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2 $x, y \in \mathbb{R}^n$.

Using Paralg. identity:

$\Rightarrow$ c) $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ (Pythagora's Th.)

$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle +$

$+ \langle y,y\rangle$ (1)

$\|x-y\|^2 = \langle x-y, x-y\rangle = \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle +$

$+ \langle y,y\rangle$ (2)

$\boxed{b) \Rightarrow c)}$

(1) + (2) $\Rightarrow$

$\|x+y\|^2 + \|x-y\|^2 = 2\langle x,x\rangle + 2\langle y,y\rangle = 2\|x\|^2 + 2\|y\|^2$

Using b) $\|x+y\| = \|x-y\|$

$\Rightarrow 2\|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad |:2$

$\|x+y\|^2 = \|x\|^2 + \|y\|^2$ (q.e.d)

$\boxed{a) \Rightarrow c)}$

Using a)

If x is orthogonal to y, $\langle x, y \rangle = 0 \Leftrightarrow$

$\Leftrightarrow \|x+y\|^2 = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle =$

$\|x\|^2 + \|y\|^2$ (q.e.d)

$\Rightarrow b) \quad \|x+y\| = \|x-y\| \quad (1)$

$\boxed{a) \Rightarrow b)}$

$(1) \Rightarrow \quad \|x+y\|^2 \stackrel{?}{=} \|x-y\|^2$

$\|x+y\|^2 = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \quad (\Leftarrow)$

from a) $\quad \langle x,y\rangle = \langle y,x\rangle = 0$

$(\Rightarrow) \quad \|x+y\|^2 = \langle x,x\rangle + \langle y,y\rangle$

$\|x+y\|^2 = \|x\|^2 + \|y\|^2$

which is true since $\langle x,y\rangle = 0$

$\boxed{(1) \Rightarrow b)}$

$\|x+y\|^2 = \|x\|^2 + \|y\|^2 = \langle x,x\rangle + \langle y,y\rangle =$

$= \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle = \|x-y\|^2$

a) $\langle x,y\rangle = 0$

b) $\Rightarrow$ a)

$$\|x+y\|^2 \overset{!}{=} \|x-y\|^2$$

$$(x_1+y_1)^2 + \dots (x_n+y_n)^2 \overset{!}{=} (x_1-y_1)^2 + \dots (x_n-y_n)^2$$

$$x_1^2+y_1^2+2x_1y_1+\dots x_n^2+y_n^2+2x_ny_2 = x_1^2+y_1^2-2x_1y_1+\dots x_n^2+y_n^2-2x_ny_n$$

identity holds true iff. $\langle x,y\rangle = \langle y,x\rangle = 0$

(q.e.d)

c) $\Rightarrow$ a)

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

$$\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle = \langle x,x\rangle + \langle y,y\rangle \Rightarrow$$

$$\Rightarrow \langle x,y\rangle + \langle y,x\rangle = 0$$

From theory $\langle x,y\rangle = \langle y,x\rangle, \forall x,y \in \mathbb{R}^n$

$\Rightarrow \langle x,y\rangle = 0$ (q.e.d.)