On Quantum Darkness

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Introduction

Dark states?

1. localized dark states in waveguide QED
2. quantum origin of classical interference
3. Implementation & Protocols & Simulations

What are Dark States and Bright States?

Collective states: multiple subsystems interact coherently

Dark (subradiant) states: cannot couple with environment/other systems

Bright (superradiant) states: can couple with environment/other systems

Dicke Model – *a single-mode light cavity & N two-level system*

$$\hat{\mathcal{H}}_{\mathrm{Di}} = \hat{\mathcal{H}}_b + \hat{\mathcal{H}}_s + \hat{\mathcal{H}}_{\mathrm{int}}$$

$$\hat{\mathcal{H}}_{\mathrm{Di}} = \omega_c \hat{a}^\dagger \hat{a} + \omega_z \sum_{j=0}^{N} \hat{\sigma}_j^z + \frac{2\lambda}{\sqrt{N}} (\hat{a} + \hat{a}^\dagger) \sum_{j=0}^{N} \hat{\sigma}_j^x$$

ω_c : cavity frequency
ω_z : energy gap of two-level system. The last term describes
λ : coupling strength
$\hat{a} + \hat{a}^\dagger$: corresponds to electric fields

Open Dicke Model: $\dot{\hat{\rho}} = -i[\hat{\mathcal{H}}_{\mathrm{Di}}, \hat{\rho}] + \sum_{\alpha} \gamma_\alpha (\hat{L}_\alpha \hat{\rho} \hat{L}_\alpha^\dagger - \frac{1}{2}[\hat{L}_\alpha^\dagger \hat{L}_\alpha, \hat{\rho}])$

$\hat{L} = \sum_j \hat{\sigma}_j$ (collective decay)

All equations on this slide are from: Roses & Dalla Torre (2020)

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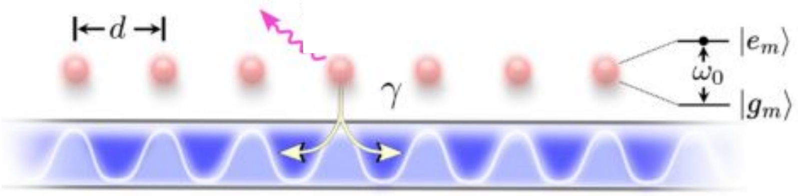
Model



R. Holzinger et al. (2022)

A chain of N two-level qubits coupled to a 1-D waveguide (mediates coherent interactions and collective dissipation)

Collective Hamiltonian:

$$\hat{\mathcal{H}}_{\text{eff}} = \sum_{m,n=1}^{N} \left(J_{m,n} - i\frac{\gamma_{m,n}}{2}\right)\hat{\sigma}_m^+\hat{\sigma}_n^-$$

Master Equation:

$$\dot{\hat{\rho}} = -i[\hat{\mathcal{H}}_{\text{eff}}, \hat{\rho}] + \sum_{m,n=1}^{N} \gamma_{m,n}\hat{\sigma}_m^-\hat{\rho}\hat{\sigma}_n^+$$

All equations on this slide are from: R. Holzinger et al. (2022)

When qubit separations is integer multiples of the guided wavelength $\lambda_0 \dots$
i.e. $d = n\lambda_0$ (the mirror configuration)

Coherent Exchange Interaction:

$$J_{m,n} = \frac{\gamma}{2}\sin(k_0|x_m - x_n|) = 0$$

Collective Dissipation Rate:

$$\gamma_{m,n} = \gamma\cos(k_0|x_m - x_n|) = \gamma$$

All equations on this slide are from: R. Holzinger et al. (2022)

Theory - (II) Dark States in Waveguide QED (3/6)

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Divide the system into two parts:
A group of M qubits and the remaining N − M qubits

Collective Operators for the 1st group:

$$\hat{\mathcal{S}}_1 = \frac{1}{\sqrt{M}} \sum_{j=1}^{M} \hat{\sigma}_j^-$$

Collective Operators for the 2nd group:

$$\hat{\mathcal{S}}_2 = \frac{1}{\sqrt{N-M}} \sum_{j=M+1}^{N} \hat{\sigma}_j^-$$

$$\rightarrow \hat{\mathcal{H}}_{\text{eff}} = -i\frac{M\gamma}{2}\hat{\mathcal{S}}_1^\dagger\hat{\mathcal{S}}_1 - i\frac{(N-M)\gamma}{2}\hat{\mathcal{S}}_2^\dagger\hat{\mathcal{S}}_2 - i\Gamma(\hat{\mathcal{S}}_1^\dagger\hat{\mathcal{S}}_2 + \hat{\mathcal{S}}_2^\dagger\hat{\mathcal{S}}_1)$$

enhanced collective coupling rate: $2\Gamma = \sqrt{M(N-M)}\gamma$

All equations on this slide are from: R. Holzinger et al. (2022)

Single Excitation Case

$$\langle \Psi_S^{(1)} | \Psi_D^{(1)} \rangle = 0 \quad \langle \Psi_{D,i}^{(1)} | \Psi_{D,j}^{(1)} \rangle = \delta_{ij} \quad \rightarrow$$ *Gram-Schmidt!*

1

Bright State:

$$|\Psi_S^{(1)}\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \hat{\sigma}_j^{+} |G\rangle$$

N-1 dimension

Dark Subspace:

(superposition of N-1 dark states)

$$|\Psi_D^{(1)}\rangle = \frac{1}{\sqrt{N}} (\sqrt{N-1}\hat{\sigma}_1^{+} - \hat{\mathcal{S}}_2^{\dagger}) |G\rangle$$

All equations on this slide are from: R. Holzinger et al. (2022)

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The coupling term under the mirror configuration…

Interaction: $J_{m,n} = 0$

Dissipation: $\langle \Psi_D^{(1)} | \sum_{m,n} \lambda_{m,n} \hat{\sigma}_m^+ \hat{\sigma}_n^- | \Psi_D^{(1)} \rangle = 0$

One can get…

$\langle \Psi_D^{(1)} | \hat{\mathcal{H}}_{\text{eff}} | \Psi_D^{(1)} \rangle = 0 \quad \rightarrow$ *Dark state is stable!*

$\langle \Psi_D^{(1)} | \hat{\sigma}_1^+ \hat{\sigma}_1^- | \Psi_D^{(1)} \rangle = 1 - \dfrac{1}{N} \quad \rightarrow$ *a large fraction of excitation is concentrated in the first qubit* $\rightarrow$ *localized!*

All equations on this slide are from: R. Holzinger et al. (2022)

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Dark state preparation via local coherent driving

A resonant coherent drive localized on the first qubit

$$\hat{\mathcal{H}}_d(t) = \Omega_d(t)(\sigma_1^- + \sigma_1^+)$$

Coupling strength:

$$\langle \Psi_S^{(1)} | \hat{\mathcal{H}}_d(t) | G \rangle = \Omega_d(t)\sqrt{1/N}$$

$$\langle \Psi_D^{(1)} | \hat{\mathcal{H}}_d(t) | G \rangle = \Omega_d(t)\sqrt{1 - 1/N}$$

All equations on this slide are from: R. Holzinger et al. (2022)

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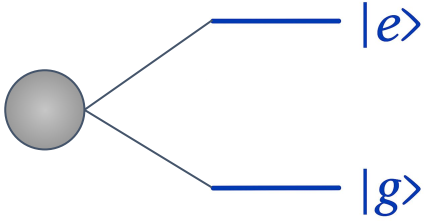
Model

Sensor - A Two-level Atom



Interaction Hamiltonian with electric field (under RWA):

$$\boxed{\hat{\mathcal{H}}_{\mathrm{fs}} = \hat{\mathbf{E}}^{(+)}(\mathbf{r},t)\,\hat{\sigma}^{+} + \hat{\mathbf{E}}^{(-)}(\mathbf{r},t)\,\hat{\sigma}^{-}}$$

Equation on this slide are from: Villas-Boas et al. (2025)

Consider photon(s) in state $|\Phi\rangle$

Absorption probability per unit time is proportional to (Glauber, 1963):

$$\langle\Phi|\hat{\mathbf{E}}^{(-)}(\mathbf{r},t)\hat{\mathbf{E}}^{(+)}(\mathbf{r},t)|\Phi\rangle$$

For single-mode (notice $\hat{\mathbf{E}}^{(+)} \propto \hat{a}$)

$$\hat{\mathbf{E}}^{(+)}|\Phi\rangle = 0 \iff |\Phi\rangle = |0\rangle$$

The sensor can detect any field except for the vacuum state!

How about considering a **two-mode** field?

Still only the vacuum state CANNOT be detected?

Consider **two-mode** field

The interaction Hamiltonian can be written as:

$$\boxed{\hat{\mathcal{H}}_{\text{fs}} = g\left(\hat{a} + \hat{b}e^{i\theta}\right)\hat{\sigma}^{+} + h.c.}$$

Define $\hat{c} = (\hat{a} + \hat{b}e^{i\theta})/\sqrt{2}$ and $\hat{d} = e^{-i\theta}(-\hat{a} + \hat{b}e^{i\theta})/\sqrt{2}$

The Hamiltonian becomes:

$$\boxed{\hat{\mathcal{H}}_{\text{fs}} = \sqrt{2}g\hat{c}\hat{\sigma}^{+} + h.c.}$$

All equations on this slide are from: Villas-Boas et al. (2025)

Theory - (III) Dark and Bright States of Light (5/10)

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For each N-excitation subspace ($n \leq N$), define

$$\left|\psi_n^N(\theta)\right\rangle \equiv \left|n, N-n\right\rangle_{c,d} = \frac{\left(\hat{a}^\dagger + \hat{b}^\dagger e^{-i\theta}\right)^n \left(-\hat{a}^\dagger e^{i\theta} + \hat{b}^\dagger\right)^{N-n}}{\sqrt{2^N n!\,(N-n)!}} \left|0,0\right\rangle_{a,b}$$

The states constitute a complete basis satisfying

$$\hat{\mathcal{H}}_{\mathrm{fs}} \left|\psi_n^N\right\rangle \left|g\right\rangle = g\sqrt{2n} \left|\psi_{n-1}^{N-1}\right\rangle \left|e\right\rangle$$

All equations on this slide are from: Villas-Boas et al. (2025)

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Recall

$$\hat{\mathcal{H}}_{\mathrm{fs}} \left|\psi_n^N\right\rangle |g\rangle = g\sqrt{2n} \left|\psi_{n-1}^{N-1}\right\rangle |e\rangle$$

or

$$\boxed{\hat{\mathbf{E}}^{(+)} \left|\psi_n^N\right\rangle = g\sqrt{2n} \left|\psi_{n-1}^{N-1}\right\rangle}$$

Observe that

$$\hat{\mathbf{E}}^{(+)} \left|\psi_0^N\right\rangle = 0$$

Dark state

$$\hat{\mathbf{E}}^{(+)} \left|\psi_N^N\right\rangle = g\sqrt{2N} \left|\psi_{N-1}^{N-1}\right\rangle$$

Bright state

All equations on this slide are from: Villas-Boas et al. (2025)

Theory - (III) Dark and Bright States of Light (7/10)

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More explicitly, recall that

$$|\psi_n^N(\theta)\rangle = \frac{\left(\hat{a}^\dagger + \hat{b}^\dagger e^{-i\theta}\right)^n \left(-\hat{a}^\dagger e^{i\theta} + \hat{b}^\dagger\right)^{N-n}}{\sqrt{2^N n!\,(N-n)!}} |0,0\rangle_{a,b}$$

There exist states, other than the vacuum state, that ***cannot be detected****!*

Use the Binomial theorem, one can obtain

$$|\psi_0^N(\theta)\rangle = \sqrt{\frac{N!}{2^N}} \sum_{m=0}^{N} \frac{(-1)^m e^{im\theta}}{\sqrt{m!\,(N-m)!}} |m, N-m\rangle_{a,b}$$

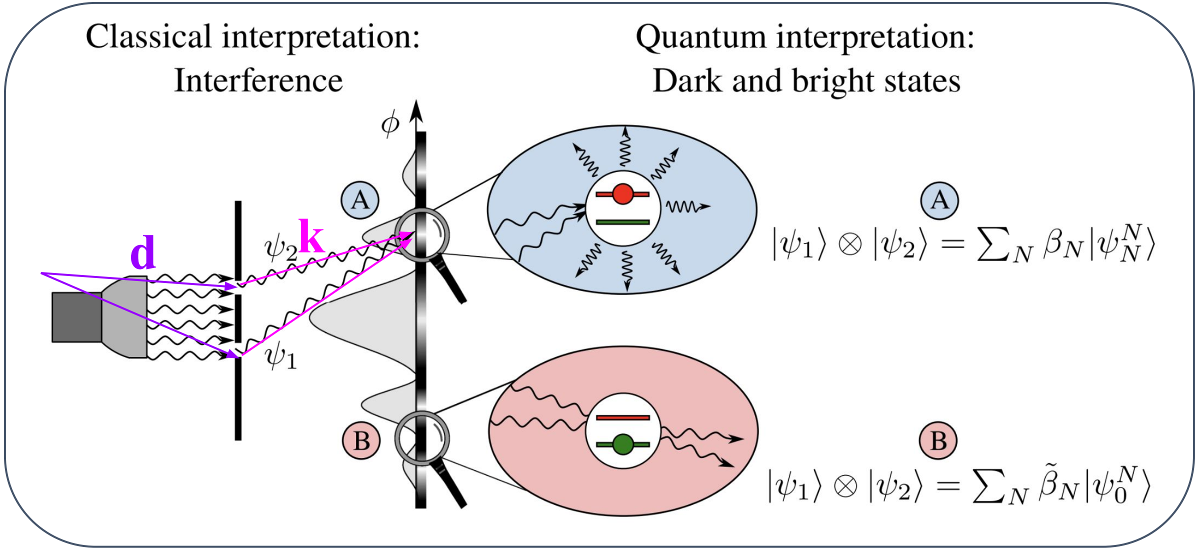
and

$$|\psi_N^N(\theta)\rangle = e^{-iN\theta} \sqrt{\frac{N!}{2^N}} \sum_{m=0}^{N} \frac{e^{im\theta}}{\sqrt{m!\,(N-m)!}} |m, N-m\rangle_{a,b}$$

All equations on this slide are from: Villas-Boas et al. (2025)

Double Slit Interference

Consider the light source being a coherent state



Villas-Boas et al. (2025)

The field on the screen is $\left|e^{-i\mathbf{k}_1\cdot\mathbf{d}_1}\alpha, e^{-i\mathbf{k}_2\cdot\mathbf{d}_2}\alpha\right\rangle$

$$\left|e^{-i\mathbf{k}_1\cdot\mathbf{d}_1}\alpha, e^{-i\mathbf{k}_2\cdot\mathbf{d}_2}\alpha\right\rangle = e^{-|\alpha|^2}\sum_{N=0}^{\infty}\sqrt{\frac{2^N}{N!}}\left(e^{-i\mathbf{k}_2\cdot\mathbf{d}_2}\alpha\right)^N\left|\chi^N(\delta\phi)\right\rangle$$

$$\left|\chi^N(\delta\phi)\right\rangle = \sqrt{\frac{N!}{2^N}}\sum_{m=0}^{N}\frac{\boxed{e^{-im\delta\phi}}e^{im\theta}}{\sqrt{m!(N-m)!}}|m, N-m\rangle$$

Recall

One observe

$$\left|\chi^N((2l+1)\pi)\right\rangle = \left|\psi_0^N(\theta)\right\rangle$$

Dark state

$$\left|\chi^N(2l\pi)\right\rangle \propto \left|\psi_N^N(\theta)\right\rangle$$

Bright state

All equations on this slide are from: Villas-Boas et al. (2025)

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One significant difference from the classical interpretation:

$$\langle \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} \rangle \, (\delta\phi) = 2|\alpha|^2$$

The photons are ***present at every point*** *on the screen!*

Equation on this slide are from: Villas-Boas et al. (2025)

Protocols (1/2) - Probing a Dark State

- General states are hard to probe due to their complicated correlations

$$|\Phi_{\mathcal{D}}^{(2)}\rangle = \frac{\sqrt{2}}{N}\Big((\hat{a}_1^\dagger)^2 - 2\sqrt{N-1}\hat{a}_1^\dagger \mathcal{S}_2^\dagger + (\mathcal{S}_2^\dagger)^2\Big)|G\rangle$$

First two transmons are driven by an external drive

$$|\Psi_{\mathcal{D}}^{(2)}\rangle = \frac{\sqrt{N-3}}{\sqrt{N-1}}\left((\mathcal{S}_1^\dagger)^2 - \frac{\sqrt{2}\mathcal{S}_1^\dagger \mathcal{S}_2^\dagger}{\sqrt{N-2}} + \frac{(\mathcal{S}_2^\dagger)^2}{N-3}\right)|G\rangle$$

superposition of dark states

(A demonstration will follow this section)

All equations on this slide are from: R. Holzinger et al. (2022)

Protocols (2/2) - An Exciting Example

- Preparation of dark state (coherent drive) $\hat{\mathcal{H}}_d(t) = \Omega_d(t)(\hat{\sigma}_1^\dagger + \hat{\sigma}_2^\dagger + \mathrm{H.c.})$
- Selectively excite single qubits
- Detuning: sudden release of energy, causing peak in intensity

$$\langle \mathcal{S}_1^\dagger \mathcal{S}_1 \rangle + \langle \mathcal{S}_2^\dagger \mathcal{S}_2 \rangle + 2\mathrm{Re}\langle \mathcal{S}_1^\dagger \mathcal{S}_2 \rangle$$

We have a protocol for storing and releasing two excitations into a waveguide

(A demonstration will follow this section)

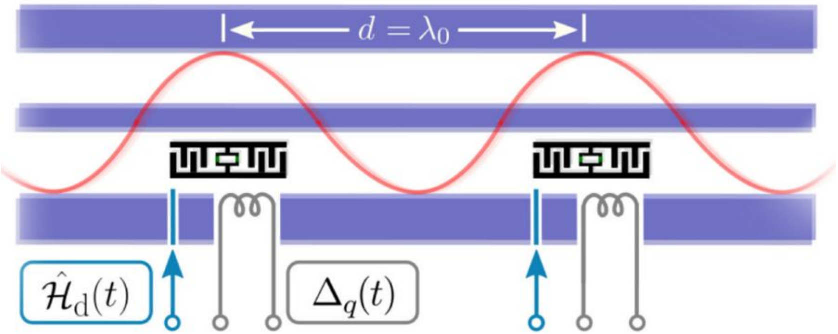
All equations on this slide are from: R. Holzinger et al. (2022)

"TRANSmission line shunted plasMa OscillatioN qubit"

- Transmons

- Waveguide QED: Transmons are coupled to a coplanar waveguide

- Designed to have reduced sensitivity to noise



R. Holzinger et al. (2022)

$$\hat{\mathcal{H}}_{\mathrm{eff}} = \sum_{m,n}\left(J_{m,n} - i\frac{\Gamma_{m,n}}{2}\right)\hat{a}_m^{\dagger}\hat{a}_n - \frac{U}{2}\sum_m \hat{n}_m(\hat{n}_m - \mathbb{I})$$

Exhibits multilevel nature

Equation on this slide are from: R. Holzinger et al. (2022)

Results (1/2) - Spatial Correlation

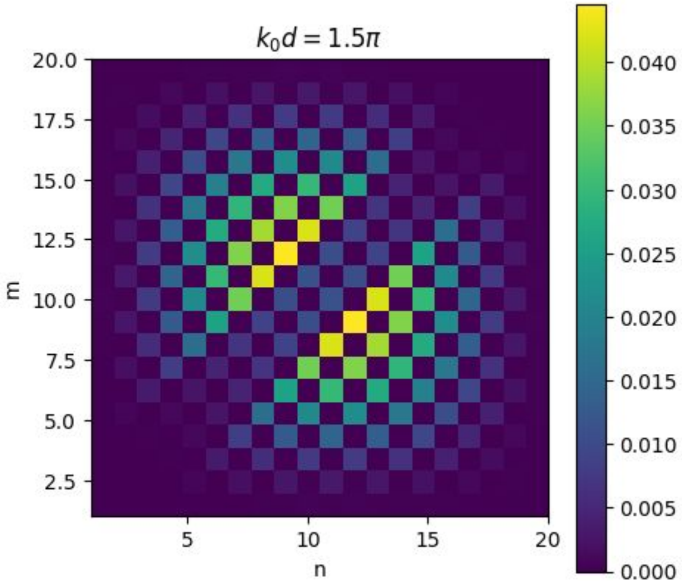
- Replicate and extend correlation graphs from the paper
- General states are usually hard to probe; dark states can be probed more easily
- Other emerging effects

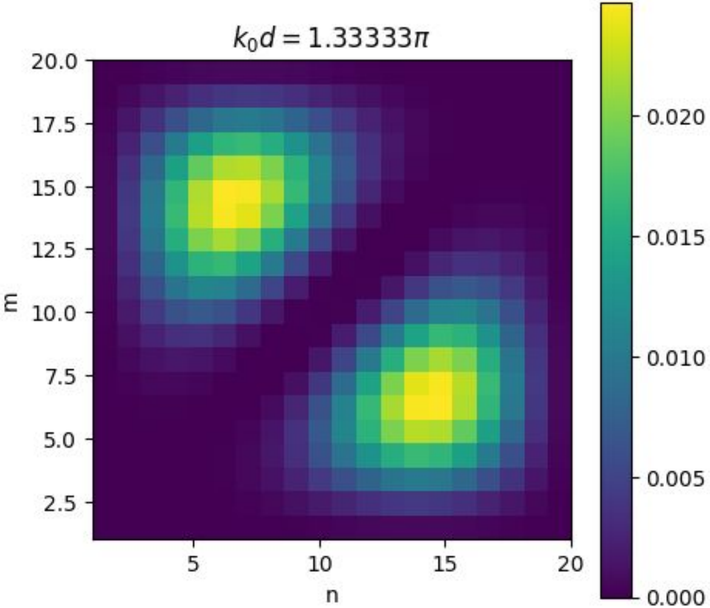
$$J_{m,n} = \frac{\gamma}{2} \sin(k_0|x_m - x_n|) \qquad \gamma_{m,n} = \gamma \cos\left(k_0|x_m - x_n|\right)$$

$$\hat{\mathcal{H}}_{\text{eff}} = \sum_{m,n=1}^{N} (J_{m,n} - i\frac{\gamma_{m,n}}{2})\hat{\sigma}_m^+ \hat{\sigma}_n^-$$

All equations on this slide are from: R. Holzinger et al. (2022)

- Correlation simulated on a lattice of *n, m* ranging from 1 to 20



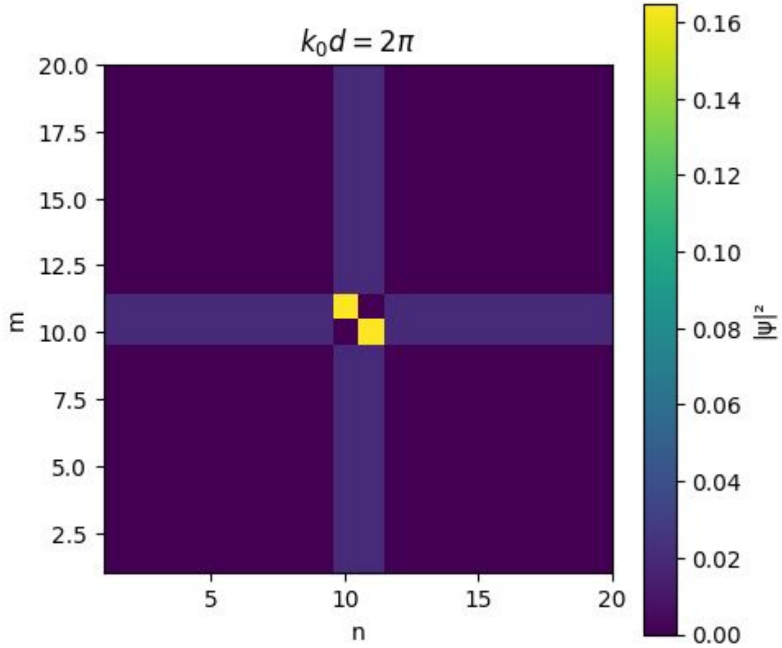


Highly entangled across specific sites. Coherent nearest-neighbor interactions are zero.

Symmetric correlation pattern

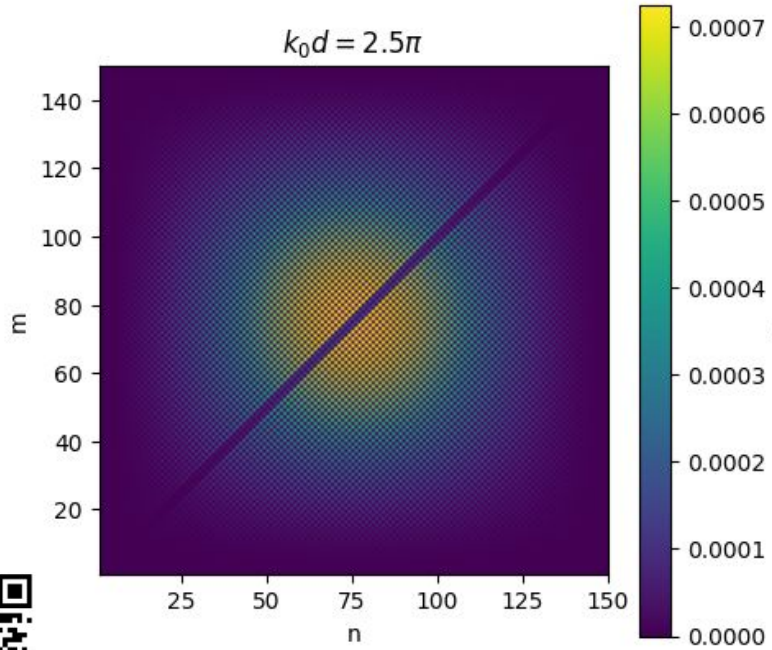
$$k_0 d \neq (2n+1)\pi/2$$

- Dark states can be probed more easily due to trivial correlation

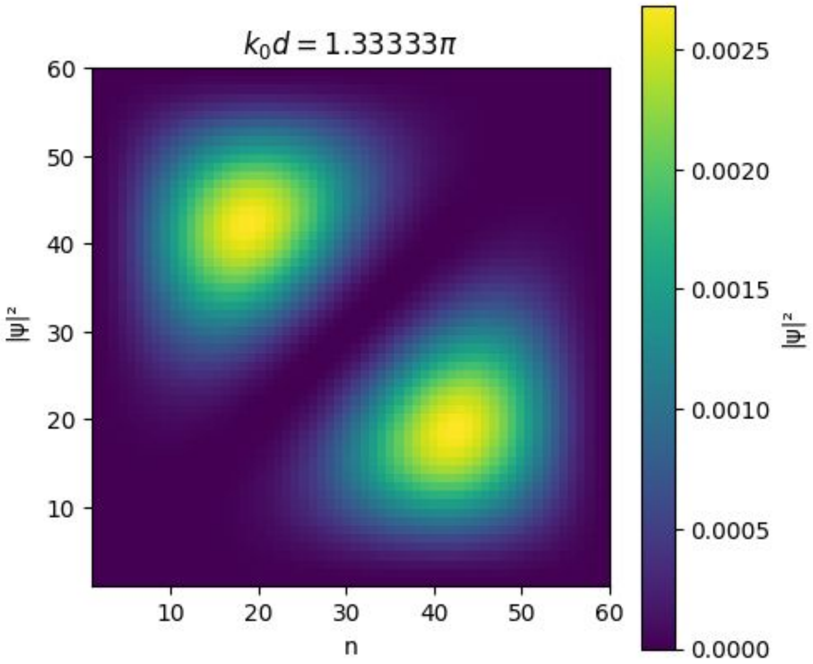


Does similar qualitative result remains as the qubit number increase?

- It is not feasible to compute the $2^N \times 2^N$ Hamiltonian matrix, so we restrict to the two-excitation subspace

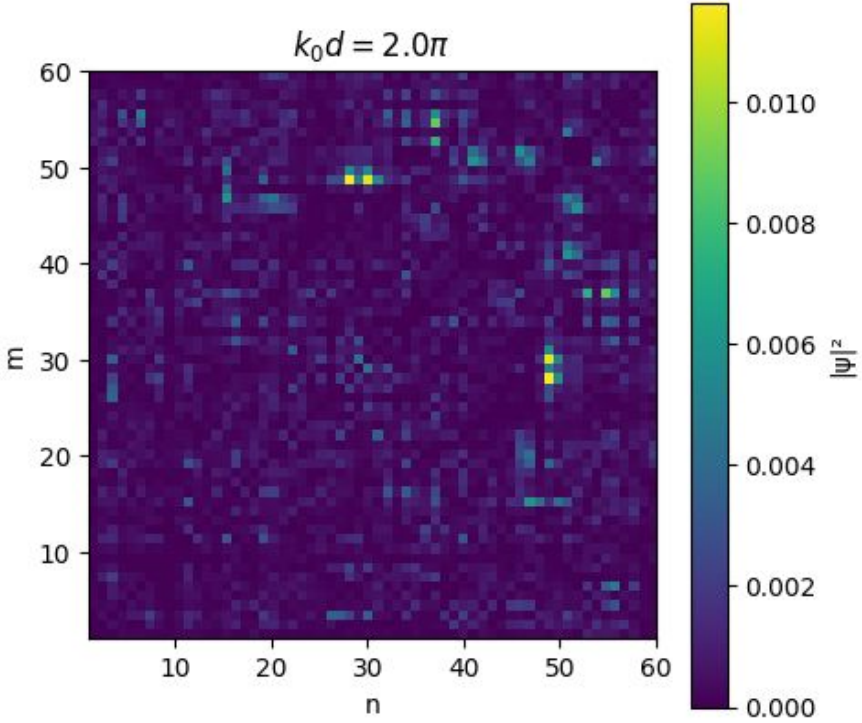


N = 150



N = 60

- <u>Observation</u>: when (*kd* = integer*pi), correlation becomes less apparent



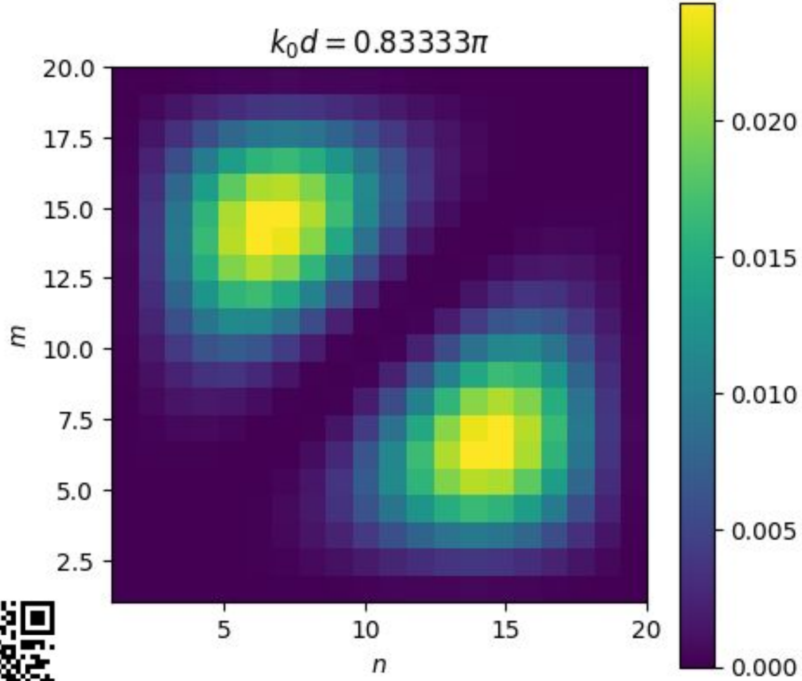
(We are not sure why this happens)

Jonathan (Shao-Kai) Huang

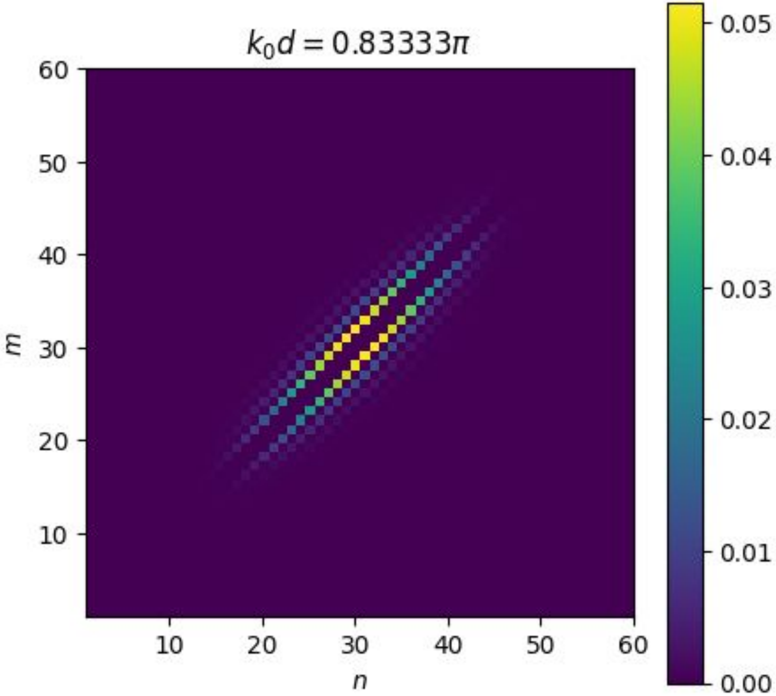
- New types of states appear as *n* increases!

$$k_0 d = (6n-1)\pi/6$$

Dimerized correlation and extremely dark



N = 20

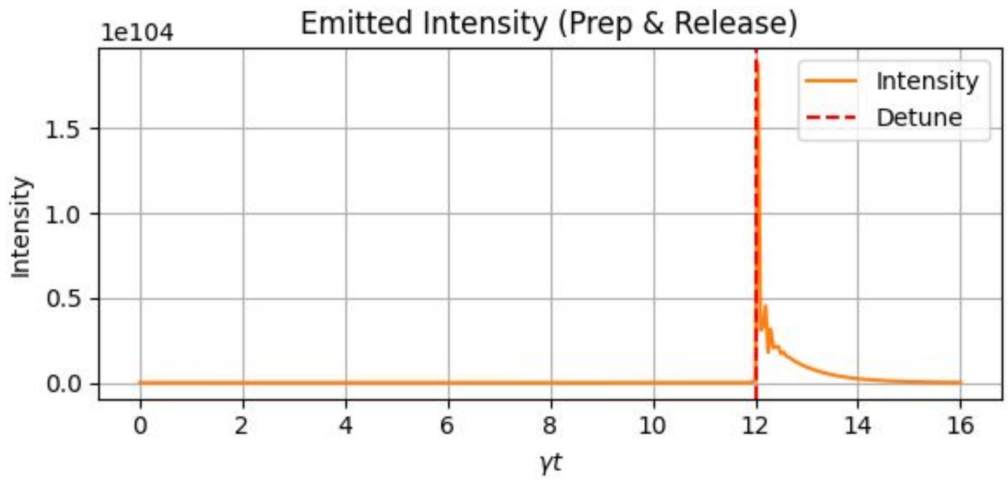


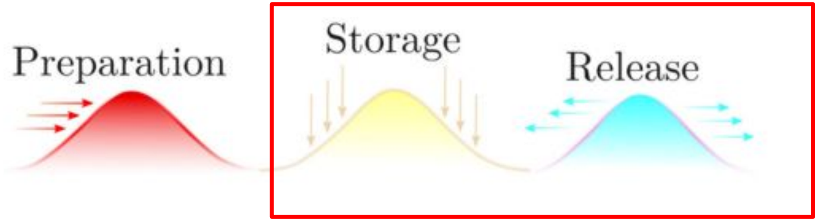
N = 60

Sudden increase in intensity and no “leaking” in between

Controlled storage and release of photons into waveguide

E.g. nonclassical multiphoton sources, memory for quantum repeater





R. Holzinger et al. (2022)

Conclusion

- **Localized dark states** are important because they trap excitations in a specific region without losing energy into the environment, making them ideal for **quantum information storage** and **precise control**.

- For photons, not only the vacuum state but also many other so-called **dark states** are **undetectable**.

- We discussed an implementation using transmon qubits, and exhibited various **spatial correlations** and a version of the **store-and-release protocol**.

Our Main Reference Paper

Phys. Rev. Lett. 129, 253601 (2022)
Control of Localized Single- and Many-Body Dark States in Waveguide QED

Phys. Rev. Lett. 134, 133603 (2025)
Bright and Dark States of Light: The Quantum Origin of Classical Interference

References

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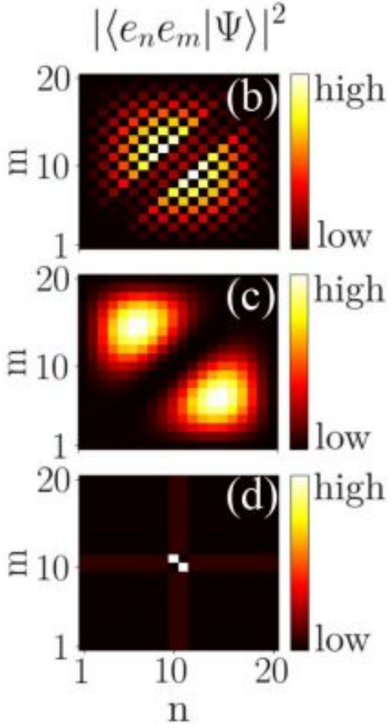
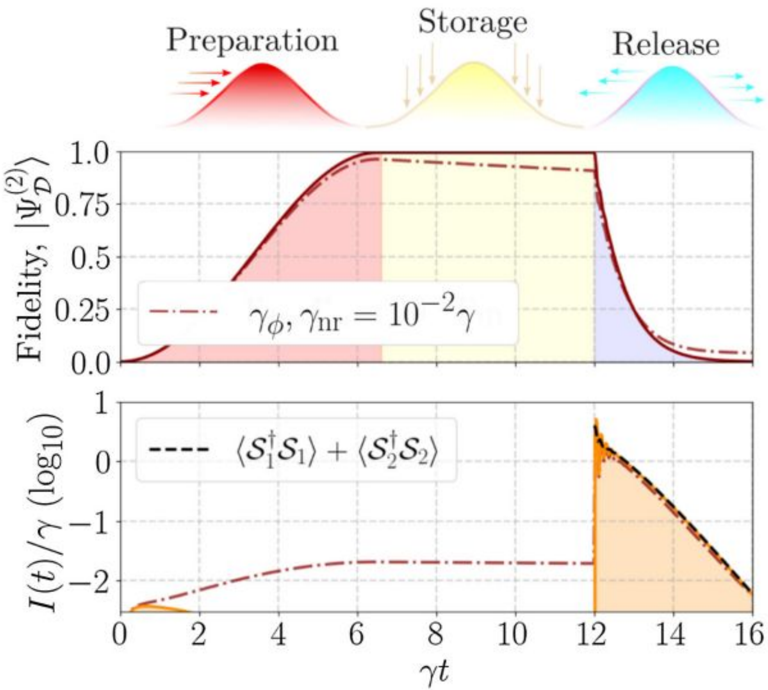
- C. J. Villas-Boas, C. E. Máximo, P. J. Paulino, R. P. Bachelard, and G. Rempe, **Bright and dark states of light: The quantum origin of classical interference**, Phys. Rev. Lett. 134, 133603 (2025).
- R. J. Glauber, **The quantum theory of optical coherence**, Phys. Rev. 130, 2529 (1963).
- R. Holzinger, R. Gutiérrez-Jáuregui, T. Hönigl-Decrinis, G. Kirchmair, A. Asenjo-Garcia, and H. Ritsch, **Control of localized single- and many-body dark states in waveguide qed**, Phys. Rev. Lett. 129, 253601 (2022).

Materials for Reference





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$$\left|\Psi_{\mathcal{D}}^{(M)}\right\rangle = \sqrt{\frac{(N-2M+1)!(N-2M)!}{(N-M+1)!(N-M)!}} \sum_{k=0}^{M} (-1)^k \binom{N-M-k}{M-k} \left[\sqrt{M}\mathcal{S}_1^\dagger\right]^{M-k} \left[\sqrt{N-M}\mathcal{S}_2^\dagger\right]^k \left|G\right\rangle.$$

This is the unique dark state which involves the operators $\mathcal{S}_1$, $\mathcal{S}_2$ and is orthogonal to $|\Psi_{\mathcal{S}}^{(M)}\rangle$. The cases for $M = 1, 2$ are already shown in the main text, the dark state for $M = 3$ excitations is given by

$$\left|\Psi_{\mathcal{D}}^{(3)}\right\rangle = \sqrt{\frac{N-5}{N-2}} \left[\frac{\sqrt{3}(\mathcal{S}_1^\dagger)^3}{2} - \frac{3(\mathcal{S}_1^\dagger)^2\mathcal{S}_2^\dagger}{2\sqrt{N-3}} + \frac{\sqrt{3}\mathcal{S}_1^\dagger(\mathcal{S}_2^\dagger)^2}{N-4} - \frac{\sqrt{N-3}(\mathcal{S}_2^\dagger)^3}{(N-4)(N-5)}\right] \left|G\right\rangle. \tag{S.6}$$

Given the expression for the M-excitation dark state, the excited state population in the first M qubits is expressed as

$$\sum_{j=1}^{M} \langle \Psi_{\mathcal{D}}^{(M)} | \hat{\sigma}_j^\dagger \hat{\sigma}_j | \Psi_{\mathcal{D}}^{(M)} \rangle = \frac{N-2M+1}{N-2M+2}. \tag{S.7}$$

Materials for Reference

I. MASTER EQUATION DERIVATION

In this Appendix, we present the approximations used to derive the superradiant master equation, the theoretical starting point for our work. The derivation is standard and follows from the conventional picture of a system coupled linearly to a one-dimensional bath formed of harmonic oscillators [1–3].

The evolution of bath and system is described by the Hamiltonian

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}_\mathrm{b} + \mathcal{H}_\mathrm{s} + \mathcal{H}_\mathrm{int}\,, \\
\mathcal{H}_\mathrm{b} &= \hbar \sum_s \int_0^\infty d\omega \omega b_s^\dagger(\omega) b_s^\dagger(\omega)\,, \\
\mathcal{H}_\mathrm{int} &= \hbar \sum_{n,s} \int d\omega \lambda_{n,s}(\omega) \left(b_s(\omega) + b_s^\dagger(\omega)\right) \left(c_n + c_n^\dagger\right)\,,
\end{aligned} \tag{S.1}$$

where $b_s(\omega)$ are boson annihilation operators for bath modes of frequency ω propagating along the $s = \pm$ directions, and c_n are system operators for the nth qubit. The coupling strength $\lambda_{n,s}(\omega)$ depends on the field amplitude evaluated at the qubit position x_n, with $\lambda_{n,s}(\omega) \propto \exp[-is\omega x_n/c]$. This rather general coupling can accurately describe atoms coupled to an electromagnetic environment under the electric dipole approximation [3] or transmon qubits capacitively coupled to a transmission line [4–6]. We restrict to the two-level approximation where system operators become $\hat{c}_n = \hat{\sigma}_n$ and the system Hamiltonian is

$$\mathcal{H}_\mathrm{s} = \sum_n \omega_0 \sigma_n^\dagger \sigma_n\,. \tag{S.2}$$

In an interaction picture with respect to the free Hamiltonians $\mathcal{H}_\mathrm{b}$ and $\mathcal{H}_\mathrm{s}$ the density operator for bath plus system $\chi(t)$ follows the equation $i\hbar\dot{\tilde{\chi}} = [\tilde{\mathcal{H}}_\mathrm{int}, \tilde{\chi}]$. This differential equation gives way to an integral equation

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$$i\hbar\dot{\rho} = -\frac{1}{\hbar^2}\int_0^t d\tau[\tilde{\mathcal{H}}_{\rm int}(t), [\tilde{\mathcal{H}}_{\rm int}(t-\tau), \tilde{\chi}(t-\tau)]] \tag{S.3}$$

where bath variables have been traced out to obtain the density matrix of the array ρ. In writting Eq. (S.3) we have assumed that system and bath are uncorrelated at an initial time.

Equation (S.3) describes the self-consistent evolution of system and bath, it thus accounting for the correlations that rise between the two. The description can be simplified for a large bath weakly coupled to the qubits $\omega_0 \gg \lambda_{n,s}(\omega_0)$ and with a smooth frequcy spectrum around the qubit resonance frequency. Futher simplification follows if we consider that the array is sufficiently small so that the only changes in a qubit as a free bath mode propagates from one end of the array to the other are given by the qubits free evolution [6, 7]. Under these conditions it is possible to make the Born-Markov approximation. This approximation neglects the correlations that arise between field and bath and sets the correlation time of the environment as the shortest time-scale of the system [1]. The upper limit in the temporal integral can be extended to infinity once this approximation has been made. After performing the temporal integral, summing over left- and right-propagating modes of the bath, and eliminating non-resonant terms under the rotating-wave approximation [8], Eq. (S.3) takes the form

$$\dot{\rho} = -i(\mathcal{H}_{\rm eff}\rho - \rho\mathcal{H}_{\rm eff}) + \sum_{n,m}\gamma\cos k_0|x_n - x_m|\sigma_m\rho\sigma_n^\dagger\,,$$

where

$$\mathcal{H}_{\rm eff} = \sum_{n,m}\tfrac{1}{2}\gamma(\sin k_0|x_n - x_m| - i\cos k_0|x_n - x_m|)\sigma_n^\dagger\sigma_m + \sum_n \Delta_{\rm rad}\sigma_n^\dagger\sigma_n \tag{S.4}$$