2023 - 14 JUNE - 2

1. $x_{k+1} = x_k + \lambda x_k (2 - x_k)$, $k \in \mathbb{N}$

Fixed points, $x_{k+1} = x_k$

$$0 = \lambda x_k (2 - x_k)$$

$$x_k = 0$$

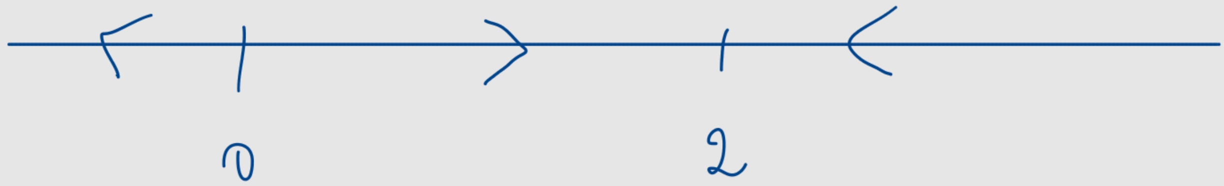
$$x_k = 2$$

$$f(x) = x + \lambda x (2 - x) = x + 2\lambda x - \lambda x^2$$

$$f'(x) = 1 + 2\lambda - 2\lambda x$$

$f'(0) = 1 + 2\lambda > 1$ unstable

$f'(2) = 1 - 2\lambda < 1$ stable

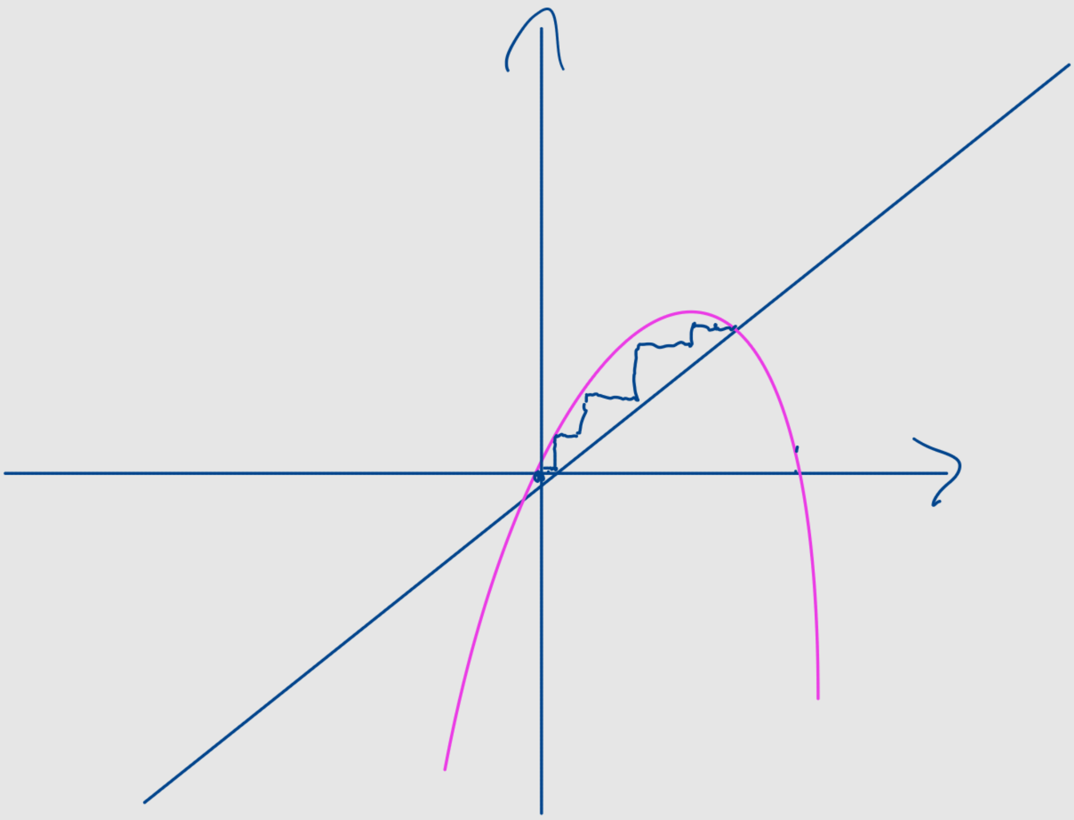


b) $\lambda = 0.5$

$$f(x) = x + 0.5 \cdot x\,(2 - x)$$

$$= x\,(1 + 1 - 0.5x)$$

$$= x\,(2 - 0.5x)$$



2.

$$\begin{cases} x' = -2x \\ y' = x - \sqrt{5}\,y \end{cases} \qquad \begin{cases} x' = -2x \\ y' = x + 3x^2 - \sqrt{5}\,(y+y^3) \end{cases}$$

a)

$$X' = \begin{pmatrix} -2 & 0 \\ 1 & -\sqrt{5} \end{pmatrix} X$$

$$\det(A - \lambda I_2) = \begin{pmatrix} -2-\lambda & 0 \\ 1 & -\sqrt{5}-\lambda \end{pmatrix} =$$

$$\Rightarrow (2+\lambda)(\sqrt{5}+\lambda) = 0$$

$$\lambda \in \{-2, -\sqrt{5}\}$$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} -2x = -2x \\ x - \sqrt{5}y = -2y \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = (-2+\sqrt{5})y \\ y = y \end{cases} \Leftrightarrow u_1 = \begin{pmatrix} \sqrt{5}-2 \\ 1 \end{pmatrix}$$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = -\sqrt{5}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} -2x = -\sqrt{5}x \\ x - \cancel{\sqrt{5}}y = -\cancel{\sqrt{5}}y \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y \in \mathbb{R} \end{cases} \Rightarrow u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$X = c_1 e^{-2t}\begin{pmatrix} \sqrt{5}-2 \\ 1 \end{pmatrix} + c_2 e^{-\sqrt{5}t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$X(0,0) = (\eta_1, \eta_2)$$

$$\begin{cases} m_1 = c_1(\sqrt{5}-2) \\ m_2 = c_1 + c_2 \end{cases} \Rightarrow \quad c_1 = \frac{m_1}{\sqrt{5}-2}$$

$$c_2 = m_2 - \frac{m_1}{\sqrt{5}-2}$$

$$\varphi(t, m_1, m_2) = \left(\frac{m_1 e^{-2t}}{\sqrt{5}-2}, \frac{m_1 e^{-2t}}{\sqrt{5}-2} + \frac{[m_2(\sqrt{5}-2) - m_1]}{\sqrt{5}-2} \cdot e^{-5t} \right)$$

b)

$$\lim_{t \to \infty} \varphi(t, n) = (0, 0)$$

c)

$$x' = 0 \Rightarrow x = 0$$

$$y' = 0 \Rightarrow y + y^3 = 0$$

$$y(1+y^2) = 0 \quad \begin{cases} y = 0 \\ y = \pm i \end{cases}$$

$\Rightarrow$ eq. points $S = \{(0,0)\}$

$$J_g(x^*) = \begin{pmatrix} -2 & 0 \\ 1+6x & -\sqrt{5}-3\sqrt{5}y^2 \end{pmatrix}$$

$$J_g((0,0)) = \begin{pmatrix} -2 & 0 \\ 1 & -\sqrt{5} \end{pmatrix}$$

$$\det(J_g(0,0)) = \begin{vmatrix} -2-\lambda & 0 \\ 1 & -\sqrt{5}-\lambda \end{vmatrix} = 0$$

$$\lambda = \{-2, -\sqrt{5}\}$$

$(0,0) \Rightarrow$ attractor, Node stable

d) $\psi(t,x)$, since $(0,0)$ is attractor

3.

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$e^{i\pi} = -1$$

$$e^{i\frac{\pi}{2}} = i$$

$$e^{\frac{\pi}{6}} \cdot e^{(-1+i)}$$

June 27, 2023

1.

a) $x_{k+2} - \sqrt{3}\, x_{k+1} + x_k = 0$

$$q^2 - \sqrt{3}\, q + 1 = 0$$

$$q \quad \sqrt{3} \pm i$$

$$r = \frac{\quad}{2}$$

$$|r_1| = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$$

$$r_1 = 1 \cdot \left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6} \right)$$

$$x_k = c_1 \cos\frac{k\pi}{6} + c_2 \sin\frac{k\pi}{6}$$

b)

$$x'' - \sqrt{3}x' + x = 0$$

$$r^2 - \sqrt{3}r + 1 = 0$$

$$r = \frac{\sqrt{3}}{2} \pm \frac{i}{2}$$

$$e^{\frac{\sqrt{3}}{2}t} \left(c_1 \cos\left(\frac{t}{2}\right) + c_2 \sin\left(\frac{t}{2}\right) \right)$$

2

$$\begin{cases} x' = y = f_1 \\ y' = -g\sin x = f_2 \end{cases}$$

a) $$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -g\sin x \end{cases} \iff \frac{dx}{dy} = \frac{y}{-g\sin x} \iff$$

$$\iff \int y\, dy = \int -g\sin x\, dx$$

$$\frac{y^2}{2} = +g\cos x + c$$

$$H(x,y) = \frac{y^2}{2} - g\cos x$$

We will check:

$H_x(x,y) \cdot \lambda_1 + H_y(x,y) \cdot \lambda_2 = 0$

$g\sin x \cdot y + y \cdot (-g\sin x) = 0$ „True"

b)

$(2\pi, 0)$ is an eq. point since $\begin{cases} x' = 0 \\ y' = -g\sin(2\pi) = 0 \end{cases}$

$$J = \begin{pmatrix} 0 & 1 \\ -g\cos(x) & 0 \end{pmatrix}$$

$$J(2\pi, 0) = \begin{pmatrix} 0 & 1 \\ -g & 0 \end{pmatrix}$$

$$\det(J - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -g & -\lambda \end{vmatrix} = \lambda^2 + g$$

$\lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i$, center

It is not hyperbolic since $\begin{cases} \text{Real}(\lambda_1) \neq 0 \\ \text{Real}(\lambda_2) \neq 0 \end{cases}$

3.

$x' = x^2 + 5x + 6$

Eq. points $\Rightarrow x^2 + 5x + 6 = 0$

$x = -2$

$x = -3$

Euler's method:

$x_{n+1} = x_n + h f(x_n)$

$0 = x_n + h(x_n^2 + 5x_n + 6)$

$x_n = -2$

$x_n = -3$

Check stability of fixed points:

$g(x) = x + h((x^2+5x+6))$

$g'(x) = 1 + h(2x+5)$

$x = -2$

$g'(-2) = 1 + h$

$|1+h| < 1$ for stability

$\left.\begin{array}{l} -2 < h < 0 \\ h > 0 \end{array}\right\} \Rightarrow$ impossible

$x = -3 \rightarrow$ attractor

$g'(-3) = 1 - h$

$|1-h| < 1 \Rightarrow 0 < h < 2$

June 28, 2023

1.

$$x' = -k(x-21)$$

$$\frac{dx}{dt} = -k(x-21)$$

$$\frac{dx}{x-21} = -k \cdot dt \quad /\int$$

$$\ln|x-21| = -kt + C$$

$$x = 21 + ce^{-kt}$$

$$x(t) = 21 + Ce^{-kt}$$

$$x(0) = 21 + C = m$$

$$C = m - 21$$

$$x(t) = 21 + (x_0 - 21)e^{-kt}$$

b)

$x(10) = 37$

I $\quad 37 = 21 + (x_0 - 21)e^{-10k}$

$$k = \frac{-1}{10} \ln\left(\frac{4}{7}\right)$$

II $\quad 37 = 21 + (x_0 - 21)e^{-20k}$

$$x_0 = 62{,}88$$

2. $\quad x'' + 9x = 0 \qquad x(0) = \frac{\pi}{2} \; ; \; x'(0) = 0$

$$r^2 + 9 = 0$$

$$r = \pm 3i$$

$$x(t) = e^{\alpha t}\left(C_1 \cos(\beta t) + C_2 \sin(\beta t)\right)$$

$$x(t) = C_1 \cdot \cos(3t) + C_2 \cdot \sin(3t)$$

$$\left.\begin{array}{l} x(0) = \frac{\pi}{2} \Rightarrow C_1 = \frac{\pi}{2} \\ x'(0) = 0 \Rightarrow C_2 = 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \quad x(t) = \frac{\pi}{2} \cos(3t)$$

$T = \frac{2\pi}{\omega}$, where ω is in $\cos(\omega t)$

$\omega = 3$

$T = \frac{2\pi}{3}$ minutes

3

Newton's method

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}$$

Taylor expansion of $g(x)$ around r:

$$g(x) = g(r) + g'(r)(x-r) + \frac{g''(\xi)}{2}(x-r)^2$$

Apply Newton

$$g(x_k) = g'(r) + g'(r)(x-r) + \frac{g''(\xi_k)}{2}(x_k - r)^2$$

$$x_{k+1} = x_k - \frac{g'(r)(x_k - r) + \frac{g''(\xi_k)(x_k - r)^2}{2}}{g'(x_k)}$$

$$g'(x_k) \approx g'(r)$$

$$x_{k+1} = x_k - (x_k - r) - \frac{g''(\xi_k) \cdot \frac{(x_k - r)^2}{2}}{g'(r)}$$

$$x_{k+1} = r - \frac{g''(\xi_k)}{2g'(r)} (x_k - r)^2$$

Since $|x_{k+1} - r|$ is proportional to $|x_k - r|^2$, for x_0 sufficiently close to r, we have $\lim_{k \to \infty} x_k = r$

(4) $y' = 1 + xy^2$

$y(0) = 0$

Euler's Method:

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

$$f(x,y) = 1 + x y^2$$

1.) $y(0) = 0$; $h = 0{,}02$

so finding $\varphi(0{,}5)$ takes 25 steps $\left(\frac{0{,}5}{0{,}02}\right)$

and for $\varphi(1)$ it take 50 steps

2) $\varphi(0{,}04)$

Step 0: $x_0 = 0$, $y_0 = 0$

$$y_1 = y_0 + h(1 + x_0 y_0^2) = 0{,}02$$

$$x_1 = x_0 + h = 0{,}02$$

Step 1 : $x_0 = 0{,}2$ $y_0 = 0{,}2$

$$y_2 = y_1 + h(1 + x_1 y_1^2) = 0{,}04000016 = \varphi(4)$$

June 14-1, 2023

1\. a) $x' = Ax$

$X(0) = X_0$

$X(t) = e^{At} X(0)$

b)

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$e \qquad = I$$

$A \begin{pmatrix} 0 & 1 \end{pmatrix}$

$$e^{t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} + 0 \dots$$

$$= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$A^0 = I, \quad A^1 = A, \quad A^2 = -I, \quad A^3 = -A, \quad A^4 = I$$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \frac{(At)^4}{4!} + \dots$$

$$e^{At} = I + At + \frac{(-I)t^2}{2!} + \frac{(-A)t^3}{3!} + \frac{I \cdot t^4}{4!} + \dots$$

$$e^{At} = I\left(1 - \frac{t^2}{2!} + \frac{t^2}{4!} - \dots\right) + At\left(1 - \frac{t^3}{3!} + \frac{t^4}{5!} - \dots\right)$$

$$e^{At} = \begin{pmatrix} \cos t & -\sin t \\ & \end{pmatrix}$$

$$\begin{pmatrix} \sin t & \cos t \end{pmatrix}$$

C) P invertible $\Rightarrow \det P \neq 0$

$$A = P J P^{-1}$$

Prove $e^{A} = P e^{J} P^{-1}$

We know $e^{A} = \sum_{k=0}^{\infty} \frac{A^{k}}{k!}$

$$A^{2} = (PJ\underbrace{P^{-1})(P}_{I_n}JP^{-1}) = PJ^{2}P^{-1}$$

...

$$A^{k} = PJ^{k}P^{-1}$$

$$e^{A} = \sum_{k=0}^{\infty} \frac{PJ^{k}P^{-1}}{k!}$$, factor P, P^{-1}

$$e^{A} = P \sum_{k=0}^{\infty} \frac{J^{k}}{k!} P^{-1}$$

$$e^{A} = P e^{J} P^{-1}$$

2.

$$\begin{cases} x' = x(1-y) \\ y' = y(2-x) \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = x - xy \\ \dfrac{dy}{dt} = 2y - xy \end{cases} \Leftrightarrow \frac{dx}{dy} = \frac{x - xy}{2y - xy} \Leftrightarrow$$

$$\int (2y - xy)\,dx = \int (x - xy)\,dy$$

$$2yx - \frac{\cancel{y}x^2}{\cancel{2}} + C = xy - \frac{\cancel{x}y^2}{\cancel{2}}$$

$y = c$!!!!

ChatGpt starts with $I(x,y) =$

$$= \ln x + \ln y + \beta x + \gamma y$$

and miraculously finds $I(x,y) = \ln x + y - x - \ln y$

b) $(2,1)$ eq. point

$$\left.\begin{array}{l} x' = x(1-y) = 0 \Rightarrow x = 0,\ y = 1 \\ y' = y(2-x) = 0 \Rightarrow y = 0,\ x = 2 \end{array}\right\} \Rightarrow$$

$(2,1)$ is eq. point

$$J = \begin{pmatrix} 1-y & -x \\ -y & 2-x \end{pmatrix}$$

$$J(2,1) = \begin{pmatrix} 0 & -2 \\ -1 & 0 \end{pmatrix} \Rightarrow \lambda = \pm i\sqrt{2}$$ $\Rightarrow$ not hyperbolic (no real parts)

stable, center

3. $x'' + 3x = 3, \quad x(0) = 0$

A. Solve the H.S.

$x'' + 3x = 0$

$r^2 + 3 = 0$

$r = \pm \sqrt{3} i$

$$x_h(t) = c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}t)$$

B. Finding the particular solution

$x_p = A$ since $R(t)$ is constant

$$3x_p = 3 \Rightarrow x_p = 1$$

$$x(t) = c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}t) + 1$$

We plug $x(0)=0$ and find c_1 but c_2 will be missing which means the eq may have been

$x' + 3x = 2$, $x(0) = 0$ (almost same)

but same solution.

b) $x_{k+1} + 3x_k = 2$, $x_0 = 0$

$$r + 3 = 0 \Rightarrow r = -3$$

$$\Rightarrow x_k = c(-3)^k$$

Finding the Part. Sol.

let $B = x_k$

$$B + 3B = 2$$

$$B = \frac{1}{2}$$

$$r_k = \frac{1}{2}$$

$$\Rightarrow \quad r_k = C(-3)^k + \frac{1}{2}$$

$$r_0 = 0 \quad \Rightarrow \quad C = -\frac{1}{2}$$

$$r_k = -\frac{1}{2}(-3)^k + \frac{1}{2}$$