

# 2025 Fall Introduction to ODE

Homework 4 (Due Sep 29 12:00, 2025)

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**Problem 1.** Comment on the difficulties that you face when trying to construct the Green's function for the boundary value problem

$$y''(x) + y(x) = f(x) \quad \text{subject to} \quad y(a) = y'(b) = 0. \quad (1)$$

**Solution 1.** First let's pick the standard fundamental solutions to the homogeneous case  $y'' + y = 0$  such that they satisfy the boundary conditions. We have:

$$\begin{aligned} y_1(x) &= \sin(x - a), & y_1(a) &= 0, \\ y_2(x) &= \cos(x - b), & y_2'(b) &= 0. \end{aligned} \quad (2)$$

The Wronskian is a constant given by

$$W = y_1 y_2' - y_2 y_1' = -\cos(a - b), \quad (3)$$

and the Green's function is given by

$$G(x, t) = \begin{cases} [y_1(x)y_2(\xi)] / W, & a \leq x < \xi \leq b, \\ [y_1(\xi)y_2(x)] / W, & a \leq \xi < x \leq b, \end{cases} \quad (4)$$

whenever  $W(x)$  is nonzero. However, if  $a - b = n\pi/2$ ,  $n \in \mathbb{Z}$ , then  $W = 0$  and the Green's function cannot be constructed. We can interpret this result by noticing that the boundary conditions are not independent when  $a - b = n\pi/2$ . In this case, the boundary value problem may not have a solution for arbitrary  $f(x)$ .

**Problem 2.** Write the generalized Legendre equation,

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \left\{ n(n+1) - \frac{m^2}{1-x^2} \right\} y = 0, \quad (5)$$

as a Sturm-Liouville equation.

**Solution 2.** Sturm-Liouville equations are of the form

$$\frac{d}{dx} \left[ p(x) \frac{dy}{dx} \right] + q(x)y(x) = -\lambda r(x)y(x) = 0, \quad (6)$$

where  $\lambda$  is the eigenvalue which depends on the boundary conditions. Furthermore, the functions  $p(x), q(x), r(x)$  are real-valued and continuous on the closed interval  $[a, b]$ ,  $p(x)$  is differentiable, and  $p(x) > 0, r(x) > 0$  on the open interval  $(a, b)$ .

From the textbook, the endpoint  $x = a$  is a **singular endpoint** if  $a = -\infty$  or if  $a < \infty$  but the above conditions do not hold on the closed interval  $[a, c]$  for some  $c \in (a, b)$ . Similar definitions hold for the other endpoint,  $x = b$ . Hence  $\pm 1$  are singular endpoints of the generalized Legendre equation. Therefore, we apply Friedrichs boundary conditions.

Then we have

$$\frac{d}{dx} \left[ (1 - x^2) \frac{dy(x)}{dx} \right] - \left( \frac{m^2}{1-x^2} \right) y(x) = -n(n+1)y(x), \quad (7)$$

which is a Sturm-Liouville equation with eigenvalue  $\lambda = n(n + 1)$  and functions

$$p(x) = 1 - x^2, \quad q(x) = -\frac{m^2}{1 - x^2}, \quad r(x) = 1 \quad (8)$$

satisfying the conditions above on the interval  $(-1, 1)$ . Since the Sturm-Liouville operator is singular at  $\pm 1$ , we assume Friedrichs boundary conditions following the description in [King & Billingham & Otto]. That is, we require that the solution  $y(x, \lambda)$  satisfies

$$|y(x, \lambda)| \leq A \text{ for } x \in (-1, 0] \text{ and } |y(x, \lambda)| \leq B \text{ for } x \in [0, 1), \quad (9)$$

for some  $A, B \in \mathbb{R}_{\geq 0}$ .

**Problem 3.** Show that

$$-(xy'(x))' = \lambda xy(x), \quad (10)$$

is self-adjoint on the interval  $(0, 1)$ , with  $x = 0$  a singular endpoint and  $x = 1$  a regular endpoint with the condition  $y(1) = 0$ .

**Solution 3.** A linear operator is said to be self-adjoint if it satisfies

$$\langle Lu, v \rangle = \langle u, Lv \rangle, \quad (11)$$

where the inner product is defined as

$$\langle u, v \rangle = \int_0^1 dx u(x)v(x). \quad (12)$$

Expanding the left-hand side, we can write the differential equation in terms of a linear operator  $L$ :

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} + \lambda xy(x) \equiv Ly = 0. \quad (13)$$

Let  $u, v$  be two functions satisfying the boundary conditions. Then, by Lemma 4.1 (Lagrange's identity) in [King & Billingham & Otto], let

$$L = x \frac{d^2}{dx^2} + \frac{d}{dx} + \lambda x \quad (14)$$

be a linear differential operator on  $(0, 1)$ , and  $u, v \in C^2(0, 1)$ , then

$$u(Lv) - v(Lu) = [p(uv' - u'v)], \quad (15)$$

thus

$$\langle Lu, v \rangle - \langle u, Lv \rangle = [x(u(x)v'(x) - u'(x)v(x))]_0^1 = 0. \quad (16)$$

where the terms at  $x = 1$  vanish due to the boundary condition  $y(1) = 0$ , and the terms at  $x = 0$  vanish by some additional regularity condition on  $y(x)$  at the singular endpoint. One sufficient and natural choice would be

$$\lim_{x \rightarrow 0^+} xy'(x) = 0. \quad (17)$$

Therefore, we have shown that

$$\langle Lu, v \rangle = \langle u, Lv \rangle, \quad (18)$$

and  $L$  is self-adjoint.