1) $\dot{x} = -3(x-21)$

$x' + 3x = 63$

We will solve using the integration method

$x' + p(t)x = g(t)$

$p(t) = 3$
$g(t) = 63$

$\mu(t) = e^{\int p(t)\,dt} = e^{3t}$

$e^{3t} \cdot x' + e^{3t} \cdot (3x) = e^{3t} \cdot 63 \quad (\int$

$\int \frac{d}{dt}\left(e^{3t} x\right) dt = \int 63 e^{3t}\, dt$

$$e^{3t} x = \cancel{63}^{21} \frac{e^{3t}}{\cancel{3}} + C$$

$$x(t) = 21 + c e^{-3t}$$

2

a)

$$x_p(t) = a\cos t + b\sin t$$

$$x_p(t)' = -a\sin t + b\cos t$$

$$x_p(t)'' = -a\cos t - b\sin t$$

$$-\cancel{a\cos t} - \cancel{b\sin t} - a\sin t + b\cos t + \cancel{a\cos t} + \cancel{b\sin t} = 2\cos t$$

$$\left.\begin{array}{l} b=2 \\ a=0 \end{array}\right| \Rightarrow x_p(t) = 2\sin t$$

b) $$\begin{cases} x'' + x' + x = 2\cos(2t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

From $x'' + x' + x = 2\cos(2t)$

$$r^2 + r + 1 = 0 \to r_{1,2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$x_h(t) = e^{-\frac{1}{2}t}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

Finding a particular solution:

$$x_p(t) = A\cos(2t) + B\sin(2t)$$

$$x_p'(t) = -2A\sin(2t) + 2B\cos(2t)$$

$$x_p''(t) = -4A\cos(2t) - 2B\sin(2t)$$

Substituting back

$$x'' + x' + x = 2\cos 2t$$

$$(-3A + 2B)\cos(2t) + (-4B - 2A)\sin(2t) = 2\cos(2t)$$

$$-3A + 2B = 2$$
$$-4B - 2A = 0$$

$\Rightarrow$ $B = \frac{1}{4}$

$A = -\frac{1}{2}$

$$\Rightarrow x_p(t) = -\frac{1}{2}\cos(2t) + \frac{1}{4}\sin(2t)$$

So:

$$x(t) = x_h(t) + x_p(t)$$

$$x(t) = e^{\,}\left[c_1 \cos\left(\frac{}{2}t\right) + c_2 \sin\left(\frac{}{2}t\right)\right] +$$

$$-\frac{1}{2}\cos(2t) + \frac{1}{4}\sin(2t)$$

We replace $x(t)$, $t=0$

$x'(t) = t=0$ $(\ldots)$

3.

$$x' + 2x = 7t^2$$

Solving the homogeneous equation:

$$x' + p(t)x = q(t)$$

$$\mu(t) = e^{\int p(t)\,dt} = e^{2t}$$

$$e^{2t}x' + e^{2t}x = e^{2t}\,7t^2 \quad / \int$$

$$e^{2t}x = \int e^{2t}\, 7t^2\, dt =$$

$$= 7 \int e^{2t} t^2\, dt =$$

$t^2 \qquad 2t$

$e^{2t} \qquad \frac{e^{2t}}{2}$

$$= \frac{e^{2t}}{2} \cdot t^2 - \int e^{2t}\, 2t =$$

$$= \frac{e^{2t}}{2} \cdot t^2 - \quad \ldots$$

$$= \frac{7}{4} \cdot (2t^2 - 2t + 1)\, e^{2t} + C$$

$\Rightarrow$

$$x(t) = \frac{7}{2}t^2 - \frac{7}{2}t + \frac{7}{4} + ce^{-2t}$$

To be polynomial $x(t)$, $c = 0$

4

a) $$x' + \frac{1}{t}x = e^{-3t}$$

First order dif. equation of the form

$$x' + p(t)\,x = q(t)$$

$$p(t) = \frac{1}{t}, \quad q(t) = e^{-3t}$$

$$\mu(t) = e^{\int p(t)\,dt} \qquad \int \frac{1}{t}\,dt$$

$\mu(t) = e \quad = e \quad = t$

$$t \cdot x' + \cancel{\frac{1}{t}} x \cdot \cancel{t} = t \cdot e^{-3t}$$

$$\frac{d}{dt}(xt) = t \cdot e^{-3t}$$

$$x \cdot t = \int t e^{-3t}$$

$$xt = -\frac{1}{3} t e^{-3t} + \frac{1}{9} e^{-3t} + C$$

$$x(t) = \left(-\frac{1}{3} + \frac{1}{9t}\right) e^{-3t} + \frac{C}{t}$$

b)

$$x' + 3t^2 x = -1$$

$$p(t) = 3t^2, \quad q(t) = -1$$

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int 3t^2\,dt} =$$

$$= e^{\frac{\cancel{3}t^3}{\cancel{3}}} = e^{t^3}$$

$$e^{t^3} x = -\int e^{t^3}\,dt$$

$$x = -e^{-t^3} \int e^{t^3}\,dt + Ce^{-t^3}$$

5 $\dot{x} = x - 2x^3 = f(x)$

$$x - 2x^3 = 0$$

$$x(1-2x^2) = 0 \quad \begin{cases} 0 \\ \pm \frac{\sqrt{2}}{2} \end{cases}$$

$f'(x) = 1 - 6x^2$

$f'\left(-\frac{\sqrt{2}}{2}\right) = -2 < 0$ attractor (stable)

$f'(0) = 1 > 0 \Rightarrow$ repeller (unstable)

$f'\left(\frac{\sqrt{2}}{2}\right) = -2 < 0 \Rightarrow$ attractor (stable)

orbits : $\left(-\infty; -\frac{\sqrt{2}}{2}\right)$, $\left\{-\frac{\sqrt{2}}{2}\right\}$ ...

... $\left(\frac{\sqrt{2}}{2}; +\infty\right)$

$\varphi(t, 0) = 0$ from definition at eq. point $x = 0$

$0,2 \in \left(0, \frac{\sqrt{2}}{2}\right) \Rightarrow$

$\varphi(t, 0,2)$ is strictly increasing

$$\lim_{t \to \infty} \varphi(t, 0,2) = \frac{1}{\sqrt{2}}$$

$$\lim_{t \to -\infty} \varphi(t, 0,2) = 0$$

$20 \in \left(\frac{\sqrt{2}}{2} ; \infty\right) \Rightarrow$ strictly decreasing

$$\lim_{t \to \infty} \varphi(t, 20) = \frac{\sqrt{2}}{2}$$

(6) $\dot{x} = x + y$

$\dot{y} = -2x + 4y$

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) + 2$$

$$= 4 - 5\lambda + \lambda^2 + 2$$

$$= \lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda_1 = 2$$

$$\lambda_2 = 3$$

$0 < \lambda_1 < \lambda_2 \rightarrow$ Node

[illegible] Repeller

(7)

$$\dot{x} = -7y$$
$$\dot{y} = 9x \quad \Rightarrow \quad A = \begin{pmatrix} 0 & -7 \\ 9 & 0 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\lambda^2 + 63 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm\, 3\sqrt{7}\, i$$

$$\frac{dx}{dt} = -7y$$

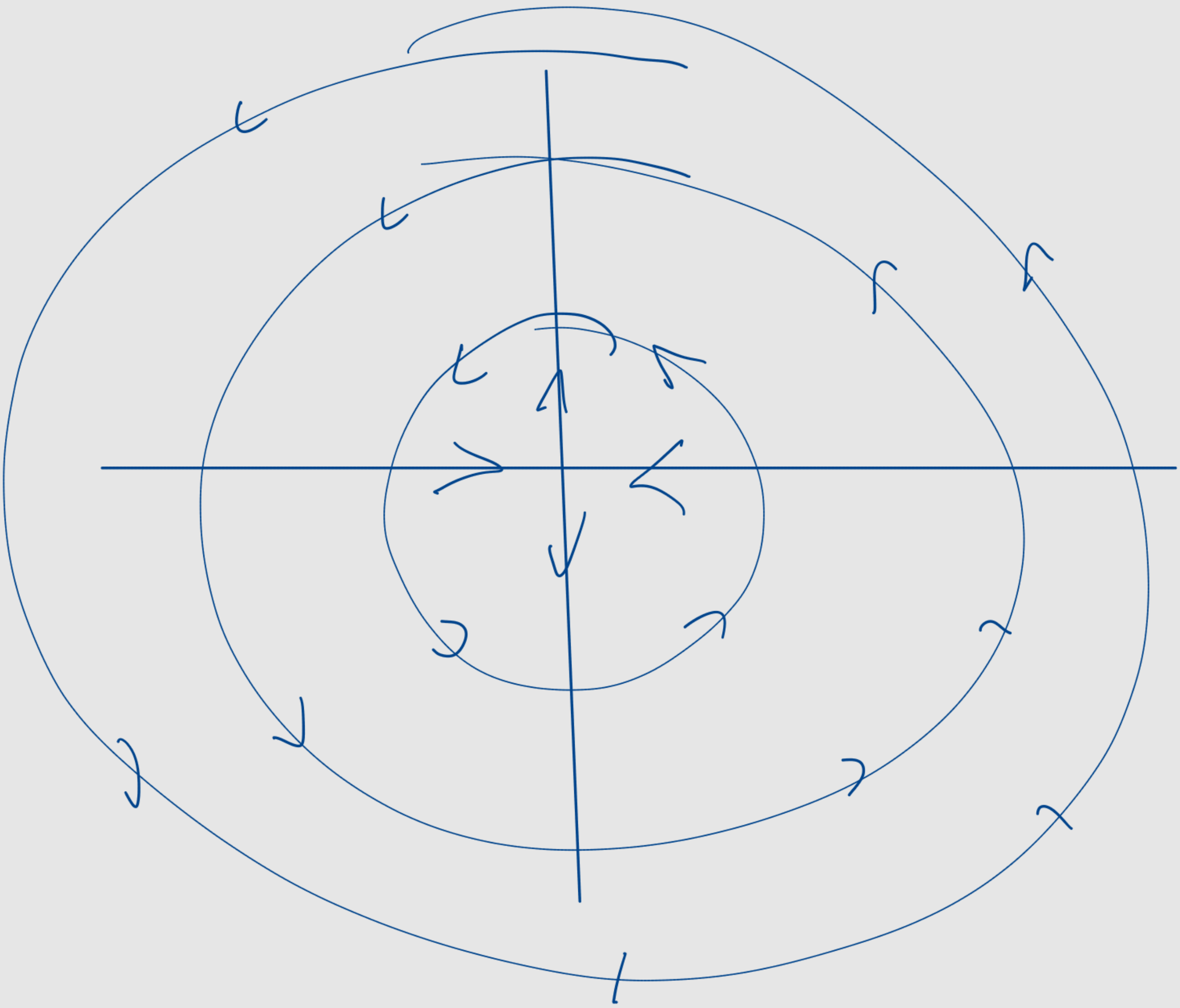
$$\frac{dy}{dt} = 9x \qquad \Leftrightarrow \qquad \frac{dx}{dy} = \frac{-7y}{9x}$$

$$9x\,dx = -7y\,dy \quad \Big|\int$$

$$9\frac{x^2}{2}\,dx + C = -7\frac{y^2}{2}$$

$$9x^2 + 7y^2 = c$$

This represents ellipses. of center C



Analysis: $\begin{cases} \dot{x} = -7y \\ \dot{y} = 9x \end{cases}$

if $x > 0$, $x' < 0$ $\leftarrow$

$x < 0$, $x' > 0$ $\rightarrow$

$\Rightarrow x > 0$
$y > 0$

if $y > 0$, $y' > 0$ $\uparrow$

$y < 0$, $y' < 0$ $\downarrow$

Cadranul I $\leftarrow\uparrow$ $\Rightarrow \nwarrow$

8

$$\begin{cases} \dot{x} = -x + xy \\ \dot{y} = -2y + 3y^2 \end{cases}$$

Finding equilibria:

$$\begin{cases} -x + xy = 0 \\ -2y + 3y^2 = 0 \end{cases} \Rightarrow$$

$$\begin{cases} y(-2+3y) = 0 \Rightarrow y = 0,\ y = \frac{2}{3} \\ x(-1+y) = 0 \Rightarrow x = 0,\ y = 1 \end{cases}$$

$$S(x,y) = \Big\{ \underset{\eta_1^*}{(0,0)},\ \underset{\eta_2^*}{\left(0,\tfrac{2}{3}\right)},\ \underset{\eta_3^*}{(1,0)}\ \underset{\eta_4^*}{\left(1,\tfrac{2}{3}\right)} \Big\}$$

$$f(x,y) = \begin{pmatrix} -x + xy \\ -2y + 3y^2 \end{pmatrix}$$

We find the Jacobian

$$J(f(x,y)) = \begin{pmatrix} f_{1_x}(x,y) & f_{1_y}(x,y) \\ f_{2_x}(x,y) & f_{2_y}(x,y) \end{pmatrix} =$$

$$= \begin{pmatrix} -1+y & x \\ 0 & -2+6y \end{pmatrix}$$

For $q^* = (0,0)$

$$J(f(0,0)) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\lambda_{1,2} = -1, -2 < 0 \Rightarrow$$

NODE, stable, atractor

For $x_2^* = (0, \frac{2}{3})$

$$J f(x_2^*) = \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{pmatrix}$$

$$\lambda_1 < 0 < \lambda_2$$

SADDLE, unstable,

For $x_3^* = (1, 0)$

$$J f(x_3^*) = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$$

$\lambda_1 = -2 < 0$

$\lambda_2 = -1 < 0$

NODE, stable, attractor

For $x^*_4 = \left(1, \frac{2}{3}\right)$

$$J_f(x^*_4) = \begin{pmatrix} -\frac{1}{3} & 1 \\ 0 & 2 \end{pmatrix}$$

$\lambda_1 = -\frac{1}{3} < 0$

$\lambda_2 = 2 > 0$

SADDLE, unstable,