Assignment 2: Probability Bayesian

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2.

Set sunny $=1$ cloudy $=2$ rainy $=3$

2.(a)

$$P(X_1=1 \cap X_2=2 \cap X_3=2 \cap X_4=3)$$

$$= P(X_4=3 \mid X_1=1 \cap X_2=2 \cap X_3=2) \times P(X_1=1 \cap X_2=2 \cap X_3=2)$$

$$= P(X_4=3 \mid X_3=2) \times P(X_3=2 \mid X_1=1 \cap X_2=2) \times P(X_1=1 \cap X_2=2)$$

$$= P(X_4=3 \mid X_3=2) \times P(X_3=2 \mid X_2=2) \times P(X_2=2 \mid X_1=1) \times P(X_1=1)$$

$$= 0.2 \times 0.4 \times 0.2 \times 1$$

$$= 0.016 \ \#$$

2.(b)

My code:

Assignment 2 > exercise_2.py > ...

```python
import random as rd

print("Exercise 2.b: ")
cnt_weath = [0,0,0]
weath = ["sunny", "cloudy", "windy"]
cnt = 0
trans = [[0.8, 1.0, 1.0],[0.4, 0.8,1.0],[0.2,0.8,1.0]]

yest = 0
today = 0
while cnt < 10000000:
    tran = trans[yest]
    p = rd.random()
    if p < tran[0]:
        today = 0
    elif p < tran[1]:
        today = 1
    else:
        today = 2
    print(weath[today])
    cnt_weath[today] = cnt_weath[today] + 1
    cnt = cnt+1
    yest = today
```

The result of the weather generation

```
cloudy
sunny
sunny
sunny
sunny
sunny
sunny
sunny
sunny
sunny
sunny
sunny
```

```
sunny
cloudy
cloudy
windy
windy
cloudy
cloudy
cloudy
windy
cloudy
sunny
sunny
cloudy
cloudy
cloudy
```

2.(c)

My code:

```
Assignment 2 > exercise_2.py > ...
import random as rd

print("Exercise 2.c: ")
cnt_weath = [0,0,0]
weath = ["sunny", "cloudy", "windy"]
cnt = 0
trans = [[0.8, 1.0, 1.0],[0.4, 0.8,1.0],[0.2,0.8,1.0]]

yest = 0
today = 0
while cnt < 10000000:
    tran = trans[yest]
    p = rd.random()
    if p < tran[0]:
        today = 0
    elif p < tran[1]:
        today = 1
    else:
        today = 2
    # print(weath[today])
    cnt_weath[today] = cnt_weath[today] + 1
    cnt = cnt+1
    yest = today

print(float(cnt_weath[0])/cnt, float(cnt_weath[1])/cnt, float(cnt_weath[2])/cnt)
```

The calculated result of the distribution

```
C:\Users\jonat\Desktop\Self-Driving-Cars\Assignment 2>python exercise_2.py
Exercise 2.c:
0.6428788 0.2857488 0.0713724
```

2(d)

Let $Y_t = \begin{bmatrix} P(X_t=1) \\ P(X_t=2) \\ P(X_t=3) \end{bmatrix}$

$$A = \begin{bmatrix} P(X_{t+1}=1 \mid X_t=1) & P(X_{t+1}=1 \mid X_t=2) & P(X_{t+1}=1 \mid X_t=3) \\ P(X_{t+1}=2 \mid X_t=1) & P(X_{t+1}=2 \mid X_t=2) & P(X_{t+1}=2 \mid X_t=3) \\ P(X_{t+1}=3 \mid X_t=1) & P(X_{t+1}=3 \mid X_t=2) & P(X_{t+1}=3 \mid X_t=3) \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 \\ 0 & 0.2 & 0.2 \end{bmatrix}$$

We already know $Y_t = A \cdot Y_{t-1}$

$$\Rightarrow \forall t \in \mathbb{N}, \; Y_t = A^n \cdot Y_0$$

The matrix A is diagonalizable, which $A = P \cdot D \cdot P^{-1}$

$$P = \begin{bmatrix} 9 & \sqrt{2}-1 & -\sqrt{2}-1 \\ 4 & -\sqrt{2} & \sqrt{2} \\ 1 & 1 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} \frac{1}{14} & \frac{1}{14} & \frac{1}{14} \\ \frac{-\sqrt{2}(\sqrt{2}-4)}{56} & \frac{-\sqrt{2}(\sqrt{2}+10)}{56} & \frac{\sqrt{2}(13\sqrt{2}+4)}{56} \\ \frac{-\sqrt{2}(\sqrt{2}+4)}{56} & \frac{-\sqrt{2}(\sqrt{2}-10)}{56} & \frac{\sqrt{2}(13\sqrt{2}-4)}{56} \end{bmatrix}$$

By $\lim_{n\to\infty} D^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \lim_{n\to\infty} Y^n = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot P^{-1} \cdot X_0 = \begin{bmatrix} \frac{9}{14} \\ \frac{2}{7} \\ \frac{1}{14} \end{bmatrix}$

(no matter what the initial distribution of X_0) #

2.(e)

Def of Entropy: $H(X) := -\sum_{x \in X} P(X) \cdot \log P(X) = E[-\log P(X)]$

Entropy of the system is $-\sum_i \bar{X}_i \cdot \log_2(\bar{X}_i) = 1.198117$ #

2.(f)

Let us compute the conditional probability of X_{t-1} given X_t

$\forall (i,j) \in [\![1,3]\!]^2$

$$P(X_{t-1} = i \mid X_t = j) = \eta\, P(X_t = j \mid X_{t-1} = i) \times P(X_{t-1} = i)$$

$$= \frac{P(X_t = j \mid X_{t-1} = i) \times P(X_{t-1} = i)}{P(X_t = j)}$$

Considering the limit distributions as $t \to \infty$

$$m_{ij} = \frac{P(X_t = j \mid X_{t-1} = i) \times \pi[j]}{\pi[j]} = \frac{a_{j,i} \times \pi[j]}{\pi[j]}$$

$\Rightarrow$ Entropy of the stationary distribution $\Rightarrow \begin{bmatrix} \frac{36}{45} & 0.45 & 0 \\ \frac{8}{45} & 0.4 & 0.8 \\ \frac{1}{45} & 0.15 & 0.2 \end{bmatrix}$

#

2(g)

The markov property states the probability law of future state conditioned on the current state does not depend on any other variables. However, the current state could be sufficient to compute the future stochastic evolution, which means the state transition function can not depent on the season. Hence, to restore the Markov property, we can incorporate the season into the state variable.

3(a)

We can obtain the status transition matrix from the previous problem

which is $A = \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 \\ 0 & 0.2 & 0.2 \end{bmatrix}$

We can get the observation matrix from the question

which is $C = \begin{bmatrix} 0.6 & 0.3 & 0 \\ 0.4 & 0.7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

According to the Bayes rule

$$\overline{bel}(X_t \mid X_{t-1}) = A \cdot bel(X_{t-1})$$

$$bel(X_t = i \mid Z_t = j) = \eta\, P(Z_t = j \mid X_t = i) \cdot P(X_t = i)$$

$$\eta = \left(\sum_{i=1}^{n} P(Z_t = j \mid X_t = i) \cdot P(X_t = i) \right)^{-1}$$

We set the initial value as

$bel(X_1) = [1 \ 0 \ 0]^T$ $\quad Z_2 = \text{cloudy}$ $\quad Z_3 = \text{rainy}$ $\quad Z_4 = \text{sunny}$

Through recursive calculation, we can get the following chart

Z_t	$\overline{bel}(X_t)$	$bel(X_t)$	η
sunny	$[1 \ 0 \ 0]^T$	$[1 \ 0 \ 0]^T$	1
cloudy	$[0.8 \ 0.2 \ 0]^T$	$[\frac{16}{23} \ \frac{7}{23} \ 0]^T$	2.1739
cloudy	$[\frac{78}{115} \ \frac{30}{115} \ \frac{7}{115}]^T$	$[\frac{312}{522} \ \frac{210}{522} \ 0]^T$	2.2031
rainy	$[0.639 \ 0.2805 \ 0.0805]^T$	$[0 \ 0 \ 1]^T$	124.2857
sunny	$[0.2 \ 0.6 \ 0.2]^T$	$[0.4 \ 0.6 \ 0.]^T$	3.3333

Through the chart above, we get the probability for Day 5 to be sunny is 0.4 #

3(b)

$P(X_2 | X_1, Z_2)$

$= \eta P(Z_2 | X_2, X_1) \cdot P(X_2 | X_1)$

$= \eta \cdot P(Z_2 | X_2) \cdot P(X_2 | X_1)$

$= \eta \cdot \begin{pmatrix} 0.6 \\ 0.3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0.8 \\ 0.2 \\ 0 \end{pmatrix} = \eta \cdot \begin{pmatrix} 0.48 \\ 0.06 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{8}{9} \\ \frac{1}{9} \\ 0 \end{pmatrix}$

Day 2: 88.9 % sunny, 11.1% cloudy, 0% rainy #

⇒ Day 2 with data from future days. 80% sunny. 20% cloudy. 0% rainy

$P(X_2 | X_1, Z_{2:4}) = \eta P(X_2|X_1) \cdot P(Z_{2:4} | X_2, X_1)$

$= \eta P(X_2 | X_1) \cdot P(Z_{2:4} | X_2)$

$= \eta \cdot P(X_2 | X_1) \cdot P(Z_2 | Z_{3:4}, X_2) \cdot P(Z_{3:4} | X_2)$

$= \eta \cdot P(X_2 | X_1) \cdot P(Z_2 | X_2) \cdot P(Z_{3:4} | X_2)$

$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot \sum_{X_3} P(X_3, Z_{3:4} | X_2)$

$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot \sum_{X_3} P(X_3 | X_2) \cdot P(Z_{3:4} | X_3, X_2)$

$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot \sum_{X_3} P(X_3|X_2) \cdot P(Z_{3:4} | X_3)$

$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot \sum_{X_3} P(X_3|X_2) \cdot P(Z_3|X_3) \cdot P(Z_4 | Z_3, X_3)$

$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot \sum_{X_3} P(X_3|X_2) \cdot P(Z_3|X_3) \cdot P(Z_4|X_3)$

$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot \sum_{X_3} P(X_3|X_2) \cdot P(Z_3|X_3) \cdot \sum_{X_4} P(X_4, Z_4 | X_3)$

$$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot \sum_{X_3} P(X_3|X_2) \cdot P(Z_3|X_3) \cdot \sum_{X_4} P(X_4|X_3) \cdot P(Z_4|X_4, X_3)$$

$$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot \sum_{X_3} P(X_3|X_2) \cdot P(Z_3|X_3) \cdot \sum_{X_4} P(X_4|X_3) \cdot P(Z_4|X_4)$$

$$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot \sum_{X_3} P(X_3|X_2) \cdot P(Z_3|X_3) \cdot \sum_{X_4} P(X_4|X_3) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{P(X_4|Z_4)}$$

$$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot \sum_{X_3} P(X_3|X_2) \cdot P(Z_3|X_3) \cdot \begin{pmatrix} 0 \\ 0.2 \\ 0.2 \end{pmatrix}_{P'(X_3|Z_4)}$$

$$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot \sum_{X_3} P(X_3|X_2) \cdot \begin{pmatrix} 0 \\ 0.06 \\ 0 \end{pmatrix}_{P'(X_3|Z_3, Z_4)}$$

$$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot \begin{pmatrix} 0.012 \\ 0.024 \\ 0.036 \end{pmatrix}_{P'(X_2|Z_3, Z_4)}$$

$$= \eta \cdot \begin{pmatrix} 0.8 \\ 0.2 \\ 0 \end{pmatrix}_{P'(X_2|X_1)} \cdot \begin{pmatrix} 0.6 \\ 0.3 \\ 0 \end{pmatrix}_{P'(X_2|Z_2)} \cdot \begin{pmatrix} 0.012 \\ 0.024 \\ 0.036 \end{pmatrix}_{P'(X_2|Z_3, Z_4)}$$

$$= \eta \cdot \begin{pmatrix} 0.00576 \\ 0.00144 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.2 \\ 0 \end{pmatrix}$$

⇒ Day 2 with data from future days : 80% sunny, 20% cloudy, 0% rainy #

$$P(X_2|X_1, Z_2, Z_3) = \eta \cdot P(X_2|X_1) \cdot P(Z_2, Z_3|X_2, X_1)$$

$$= \eta \cdot P(X_2|X_1) \cdot P(Z_2, Z_3|X_2)$$

$$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|Z_3, X_2) \cdot P(Z_3|X_2)$$

$$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot P(Z_3|X_2)$$

$$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot \sum P(X_3, Z_3|X_2)$$

$$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot \sum P(X_3|X_2) \cdot P(Z_3|X_3, X_2)$$

$$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot \sum P(X_3|X_2) \cdot P(Z_3|X_3)$$

$$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot \sum P(X_3|X_2) \cdot \begin{pmatrix} 0.6 \\ 0.3 \\ 0 \end{pmatrix}_{P(X_3|Z_3)}$$

$$= \eta \cdot P(X_2|X_1) \cdot P(Z_2|X_2) \cdot \begin{pmatrix} 0.60 \\ 0.36 \\ 0 \end{pmatrix}_{P'(X_2|Z_3)}$$

$$= \eta \cdot \begin{pmatrix} 0.8 \\ 0.2 \\ 0 \end{pmatrix}_{P'(X_2|X_1)} \cdot \begin{pmatrix} 0.6 \\ 0.3 \\ 0 \end{pmatrix}_{P'(X_2|Z_2)} \cdot \begin{pmatrix} 0.54 \\ 0.36 \\ 0 \end{pmatrix}_{P'(X_2|X_3)}$$

$$= \eta \cdot \begin{pmatrix} \frac{2592}{2808} \\ \frac{216}{2808} \\ 0 \end{pmatrix} = \begin{pmatrix} 0.9231 \\ 0.0769 \\ 0 \end{pmatrix}$$

⇒ Probability of day 2 given data from only day 3

is 92.3 % sunny, 7.7 % cloudy, 0 % rainy #

$$P(X_3|X_1, Z_{2:3}) = \eta \cdot P(Z_3|X_3, X_1, Z_2) \cdot P(X_3|X_1, Z_2)$$

$$= \eta \cdot P(Z_3|X_3) \cdot \sum_{X_2} P(X_3, X_2 | X_1, Z_2)$$

$$= \eta \cdot P(Z_3|X_3) \cdot \sum_{X_2} P(X_3|X_2) \cdot P(X_2|X_1, Z_2)$$

$$= \eta \cdot P(Z_3|X_3) \cdot \sum_{X_2} P(X_3|X_2) \cdot \begin{pmatrix} 8 \\ 1 \\ 0 \end{pmatrix}_{P'(X_2|X_1, Z_2)}$$

$$= \eta \cdot P(Z_3|X_3) \cdot \begin{pmatrix} 6.8 \\ 2.0 \\ 0.2 \end{pmatrix}_{P'(X_3|X_1, Z_2)}$$

$$= \eta \cdot \begin{pmatrix} 0.6 \\ 0.3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 6.8 \\ 2.0 \\ 0.2 \end{pmatrix} = \begin{pmatrix} \frac{408}{468} \\ \frac{60}{468} \\ 0 \end{pmatrix} = \begin{pmatrix} 0.8718 \\ 0.1282 \\ 0 \end{pmatrix}$$

$\Rightarrow$ Day 3 : 87.2 % sunny, 12.8 % cloudy, 0% rainy.

$$P(X_3 | X_1, Z_{2:4}) = \eta \cdot P(X_3 | X_1, Z_{2:3}) \cdot P(Z_4 | X_3, X_1, Z_{2:3})$$

$$= \eta \cdot P(X_3 | X_2, Z_3) \cdot P(Z_4 | X_3)$$

$$= \eta \cdot \begin{pmatrix} 0.8395 \\ 0.1605 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0.2 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\rightarrow$ Day 3 with data from future days: 0 % sunny, 100% cloudy, 0 % rainy #

$$P(X_4 | X_1, Z_{2:4}) = P(X_4 | X_3, Z_4)$$

$$= \eta \cdot P(Z_4 | X_4) \cdot P(X_4 | X_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\rightarrow$ Day 4 : 0% sunny, 0 % cloudy, 100 % rainy #

3(c)

The probability of the sequence of weather is given by

$$P(X_{2:4} | X_1, Z_{2:4}) = \eta \cdot P(Z_{2:4} | X_1, X_{2:4}) \cdot P(X_{2:4} | X_1)$$

where

$$P(X_{2:4} | X_1) = P(X_4 | X_3) \cdot P(X_3 | X_2) \cdot P(X_2 | X_1)$$

$$P(Z_{2:4} | X_1, X_{2:4}) = P(Z_4 | X_4) \cdot P(Z_3 | X_3) \cdot P(Z_2 : X_2)$$

Hence, the "sunny, cloudy, rainy" probability is

$$\frac{0.00576}{0.00576 + 0.00144} = 80\%$$

which there is a 20% probability of "cloudy, cloudy, rainy" and 0% of probability for all the other sequences.