915 – MOGOVAN JONATHAN

2. $\sum_{n\geq 2} \ln\left(1-\frac{1}{n^2}\right) = \sum_{n\geq 2} \ln\left(\frac{n^2-1}{n^2}\right) =$

$= \sum_{n\geq 2} \ln\left(\frac{(n-1)(n+1)}{n\cdot n}\right) = \sum_{n\geq 2} \ln\left(\frac{n-1}{n}\right) + \sum_{n\geq 2} \ln\left(\frac{n+1}{n}\right)$

$= \ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) + \dots + \ln\left(\frac{n-1}{n}\right) +$

$+ \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \dots + \ln\left(\frac{n}{n-1}\right) + \ln\left(\frac{n+1}{n}\right) =$

$= \ln\left(\frac{1}{2}\right) + \ln\left(\frac{n+1}{n}\right)$

$\lim_{n\to\infty} \sum_{n\geq 2} \ln\left(1-\frac{1}{n^2}\right) = \ln\left(\frac{1}{2}\right) + \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) =$

$= \ln\left(\frac{1}{2}\right) = -\ln(2)$

b) $\sum_{n\geq 1} \frac{n+1}{3^n} =$

$$S = \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \dots + \frac{n+1}{3^n}$$

$$\frac{S}{3} = \frac{2}{3^2} + \frac{3}{3^3} + \frac{4}{3^4} + \dots + \frac{n+1}{3^{n+1}}$$

$$S - \frac{S}{3} = \frac{2}{3} + \underbrace{\frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^n}} - \frac{n+1}{3^{n+1}}$$

$$1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^n} = \frac{1 - \left(\frac{1}{3}\right)^n}{1 - \frac{1}{3}} \qquad \left(\frac{1}{3}\right)^n \to 0$$

$$\frac{1}{3^2} + \dots + \frac{1}{3^n} = \frac{1}{\frac{2}{3}} - 1 - \frac{1}{3}$$

$$S' = \frac{3}{2} - 1 - \frac{1}{3} = \frac{1}{6}$$

$$\frac{2S}{3} = \frac{2}{3} + \frac{1}{6} - \frac{n+1}{3^{n+1}}$$

$$2S = 2 + \frac{1}{2} - \frac{n+1}{3^n}$$

$$S = 1 + \frac{1}{4} - \frac{n+1}{3^n} \quad \Big/ \lim_{n \to \infty}$$

$$S = \frac{5}{4}$$

c) $$\sum_{n \geq 1} \frac{n}{n^4 + n^2 + 1} = \sum_{n \geq 1} \frac{n}{(n^2 - n + 1)(n^2 + n + 1)} =$$

$$= \frac{1}{2} \sum_{n \geq 1} \left[\frac{1}{n^2 - n + 1} - \frac{1}{n^2 + n + 1} \right] =$$

$$= \frac{1}{2} \left(\frac{1}{1} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{7}} + \dots + \cancel{\frac{1}{n^2 - n + 1}} \right) -$$

$$\frac{1}{2} \left(\cancel{\frac{1}{3}} + \cancel{\frac{1}{7}} + \dots \cancel{+} \frac{1}{n^2 + n + 1} \right) =$$

$$= \frac{1}{2} - \frac{1}{2(n^2+n+1)}$$

$$\lim_{n \to +\infty} \sum_{n=1} \frac{n}{n^4+n^2+1} = \frac{1}{2} - \lim_{n\to+\infty} \underset{\downarrow \atop 0}{\frac{1}{2(n^2+n^2+1)}} = \frac{1}{2}$$

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b) $\sum_{n\geq 2} \frac{1}{(\ln n)^{\ln n}}$ $-$

$$(\ln n)^{\ln n} = n^{\ln(\ln(n))}$$

$$= \sum_{n \geq 2} \frac{1}{n^{\ln(\ln(n))}}$$

for $n \geq e^{e^2}$, $\ln(\ln(e^{e^2})) \geq 2$

$p > 1$, by P-Test we

have that $\displaystyle\sum_{n \geq 2} \frac{1}{n^{\ln(\ln n)}}$ converges

Thus: $\displaystyle\sum_{n \geq 2} \frac{1}{\ln(n)^{\ln(n)}}$ converges

c) $\displaystyle\sum_{n \geq 1} \left(\sqrt[n]{n} - 1 \right) = \lambda_n$

$$\lim_{n \to \infty} \left(\sqrt[n]{n} - 1 \right) = \lim_{n \to \infty} \left(e^{\frac{\ln(n)}{n}} - 1 \right) \geq$$

$$\Rightarrow \lim_{n\to\infty} \frac{\ln(n)}{n} \geq \lim_{n\to\infty} \frac{1}{n} \Rightarrow$$

$\Rightarrow x_n$ is divergent

5\. let N — number of sides at iteration

$a \Rightarrow N = 3 \cdot 4^a$

if L is the length of one side

$L = \left(\frac{1}{3}\right)^a$, a = iteration no.

P - perimeter is $P = N \cdot L =$

$$= 3 \cdot 4^a \cdot \left(\frac{1}{3}\right)^a = 3 \cdot \left(\frac{4}{3}\right)^a$$

$$\lim_{a \to \infty} 3 \cdot \left(\frac{4}{3}\right)^a = \infty$$

area of each triangle added in iteration

$a \Rightarrow A'_a = \frac{\sqrt{3}}{4} \cdot 9^{a}$

original area of side 1 Δ

let A''_a — total area added for all triangles at iteration a

$$A''_a = \frac{3}{4} \cdot \left(\frac{4}{9}\right)^a \cdot \frac{\sqrt{3}}{4}$$

let A_a — total area including the triangles already added

$$A_n = \frac{\sqrt{3}}{4} + \sum_{k=1} A'_k =$$

$$= \frac{\sqrt{3}}{4}\left(1 + \frac{1}{3}\sum_{k=0}^{n-1}\left(\frac{4}{9}\right)^k\right) =$$

$$= \frac{\sqrt{3}}{4}\left(1 + \frac{3}{5}\left(1 - \left(\frac{4}{9}\right)^n\right)\right) = \frac{\sqrt{3}}{4\cdot 5}\left(8 - 3\left(\frac{4}{9}\right)^n\right)$$

$$\lim_{n\to\infty} A_n = \frac{\sqrt{3}}{20}\left(8 - 3\left(\frac{4}{5}\right)^n\right) =$$

$$= \frac{\cancel{8}^{2}}{5}\cdot\frac{\sqrt{3}}{\cancel{4}} = \frac{2\sqrt{3}}{5}$$

4 a) The problem can only be solved analyzing cases for $x<1$, $x=1$, $x>1$

$$S = \sum_{n\geq 1} \frac{x^n}{n^p}, \quad x>0, \quad p \in \mathbb{N}$$

Case I: $x<1 \Rightarrow$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{\cancel{x^{n+1}}\, x}{\cancel{(n+1)^p}} \cdot \frac{\cancel{n^p}}{\cancel{x^n}} < 1$$

from ratio test $\Rightarrow \sum\limits_{n\geq 1} \frac{x^n}{n^p}$ is convergent

Case II: $x=1 \Rightarrow$

$$S = \sum_{n\geq 1} \frac{1}{n^p}, \quad \begin{matrix} p>1, & S \text{ converges} \\ p\leq 1, & S \text{ diverges} \end{matrix}$$

Case III: $x>1$

Case VII $x > 1$

it is easy to prove that S diverges as

$\lim\limits_{n \to \infty} x^n > \lim\limits_{n \to \infty} n^p$, $p \in \mathbb{N}$, hence the

conclusion.