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915

(3) (d)

$$\int_0^\infty e^{-x} \sin x \, dx = ?$$

$u = \sin x$, $du = \cos x$
$dv = e^{-x}$, $v = -e^{-x}$ $\Bigg\} = 1$

$\int e^{-x} \sin x \, dx = -e^{-x} \sin(x) - \int -e^{-x} \cos(x) \, dx =$

$u = \cos x$, $du = -\sin x$
$dv = -e^{-x}$, $v = e^{-x}$ $\Bigg\} = 1$

$$= -e^{-x}\sin x - \left(e^{-x}\cos(x) - \int -e^{-x}\sin x\,dx \right) =$$

$$\Rightarrow I = \frac{-e^{-x}\sin(x) - e^{-x}\cos x}{2} + C$$

$$= -\frac{e^{-x}(\sin x + \cos x)}{2} + C$$

But we need to find

$\rightarrow$ for ∞, $e^{-x} \to 0$

$$\int_0^\infty e^{-x}\sin x\,dx = -\frac{e^{-x}(\sin x + \cos x)}{2}\Bigg|_0^\infty = \frac{1}{2}$$

(1) b)

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{k \cdot e^{\frac{k}{n}}}{n} =$$

Applying Riemann Sum

$$= \int_0^1 x e^x = (x-1)e^x \Big|_0^1 = 1$$

d)

$$\text{let } \ell = \lim_{n\to\infty} \left(\sin\frac{\pi}{2n} \cdot \ldots \sin\frac{(n-1)\pi}{2n} \right)^{\frac{1}{n}}$$

$$\log(\ell) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sin\frac{\pi}{2n} \cdot \ldots \sin\frac{(n-1)\pi}{2n} \right)$$

$$\log(\ell) = \lim_{n\to\infty} \frac{1}{n} \left[\log\left(\sin\frac{\pi}{2n}\right) + \ldots + \log\left(\sin\frac{(n-1)\pi}{2n}\right) \right]$$

$$\log(\ell) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \log\left(\sin\frac{k\pi}{2n}\right)$$

$$\log(l) = \int_0^1 \log\left(\sin \frac{x\pi}{2}\right), \quad \text{let } t = \frac{x\pi}{2} = t$$

$$dx = \frac{2}{\pi}\, dt$$

$$\log(l) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \log(\sin t)\, dt$$

$$\text{let } I = \int_0^{\frac{\pi}{2}} \log(\sin t)\, dt$$

$$I = \int_0^{\frac{\pi}{2}} \log(\cos t)\, dt$$

$$2I = \int_0^{\frac{\pi}{2}} \left[\log \sin t + \log \cos t\right] dt =$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{2 \sin t \cos t}{2}\right) dt =$$

$$= \int_0^{\frac{\pi}{2}} \log \sin(2t)\,dt - \int_0^{\frac{\pi}{2}} \log 2\,dt$$

$$2t = y \,,\quad dt = \frac{y}{2}$$

$$2I = \frac{1}{2}\int_0^{\pi} \log \sin y\,dy - \log_2 (t)\Big|_0^{\frac{\pi}{2}}$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin t)\,dt - \frac{\pi}{2}\log 2$$

$$2I = I - \frac{\pi}{2}\log 2 \Rightarrow$$

$$\Rightarrow \quad \log(\ell) = \frac{2}{\cancel{\pi}}\left(-\frac{\cancel{\pi}}{2}\log 2\right)$$

$$\Rightarrow \quad \log\left(\frac{1}{2}\right) \quad (\Leftrightarrow) \quad p = \frac{1}{2}$$