	D_1	M_x	M_y
R	R	R_-	R_+	E	M_y	M_x	D_2	D_1
M_x	M_x	D_2	D_1	M_y	E	R	R_-	R_+
M_y	M_y	D_1	D_2	M_x	R	E	R_+	R_-
D_1	D_1	M_y	M_x	D_2	R_-	R_+	E	R
D_2	D_2	M_x	M_y	D_1	R_+	R_-	R	E

(b) The elements of the D_4 symmetry group have a group representation in $\mathcal{M}_{2\times 2}(\mathbb{R})$, which will aid the calculation of symmetry transformations on the permittivity tensor. The representation is listed below:

E	identity	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$
R_+	rotation by $\pi/2$	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$
R_-	rotation by $-\pi/2$	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$
R	rotation by π	$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$
M_x	mirror symmetry about the x axis	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
M_y	mirror symmetry about the y axis	$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$
D_1	mirror symmetry about diagonal D_1 ($y = x$)	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
D_2	mirror symmetry about diagonal D_2 ($y = -x$)	$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$

The permittivity tensor ϵ must be invariant under all symmetry transformations in the D_4 group. Therefore

$$A\epsilon A^{-1} = A\epsilon A^{\intercal} = \epsilon, \quad \forall A \in D_4, \tag{3}$$

as elements of $D_4 \subseteq O(2)$ are orthogonal.

$$R_+\epsilon R_+^{\intercal} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\epsilon_{xx} & \epsilon_{xy}\\ \epsilon_{yx} & \epsilon_{yy}\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}\epsilon_{yy} & -\epsilon_{yx}\\ -\epsilon_{xy} & \epsilon_{xx}\end{pmatrix} \tag{4}$$

Then

$$\begin{pmatrix}\epsilon_{yy} & -\epsilon_{yx}\\ -\epsilon_{xy} & \epsilon_{xx}\end{pmatrix} = \begin{pmatrix}\epsilon_{xx} & \epsilon_{xy}\\ \epsilon_{yx} & \epsilon_{yy}\end{pmatrix}, \tag{5}$$

and thus $\epsilon_{xx} = \epsilon_{yy} \equiv \epsilon$ and $\epsilon_{xy} = -\epsilon_{yx}$. Furthermore,

$$M_x\epsilon M_x^{\intercal} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\epsilon & \epsilon_{xy}\\ -\epsilon_{xy} & \epsilon\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}\epsilon & -\epsilon_{xy}\\ \epsilon_{xy} & \epsilon\end{pmatrix}, \tag{6}$$

therefore $\epsilon_{xy} = -\epsilon_{xy} = 0$, and

$$\epsilon = \begin{pmatrix}\epsilon & 0\\ 0 & \epsilon\end{pmatrix}. \tag{7}$$

Problem 2. By using Maxwell's equation (assuming no loss and $J = 0$) and the Lorentz reciprocity theorem, derive the following equation:

$$\epsilon \cdot \mathbf{E}^{(a)}\mathbf{E}^{(b)} = \epsilon \cdot \mathbf{E}^{(b)}\mathbf{E}^{(a)}. \tag{8}$$

Solution 2. For two solutions at $p \in \{a, b\}$ at some frequency ω, the sourceless Maxwell equations give

$$\begin{aligned}\nabla \times \mathbf{E}^{(p)} &= -i\omega\mu_0\mathbf{H}^{(p)},\\ \nabla \times \mathbf{H}^{(p)} &= i\omega\epsilon \cdot \mathbf{E}^{(p)}.\end{aligned} \tag{9}$$

Consider the following divergence:

$$\begin{aligned}\nabla \cdot \left(\mathbf{E}^{(a)} \times \mathbf{H}^{(b)} - \mathbf{H}^{(a)} \times \mathbf{E}^{(b)}\right) = \mathbf{H}^{(b)} \cdot (\nabla \times \mathbf{E}^{(a)}) - \mathbf{E}^{(a)} \cdot (\nabla \times \mathbf{H}^{(b)})\\ - \mathbf{H}^{(a)} \cdot (\nabla \times \mathbf{E}^{(b)}) + \mathbf{E}^{(b)} \cdot (\nabla \times \mathbf{H}^{(a)}).\end{aligned} \tag{10}$$

Substitute the Maxwell equations to get

$$
\begin{aligned}
\nabla \cdot \left(\mathbf{E}^{(a)} \times \mathbf{H}^{(b)} - \mathbf{H}^{(a)} \times \mathbf{E}^{(b)}\right) &= -i\omega\mu_0 \mathbf{H}^{(b)} \cdot \mathbf{H}^{(a)} - i\omega \mathbf{E}^{(a)} \cdot (\epsilon \cdot \mathbf{E}^{(b)}) \\
&\quad + i\omega\mu_0 \mathbf{H}^{(a)} \cdot \mathbf{H}^{(b)} + i\omega \mathbf{E}^{(b)} \cdot (\epsilon \cdot \mathbf{E}^{(a)}) \\
&= i\omega \left[\mathbf{E}^{(b)} \cdot (\epsilon \cdot \mathbf{E}^{(a)}) - \mathbf{E}^{(a)} \cdot (\epsilon \cdot \mathbf{E}^{(b)})\right].
\end{aligned} \tag{11}
$$

Integrate over a volume Ω with differentiable boundary $\partial\Omega \in C^1$, then apply the divergence theorem in three dimensions to get

$$
\begin{aligned}
\int_\Omega \mathrm{d}V\, \nabla \cdot \left(\mathbf{E}^{(a)} \times \mathbf{H}^{(b)} - \mathbf{H}^{(a)} \times \mathbf{E}^{(b)}\right) &= \oint_{\partial\Omega} \mathrm{d}\mathbf{S} \cdot \left(\mathbf{E}^{(a)} \times \mathbf{H}^{(b)} - \mathbf{H}^{(a)} \times \mathbf{E}^{(b)}\right) \\
&= i\omega \int_\Omega \mathrm{d}V\, \left[\mathbf{E}^{(b)} \cdot (\epsilon \cdot \mathbf{E}^{(a)}) - \mathbf{E}^{(a)} \cdot (\epsilon \cdot \mathbf{E}^{(b)})\right].
\end{aligned} \tag{12}
$$

The fields will vanish at infinity, since $\mathrm{d}\mathbf{S}$ grows as r^2 while the fields decay as $1/r^2$. Therefore, the surface integral evaluates to zero. Then we have

$$
\mathbf{E}^{(a)} \cdot (\epsilon \cdot \mathbf{E}^{(b)}) = \mathbf{E}^{(b)} \cdot (\epsilon \cdot \mathbf{E}^{(a)}), \tag{13}
$$

or, equivalently,

$$
\epsilon \cdot \mathbf{E}^{(a)}\mathbf{E}^{(b)} = \epsilon \cdot \mathbf{E}^{(b)}\mathbf{E}^{(a)}. \tag{14}
$$

Problem 3. Determine the characteristic waves (eigenvectors) and their phase velocities (eigenvalues) for wave propagation in a biaxial crystal. Please describe the direction of D and E fields, and the eigenmode refractive indices as a function of θ and ϕ. Describe your solution in the **simplest** form. You might want to assume B and H are in the same direction.

Solution 3. We can choose a representation of the permittivity tensor for a biaxial crystal as a 3×3 diagonal matrix by the discussion in lecture notes. Suppose the principal axes of the crystal are aligned with the x, y, and z axes, then

$$
\epsilon = \begin{pmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{pmatrix}, \tag{15}
$$

for $\epsilon_x, \epsilon_y, \epsilon_z \in \mathbb{R}_{>0}$. It has the inverse $\kappa \equiv \epsilon^{-1}$,

$$
\kappa = \begin{pmatrix} \kappa_x & 0 & 0 \\ 0 & \kappa_y & 0 \\ 0 & 0 & \kappa_z \end{pmatrix} = \begin{pmatrix} 1/\epsilon_x & 0 & 0 \\ 0 & 1/\epsilon_y & 0 \\ 0 & 0 & 1/\epsilon_z \end{pmatrix}. \tag{16}
$$

By a transformation, we can switch to the (k, D, B) space, where $D = \epsilon \cdot E$ and $B = \mu_0 H$. The transformed tensor is

$$
T = T_{e_1}(-\theta)\, T_{x_3}(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\cos\theta\sin\phi & \cos\theta\cos\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \end{pmatrix}, \tag{17}
$$

$$
T^{-1} = T^{\intercal} = \begin{pmatrix} \cos\phi & -\cos\theta\sin\phi & -\sin\theta\sin\phi \\ \sin\phi & \cos\theta\cos\phi & \sin\theta\cos\phi \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{18}
$$

Then

$$
\tilde{\kappa} = T\kappa T^{-1} = \begin{pmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{21} & \kappa_{22} & \kappa_{23} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} \end{pmatrix} \tag{19}
$$

where

$$\begin{aligned}
\kappa_{11} &= \kappa_x \sin^2\phi + \kappa_y \cos^2\phi, \\
\kappa_{22} &= \cos^2\theta(\kappa_x \cos^2\phi + \kappa_y \sin^2\phi) + \kappa_z \sin^2\theta, \\
\kappa_{33} &= \sin^2\theta(\kappa_x \cos^2\phi + \kappa_y \sin^2\phi) + \kappa_z \cos^2\theta, \\
\kappa_{12} &= \kappa_{21} = (\kappa_x - \kappa_y)\sin\phi\cos\phi\cos\theta, \\
\kappa_{13} &= \kappa_{31} = (\kappa_x - \kappa_y)\sin\phi\cos\phi\sin\theta, \\
\kappa_{23} &= \kappa_{32} = \sin\theta\cos\theta((\kappa_x - \kappa_y)\sin\phi\cos\phi - \kappa_z).
\end{aligned} \tag{20}$$

We will use the following equations, assuming that B and H are in the same direction, and further noticing that only the transverse direction matters:

$$\begin{aligned}
\begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} &= \begin{pmatrix} 0 & \omega/k \\ -\omega/k & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \\
\gamma \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} &= \begin{pmatrix} 0 & -\omega/k \\ \omega/k & 0 \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix},
\end{aligned} \tag{21}$$

where $\gamma = 1/\mu_0$. Substitute the second equation into the first to get

$$\begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \frac{(\omega/k)^2}{\gamma} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \frac{(\omega/k)^2}{\gamma} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}. \tag{22}$$

Therefore, the eigenvalue equation is given by

$$\begin{pmatrix} \kappa_{11} - (\omega^2/\gamma k^2) & \kappa_{12} \\ \kappa_{21} & \kappa_{22} - (\omega^2/\gamma k^2) \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = 0. \tag{23}$$

If D_1, D_2 are both zero, then the equation is trivially true. If not, then the determinant of the matrix must be zero, i.e. we have a **dispersion relation**:

$$\det \begin{pmatrix} \kappa_{11} - (\omega^2/\gamma k^2) & \kappa_{12} \\ \kappa_{21} & \kappa_{22} - (\omega^2/\gamma k^2) \end{pmatrix} = 0. \tag{24}$$

The allowed propagation wavevectors are identified by solving for k to obtain the phase velocity eigenvalues and their characteristic wave modes.

$$\begin{aligned}
k_1^{\pm} &= \pm\sqrt{\frac{2}{\gamma}}\omega \left\{ (\kappa_{11} + \kappa_{22}) + \sqrt{\left(\kappa_{22} + \kappa_{11}\right)^2 + 4(\kappa_{12}\kappa_{21} - \kappa_{11}\kappa_{22})} \right\}^{-1/2}, \\
k_2^{\pm} &= \pm\sqrt{\frac{2}{\gamma}}\omega \left\{ (\kappa_{11} + \kappa_{22}) - \sqrt{\left(\kappa_{22} + \kappa_{11}\right)^2 + 4(\kappa_{12}\kappa_{21} - \kappa_{11}\kappa_{22})} \right\}^{-1/2}.
\end{aligned} \tag{25}$$

There are two modes of propagation, corresponding to the two positive roots k_1 and k_2. The solutions are discussed below.

(1) For the first mode $k = k_1$, the refractive index is given by

$$n_1(\theta, \phi) \equiv \frac{\omega}{k_1} = \sqrt{\frac{\gamma}{2}} \sqrt{(\kappa_{11} + \kappa_{22}) + \sqrt{\left(\kappa_{22} + \kappa_{11}\right)^2 + 4(\kappa_{12}\kappa_{21} - \kappa_{11}\kappa_{22})}}, \tag{26}$$

where

$$\begin{aligned}
\kappa_{11} &= \kappa_x \sin^2\phi + \kappa_y \cos^2\phi, \\
\kappa_{22} &= \cos^2\theta(\kappa_x \cos^2\phi + \kappa_y \sin^2\phi) + \kappa_z \sin^2\theta, \\
\kappa_{12} &= \kappa_{21} = (\kappa_x - \kappa_y)\sin\phi\cos\phi\cos\theta,
\end{aligned} \tag{27}$$

The electric displacement eigenvector is given by

$$\begin{pmatrix} D_1^{(1)} \\ D_2^{(1)} \end{pmatrix} = \left[\kappa_{12}^2 + \left(k_1 - \kappa_{11} + (\omega^2/\gamma k_1^2)\right)^2\right]^{-1/2} \begin{pmatrix} \kappa_{12} \\ k_1 - \kappa_{11} + (\omega^2/\gamma k_1^2) \end{pmatrix} e^{i(\omega t - k_1 z)}. \tag{28}$$

The electric field for this mode is given by

$$\begin{pmatrix} E_1^{(1)} \\ E_2^{(1)} \\ E_3^{(1)} \end{pmatrix} = \kappa \cdot \begin{pmatrix} D_1^{(1)} \\ D_2^{(1)} \\ 0 \end{pmatrix} = \begin{pmatrix} D_1^{(1)}/\epsilon_x \\ D_2^{(1)}/\epsilon_y \\ 0 \end{pmatrix} e^{i(\omega t - k_1 z)}. \tag{29}$$

Notice that $\mathbf{D}^{(1)}$ and $\mathbf{E}^{(1)}$ are not parallel.

(2) For the second mode, the refractive index is given by

$$n_2(\theta, \phi) \equiv \frac{\omega}{k_2} = \sqrt{\frac{\gamma}{2}} \sqrt{(\kappa_{11} + \kappa_{22}) - \sqrt{\left(\kappa_{22} + \kappa_{11}\right)^2 + 4(\kappa_{12}\kappa_{21} - \kappa_{11}\kappa_{22})}}\,, \tag{30}$$

where

$$\begin{aligned} \kappa_{11} &= \kappa_x \sin^2\phi + \kappa_y \cos^2\phi, \\ \kappa_{22} &= \cos^2\theta(\kappa_x \cos^2\phi + \kappa_y \sin^2\phi) + \kappa_z \sin^2\theta, \\ \kappa_{12} &= \kappa_{21} = (\kappa_x - \kappa_y)\sin\phi\cos\phi\cos\theta, \end{aligned} \tag{31}$$

The electric displacement eigenvector is given by

$$\begin{pmatrix} D_1^{(2)} \\ D_2^{(2)} \end{pmatrix} = \left[\kappa_{12}^2 + \left(k_2 - \kappa_{11} + (\omega^2/\gamma k_2^2)\right)^2\right]^{-1/2} \begin{pmatrix} \kappa_{12} \\ k_2 - \kappa_{11} + (\omega^2/\gamma k_2^2) \end{pmatrix} e^{i(\omega t - k_2 z)}. \tag{32}$$

The electric field for this mode is given by

$$\begin{pmatrix} E_1^{(2)} \\ E_2^{(2)} \\ E_3^{(2)} \end{pmatrix} = \kappa \cdot \begin{pmatrix} D_1^{(2)} \\ D_2^{(2)} \\ 0 \end{pmatrix} = \begin{pmatrix} D_1^{(2)}/\epsilon_x \\ D_2^{(2)}/\epsilon_y \\ 0 \end{pmatrix} e^{i(\omega t - k_2 z)}. \tag{33}$$

Notice that $\mathbf{D}^{(2)}$ and $\mathbf{E}^{(2)}$ are not parallel.

Refer to the following paper for a more general discussion: J. Massman and M. Havrilla, ”Analysis of General Plane Wave Propagation in Biaxial Media Using the kDB System,” 2022. doi: `10.1109/Metamaterials54993.2022.9920911`.