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(915)

Extra Seminar IV

1. Since $\sum_{n=1}^{\infty} \frac{1}{n} \to \infty \Rightarrow$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2n} \to \infty$

let $\alpha_1 \in \mathbb{N}$ such that

$2 < \frac{1}{2} + \frac{1}{4} + \dots \frac{1}{2\alpha_1} < 3 \quad | -1$

$1 < \frac{1}{2} + \frac{1}{4} + \dots \frac{1}{2\alpha_1} - 1 < 2$

let $\alpha_2 \in \mathbb{R}$ such that

$$3 < \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2\alpha_1} - 1 + \dots + \frac{1}{2\alpha_2} < 4$$

$\checkmark$

$$\Big| -\frac{1}{3}$$

$$2 < \frac{8}{3} < \frac{1}{2} + \frac{1}{4} + \dots \frac{1}{2\alpha_1} - 1 + \frac{1}{2(\alpha_1+1)} + \dots +$$

$$+ \frac{1}{2\alpha_2} - \frac{1}{3}$$

Let $\alpha_3 \in \mathbb{N}$ such that

$$4 < \frac{1}{2} + \frac{1}{4} + \dots \frac{1}{2\alpha_1} - 1$$

$$+ \frac{1}{2(\alpha_1+1)} + \dots + \frac{1}{2\alpha_2} - \frac{1}{3}$$

$$+ \frac{1}{2(\alpha_2+1)} + \dots \frac{1}{2\alpha_3} < 5 \quad \Big| -\frac{1}{5}$$

3c $\frac{19}{5} < \frac{1}{2} + \frac{1}{4} + \dots \frac{1}{2\alpha_1} - 1$

$+ \frac{1}{2(\alpha_1+1)} + \dots + \frac{1}{2\alpha_2} - \frac{1}{3}$

$+ \frac{1}{2(\alpha_2+1)} + \dots \frac{1}{2\alpha_3} - \frac{1}{5}$

We can prove by induction that applying this process will give us a series with limit $\in \overline{\mathbb{R}}$, $x'_n \to \infty$

2. I(i) C_n - number of binary trees with $n+1$ leaves.

$C_0 = 1$

$C_{n+1} = ?$

If we have a root node,

we need $n+1-k$ leaves on one side and k leaves on the other side

We know that there are C_k ways to choose for one side and C_{n+1-k}

we get

$$C_{n+1} = \sum_{n=1}^{k} C_k \cdot C_{n-k}$$

b)

$$g(x) = \sum_{n}^{\infty} C_n x^n =$$

Let $y = \sum_{n \geq 0} c_n x^n$

Multiply x^n, $n \geq 0$

$$\sum_{n \geq 0} c_{n+1} x^n = \sum_{n \geq 0} \left(\sum_{k=0}^{n} c_k \cdot c_{n-k} \right) \cdot x^n$$

Now

$$x \sum_{n \geq 0} c_{n+1} x^n = \sum_{n \geq 1} c_n x^n = y - 1$$

Also

$\sum_{k=0}^{n} c_n c_{n-k}$ is the coefficient of x^n

in $\left(\sum_{n \geq 0} c_n x^n \right)^2 = y^2 \Rightarrow$

$$\Rightarrow \frac{y-1}{x} = y^2 \Rightarrow xy^2 - y + 1 = 0 \Leftrightarrow$$

$$(\Rightarrow) \quad y = \frac{1 \pm \sqrt{1-4x}}{2x}$$

We need to find which sign $(+, -)$ is correct:

From Taylor Series we get

$$\sqrt{1-4x} = 1 - 2x - 2x^2 + \dots$$

For $y = \sum C_n x^n$

$n \geq 0$

the + sign gives:

$$\frac{1+(1-2x-2x^2+\dots)}{2x} = \frac{1}{x} - 1 - x + \dots$$

which doesn't work for y, so it follows that $-$ sign should be correct.

$$\frac{1-(1-2x-2x^2+\dots)}{2x} = 1 + x + \dots + \dots$$

minus sign

$$y = \frac{1}{2x}\left(1-\sqrt{1-4x}\right) =$$

$$= \frac{1}{2x}\left(1 - \sum \binom{\frac{1}{2}}{n}(-4x)^n\right)$$

and $\displaystyle \binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-3}{2}\right)}{n!}$

which, simplified gives us:

$$y = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n$$

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!}$$

(q.e.d)