Seminar Test 1.

MJ

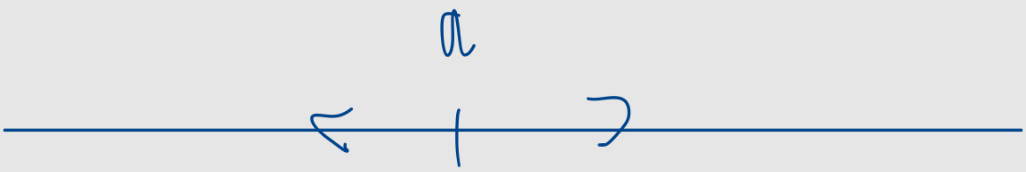
Exercise 1

$\dot{x} = ax - 1 = f(x)$, $f'(x) = a$

$f(x) = 0 \iff ax = 1$

$x = \frac{1}{a}$

if $a > 0$, $f'(\frac{1}{a}) > 0$, repeller (unstable)



if $a < 0$, $f'(\frac{1}{a}) < 0$, attractor (stable)

ii) $a = 0$, $x = -1 \neq 0 \Rightarrow$ no eq. points

Finding the general solution:

$$x' - ax = -1$$

$$x' + p(t)x = q(t) \quad , \quad p(t) = -a$$
$$q(t) = -1$$

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int -a\,dt} = e^{-at}$$

$$e^{-at} x' - e^{-at} \cdot ax = -e^{-at}$$

$$\frac{d}{dt}\left(e^{-at} x\right) = -e^{-at} \quad / \int$$

$$e^{-at}x = -\int e^{-at}$$

$$e^{-at}x = \frac{-e^{-at}}{a} + C \quad \Big/ : e^{-at}$$

$$x(t) = \frac{-1}{a} + C \cdot e^{at}$$

Applying the initial condition:

$$x(0) = x_0$$

$$x(0) = \frac{-1}{a} + C = x_0$$

$$C = x_0 + \frac{1}{a}$$

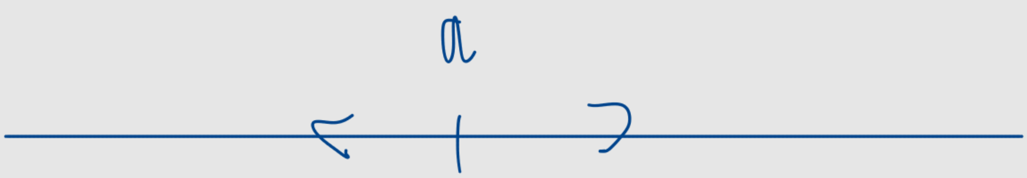
$$x(t) = \frac{-1}{a} + \left(x_0 + \frac{1}{a}\right) e^{at}$$

$$\varphi\left(t, \frac{1}{a}\right) = \frac{-1}{a} + \frac{2}{a} e^{at}$$

$$\varphi(t, 1) = \frac{-1}{a} + \left(1 + \frac{1}{a}\right) e^{at}$$

According to prev analysis:

if $a > 0$, $f'\left(\frac{1}{a}\right) > 0$, repeller (unstable)



- if $1 > a$, increasing

$\mathrm{Im}(\varphi(\cdot,1))$ is $(a, +\infty)$

$$\lim_{t\to\infty} \varphi(\cdot,1) = \infty$$

$$\lim_{t\to-\infty} \varphi(\cdot,1) = a$$

- if $1 < a$, decreasing

$\mathrm{Im}(\varphi(\cdot,1))$ is $(-\infty, a)$

$$\lim_{t\to\infty} \varphi(\cdot,1) = -\infty$$

$$\lim_{t\to-\infty} \varphi(\cdot,1) = a$$

if $a < 0$, $f'\left(\frac{1}{a}\right) < 0$, attractor (stable)

- if $1 > a$, decreasing

$\mathrm{Im}(\varphi(\cdot,1))$ is $(a,+\infty)$

$\lim_{t\to\infty} \varphi(\cdot,1) = a$

$\lim_{t\to-\infty} \varphi(\cdot,1) = \infty$

- if $1<a$, increasing

$\mathrm{Im}(\varphi(\cdot,1))$ is $(-\infty, a)$

$\lim_{t\to\infty} \varphi(\cdot,1) = a$

$\lim_{t\to-\infty} \varphi(\cdot,1) = -\infty$

if $a=0$, $\dot{x} = -1 \neq 0 \Rightarrow$ no eq. points

$\boxed{\text{Exercise 2}}$

EASIER SOLUTION at the end of ex!!

a) $\begin{cases} \dot{x} = 4y \\ \dot{y} = -x \end{cases} \Rightarrow A = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix}$

$\det(A - \lambda I) = 0$

$\begin{vmatrix} -\lambda & 4 \end{vmatrix}$

$$\begin{vmatrix} \cdot & \cdot \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 = -4$$

$$\lambda = \pm 2i$$

Finding the eigenvectors:

for $\lambda_1 = 2i$

$$\begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2i \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{cases} 4v_2 = 2i v_1 \\ -v_1 = 2i v_2 \end{cases} \Rightarrow \begin{cases} 2v_2 = i v_1 \\ v_1 = -2i v_2 \end{cases} \Rightarrow$$

for $v_1 = 1 \Rightarrow \quad v_2 = \frac{-i}{2} \Rightarrow \begin{pmatrix} 1 \\ -\frac{i}{2} \end{pmatrix}$

$\downarrow$

eigenvector

for $\lambda_2 = -2i$

$$\begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -2i \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{cases} 4v_2 = -2i v_1 \\ -v_1 = -2i v_2 \end{cases}$$

for $v_1 = 1 \Rightarrow v_2 = \frac{i}{2} \Rightarrow \begin{pmatrix} 1 \\ \frac{i}{2} \end{pmatrix}$

$$\theta(t) = c_1 e^{2it} \begin{pmatrix} 1 \\ -\frac{i}{2} \end{pmatrix} + c_2 e^{-2it} \begin{pmatrix} 1 \\ \frac{i}{2} \end{pmatrix}$$

From euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$

$$\varphi(t) = c_1 \begin{pmatrix} \cos(2t) \\ -\frac{1}{2}\sin(2t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(2t) \\ \frac{1}{2}\cos(2t) \end{pmatrix}$$

Finding the flow:

$x(0) = x_0$

$y(0) = y_0$

$$\varphi\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 \\ \frac{1}{2} c_2 \end{pmatrix}$$

$C_1 = x_0$

$C_2 = 2y_0$

$$\Phi\left(t, (x_0, y_0)\right) = \begin{pmatrix} x_0 \cos(2t) + 2y_0 \sin(2t) \\ -\frac{1}{2} x_0 \sin(2t) + y_0 \cos(2t) \end{pmatrix}$$

b) $\eta_1^{\cancel{*}} = (0, 0)$

$$f(x, y) = \begin{pmatrix} 4y \\ -x \end{pmatrix}$$

We find the Jacobian

$$J(f(x,y)) = \begin{pmatrix} f_{1_x}(x,y) & f_{1_y}(x,y) \\ f_{2_x}(x,y) & f_{2_y}(x,y) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix}$$

$$J(f(0,0)) = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix}$$

$\lambda_{1,2} = \pm 2i$, center , stable

c) $\dfrac{dx}{dt} = 4y$

$$\begin{cases} \dfrac{dx}{dt} \\[2ex] \dfrac{dy}{dt} = -x \end{cases} \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \frac{dx}{dy} = \frac{4y}{-x} \quad \Leftrightarrow$$

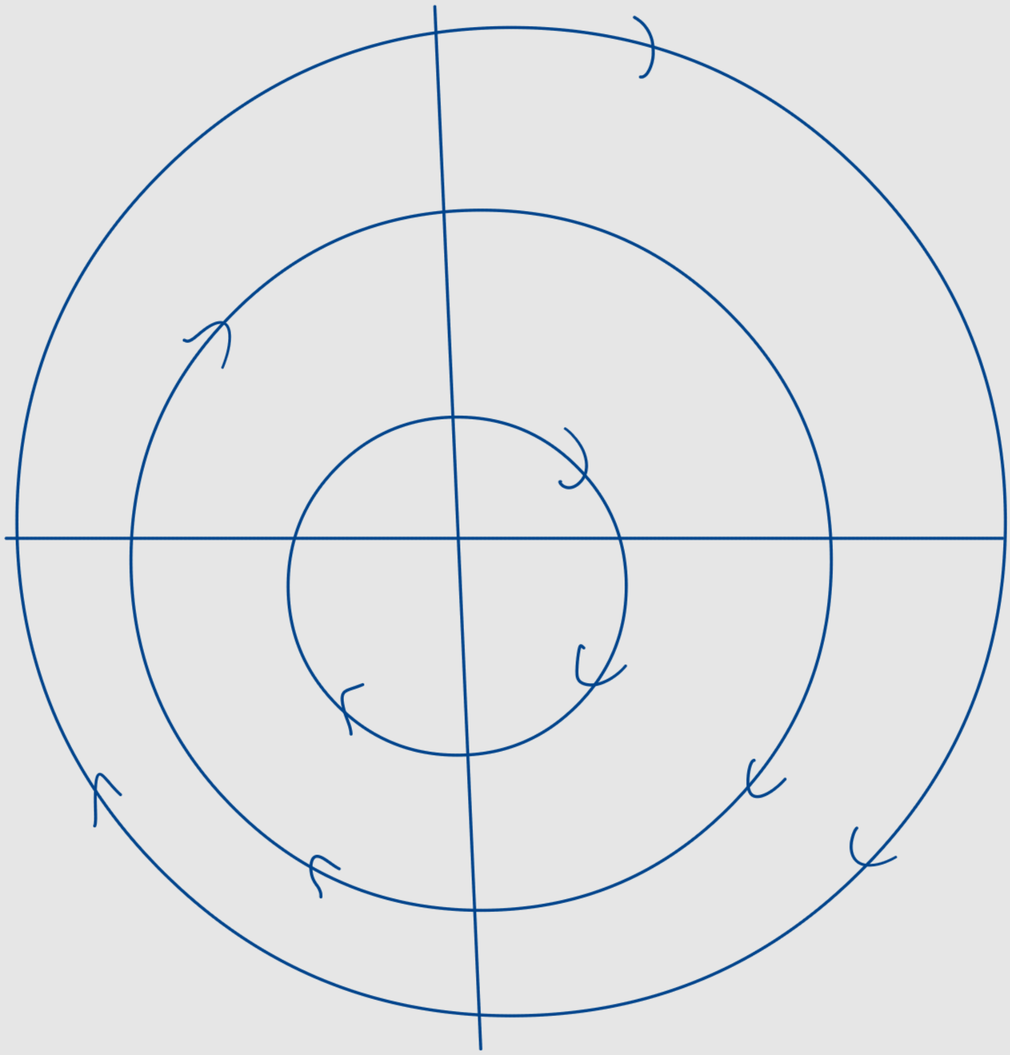
$$4y\,dy = -x\,dx \;\Big/ \int$$

$$4\frac{y^2}{2} + C_1 = -\frac{x^2}{2}$$

$$4y^2 + x^2 = C_2$$

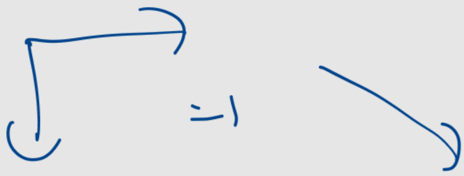
Orbits are ellipses

d)



Cadran I : $x > 0$, $x' > 0$ $\longrightarrow$

$y > 0$, $y' < 0$ $\downarrow$



a) $\begin{cases} x' = 4y \\ y' = -x \end{cases}$

$\begin{cases} x'' = 4y' \\ y' = -x \end{cases} \Rightarrow x'' = -4x$

$$x'' + 4x = 0$$

$$r^2 + 4 = 0$$

$$r = \pm 2i$$

$$x(t) = c_1 \cos(2t) + c_2 \sin(2t)$$

Finding $y(t)$:

$$\frac{dx}{dt} = -2c_1 \sin(2t) + 2c_2 \cos(2t) = 4y$$

$$y(t) = \frac{-c_1}{2}\sin(2t) + \frac{c_2}{2}\cos(2t)$$

b, c, d the same solution.

$$\boxed{\text{Exercise } 3}$$

a) $x_p(t) = at^2 e^t$

$x'_p(t) = at(t+2)e^t$

$x''_p(t) = a(t^2+4t+2)e^t$

$x'' - 2x' + x = e^t$

$$a(t^2+4t+2)\cancel{e^t} - 2at(t+2)\cancel{e^t} + at^2\cancel{e^t} = \cancel{e^t}$$

$$\cancel{at^2} + \cancel{4at} + 2a - \cancel{2at^2} - \cancel{4at} + \cancel{at^2} = 1$$

$$a = \frac{1}{2}$$

d)

$$\begin{cases} x'' - 2x' + x = 0 \\ x(0) = 5 \\ x'(0) = 0 \end{cases}$$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$r = 1 \quad = 1$$

$$x(t) = (c_1 + c_2 t)\, e^t$$

$$x(0) = 5 \Rightarrow c_1 = 5$$

$$x'(0) = 0 \Rightarrow \quad c_2 + c_1 = 0$$

$$c_2 = -5$$

$$\Rightarrow \quad x(t) = (5 - 5t)\, e^{t}$$