Bayes Theorem Over Time

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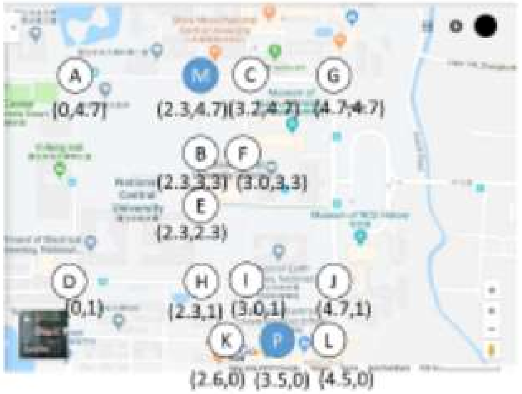
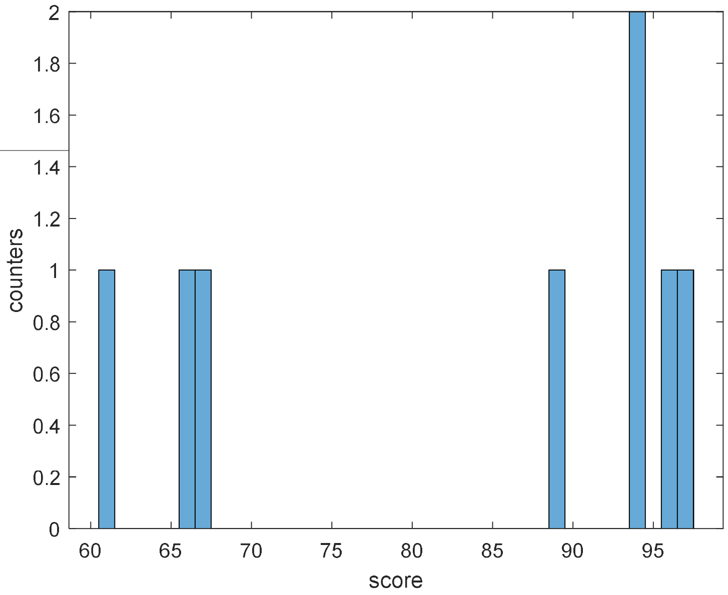
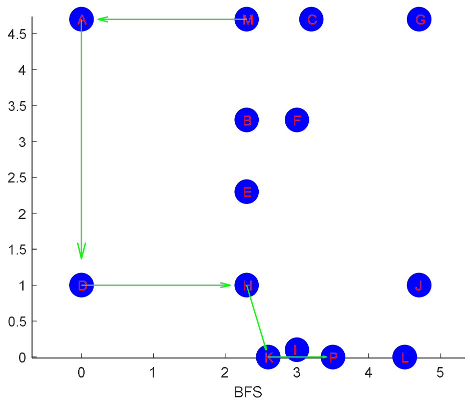
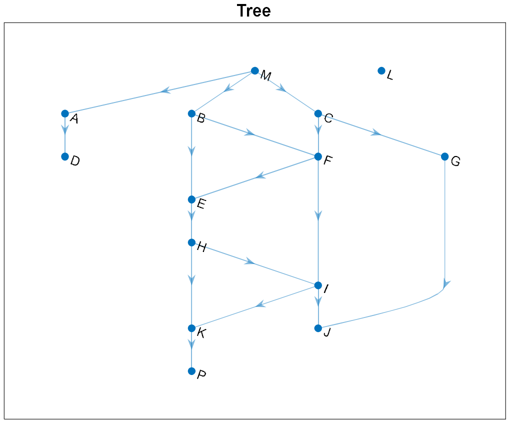
Course Announcement

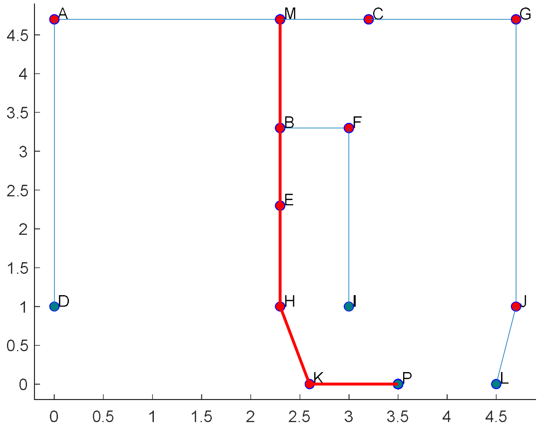
- HW1 was graded.
  - Search (50%)
  - LRTA* (25%)
  - MCTS* (25%)











Course Announcement

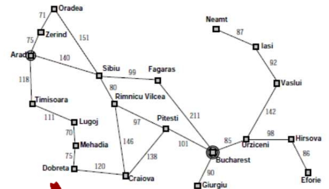
- HW2 was released today. The deadline is 4/14(Wed).
  - Bayesian inference
  - MDP solver
  - A MDP problem
- You should start to work on HW2 and implement Bayesian filter.

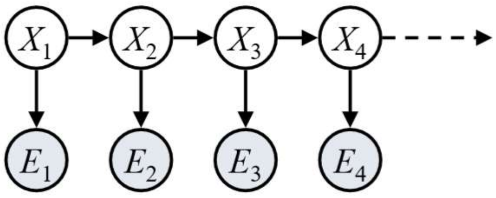
Outline

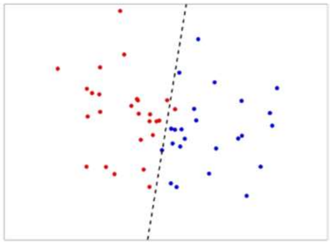
- Inference
  - Filtering
  - Prediction
  - Smoothing
- The Most Likely Sequence— Viterbi algorithm
- Hidden Markov Models
- EX: Gaze patterns
- Kalman Filter and Particle Filter
- State of the art perception technology

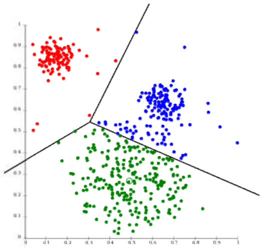
Outline

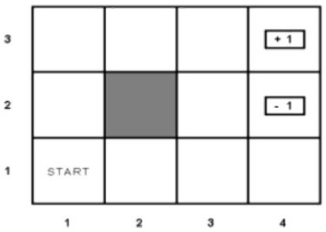
- [Problem solving]
  - Search problems
    - Adversarial Search
- [Perception and Uncertainty]
  - Bayes Theorem
    - Bayes Filter and Smoothing
- [Learning and Decision-making]
  - Supervised learning
  - Unsupervised learning
  - Reinforcement learning





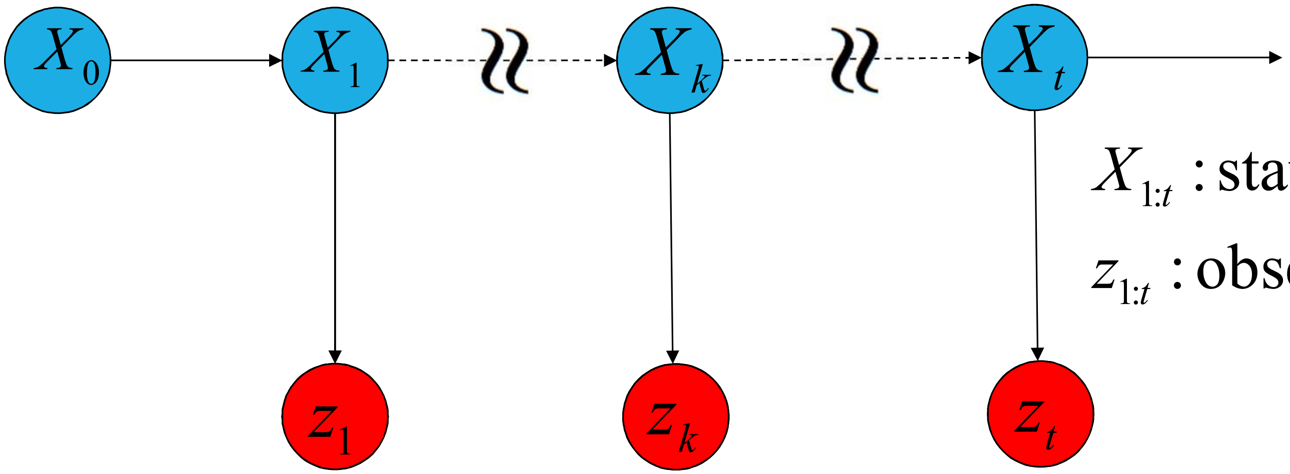






Inference

- Filering $P(X_t \mid z_{1:t})$
- Prediction $P(X_{t+k} \mid z_{1:t})$
- Smoothing $P(X_k \mid z_{1:t}), 0 \leq k \leq t$
- Most likely explanation $\arg\max_{x_{1:t}} P(x_{1:t} \mid z_{1:t})$

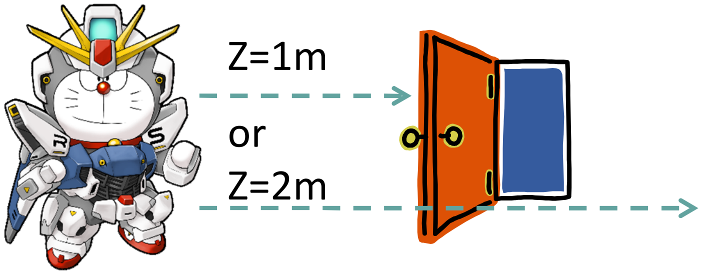


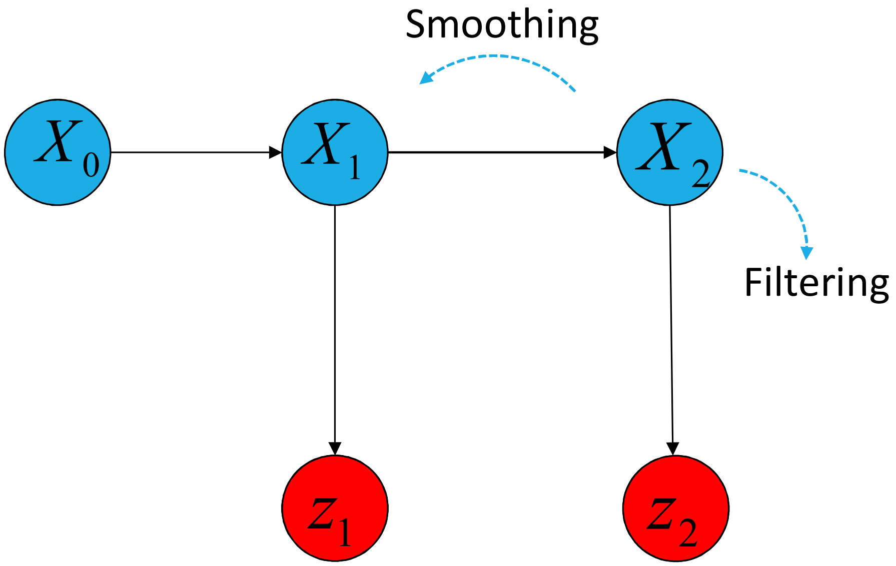
$X_{1:t}$: state from time 1 to t

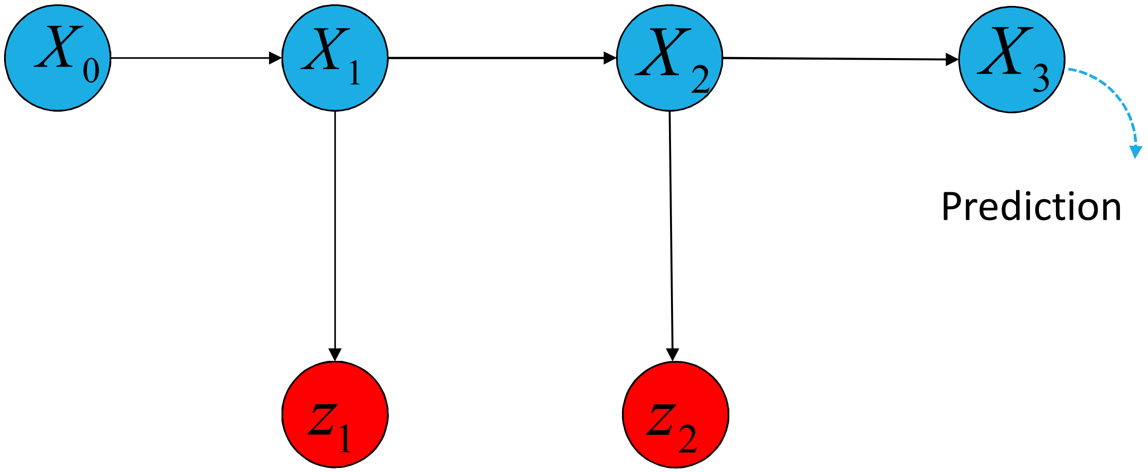
$z_{1:t}$: observation from time 1 to t

Inference

- Filtering $P(X_t \mid z_{1:t})$
- Prediction $P(X_{t+k} \mid z_{1:t})$
- Smoothing $P(X_k \mid z_{1:t}), 0 \le k \le t$

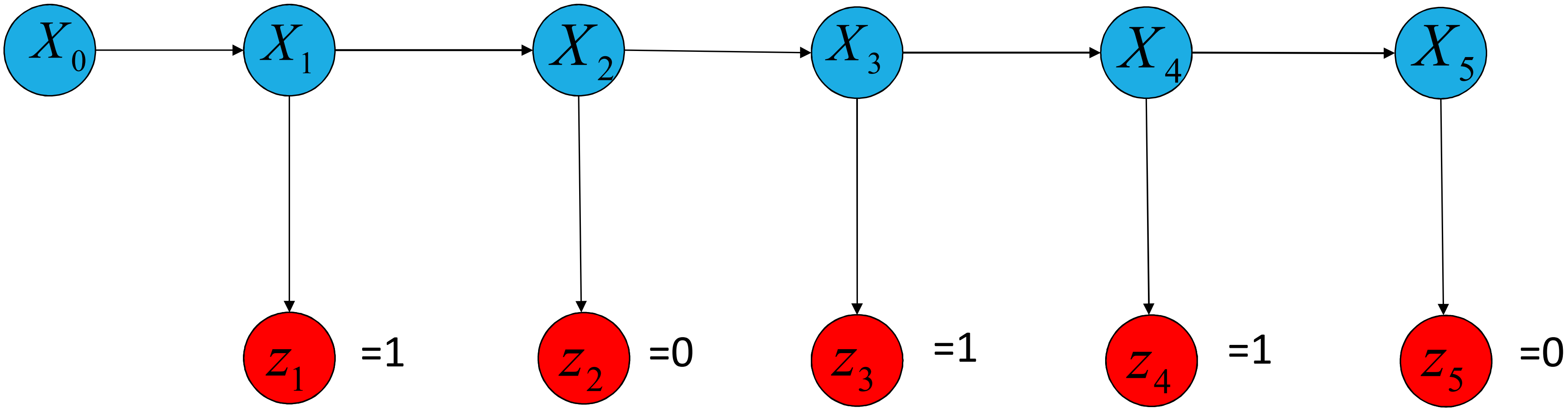






Inference

- Most likely explanation $\arg\max_{x_{1:t}} P(x_{1:t} \mid z_{1:t})$
- Given a sequential measurement, what’s the most likely states?



Inference

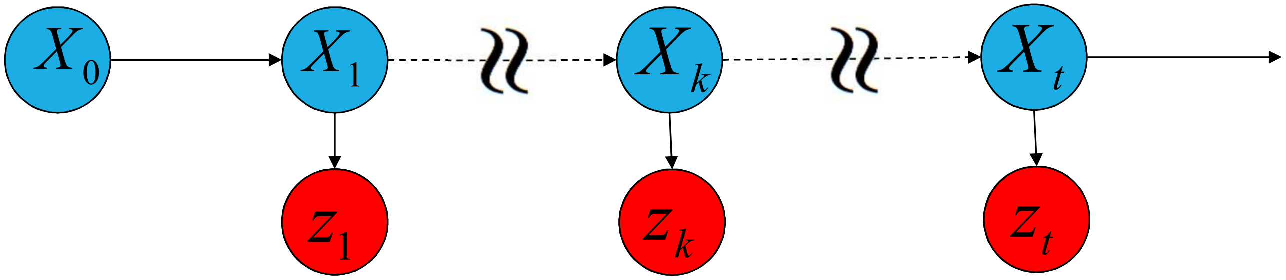
- The assumptions of Bayesian inference

1. States satisfy Markov chain $P(X_t \mid X_{t-1}, X_{t-2}) = P(X_t \mid X_{t-1})$

2. Given motion model $P(X_{t+1} \mid X_t)$

3. Given sensor model $P(z_t \mid X_t)$ and sensor independence

4. Dynamic bayesian network is shows as follows:



Inference—Filtering

- Filtering $P(X_t \mid z_{1:t})$

$$P(X_{t+1} \mid z_{1:t+1}) = P(X_{t+1} \mid z_{t+1}, z_{1:t})$$

$$= \eta P(z_{t+1} \mid X_{t+1}, z_{1:t}) P(X_{t+1} \mid z_{1:t})$$

$$= \eta P(z_{t+1} \mid X_{t+1}) P(X_{t+1} \mid z_{1:t})$$

$$= \eta P(z_{t+1} \mid X_{t+1}) \sum_{x_t} P(X_{t+1} \mid x_t, z_{1:t}) P(x_t \mid z_{1:t})$$

$$= \eta P(z_{t+1} \mid X_{t+1}) \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t \mid z_{1:t})$$

P(A|B,C)=P(B,C|A)P(A)/P(B,C)
=P(B|A,C)P(A|C)/P(B|C)

(Sensor independence)

(Total probability)

(Markov chain)

Inference—Filtering

- EX:

$$P(Z_{t_0}=2\mid x_{t_0}=o)=0.9 \qquad P(Z_{t_0}=2\mid x_{t_0}=c)=0.2$$

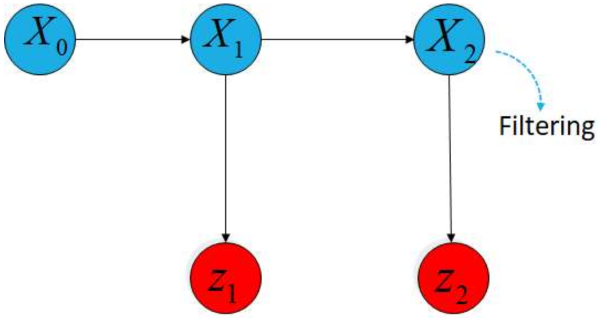
$$P(x_{t_0}=o)=P(x_{t_0}=c)=0.5\ \text{(by experiments)}$$

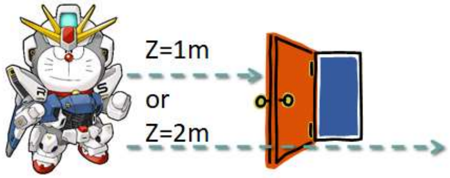
$$\left(Z_{t_1},Z_{t_2}\right)=\{2,2\}$$

$$P(x_t=o\mid x_{t-1}=o)=P(x_t=c\mid x_{t-1}=c)=0.7$$

$$P(x_t=o\mid x_{t-1}=c)=P(x_t=c\mid x_{t-1}=o)=0.3$$

$$P\left(x_2\mid Z_{1:2}=\{2,2\}\right)=?$$





Inference—Filtering

t 1, Z=2m

Given: $P(Z_t = 2 \mid x_t = o) = 0.9 \qquad P(Z_t = 2 \mid x_t = c) = 0.2$

$$\boxed{\begin{aligned} &P(x_t = o \mid x_{t-1} = o) = P(x_t = c \mid x_{t-1} = c) = 0.7 \\ &P(x_t = o \mid x_{t-1} = c) = P(x_t = c \mid x_{t-1} = o) = 0.3 \end{aligned}}$$

Transition matrix

$$P(x_t \mid x_{t-1}, z_{t-1}) = \int P(x_t \mid x_{t-1}) P(x_{t-1} \mid z_{t-1}) dx_{t-1}$$

$$\begin{cases} P(x_t = o \mid x_{t-1}, z_{t-1}) = \underset{0.7}{P(x_t = o \mid x_{t-1} = o)} \underset{0.5}{P(x_{t-1} = o \mid z_{t-1})} + \underset{0.3}{P(x_t = o \mid x_{t-1} = c)} \underset{0.5}{P(x_{t-1} = c \mid z_{t-1})} = 0.5 \\ P(x_t = c \mid x_{t-1}, z_{t-1}) = \underset{0.3}{P(x_t = c \mid x_{t-1} = o)} \underset{0.5}{P(x_{t-1} = o \mid z_{t-1})} + \underset{0.7}{P(x_t = c \mid x_{t-1} = c)} \underset{0.5}{P(x_{t-1} = c \mid z_{t-1})} = 0.5 \end{cases}$$

$$P(x_t \mid z_t) = \eta \bullet P(z_t \mid x_t) P(x_t \mid x_{t-1}, z_{t-1})$$

$$\begin{cases} P(x_t = o \mid z_t) = \eta \bullet \underset{0.9}{P(z_t \mid x_t = o)} \underset{0.5}{P(x_t = o \mid x_{t-1}, z_{t-1})} = 0.818 \\ P(x_t = c \mid z_t) = \eta \bullet \underset{0.2}{P(z_t \mid x_t = c)} \underset{0.5}{P(x_t = c \mid x_{t-1}, z_{t-1})} = 0.182 \end{cases}$$

t 2, Z=2m

Inference—Filtering

Given: $P(Z_t = 2 \mid x_t = o) = 0.9 \qquad P(Z_t = 2 \mid x_t = c) = 0.2$

$$\boxed{\begin{aligned} &P(x_t = o \mid x_{t-1} = o) = P(x_t = c \mid x_{t-1} = c) = 0.7 \\ &P(x_t = o \mid x_{t-1} = c) = P(x_t = c \mid x_{t-1} = o) = 0.3 \end{aligned}}$$

Transition matrix

$$P(x_t \mid x_{t-1}, z_{t-1}) = \int P(x_t \mid x_{t-1}) P(x_{t-1} \mid z_{t-1}) dx_{t-1}$$

$$\begin{cases} P(x_t = o \mid x_{t-1}, z_{t-1}) = \underset{0.7}{P(x_t = o \mid x_{t-1} = o)}\underset{0.818}{P(x_{t-1} = o \mid z_{t-1})} + \underset{0.3}{P(x_t = o \mid x_{t-1} = c)}\underset{0.182}{P(x_{t-1} = c \mid z_{t-1})} = 0.627 \\ P(x_t = c \mid x_{t-1}, z_{t-1}) = \underset{0.3}{P(x_t = c \mid x_{t-1} = o)}\underset{0.818}{P(x_{t-1} = o \mid z_{t-1})} + \underset{0.7}{P(x_t = c \mid x_{t-1} = c)}\underset{0.182}{P(x_{t-1} = c \mid z_{t-1})} = 0.373 \end{cases}$$

$$P(x_t \mid z_t) = \eta \bullet P(z_t \mid x_t) P(x_t \mid x_{t-1}, z_{t-1})$$

$$\begin{cases} P(x_t = o \mid z_t) = \eta \bullet \underset{0.9}{P(z_t \mid x_t = o)}\underset{0.627}{P(x_t = o \mid x_{t-1}, z_{t-1})} = 0.883 \\ P(x_t = c \mid z_t) = \eta \bullet \underset{0.2}{P(z_t \mid x_t = c)}\underset{0.373}{P(x_t = c \mid x_{t-1}, z_{t-1})} = 0.117 \end{cases}$$

Inference—Prediction

- Prediction $P(X_{t+k} \mid z_{1:t})$

$$P(X_{t+k} \mid z_{1:t}) = P(X_{t+1:t+k}, X_t \mid z_{1:t})$$
$$= P(X_{t+1:t+k} \mid X_t, z_{1:t}) P(X_t \mid z_{1:t})$$
$$= \underbrace{P(X_{t+1:t+k} \mid X_t)}_{Motion\ \mathrm{mod}\, el} \underbrace{P(X_t \mid z_{1:t})}_{filtering}$$

(Markov chain)

$$P(X_{t+1:t+k} \mid X_t)$$
$$= P(X_{t+k} \mid X_{t+k-1}) \ldots P(X_{t+2} \mid X_{t+1}) P(X_{t+1} \mid X_t)$$

(Markov chain)

Inference—Prediction

- EX:

$$P(Z_{t_0}=2 \mid x_{t_0}=o)=0.9 \qquad P(Z_{t_0}=2 \mid x_{t_0}=c)=0.2$$

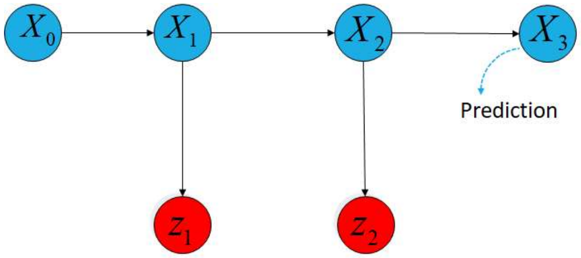
$$P(x_{t_0}=o)=P(x_{t_0}=c)=0.5 \text{ (by experiments)}$$

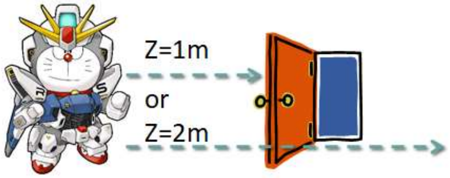
$$\left(Z_{t_1}, Z_{t_2}\right)=\{2,2\}$$

$$P(x_t=o \mid x_{t-1}=o)=P(x_t=c \mid x_{t-1}=c)=0.7$$

$$P(x_t=o \mid x_{t-1}=c)=P(x_t=c \mid x_{t-1}=o)=0.3$$

$$P\left(x_3 \mid Z_{1:2}=\{2,2\}\right)=?$$





t 3, Z=?

Inference—Prediction

Given: $P(Z_t = 2 \mid x_t = o) = 0.9 \qquad P(Z_t = 2 \mid x_t = c) = 0.2$

$$P(x_t = o \mid x_{t-1} = o) = P(x_t = c \mid x_{t-1} = c) = 0.7$$

$$P(x_t = o \mid x_{t-1} = c) = P(x_t = c \mid x_{t-1} = o) = 0.3$$

Transition matrix

$$P(x_t \mid x_{t-1}, z_{t-1}) = \int P(x_t \mid x_{t-1}) P(x_{t-1} \mid z_{t-1}) dx_{t-1}$$

$$\begin{cases} P(x_t = o \mid x_{t-1}, z_{t-1}) = \underset{0.7}{P(x_t = o \mid x_{t-1} = o)}\,\underset{0.883}{P(x_{t-1} = o \mid z_{t-1})} + \underset{0.3}{P(x_t = o \mid x_{t-1} = c)}\,\underset{0.117}{P(x_{t-1} = c \mid z_{t-1})} = 0.653 \\ P(x_t = c \mid x_{t-1}, z_{t-1}) = \underset{0.3}{P(x_t = c \mid x_{t-1} = o)}\,\underset{0.883}{P(x_{t-1} = o \mid z_{t-1})} + \underset{0.7}{P(x_t = c \mid x_{t-1} = c)}\,\underset{0.117}{P(x_{t-1} = c \mid z_{t-1})} = 0.347 \end{cases}$$

Inference—Smoothing

- Smoothing $P(X_k \mid z_{1:t}), 0 \le k \le t$

$$P(X_k \mid z_{1:t}) = P(X_k \mid z_{1:k}, z_{k+1:t})$$

$$= \eta P(X_k \mid z_{1:k}) P(z_{k+1:t} \mid X_k, z_{1:k})$$

$$= \eta \underbrace{P(X_k \mid z_{1:k})}_{\text{filtering}} \underbrace{P(z_{k+1:t} \mid X_k)}_{\text{backward}}$$

P(A|B,C)
=P(B|A,C)**P(A|C)**/P(B|C)

(Sensor independence)

$$P(z_{k+1:t} \mid X_k) = \sum_{x_{k+1}} P(z_{k+1:t} \mid X_k, x_{k+1}) P(x_{k+1} \mid X_k)$$

(Total probability)

$$= \sum_{x_{k+1}} P(z_{k+1:t} \mid x_{k+1}) P(x_{k+1} \mid X_k)$$

$$= \sum_{x_{k+1}} P(z_{k+1}, z_{k+2:t} \mid x_{k+1}) P(x_{k+1} \mid X_k)$$

(Sensor independence)

$$= \sum_{x_{k+1}} \underbrace{P(z_{k+1} \mid x_{k+1})}_{\text{sensor model}} \underbrace{P(z_{k+2:t} \mid x_{k+1})}_{\text{recursive call}} \underbrace{P(x_{k+1} \mid X_k)}_{\text{motion model}}$$

P(A,B|C)
=P(A|B,C)P(B|C)

Inference—Smoothing

- EX:

$$P(Z_{t_0}=2\mid x_{t_0}=o)=0.9 \qquad P(Z_{t_0}=2\mid x_{t_0}=c)=0.2$$

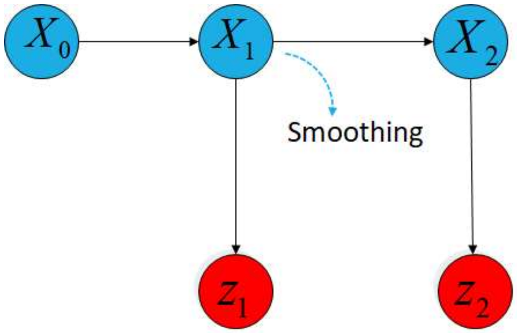
$$P(x_{t_0}=o)=P(x_{t_0}=c)=0.5\ \text{(by experiments)}$$

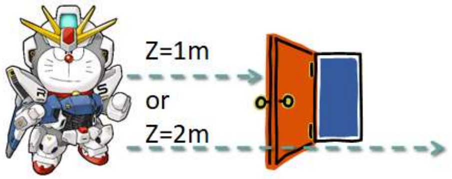
$$\left(Z_{t_1},Z_{t_2}\right)=\{2,2\}$$

$$P(x_t=o\mid x_{t-1}=o)=P(x_t=c\mid x_{t-1}=c)=0.7$$

$$P(x_t=o\mid x_{t-1}=c)=P(x_t=c\mid x_{t-1}=o)=0.3$$

$$P\left(x_1\mid Z_{1:2}=\{2,2\}\right)=?$$





Inference—Smoothing

- EX: $\boxed{P\left(x_1 \mid Z_{1:2} = \{2,2\}\right) = ?}$

Given: $P(Z_t = 2 \mid x_t = o) = 0.9 \qquad P(Z_t = 2 \mid x_t = c) = 0.2$

$P(x_t = o \mid x_{t-1} = o) = P(x_t = c \mid x_{t-1} = c) = 0.7$

$P(x_t = o \mid x_{t-1} = c) = P(x_t = c \mid x_{t-1} = o) = 0.3$

Transition matrix

$$P(X_k \mid z_{1:t}) = \eta P(X_k \mid z_{1:k}) P(z_{k+1:t} \mid X_k)$$

$$P(x_1 \mid z_{1:2}) = \eta \underbrace{P(x_1 \mid z_1)}_{\text{filtering}} \underbrace{P(z_2 \mid x_1)}_{\text{backward}}$$

$$P(x_1 \mid z_1) = \begin{bmatrix} 0.818 \\ 0.182 \end{bmatrix}$$

Inference—Smoothing

- EX:

$$P(z_{k+1:t} \mid X_k) = \sum_{x_{k+1}} \underbrace{P(z_{k+1} \mid x_{k+1})}_{\text{sensor model}} \underbrace{P(z_{k+2:t} \mid x_{k+1})}_{\text{recursive call}} \underbrace{P(x_{k+1} \mid X_k)}_{\text{motion model}}$$

$$P(z_2 \mid x_1) = \sum_{x_2} \underbrace{P(z_2 \mid x_2)}_{\text{sensor model}} \underbrace{P(z_{3:2} \mid x_2)}_{\text{recursive call}} \underbrace{P(x_2 \mid x_1)}_{\text{motion model}}$$

$$= P(z_2 \mid x_2 = o)P(z_{3:2} \mid x_2 = o)P(x_2 = o \mid x_1) + P(z_2 \mid x_2 = c)P(z_{3:2} \mid x_2 = c)P(x_2 = c \mid x_1)$$

$$= 0.9 \times 1 \times \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} + 0.2 \times 1 \times \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.69 \\ 0.41 \end{bmatrix}$$

$$\boxed{\begin{array}{l} P(z_{3:2} \mid x_2 = o) = ? \\ P(z_{3:2} \mid x_2 = c) = ? \end{array}}$$

Inference—Smoothing

- EX:

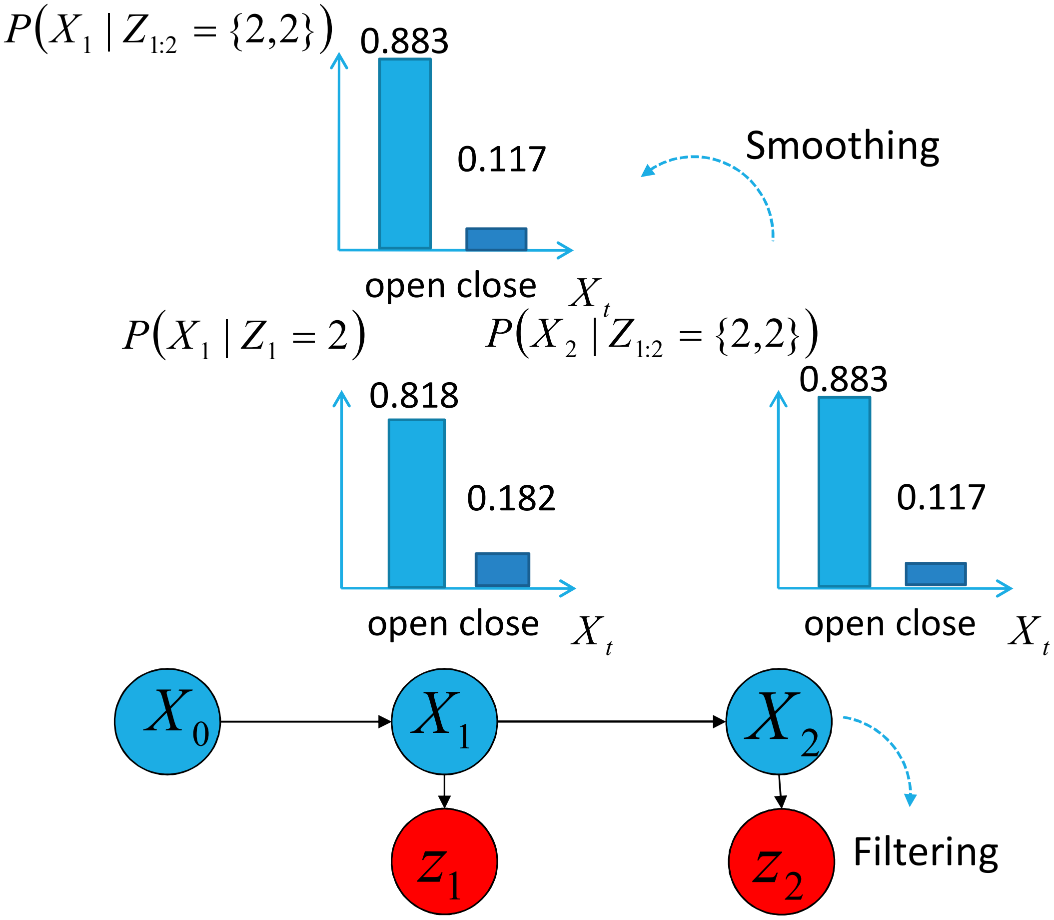
$$P(X_k \mid z_{1:t}) = \eta P(X_k \mid z_{1:k}) P(z_{k+1:t} \mid X_k)$$

$$P(x_1 \mid z_{1:2}) = \eta \underbrace{P(x_1 \mid z_1)}_{\text{filtering}} \underbrace{P(z_2 \mid x_1)}_{\text{backward}}$$

$$P(x_1 \mid z_1) = \begin{bmatrix} 0.818 \\ 0.182 \end{bmatrix} \qquad P(z_2 \mid x_1) = \begin{bmatrix} 0.69 \\ 0.41 \end{bmatrix}$$

$$P(x_1 \mid z_{1:2}) = \eta \begin{bmatrix} 0.818 \\ 0.182 \end{bmatrix} \begin{bmatrix} 0.69 \\ 0.41 \end{bmatrix} = \begin{bmatrix} 0.883 \\ 0.117 \end{bmatrix}$$

Inference—Smoothing



Inference—Smoothing

- Forward–backward algorithm: the smoothing equations can be reformulated as forward and backward messages.

$$P(X_{t+1} \mid z_{1:t+1}) = \eta P(z_{t+1} \mid X_{t+1}) \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t \mid z_{1:t})$$

$$\Rightarrow \mathbf{f}_{1:t+1} = \eta FORWARD(\mathbf{f}_{1:t}, z_{t+1})$$

Given

$$(1)\ \mathbf{f}_{1:0} = P(X_0)$$

$$(2)\ \text{sensor and motion models} \quad \Rightarrow \quad P(X_{t+1} \mid z_{1:t+1})$$

$$(3)\ z_{1:t+1}$$

The probability is like a forward message!

Inference—Smoothing

- Forward–backward algorithm: the smoothing equations can be reformulated as forward and backward messages.

$$P(z_{k+1:t} \mid X_k) = \sum_{x_{k+1}} \underbrace{P(z_{k+1} \mid x_{k+1})}_{\text{sensor model}} \underbrace{P(z_{k+2:t} \mid x_{k+1})}_{\text{recursive call}} \underbrace{P(x_{k+1} \mid X_k)}_{\text{motion model}}$$

$$\Rightarrow \mathbf{b}_{k+1:t} = BACKWARD(\mathbf{b}_{k+2:t}, z_{k+1})$$

Given

(1) sensor and motion models $\Rightarrow$ $P(z_{k+1:t} \mid X_k)$

(2) $z_{k+1:t}$

The probability is like a backward message!

Inference—Smoothing

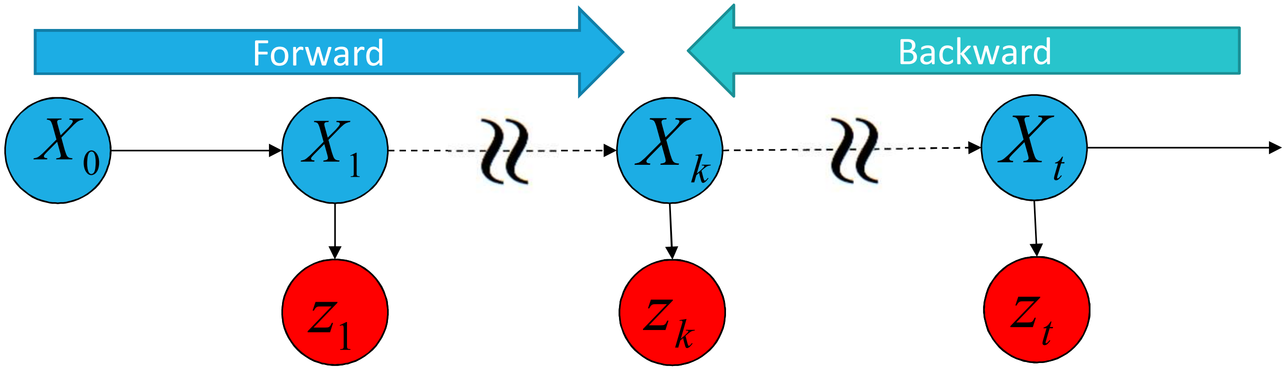
- Forward–backward algorithm: the smoothing equations can be reformulated as forward and backward messages.

$$P(X_{t+1} \mid z_{1:t+1}) \Rightarrow \mathbf{f}_{1:t+1} = \eta FORWARD(\mathbf{f}_{1:t}, z_{t+1})$$

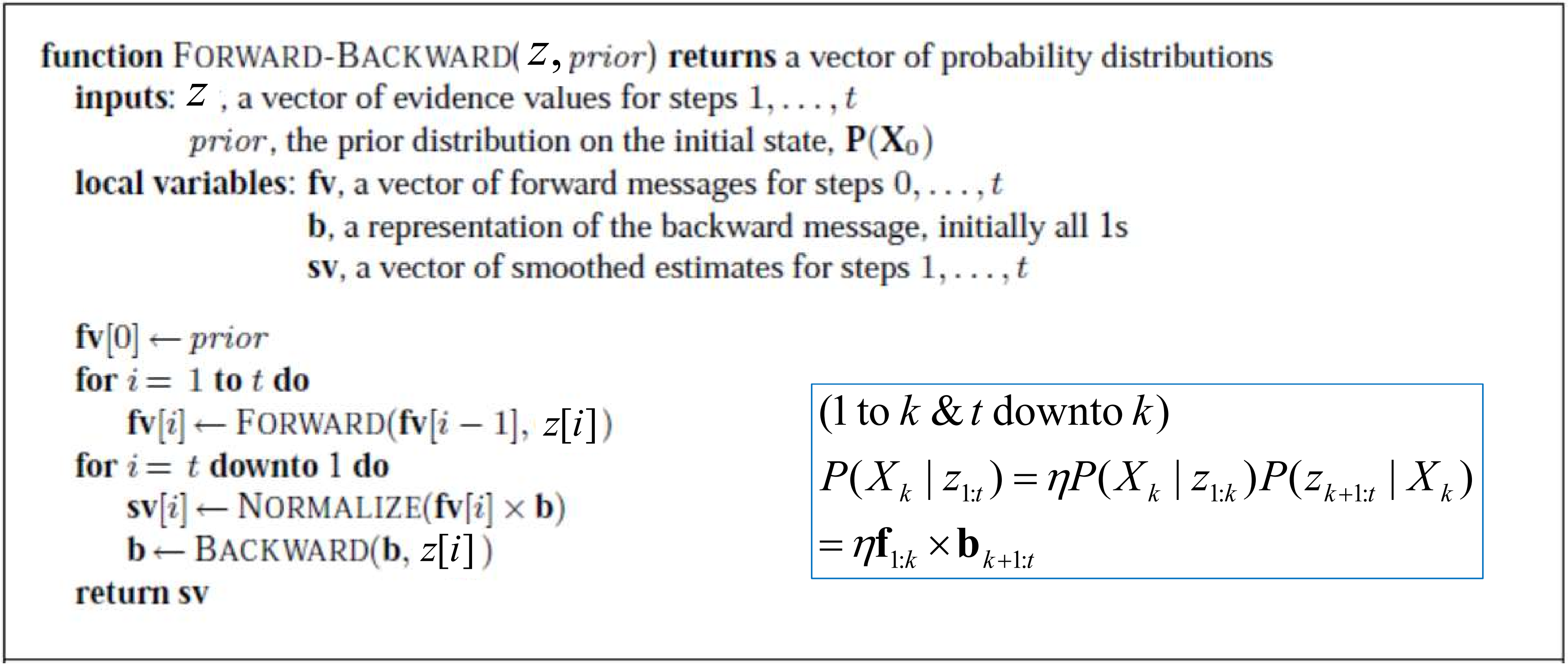
$$P(z_{k+1:t} \mid X_k) \Rightarrow \mathbf{b}_{k+1:t} = BACKWARD(\mathbf{b}_{k+2:t}, z_{k+1})$$

$$P(X_k \mid z_{1:t}) = \eta P(X_k \mid z_{1:k}) P(z_{k+1:t} \mid X_k)$$

$$= \eta \mathbf{f}_{1:k} \times \mathbf{b}_{k+1:t}$$



Inference—Smoothing



```
function FORWARD-BACKWARD($z$, prior) returns a vector of probability distributions
    inputs: $z$, a vector of evidence values for steps $1, \ldots, t$
            prior, the prior distribution on the initial state, $\mathbf{P}(\mathbf{X}_0)$
    local variables: fv, a vector of forward messages for steps $0, \ldots, t$
                     b, a representation of the backward message, initially all 1s
                     sv, a vector of smoothed estimates for steps $1, \ldots, t$

    fv[0] ← prior
    for $i = 1$ to $t$ do
        fv[$i$] ← FORWARD(fv[$i-1$], $z[i]$)
    for $i = t$ downto 1 do
        sv[$i$] ← NORMALIZE(fv[$i$] × b)
        b ← BACKWARD(b, $z[i]$)
    return sv
```

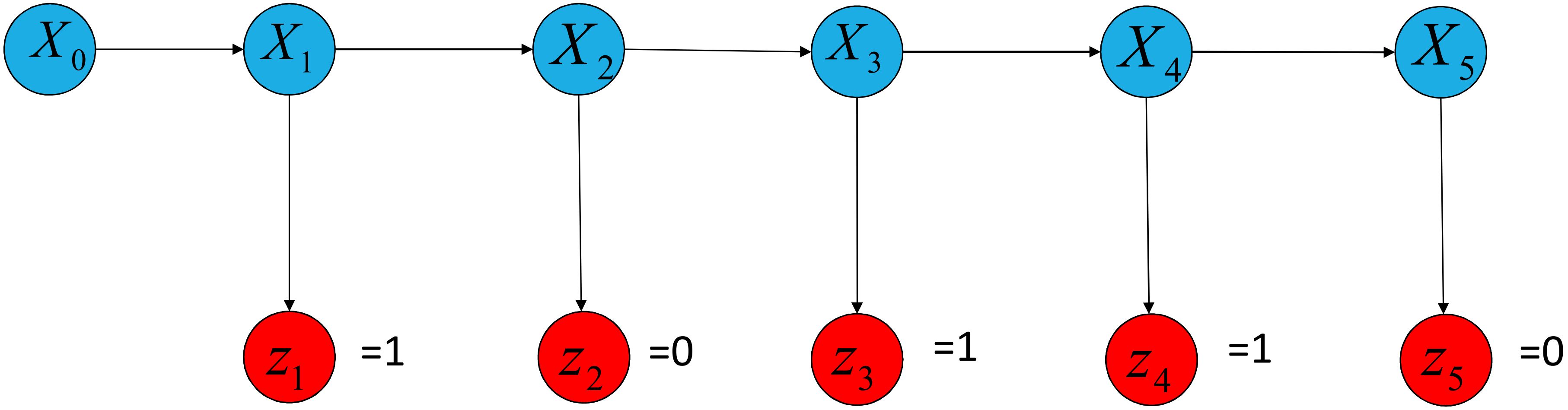
(1 to k & t downto k)

$$P(X_k \mid z_{1:t}) = \eta P(X_k \mid z_{1:k}) P(z_{k+1:t} \mid X_k)$$

$$= \eta \mathbf{f}_{1:k} \times \mathbf{b}_{k+1:t}$$

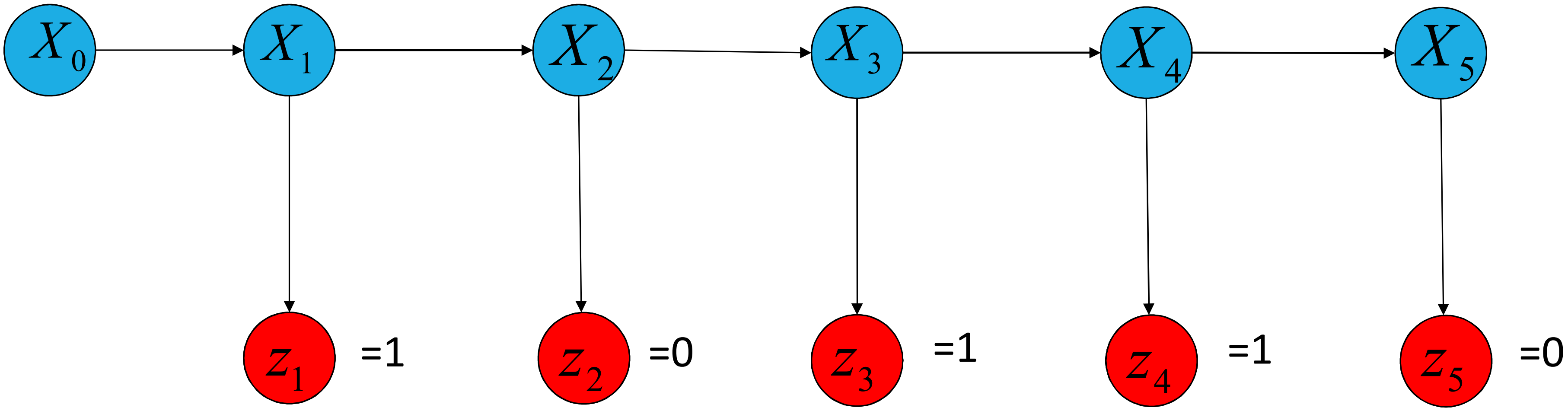
The Most Likely Sequence— Viterbi algorithm

- The Most Likely Sequence $\underset{x_{1:t}}{\arg\max}\, P(x_{1:t} \mid z_{1:t})$
- Given a sequential measurement, what’s the most likely states?



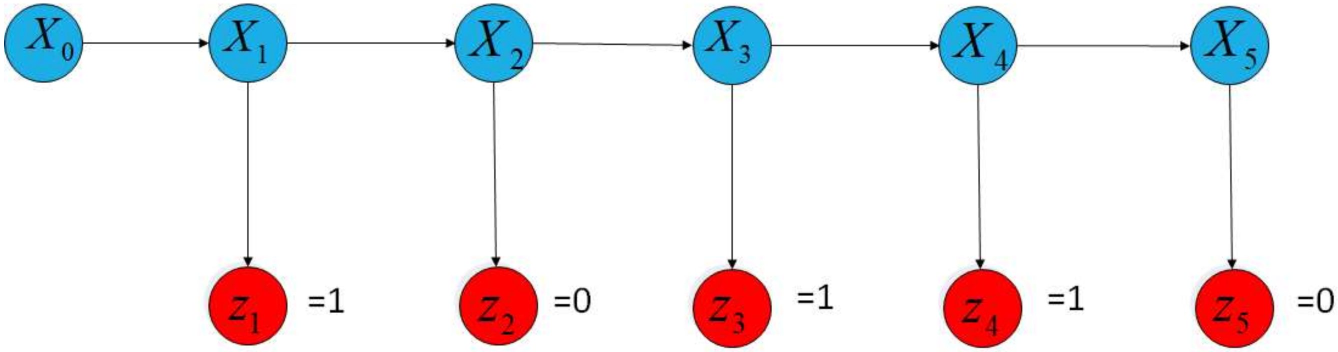
The Most Likely Sequence— Viterbi algorithm

- Viterbi algorithm is applied to “decode” noisy data.
- For example,
- The receiver got “0 1 0 1 0”. What’s the most likely sequence?
- The DNA sensor got “A G C T G”. What’s the most likely sequence?



The Most Likely Sequence— Viterbi algorithm

- To find the most likely sequence, there are 2^5 possible sequence.
- We can adopt smoothing technology but it only consider one time step. If we try to find the most likely sequence, it will involved in jointly distribution over time.
- Prof. Andrew Viterbi proposed an algorithm for it based on Markov property. (finding the most likely path over a graph)



The Most Likely Sequence— Viterbi algorithm

- The Viterbi algorithm is similar to the filtering algorithm, except

1. The forward message $\mathbf{f}_{1:t} = P(X_t \mid z_{1:t})$ is replaced by the message

$$\mathbf{m}_{1:t} = \max_{x_1 \ldots x_{t-1}} P(x_1, \ldots, x_{t-1}, X_t \mid z_{1:t})$$

2. The summation over x_t in filtering is replaced by the maximization over x_t in viterbi.

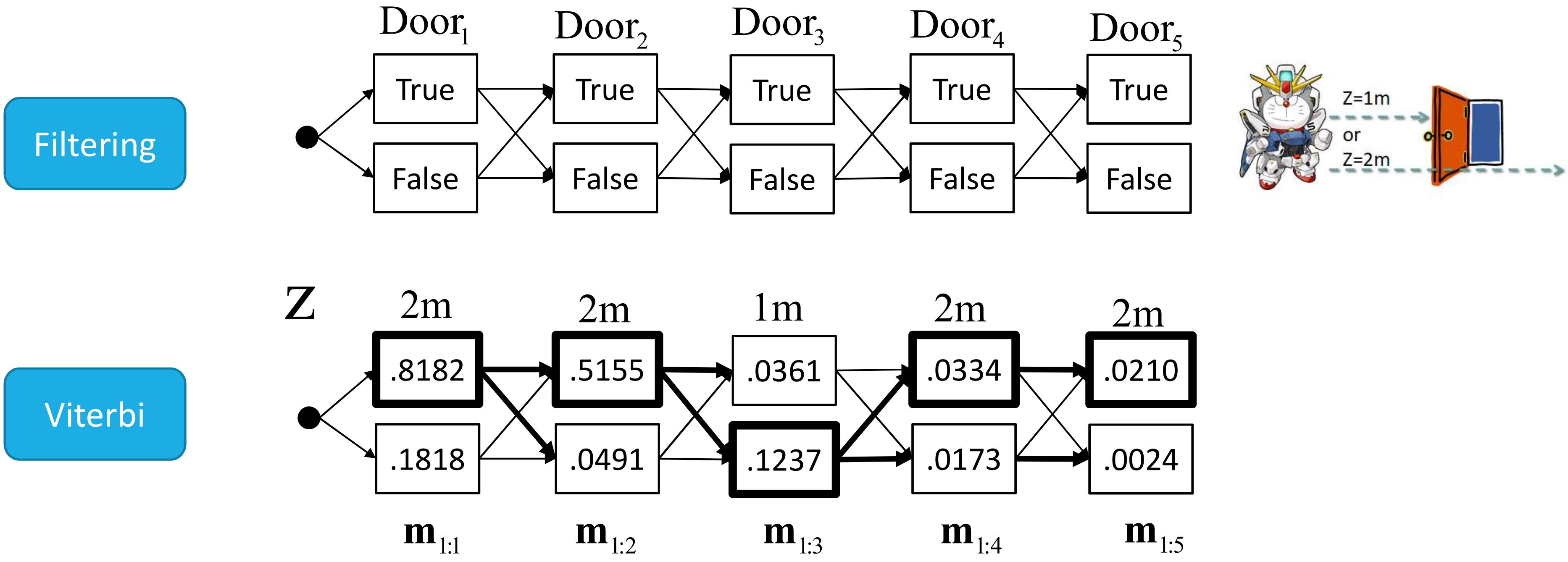
$$P(X_{t+1} \mid z_{1:t+1}) = \eta P(z_{t+1} \mid X_{t+1}) \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t \mid z_{1:t}) \ldots (filtering)$$

$$\max_{x_1 \ldots x_t} P(x_1, \ldots, x_t, X_{t+1} \mid z_{1:t+1})$$

$$= \eta P(z_{t+1} \mid X_{t+1}) \max_{x_t} \left(P(X_{t+1} \mid x_t) \max_{x_1 \ldots x_{t-1}} P(x_1, \ldots, x_t \mid z_{1:t}) \right) \ldots (viterbi)$$

The Most Likely Sequence— Viterbi algorithm

- An illustration of Viterbi algorithm



The Most Likely Sequence— Viterbi algorithm

Given: $P(Z_t = 2 \mid x_t = o) = 0.9 \qquad P(Z_t = 2 \mid x_t = c) = 0.2$

$$\boxed{\begin{array}{l} P(x_t = o \mid x_{t-1} = o) = P(x_t = c \mid x_{t-1} = c) = 0.7 \\ P(x_t = o \mid x_{t-1} = c) = P(x_t = c \mid x_{t-1} = o) = 0.3 \end{array}}$$

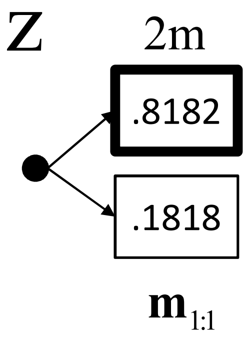
Transition matrix

$$P(x_t \mid x_{t-1}, z_{t-1}) = \int P(x_t \mid x_{t-1}) P(x_{t-1} \mid z_{t-1}) dx_{t-1}$$

$$\begin{cases} P(x_t = o \mid x_{t-1}, z_{t-1}) = \underset{0.7}{P(x_t = o \mid x_{t-1} = o)}\,\underset{0.5}{P(x_{t-1} = o \mid z_{t-1})} + \underset{0.3}{P(x_t = o \mid x_{t-1} = c)}\,\underset{0.5}{P(x_{t-1} = c \mid z_{t-1})} = 0.5 \\ P(x_t = c \mid x_{t-1}, z_{t-1}) = \underset{0.3}{P(x_t = c \mid x_{t-1} = o)}\,\underset{0.5}{P(x_{t-1} = o \mid z_{t-1})} + \underset{0.7}{P(x_t = c \mid x_{t-1} = c)}\,\underset{0.5}{P(x_{t-1} = c \mid z_{t-1})} = 0.5 \end{cases}$$

$$P(x_t \mid z_t) = \eta \bullet P(z_t \mid x_t) P(x_t \mid x_{t-1}, z_{t-1})$$

$$\begin{cases} P(x_t = o \mid z_t) = \eta \bullet \underset{0.9}{P(z_t \mid x_t = o)}\,\underset{0.5}{P(x_t = o \mid x_{t-1}, z_{t-1})} = 0.818 \\ P(x_t = c \mid z_t) = \eta \bullet \underset{0.2}{P(z_t \mid x_t = c)}\,\underset{0.5}{P(x_t = c \mid x_{t-1}, z_{t-1})} = 0.182 \end{cases}$$



The Most Likely Sequence— Viterbi algorithm

Given: $P(Z_t = 2 | x_t = o) = 0.9 \qquad P(Z_t = 2 | x_t = c) = 0.2$

$$\boxed{\begin{array}{l} P(x_t = o | x_{t-1} = o) = P(x_t = c | x_{t-1} = c) = 0.7 \\ P(x_t = o | x_{t-1} = c) = P(x_t = c | x_{t-1} = o) = 0.3 \end{array}}$$

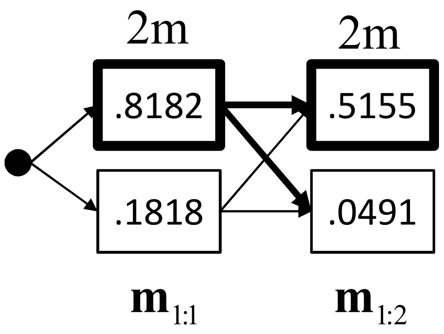
Transition matrix

$$P(x_t | x_{t-1}, z_{t-1}) = \max_{x_t} P(x_t | x_{t-1}) P(x_{t-1} | z_{t-1})$$

$$\begin{cases} P(x_t = o | x_{t-1}, z_{t-1}) = \underline{\max}[\underset{0.7}{P(x_t = o | x_{t-1} = o)}\underset{0.818}{P(x_{t-1} = o | z_{t-1})}, \underset{0.3}{P(x_t = o | x_{t-1} = c)}\underset{0.182}{P(x_{t-1} = c | z_{t-1})}] = 0.5727 \\ P(x_t = c | x_{t-1}, z_{t-1}) = \underline{\max}[\underset{0.3}{P(x_t = c | x_{t-1} = o)}\underset{0.818}{P(x_{t-1} = o | z_{t-1})}, \underset{0.7}{P(x_t = c | x_{t-1} = c)}\underset{0.182}{P(x_{t-1} = c | z_{t-1})}] = 0.2454 \end{cases}$$

$$P(x_t | z_t) = \eta \bullet P(z_t | x_t) P(x_t | x_{t-1}, z_{t-1})$$

$$\begin{cases} P(x_t = o | z_t) = \eta \bullet \underset{0.9}{P(z_t | x_t = o)}\underset{0.5727}{P(x_t = o | x_{t-1}, z_{t-1})} = 0.5155\eta \\ P(x_t = c | z_t) = \eta \bullet \underset{0.2}{P(z_t | x_t = c)}\underset{0.2454}{P(x_t = c | x_{t-1}, z_{t-1})} = 0.0491\eta \end{cases}$$



The Most Likely Sequence— Viterbi algorithm

Given: $P(Z_t=2\,|\,x_t=o)=0.9 \qquad P(Z_t=2\,|\,x_t=c)=0.2$

$$\boxed{\begin{aligned}&P(x_t=o\,|\,x_{t-1}=o)=P(x_t=c\,|\,x_{t-1}=c)=0.7\\&P(x_t=o\,|\,x_{t-1}=c)=P(x_t=c\,|\,x_{t-1}=o)=0.3\end{aligned}}$$

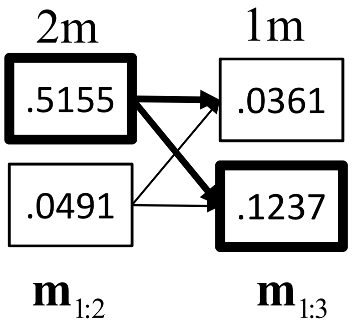
Transition matrix

$$P(x_t\,|\,x_{t-1},z_{t-1})=\max_{x_t} P(x_t\,|\,x_{t-1})P(x_{t-1}\,|\,z_{t-1})$$

$$\begin{cases}P(x_t=o\,|\,x_{t-1},z_{t-1})=\underline{\max}[\underset{0.7}{P(x_t=o\,|\,x_{t-1}=o)}\underset{0.5155}{P(x_{t-1}=o\,|\,z_{t-1})},\underset{0.3}{P(x_t=o\,|\,x_{t-1}=c)}\underset{0.0491}{P(x_{t-1}=c\,|\,z_{t-1})}]=0.3608\\ P(x_t=c\,|\,x_{t-1},z_{t-1})=\underline{\max}[\underset{0.3}{P(x_t=c\,|\,x_{t-1}=o)}\underset{0.5155}{P(x_{t-1}=o\,|\,z_{t-1})},\underset{0.7}{P(x_t=c\,|\,x_{t-1}=c)}\underset{0.0491}{P(x_{t-1}=c\,|\,z_{t-1})}]=0.1546\end{cases}$$

$$P(x_t\,|\,z_t)=\eta\bullet P(z_t\,|\,x_t)P(x_t\,|\,x_{t-1},z_{t-1})$$

$$\begin{cases}P(x_t=o\,|\,z_t)=\eta\bullet \underset{\mathbf{0.1}}{P(z_t\,|\,x_t=o)}\underset{0.5727}{P(x_t=o\,|\,x_{t-1},z_{t-1})}=0.0361\eta\\ P(x_t=c\,|\,z_t)=\eta\bullet \underset{\mathbf{0.8}}{P(z_t\,|\,x_t=c)}\underset{0.2454}{P(x_t=c\,|\,x_{t-1},z_{t-1})}=0.1237\eta\end{cases}$$



The Most Likely Sequence— Viterbi algorithm

Given: $P(Z_t = 2 \mid x_t = o) = 0.9 \qquad P(Z_t = 2 \mid x_t = c) = 0.2$

$$\boxed{\begin{aligned} P(x_t = o \mid x_{t-1} = o) = P(x_t = c \mid x_{t-1} = c) = 0.7 \\ P(x_t = o \mid x_{t-1} = c) = P(x_t = c \mid x_{t-1} = o) = 0.3 \end{aligned}}$$

Transition matrix

$$P(x_t \mid x_{t-1}, z_{t-1}) = \max_{x_t} P(x_t \mid x_{t-1}) P(x_{t-1} \mid z_{t-1})$$

$$\begin{cases} P(x_t = o \mid x_{t-1}, z_{t-1}) = \underline{\max}[P(x_t = o \mid x_{t-1} = o) P(x_{t-1} = o \mid z_{t-1}), P(x_t = o \mid x_{t-1} = c) P(x_{t-1} = c \mid z_{t-1})] = 0.0371 \\ P(x_t = c \mid x_{t-1}, z_{t-1}) = \underline{\max}[P(x_t = c \mid x_{t-1} = o) P(x_{t-1} = o \mid z_{t-1}), P(x_t = c \mid x_{t-1} = c) P(x_{t-1} = c \mid z_{t-1})] = 0.0865 \end{cases}$$

0.7 0.0361 0.3 0.1237

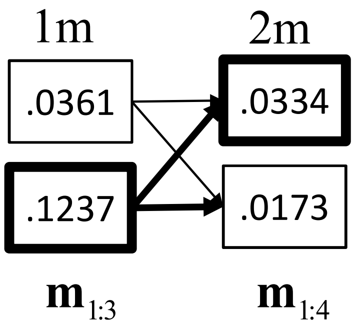
0.3 0.0361 0.7 0.1237

$$P(x_t \mid z_t) = \eta \bullet P(z_t \mid x_t) P(x_t \mid x_{t-1}, z_{t-1})$$

$$\begin{cases} P(x_t = o \mid z_t) = \eta \bullet P(z_t \mid x_t = o) P(x_t = o \mid x_{t-1}, z_{t-1}) = 0.0334\eta \\ P(x_t = c \mid z_t) = \eta \bullet P(z_t \mid x_t = c) P(x_t = c \mid x_{t-1}, z_{t-1}) = 0.0173\eta \end{cases}$$

0.9 0.0371

0.2 0.0865



The Most Likely Sequence— Viterbi algorithm

Given: $P(Z_t=2\,|\,x_t=o)=0.9 \qquad P(Z_t=2\,|\,x_t=c)=0.2$

$$\boxed{\begin{aligned} &P(x_t=o\,|\,x_{t-1}=o)=P(x_t=c\,|\,x_{t-1}=c)=0.7\\ &P(x_t=o\,|\,x_{t-1}=c)=P(x_t=c\,|\,x_{t-1}=o)=0.3\end{aligned}}$$

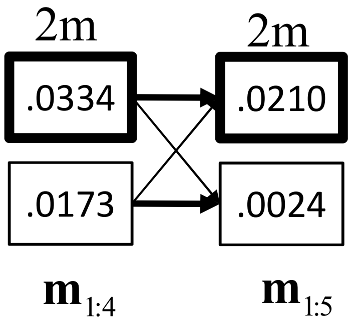
Transition matrix

$$P(x_t\,|\,x_{t-1},z_{t-1})=\max_{x_t} P(x_t\,|\,x_{t-1})P(x_{t-1}\,|\,z_{t-1})$$

$$\begin{cases} P(x_t=o\,|\,x_{t-1},z_{t-1})=\underline{\max}[\underset{0.7}{P(x_t=o\,|\,x_{t-1}=o)}\underset{0.0334}{P(x_{t-1}=o\,|\,z_{t-1})},\underset{0.3}{P(x_t=o\,|\,x_{t-1}=c)}\underset{0.0173}{P(x_{t-1}=c\,|\,z_{t-1})}]=0.0233\\ P(x_t=c\,|\,x_{t-1},z_{t-1})=\underline{\max}[\underset{0.3}{P(x_t=c\,|\,x_{t-1}=o)}\underset{0.0334}{P(x_{t-1}=o\,|\,z_{t-1})},\underset{0.7}{P(x_t=c\,|\,x_{t-1}=c)}\underset{0.0173}{P(x_{t-1}=c\,|\,z_{t-1})}]=0.0121\end{cases}$$

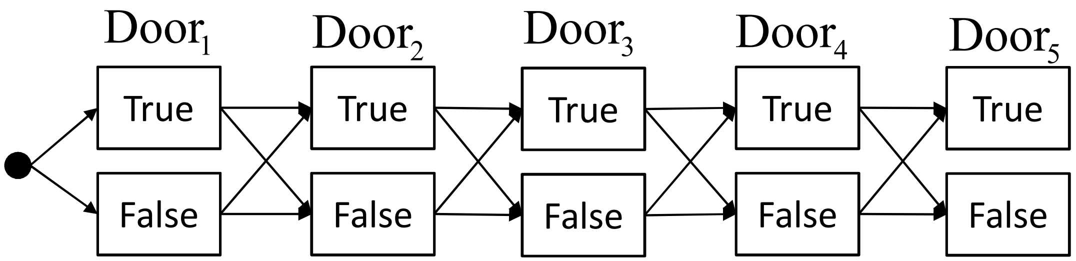
$$P(x_t\,|\,z_t)=\eta\bullet P(z_t\,|\,x_t)P(x_t\,|\,x_{t-1},z_{t-1})$$

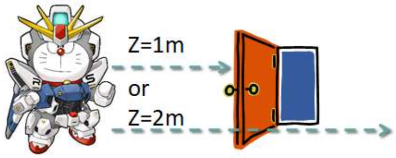
$$\begin{cases} P(x_t=o\,|\,z_t)=\eta\bullet \underset{\mathbf{0.9}}{P(z_t\,|\,x_t=o)}\underset{0.0233}{P(x_t=o\,|\,x_{t-1},z_{t-1})}=0.0210\eta\\ P(x_t=c\,|\,z_t)=\eta\bullet \underset{\mathbf{0.2}}{P(z_t\,|\,x_t=c)}\underset{0.0121}{P(x_t=c\,|\,x_{t-1},z_{t-1})}=0.0024\eta\end{cases}$$



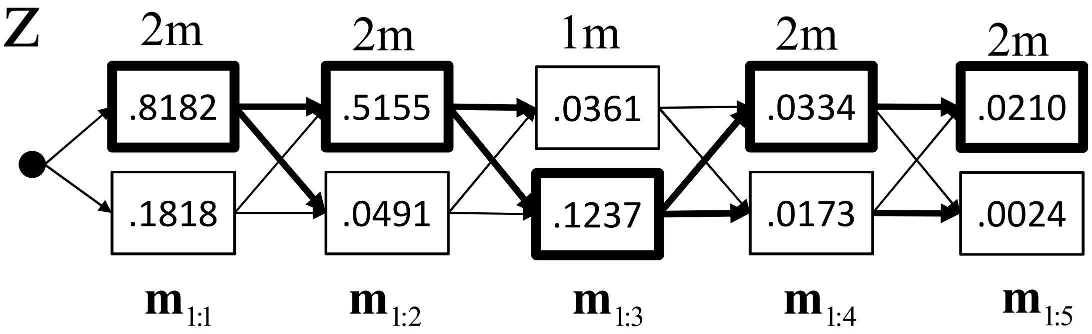
The Most Likely Sequence— Viterbi algorithm

Filtering





Viterbi

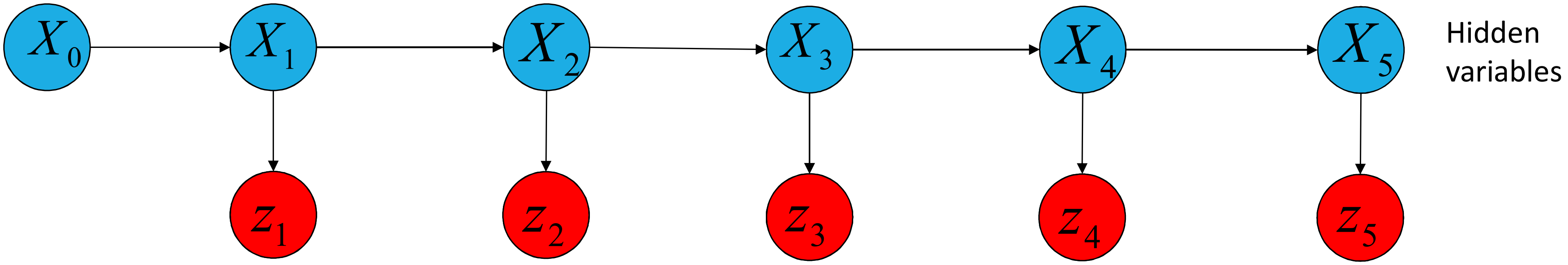


$$P(x_t \mid x_{t-1}, z_{t-1}) = \max_{x_t} P(x_t \mid x_{t-1}) P(x_{t-1} \mid z_{t-1})$$

$$P(x_t \mid z_t) = \eta \bullet P(z_t \mid x_t) P(x_t \mid x_{t-1}, z_{t-1})$$

Hidden Markov Models

- Hidden Markov model (HMM) is a temporal probabilistic model in which the state of the process is described by a single discrete random variable.
- Filtering, prediction, smoothing and Viterbi algorithms can be represented as a HMM.



Hidden Markov Models

Let the state variable $X_t \in \{1,\ldots,n\}$

The transition model $P(X_t \mid X_{t-1})$ is a $n \times n$ matrix $\mathbf{T}$,

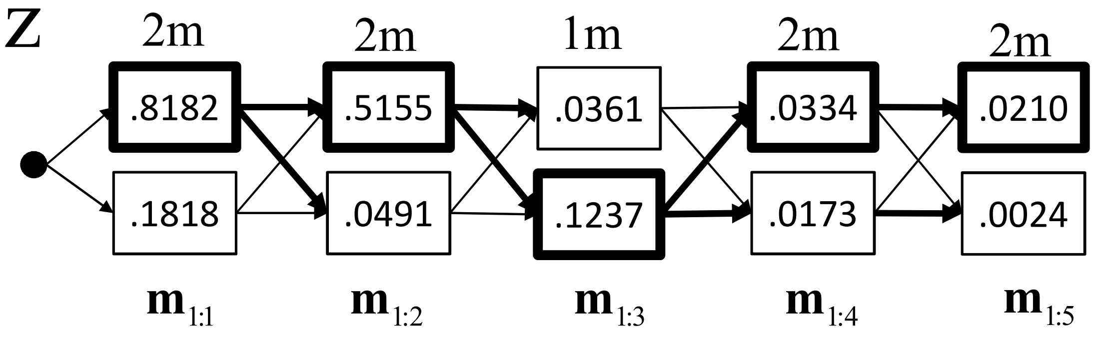
where $\mathbf{T}_{ij} = P(X_t = j \mid X_{t-1} = i)$

The sensor model $P(z_t \mid X_t)$ is a $n \times n$ matrix $\mathbf{O}_t$,

whose ith diagonal entry is $P(z_t \mid X_t = i)$ and others are 0.

$$\mathbf{T} = P(X_t \mid X_{t-1}) = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

$$\mathbf{O}_1 = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.2 \end{bmatrix}, \mathbf{O}_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.8 \end{bmatrix}$$



Hidden Markov Models

Let the state variable $X_t \in \{1,\ldots,n\}$

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The sensor model $P(z_t \mid X_t)$ is a $n \times n$ matrix $\mathbf{O}_t$,

whose ith diagonal entry is $P(z_t \mid X_t = i)$ and others are 0.

[illegible] $\begin{bmatrix} 0.9 & 0 \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} .7 & .3 \\ .3 & .7 \end{bmatrix} \begin{bmatrix} 0.5 \\ .5 \end{bmatrix}$

$$P(X_{t+1} \mid z_{1:t+1}) \Rightarrow \mathbf{f}_{1:t+1} = \eta FORWARD(\mathbf{f}_{1:t}, z_{t+1}) \quad \Longrightarrow \quad \mathbf{f}_{1:t+1} = \eta \mathbf{O}_{t+1} \mathbf{T}^T \mathbf{f}_{1:t}$$

$$P(z_{k+1:t} \mid X_k) \Rightarrow \mathbf{b}_{k+1:t} = BACKWARD(\mathbf{b}_{k+2:t}, z_{k+1}) \quad \Longrightarrow \quad \mathbf{b}_{k+1:t} = \mathbf{T}\mathbf{O}_{k+1}\mathbf{b}_{k+2:t}$$

$$P(z_{k+1:t} \mid X_k) = \sum_{x_{k+1}} \underbrace{P(z_{k+1} \mid x_{k+1})}_{\text{sensor model}} \underbrace{P(z_{k+2:t} \mid x_{k+1})}_{\text{recursive call}} \underbrace{P(x_{k+1} \mid X_k)}_{\text{motion model}}$$

$$\Rightarrow \mathbf{b}_{k+1:t} = BACKWARD(\mathbf{b}_{k+2:t}, z_{k+1})$$

EX: Gaze patterns

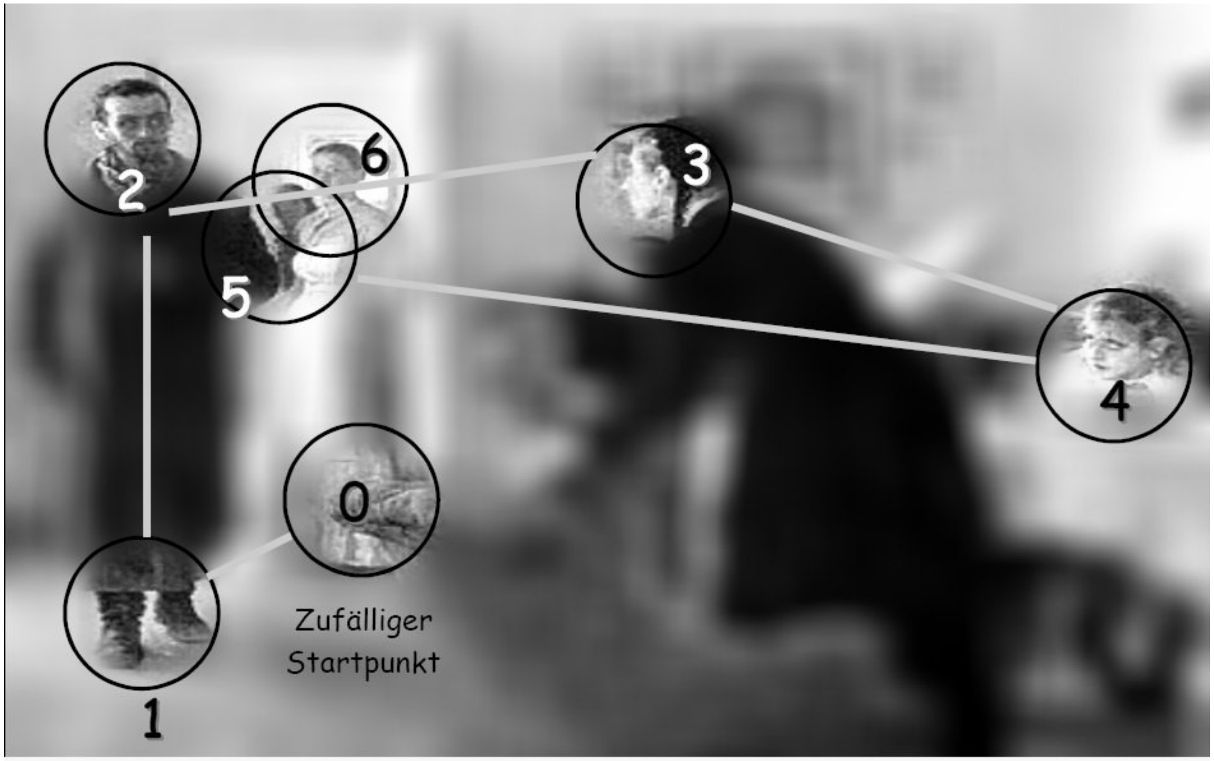


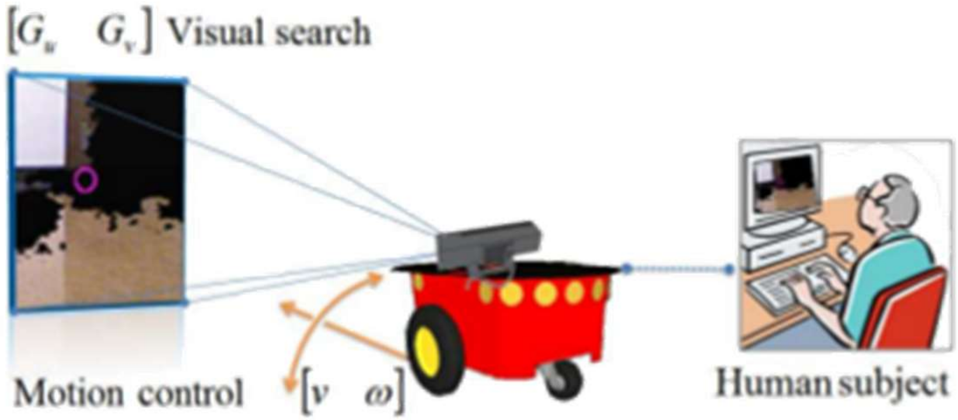
Bild 11: Foveale Ergänzung durch die ersten 6 Fixationen (nach Daten von *Yarbus, 1967*)

https://en.wikipedia.org/wiki/Eye_movement

1. Fixation
2. Smooth pursuit
3. Saccades

EX: Gaze patterns

- A human subject remotely control a robot to search for objects. The gaze tracker can detect where the subject is looking.
- We want to know the gaze patterns from noisy data.

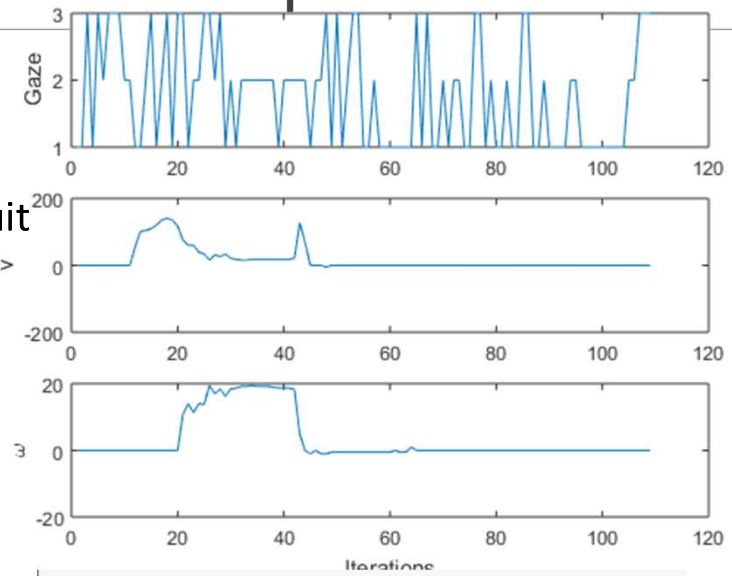


[1] Kuo-Shih Tseng and Bérénice Mettler, “Analysis of Coordination Patterns between Gaze and Control in Human Spatial Search”, 2nd IFAC Conference on Cyber-Physical and Human-Systems, 2018.

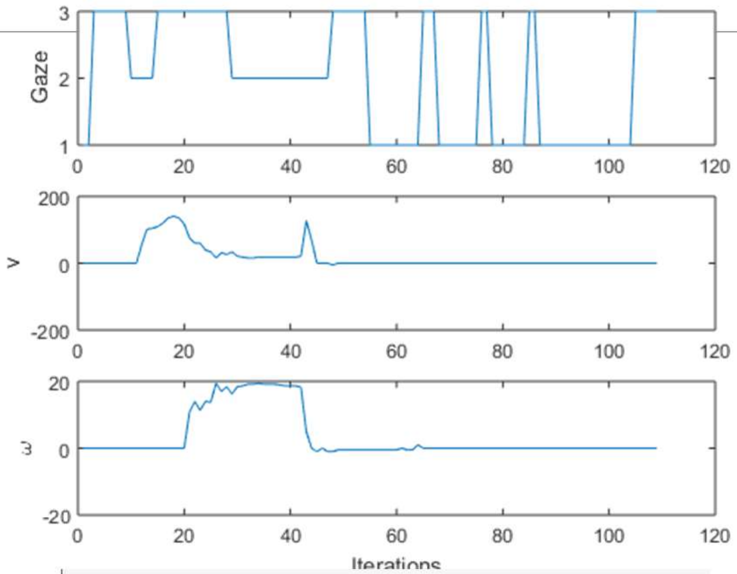
EX: Gaze patterns

Zt={1,2,3}

1. Fixation
2. Smooth pursuit
3. Saccades

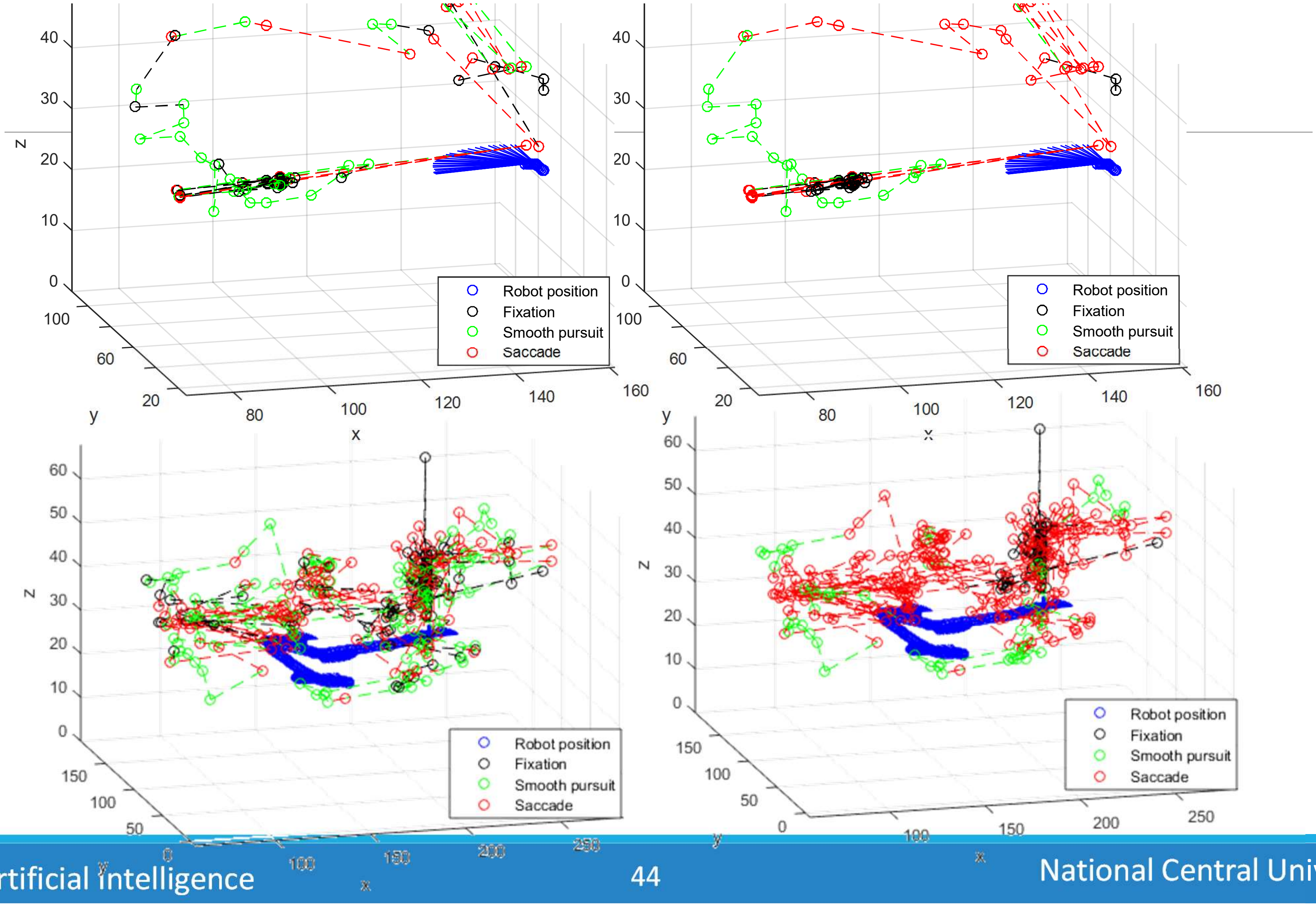


HMM+Viterbi



Xt={1,2,3}
Via
HMM+Viterbi

○ Robot position
○ Fixation
○ Smooth pursuit
○ Saccade


Robot position
Fixation
Smooth pursuit
Saccade
z
y
x
0
10
20
30
40
50
60
20
60
100
150
80
100
120
140
160
200
250

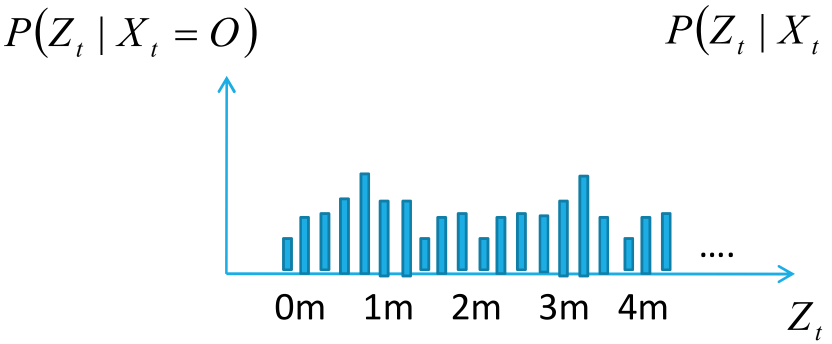
Kalman Filter and Particle Filter

- Prediction

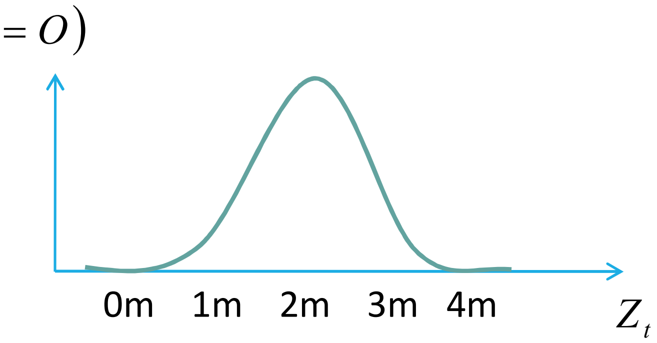
$$P(x_t \mid x_{t-1}, u_t, z_{t-1}) = \int P(x_t \mid x_{t-1}, u_t) P(x_{t-1} \mid z_{t-1}) dx_{t-1}$$

- Correction

$$P(x_t \mid z_t) = \eta \bullet P(z_t \mid x_t) P(x_t \mid x_{t-1}, u_t, z_{t-1})$$



Particle Filter 194X~



Kalman Filter 1960~

Kalman Filter and Particle Filter

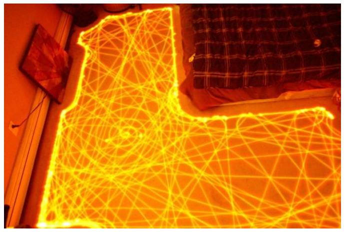
- Both of KF and PF can be applied to filtering, prediction and smoothing. In robotics, we usually adopt filtering since the robots needs to deal with real-time data.
- How to design KF and PF is the key to perception problems.
- If you are interested in perception problems, please take "MA3131: Introduction to data science"
- This course will cover
  - Bayesian estimation
  - Given sensors, how to fuse data to estimate the states.

State of the art perception technology

- SLAM for cleaner robots

2002 2006 2010 2016

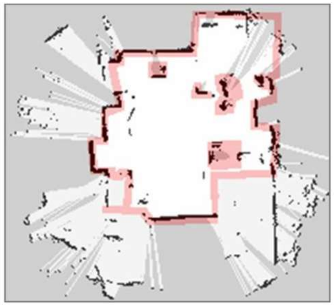
Roomba



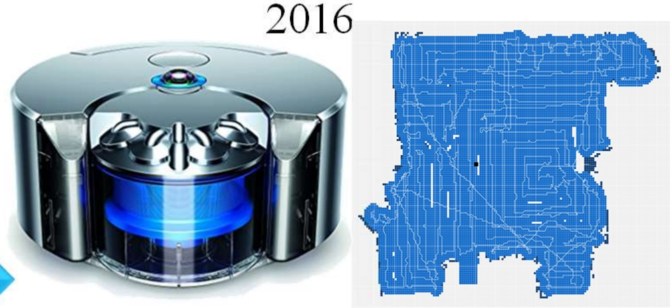
Path

Hokuyu laser

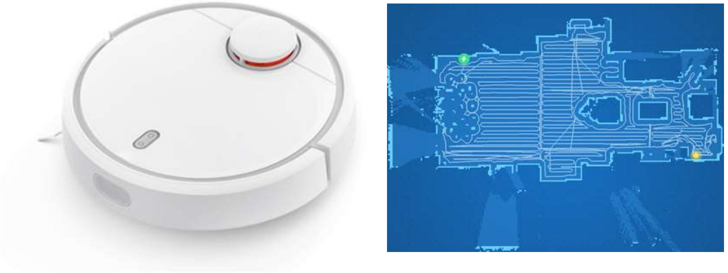
vSLAM



Laser-based SLAM



Dyson cleaner robot
iRobot cleaner robot



Mi cleaner robot

State of the art perception technology

- Localization for VR/AR head mounted device (HMD)— Meta 2

https://www.metavision.com/

Conclusion

Perception

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Decision

$$\frac{P(X = open \mid Z)}{P(X = close \mid Z)} > \frac{(R_{CC} - R_{OC})}{(R_{OO} - R_{CO})}$$

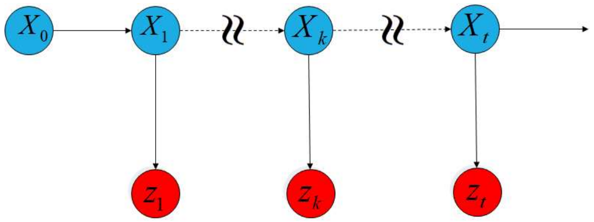
Feedback

$$\begin{cases} P(x_t \mid x_{t-1}, u_t, z_{t-1}) = \int P(x_t \mid x_{t-1}, u_t) P(x_{t-1} \mid z_{t-1}) dx_{t-1} \\ P(x_t \mid z_t) = \eta \bullet P(z_t \mid x_t) P(x_t \mid x_{t-1}, u_t, z_{t-1}) \end{cases}$$

Particle filter

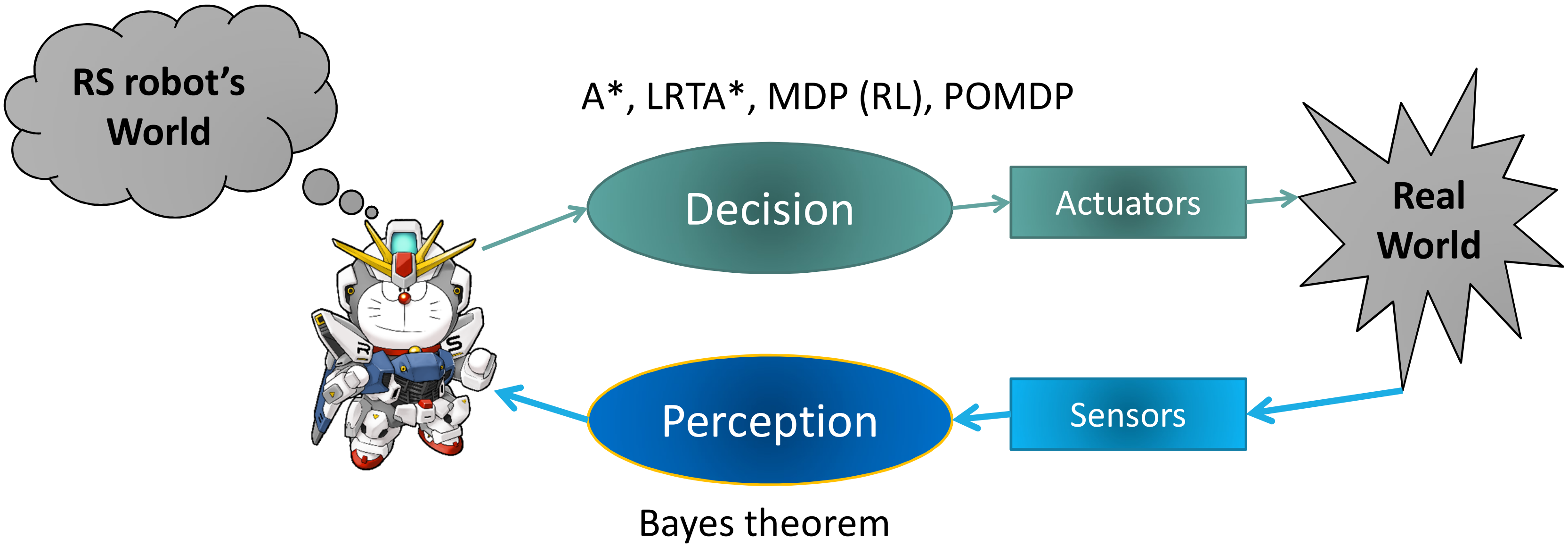
Kalman filter

Filtering, Prediction, Smoothing, Most likely explanation



Conclusion

- The challenge of perception problems is that ***nothing is true***. Robots only can estimate the states via an objective function (e.g., maximal likelihood or minimal mean square error etc.).
- The complexity of perception problems is polynomial time.



RS robot's World

A*, LRTA*, MDP (RL), POMDP

Decision

Actuators

Real World

Sensors

Perception

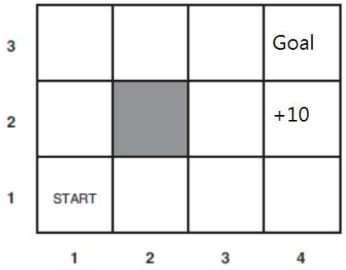
Bayes theorem

Q&A

LRTA* and Reinforcement Learning

- LRTA*

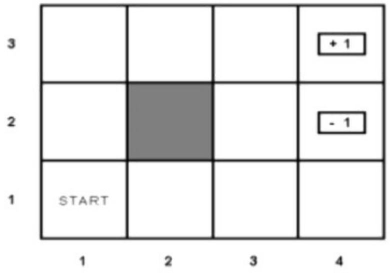
Deterministic action



$s, a \rightarrow s'$

MDP (RL)

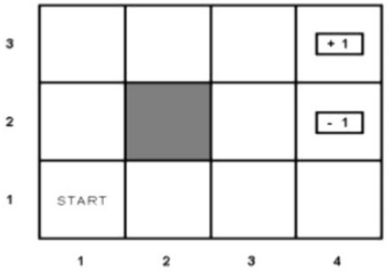
Probabilistic actions



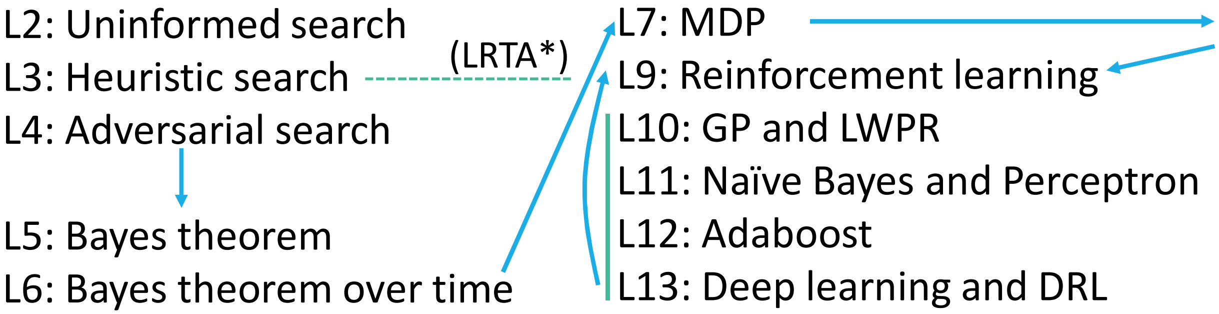
$P(s' | s, a)$

POMDP

Probabilistic actions and states



$P(s' | s, a), P(s)$



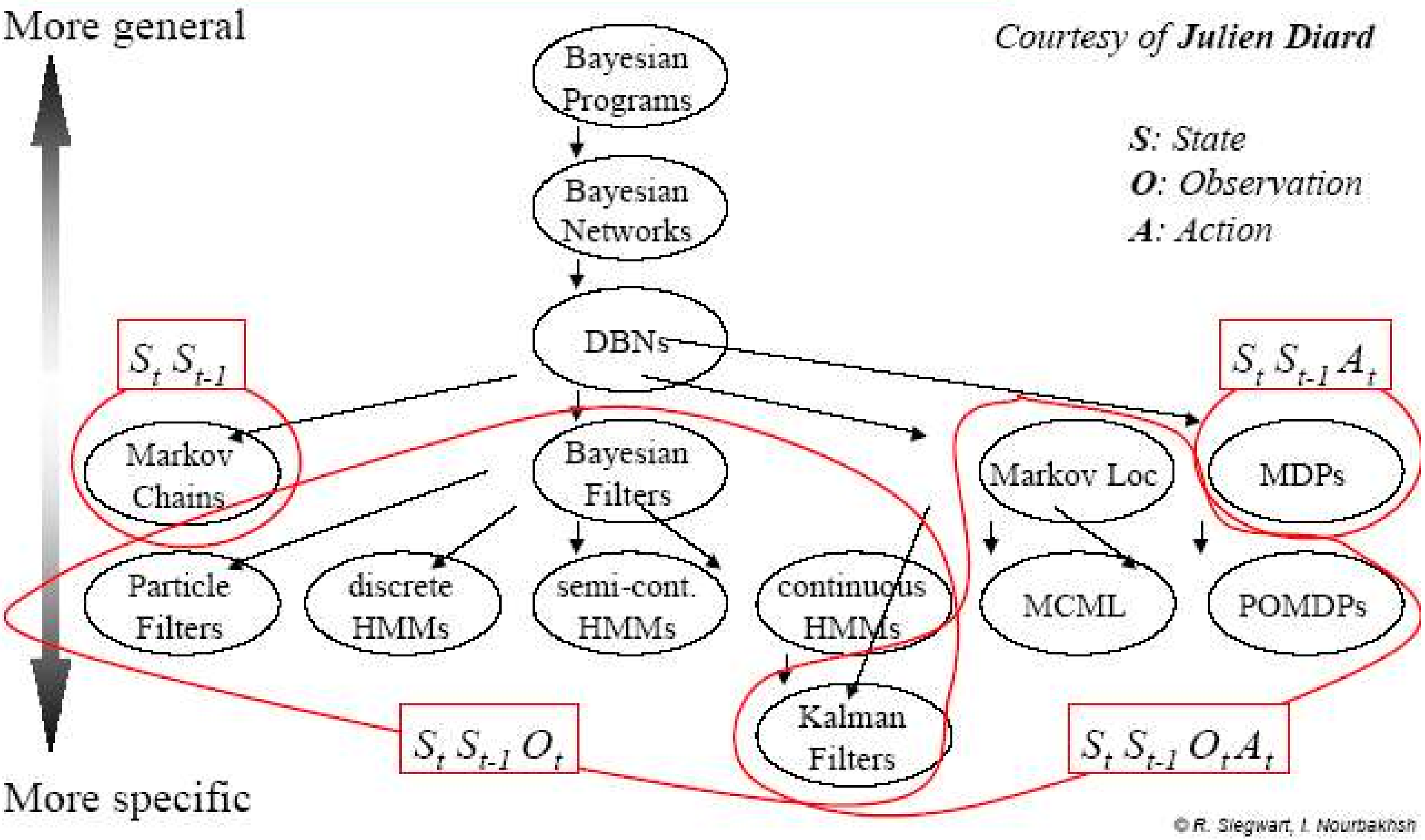
L2: Uninformed search
L3: Heuristic search
L4: Adversarial search
L5: Bayes theorem
L6: Bayes theorem over time

(LRTA*)

L7: MDP
L9: Reinforcement learning
L10: GP and LWPR
L11: Naïve Bayes and Perceptron
L12: Adaboost
L13: Deep learning and DRL

L8: POMDP

Bayesian Approach: A taxonomy of probabilistic models



Courtesy of ***Julien Diard***

***S**: State*
***O**: Observation*
***A**: Action*

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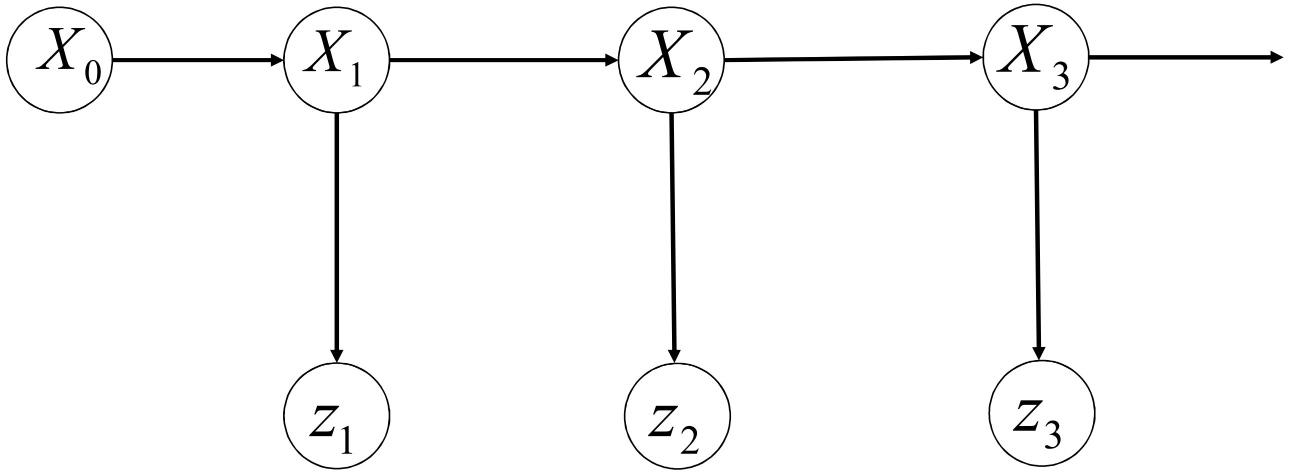
X_0
X_1
X_2
X_3
z_1
z_2
z_3