5. ★ Let $a, b \in \mathbb{R}$ with $a > 0$. If S is nonempty and bounded above, prove that

$$\sup_{x \in S}(ax + b) = a \sup(S) + b.$$

5.

let $M = \sup(S)$;

$H = \{ax+b \mid x \in S\}$;

$S_H = \sup(H) \Rightarrow S_H \geq x, \forall x \in H$ (1)

from the definition of supremum $\Rightarrow$

$M \geq x, \forall x \in S \Leftrightarrow$

$\Leftrightarrow aM + b \geq x, \forall x \in H$ (2)

(1) + (2) $\Rightarrow aM + b \geq S_H$

$$m \geq \frac{S_H - b}{a} \geq x, \forall x \in S \Rightarrow$$

$$\Rightarrow \frac{S_H - b}{a} = m \Rightarrow S_H = am + b$$

8. $a, b \in \mathbb{R}$

$U \in \mathcal{V}(a), V \in \mathcal{V}(b)$ s.t.

$$U \cap V = \emptyset$$

$a = b$ is impossible by default.

so for $a \neq b$:

$V(a) = (a - \varepsilon_1 ; a + \varepsilon_1)$

$V(b) = (b - \varepsilon_2 ; b + \varepsilon_2)$

Since $a \neq b$ and using the density property ($\mathbb{Q}$ is dense in $\mathbb{R}$), there will always exist $\varepsilon_1, \varepsilon_2 > 0$ such that $U \cap V = \emptyset$.

Another solution would be using and setting $\varepsilon = \frac{b-a}{2}$ and we prove $U \cap V = \emptyset$

b.

$A = [0,1] \cap \mathbb{Q}$

- Starting with $\mathbb{Q}$ is dense in $\mathbb{R}$, we have $\inf(A) = 0$, $0 \leq x, \forall x \in A$
- The same for $\sup(A) = 1$, 1 is the smallest lowerbound integer $(1 \geq x, \forall x \in A)$
- $\text{int}(A)$: $\{x \in \mathbb{R} \mid \exists V \in \mathcal{V}(x) \text{ s.t. } V \subseteq A\}$

$\mathbb{Q}$ is dense in $\mathbb{R}$ $\Rightarrow$ $\nexists V \in \mathcal{V}(x)$ s.t. $V \subseteq A$, due to the fact that

A is not an interval and a finite set!

$\Rightarrow \operatorname{int}(A) = \emptyset$

- $cl(A) := \{x \in \mathbb{R} \mid \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset\}$

$\mathbb{Q}$ is dense in $\mathbb{R}$

$$\left.\begin{array}{l} 1^\circ\ cl(A) \subseteq [0,1] \\ 2^\circ\ [0,1) \subseteq cl(A) \end{array}\right\} \Rightarrow$$

$\Rightarrow cl(A) = [0,1]$