for a given initial state η,

the <u>orbit</u> of the initial state η is:

$$\gamma(\eta) = \{ \varphi(t, \eta) : t \in I_\eta \}$$

Orbit (image?)

$$\gamma^+(\eta) = \{ \varphi(t, \eta) : t \in I_\eta, t > 0 \}$$

$$\gamma^-(\eta) = \{ \varphi(t, \eta) : t \in I_\eta, t < 0 \}$$

<u>ORBIT OF AN EQUILIBRIUM POINT</u>:

→ formed only by the point itself

$$\gamma(\eta^*) = \{ \eta^* \} \; ; \; \eta^* \to \text{eq. point}$$

$$\lim_{t \to \infty} \| \varphi(t, \eta) - \eta^* \| = 0. \Rightarrow \text{attractor}$$

$$A_{\eta^*} = \{ \eta \in \mathbb{R}^n : \lim_{t \to \infty} \| \varphi(t, \eta) - \eta^* \| = 0. \}$$

↳ basin of attraction.

$\dot{x} = f(x)$

$$\begin{cases} x' = f(x) \\ x(0) = \eta \end{cases}$$

The flow: $\varphi(t, \eta)$ the unique solution of
$\varphi(\cdot, \eta)$ the IVP

$\varphi(0, \eta) = \eta$ because $\begin{cases} \eta - \text{initial state } (t=0) \\ \varphi(t, \eta) - \text{state at time } t \end{cases}$

The space ~~to~~ which the states belong

⇓

STATE/PHASE space

Equilibrium points: (states)

$\exists! \; \varphi(t, \eta^*) = \eta^*, \forall t \in \mathbb{R}$

η^* is given by $f(\eta^*) = 0$.

η^* - equilibrium points

$x' = f(x)$

$$\begin{cases} f'(\eta^*) < 0 \Rightarrow \text{attractor} \\ f'(\eta^*) > 0 \Rightarrow \text{repeller} \end{cases}$$

η^* eq. point — unstable
— stable
— asymptotically stable: stable + attractor

$\dot{x} = Ax$

$\det(A - \lambda I_n) = 0 \Rightarrow$ eigenvalues

$\sigma(A)$ – all eigenvalues

① $Re(\lambda) < 0$ for all $\Rightarrow$ asymptotically stable.

② any $Re(\lambda) > 0 \Rightarrow$ unstable

$\hookrightarrow$ $\dot{x} = Ax$ unstable

<u>Linearization method</u>:

η^* hyperbolic when $Re(\lambda) \neq 0$, ~~for all~~ $\forall \lambda \in \sigma(A)$.

$\dot{x} = f(x)$

$f'(\eta^*) < 0$ asymptotically stable

$f'(\eta^*) > 0$ unstable

$\dot{x} = Ax$

3. $x^* = 0$. eq. point of $\dot{x} = Ax$

1) node: same sign: asymptotically stable ($\lambda_1 < \lambda_2 < 0$) global attractor; unstable ($0 < \lambda_1 < \lambda_2$) global repeller
2) center: $\lambda_{1,2} = \pm i\beta$: always stable, never asymptotical.
3) focus: $\lambda_{1,2} = \alpha \pm i\beta$: asymptotically stable: $\alpha < 0$; unstable: $\alpha > 0$.
4) saddle: different sign: always unstable (~~at least~~ one $\lambda > 0$).

NODE: same sign $\in \mathbb{R}$
- $\lambda_1, \lambda_2 < 0$ global attractor asymtotically stable
- $\lambda_1, \lambda_2 > 0$ global repeller unstable

SADDLE: different sign $\in \mathbb{R}$ $\rightarrow$ always unstable

CENTER: $\lambda_1, \lambda_2 = \pm i\beta$, $\beta > 0$. $\beta \in \mathbb{R}$ always stable never asymtotically
$\rightarrow$ never hyperbolic

FOCUS: $\lambda_1, \lambda_2 = \alpha \pm i\beta$
- asymtotically stable $\alpha < 0$
- unstable $\alpha > 0$

Planar dynamical systems

$$\dot{X} = f(X), \quad f: \mathbb{R}^2 \to \mathbb{R}^2, \quad f \in C^1(\mathbb{R}^2)$$

$$\begin{cases} \dot{X} = f(X) \\ X(0) = \eta \end{cases} \xrightarrow[\text{Uniqueness}]{\text{Existence and}} \varphi(t,\eta) \text{ is the unique solution}$$

(η is the initial state)

$$\gamma_\eta = \{ \varphi(t,\eta) : t \in J_\eta \}$$

$\varphi(t,\eta) \to$ the FLOW

~~$U \subset \mathbb{R}^2$~~

~~$H: U \to \mathbb{R}$ continuous~~

the orbits are contained in the level curves of a first integral.

$U \subset \mathbb{R}^2$

$H: U \to \mathbb{R}$, $H \in C^1(U)$

$U = \mathbb{R}^2$ ↳ H is a global first integral

$c \in \mathbb{R}$, $\Gamma_c = \{ H(X) = c \mid X \in U \}$

the c-level curve

and $H(\varphi(t,\eta)) = H(\eta)$, $\forall \eta \in U$, $\forall t \in J_\eta$, $\varphi(t,\eta) \in U$

3 $\dot{X} = AX$ or $\begin{cases} \dot{x} = a_{11}x + a_{12}y \\ \dot{y} = a_{21}x + a_{22}y \end{cases}$

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} ; \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} ; \quad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$\det(A) = 0 \rightarrow$ infinite number of solutions, however Existence and uniqueness states that $\varphi(t, \eta)$ is unique.

$AX = 0$; X equilibrium point

$\dot{x} = f(x)$

η^* eq. point, $f(\eta^*) = 0$.

Study the stability: $\rightarrow$ decide $\begin{cases} \text{attractor / repeller} \\ \text{stable / unstable} \end{cases}$

$\dot{x} = Ax$, $\det(A) \neq 0 \longrightarrow$ the origin is the only eq. point.

$\det(A) = \lambda_1 \cdot \lambda_2$

---

node : λ_1, λ_2 same sign $\begin{cases} \lambda_1 \leq \lambda_2 < 0 : \text{global attractor, asym. stable} \\ 0 < \lambda_1 \leq \lambda_2 : \text{global repeller, unstable} \end{cases}$

saddle : λ_1, λ_2 different sign : ~~always~~ unstable

center : $\pm \beta i$: always stable, never asymptotically

focus : $\alpha \pm \beta i$: $\begin{cases} \alpha < 0 \text{ asym. stable, attractor} \\ \alpha > 0 \text{ unstable} \end{cases}$

$\dot{x} = f(x)$

$f : \mathbb{R}^2 \to \mathbb{R}^2$

find first integral

$\dot{x} = f_1(x,y)$; $\frac{dy}{dx} = \frac{f_2(x,y)}{f_1(x,y)}$

$\dot{y} = f_2(x,y)$

η^* eq. point ; $J_f(\eta^*) \to$ Jacobian Matrix of f computed in η^*

$\dot{X} = J_f(\eta^*) X$

$\to$ the linearization of ~~f~~ $\dot{x} = f(x)$ around η^*

η^* hyperbolic $\Leftrightarrow$ $Re(\lambda_1) \neq 0$
$Re(\lambda_2) \neq 0$.

origin : repeller / attractor $\to$ no first integral

$H : U \to \mathbb{R}$

$\dot{x} = f_1(x,y)$

$\dot{y} = f_2(x,y)$

$\Rightarrow f_1(x,y)\frac{\partial H}{\partial x}(x,y) + f_2(x,y)\frac{\partial H}{\partial y}(x,y) = 0 \Leftrightarrow$

H first integral

Study the stability of the equilibrium points of the non-linear system:

$$\begin{cases} x' = x(1-x) \\ y' = y(3-y) \end{cases}$$

<u>EQUILIBRIUM POINTS:</u>

$\cancel{\left\{ x(1-x) \right.}$ First we find the equilibrium points by finding the solution of the system $f(X) = 0$.

$$\begin{cases} \cancel{x'} \; x(1-x) = 0 \\ y(3-y) = 0 \end{cases} \Leftrightarrow \begin{cases} x \in \{0, 1\} \\ y \in \{0, 3\}. \end{cases}$$

$$\rightarrow S^{(x,y)} = \{ \underset{z_1^*}{(0,0)}, \underset{z_2^*}{(0,3)}, \underset{z_3^*}{(1,0)}, \underset{z_4^*}{(1,3)} \}$$

$$f(X) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} ; \; X = \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\begin{cases} x' = -x + xy \\ y' = -2y + 3y^2 \end{cases} ; \quad \dot{X} = f(X), \; f(X) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

nonlinear differential equation

Step 1: find equilibrium points by solving

$$f(X) = 0 = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

$$\begin{cases} -x + xy = 0 \\ -2y + 3y^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x(y-1) = 0 \\ y(3y-2) = 0 \end{cases} \Rightarrow$$

$$\Rightarrow S(x,y) = \left\{ (0,0), \left(0, \frac{2}{3}\right) \right\}$$

let $\eta_1^* = (0,0)$ and $\eta_2^* = \left(0, \frac{2}{3}\right)$ be the equilibrium points, $\eta_1^*, \eta_2^* \in \mathbb{R}^2$

Step 2: Compute Jacobi Matrix

for $\dot{X} = f(X)$, where $f(X) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$; $J_f(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{bmatrix}$;

$X = \begin{bmatrix} x \\ y \end{bmatrix}$; $X \in \mathbb{R}^2$

2*

$$\left.\begin{aligned} f_1(x,y) &= -x + xy \\ f_2(x,y) &= -2y + 3y^2 \end{aligned}\right| \Rightarrow J_f(x,y) = \begin{bmatrix} y-1 & x \\ 0 & 6y-2 \end{bmatrix}$$

Step 3: Check stability of equilibrium points, using linearization method, for the system:

$$\dot{X} = J_f(\xi^*) \cdot X, \quad \xi^* \text{ eq. point.}$$

~~let $\sigma($ λ_1, λ_2~~

~~for given ξ^*~~,

~~let~~ ~~let~~ for fixed ξ^*, let $\lambda_1, \lambda_2 \in \sigma(\lambda)$,

$$\sigma(\lambda) = \{ \lambda \in \mathbb{C} : \det(J_f(\xi^*) - \lambda I_n) = 0 \}$$

i) $\xi_1^* = (0,0)$

$$J_f(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}; \det(J_f(0,0) - \lambda I_2) = \begin{vmatrix} -1-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} =$$

$$= (-1-\lambda)(-2-\lambda) \Rightarrow \lambda_1 = -1, \quad \lambda_2 = -2$$

λ_1, λ_2 have same sign, $\lambda_1 \leq \lambda_2 < 0$ $\Rightarrow$ stable node, global attractor

Study the stability of the equilibrium points of the non-linear system:

$$\begin{cases} x' = x(1-x) \\ y' = y(3-y) \end{cases}$$

EQUILIBRIUM POINTS:

First we find the equilibrium points by finding the solution of the system $f(X) = 0$.

$$\begin{cases} \cancel{x'} \; x(1-x) = 0 \\ y(3-y) = 0 \end{cases} \Leftrightarrow \begin{cases} x \in \{0, 1\} \\ y \in \{0, 3\}. \end{cases}$$

$$\rightarrow S^{(x,y)} = \{\underbrace{(0,0)}_{z_1^*}, \underbrace{(0,3)}_{z_2^*}, \underbrace{(1,0)}_{z_3^*}, \underbrace{(1,3)}_{z_4^*}\}$$

$$f(X) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} ; \quad X = \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$J_f(z^*) = \begin{bmatrix} \frac{\partial f_1(x,y)}{\partial x}(z^*) & \frac{\partial f_1(x,y)}{\partial y}(z^*) \\ \frac{\partial f_2(x,y)}{\partial x}(z^*) & \frac{\partial f_2(x,y)}{\partial y}(z^*) \end{bmatrix} \; ; \; z^* \in \mathbb{R}.$$

$$= \begin{bmatrix} 1-x-x & 0 \\ 0 & 3-y-y \end{bmatrix} = \begin{bmatrix} 1-2x & 0 \\ 0 & 3-2y \end{bmatrix}$$

for z^* eq. point, $\lambda_1, \lambda_2 \in \sigma(J_f(z^*))$, $\sigma(J_f(z^*)) = \{\lambda \in \mathbb{R} \mid \det(J_f(z^*) - \lambda I_2) = 0\}$

for $z_1^* = (0,0)$

$$J_f(z_1^*) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$\lambda_1, \lambda_2 \in \sigma(J_f(z_1^*))$; $\sigma(J_f(z_1^*)$

$$\det(J_f(0,0) - \lambda I_2) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) = \cancel{3 - \lambda - 3\lambda + \lambda^2} = \cancel{\lambda^2 - 4\lambda + 3}$$

$\Rightarrow \lambda_1 = 1$, $\lambda_2 = 3$ same sign $\downarrow$ node, $(0 < \lambda_1 \leq \lambda_2 \rightarrow$ repeller

Seminar Test

1)
$$\dot{x} = -3(x-21)$$
$$\dot{x} = f(x) \Rightarrow f(x) = -3(x-21)$$

The flow → the unique solution of the IVP:

$$\begin{cases} \dot{x} = -3(x-21) \\ x(0) = \eta \end{cases}$$

$$\dot{x} = -3(x-21) \Rightarrow$$

$$\dot{x} = -3x + 63$$

$$\dot{x} + 3x = 63. \; ; \; x = ?$$

① x_h:

$$x' + 3x = 0$$

$$r + 3 = 0 \Rightarrow r = -3 \Rightarrow c \cdot e^{-3t} \text{ solution,}$$

$c \in \mathbb{R}$.

② x_p: $\quad x_p' + 3x_p = 63$

$$\Rightarrow x_p = 21.$$

$$x = x_h + x_p = c \cdot e^{-3t} + 21$$

$$x(0) = \eta \Rightarrow c \cdot e^{-3\cdot 0} + 21 = \eta \Rightarrow$$

$$\Rightarrow c + 21 = \eta \Rightarrow$$

$$\Rightarrow c = \eta - 21.$$

<u>The flow</u>:

$$\varphi(t, \eta) = (\eta - 21) \cdot e^{-3t} + 21.$$

2.

a) $x_p = a\cos t + b\sin t$

$x'' + x' + x = 2\cos t$

~~a cost~~ $x'_p = -a\sin t + b\cos t$

$x''_p = -a\cos t + (-b\sin t) =$

$= -a\cos t - b\sin t$

$x''_p + x'_p + x_p = 2\cos t \Leftrightarrow$

$\Leftrightarrow -a\sin t + b\cos t - a\cos t - b\sin t = 2\cos t \Leftrightarrow$

$\Leftrightarrow \sin t(-a-b) + \cos t(b-a) = 2\cos t.$

$\sin t, \cos t$ linearly independent $\Rightarrow$

$\Rightarrow \begin{cases} -a-b=0 \overset{(1)}{\Rightarrow} -a-2-a=0 \Rightarrow 2a=-2 \Rightarrow a=-1 \Rightarrow \\ b-a=2 \Rightarrow b=2+a \ (1) \qquad \Rightarrow b=1 \end{cases}$

Sk $x_p = -\cos t + \sin t, \ t \in \mathbb{R}.$

b) $\begin{cases} x'' + x' + x = 2\cos t \\ x(0) = 0 \\ x'(0) = 0. \end{cases}$

① x_H

$r^2 + r + 1 = 0.$

$\Delta = 1 - 4\cdot 1 = -3; \quad \sqrt{\Delta} = i\sqrt{3}$

$r_1 = \dfrac{-1 + i\sqrt{3}}{2} = -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$

$r_2 = \dfrac{-1 - i\sqrt{3}}{2} = -\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$

$\Rightarrow e^{-\frac{1}{2}t}\left(c_1 \cos\dfrac{\sqrt{3}}{2}t + c_2 \sin\dfrac{\sqrt{3}}{2}\right)$ is a solution, $c_1, c_2 \in \mathbb{R}$.

$x'(0) = 0$

$x'(t) = -\dfrac{1}{2}e^{-\frac{1}{2}t}\left(\cos\dfrac{\sqrt{3}}{2}t + c_2 \cdot \sin\dfrac{\sqrt{3}}{2}t\right)$

$+ \sin t + \cos t +$

$+ e^{-\frac{1}{2}t}\left(-\dfrac{\sqrt{3}}{2}\sin t + \dfrac{\sqrt{3}}{2} c_2 \cos\dfrac{\sqrt{3}}{2}t\right)$

$x'(0) = -\dfrac{1}{2}(1 + c_2 \cdot 0) + 0 + 1$

$+ c_2 \cdot \dfrac{\sqrt{3}}{2} = 0 \Rightarrow$

$\Rightarrow -\dfrac{1}{2} + c_2 \cdot \dfrac{\sqrt{3}}{2} = 0 \Rightarrow$

$\Rightarrow c_2 = \dfrac{1}{2} \cdot \dfrac{2}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

<u>probably wrong</u>

~~$x_? = \varphi(t) \cdot e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + \sin\frac{\sqrt{3}}{2}t\right); \quad \varphi(t) \in C^1(\mathbb{R})$~~

$x_p = -\cos t + \sin t$

$\Rightarrow x = e^{-\frac{1}{2}t}\left(c_1 \cdot \cos\dfrac{\sqrt{3}}{2}t + c_2 \cdot \sin\dfrac{\sqrt{3}}{2}t\right) - \cos t + \sin t$

$x(0) = 1 \cdot \left(c_1 \cdot \cos\dfrac{\sqrt{3}}{2}\cdot 0 + c_2 \cdot \sin\dfrac{\sqrt{3}}{2}\cdot 0\right) - \cos 0 + \sin 0 =$

$= c_1 + 0 \cdot c_2 - 1 + 0 =$

$= c_1 - 1$

$x(0) = 0$

$\Rightarrow c_1 = 1$

$\Rightarrow \left\{ x = e^{-\frac{1}{2}t}\left(\cos\dfrac{\sqrt{3}}{2}t + \dfrac{\sqrt{3}}{3}\sin t\right) - \cos t + \sin t. \right.$

3.

$$x' = -2x + 7t^2$$

$$x' + 2x = 7t^2$$

$f(x) = 7t^2 \Rightarrow$ looking for a second degree polynomial solution.

$x_p = at^2 + bt + c$ such solution $\Rightarrow$

$$\Rightarrow 2at + b + 2at^2 + 2bt + 2c = 7t^2 \Rightarrow$$

$$\Rightarrow t^2(2a-7) + t(2a+2b) + b+2c = 0.$$

$1, t, t^2$ lin. indep $\Rightarrow \begin{cases} 2a-7=0 \Rightarrow a=\frac{7}{2} \\ 2a+2b=0 \Rightarrow b=-\frac{7}{2} \\ b+2c=0 \Rightarrow c=\frac{7}{4}. \end{cases}$

$$x_p = \frac{7}{2}t^2 + -\frac{7}{2}t + \frac{7}{4}$$

4.

a) $x' + \frac{1}{t}x = e^{-3t}$

① $x' + \frac{1}{t}x = 0$

$\frac{x'}{x} = -\frac{1}{t} \quad / \int .$

$\ln|x| = -\ln t + c$

$x = e^{-\ln t + c}$

$x = \frac{1}{t} \cdot c \Rightarrow x_H = \frac{c}{t}, c \in \mathbb{R}.$

$\int t \cdot e^{-3t} dt = \int t \cdot \left(-\frac{1}{3} \cdot e^{-3t}\right)' dt = \dots$

$= -\frac{t e^{-3t}}{3} - \int -\frac{e^{-3t}}{3} dt = \dots$

$= -\frac{t e^{-3t}}{3} - \frac{e^{-3t}}{9} = \dots$

$= -\frac{(3t+1) \cdot e^{-3t}}{9} + c$

$\Rightarrow x_p = \frac{-(3t+1) \cdot e^{-3t}}{9t}$

$x = \frac{c}{t} - \frac{(3t+1) \cdot e^{-3t}}{9t}$?

②. $x' + \frac{1}{t}x = e^{-3t}$

find $x_p = \varphi(t) \cdot \frac{1}{t}$, $\varphi(t) \in C^1(\mathbb{R})$

$x_p' = \varphi'(t) \cdot \frac{1}{t} + \varphi(t) \cdot \left(-\frac{1}{t^2}\right)$

$\varphi'(t) \cdot \frac{1}{t} + \cancel{\varphi(t) \cdot \frac{-1}{t^2}} + \cancel{\frac{1}{t} \cdot \varphi(t) \cdot \frac{1}{t}} = e^{-3t}$

$\varphi'(t) \cdot \frac{1}{t} = e^{-3t}$

$\varphi'(t) = t \cdot e^{-3t}$

b) $x' + 3t^2 x = -1$

$$x' = -3t^2 x \quad / :x$$

$$\frac{x'}{x} = -3t^2 \quad / \int$$

$$\ln|x| = -\cancel{3}\cdot\frac{1}{\cancel{3}}t^3$$

$$x = c\cdot e^{-t^3}$$

$$x_H = c\cdot e^{-t^3}$$

$$x_p = \varphi(t)\cdot e^{-t^3}$$

$$x_p' = \varphi'(t)\cdot e^{-t^3} + \varphi(t)\cdot(-3t^2)\cdot e^{-t^3}$$

$$\varphi'(t)\cdot e^{-t^3} - 3\cancel{t^2\cdot\varphi(t)\cdot e^{-t^3}} + \cancel{3t^2\cdot\varphi(t)\cdot e^{-t^3}} = -1.$$

$$\varphi'(t) = -e^{t^3} \quad ???$$

$\cancel{\varphi(t)} =$

5.

$$\dot{x} = x - 2x^3$$

$$f(x) = x - 2x^3 = 0 \Leftrightarrow x(1-2x^2) = 0 \Leftrightarrow$$

$$\Leftrightarrow x = 0 \text{ or } x^2 = \frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow x = \pm \frac{\sqrt{2}}{2}$$

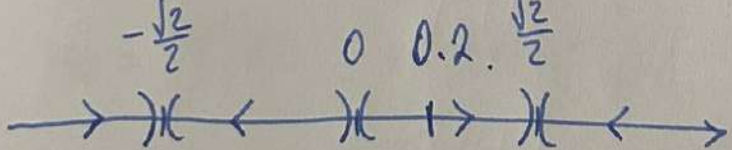
η^* eq points: $\left\{ -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \right\}$

$$f'(x) = 1 - 6x^2$$

$f'\left(-\frac{\sqrt{2}}{2}\right) = 1 - \cancel{6}^{3} \cdot \frac{\cancel{2}}{\cancel{4}_{2}} = -2 < 0 \Rightarrow$ attractor (stable)

$f'(0) = 1 - 6 \cdot 0 = 1 > 0 \Rightarrow$ repeller (unstable)

$f'\left(\frac{\sqrt{2}}{2}\right) = -2 < 0$ attractor (stable)



orbits: $\left(-\infty, -\frac{\sqrt{2}}{2}\right), \left\{-\frac{\sqrt{2}}{2}\right\}, \left(-\frac{\sqrt{2}}{2}, 0\right), \{0\}, \left(0, \frac{\sqrt{2}}{2}\right), \left\{\frac{\sqrt{2}}{2}\right\},$
$\left(\frac{\sqrt{2}}{2}, +\infty\right)$

$\varphi(t, 0) = 0$

- $\eta = 0.2 \to$ moves towards $\frac{\sqrt{2}}{2}$ as $t \to \infty$?
- $\eta = 20 \to$ same ?

?