Seminar 8: 1, 2, 4, 5, 11, 3, 12, 13, 14, 16

Chapter 5

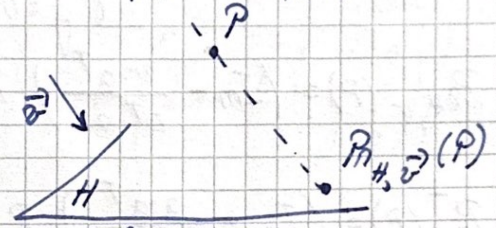
$\varphi : \mathbb{A}^n \to \mathbb{A}^m$; $n, m \in \mathbb{N}$

affine morphism if: $\varphi(\overrightarrow{AB}) = \overrightarrow{\varphi(A)\varphi(B)}$

$\forall \varphi : \mathbb{A}^n \to \mathbb{A}^m$, $\varphi(P) = A \cdot P + b$; $A \in \mathcal{M}_{m,n}(\mathbb{R})$, $b \in \mathbb{R}^m$

$(\operatorname{lin} \varphi)(P) = A \cdot P$

→ Projection on a hyperplane $H : a_1 x_1 + \ldots + a_n x_n = 0$ parallel to $\vec{v}(v_1, \ldots, v_n)$



$$Pr_{H,v}(P) = \left(I_n - \frac{v \cdot a^T}{v^T \cdot a}\right) \cdot P - \frac{a_{n+1}}{v^T \cdot a} \cdot v = \left(I_n - \frac{v \otimes a}{\langle a, a \rangle}\right) \cdot P - \frac{a_{n+1}}{\langle v, a \rangle} \cdot v$$

$v = (x_1, \ldots, x_n)^T$; $w = (y_1, \ldots, y_n)^T$

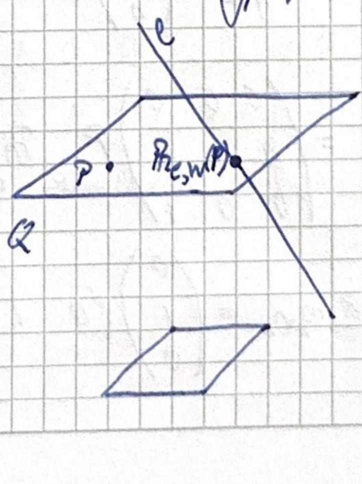
$$v \otimes w = v \cdot w^T = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot (y_1 \ldots y_n) = \begin{pmatrix} x_1 y_1 & \ldots & x_1 y_n \\ \vdots & & \vdots \\ x_n y_1 & \ldots & x_n y_n \end{pmatrix}$$

$$\langle v, w \rangle = v^T \cdot w = (x_1 \ldots x_n) \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + \ldots + x_n y_n$$

→ Projection on a line parallel to a hyperplane

$W : a_1 x_1 + \ldots + a_n x_n + a_{n+1} = 0$

$Q(q_1, \ldots, q_n) \in \ell$, $v \in D(\ell)$



$$Pr_{\ell,W} = \frac{v \cdot a^T}{v^T \cdot a} \cdot P + \left(I_n - \frac{v \cdot a^T}{v^T \cdot a}\right) \cdot Q$$

5.1. Consider an orthonormal coordinate system $\mathcal{K}$ of $\mathbb{E}^n$ where $n = 2, 3$. Deduce the matrices of the orthogonal projections on the coordinate axes and the coordinate hyperplanes of $\mathcal{K}$.

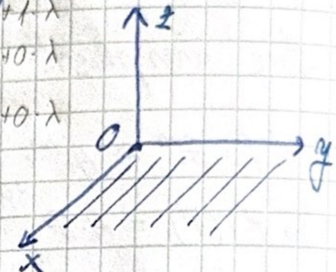
$n = 3$

$(xOy): z = 0$

$(yOz): x = 0$

$(zOx): y = 0$

$Ox: \begin{cases} y = 0 \\ z = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 + 1 \cdot \lambda \\ y = 0 + 0 \cdot \lambda \\ z = 0 + 0 \cdot \lambda \end{cases}$



$Oy: \begin{cases} z = 0 \\ x = 0 \end{cases}$

$Oz: \begin{cases} x = 0 \\ y = 0 \end{cases}$

$$Pr_{H,v}(P) = \left(I_n - \frac{v \cdot a^T}{v^T \cdot a}\right) \cdot P - \frac{a_{n+1}}{v^T \cdot a} \cdot v$$

$$Pr_H^{\perp}(P) = \left(I_n - \frac{a \cdot a^T}{a^T \cdot a}\right) \cdot P - \frac{a_{n+1}}{a^T \cdot a} \cdot a$$

$n_{xOy}(0, 0, 1)$; $n_{yOz}(1, 0, 0)$; $n_{zOx}(0, 1, 0)$

$$n_{xOy} \otimes n_{xOy} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (0\ 0\ 1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\langle n_{xOy}, n_{xOy} \rangle = \langle (0,0,1), (0,0,1) \rangle = 1$$

$$Pr_{xOy}^{\perp}(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot P$$

$$n_{yOz} \otimes n_{yOz} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1\ 0\ 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Pr_{yOz}^{\perp} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot P; \quad Pr_{yOz}^{\perp} = I_3 - n_{yOz} \otimes n_{yOz}$$

$$n_{zOx} \otimes n_{zOx} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (0\ 1\ 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Pr_{zOx}^{\perp}(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} P$$

$$\begin{cases} Pr_{e,H}(P) = \dfrac{v \cdot a^T}{v^T \cdot a} \cdot P + \left(I_m - \dfrac{v \cdot a^T}{v^T \cdot a}\right) \cdot Q \\ \text{orthogonal projection} : v = a \\ Pr_e^{\perp}(P) = \dfrac{a \cdot a^T}{a^T \cdot a} \cdot P + \left(I_m - \dfrac{a \cdot a^T}{a^T \cdot a}\right) \cdot Q \end{cases}$$

$\rightarrow$ $\vec{v}(1, 0, 0) = \vec{a}$ $\left(Ox : \begin{cases} y = 0 \\ z = 0 \end{cases} \right)$

$$Pr_e^{\perp}(P) = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \frac{1}{1} \right] \cdot P + \left[I_m - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] Q$$

$Q(0, 0, 0) \in \ell$

$$Pr_e^{\perp}(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P$$

$$Q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\rightarrow$ $Oy : \vec{a}(0, 1, 0)$

$$Pr_{Oy}^{\perp}(P) = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot 1 \right] \cdot P + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot P$$

$\rightarrow$ $Oz : \vec{a}(0, 0, 1)$

$$Pr_{Oz}^{\perp}(P) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot P + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot P$$

5.4. Determine the orthogonal projection of the point $P(6,-5,5)$ in the plane $\Pi: 2x-3y+z-4=0$ by determing the matrix form of the reflection.

$$Pr_{\Pi}^{\perp}(P) = \left(I_n - \frac{a \otimes a}{\langle a, a\rangle}\right)\cdot P - \frac{a_{n+1}}{\langle a, a\rangle}\cdot a$$

$a = (2,-3,1)$

$$a \otimes a = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}\begin{pmatrix} 2 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -6 & 2 \\ -6 & 9 & -3 \\ 2 & -3 & 1 \end{pmatrix}$$

$\langle a, a\rangle = 4+9+1 = 14$

$$Pr_{\Pi}^{\perp}(P) = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{14}\cdot\begin{pmatrix} 4 & -6 & 2 \\ -6 & 9 & -3 \\ 2 & -3 & 1 \end{pmatrix}\right]\begin{pmatrix} 6 \\ -5 \\ 5 \end{pmatrix} - \frac{-4}{14}\cdot\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1-\frac{4}{14} & \frac{6}{14} & -\frac{1}{7} \\ +\frac{6}{14} & 1-\frac{9}{14} & \frac{3}{14} \\ -\frac{1}{7} & \frac{3}{14} & 1-\frac{1}{14} \end{pmatrix}\cdot\begin{pmatrix} 6 \\ -5 \\ 5 \end{pmatrix} + \frac{2}{7}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \frac{1}{14}\left[\begin{pmatrix} 10 & 6 & -2 \\ 6 & 5 & 3 \\ -2 & 3 & 13 \end{pmatrix}\right.$$

$$\left.\cdot\begin{pmatrix} 6 \\ -5 \\ 5 \end{pmatrix} + \begin{pmatrix} 8 \\ -12 \\ 4 \end{pmatrix}\right] = \frac{1}{14}\left[\begin{pmatrix} 20 \\ 26 \\ 28 \end{pmatrix} + \begin{pmatrix} 8 \\ -12 \\ 4 \end{pmatrix}\right] = \frac{1}{14}\cdot\begin{pmatrix} 28 \\ 14 \\ 32 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ \frac{16}{7} \end{pmatrix}$$

$$\begin{cases} Pr_{H,v}(P) = \left(I_n - \frac{v \otimes a}{\langle v, a\rangle}\right)\cdot P - \frac{a_{n+1}}{\langle v, a\rangle}\cdot a \\ Ref_{H,v}(P) = 2\cdot Pr_{H,v}(P) - \underbrace{P}_{I_n\cdot P} = \left(I_n - 2\cdot\frac{v \otimes a}{\langle v, a\rangle}\right)\cdot P - \frac{2a_{n+1}}{\langle v, a\rangle}\cdot a \end{cases}$$

5.12. H-hyperplane, $v \in V^n$, $v \nparallel H$. Use ~~that~~ the matrix forms to show that:

a) $Pr_{H,v} \circ Pr_{H,v} = Pr_{H,v}$

b) $Ref_{H,v} \circ Ref_{H,v} = Id$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

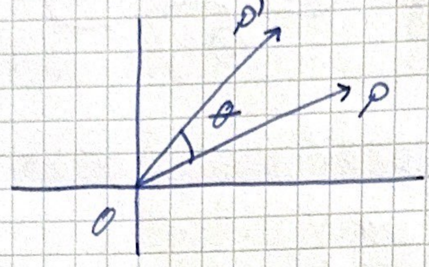
a) We have to show that $\forall p: Pr_{H,v}\left(Pr_{H,v}(p)\right) = Pr_{H,v}(p)$

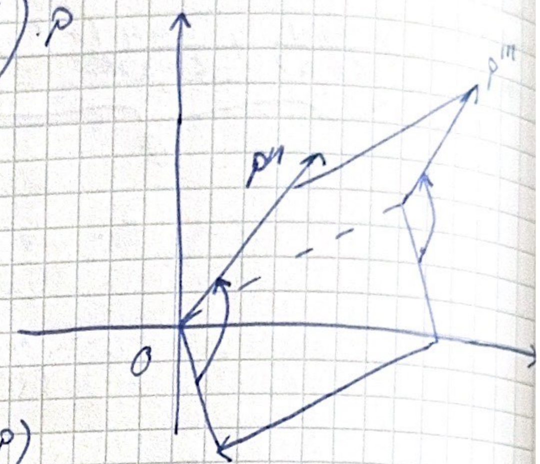
$$\left(I_m - \frac{v \otimes a}{\langle v, a\rangle}\right)\left(\left(I_m - \frac{v \otimes a}{\langle v, a\rangle}\right)\cdot p - \frac{a_{m+1}}{\langle v, a\rangle}\cdot a\right) - \frac{a_{m+1}}{\langle v, a\rangle}\, a =$$

$$= \underbrace{-2\,\frac{a_{m+1}}{\langle v, a\rangle} + \frac{a_{m+1}\,(v \otimes a)\cdot a}{\langle v, a\rangle^2}}_{1} + \underbrace{\left(I_m - \frac{v \otimes a}{\langle v, a\rangle}\right)^2 p}_{2}$$

5.14. $A(1,1)$, $B(4,1)$, $C(2,3)$. Determine the image of ABC under $Rot_{C,\frac{\pi}{2}}$ followed by an orthogonal reflection relative to

$$Rot_\theta(P) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot P$$





$$Rot_{C,\theta}(P) = \overset{P'}{T_{\vec{OC}}} \circ Rot_\theta \circ T_{\vec{CO}}(P)$$

$\varphi : \mathbb{A}^n \to \mathbb{A}^n$, $\varphi(P) = A \cdot P + b$

$\vec{P} \to$ projective coordinates

$\vec{P} = (P, 1)$

$$\bar{\varphi}(\vec{P}) = \left(\begin{array}{c|c} A & b \\ \hline 0 & 1 \end{array}\right) \cdot \vec{P} \to \text{projective version of } \varphi$$

$C(x_C, y_C)$

$$T_{\vec{CO}}(P) = \left(\begin{array}{cc|c} 1 & 0 & -x_C \\ 0 & 1 & -y_C \\ \hline 0 & 0 & 1 \end{array}\right) \cdot P$$

$$Rot_\theta(P) = \left(\begin{array}{cc|c} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \cdot P$$

$$T_{\vec{OC}}(P) = \left(\begin{array}{cc|c} 1 & 0 & x_C \\ 0 & 1 & y_C \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$\Rightarrow Rot_{c,\theta}(P) = \begin{pmatrix} 1 & 0 & x_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{pmatrix} \cdot \left(\begin{matrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & \quad 0 \end{matrix} & 1 \end{matrix} \right) \begin{pmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ 0 & 0 & 1 \end{pmatrix} \cdot P =$$

$$Rot_{c,\frac{\pi}{2}}(P) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -2 \\ & & -3 \\ & & \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} =$$

$= \_ \ \_ \ \_$

$\Rightarrow Rot_{c,\frac{\pi}{2}}(A) = (4, 2) + \ldots$

Seminar 9: Chapter 5

16, 18, 19, 20.3, 22, 23, 24

$f: \mathbb{E}^m \to \mathbb{E}^m$, $f(P) = A \cdot P + b$

$(\text{lin } f)(P) = A \cdot P$ – Affine morphism

f isometry $\Leftrightarrow \forall A, B \in \mathbb{E}^m : dist(A, B) = dist(f(A), f(B))$

$\Leftrightarrow f(P) = A \cdot P + b$ and $A \in O(m) = \{X \in \mathcal{M}_m(\mathbb{R}) : X^{-1} = X^T\}$

$\Leftrightarrow f(P) = A \cdot P + b$ and $AA^T = I_m$

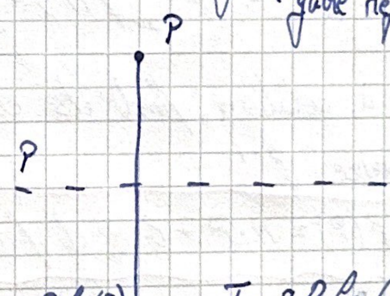
Classification:

f isometry $<$ direct : $\det f = 1$; indirect : $\det f = -1$

$\boxed{\text{For } m = 2:}$

direct isometry $<$
- translative, $Fix(f) = \emptyset$
- rotation, $Fix(f) = \{R\}$
- identity, $Fix(f) = \mathbb{E}^m$

indirect isometry $<$
- reflection w.r.t. to a line, $Fix(f) = e$
- glide reflection



$f: \mathbb{E}^m \to \mathbb{E}^m$, $Fix(f) = \{P \in \mathbb{E}^m \mid f(P) = P\}$

$Ref_e(P) \to T_{\vec{v}} \circ Ref_e(P)$, $\vec{v} \in D(e)$

5.16 Let F be the ~~rotated~~ isometry obtained by applying $Rot_{-\frac{\pi}{3}}$ after a translation $T_{(-2,5)}$. Determine the inverse transformation F^{-1}.

$F = Rot_{-\frac{\pi}{3}} \circ T_{(-2,5)} : \mathbb{E}^2 \to \mathbb{E}^2$.

$$\left(\begin{array}{cc|c} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \cdot \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \\ \hline 0 & 0 & 1 \end{array}\right) \cdot \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & -1 + \frac{5\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \sqrt{3} + \frac{5}{2} \\ 0 & 0 & 1 \end{pmatrix}}_{M} \cdot$$

$$\cdot \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \Rightarrow$$

$$M = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} & -2+5\sqrt{3} \\ -\sqrt{3} & 1 & 2\sqrt{3}+5 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\left[\widehat{Rot_{-\frac{\pi}{3}}}\right]^{-1} = \left[\widehat{Rot_{\frac{\pi}{3}}}\right] \Leftrightarrow \left[T_{(-2,5)}\right]^{-1} = \left[T_{(2,-5)}\right] \Big\} \Rightarrow$$

$$\Rightarrow F^{-1} = \left(Rot_{-\frac{\pi}{3}} \circ T_{(-2,5)}\right)^{-1} \Rightarrow$$

$$\Rightarrow F^{-1} = T_{(2,-5)} \circ Rot_{\frac{\pi}{3}} ; \left[\widehat{F^{-1}}\right] = \left(\begin{array}{cc|c} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -5 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 2 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -5 \\ 0 & 0 & 1 \end{pmatrix}$$

5.18 $f: \mathbb{E}^2 \to \mathbb{E}^2$, $f(P) = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \cdot P + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Show that f is a rotation, find its centre, its angle of rotation and the inverse f^{-1}!

$$\boxed{f(P) = A \cdot P + b \text{ ; if } f \text{ rotation} \Rightarrow Tr(A) = 2\cos\theta}$$

$$M = [\lim f] = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} ; M \cdot M^T = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \cdot \frac{1}{\sqrt{13}} =$$

$$= \frac{1}{13}\begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix} = I_2 \Rightarrow M \in O(2) \Rightarrow \text{isometry}$$

$$\det(M) = \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} \cdot \left(\frac{1}{\sqrt{13}}\right)^2 = (4+9) \cdot \frac{1}{13} = 1 \Rightarrow M \in SO(2) \Rightarrow$$

$\Rightarrow$ direct isometry

$P(x, y)$, $f(P) = P$ (to find the center)

$$\frac{1}{\sqrt{13}}\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} F$$

$$\frac{1}{\sqrt{13}}\begin{pmatrix} 2x - 3y + 1 \\ 3x + 2y - 2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} \sqrt{13}x = 2x - 3y + 1\sqrt{13} \\ \sqrt{13}y = 3x + 2y - 2\sqrt{13} \end{cases} \Rightarrow$$

$\sqrt{13}$

$$\Rightarrow \begin{cases} y = \dfrac{x(2-\sqrt{13})+\sqrt{13}}{3} \\ \sqrt{13}\cdot\dfrac{x(2-\sqrt{13})+\sqrt{13}}{3} = 3x + 2\cdot\dfrac{x(2-\sqrt{13})+\sqrt{13}}{3} - 2 \; \Big/ \cdot 3 \end{cases}$$

$$x[\sqrt{13}(2-\sqrt{13}) - 9 - 2(2-\sqrt{13})] = -\sqrt{13} + 2 - 6$$

$$x[2\sqrt{13} - 13 - 9 - 4 + 2\sqrt{13}] = -\sqrt{13} - 4$$

$$x[4\sqrt{13} - 26] = -\sqrt{13} - 4 \Rightarrow x = \frac{-\sqrt{13}-4}{4\sqrt{13}-26} \Rightarrow y = \dots$$

$$Tr_{\ell} = \frac{4}{\sqrt{13}} = 2\cos\theta \Rightarrow \cos\theta = \frac{2}{\sqrt{13}} \qquad \Rightarrow \text{rot.}$$

$\boxed{\text{For } n = 3:}$

→ direct: → rotation around an axis $Rot_{\ell,\theta}$

$Fix(f) = \ell$

→ identity, $Fix(f) = \mathbb{E}^3$

→ translation, $Fix(\ell) = \emptyset$

→ glide rotation

$T_{\vec{v}} \circ Rot_{\ell,\theta}$, where $\vec{v} \in \mathcal{D}(\ell)$

indirect:
- reflection w.r. to a plane π, $Fix(f) = \pi$
- glide reflection. $T_{\vec{v}} \circ Ref_{\pi}$, where $\vec{v} \in \mathcal{D}(\pi)$
- rotation reflection $Rot_{\ell,\theta} \circ Ref_{\pi}$, where $\ell \perp \pi$

If $f: \mathbb{E}^3 \to \mathbb{E}^3$ rotation, then $Tr([\text{lin} f]) = 2\cos\theta + 1$

5.19. EXAM Verify that $A = \frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix} \in SO(3)$. Determine the axis and the angle of rotation.

$$A \cdot A^T = \frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & -2 & -2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix} \cdot \frac{1}{3} = \frac{1}{9}\begin{pmatrix} 1+4+4 & 2-4+2 & 2+2-4 \\ 2-4+2 & 4+4+1 & 4-2-2 \\ 2+2-4 & 4-2-2 & 4+1+4 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = I_3 \Rightarrow A \in SO(3) \Rightarrow \text{isomorphism}; \det(A) = \left(\frac{1}{3}\right)^3 \cdot \begin{vmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{vmatrix} =$$

$\Rightarrow \det(A) = 1 \Rightarrow A \in SO(3) \Rightarrow$ direct isomorphism

$$F^{\text{not } A} = \frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix}$$

$\det(A) = \frac{1}{27}(4+4+4+8-1+8) = \frac{1}{\cancel{27}} \cdot \cancel{27} = \underline{1} \Rightarrow$

$\Rightarrow A \in SO(3) \Rightarrow$ direct isomorphism

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3} \cdot \begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -x+2y-2z \\ -2x-2y-z \\ -2x+y+2z \end{pmatrix}$$

$$\Rightarrow \begin{cases} 3x = -x+2y-2z \\ 3y = -2x-2y-z \\ 3z = -2x+y+2z \end{cases} \Rightarrow \begin{cases} 2x = y-z \\ 5y = -2x-z \\ z = -2x+y \end{cases} \Rightarrow 5y = -y+z-z \Rightarrow$$

$\Rightarrow y = 0$

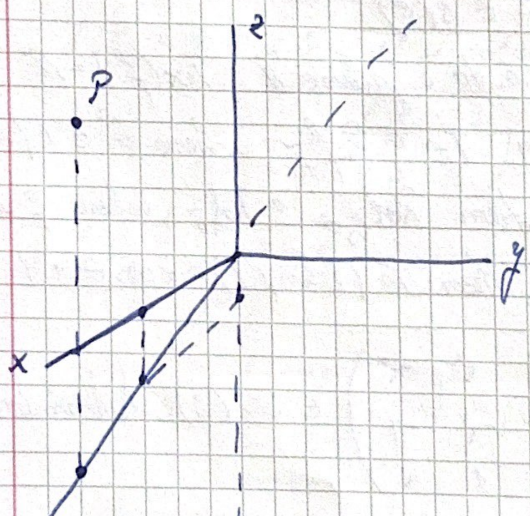
$$\Rightarrow \begin{cases} 2x = -z \Rightarrow x = -\frac{1}{2}z \\ z = -2x \\ y = 0 \end{cases}$$

$\Rightarrow Fix(A) = \{(\alpha, 0, -2\alpha) \mid \alpha \in \mathbb{R}\}$

$$\Rightarrow \ell : \begin{cases} x = \alpha \\ y = 0 \\ z = -2\alpha \end{cases}$$ is the axis of rotation

$Tr(A) = 2\cos\theta + 1 \Rightarrow$

$\Rightarrow -1 = 2\cos\theta + 1 \Rightarrow -2 = 2\cos\theta \Rightarrow \cos\theta = -1 \Rightarrow \theta = -\pi$



Euler Rodriguez:

$Rot_{\ell,\theta}(P) = \cos\theta \cdot P + \sin\theta \cdot (\vec{v} \times P) + \cos\theta \cdot \langle v, p \rangle \cdot v$,

where $\vec{v} \in D(\ell)$, $|\vec{v}| = 1$

5.22 Using the Euler Rodrigues formula write the matrix form of a rotation around the axis $\mathbb{R}\vec{v}$, where $\vec{v} = (1, 1, 0)$. Use this matrix form to give a parametrization of a cylinder with axis $\mathbb{R}\vec{v}$ and diameter $\sqrt{2}$. $\Big\} \Rightarrow$

$$\vec{w} = \frac{1}{|\vec{v}|} \cdot \vec{v} = \frac{(1, 1, 0)}{\sqrt{2}} \quad ; \quad e: \begin{cases} x = \frac{1}{\sqrt{2}}\lambda \\ y = \frac{1}{\sqrt{2}}\lambda \\ z = 0 \end{cases}$$

$$Rot_{l,\theta}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \cos\theta \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \sin\theta \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ x & y & z \end{vmatrix} + (1-\cos\theta)\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} =$$

$$= \cos\theta \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \sin\theta\left(\frac{1}{\sqrt{2}}z - \frac{1}{\sqrt{2}}z + \frac{1}{\sqrt{2}}y - \frac{1}{\sqrt{2}}x\right) + (1-\cos\theta) \cdot \frac{1}{2}\begin{pmatrix} x+y \\ x+y \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -\frac{1}{\sqrt{2}}\sin\theta & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} +$$

$$+ \begin{pmatrix} \frac{1-\cos\theta}{2} & \frac{1-\cos\theta}{2} & 0 \\ \frac{1-\cos\theta}{2} & \frac{1-\cos\theta}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta + \frac{1-\cos\theta}{2} - \frac{1}{\sqrt{2}}\sin\theta \\ \frac{1-\cos\theta}{2} \\ 0 \end{pmatrix}$$

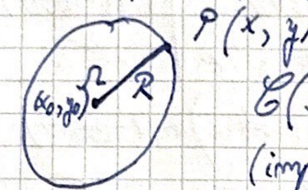
Seminar 10:

Chapter 6: 1, 3, 4, 5, 6, 7, 8, 9, 10

Quadratic curves (ellipses)

$Q: a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{01}y + a_{00}$

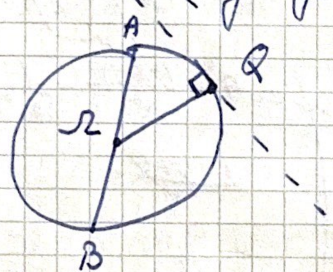
circle = locus of points in the plane whose distance to a fixed point $\Omega(x_0, y_0)$ is $R > 0$ (the radius)



$\mathcal{C}(\Omega, R): (x - x_0)^2 + (y - y_0)^2 = R^2$

(implicit form)

$$\mathcal{C}(\Omega, R): \begin{cases} x = x_0 + R\cos t \\ y = y_0 + R\sin t \end{cases} \quad \text{(parametric equation)} \quad t \in [0, 2\pi)$$



6.1. Find the eq. of the circle:

a) of diameter $[AB]$ with $A(1, 2)$, $B(-3, -1)$

e) passing through $A(3, 1)$ and $B(-1, 3)$ and having the center on line $\ell: 3x - y - 2 = 0$

a) $AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} = \sqrt{(-3-1)^2 + (-1-2)^2} =$

$= \sqrt{(-4)^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5 \Rightarrow R = \frac{5}{2}$

$x_0 = \frac{x_A + x_B}{2} = \frac{1-3}{2} = \frac{-2}{2} = -1$

$y_0 = \frac{y_A + y_B}{2} = \frac{2-1}{2} = \frac{1}{2}$

$\mathcal{C}(\Omega, R): (x+1)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$

e) $\begin{cases} 3x - y - 2 = 0 \\ (3 - x_0)^2 + (1 - y_0)^2 = R^2 \\ (-1 - x_0)^2 + (3 - y_0)^2 = R^2 \end{cases} \Rightarrow \begin{cases} 3x - y - 2 = 0 \\ 3 - 6x_0 + x_0^2 + 1 - 2y_0 + y_0^2 = R^2 \\ 1 + 2x_0 + x_0^2 + 9 - 6y_0 + y_0^2 = R^2 \end{cases} \Rightarrow$

$\Rightarrow 2 - 8x_0 - 8 + 4y_0 = 0 \Rightarrow -8x_0 + 4y_0 - 6 = 0 \Rightarrow$

$\Rightarrow \begin{cases} 3x - y - 2 = 0 \quad | \cdot 2 \\ 4x_0 - 2y_0 + 3 = 0 \end{cases} \Rightarrow \begin{cases} 6x - 2y - 4 = 0 \\ 4x - 2y + 3 = 0 \end{cases} \; (-)$

$2x \quad / \quad -7 = 0 \Rightarrow x = \frac{7}{2}$

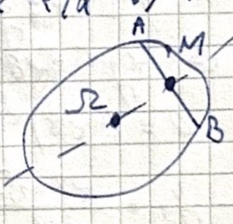
$x_M = \frac{x_A + x_B}{2} = \frac{3 - 1}{2} = \frac{2}{2} = 1; \quad y_M = \frac{y_A + y_B}{2} = \frac{1 + 3}{2} = \frac{4}{2} = 2$

M - midpoint of AB $\Rightarrow M(1, 2)$

$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{+3 - 1}{-1 - 3} = \frac{+2}{-4} = -\frac{1}{2} \Rightarrow m_{\Omega M} = 2$

$\Omega M: (y - y_M) = m_{\Omega M}(x - x_M) \Leftrightarrow y - 2 = 2(x - 1) \Rightarrow$

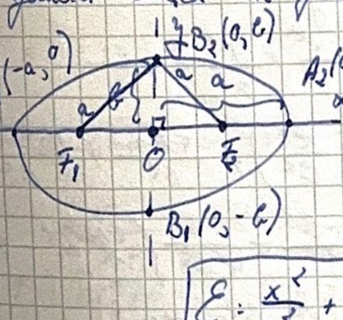
$\Rightarrow \Omega M: 2x - y = 0$



$\Omega: \begin{cases} 3x - y - 2 = 0 \\ 2x - y = 0 \end{cases} \Rightarrow \Omega(2, 4)$

$R = \sqrt{1 + 9} = \sqrt{10} \Rightarrow \mathcal{C}(\Omega, R): (x - 2)^2 + (y - 4)^2 = 10$

Ellipse: locus of point in the plane whose sum of distances to two distinct fixed points F_1 and F_2 (~~focused~~ called the focal points or foci) is a constant $2a$.



If we fix the reference system $\mathcal{K} = (O, \vec{i}, \vec{j})$, where O is the midpoint of $[F_1, F_2]$ and $\vec{i} \in \Delta(F_1 F_2)$ then the equation is of the form:

$$\boxed{\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

$a, b > 0, \; a \geq b$

$c = \sqrt{a^2 - b^2}$

$F_1(-c, 0), \; F_2(c, 0)$

$e = \frac{c}{a}$ = eccentricity

6.3. Det. the foc of the ellipse $\mathcal{E}: 9x^2 + 25y^2 - 225 = 0$

$9x^2 + 25y^2 = 225 \ /: 225$

$\frac{9}{225}x^2 + \frac{25}{225}y^2 = 1 \Leftrightarrow \frac{x^2}{25} + \frac{y^2}{9} = 1 \Leftrightarrow \frac{x^2}{5^2} + \frac{y^2}{3^2} = 1 \Rightarrow$

$\Rightarrow a = 5, \ b = 3 \Rightarrow c = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4$

$\Rightarrow F_1(-4, 0); \ F_2(4, 0)$

6.4. Det. the intersection of the line $\ell: x + 2y - 7 = 0$ and the ellipse $\mathcal{E}: x^2 + 3y^2 - 25 = 0$

28×28: 224, 56, 784

28×24: 112, 56, 672

$784 - 672 = 112$

$\begin{cases} x + 2y - 7 = 0 \Rightarrow x = 7 - 2y \\ x^2 + 3y^2 - 25 = 0 \Rightarrow (7 - 2y)^2 + 3y^2 - 25 = 0 \end{cases}$

$\Leftrightarrow 49 - 28y + 4y^2 + 3y^2 - 25 = 0$

$\Leftrightarrow 7y^2 - 28y + 24 = 0$

$\Delta = (-28)^2 - 4 \cdot 7 \cdot 24 = 784 - 672 = 112 \Rightarrow \sqrt{\Delta} = 4\sqrt{7}$

$y_1 = \frac{28 + 4\sqrt{7}}{2 \cdot 7} = \frac{14 + 2\sqrt{7}}{7} \Rightarrow x_1 = 7 - 2\left(\frac{14 + 2\sqrt{7}}{7}\right) = \frac{-21 - 4\sqrt{7}}{7}$

$y_2 = \frac{28 - 4\sqrt{7}}{2 \cdot 7} = \frac{14 - 2\sqrt{7}}{7} \Rightarrow x_2 = 7 - 2\left(\frac{14 - 2\sqrt{7}}{7}\right) = \frac{-21 + 4\sqrt{7}}{7}$

$F_1\left(\frac{-21 - 4\sqrt{7}}{7}, \frac{14 + 2\sqrt{7}}{7}\right), \ F_2\left(\frac{-21 + 4\sqrt{7}}{7}, \frac{14 - 2\sqrt{7}}{7}\right)$

$\ell: y = kx + m$

$\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\quad \ell \cap \mathcal{E}: \begin{cases} y = kx + m \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \end{cases}$

$\frac{x^2}{a^2} + \frac{(kx + m)^2}{b^2} = 1 \Leftrightarrow x^2\left(\frac{1}{a^2} + \frac{k^2}{b^2}\right) + \frac{2km}{b^2} + \frac{m^2 - b^2}{b^2} = 0$

$\Delta = \frac{4k^2m^2}{b^2} - 4 \cdot \frac{b^2 + k^2a^2}{a^2b^2} \cdot \frac{m^2 - b^2}{b^2} = \frac{4}{a^2b^2}(a^2\cancel{k^2}m^2 - b^2m^2 -$

$- k^2\cancel{a^2}m^2 + b^4 + k^2a^2b^2) = \frac{4}{a^2b^2}(b^2 + k^2a^2 - m^2)$

$b^2+k^2a^2-m^2$	intersection
<0	$\emptyset$
$=0$	one point (ℓ-tangent)
>0	two points (ℓ-secant)

So if ℓ is tangent, then $m = \pm\sqrt{b^2+k^2a^2}$. So:

$\ell: y = kx \pm \sqrt{a^2k^2+b^2}$

6.6. Determine an equation of a line which is orthogonal to $\ell: 2x-2y-13=0$ and tangent to the ellipse $\mathcal{E}: x^2+4y^2-20=0$.

$\ell: 2x-2y-13=0 \Rightarrow 2y = 2x-13 \Rightarrow y = x - \frac{13}{2} \Rightarrow m_\ell = 1$

* Tangent to $\mathcal{E}$ in (x_0, y_0): $T_{(x_0,y_0)}\mathcal{E}: \frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$

$T_{(x_0,y_0)}\mathcal{E}: \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) = 0$

$f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$

$T\mathcal{E}: \frac{x_0}{a^2}(x-x_0) + \frac{y_0}{b^2}(y-y_0) = 0$

$\ell: y = x - \frac{13}{2} \Rightarrow m_\ell = 1$; ~~$k=1$; $y=$~~

$t \perp \ell \Rightarrow m_t \cdot m_\ell = -1 \Rightarrow m_t = -1 \Rightarrow k = -1 \Rightarrow \begin{cases} t: y = kx+m \\ t: y = -x+m \end{cases}$

$\mathcal{E}: x^2+4y^2 = 20 \,/ :20 \Rightarrow \frac{x^2}{20} + \frac{4}{20}y^2 = 1 = \frac{x^2}{20} + \frac{y^2}{5} = 1$

~~$a = 2\sqrt{5}$, $b = \sqrt{5} \Rightarrow c = \sqrt{20-5} = \sqrt{15}$~~

~~$20^2 \quad 5^2 \quad m = \pm\sqrt{b^2+k^2a^2} = \sqrt{25+400} = \sqrt{425} =$~~

$t \cap \mathcal{E} = \{A\}$ $\begin{cases} y = -x+m \\ x^2+4y^2-20=0 \end{cases}$ $\Rightarrow x^2+4(-x+m)^2-20=0$

$\Rightarrow x^2+4x^2-8xm+4m^2-20=0$

$\Rightarrow 5x^2-8xm+4m^2-20=0$

$\Delta = (8m)^2 - 4\cdot 5\cdot(4m^2 - 20) = 64m^2 - 8m^2 + 400$

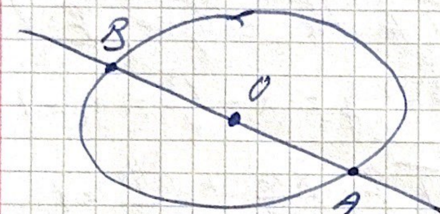
$\Delta = 400 - 16m^2 \Rightarrow \sqrt{\Delta} = \sqrt{400 - 16m^2}$

e - tangent $\Rightarrow \Delta = 0 \Rightarrow 400 - 16m^2 = 0 \Rightarrow m^2 = \frac{400}{16} = 25 \Rightarrow$

$\Rightarrow m = \pm 5$

$t: y = -x \pm 5$

6.7. diameter for an ellipse = line segment obtained by intersecting a line through the center with the ellipse



Show that the tangent lines to an ellipse at the endpoints of a diameter are parallel.

$e: y = kx \;;\quad e \cap \mathcal{E} = \{A, B\}$

$AB: \begin{cases} y = kx \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \end{cases} \Rightarrow \frac{x^2}{a^2} + \frac{k^2x^2}{b^2} = 1$

$x^2\left(\frac{1}{a^2} + \frac{k^2}{b^2}\right) = 1 \Rightarrow$

$\Leftrightarrow x^2 \cdot \frac{b^2 + a^2k^2}{a^2b^2} = 1 \Rightarrow x^2 = \frac{a^2b^2}{b^2 + a^2k^2} \Rightarrow$

$\Rightarrow x = \pm\sqrt{\frac{a^2b^2}{b^2 + a^2k^2}}$

$\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$e \cap \mathcal{E}: \begin{cases} y = kx \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \end{cases} \Rightarrow x_{1,2} = \pm\frac{ab}{\sqrt{b^2 + a^2k^2}}$

$y_{1,2} = \pm\frac{abk}{\sqrt{a^2 + a^2k^2}}$

Let $A(x_1, y_1)$, $B(x_2, y_2) \Rightarrow x_B = -x_A,\ y_B = -y_A$

$T_A\mathcal{E}: \frac{x_A \cdot x}{a^2} + \frac{y_A \cdot y}{b^2} = 1$

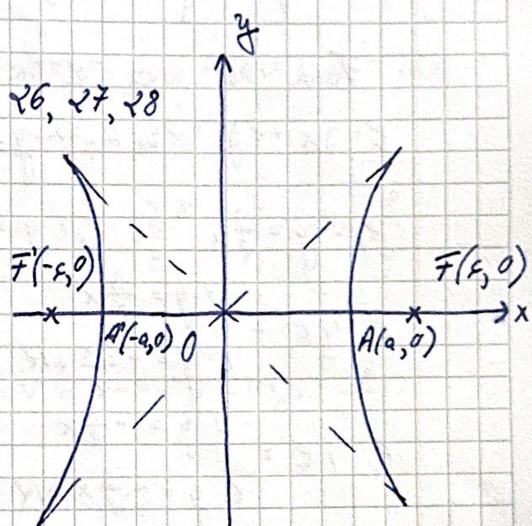
$$m_{T_A \mathcal{E}} = \frac{-\frac{x_A}{a^2}}{\frac{y_A}{b^2}} = -\frac{b^2}{a^2}\cdot\frac{x_A}{y_A}$$

$$m_{T_B \mathcal{E}} = \frac{-b^2}{a^2}\cdot\frac{x_B}{y_B} = \frac{-b^2}{a^2}\cdot\frac{-x_A}{-y_A} = -\frac{b^2}{a^2}\cdot\frac{x_A}{y_A} = m_{T_A\mathcal{E}} \Rightarrow$$

$$\Rightarrow T_A\mathcal{E} \parallel T_B\mathcal{E}$$

Seminar 11:

Section 6.4: 2, 8, 9, 10, 18, 20, 26, 27, 28



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

~~Asim~~ Assymptotes: $y = \frac{b}{a}x$

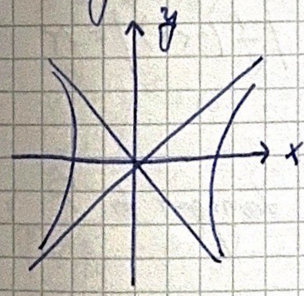
$y = -\frac{b}{a}x$

$c^2 = a^2 + b^2$

$$\frac{x^2}{25} - \frac{y^2}{49} = -1 \quad !$$

6.18. $\mathcal{H}: x^2 - y^2 = 16$. Find the tangent through $M(-1,7)$

$x^2 - y^2 = 16 \,/ :16 \Rightarrow \frac{x^2}{16} - \frac{y^2}{16} = 1 \Rightarrow a = b = 4$



$M(-1,7) \overset{?}{\in} \mathcal{H}$

$(-1)^2 - 7^2 = 1 - 49 = -48 \neq 16 \Rightarrow M \notin \mathcal{H}$

$$T_P: \boxed{\frac{x\cdot x_0}{a^2} - \frac{y\cdot y_0}{b^2} = 1}$$

Let $P(x_0, y_0)$ be a point on $\mathcal{H}$

$$T_P = \frac{x\cdot x_0}{16} - \frac{y\cdot y_0}{16} = 1 \Leftrightarrow \frac{-1\cdot x_0}{16} - \frac{7y_0}{16} = 1 \,/\cdot 16 \Leftrightarrow \boxed{-x_0 - 7y_0 = 16} \; (1)$$

$$\begin{cases} -x_0 - 7y_0 = 16 \; |(\,)^2 \\ x_0^2 - y_0^2 = 16 \end{cases} \Rightarrow \begin{cases} x_0^2 + 14x_0y_0 + 49y_0^2 = 256 \\ x_0^2 - y_0^2 = 16 \end{cases} (-)$$

$$14x_0y_0 + 48y_0^2 = 240$$

$y_0 \in \left\{-\frac{5}{3}, -3\right\}$; $x_0 \in \left\{-\frac{13}{3}, 5\right\}$

The equation of a tangent line : T_P

$$T_P : \begin{cases} y - 7 = m(x+1) \\ x^2 - y^2 = 16 \end{cases}$$

$\boxed{\Delta = 0}$

6.20 $H : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and $\ell : 9x + 2y - 24 = 0$

Find the area of the triangle det. by the asymptotes of the

$\ell : 9x + 2y - 24 = 0 \Rightarrow y = \frac{-9x}{2} + 12$

$\ell_1 : y = \frac{3}{2}x$; $\ell_2 : y = -\frac{3}{2}x$

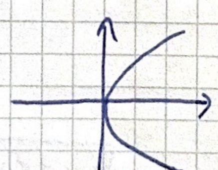
$$\ell_1 \cap \ell = \begin{cases} y = \frac{3}{2}x \\ y = -\frac{9x}{2} + 12 \end{cases} \Rightarrow 3x = -9x + 24 \Rightarrow x = 2,\ y = 3$$

$$\ell_2 \cap \ell = \begin{cases} y = -\frac{3}{2}x \\ y = -\frac{9}{2}x + 12 \end{cases} \Rightarrow -3x = -9x + 24 \Rightarrow x = 4,\ y = -6$$

$$\ell_1 \cap \ell_2 = \begin{cases} y = \frac{3}{2}x \\ y = -\frac{3}{2}x \end{cases} \Rightarrow x = 0,\ y = 0$$

$$A = \frac{1}{2}\left|\begin{vmatrix} 0 & 0 & 1 \\ 2 & 3 & 1 \\ 4 & -6 & 1 \end{vmatrix}\right| = \left|\frac{1}{2}\begin{vmatrix} 2 & 3 \\ 4 & -6 \end{vmatrix}\right| = \left|\frac{1}{2}(-24)\right| = 12$$

6.26. For which k is the line $y = kx + 2$ tangent to $P : y^2 = 4x$?



$$\begin{cases} y = kx + 2 \\ y^2 = 4x \end{cases} \Rightarrow k^2x^2 + 4kx + 4 = 4x \Rightarrow$$

$$\Rightarrow k^2x^2 + 4(k-1)x + 4 = 0$$

$\Delta = 16(k^2 - 2k + 1) - 4k^2 \cdot 4 = \cancel{16k^2} - 32k + 16 - \cancel{16k^2} = -32k + 16$

The system has one sol $\Rightarrow \Delta = 0 \Rightarrow k = \frac{1}{2} \neq 0$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow T_P: \frac{x\cdot x_0}{a^2}+\frac{y\cdot y_0}{b^2}=1$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow T_P: \frac{x\cdot x_0}{a^2}-\frac{y\cdot y_0}{b^2}=1$$

$$y^2=2px \Rightarrow T_P: y\cdot y_0=p(x+x_0)$$

27 $P: y^2=16x$. Det. the tangents to P: e' s.t.

a) $e' \parallel e$, $e: 3x-2y+30=0 \Rightarrow m=\frac{3}{2}$

$e': y=mx+k=\frac{3}{2}x+k$

$$\begin{cases} y=\frac{3}{2}x+k \\ y^2=16x \end{cases} \Rightarrow \left(\frac{3}{2}x+k\right)^2=16x \Rightarrow \frac{9}{4}x^2+3kx+k^2=16x \Rightarrow$$

$$\Rightarrow \frac{9}{4}x^2+(3k-16)x+k^2=0$$

$$\Delta=(3k-16)^2-9k^2=\cancel{9k^2}-96k+256-\cancel{9k^2} \quad \left.\begin{matrix}\Delta=0\end{matrix}\right\} \Rightarrow k=\frac{8}{3}$$

$\Rightarrow e': y=\frac{3}{2}x+\frac{8}{3}$

b) $e' \perp e$, $e: 4x+2y+7=0 \Rightarrow m_e=-2 \Rightarrow m_{e'}=\frac{1}{2}$

$e': y=\frac{1}{2}x+k$

$$\begin{cases} y=\frac{1}{2}x+k \\ y^2=16x \end{cases} \qquad \begin{matrix}\frac{1}{4}x^2+kx+k^2=16x \\ \frac{1}{4}x^2+(k-16)x+k^2=0\end{matrix}$$

$\Delta=(k-16)^2-k^2=-32k+256 \Rightarrow k=8 \Rightarrow e': y=\frac{1}{2}x+8$

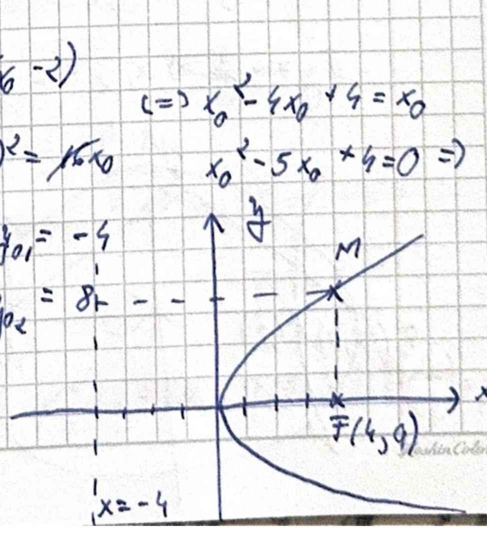
6.28 $P: y^2=16x \Rightarrow p=8$

$P(-2,2)\in T_P$, $T_P=?$

$$\begin{cases} 2y_0=8(x_0-2) \\ y_0^2=16x_0 \end{cases} \Leftrightarrow \begin{cases} y_0=4(x_0-2) \\ \cancel{16}(x_0-2)^2=\cancel{16}x_0 \end{cases} \Leftrightarrow x_0^2-4x_0+4=x_0$$

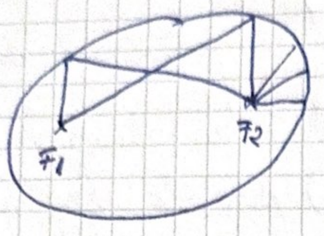
$$x_0^2-5x_0+4=0 \Rightarrow$$

$$\Rightarrow (x_0-1)(x_0-4)=0 \Rightarrow \begin{cases} x_{01}=1 \Rightarrow y_{01}=-4 \\ x_{02}=4 \Rightarrow y_{02}=8 \end{cases}$$



$T_1: -4y=8(x+1)$

$T_2: 8y=8(x+4) \Rightarrow y=x+4$



6.10. Det. the common tangents to the ellipses:

$\frac{x^2}{45}+\frac{y^2}{9}=1$ and $\frac{x^2}{9}+\frac{y^2}{18}=1$

$$\left.\begin{array}{l} y=mx\pm\sqrt{45m^2+9} \\ y=mx\pm\sqrt{9m^2+18} \end{array}\right\} \Rightarrow 45m^2+9=9m^2+18$$

$$36m^2=9 \Rightarrow m=\pm\frac{1}{2}$$

$$y=\pm\frac{1}{2}x+\sqrt{45\cdot\frac{1}{4}+9}=\pm\frac{1}{2}x+\frac{9}{2}$$

6.23 $\mathcal{H}: x^2-\frac{y^2}{4}-1=0$ with F_1 and F_2. Find $M\in\mathcal{H}$ s.t.:

a) $\sphericalangle F_1MF_2$ is right

$a=1,\ b=2 \Rightarrow c=\sqrt{5}$

$\mathcal{C}: x^2+y^2=5$

$$\begin{cases} x^2+y^2=5 \Rightarrow x^2=5-y^2 \Rightarrow 5y^2=16 \Rightarrow y_{1,2}=\pm\frac{4}{\sqrt5} \\ x^2-\frac{y^2}{4}=1 \end{cases}$$

$5-y^2-\frac{y^2}{4}=1 \Leftrightarrow 5-\frac{5y^2}{4}=1\ /\cdot 4 \Rightarrow 20-5y^2=4 \Rightarrow$

I $y=\frac{4}{\sqrt5} \Rightarrow x_{1,2}=\pm\frac{3}{\sqrt5}$

II $y=-\frac{4}{\sqrt5} \Rightarrow x_{1,2}=\pm\frac{3}{\sqrt5}$