Special Function for Michaelis-Menten Calculations

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1 Special Function

As defined by Dr. Lin, the equation

$$1 - mx = \frac{cx}{k + x} \tag{1}$$

admits the nonnegative solution given by

$$p(c, k, m) = \frac{1}{2m}\left[(1 - c - mk) + \sqrt{(1 - c - mk)^2 + 4mk}\right]. \tag{2}$$

The quantity $p(c, k, m)$ always lies between 0 and $1/m$, is an increasing function of c, m, and a decreasing function of k. For our system, we can express the fixed point solution as

$$Y_1^* = p\left(\frac{a_1\theta_2}{b\theta_1}, k_1, 1\right) \equiv p(c_1, k_1, m_1), \tag{3a}$$

$$Y_2^* = p\left(\frac{a_2\theta_3}{a_1\theta_2}\frac{k_1 + Y_1^*}{Y_1^*}, k_2, \frac{b\theta_1}{a_1\theta_2}\frac{k_1 + Y_1^*}{Y_1^*}\right) \equiv p(c_2, k_2, m_2), \tag{3b}$$

$$Y_{k+2}^* = \theta_k W^*, \quad W^* \equiv 1 - Y_1^* - Y_2^*, \quad k = 1, 2, 3. \tag{3c}$$

Let's consider the relevant limits of $p(c, k, m)$ for the $b \to 0$ and $b \to \infty$ limits. When $b \to 0$, we have $c_1 \to \infty$ and $c_2 \to \infty$. In this case, even though it seems like $m_2 \to 0$, we simultaneously have $Y_1^* \to 0$, and hence $m_2 = O(1)$. It feels handwavy to conclude so, but a full expansion shown later will verify this fact. On the other hand, when $b \to \infty$, we have $c_1 \to 0$, $Y_1^* \to 1$, and hence $m_2 \to \infty$.

(1) $c \to 0$: By direct expansion of $p(c, k, m)$ about $c = 0$, we have

$$\boxed{p(c,k,m) = \frac{1}{m} - \frac{c}{m\,(km+1)} + \frac{k}{(km+1)^3}\,c^2 + \frac{k\,(1-km)}{(km+1)^5}\,c^3 + O(c^4). \qquad (4)}$$

(2) $c \to \infty$: By direct expansion of $p(c,k,m)$ about $c = \infty$, we have

$$\boxed{p(c,k,m) = \frac{k}{c} + \frac{k - k^2 m}{c^2} + \frac{k\,(k^2m^2 - 3km + 1)}{c^3} + O(c^{-4}). \qquad (5)}$$

(3) $m \to 0$: By direct expansion of $p(c,k,m)$ about $m = 0$, we have

$$p(c,k,m) = \frac{|c-1| - (c-1)}{2m} + \frac{k}{2}\left(\frac{1+c}{|c-1|} - 1\right) - \frac{c\,k^2}{|c-1|^3}\,m + \frac{k^3 c(c+1)}{|c-1|^5}\,m^2 + O(m^3).$$

Assuming $c < 1$, this becomes

$$\boxed{p(c,k,m) = \frac{1-c}{2m} + \frac{ck}{1-c} - \frac{c\,k^2}{(1-c)^3}\,m + \frac{k^3 c(c+1)}{(1-c)^5}\,m^2 + O(m^3). \qquad (6)}$$

If we instead assume $c > 1$, then

$$\boxed{p(c,k,m) = \frac{k}{c-1} - \frac{c\,k^2}{(c-1)^3}\,m + \frac{k^3 c(c+1)}{(c-1)^5}\,m^2 + O(m^3). \qquad (7)}$$

(4) $m \to \infty$: By direct expansion of $p(c,k,m)$ about $m = \infty$, we have

$$\boxed{p(c,k,m) = \frac{1}{m} - \frac{c}{k\,m^2} + \frac{c(1+c)}{k^2\,m^3} + O(m^{-4}). \qquad (8)}$$

Notice that p diverges when $m \to 0$, but fortunately we do not need this limit in our analysis at all. Using these expansions, we can now give a preliminary analysis of the starvation and overabundance limits.

2 Starvation Limit

When $b \to 0$, we have $c_1 \to \infty$, $c_2 \to \infty$, and $m_2 = O(1)$. Therefore, using the $c \to \infty$ expansion on c_1, we have

$$\boxed{Y_1 = k_1 \left(\frac{b\theta_1}{a_1\theta_2} \right) + k_1(1-k_1) \left(\frac{b\theta_1}{a_1\theta_2} \right)^2 + O\left(n_1^3\right), \qquad (9)}$$

and

$$\frac{Y_1^*}{k_1 + Y_1^*} = \left(\frac{\theta_1}{a_1\theta_2} \right) b - (2-k_1) \left(\frac{\theta_1}{a_1\theta_2} \right)^2 b^2 + O\left(\left(\frac{b\theta_1}{a_1\theta_2} \right)^3 \right), \tag{10}$$

$$\frac{k_1 + Y_1^*}{Y_1^*} = \left(\frac{a_1\theta_2}{\theta_1} \right) \frac{1}{b} + (2-k_1) + O\left(\frac{b\theta_1}{a_1\theta_2} \right). \tag{11}$$

Then, using the same expansion on c_2, we have

$$\frac{1}{c_2} = \left(\frac{a_1\theta_2}{a_2\theta_3} \right) \left(\frac{Y_1^*}{k_1 + Y_1^*} \right) = \left(\frac{\theta_1}{a_2\theta_3} \right) b - (2-k_1) \left(\frac{\theta_1^2}{a_1 a_2 \theta_2 \theta_3} \right) b^2 + O\left(\left(\frac{b\theta_1}{a_1\theta_2} \right)^3 \right), \tag{12}$$

$$m_2 = 1 + (2-k_1) \left(\frac{\theta_1}{a_1\theta_2} \right) b + O\left(\frac{b\theta_1}{a_1\theta_2} \right)^2. \tag{13}$$

It is as expected that $m_2 = O(1)$. Then, for Y_2^* we have

$$Y_2^* = k_2(2-k_2) \left(\frac{\theta_1}{a_2\theta_3} \right) b - 2k_2(2-k_1) \left(\frac{\theta_1^2}{a_1 a_2 \theta_2 \theta_3} \right) b^2 + O\left(\left(\frac{b\theta_1}{a_1\theta_2} \right)^3 \right). \tag{14}$$

This does not match the full expansion given in the next section. This may be because of the assumption that $m_2 = O(1)$, which is not fully justified. A more rigorous derivation is given in the corresponding Subsection.

3 Full Calculation

We also show the result of a full on expansion for comparision with results from the special function method. Consider the series expansion of Y_1, Y_2, Y_3 in the limit $b \to 0$. In effect, we are computing the series expansion with respect to the dimensionless quantity $n_1 = b\theta_1/a_1\theta_2$ and $n_2 = b\theta_1/a_2\theta_3$. We have

$$Y_1 = k_1 \left(\frac{b\theta_1}{a_1\theta_2} \right) + \frac{k_1\theta_2(1-k_1)}{\theta_2} \left(\frac{b\theta_1}{a_1\theta_2} \right)^2 + O\left(n_1^3\right), \tag{15}$$

$$Y_2 = k_2 \left(\frac{b\theta_1}{a_2\theta_3} \right) - \frac{k_2(a_1k_2\theta_2 - a_1\theta_2 + a_2k_1\theta_3)}{a_1\theta_2} \left(\frac{b\theta_1}{a_2\theta_3} \right)^2 + O\left(n_2^3\right), \tag{16}$$

$$Y_3 = \theta_1 - b\theta_1^2 \left(\frac{k_1}{a_1\theta_2} + \frac{k_2}{a_2\theta_3} \right) + O\left(\min\{n_1^2, n_2^2\}\right). \tag{17}$$