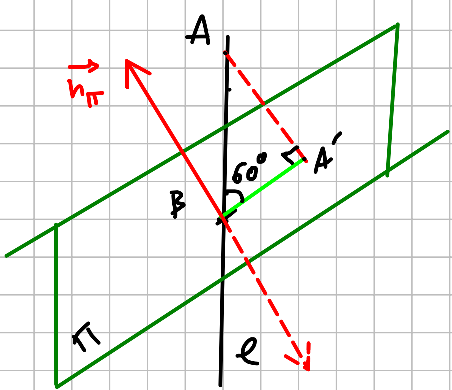
3.41. Determine the planes which pass through $P(0,2,0)$ and $Q(-1,0,0)$ and which form an angle of $60°$ with the z-axis.



$\ell \cap \pi = \{B\}$

$A' = pr_\pi(A)$

$m(\widehat{\ell, \pi}) = \frac{\pi}{2} - m(\widehat{\vec{n}_\pi, \vec{u}})$

unde $\vec{u} \in D(\ell)$

Pt. noi $m(\widehat{\vec{n}_\pi, \vec{u}}) = 30°$

$\vec{n}_\pi(a, b, c) \Rightarrow m(\widehat{(a,b,c),(0,0,1)}) = 30°$

$$\Rightarrow \cos 30° = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{\sqrt{3}}{2}$$

$\pi : ax + by + cz + d = 0$

$P(0,2,0) \in \pi \Rightarrow 2b + d = 0$

$Q(-1,0,0) \in \pi \Rightarrow -a + d = 0$

$\Rightarrow d = a,\quad b = -\frac{a}{2}$

$$\frac{c^2}{a^2+b^2+c^2} = \frac{3}{4} \Rightarrow \frac{c^2}{a^2+\frac{a^2}{4}+c^2} = \frac{3}{4} \Rightarrow 4c^2 = 3c^2 + \frac{15a^2}{4}$$

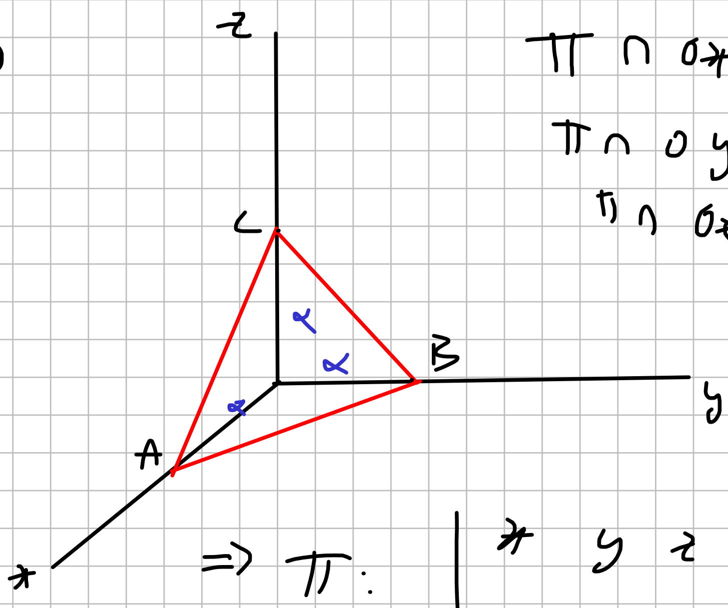
$$\Rightarrow c = \frac{15a^2}{4} \Rightarrow c = \pm\frac{\sqrt{15}\,a}{2}$$

$$\Rightarrow \pi : ax - \frac{a}{2}y \pm \frac{\sqrt{15}\,a}{2}z + a = 0 \Leftrightarrow a\left(x - \frac{1}{2}y \pm \frac{\sqrt{15}}{2}z + 1\right) = 0$$

$$\Rightarrow \pi : x - \frac{1}{2}y \pm \frac{\sqrt{15}}{2}z + 1 = 0$$

3.29. Determine an equation for each plane passing through $P(3,5,-7)$ and intersecting the coordinate axes in congruent segments.

$\alpha > 0$



$\pi \cap Ox = A(s_1\alpha, 0, 0)$

$\pi \cap Oy = B(0, s_2\alpha, 0)$

$\pi \cap Oz = C(0, 0, s_3\alpha)$

$s_1, s_2, s_3 \in \{-1, 1\}$

$$\Rightarrow \pi : \begin{vmatrix} x & y & z & 1 \\ s_1\alpha & 0 & 0 & 1 \\ 0 & s_2\alpha & 0 & 1 \\ 0 & 0 & s_3\alpha & 1 \end{vmatrix} = 0$$

$$\pi : \quad x \cdot \begin{vmatrix} 0 & 0 & 1 \\ s_2\alpha & 0 & 1 \\ 0 & s_3\alpha & 1 \end{vmatrix} - s_1\alpha \cdot \begin{vmatrix} y & z & 1 \\ s_2\alpha & 0 & 1 \\ 0 & s_3\alpha & 1 \end{vmatrix} = 0$$

$$x\, s_2\alpha\, s_3\alpha - s_1\alpha\big(y\cdot(-s_3\alpha) - z\cdot s_2\alpha + 1\cdot s_2\alpha\cdot s_3\alpha\big) = 0$$

$$x \cdot s_2 s_3 \cdot \alpha^2 + y \cdot s_1 s_3 \alpha^2 + z \cdot s_1 s_2 \alpha^2 - s_1 s_2 s_3 \alpha^3 =$$

$$\Rightarrow \frac{x}{s_1\alpha} + \frac{y}{s_2\alpha} + \frac{z}{s_3\alpha} = 1$$

$$P \in \pi \Rightarrow \quad \frac{3}{s_1} + \frac{5}{s_2} + \frac{7}{s_3} = \alpha$$

În general planele sunt:

$$\Pi_{s_1,s_2,s_3} \quad \frac{x}{s_1} + \frac{y}{s_2} + \frac{z}{s_3} = \frac{3}{s_1} + \frac{5}{s_2} + \frac{7}{s_3}$$

$$\Pi_{s_1,s_2,s_3} \quad \frac{x-3}{s_1} + \frac{y-5}{s_2} + \frac{z-7}{s_3} = 0$$

ex.: $\Pi_{1,1,1}$: $x+y+z-15=0$

$\Pi_{1,-1,1}$: $x-3+5-y+z-7=0$

$x-y+z-5=0$

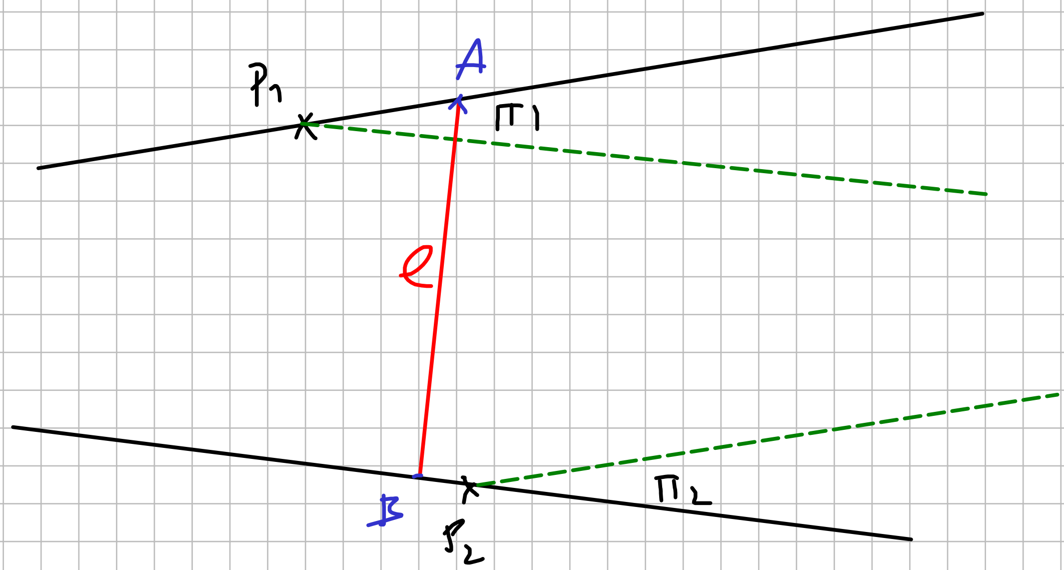
Planul determinat de 3 puncte $A_i=(x_i, y_i, z_i)$, $i=1,2,3$

este Π : $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$

4.19. Consider two lines ℓ_1 and ℓ_2 in $\mathbb{E}^3$. Suppose that the common perpendicular line is

$$\ell : \begin{cases} x = 1 + t \\ y = 2 - t \\ z = t \end{cases},$$

that $P_1(1,0,1) \in \ell_1$ and that $P_2(-1,1,0) \in \ell_2$. Determine the two lines.



$\vec{v}(1,-1,1)$ vector normal pt. π_1 și π_2

$$\pi_1 : 1\cdot(x-1) + (-1)(y-0) + 1\cdot(z-1) = 0$$

$$x - y + z - 2 = 0$$

$$\pi_1 : 1\cdot(x+1) + (-1)(y-1) + 1\cdot(z-0) = 0$$

$$x - y + z + 2 = 0$$

Fie punctul $\{A\} = \ell \cap \pi_1$: $\begin{cases} x - y + z - 2 = 0 \\ x = 1+t \\ y = 2-t \\ z = t \end{cases}$

$$1+t-(2-t)+t-2=0$$

$$3t - 3 = 0 \Rightarrow t = 1$$

$$\Rightarrow A(2, 1, 1) \qquad P_1(1,0,1)$$

$$\Rightarrow \ell_1 : \frac{x-2}{1-2} = \frac{y-1}{0-1} = \frac{z-1}{1-1} \Rightarrow \ell_1 : \begin{cases} \frac{x-2}{-1} = \frac{y-1}{-1} \\ z = 1 \end{cases} \Rightarrow \begin{cases} x - y - 1 = 0 \\ z = 1 \end{cases}$$

Analog pt ℓ_2

4.17. With respect to an orthonormal system consider the vectors $\mathbf{a}(8,4,1)$, $\mathbf{b}(2,2,1)$ and $\mathbf{c}(1,1,1)$. Determine a vector $\mathbf{d}$ satisfying the following properties

a) the angles $\measuredangle(\mathbf{d},\mathbf{a})$ and $\measuredangle(\mathbf{d},\mathbf{b})$ are equal,

b) $\mathbf{d}$ is orthogonal to $\mathbf{c}$,

c) $(\mathbf{a},\mathbf{b},\mathbf{c})$ and $(\mathbf{a},\mathbf{b},\mathbf{d})$ have the same orientation.

$\vec{d}(x,y,z)$, $x^2+y^2+z^2=1$

a) $\Rightarrow \dfrac{\vec{d}\cdot\vec{a}}{|\vec{a}|} = \dfrac{\vec{d}\cdot\vec{b}}{|\vec{b}|} \Rightarrow \dfrac{8x+4y+z}{\sqrt{64+16+1}} = \dfrac{2x+2y+z}{\sqrt{9}}$

$\Rightarrow 3(8x+4y+z) - 9(2x+2y+z) = 0$

$6x - 6y - 6z = 0$

$\Rightarrow x - y - z = 0$

b) $\vec{d} \perp \vec{c} \Leftrightarrow \vec{d}\cdot\vec{c} = 0 \Leftrightarrow x+y+z=0$

c) (a,b,c) şi (a,b,d) au aceaşi orientare

$\Leftrightarrow [a,b,c]\cdot[a,b,d] > 0$

$\begin{vmatrix} 8 & 4 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \cancel{16} + 4 + \cancel{2} - \cancel{2} - \cancel{8} - 8 = 4 > 0$

$\Rightarrow \begin{vmatrix} 8 & 4 & 1 \\ 2 & 2 & 1 \\ x & y & z \end{vmatrix} > 0 \Rightarrow 2x - 6y + 8z > 0$

$x - 3y + 4z > 0$

$\begin{cases} x - y - z = 0 \\ x + y + z = 0 \\ x - 3y + 4z > 0 \end{cases}$ $\begin{cases} x = y + z \Rightarrow x = 0 \\ y = -z \\ 3z + 4z > 0 \Rightarrow z > 0 \end{cases}$

$x^2+y^2+z^2=1$

$z^2+z^2=1 \Rightarrow 2z^2 = 1 \Rightarrow z = \frac{1}{\sqrt{2}}$

$\Rightarrow \vec{d}\left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

2.35. In $\mathbb{A}^4$ consider the affine subspaces

$$\alpha = \begin{bmatrix}2\\1\\2\\1\end{bmatrix} \quad \beta = \langle\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\rangle + \begin{bmatrix}1\\3\\0\\0\end{bmatrix} \quad \gamma = \langle\begin{bmatrix}2\\1\\3\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\2\\-2\end{bmatrix}\rangle + \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \quad \delta = \langle\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\rangle.$$

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Which of the following is true?

a) $\alpha \in \beta$ d) $\beta \parallel \gamma$ g) $\beta \subseteq \gamma$

b) $\alpha \in \gamma$ e) $\beta \parallel \delta$ h) $\gamma \subseteq \delta$

c) $\alpha \in \delta$ f) $\gamma \parallel \delta$ i) $\beta \subseteq \delta$

$$\alpha \in \gamma \iff \begin{pmatrix}2\\1\\2\\1\end{pmatrix} \in \begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \left\langle \begin{pmatrix}2\\1\\3\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\2\\-2\end{pmatrix} \right\rangle$$

$$\iff \exists\, \alpha, \beta \in \mathbb{R} \text{ a.î.}$$

$$\begin{pmatrix}2\\1\\2\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \alpha \begin{pmatrix}2\\1\\3\\-1\end{pmatrix} + \beta \cdot \begin{pmatrix}1\\0\\2\\-2\end{pmatrix} \iff$$

$$\iff \exists\, \alpha, \beta : \begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \alpha \begin{pmatrix}2\\1\\3\\-1\end{pmatrix} + \beta \cdot \begin{pmatrix}1\\0\\2\\-2\end{pmatrix}$$

$$M = \begin{pmatrix}2 & 1\\1 & 0\\3 & 2\\-1 & -2\end{pmatrix} \qquad \overline{M} = \begin{pmatrix}2 & 1 & 1\\1 & 0 & 1\\3 & 2 & 1\\-1 & -2 & 1\end{pmatrix}$$

$$\begin{cases} 2\alpha + \beta = 1 & \Rightarrow \beta = -1 \\ \alpha = 1 & \\ 3\alpha + 2\beta = 1 & \Rightarrow 3 - 2 = 1 \\ -\alpha - 2\beta = 1 & \Rightarrow -1 + 2 = 1 \end{cases} \quad \checkmark$$

$$\Rightarrow \alpha \in \gamma$$

$$\beta \parallel \gamma \Leftrightarrow D(\beta) \subseteq D(\gamma) \Leftrightarrow \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle \subseteq \left\langle \begin{pmatrix} 2 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ -2 \end{pmatrix} \right\rangle$$

$\hookrightarrow \dim \beta = 1$

$\dim \gamma = 2$

$$\Leftrightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \in \left\langle \begin{pmatrix} 2 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ -2 \end{pmatrix} \right\rangle \Leftrightarrow$$

$\Leftrightarrow \exists\, \alpha, \beta \in \mathbb{R}$ a.i.

$$\begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 1 \\ 3 \\ -1 \end{pmatrix} + \beta \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \\ -2 \end{pmatrix}.$$