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(918)

①

$\lim_{n \to \infty} \frac{n+1}{2n+3} = \frac{1}{2}$ using ε-definition

$\left| \frac{n+1}{2n+3} - \frac{1}{2} \right| < \varepsilon \rightarrow$

$\Leftrightarrow \left| \frac{2n+2-2n-3}{4n+6} \right| < \varepsilon \Leftrightarrow$

$\Leftrightarrow \left| \frac{1}{4n+6} \right| < \varepsilon$

$\frac{1}{4n+6} < \varepsilon$

$\frac{1}{\varepsilon} < 4n+6$

$\frac{\frac{1}{\varepsilon} - 6}{4} < n$

$$\underbrace{\frac{1}{4\varepsilon} - \frac{3}{2}}_{\shortparallel} < n$$

$$\frac{1}{4\varepsilon} = \frac{3}{2} \cdot 1 \,,\quad 12\varepsilon = 2 \qquad \varepsilon = \frac{1}{6}$$

$$\therefore \; n_\varepsilon = \begin{cases} 1 \,, & \varepsilon \geq \frac{1}{6} \\ \left\lfloor \frac{1}{4\varepsilon} - \frac{3}{2} \right\rfloor + 1 \,, & \varepsilon < \frac{1}{6} \end{cases}$$

(2)

$$x_n = \frac{\sin(n)}{n}$$

$$-1 \leq \sin n \leq 1 \quad \Big| \; \frac{1}{n}$$

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \quad \Big| \; \frac{1}{n} \to 0$$

$$0 \leq \frac{\sin}{n} \leq 0 \quad (\text{Sandwich Theorem})$$

convergent to 0, bounded but not monotone as $\sin n \in (-1, 1)$ with $T = 2\pi$

(b)

$$x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n$$

$$x_n - x_{n-1} = \frac{1}{n} - \ln(n) + \ln(n-1) =$$

$$= \frac{1}{n} + \ln\left(1 - \frac{1}{n}\right) < 0 \quad , \forall n \in \mathbb{N}$$

(since $\ln(1-x)$ is a concave function)

Proving bounded:

$$\sum_{k=1}^{n} \frac{1}{k} > \int_{1}^{n+1} \frac{1}{x}\,dx = \ln(n+1) > \ln(n)$$

$\forall n \in \mathbb{N}$

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(b)

$$\lim \frac{n^n}{1 + 2^2 + 3^3 + \dots + n^n} \ominus$$

We use Stolz - Cezaro

$$\circledS \lim_{n\to\infty} \frac{(n+1)^{n+1} - n^n}{(n+1)^{n+1}} =$$

$$\circledS \lim_{n\to\infty} \left(1 - \frac{n^n}{(n+1)^{n+1}}\right) = 0$$

3) d)

$$\lim_{n\to\infty} \sqrt[n]{1+2+\dots+n} =$$

Using „Criteriul Radicalului"

$$\lim \frac{1+2+\dots+(n+1)}{} = 1 \ \ \lambda$$

$$\lim_{n\to\infty} \frac{\ldots}{1+2+\ldots+n}$$

$$= \lim_{n\to\infty} \sqrt[n]{1+2+\ldots+n} = 1$$

5.

$$\lim_{n\to\infty} \left(\frac{\ln(n+1)}{\ln(n)}\right)^n = \left(\frac{\ln(n+\frac{1}{n})}{\ln n} + 1\right)^n =$$

$$\left(\left(\ldots\right)^{\ldots}\right)^{\frac{\ln(1+\frac{1}{n})n}{\ln n}}$$

$$\lim \frac{\ln(1+\frac{1}{n})}{\ln n} \cdot n = \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} \cdot \frac{1}{n} \cdot n \cdot \frac{1}{\ln n} =$$

$$= 1$$

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Now, we need to prove that the limit exists and $\in \mathbb{R}$:

1 we first prove using induction that x_n is strictly decreasing

We prove:

$x_3 < x_2$, $x_2 = \frac{1+a}{2}$

$$x_3 = \frac{1}{2}\left(\frac{1+a}{2} + \frac{a}{\frac{1+a}{2}}\right)$$

$$\frac{1}{2}\left(\frac{1+a}{2} + \frac{a}{\frac{1+a}{2}}\right) \overset{?}{<} \frac{1+a}{2}$$

$$\frac{1+a}{2} + \frac{a}{1+a} < 1+a \quad / \cdot 2$$

$$1+a+\frac{4}{1+a} < 2+2a$$

$$\frac{4}{1+a} < 1+a$$

$$4 < (1+a)^2$$

$2 < 1+a$, since $a > 1$ this is true so $x_3 < x_2$

$P(n)$: $x_n > x_{n+1}$

$$x_n > \frac{1}{2}x_n + \frac{a}{2x_n}$$

$P(n+1)$: $x_{n+1} > x_{n+2}$

$$x_{n+1} > \frac{1}{2}x_{n+1} + \frac{a}{2x_{n+1}}$$

$\frac{1}{2}x + \frac{a}{\ldots} > \frac{1}{2}\left(\frac{1}{\ldots} \ldots \frac{a}{\ldots}\right)$

$$\frac{1}{2}x_n + \frac{a}{2x_n} > \frac{1}{4}\left(x_n + \frac{a}{x_n}\right) + \frac{a}{x_n + \frac{a}{x_n}}$$

but :

$$\frac{1}{2}x_n > \frac{1}{4}\left(x_n + \frac{a}{x_n}\right)$$

$$\frac{a}{2x_n} > \frac{a}{x_n + \frac{a}{x_n}}$$

so $x_n > x_{n+1}$ and $(x_n)_{n \geq 1}$ is

strictly decreasing

Now we prove that it is bounded.

$P(n): x_n > 1$, $\forall n \in \mathbb{N}$, $n \geq 2$

$$P(2): \quad x_2 \overset{?}{>} 1$$

$$1 + \frac{a}{2} > 1 \quad \Rightarrow P(2) \text{ is "1"}$$

$$a > 1$$

$$P(n) \to P(n+1)$$

$P(n)$: $x_n > 1$, $n \geq 2$, $\forall n \in \mathbb{N}$

$$P(n+1): \quad x_{n+1} > 1$$

$$\frac{1}{2}\left(x_n + \frac{a}{x_n}\right) > 1$$

$$\frac{1}{2}x_n + \frac{a}{2x_n} > \frac{1}{2} + \frac{a}{2} \quad \text{since } a > 1$$

We have to prove

$\frac{1}{2} + \frac{a}{2} > 1$

$\frac{a}{2} > \frac{1}{2}$ " 1 " since the hypothesis of the pb $a>1$

We proved that $x_n > 1$, $\forall n \in \mathbb{N}, n \geq 1$

and x_n is strictly decreasing.

From Weierstrass Theorem, x_n is convergent and has a limit.

Let l be the limit, $l \in \mathbb{R}$:

$$l = \frac{1}{2}\left(l + \frac{a}{l}\right)$$

$$l = \frac{1}{2}l + \frac{a}{\ldots}$$

$$l = \frac{l}{2} + \frac{a}{2l}$$

$$\frac{l}{2} = \frac{a}{2l} \implies 2l^2 = 2a$$

$$l = \sqrt{a}$$

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Conclusión:

$$\lim_{n \to \infty} x_n = \sqrt{a} ,$$

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