Midterm:

w7 13.04 G_1

w14 8.07 G_2

$G_{final} = G_1 + G_2 + B$

$G_1 + G_2 \geq 5$

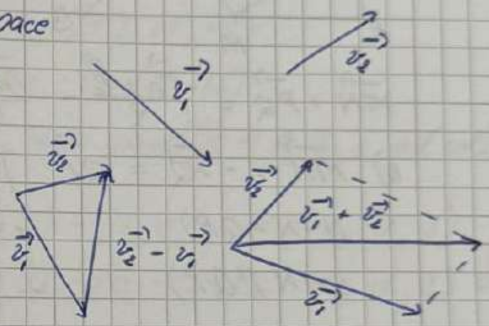
Retake: $G_{final} \geq 5$

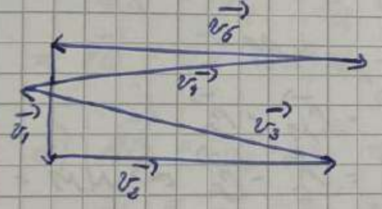
Sem 1:

$\mathbb{E}^m$ Euclidean space

usually $m = 2$ or $m = 3$

$\mathbb{V}^m$ the vector space of vectors in $\mathbb{E}^m$





$\vec{v_1} + \vec{v_2} + \vec{v_3} + \vec{v_4} + \vec{v_5} = \vec{0}$

Fix $O \in \mathbb{E}^m$

We denote $\forall A \in \mathbb{E}^m$ $\vec{OA} = \vec{r_A}$

the position vector of A w.r. to O

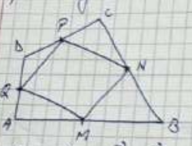
$A, B \in \mathbb{E}^m$: $\vec{AB} = \vec{r_B} - \vec{r_A}$

M midpoint of [AB]

$\vec{r_M} = \frac{\vec{r_A} + \vec{r_B}}{2}$

1.3. ABCD quadrilateral M, N, P, Q midpoints of [AB], [BC], [CD], [DA]. Show that $\vec{MN} + \vec{PQ} = \vec{0}$

b) deduce that the midpoints of the sides of ABCD form a parallelogram



$\vec{MN} = \vec{r_N} - \vec{r_M}$

$\vec{PQ} = \vec{r_Q} - \vec{r_P}$

$\vec{r_N} = \frac{\vec{r_B} + \vec{r_C}}{2}$; $\vec{r_M} = \frac{\vec{r_A} + \vec{r_B}}{2}$

$\vec{r_Q} = \frac{\vec{r_A} + \vec{r_D}}{2}$; $\vec{r_P} = \frac{\vec{r_D} + \vec{r_C}}{2}$

$\vec{MN} + \vec{PQ} = \frac{\vec{r_B} + \vec{r_C}}{2} - \frac{\vec{r_A} + \vec{r_B}}{2} + \frac{\vec{r_A} + \vec{r_D}}{2} - \frac{\vec{r_D} + \vec{r_C}}{2} = \vec{0}$

b) $\left.\begin{array}{l} \vec{MN} = -\vec{PQ} = \vec{QP} \\ (MN = QP) \\ (MN \parallel QP) \end{array}\right\} \Rightarrow MNPQ$ - parallelogram

MNPQ - parall.

$\Leftrightarrow \vec{r_N} + \vec{r_Q} = \vec{r_M} + \vec{r_P}$

$\Leftrightarrow \vec{r_M} - \vec{r_N} = \vec{r_Q} - \vec{r_P} \Leftrightarrow \vec{NM} = \vec{PQ}$

[MN] - line segment

$MN <$ line / length of [MN]

$\vec{v} \parallel \vec{w} \Leftrightarrow \exists \alpha \in \mathbb{R},\ \vec{v} = \alpha \vec{w}$

1.8. (Fix a reference system)

Decide if the given points are collinear

a) $P(3, -5)$, $Q(-1, 2)$, $R(-5, 9)$

c) $P(1, 0, -1)$, $Q(0, -1, 2)$, $R(-1, -2, 5)$

b) $A(11, 2)$, $B(1, -3)$, $C(31, 13)$

d) $A(-1, -1, -4)$, $B(1, 1, 0)$, $C(2, 2, 2)$

A, B, C - collinear $\Leftrightarrow$ rank$(\vec{AB}, \vec{BC}, \vec{CA}) = 1 \Leftrightarrow$

$\Leftrightarrow \exists \alpha \in \mathbb{R} : \vec{AB} = \alpha \cdot \vec{BC}$

a) $\left.\begin{array}{l} \vec{PQ} = \vec{r_Q} - \vec{r_P} = (-1, 2) - (3, -5) = (-4, 7) \\ \vec{QR} = \vec{r_R} - \vec{r_Q} = (-5, 9) - (-1, 2) = (-4, 7) \end{array}\right\} \Rightarrow$

$\Rightarrow \vec{PQ} = \vec{QR} \Rightarrow P, Q, R$ - collin.

b) $\vec{AB} = \vec{r_B} - \vec{r_A} = (1, -3) - (11, 2) = (-10, -5)$

$\vec{BC} = \vec{r_C} - \vec{r_B} = (31, 13) - (1, -3) = (30, 16)$

$\Rightarrow \frac{30}{-10} \neq \frac{16}{-5} \Rightarrow \vec{AB}, \vec{BC}$ not parallel $\Rightarrow A, B, C$ not collinear

c) $\left.\begin{array}{l} \vec{PQ} = \vec{r_Q} - \vec{r_P} = (0, -1, 2) - (1, 0, -1) = (-1, -1, 3) \\ \vec{QR} = \vec{r_R} - \vec{r_Q} = (-1, -2, 5) - (0, -1, 2) = (-1, -1, 3) \end{array}\right\} \Rightarrow$

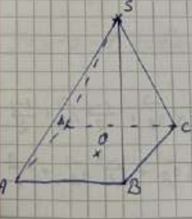
$\Rightarrow \vec{PQ} = \vec{QR} \Rightarrow P, Q, R$ - collin.

d) $\vec{AB} = \vec{r_B} - \vec{r_A} = (1, 1, 0) - (-1, -1, -4) = (2, 2, 4)$

$\vec{BC} = \vec{r_C} - \vec{r_B} = (2, 2, 2) - (1, 1, 0) = (1, 1, 2)$

$\frac{2}{1} = \frac{2}{1} = \frac{4}{2} = 2 \Rightarrow AB \parallel BC \Rightarrow A, B, C$ - collin.

1.11. SABCD pyramid with apex S and base the parallelogram ABCD with centre O. Show that $\vec{SA} + \vec{SB} + \vec{SC} + \vec{SD} = 4\vec{SO}$



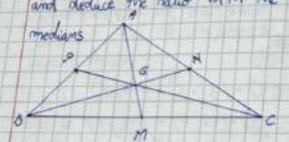
$\vec{SA} + \vec{SB} + \vec{SC} + \vec{SD} = \vec{SO} + \vec{OA} + \vec{SO} + \vec{OB} + \vec{SO} + \vec{OC} + \vec{SO} + \vec{OD} = 4\vec{SO} + (\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD})$

ABCD - parall. $\Rightarrow \vec{OA} = \vec{CO} \Rightarrow \vec{OA} + \vec{OC} = \vec{0}$

$\vec{OB} = \vec{DO} \Rightarrow \vec{OB} + \vec{OD} = \vec{0}$

$\Rightarrow \vec{SA} + \vec{SB} + \vec{SC} + \vec{SD} = 4\vec{SO}$

1.7. Show that the medians of a triangle intersect in one point and deduce the ratio with the common intersection divides the medians



Let $\{G\} = AM \cap BN$

We must show that C, G, P are collinear

$A, B \in \mathbb{E}^n$

$T \in AB \Leftrightarrow \vec{r}_T = \alpha \vec{r}_A + (1-\alpha)\vec{r}_B$ for $\alpha \in \mathbb{R}$

$T \in [AB] \Leftrightarrow \vec{r}_T = \alpha \vec{r}_A + (1-\alpha)\vec{r}_B$, $\alpha \in [0, 1]$

Let M_1 be the midpoint of AG and N_1 the midpoint of BG.

MN - midline in $\triangle ABC \Rightarrow MN \parallel AB$, $MN = \frac{AB}{2}$

M_1N_1 - midline in $\triangle AGB \Rightarrow M_1N_1 \parallel AB$, $M_1N_1 = \frac{AB}{2}$ $\Bigg\} \Rightarrow$

$\Rightarrow M_1N_1 \overset{\parallel}{=} MN \Rightarrow MNN_1M_1$ - parallelogram $\Rightarrow$

$\Rightarrow G$ midpoint of

Say $CP \cap AM = \{G_1\}$ by the same argument G_1 is the midpoint of $[MM_1]$

$\Rightarrow G = G_1$

$\Rightarrow AM \cap BN \cap CP = \{G\}$, G midpoint of MM_1

M_1 midpoint of $AG \Rightarrow \frac{GM}{AG} = \frac{1}{2}$

2nd method

$AM \cap BN = \{G\}$

$G \in AM \Rightarrow \vec{r}_G = \alpha \vec{r}_A + (1-\alpha)\vec{r}_M = \alpha \vec{r}_A + \frac{1-\alpha}{2}\vec{r}_B + \frac{1-\alpha}{2}\vec{r}_C$

$G \in BN \Rightarrow \vec{r}_G = \beta \vec{r}_B + (1-\beta)\vec{r}_N = \beta \vec{r}_B + \frac{1-\beta}{2}\vec{r}_A + \frac{1-\beta}{2}\vec{r}_C$

$\Rightarrow \left(\alpha - \frac{1-\beta}{2}\right)\vec{r}_A + \left(\frac{1-\alpha}{2} - \beta\right)\vec{r}_B + \left(\frac{1-\alpha}{2} - \frac{1-\beta}{2}\right)\vec{r}_C = \vec{0}$

$\vec{AB} = \vec{v}$, $\vec{AC} = \vec{w}$ are lin. indep.

$\vec{r}_B = \vec{v} + \vec{r}_A$

$\vec{r}_C = \vec{w} + \vec{r}_A$

$\Rightarrow \left(\alpha - \frac{1-\beta}{2}\right)\vec{r}_A + \left(\frac{1-\alpha}{2} - \beta\right)(\vec{v} + \vec{r}_A) + \left(\frac{\beta-\alpha}{2}\right)(\vec{w} + \vec{r}_A) = \vec{0}$

$\Rightarrow \underbrace{\left(\alpha - \frac{1-\beta}{2} + \frac{1-\alpha}{2} - \beta + \frac{\beta-\alpha}{2}\right)}_{=0}\vec{r}_A + \left(\frac{1-\alpha}{2} - \beta\right)\vec{v} +$

$+ \left(\frac{\beta-\alpha}{2}\right)\vec{w} = \vec{0}$

$\Rightarrow \left(\frac{1-\alpha}{2} - \beta\right)\vec{v} + \frac{\beta-\alpha}{2}\vec{w} = \vec{0}$

$\vec{v}, \vec{w}$ lin. indep. $\Rightarrow \begin{cases} \beta = \frac{1-\alpha}{2} \\ \frac{\beta-\alpha}{2} = 0 \end{cases} \Rightarrow \begin{array}{l} \alpha = \beta \\ \alpha = \frac{1-\alpha}{2} \Rightarrow 1-\alpha = 2\alpha \Rightarrow \\ \Rightarrow \alpha = \frac{1}{3} \end{array}$

$\Rightarrow \vec{r}_G = \frac{1}{3}\vec{r}_A + \frac{1-\frac{1}{3}}{2}\vec{r}_B + \frac{1-\frac{1}{3}}{2}\vec{r}_C = \frac{1}{3}(\vec{r}_A + \vec{r}_B + \vec{r}_C)$

$\vec{GP} = \vec{r}_P - \vec{r}_G = \frac{1}{2}\vec{r}_A + \frac{1}{2}\vec{r}_B - \frac{1}{3}\vec{r}_A - \frac{1}{3}\vec{r}_B - \frac{1}{3}\vec{r}_C =$

$= \frac{1}{6}\vec{r}_A + \frac{1}{6}\vec{r}_B - \frac{1}{3}\vec{r}_C$

$\vec{CP} = \vec{r}_P - \vec{r}_C = \frac{1}{2}\vec{r}_A + \frac{1}{2}\vec{r}_B - \vec{r}_C = \frac{1}{3}\vec{GP} \Rightarrow$

$\Rightarrow C - G - P \rightarrow$ collinear

Lecture 1:

Geometric vectors

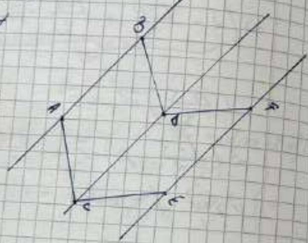
The equipollence relation is an equivalence relation.

$\to$ refl. $(A, B) \sim (A, B)$

$\to$ sym. $(A, B) \sim (C, D) \Rightarrow (C, D) \sim (A, B)$

$\to$ transitive $\left.\begin{array}{l}(A, B) \sim (C, D) \\ (C, D) \sim (E, F)\end{array}\right\} \Rightarrow (A, B) \sim (E, F)$

case AB, CD, EF distinct



Cartesian coordinate systems

Seminar 2:

$\mathcal{K} = (O, \mathcal{B})$ reference frame

$O \in \mathbb{E}^n$, $\mathcal{B}$ basis of $\mathbb{V}^n$

We define $[P]_{\mathcal{K}} := [\underbrace{\overrightarrow{OP}}_{=\vec{r}_P}]_{\mathcal{B}}$

Notations: $\varphi: V_1 \to V_2$ linear map

$\mathcal{B}_1$, $\mathcal{B}_2$ bases of V_1, V_2

$M_{\mathcal{B}_2, \mathcal{B}_1}(\varphi) := [\varphi]_{\mathcal{B}_1, \mathcal{B}_2} = ([\varphi(v_1)]_{\mathcal{B}_2} \dots [\varphi(v_n)]_{\mathcal{B}_2})$

The base-change matrix from $\mathcal{B}$ to $\mathcal{B}'$ is $M_{\mathcal{B}'; \mathcal{B}} = [id]_{\mathcal{B}, \mathcal{B}'}$

V, $\mathcal{B}$, $\mathcal{B}'$ bases of V

We use it as follows: $\forall v \in V$: $[v]_{\mathcal{B}'} = M_{\mathcal{B}', \mathcal{B}} \cdot [v]_{\mathcal{B}}$

$\mathcal{K} = (O, \mathcal{B})$; $\mathcal{K}' = (O', \mathcal{B}')$ reference frames for $\mathbb{E}^n$

We will denote: $M_{\mathcal{K}'; \mathcal{K}} := M_{\mathcal{B}'; \mathcal{B}}$

We want $[P]_{\mathcal{K}'}$ in terms of $[P]_{\mathcal{K}}$

$[P]_{\mathcal{K}'} = [\overrightarrow{O'P}]_{\mathcal{B}'} = M_{\mathcal{K}'; \mathcal{K}} \cdot [\overrightarrow{O'P}]_{\mathcal{B}} = M_{\mathcal{K}', \mathcal{K}} \cdot [\overrightarrow{OP} - \overrightarrow{OO'}]_{\mathcal{B}} =$

$= M_{\mathcal{K}'; \mathcal{K}} \cdot [\overrightarrow{OP}]_{\mathcal{K}} - M_{\mathcal{K}'; \mathcal{K}} \cdot [\overrightarrow{OO'}]_{\mathcal{K}} = M_{\mathcal{K}'; \mathcal{K}} \cdot ([P]_{\mathcal{K}} - [O']_{\mathcal{K}})$

$M_{\mathcal{K}'; \mathcal{K}} \cdot [\overrightarrow{OO'}]_{\mathcal{K}} = [\overrightarrow{OO'}]_{\mathcal{K}'} = -[\overrightarrow{O'O}]_{\mathcal{K}'} = -[O]_{\mathcal{K}'}$

$[P]_{\mathcal{K}'} = M_{\mathcal{K}'; \mathcal{K}} \cdot [P]_{\mathcal{K}} + [O]_{\mathcal{K}'}$

1.16. $\mathcal{K} = (O, \vec{i}, \vec{j})$, $\mathcal{K}' = (O', \vec{i}', \vec{j}')$, $[O']_{\mathcal{K}} = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$

$\vec{i}' = -2\vec{i} + \vec{j}$, $\vec{j}' = \vec{i} + 2\vec{j}$

Determine the base-change matrix from K to K' and the coordinates of the points A, B, C in K'

$[A]_K = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $[B]_K = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $[C]_K = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Determine $M_{K, K'}$ and check the previous result

$(M_{K, K'}^{-1} = M_{K'; K})$

$M_{K, K'} = [id]_{K'; K} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$

$[A]_{K'} = [\overrightarrow{O'A}]_{K'} = [\overrightarrow{OA} - \overrightarrow{OO'}]_{K'} = [\overrightarrow{OA}]_{K'} + [O]_{K'} =$

$= M_{K'; K} \cdot [\overrightarrow{OA}]_K + M_{K'; K} \cdot [\overrightarrow{O'O}]_K$

$M_{K'; K} = M_{K, K'}^{-1} = \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$

$[A]_{K'} = \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \cdot \begin{pmatrix} -7 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} + \frac{2}{5} \\ \frac{1}{5} + \frac{4}{5} \end{pmatrix} +$

$\begin{pmatrix} \frac{15}{5} + \frac{1}{5} \\ -\frac{7}{5} + \frac{2}{5} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$[B]_{K'} = [\vec{O'B}]_{K'} = [\vec{OB} - \vec{OO'}]_{K'} = M_{K';K}([B]_K - [O']_K) =$

$= M_{K';K}\left(\begin{pmatrix}2\\4\end{pmatrix} - \begin{pmatrix}7\\-1\end{pmatrix}\right) = \begin{pmatrix}-\frac{2}{5} & \frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix}\begin{pmatrix}-5\\5\end{pmatrix} = \begin{pmatrix}2+1\\-1+2\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$

$= [\vec{OO'} + \vec{O'C}]_{K'} = [\vec{OC}]_{K'} - [\vec{OO'}]_{K'} =$

$[C]_{K'} = [\vec{O'C}]_{K'} = [\vec{OC} - \vec{OO'}]_{K'} = M_{K';K}([C]_K - [O']_K) =$

$= \begin{pmatrix}-\frac{2}{5} & \frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix} \cdot \begin{pmatrix}3-7\\1+1\end{pmatrix} = \begin{pmatrix}-\frac{2}{5} & \frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix}\begin{pmatrix}-4\\2\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$

$M_{K;K'} = \begin{pmatrix}-2 & 1\\ 1 & 2\end{pmatrix}$; $M_{K';K} = \begin{pmatrix}-\frac{2}{5} & \frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix}$

$[O']_K = \begin{pmatrix}7\\-1\end{pmatrix}$, $[A]_{K'} = \begin{pmatrix}3\\0\end{pmatrix}$, $[B]_{K'} = \begin{pmatrix}3\\1\end{pmatrix}$, $[C]_{K'} = \begin{pmatrix}2\\0\end{pmatrix}$

$[A]_K = [\vec{OA}]_K = [\vec{OO'} + \vec{O'A}]_K = [\vec{O'A}]_K - [\vec{O'O}]_K = M_{K,K'}([A]_{K'} - [O]_{K'}) =$

$[O]_{K'} = [\vec{O'O}]_{K'} = -[\vec{OO'}]_{K'} = -(M_{K',K} \cdot [O']_K) = -\begin{pmatrix}-\frac{2}{5} & \frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix}\begin{pmatrix}7\\-1\end{pmatrix} =$

$= -\begin{pmatrix}-\frac{14}{5} & -\frac{1}{5}\\ \frac{7}{5} & -\frac{2}{5}\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix}$

$[A]_K = \begin{pmatrix}-2 & 1\\ 1 & 2\end{pmatrix}\left(\begin{pmatrix}3\\0\end{pmatrix} - \begin{pmatrix}3\\-1\end{pmatrix}\right) = \begin{pmatrix}-2 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$

$[B]_K = [\vec{OB}]_K = [\vec{O'B} - \vec{O'O}]_K = M_{K,K'}([B]_{K'} - [O]_{K'}) =$

$= \begin{pmatrix}-2 & 1\\ 1 & 2\end{pmatrix}\left(\begin{pmatrix}3\\1\end{pmatrix} - \begin{pmatrix}3\\-1\end{pmatrix}\right) = \begin{pmatrix}-2 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$

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$[C]_K = [\vec{OC}]_K = [\vec{O'C} - \vec{O'O}]_K = M_{K,K'}([C]_{K'} - [O]_K) =$

$= \begin{pmatrix}-2 & 1\\ 1 & 2\end{pmatrix}\left(\begin{pmatrix}2\\0\end{pmatrix} - \begin{pmatrix}3\\-1\end{pmatrix}\right) = \begin{pmatrix}-2 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$

1.17 Consider the tetrahedron ABCD and the coordinate systems:
$K_A = (A, \vec{AB}, \vec{AC}, \vec{AD})$; $K'_A = (A, \vec{AB}, \vec{AD}, \vec{AC})$; $K_B = (B, \vec{BA}, \vec{BC}, \vec{BD})$
M midpoint of [BC]

a) find the coordinates of A, B, C, D, M in the three coordinate systems

b) the base change matrix from K_A to K'_A

c) the base change matrix from K_B to K_A

$[A]_{K_A} = \begin{pmatrix}0\\0\\0\end{pmatrix}$; $[B]_{K_A} = [\vec{AB}]_{K_A} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$[C]_{K_A} = [\vec{AC}]_{K_A} = \begin{pmatrix}0\\1\\0\end{pmatrix}$; $[D]_{K_A} = [\vec{AD}]_{K_A} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

M midpoint [BC] ; $\vec{BM} = \frac{1}{2}\vec{BC}$

$\vec{BC} = \vec{BA} + \vec{AC} = -\vec{AB} + \vec{AC}$; $\vec{BM} = -\frac{1}{2}\vec{AB} + \frac{1}{2}\vec{AC}$

$\vec{AM} = \vec{BM} - \vec{BA} = -\frac{1}{2}\vec{AB} + \vec{AB} + \frac{1}{2}\vec{AC} = \frac{1}{2}\vec{AB} + \frac{1}{2}\vec{AC}$

$[M]_{K_A} = [\vec{AM}]_{K_A} = \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\\ 0\end{pmatrix}$

$[A]_{K'_A} = \begin{pmatrix}0\\0\\0\end{pmatrix}$; $[B]_{K'_A} = \begin{pmatrix}1\\0\\0\end{pmatrix}$; $[C]_{K'_A} = \begin{pmatrix}0\\0\\1\end{pmatrix}$; $[D]_{K'_A} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$[M]_{K'_A} = \begin{pmatrix}\frac{1}{2}\\ 0\\ \frac{1}{2}\end{pmatrix}$

$[A]_{K_B} = [\vec{BA}] = \begin{pmatrix}1\\0\\0\end{pmatrix}$; $[B]_{K_B} = \begin{pmatrix}0\\0\\0\end{pmatrix}$; $[C]_{K_B} = [\vec{BC}] = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$[D]_{K_B} = [\vec{BD}] = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$[M]_{K_B} = \frac{[\vec{BD} + \vec{BC}]}{2} = \left(\begin{pmatrix}0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}\right) \cdot \frac{1}{2} = \begin{pmatrix}0\\ \frac{1}{2}\\ 0\end{pmatrix}$

$$M_{K_B', K_A} = \left([\vec{AB}]_{K_A'} \; [\vec{AC}]_{K_A'} \; [\vec{AD}]_{K_A'} \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$M_{K_A, K_B} = \left([\vec{BA}]_{K_A} \; [\vec{BC}]_{K_A} \; [\vec{BD}]_{K_A} \right) = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Seminar 3

affine varieties:

$a + U$, $a \in \mathbb{E}^n$, $U \leq V^n$

dim 1: line ; dim 2: plane

Vector equation

Line: $A(x_A, y_A)$, $\vec{v}(x_{\vec{v}}, y_{\vec{v}})$

line ℓ in $\mathbb{E}^2$ (consider a reference system $K = (O, B)$)

$\forall M \in \mathbb{E}^2 : \exists \lambda \in \mathbb{R} : \underbrace{\vec{OM}}_{\vec{r}_M} = \underbrace{\vec{OA}}_{\vec{r}_A} + \lambda \vec{v}$ (vector equation)

$\Leftrightarrow \begin{cases} x = x_A + \lambda x_{\vec{v}} \\ y = y_A + \lambda y_{\vec{v}} \end{cases}$ (parametric equation of ℓ)

$\Leftrightarrow \dfrac{x - x_A}{x_{\vec{v}}} = \dfrac{y - y_A}{y_{\vec{v}}}$ if $x_{\vec{v}}, y_{\vec{v}} \neq 0$ (symmetric form)

If $x_{\vec{v}} = 0$: $\ell: x = x_A$

Implicit form: $y_{\vec{v}}(x - x_A) - x_{\vec{v}}(y - y_A) = 0$

Explicit form: $y = \underbrace{m}_{\text{slope}} x + n$

2.1, 2.2 Determine parametric and cartesian equations for the line $\ell \subseteq \mathbb{E}^2$ and describe its direction vectors:

a) $\ell \ni A(1, 2)$ and $\ell \parallel \vec{a}(3, -1)$

b) $\ell \ni O$, $\ell \parallel \vec{b}(4, 5)$

c) $\ell \ni M(1, 7)$, $\ell \parallel Oy$

d) $\ell \ni M(2, 4)$, $N(2, -5)$, $\ell \parallel Oy$

a) Parametric eq: $\begin{cases} x = 1 + 3\lambda \\ y = 2 - \lambda \end{cases}$ $\quad \delta(\ell) = \langle (3, -1) \rangle$

Sym. eq: $\dfrac{x - 1}{3} = \dfrac{y - 2}{-1}$

Implicit form: $-1(x - 1) - 3(y - 2) = 0 \Leftrightarrow$

$\Leftrightarrow -x + 1 - 3y + 6 = 0 \Leftrightarrow 3y = 7 - x \Rightarrow$

$y = \dfrac{7}{3} - \dfrac{1}{3}x$

b) $\begin{cases} x = 0 + 4\lambda \\ y = 0 + 5\lambda \end{cases}$ Param eq $\quad \delta(\ell) = \langle (4, 5) \rangle$

Sym eq: $\dfrac{x}{4} = \dfrac{y}{5}$

Implicit form: $5x = 4y \Rightarrow y = \dfrac{5}{4}x \rightarrow$ explicit form

$5x - 4y = 0$

c) Param eq: $\begin{cases} x = 1 + \lambda \cdot 0 \\ y = 7 + \lambda \cdot 1 \end{cases}$ $\quad \delta(\ell) = \langle (0, 1) \rangle$

Sym. eq: $x_{\vec{v}} = 0 \Rightarrow \ell: x = 1$

Implicit form: $x - 1 = 0$

d) Param eq: $\begin{cases} x = 2 + 0 \cdot \lambda \\ y = 4 - 9\lambda \end{cases}$ $\quad \vec{MN} = \vec{r}_N - \vec{r}_M = (2 - 2, -5 - 4) = (0, -9)$

$\delta(\ell) = \langle (0, -9) \rangle$

Cart. eq: $x_{\vec{v}} = 0 \Rightarrow \ell: x = 2$

Implicit form: $x - 2 = 0$

2.5. Determine the equation of the line parallel to $\vec{v}$ and passing through $S \cap T$ if: a) $\vec{v} = (2, 2)$, $S: 3x - 2y - 7 = 0$, $T: 2x + 3y = 0$

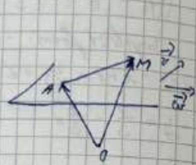
$S \cap T: \begin{cases} 3x - 2y - 7 = 0 \mid \cdot 3 \\ 2x + 3y = 0 \mid \cdot 2 \end{cases} \Rightarrow \begin{cases} 9x - 6y - 21 = 0 \\ 4x + 6y \end{cases}$ (+)

$13x \quad / \quad -21 = 0 \Rightarrow x = \dfrac{21}{13}$

$y = -\frac{43}{13} \cdot \frac{1}{3} = -\frac{42}{39}^{13} = -\frac{14}{13}$

$s \cap T = v\left(\frac{21}{13}, -\frac{14}{13}\right)$

Param eq: $e: \begin{cases} x = \frac{21}{13} + 2\lambda \\ y = -\frac{14}{13} + 4\lambda \end{cases}$



π plane (ie $\dim_{\mathbb{R}} \delta(\pi) = 2$)

$A \in \pi$, $\vec{v}, \vec{w} \parallel \pi$ ($\vec{v}, \vec{w} \in \delta(\pi)$)

$\vec{r_M} = \vec{r_A} + \vec{AM}$

$\vec{AM} \in \delta(\pi)$; $\vec{v}, \vec{w}$ lin. indep., so they are a basis of $\delta(\pi)$

$\Rightarrow \exists!\, \lambda, \mu \in \mathbb{R}: \vec{r_M} = \vec{r_A} + \lambda \vec{v} + \mu \vec{w}$ (vector equation)

Parametric equation: $(s): \begin{cases} x = x_A + \lambda x_{\vec{v}} + \mu x_{\vec{w}} \\ y = y_A + \lambda y_{\vec{v}} + \mu y_{\vec{w}} \\ z = z_A + \lambda z_{\vec{v}} + \mu z_{\vec{w}} \end{cases}$; $\lambda, \mu \in \mathbb{R}$

(s) compatible (w. r. to the variable λ, μ) $\Leftrightarrow$

Cartesian equ:
$$0 = \begin{vmatrix} x - x_A & y - y_A & z - z_A \\ x_{\vec{v}} & y_{\vec{v}} & z_{\vec{v}} \\ x_{\vec{w}} & y_{\vec{w}} & z_{\vec{w}} \end{vmatrix}$$

M, N, P, Q coplanar iff
$$\begin{vmatrix} x_M & y_M & z_M & 1 \\ x_N & y_N & z_N & 1 \\ x_P & y_P & z_P & 1 \\ x_Q & y_Q & z_Q & 1 \end{vmatrix} = 0$$

Implicit form: $Ax + By + Cz + D = 0$

$\hookrightarrow$ normal vectors of π are the ones in $\langle (A, B, C) \rangle$

2.10. Determine cartesian equ. for the plane π:

a) $\pi: \begin{cases} x = 2 + 3u - 4v \\ y = 4 - v \\ z = 2 + 3u \end{cases}$ $\quad \begin{vmatrix} x-2 & y-4 & z-2 \\ 3 & 0 & 3 \\ -4 & -1 & 0 \end{vmatrix} = 0$

b) $\pi: \begin{cases} x = u + v \\ y = u - v \\ z = 5 + 6u - 4v \end{cases}$ $\quad \begin{vmatrix} x-0 & y-0 & z-5 \\ 1 & 1 & 6 \\ 1 & -1 & -4 \end{vmatrix} = 0$

a) $0 + (-3)(z-2) - 12(y-4) - 0 + 3(x-2) - 0 = 0$

$-3z + \cancel{6} - 12y + 48 + 3x - \cancel{6} = 0$

$3x - 12y - 3z + 48 = 0 \,/ : 3 \Leftrightarrow x - 4y - z + 16 = 0$

b) $-4x - z + 5 + 6y - z + 5 + 6x + 4y = 0$

$2x + 10y - 2z + 10 = 0 \,/ : 2 \Rightarrow x + 5y - z + 5 = 0$

2.11. Determine param. equ. for π: Find $\delta(\pi)$

a) $\pi: 3x - 6y + z = 0$

b) $\pi: 2x - y - z - 3 = 0$

a) $x = \lambda$
$y = \mu$
$3\lambda - 6\mu + z = 0$
$\begin{cases} x = \lambda \\ y = \mu \\ z = -3\lambda + 6\mu \end{cases}$ $\quad \delta(\pi) = \langle (1, 0, -3), (0, 1, 6) \rangle$

b) $x = \lambda$
$y = \mu$
$2\lambda - \mu - z - 3 = 0$
$\begin{cases} x = \lambda \\ y = \mu \\ z = 2\lambda - \mu - 3 \end{cases}$ $\quad \delta(\pi) = \langle (1, 0, 2), (0, 1, -1) \rangle$

2.14. Determine the relative positions of the planes:

skew ($\nexists$ intersection) — incident ($\exists$ intersection)

a) $\pi_1: x + 2y + 3z - 1 = 0$

$\pi_2: x + 2y - 3z - 1 = 0$

c) $\pi_1: x+2y-3z-1=0$
$\pi_2: 2x+y+3z-2=0$
$\pi_3: x+2y+3z+2=0$

a) $\pi_1 \cap \pi_2 = \begin{cases} x+2y+3z-1=0 \\ x+2y-3z-1=0 \end{cases}$ (−)

$6z = 0 \Rightarrow z = 0$

$x+2y-1=0 \Rightarrow x = 1-2y$

$\pi_1 \cap \pi_2 = \{(1-2a, a, 0) \mid a \in \mathbb{R}\}$

<u>Seminar 4:</u>

$\vec{v} \cdot \vec{w} = \begin{cases} 0, \text{ if } \vec{v} \cdot \vec{w} \text{ is } 0 \\ |\vec{v}| \cdot |\vec{w}| \cos(\vec{v}, \vec{w}) \end{cases}$

↳ the dot product (the scalar product)

Alternate notation: $\vec{v} \cdot \vec{w} = \langle \vec{v}, \vec{w} \rangle$

<u>Properties:</u>

- bilinearity: $\forall \alpha, \beta \in \mathbb{R}, \forall \vec{v_1}, \vec{v_2}, \vec{w} \in V$
  $(\alpha v_1 + \beta v_2) \cdot w = \alpha \cdot \vec{v_1} \cdot \vec{w} + \beta \cdot \vec{v_2} \cdot \vec{w}$
- symmetry: $\forall \vec{v}, \vec{w}: \vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
- positive definiteness: $\forall \vec{v} \in V, \vec{v} \cdot \vec{v} \in \mathbb{R}_{\geq 0}$
  if $\vec{v} \neq \vec{0}: \vec{v} \cdot \vec{v} > 0$

Consequences:

- $\vec{v} \cdot \vec{v} = |\vec{v}|^2$
- $\vec{v}^2 - \vec{w}^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} + \vec{w})$

If we have $\mathcal{K}$ an orthonormal reference system and $(\vec{v})_{\mathcal{K}} = \begin{pmatrix} x_v \\ y_v \end{pmatrix}$ $(w)_{\mathcal{K}} = \begin{pmatrix} x_w \\ y_w \end{pmatrix} \Rightarrow \vec{v} \cdot \vec{w} = x_v x_w + y_v y_w$

$\mathcal{K} = (O, \vec{v_1}, \ldots, \vec{v_m})$ orthonormal if $\forall i, j, i \neq j: \vec{v_i} \cdot \vec{v_j} = 0$

$\forall i: \vec{v_i} \cdot \vec{v_i} = |v_i| = 1$

3.2 $\vec{m}$ and $\vec{n}$ unit vectors s.t. $\sphericalangle(\vec{m}, \vec{n}) = 120°$. Determine $\sphericalangle(\vec{a}, \vec{b})$, where $\vec{a} = 2\vec{m} + 4\vec{n}$; $\vec{b} = \vec{m} - \vec{n}$

$\vec{a} \cdot \vec{b} = (2\vec{m} + 4\vec{n}) \cdot (\vec{m} - \vec{n}) = 2|\vec{m}|^2 + 2\vec{m} \cdot \vec{n} - 4|\vec{n}|^2 =$
$= -2 + 2 \cdot \vec{m} \cdot \vec{n}$

$\vec{m} \cdot \vec{n} = |m| \cdot |n| \cdot \cos 120° = -\frac{1}{2}$

$\vec{a} \cdot \vec{b} = -2 - 1 = -3$

$|\vec{a}|^2 = (2\vec{m} + 4\vec{n})(2\vec{m} + 4\vec{n}) = 4|\vec{m}|^2 + 16\vec{m}\vec{n} + 16|\vec{n}|^2 =$
$= 20 - 8 = 12 \Rightarrow |\vec{a}| = 2\sqrt{3}$

$|\vec{b}| = (\vec{m} - \vec{n})(\vec{m} - \vec{n}) = |\vec{m}|^2 - 2\vec{m} \cdot \vec{n} + |\vec{n}|^2 = 2 + 1 = 3 \Rightarrow$
$\Rightarrow |\vec{b}| = \sqrt{3}$

$\cos(\vec{a}, \vec{b}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{-3}{2\sqrt{3} \cdot \sqrt{3}} = -\frac{3}{6} = -\frac{1}{2} \Rightarrow m(\vec{a}, \vec{b}) = 120°$

3.3. Fix an orthonormal basis: $\vec{a}(2, 1, 0)$; $\vec{b}(0, -2, 1)$

Find the angle between the diagonals of the parallelogram spanned by $\vec{a}$ and $\vec{b}$

$\vec{d_1} = \vec{a} + \vec{b}$; $\vec{d_2} = \vec{a} - \vec{b}$

$\vec{d_1} \cdot \vec{d_2} = (\vec{a} + \vec{b})(\vec{a} - \vec{b}) = \vec{a}^2 - \vec{b}^2 = |\vec{a}|^2 - |\vec{b}|^2 =$
$= 2^2 + 1^2 + 0^2 - 0^2 - 2^2 - 1^2 = 0 \Rightarrow \vec{d_1} \perp \vec{d_2}$

3.7 ABCD tetrahedron. Show that: $\cos(\vec{AB}, \vec{CD}) = \frac{AD^2 + BC^2 - AC^2 - BD^2}{2 \cdot AB \cdot CD}$

(3D ~~law~~ ~~form~~ law of cosinus)

<u>Proof:</u>

$\vec{AB}^2 + \vec{AC}^2 - \vec{BC}^2 = (\vec{AB}^2 - \vec{BC}^2) + \vec{AC}^2 = (\vec{AB} - \vec{BC})(\vec{AB} + \vec{BC}) + \vec{AC}^2 =$
$= (\vec{AB} - \vec{BC}) \cdot \vec{AC} + \vec{AC}^2 = (\vec{AB} - \vec{BC} + \vec{AC}) \cdot \vec{AC} = (\vec{r_B} - \vec{r_A} - \cancel{\vec{r_C}} + \vec{r_B} + \cancel{\vec{r_C}} - \vec{r_A}) \cdot$
$\cdot \vec{AC} = (2\vec{r_B} - 2\vec{r_A}) \cdot \vec{AC} = 2\vec{AB} \cdot \vec{AC}$

Law of cosines in 2D: ΔABC

$BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cdot \cos(\widehat{BAC})$

$\vec{BC}^2 = \vec{AB}^2 + \vec{AC}^2 - 2 \cdot \vec{AB} \cdot \vec{AC}$

2.18, 19, 26, 27, 30 ☺ SMILE

$AD^2 + BC^2 - AC^2 - BD^2 = (AD^2 - AC^2) + (BC^2 - BD^2) =$

$= (\vec{AD} - \vec{AC})(\vec{AD} + \vec{AC}) + (\vec{BC} - \vec{BD}) \cdot (\vec{BC} + \vec{BD}) =$

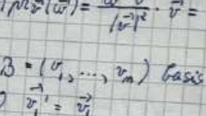
$= (\vec{CA} + \vec{AD})(\vec{AD} + \vec{AC}) + (\vec{DB} + \vec{BC})(\vec{BC} + \vec{BD}) =$

$= \vec{CD}(\vec{AD} + \vec{AC}) + \vec{DC}(\vec{BC} + \vec{BD}) = \vec{CD}(\vec{AD} + \vec{AC} - \vec{BC} - \vec{BD}) =$

$= \vec{CD}(\vec{AB} + \vec{AB}) = \vec{CD} \cdot 2 \cdot \vec{AB}$

$$\frac{AB^2 + BC^2 - AC^2 - BD^2}{2 \cdot AB \cdot CD} = \cos(\vec{AB}, \vec{CD})$$

To get an orthonormal basis from a general one → Gram-Schmidt



$\vec{w} \cdot \vec{v} = |\vec{w}| \cdot \cos(\vec{v}, \vec{w}) \cdot |\vec{v}| = |pr_{\vec{v}}(\vec{w})| \cdot |\vec{v}| \Rightarrow$

$\Rightarrow pr_{\vec{v}}(\vec{w}) = \frac{\vec{w} \cdot \vec{v}}{|\vec{v}|^2} \cdot \vec{v} = \frac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \cdot \vec{v}$

$B = (v_1, \dots, v_m)$ basis of V^n to get an orthonormal basis:

① $\vec{v_1}' = \vec{v_1}$

$\vec{v_2}' = \vec{v_2} - pr_{\vec{v_1}}(\vec{v_2})$

$\vec{v_3}' = \vec{v_3} - pr_{\langle \vec{v_1}, \vec{v_2} \rangle}(\vec{v_3}) = \vec{v_3} - \frac{\vec{v_3} \cdot \vec{v_1}'}{\vec{v_1}' \cdot \vec{v_1}'} \cdot \vec{v_1}' - \frac{\vec{v_3} \cdot \vec{v_2}'}{\vec{v_2}' \cdot \vec{v_2}'} \cdot \vec{v_2}'$

Repeat until we have: v_m'

② Normalize: $\vec{v_i}'' := \frac{\vec{v_i}'}{|\vec{v_i}'|} \Rightarrow B'' = (\vec{v_1}'', \dots, \vec{v_m}'')$ orthonormal basis

3.1.0. $\vec{v_1}(0, 1, 0)$; $\vec{v_2}(2, 1, 0)$; $\vec{v_3}(-1, 0, 1)$. Use Gram-Schmidt to get an orthonormal basis containing

① $\vec{v_1} = \vec{v_1}' \Rightarrow \vec{v_1}' = (0, 1, 0)$

② $\vec{v_2}' = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{2 \cdot 0 + 1 \cdot 1 + 0 \cdot 0}{1} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \vec{v_2}'$

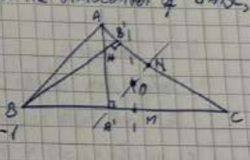
$\vec{v_3}' = \vec{v_3} - pr_{\vec{v_1}'}(v_3) - pr_{\vec{v_2}'}(v_3) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{-1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0}{1} \cdot \vec{v_1}' -$

$- \frac{2 \cdot (-1) + 0 \cdot 0 + 1 \cdot 0}{4} \cdot \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{v_3}'$

$\vec{v_1}'' = \frac{\vec{v_1}'}{|\vec{v_1}'|} = \vec{v_1}'$ $\quad$ $\vec{v_3}'' = \frac{\vec{v_3}'}{|\vec{v_3}'|} = \vec{v_3}'$

$\vec{v_2}'' = \frac{\vec{v_2}'}{|\vec{v_2}'|} = \frac{1}{2} \cdot \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

3.18 Determine the circumcenter and the orthocenter of $\triangle ABC$, with $A(1, 2)$, $B(3, -2)$, $C(5, 6)$



$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{6 - 2}{5 - 1} = \frac{4}{4} = 1$

$NO \perp AC \Rightarrow m_{NO} \cdot m_{AC} = -1 \Rightarrow m_{NO} = -1$

N mid $AC \Rightarrow N(3, 4)$

$NO: (y - y_N) = (x - x_N)\, m_{NO}$

$NO: y - 4 = -x + 3$

$NO: y = 7 - x$

M mid $BC \Rightarrow M(4, 2)$, $m_{BC} = \frac{6 + 2}{5 - 3} = 4$

$m_{MO} = -\frac{1}{4} \Rightarrow MO: y - 2 = (x - 4)\left(-\frac{1}{4}\right)$

$MO: y - 2 = -\frac{1}{4}x + 1 \quad | + 2$

$MO: y = -\frac{1}{4}x + 3$

$\{O\} \in MO \cap NO \Rightarrow 7 - x_0 = -\frac{1}{4}x_0 + 3 \Rightarrow \frac{3}{4}x_0 = 4 \Rightarrow x_0 = \frac{16}{3}$

$\Rightarrow y_0 = 7 - \frac{16}{3} = \frac{5}{3}$

$m_{AA'} = m_{NO}$

3: 1, 4, 9, 12, 20

Hello Visinel

$\Rightarrow AA': \ y - y_A = m_{AA'}(x - x_A)$

$y - 2 = \left(-\frac{1}{4}\right)(x - 1)$

$m_{BB'} = m_{DN}$

$BB': \ y + 2 = (-1)(x - 3)$

$y + 2 = -x + 3$

$$H: \begin{cases} y - 2 = -\frac{1}{4}x + \frac{1}{4} \\ y + 2 = -x + 3 \end{cases}$$

$y_H = -\frac{1}{4}x_H + \frac{9}{4} \Rightarrow -x_H + 3$

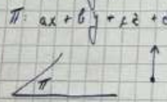
Seminar 5:

Today: from chapter 3: 26, 29, 30, 31, 32, 33, 35, 37, 38, 39, 42

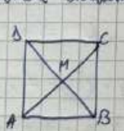
We work with respect to an orthonormal system

$e: ax + by + c = 0 \Rightarrow \vec{v}(a, b)$ normal vector of e (in 2D)

$\pi: ax + by + cz + d = 0 \Rightarrow \vec{v}(a, b, c)$ normal vector of π (in 3D)



3.26. The point $A(3, -2)$ is the vector of a square ABCD and $M(1, 1)$ is the intersection point of the diagonals. Determine Cartesian equation for the sides of the square.



$$\begin{cases} \frac{x_A + x_C}{2} = 1 \Rightarrow x_C = -1 \\ \frac{y_A + y_C}{2} = 1 \Rightarrow y_C = 4 \end{cases} \Rightarrow C(-1, 4)$$

$AC^2 = (x_C - x_A)^2 + (y_C - y_A)^2 = 4^2 + 6^2 = 52$

$BD^2 = (x_D - x_B)^2 + (y_D - y_B)^2 = x_D^2 + x_B^2 - 2x_D x_B + y_D^2 + y_B^2 - 2y_D y_B$

$\Rightarrow x_B^2 + y_B^2 + x_D^2 + y_D^2 - 2x_B x_D - 2y_B y_D = 52$

$MA^2 = \left(\frac{1}{2}AC\right)^2 = \frac{52}{4} = 13$

$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{4+2}{-1-3} = -\frac{6}{4} = -\frac{3}{2}$

$m_{BD} = \frac{2}{3} \Rightarrow BD: \ y - 1 = \frac{2}{3}(x - 1) \ | \cdot 3$

$3y - 3 = 2x - 2 \Rightarrow 2x - 3y + 1 = 0$

$$B, D: \begin{cases} 2x - 3y + 1 = 0 \\ (x-1)^2 + (y-1)^2 = 13 \end{cases} \Leftrightarrow \begin{cases} 2x - 3y + 1 = 0 \\ x^2 + 1 - 2x + y^2 + 1 - 2y = 0 \end{cases}$$

$x^2 + 1 - 2x + \left(\frac{2}{3}x + \frac{1}{3}\right)^2 + 1 - 2 \cdot \left(\frac{2}{3}x + \frac{1}{3}\right) = 0$

$x^2 - 2x + 1 + \frac{4}{9}x^2 + \frac{4}{9}x + \frac{1}{9} + 1 - \frac{4}{3}x - \frac{2}{3} = 0$

$\frac{13}{9}x^2 - \frac{26}{9}x + \frac{13}{9} = 13 \Rightarrow x^2 - 2x + 1 = 9 \Rightarrow x \in \{4, -2\}$

$\Rightarrow y = \frac{2}{3} \cdot \pm 1 + \frac{1}{3} \Rightarrow \begin{cases} y_1 = 3 \\ y_2 = -1 \end{cases}$

$\Rightarrow B(4, 3), D(-2, -1)$ or $B(4, 3), D(-2, -1)$

$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{3+4}{4-1} = \frac{4}{}$

$AB: \ y + 2 = - \ (x - 3) \ | \cdot 2 \Rightarrow 5x - y + 17 = 0$

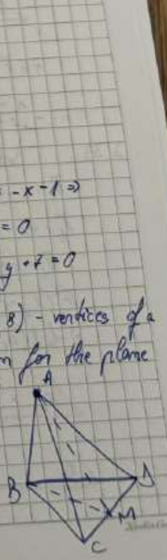
$CD: \ y - 4 = 5(x + 1) \Rightarrow 5x + y - 9 = 0$

$m_{BC} = -\frac{1}{5} \quad BC: \ y - 4 = -\frac{1}{5}(x + 1) \Rightarrow 5y - 20 = -x - 1 \Rightarrow$

$\Rightarrow x + 5y - 19 = 0$

$AD: \ y + 2 = -\frac{1}{4}(x - 3) \Rightarrow 5y + 10 = x + 3 \Rightarrow x + y + 7 = 0$

3.30. $A(2, 1, 0)$; $B(1, 3, 5)$; $C(6, 3, 4)$; $D(0, -7, 8)$ - vertices of a tetrahedron. Determine a Cartesian equation for the plane containing $[AB]$ and the midpoint of $[CD]$



$M\left(\frac{x_C + x_D}{2}, \frac{y_C + y_D}{2}, \frac{z_C + z_D}{2}\right)$

$M(3, -2, 6)$

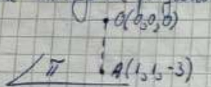
$$\begin{vmatrix} x & y & z & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 5 & 1 \\ 3 & -2 & 6 & 1 \end{vmatrix} = 0 \Leftrightarrow x\cdot(-1)^2 \begin{vmatrix} 1 & 0 & 1 \\ 3 & 5 & 1 \\ -2 & 6 & 1 \end{vmatrix} + y\cdot(-1)^3 \begin{vmatrix} 2 & 0 & 1 \\ 1 & 5 & 1 \\ 3 & 6 & 1 \end{vmatrix} +$$

$$+ z\cdot(-1)^4 \cdot \begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 3 & -2 & 1 \end{vmatrix} + (-1)^5 \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 5 \\ 3 & 2 & 6 \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow x(5+18+10-6) - y(10+6-15-12) + z(6+3-2-9+4-1) -$$

$$-(36+15+20-6) = 0 \Leftrightarrow 27x + 11y + z - 65 = 0$$

3.32 Determine Cartesian equation of the plane $\tilde{\pi}$ if $A(1, -1, 3)$ is the orthogonal projection of the origin on $\tilde{\pi}$



$OA \perp \tilde{\pi}$, $\vec{OA} \perp \tilde{\pi}$

$\vec{OA}$ is a normal vector of $\tilde{\pi}$

$ax + by + cz + d = 0 \Leftrightarrow 1\cdot x + (-1)\cdot y + 3z + d = 0$

$A \in \tilde{\pi} \Rightarrow 1\cdot 1 + 1 + 9 + d = 0 \Rightarrow 11 + d = 0 \Rightarrow d = -11$

$x - y + 3z - 11 = 0$

3.33 Determine the distance between the planes:

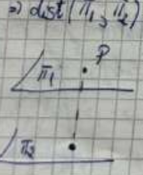
$\tilde{\pi}_1 : x - 2y - 2z + 7 = 0$

$\tilde{\pi}_2 : 2x - 4y - 4z + 17 = 0$

$\tilde{\pi}_2 : x - 2y - 2z + \frac{17}{2} = 0$

$\vec{v}(1, -2, -2)$ is a normal vector for $\tilde{\pi}_1$ and $\tilde{\pi}_2 \Rightarrow \tilde{\pi}_1 \parallel \tilde{\pi}_2$

$\Rightarrow dist(\tilde{\pi}_1, \tilde{\pi}_2) = dist(P, \tilde{\pi}_2)$, $P \in \tilde{\pi}_1$



let $P(9, 1, 0) \in \tilde{\pi}_1$

$d(P, \tilde{\pi}_2) = \frac{|9 - 2 + \frac{17}{2}|}{\sqrt{1+4+4}} = \frac{-2 + \frac{17}{2}}{3} = \frac{3}{6} = \frac{1}{2}$

$\Rightarrow d(\tilde{\pi}_1, \tilde{\pi}_2) = \frac{1}{2}$

3.37 Determine the values a and c for which the line e: $3x - 2y + z + 3 = 0 \cap 4x - 3y + 4z + 1 = 0$ is perpendicular to the plane $ax + 8y + cz + 2 = 0$.

$$e: \begin{cases} 3x - 2y + z + 3 = 0 \quad \Big| \cdot \frac{4}{3} \\ 4x - 3y + 4z + 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 3x - 2y + z + 3 = 0 \\ 0 + \frac{1}{3}y + \frac{9}{3}z + 3 = 0 \end{cases} \Rightarrow \begin{cases} 3x - 2y + \frac{1}{8}y + \frac{9}{8} + 3 = 0 \\ z = \frac{1}{8}y + \frac{9}{8} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 3x - \frac{15}{8}\alpha + \frac{33}{8} = 0 \\ z = \frac{1}{8}\alpha + \frac{9}{8} \\ y = \alpha \end{cases} \Rightarrow \begin{cases} x = \frac{5}{8}\alpha - \frac{11}{8} \\ z = \frac{1}{8}\alpha + \frac{9}{8} \\ y = \alpha \end{cases}$$

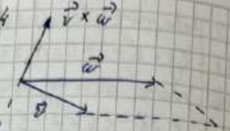
$\Delta(e) = \langle (\frac{5}{8}, 1, \frac{1}{8}) \rangle = \langle (5, 8, 1) \rangle$

$e \perp \tilde{\pi} \Leftrightarrow (5, 8, 1) \parallel \vec{n_{\tilde{\pi}}}$

$\vec{n_{\tilde{\pi}}}(a, 8, c)$; $k \cdot \vec{n_{\tilde{\pi}}} = (5, 8, 1)$

$$\begin{cases} k \cdot a = 5 \Rightarrow a = 5 \\ k \cdot 8 = 8 \Rightarrow k = 1 \\ k \cdot c = 1 \Rightarrow c = 1 \end{cases}$$

Seminar 6: Chapter 4



2, 3, 4, 10a, 11a, 13, 16, 17

$\vec{v}, \vec{w} \in V^3$

$|\vec{v} \times \vec{w}| = |\vec{v}| \cdot |\vec{w}| \cdot \sin(\widehat{\vec{v}, \vec{w}})$

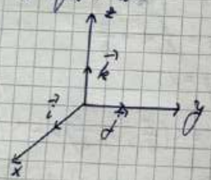
Properties:

- bilinear: $(\alpha_1 \vec{v_1} + \alpha_2 \vec{v_2}) \times \vec{w} = \alpha_1 \cdot (\vec{v_1} \times \vec{w}) + \alpha_2 \cdot (\vec{v_2} \times \vec{w})$ (same for the other variable)
- skew-symmetry: $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

if $\vec{v}, \vec{w}$ lin. dep. $\Rightarrow \vec{v} \times \vec{w} = \vec{0}$

If we work with a right oriented orthonormal system,
$\vec{v} = (x_{\vec{v}}, y_{\vec{v}}, z_{\vec{v}})$; $\vec{w} = (x_{\vec{w}}, y_{\vec{w}}, z_{\vec{w}})$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_{\vec{v}} & y_{\vec{v}} & z_{\vec{v}} \\ x_{\vec{w}} & y_{\vec{w}} & z_{\vec{w}} \end{vmatrix}$$



$\vec{i}, \vec{j}, \vec{k}$ right oriented

$\vec{k} = \vec{i} \times \vec{j} = -\vec{j} \times \vec{i}$

$\vec{i} = \vec{j} \times \vec{k} = -\vec{k} \times \vec{j}$

$\vec{j} = \vec{k} \times \vec{i} = -\vec{i} \times \vec{k}$

2. With respect to a right oriented orthonormal basis of V^3 consider the vectors $a(3, -1, -2)$ and $b(1, 2, -1)$. Calculate $a \times b$, $(2a + b) \times b$, $(2a + b) \times (2a - b)$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 3 & -1 & -2 \\ 1 & 2 & -1 \end{vmatrix} = \vec{i} + 6\vec{k} - 2\vec{j} + \vec{k} + 4\vec{i} + 3\vec{j} = 5\vec{i} + 7\vec{k} + \vec{j}$$

$(2\vec{a} + \vec{b}) \times \vec{b} = 2 \cdot (\vec{a} \times \vec{b}) + 1 \cdot (\vec{b} \times \vec{b})$

$(2\vec{a} + \vec{b}) \times \vec{b} = ((6, -2, -4) + (1, 2, -1)) \times (1, 2, -1) =$

$$= (7, 0, -5) \times (1, 2, -1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 0 & -5 \\ 1 & 2 & -1 \end{vmatrix} = 14\vec{k} - 5\vec{j} + 10\vec{i} + 7\vec{j} = 10\vec{i} + 2\vec{j} + 14\vec{k}$$

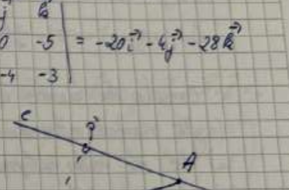
$2\vec{a} - \vec{b} = 6\vec{i} - 2\vec{j} - 4\vec{k} - \vec{i} - 2\vec{j} + \vec{k} = 5\vec{i} - 4\vec{j} - 3\vec{k} \Rightarrow$

$\Rightarrow 2\vec{a} - \vec{b} = (5, 4, -3)$

$$(2\vec{a} + \vec{b}) \times (2\vec{a} - \vec{b}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 0 & -5 \\ 5 & -4 & -3 \end{vmatrix} = -20\vec{i} - 4\vec{j} - 28\vec{k}$$

ℓ line in $\mathbb{E}^3$

P point in $\mathbb{E}^3$, $P \notin \ell$

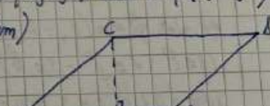


$PP' = PA \cdot \sin(\widehat{PAP'})$

$|\vec{PA} \times \vec{P'A}| = |\vec{PA}| \cdot |\vec{P'A}| \cdot \sin(\widehat{PAP'}) = |\vec{PA}| \cdot |\vec{PP'}|$

$\Rightarrow dist(P, \ell) = \dfrac{|\vec{PA} \times \vec{P'A}|}{|\vec{P'A}|} = \dfrac{|\vec{PA} \times \vec{v}|}{|\vec{v}|}, \forall \vec{v} \in D(\ell)$

3 Det. the distances between opposite sides of the parallelogram ABDC spanned by the vectors $\vec{AB}(6, 0, 1)$ and $\vec{AC}(\frac{3}{2}, 2, 1)$. (right oriented orthonormal system)



$dist(AB, CD) = dist(C, AB)$

$dist(C, AB) = \dfrac{|\vec{CA} \times \vec{C'A}|}{|\vec{C'A}|} = \dfrac{|\vec{CA} \times \vec{v}|}{|\vec{v}|}$

$\vec{CA} = -\vec{AC} \Rightarrow \vec{CA}(-\frac{3}{2}, -2, -1)$; $\vec{BA} = (-6, 0, -1)$

$pr_{AB} AC = AC'$

$dist(C, AB) = \frac{|\vec{AC} \times \vec{AB}|}{|\vec{AB}|}$

$\vec{AC} \times \vec{AB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{7}{2} & 2 & 1 \\ 6 & 0 & 1 \end{vmatrix} = 2\vec{i} - \frac{9}{2}\vec{j} - 12\vec{k} = (2, -\frac{9}{2}, -12)$

$|\vec{AC} \times \vec{AB}| = \sqrt{4 + \frac{81}{4} + 144} = \sqrt{\frac{592+81}{4}} = \sqrt{\frac{673}{4}} = \frac{\sqrt{673}}{2}$

$|\vec{AB}| = \sqrt{36+0+1} = \sqrt{37}$

$dist(AB, C) = CC' = \frac{\sqrt{673}}{2\sqrt{37}}$

4. Consider the vectors $\vec{a}(2, 3, -1)$ and $\vec{b}(1, -1, 3)$ (right oriented orthonormal ~~system~~ basis)

a) Det. the vector subspace $\langle a, b \rangle^{\perp} = \langle \vec{a} \times \vec{b} \rangle$

b) $\vec{p} \in V^3$ which is $\perp$ ~~to~~ $\vec{a}$ and $\vec{b}$ and for which $\vec{p} \cdot (2\vec{i} - 3\vec{j} + 4\vec{k}) = 51$

$S \subseteq V^3$; $S^{\perp} = \{ v \in V^3 \mid \vec{v} \cdot \vec{w} = 0, \forall w \in S \}$

a) $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 8\vec{i} - 7\vec{j} - 5\vec{k}$

$\Rightarrow \langle \vec{a}, \vec{b} \rangle^{\perp} = \langle (8, -7, -5) \rangle$

b) Let $\vec{p} = t(8, -7, -5)$

$(8t\vec{i} - 7t\vec{j} - 5t\vec{k}) \cdot (2\vec{i} - 3\vec{j} + 4\vec{k}) = 16t + 21t - 20t = 17t$

$17t = 51 \Rightarrow t = 3 \Rightarrow p = (24, -21, -15)$

10. a) (i, j, k) right oriented orthonormal basis, $\vec{w} = -\vec{i} + 3\vec{j} + \vec{k}$.

Det. the matrices of the linear maps:

$\varphi : V^3 \to V^3$, $\vec{v} \longmapsto \vec{w} \times \vec{v}$

$\psi : V^3 \to V^3$, $\vec{v} \longmapsto \vec{v} \times \vec{w}$

$M_b(\varphi) = ?$

$\varphi(\vec{i}) = \vec{w} \times \vec{i} = (-\vec{i} + 3\vec{j} + \vec{k}) \times \vec{i} = -\vec{i} \times \vec{i} + 3\vec{j} \times \vec{i} + \vec{k} \times \vec{i} =$
$= \vec{0} - 3\vec{k} + \vec{j} \Rightarrow [\varphi(\vec{i})]_b = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$; $\varphi(\vec{j}) = \vec{w} \times \vec{j} =$

$(-\vec{i} + 3\vec{j} + \vec{k}) \times \vec{j} = -\vec{i} \times \vec{j} + 3\vec{j} \times \vec{j} + \vec{k} \times \vec{j} = -\vec{k} + \vec{0} - \vec{i} \Rightarrow$

$\Rightarrow [\varphi(\vec{j})]_b = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$

$\varphi(\vec{k}) = \vec{w} \times \vec{k} = (-\vec{i} + 3\vec{j} + \vec{k}) \times \vec{k} = -\vec{i} \times \vec{k} + 3\vec{j} \times \vec{k} + \vec{k} \times \vec{k} =$
$= \vec{j} + 3\vec{i} + \vec{0} \Rightarrow [\varphi(\vec{k})]_b = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$; $M_b(\varphi) = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & 1 \\ -3 & -1 & 0 \end{pmatrix}$

$M_b(\psi) = \begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & -1 \\ 3 & 1 & 0 \end{pmatrix}$

11. a) Prove the Grassmann identity: $v_1, v_2, v_3 \in V^3$

$(\vec{v_1} \times \vec{v_2}) \times \vec{v_3} = \begin{vmatrix} \vec{v_2} & \vec{v_1} \\ \vec{v_2} \cdot \vec{v_3} & \vec{v_1} \cdot \vec{v_3} \end{vmatrix} = (\vec{v_1} \cdot \vec{v_3}) \cdot \vec{v_2} - (\vec{v_2} \cdot \vec{v_3}) \cdot \vec{v_1}$

Let $v_i(x_i, y_i, z_i)$

$(\vec{v_1} \times \vec{v_2}) \times \vec{v_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_{\vec{v_1} \times \vec{v_2}} & y_{\vec{v_1} \times \vec{v_2}} & z_{\vec{v_1} \times \vec{v_2}} \\ x_3 & y_3 & z_3 \end{vmatrix}$; $\vec{v_1} \times \vec{v_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \Rightarrow$

$\Rightarrow (\vec{v_1} \times \vec{v_2}) \times \vec{v_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} & \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} & \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\ x_3 & y_3 & z_3 \end{vmatrix} = \vec{i}\left(z_3 \cdot \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} - y_3 \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\right) + \vec{j} \cdot (\dots) + \vec{k} \cdot (\dots)$

$$x_{(\vec{v_1}\times\vec{v_2})\times\vec{v_3}} = z_3(z_1x_2 - x_1z_2) - y_3(x_1y_2 - x_2y_1) = x_2(z_3z_1 + y_3y_1) -$$

$$- x_1(z_3z_2 + y_2y_3) = x_2(x_1x_3 + y_1y_3 + z_1z_3) - x_1(x_2x_3 + y_2y_3 + z_2z_3) =$$

$$= (\vec{v_1}\cdot\vec{v_3})\cdot x_{\vec{v_2}} - (\vec{v_2}\cdot\vec{v_3})\cdot x_{\vec{v_1}} \Rightarrow x_{(\vec{v_1}\times\vec{v_2})\times\vec{v_3}} = x_{(\vec{v_1}\cdot\vec{v_3})\vec{v_2} - (\vec{v_2}\cdot\vec{v_3})\vec{v_1}}$$

Prove the same for y and z.

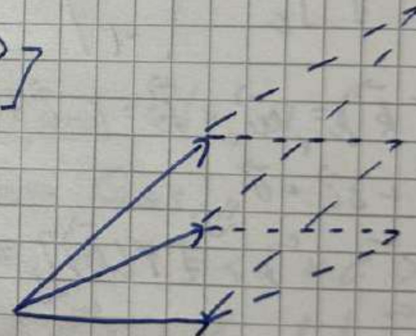
16. $Vol(ABC\Delta) = 5$, $ABC\Delta$ - tetrahedron

$A(2, 1, -1)$, $B(3, 0, 1)$, $C(2, -1, 3)$, $\Delta \in Oy$

Det. the coordinates of Δ

$$Vol(ABC\Delta) = \frac{1}{6}[\vec{AB}, \vec{AC}, \vec{A\Delta}]$$

$$[\vec{v_1}, \vec{v_2}, \vec{v_3}] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$



$$\Delta \in Oy \Rightarrow \Delta(0, a, 0) \Rightarrow \frac{1}{6}\cdot\begin{vmatrix} 1 & -1 & 2 \\ 0 & -2 & 4 \\ -2 & a-1 & 1 \end{vmatrix} = 5 \quad /\cdot 6 \Rightarrow$$

$$\Rightarrow -2 + \cancel{8} - \cancel{8} - 4(a-1) = 30 \Rightarrow -4a - 2 + 4 = 30 \Rightarrow 4a = -28 \Rightarrow a = -7$$