Final Exam #1

1. $u, v \in \mathbb{R}^n$

$$\|u\| = \|v\| = \|u - v\| = 1$$

a) $$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$

$$\|u+v\| = \sqrt{3}$$

b)

$n \in \{2, 3\} \Rightarrow$

$$\Rightarrow xy = \|x\| \, \|y\| \cos \angle(x, y)$$

$$\cos(\angle u, v) = \frac{u \cdot v}{\|u\| \cdot \|v\|} = u \cdot v$$

$$\Rightarrow \angle(u, v) = \frac{\pi}{3}$$

$$\|u-v\|^2 = 1 \Rightarrow \|u\|^2 + \|v^2\| - 2uv = 1$$

$$uv = \frac{1}{2}$$

②

$$f(x,y) = e^{x^2+y} \quad , \quad (a,b) = (1,0)$$

$$T_2(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$+ \frac{1}{2} f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{1}{2} f_{yy}(a,b)$$

$$(y-b)^2$$

$$f(a,b) = e$$

$$f_x(x,y) = 2x e^{x^2+y}$$

$$f_x(a,b)(x-a) = 2e(x-1)$$

$$f_y(x,y) = e^{x^2+y}$$

$$\Rightarrow f_y(a,b)(y-b) = e(y)$$

$$f_{xx}(x,y) = 2e^{x^2+y} + 4x^2 e^{x^2+y} = 2(2x^2+1)e^{x^2+y}$$

$$\Rightarrow \frac{1}{2} f_{xx}(a,b)(x-a)^2 = \frac{1}{2} 6e(x-1)^2$$

$f_{xy}(x,y) = 2xe$

$\to f_{xy}(a,b)(x-a)(y-b) = 2e(x-1)y$

$f_{yy}(x,y) = e^{x^2+y}$

$\to \frac{1}{2} f_{yy}(a,b)(y-b)^2 = \frac{1}{2} e y^2$

③ $f: \mathbb{R}^n \to \mathbb{R}$, differentiable at a local extrem.

~~Suppose~~ S

~~Let~~ $\cancel{i \in \{1, \dots, n\}}$

For small ε, we can define $g: (-\varepsilon, \varepsilon) \to \mathbb{R}^n$

$g(t) = x + t\nabla f(x)$.

$f \circ g$ has a local extremum at 0, so

$(f \circ g)'(0) = 0$, by the chain rule and Fermat th.

$0 = (f \circ g)'(0) = \nabla f(x) \cdot \nabla f(x) \Rightarrow$

$\Rightarrow \nabla f(x) = 0$

(4) $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x^T A x$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

a)

$$f(x) = (x_1 \ \ x_2) \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$$

$$= (x_1 + 2x_2 \quad 2x_1 - x_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 - x_2^2 + 4x_1 x_2$$

$$\left.\begin{aligned} f_{x_1}(x) &= 2x_1 + 4x_2 \Rightarrow f(1,0) = 2 \\ f_{x_2}(x) &= -2x_2 + 4x_1 \Rightarrow f(1,0) = 4 \end{aligned}\right| \Rightarrow$$

$$\Rightarrow \nabla f(1,0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

steepest dex $-\nabla f(x)$ $\Big| \Rightarrow \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

b) $$\left.\begin{aligned} f_{x_1}(x) = 0 &\Leftrightarrow 2x_1 + 4x_2 = 0 \\ f_{x_2}(x) = 0 &\Leftrightarrow -2x_2 + 4x_1 = 0 \end{aligned}\right\} \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ - critical point

$$\left.\begin{aligned} f_{x_1x_1}(x) &= 2 \\ f_{x_2x_2}(x) &= -2 \\ f_{x_1x_2}(x) &= 4 \end{aligned}\right| \Rightarrow D(f(x,y)) = f_{xx} \cdot f_{yy} - f_{xy}^2 =$$

$$= -4 - 16 = -20 < 0 \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ - saddle point

$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

c) $\|x\|^2 = 1 \Rightarrow x_1^2 + x_2^2 = 1$

$L(x_1, x_2, \lambda) = x_1^2 - x_2^2 + 4x_1x_2 - \lambda(x_1^2 + x_2^2 - 1)$

$L_{x_1}(x_1, x_2, \lambda) = 2x_1 + 4x_2 - 2\lambda x_1$

$L_{x_2}(x_1, x_2, \lambda) = -2x_2 + 4x_1 - 2\lambda x_2$

$L_\lambda = x_1^2 + x_2^2 - 1$

$\Rightarrow$

$\Rightarrow \quad x_1 + 2x_2 - \lambda x_1 = 0 \quad \Rightarrow \lambda = \frac{x_1 + 2x_2}{x_1}$

$-x_2 + 2x_1 - \lambda x_2 = 0 \quad \Rightarrow$

$x_1^2 + x_2^2 - 1 = 0$

$\Rightarrow \quad -x_2 + 2x_1 - \frac{(x_1 + 2x_2)x_2}{x_1} = 0$

$-\cancel{x_2} + x_1 - \frac{x_2^2}{x_1} = 0 \Leftrightarrow x_1 - \frac{x_2^2}{x_1} = \sqrt{1 - x_1^2} \Leftrightarrow$

$\Leftrightarrow 2x^2 - 1 = \sqrt{1 - x^2}\, x \Leftrightarrow 4x^4 - 4x^2 + 1 = x^2 - x^4 \Leftrightarrow$

$\Leftrightarrow 5x^4 - 5x^2 + 1 = 0 \quad , t = x^2 \Rightarrow$

$\Rightarrow x = \left(\pm \frac{\sqrt{50 + 10\sqrt{5}}}{10}, \pm \frac{\sqrt{50 - 10\sqrt{5}}}{10} \right) \Rightarrow$

$$\frac{1-\sqrt{5}}{20}\sqrt{10\sqrt{5}+50}, \sqrt{\frac{\sqrt{5}}{10}+\frac{1}{2}}\Big)$$

$$P_3\left(\frac{-(1+\sqrt{5})}{20}\sqrt{50-10\sqrt{5}}\,;\ -\sqrt{\frac{1}{2}-\frac{\sqrt{5}}{10}}\right)$$

$$P_4=\left(\frac{1+\sqrt{5}}{20}\sqrt{50-10\sqrt{5}}\,;\ \sqrt{\frac{1}{2}-\frac{\sqrt{5}}{10}}\right)$$

Calculating $f(x)$ for P_1, P_2, P_3, P_4, we get P_1 - minimum
P_3 - maximum

d) $\|x\|^2 = x^T x = 1$

$$L(x) = x^T B x - \lambda(x^T x - 1)$$

$$L_x'(x) = 0 \iff 2x^T B - 2\lambda x^T = 0$$

$$Bx - \lambda x = 0 \Rightarrow Bx = \lambda x$$

$x_1, \dots x_n$ - eigenvectors of B are the critical points
their eigenvalues $\lambda_1, \dots \lambda_n$ are the values of Langrange Mult.
Hence, $\max x^T Bx$ subject to $\|x\|^2=1$ is attained
by the eigenvector of B corresponding to the largest
eigenvalue. (q.e.d)

5.

a) $\int_0^\infty e^{-x} x^n \, dx \quad , n \in \mathbb{N}$

$$\int_0^\infty e^{-x} x^n = \underbrace{-e^{-x} \cdot x^n}_{0} + n \underbrace{\int_0^\infty x^{n-1} e^{-x} \, dx}_{I_{n-1} \dots} =$$

$$= n!$$

b) $\int_0^1 \int_0^1 \max(x,y) \, dx \, dy = \int_0^1 \int_0^y y \, dx \, dy + \int_0^1 \int_y^1 x \, dx \, dy$

$$= \int_0^1 \left[\int_0^x x \, dy + \int_{\cancel{x}}^1 y \, dy \right] dx = \int_0^1 \left[x^2 + \frac{1}{2} - \frac{1}{2} x^2 \right] dx =$$

$$= \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

, only space above xy-plane

$x^2+y^2 \leq 1$, circle with $R=1$

$x^2+y^2+z^2 \leq 4$, sphere with $R=2$

$x = r\cos\theta$, $y = r\sin\theta$, $z=z$

$\Rightarrow$

$$\Rightarrow \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq \sqrt{4-r^2} \end{array} \Bigg| \Rightarrow V = \int_0^{2\pi}\int_0^{1}\int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^{1} r\sqrt{4-r^2} \, dr \, d\theta \quad , \quad u = 4-r^2 \quad =$$

$$= \int_0^{2\pi} -\frac{(4-r^2)^{\frac{3}{2}}}{3}\Bigg|_0^1 d\theta =$$

$$= \int_0^{2\pi}\left(\frac{8}{3}-\sqrt{3}\right) d\theta = \left(\frac{8}{3}-\sqrt{3}\right)2\pi$$