Bayes Theorem

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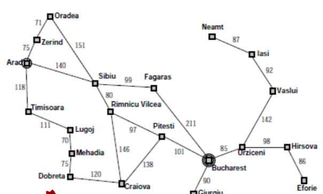
Course Announcement

- HW2 was released today. The deadline is 4/14(Wed).
  - Bayesian inference
  - MDP solver
  - A MDP problem
- You should start to work on HW2 and implement Bayesian filter.

Outline

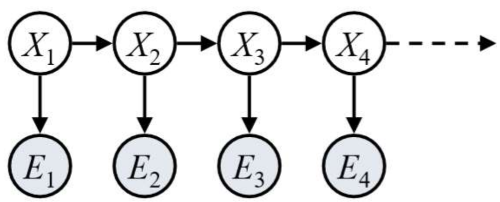
[Problem solving]

- Search problems
- Adversarial Search



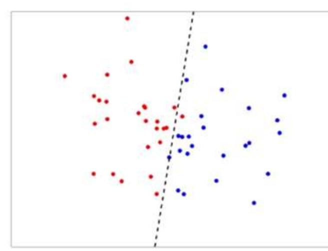
[Perception and Uncertainty]

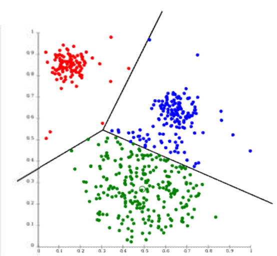
- Bayes Theorem
- Bayes Filter and Smoothing

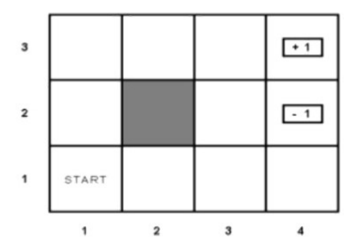


[Learning and Decision-making]

- Supervised learning
- Unsupervised learning
- Reinforcement learning





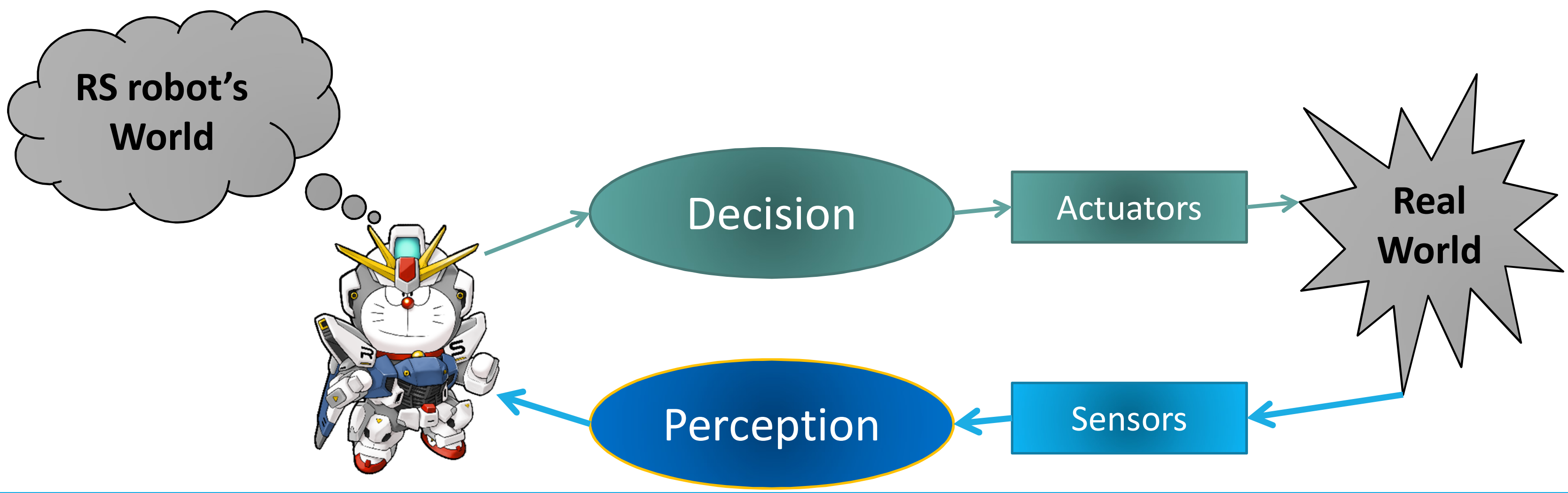


Outline

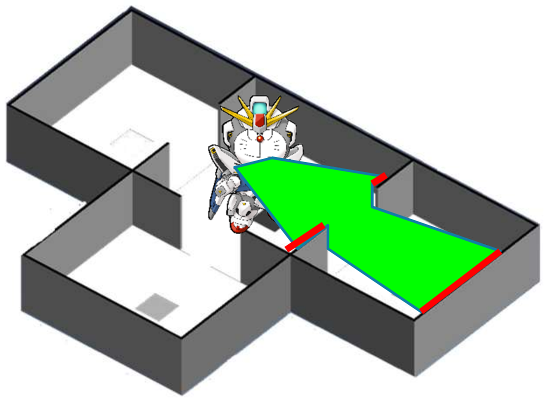
- **Need**
- Bayes theorem
- Bayes theorem for perception
- Bayes theorem for decision
- Bayes filter
- Conclusion

Need

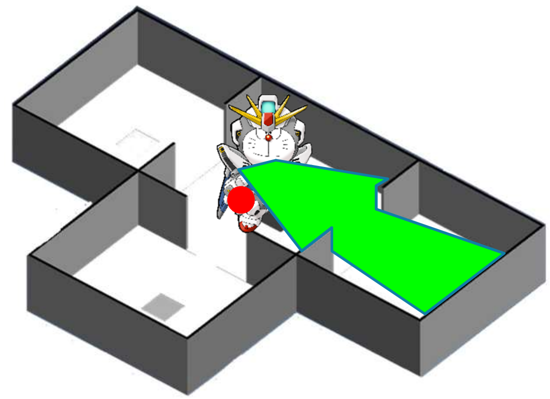
- Perception
- Decision making
- Feedback



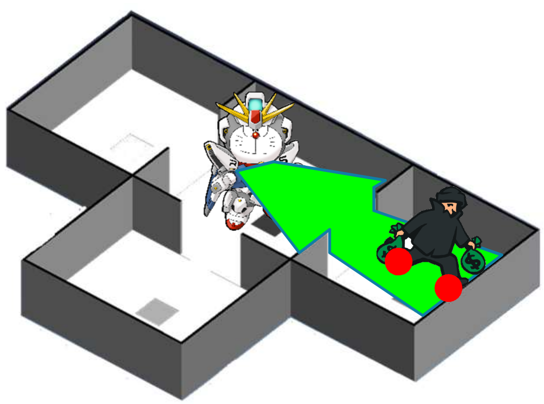
Need



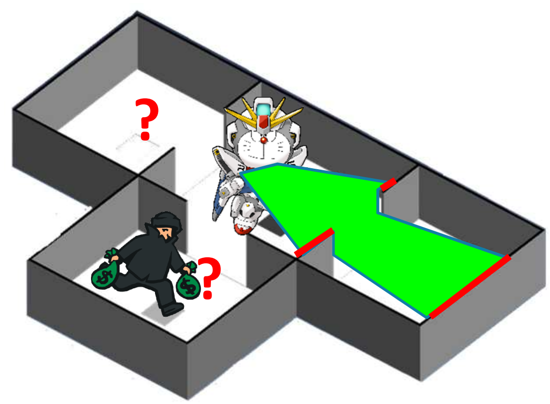
Mapping



Localization



Tracking



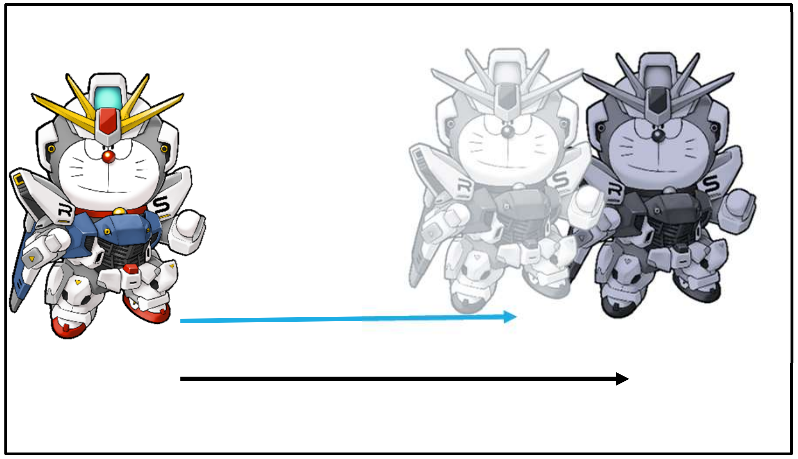
Searching

Need

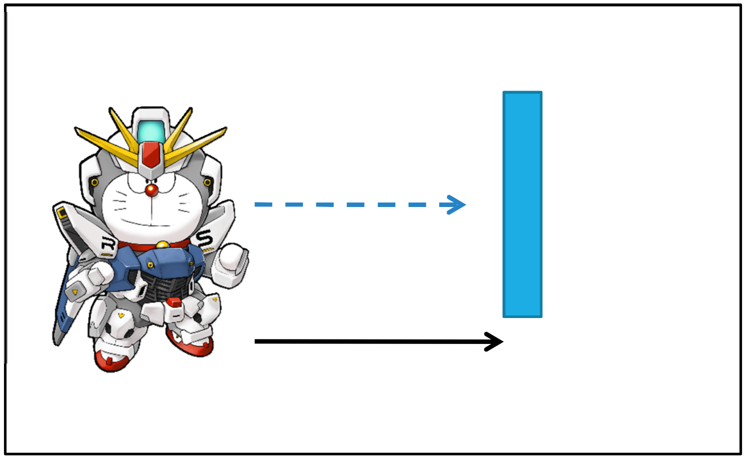
However, sensors are with uncertainty

- Motion is with accumulated error (inertial sensors, encoders)
- Measurement is with error (distance sensors)

Probability → Uncertainty



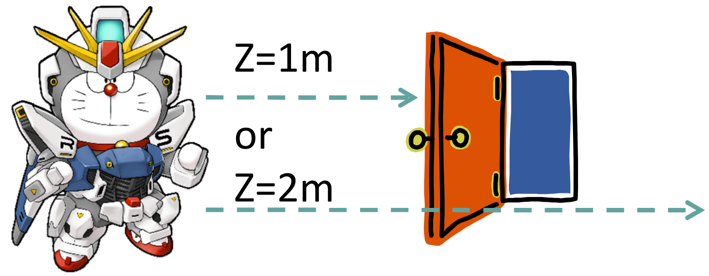
Motion case



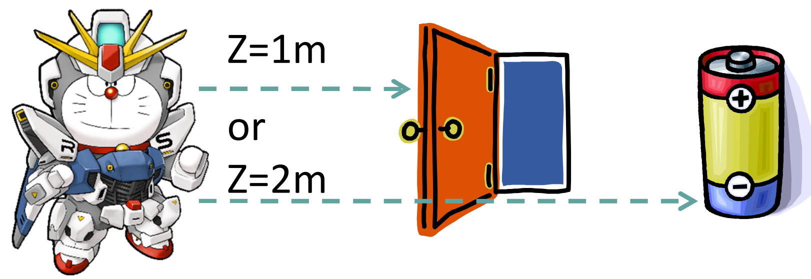
Measurement case

Outline

Perception



Decision



Feedback

t=0,1,2

Bayes Theorem

- Bayes equation/**inference**

Thomas Bayes
1702-1761
Bayes theorem was published in 1763!

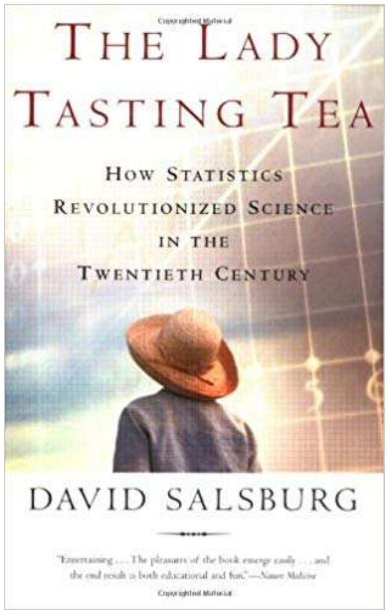
$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

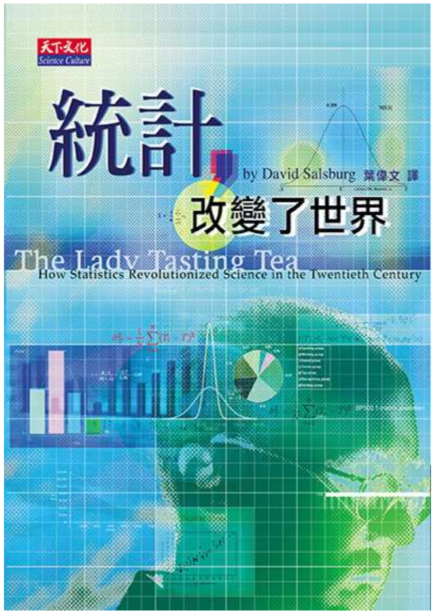
$A \rightarrow B$ $\quad P(B \mid A)$ Given A, what's the probability of B event?

$A \leftarrow B$ $\quad P(A \mid B)$ Given B, what's the probability of A event?

Bayes Theorem

- The Lady Tasting Tea:
  How Statistics Revolutionized Science in the Twentieth Century





Bayesian Search for USS Scorpion:
https://www.youtube.com/watch?v=WH-HWWkCMeU

Bayes Theorem

$P(A) = A$ area in Asia

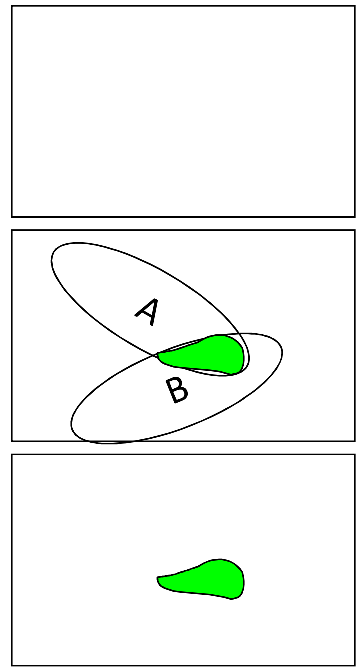
$P(B) = B$ area in Asia

$P(A \cap B)$

$=$ The intersection of A and B in Asia

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$=$ The intersection of A and B in B area



Asia area

A: Country A area

B: Country B area

$P(A \cap B)$

Bayes Theorem

- Set problems?

1 $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

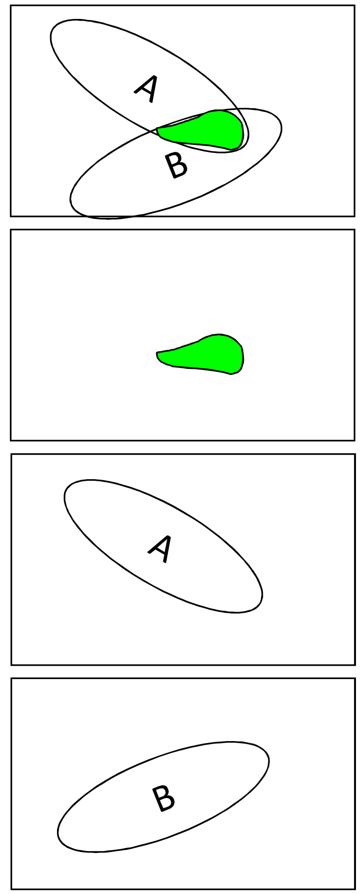
$= \dfrac{P(B|A) * 0.3}{0.15} = \dfrac{(1/6) * 0.3}{0.15} = \dfrac{1}{3}$

$P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{0.05}{0.3} = \dfrac{1}{6}$

2 $P(A|B) \neq P(B|A)$

$$\boxed{\underset{\frac{1}{3}}{\dfrac{P(A \cap B)}{P(B)}} \neq \underset{\frac{1}{6}}{\dfrac{P(A \cap B)}{P(A)}}}$$

Reasonable !



A: Country A area

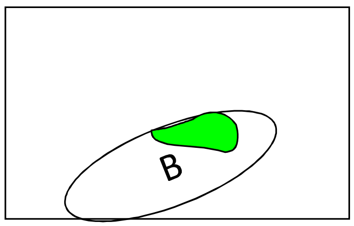
B: Country B area

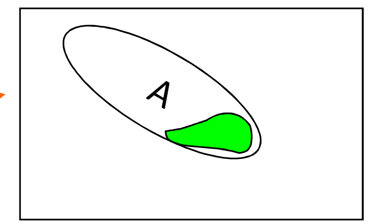
$P(A \cap B) = 0.05$

$P(A) = 0.3$

$P(B) = 0.15$

Bayes Theorem

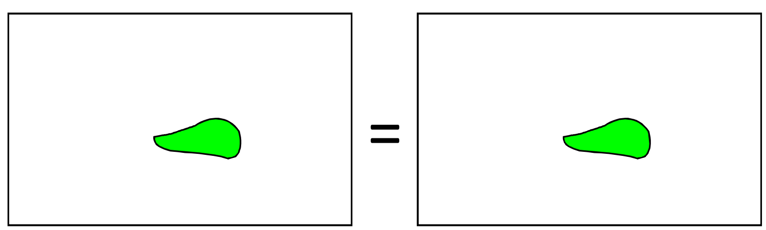


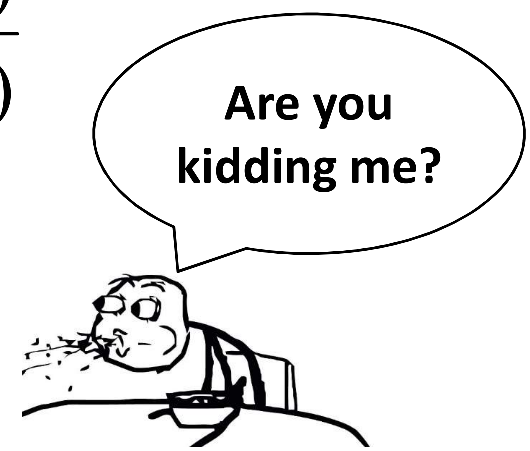


$$\boxed{P(A \mid B)} = \frac{\boxed{P(B \mid A)}\,P(A)}{P(B)}$$

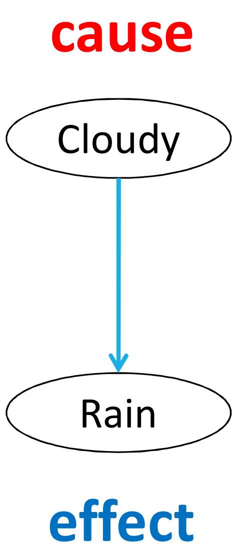
$$\boxed{\frac{P(A \cap B)}{P(B)}} = \boxed{\frac{P(B \cap A)}{P(A)}}\,\frac{P(A)}{P(B)}$$

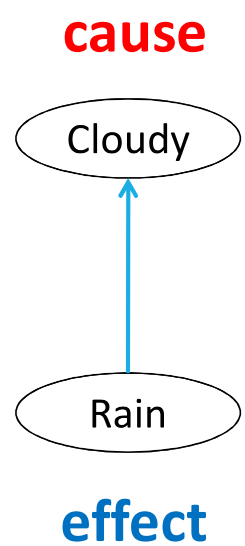
$$P(A \cap B) = P(B \cap A)$$





Bayes Theorem





When it's raining, what's the probability of it's cloudy?

$$P(C \mid R) = \frac{P(R \mid C)P(C)}{P(R)}$$

Bayes Theorem for Perception

- **When it's raining, what's the probability of it's cloudy?**
  - *When we see R=1*
  - ***P(R=1|C=1) = 0.8*** ***P(R=0|C=1) =0.2***
  - ***P(C=0) = P(C=1) = 0.5*** **(by experiments)**

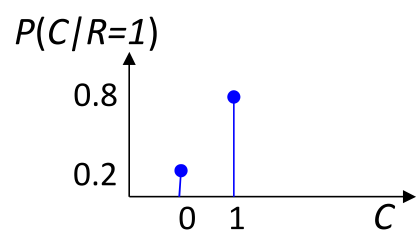
P(C=F)	P(C=T)
0.5	0.5

C	P(R=F)	P(R=T)
F	0.8	0.2
T	0.2	0.8

$$P(C=1\,|\,R=1)=\frac{P(R=1\,|\,C=1)P(C=1)}{P(R=1)}=\frac{P(R=1\,|\,C=1)P(C=1)}{P(R=1\,|\,C=1)P(C=1)+P(R=1\,|\,C=0)P(C=0)}$$

$$=\frac{0.8*0.5}{0.8*0.5+0.2*0.5}=0.8$$

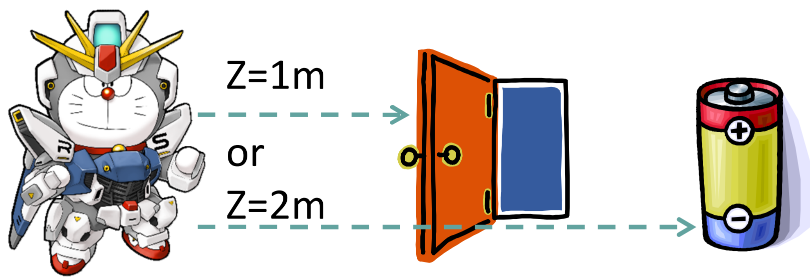
$$P(C\,|\,R=1)=?$$

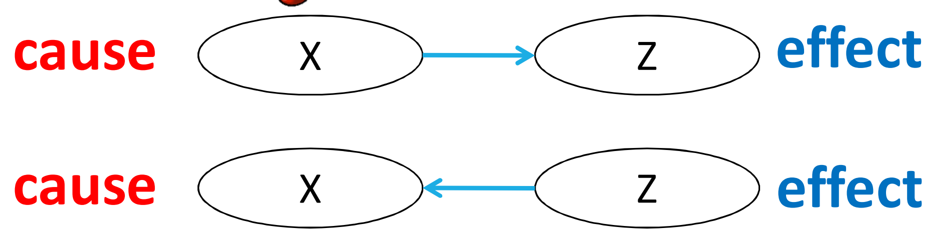


cause and effect ????

Bayes Theorem for Perception

- **Robots can use Bayes theorem to infer the world.**



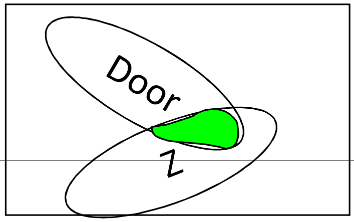


$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

$$X = open \text{ or } close$$

$$Z = z_1(1m) \ or \ z_2(2m)$$

Bayes Theorem for Perception

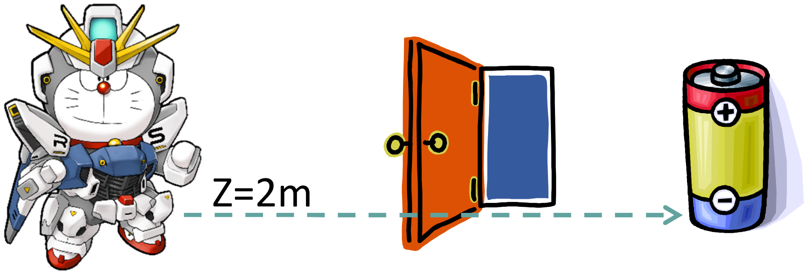


- Ex1: Is the door open?

$$P(X = open \mid Z = 2) = ?$$

What is the probability of open door, when $Z = 2\text{m}$?

To solve this problem using Bayes theorem



Bayes Theorem for Perception

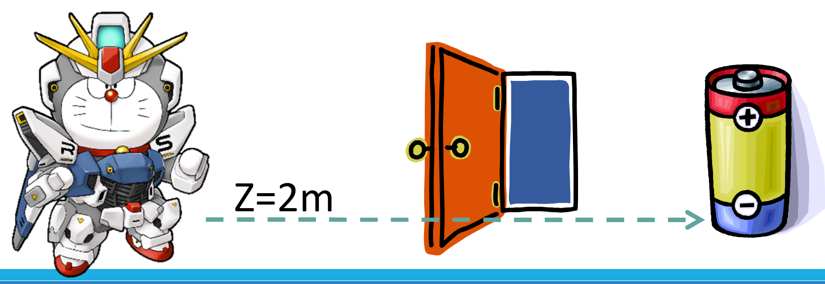
- ***P(X=open|Z=2m)=?***

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

  - *z=2m*
  - ***P(z|open) = 0.6*** ***P(z|close) =0.3***
  - ***P(open) = P(close) = 0.5*** **(by experiments)**

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z \mid open)\,p(open) + P(z \mid close)\,p(close)}$$

$$P(open \mid z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.66$$



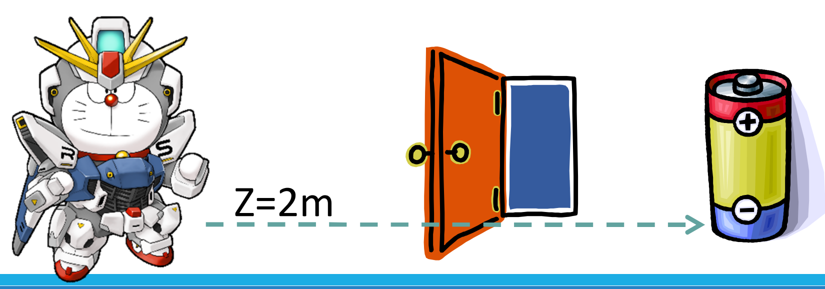
Robot can utilize Bayes Theorem to infer the world!

Bayes Theorem for Perception

- ***P(X=open|Z=2m)=?***
  - *z=2m*
  - ***P(z|open) = 0.6*** ***P(z|close) =0.3***
  - ***P(open) = P(close) = 0.5 (by experiments)***

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z \mid open)\,p(open) + P(z \mid close)\,p(close)}$$

$$P(open \mid z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.66$$



66%?!

Move or not ?

Bayes Theorem for Decision Making

- Probability + Utility theory ➔ Decision making

$$r_{move} = R_{OO} \cdot P(open \mid Z) + R_{OC} \cdot P(close \mid Z)$$

$$r_{stop} = R_{CO} \cdot P(open \mid Z) + R_{CC} \cdot P(close \mid Z)$$

The reward that decides X ∈ close when X ∈ open

The expected reward that decides to *stop*

$R_{OC} = -20 \text{(hit)}$

$R_{CC} = +0 \text{ (wait)}$

$R_{CO} = \text{-}2 \text{ (waste power)}$

$R_{OO} = +6 \text{ (charging)}$

- The decision rule:

$$r_{move} > r_{stop} \Rightarrow Decision : Move$$

$$r_{move} < r_{stop} \Rightarrow Decision : Stop$$

A simple version

$$r_{stop} = R_{CO} \cdot P(open) + R_{CC} \cdot P(close)$$

Bayes Theorem for Decision Making

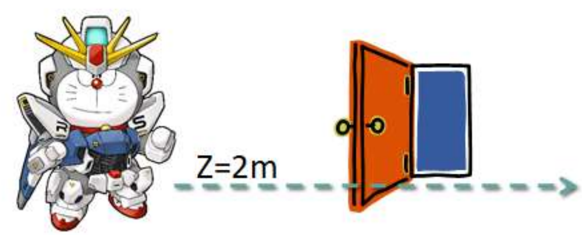
$$r_{move} > r_{stop} \Rightarrow Decision : Move$$

$$R_{OO} \cdot P(open \mid Z) + R_{OC} \cdot P(close \mid Z)$$
$$> R_{CO} \cdot P(open \mid Z) + R_{CC} \cdot P(close \mid Z)$$

$$\frac{P(open \mid Z)}{P(close \mid Z)} > \frac{(R_{CC} - R_{OC})}{(R_{OO} - R_{CO})} \Rightarrow Decision : Move$$
$$\frac{P(open \mid Z)}{P(close \mid Z)} < \frac{(R_{CC} - R_{OC})}{(R_{OO} - R_{CO})} \Rightarrow Decision : Stop$$

Bayes Theorem for Decision Making

- Ex2: Should it move?



$$p(open \mid z_2) = 0.66 \text{ (by Bayes theorem)}$$

$$\frac{p(open \mid z_2)}{p(close \mid z_2)} > \frac{(R_{CC} - R_{OC})}{(R_{OO} - R_{CO})} \Rightarrow Decision : Move!$$

$$R_{OC} = -20 \text{(hit)}$$

$$R_{CC} = +0 \text{ (wait)}$$

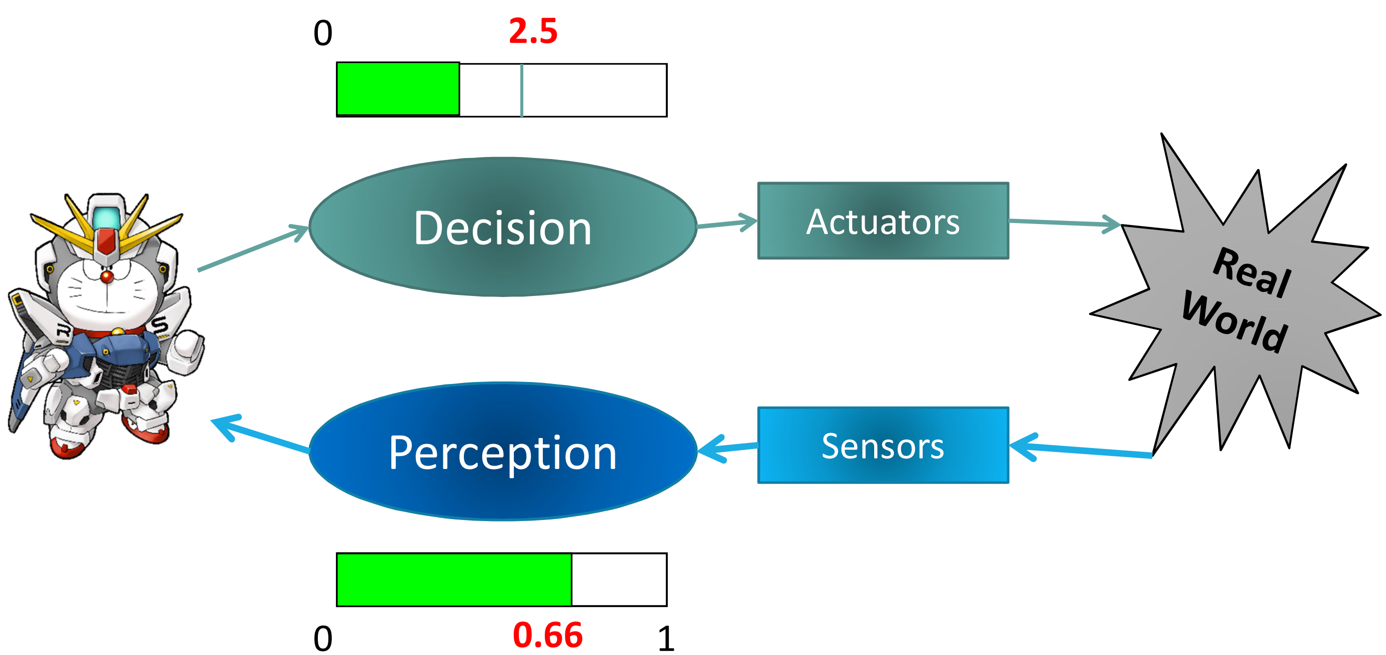
$$R_{CO} = \text{-2} \text{ (waste power)}$$

$$R_{OO} = +6 \text{ (charging)}$$

$$\frac{p(open \mid z_2)}{p(close \mid z_2)} = \frac{0.66}{0.33} < \frac{0-(-20)}{6-(-2)} = 2.5(Stop!)$$

Bayes Theorem for Decision Making

- What happened next second?
  - State (X) will change
  - Move (u), sense (Z) and making decision again



Bayes filter

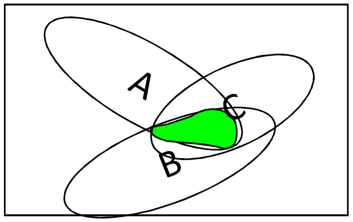
- Bayes theorem

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

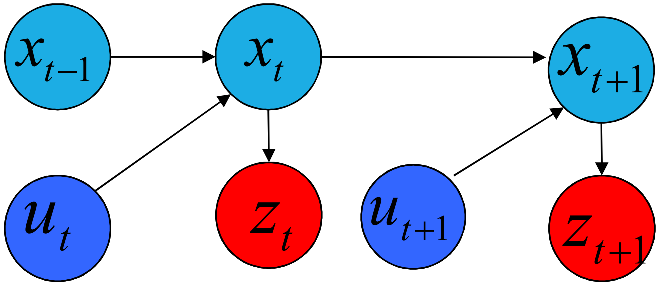
$$P(A \mid B, C) = \frac{P(B, C \mid A)P(A)}{P(B, \mathrm{C})}$$

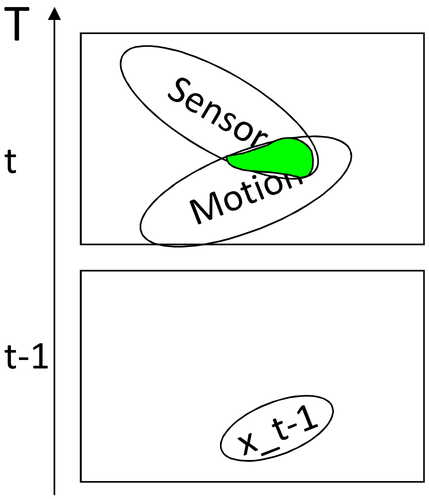
P(A|B,C)=P(B,C|A)P(A)/P(B,C)
=P(B|A,C)P(A|C)/P(B|C)

$$P(x_t \mid \underline{x_{t-1}}, \underline{u_t, z_t})$$



Set





Bayes filter

$$P(A \mid B, C) = \frac{P(B, C \mid A)P(A)}{P(B, \mathrm{C})}$$

$$P(x_t \mid u_{1:t}, z_{1:t}) = \frac{P(z_t \mid u_{1:t}, z_{1:t-1}, x_t)P(x_t \mid u_{1:t}, z_{1:t-1})}{P(z_t \mid u_{1:t}, z_{1:t-1})}$$

$$= \underline{bel(x_t) = \eta \bullet P(z_t \mid x_t)\overline{bel}(x_t)}$$

$$\underline{\overline{bel}(x_t)} = P(x_t \mid u_{1:t}, z_{1:t-1})$$

By total probability theorem

$$= \int P(x_t \mid x_{t-1}, u_{1:t}, z_{1:t-1})P(x_{t-1} \mid u_{1:t}, z_{1:t-1})dx_{t-1}$$

$$= \underline{\int P(x_t \mid x_{t-1}, u_{1:t})bel(x_{t-1})dx_{t-1}}$$

Bayes filter

$$P(A \mid B, C) = \frac{P(B, C \mid A)P(A)}{P(B, \mathrm{C})}$$

- Bayes theorem

$$P(x_t \mid u_{1:t}, z_{1:t}) = \frac{P(z_t, z_{1:t-1}, u_{1:t}, \mid x_t)P(x_t)}{P(z_t, z_{1:t-1}, u_{1:t})}$$

$$= \frac{P(z_t \mid z_{1:t-1}, u_{1:t}, x_t)P(z_{1:t-1}, u_{1:t} \mid x_t)P(x_t)}{P(z_t \mid z_{1:t-1}, u_{1:t})P(z_{1:t-1}, u_{1:t})}$$

$$= \frac{P(z_t \mid z_{1:t-1}, u_{1:t}, x_t)P(z_{1:t-1}, u_{1:t}, x_t)}{P(z_t \mid z_{1:t-1}, u_{1:t})P(z_{1:t-1}, u_{1:t})}$$

$$= \frac{P(z_t \mid z_{1:t-1}, u_{1:t}, x_t)P(x_t \mid z_{1:t-1}, u_{1:t})\cancel{P(z_{1:t-1}, u_{1:t})}}{P(z_t \mid z_{1:t-1}, u_{1:t})\cancel{P(z_{1:t-1}, u_{1:t})}}$$

$$= \frac{P(z_t \mid \cancel{z_{1:t-1}}, \cancel{u_{1:t}}, x_t)P(x_t \mid u_{1:t}, z_{1:t-1})}{P(z_t \mid u_{1:t}, z_{1:t-1})}$$

$$= bel(x_t) = \eta \bullet P(z_t \mid x_t)\overline{bel}(x_t)$$

Utilize conditional independence

$$P(z_t \mid z_{t-1}, x_t)$$
$$= P(z_t \mid x_t)$$

Derive "sensor model"

$$P(z_t \mid x_t)$$

Bayes filter

- Total probability

$$P(B) = \int P(B, A)dA$$

$$P(A \mid B) = \int P(A \mid B, C)P(C)dC$$

$$P(B) = \sum_{x=1}^{x=n} P(B \cap A_{x=i}) = \sum_{x=1}^{x=n} P(B \mid A_i)P(A_i)$$

Example:

$$\frac{P(R=1 \mid C=1)P(C=1)}{P(R=1)} = \frac{P(R=1 \mid C=1)P(C=1)}{P(R=1 \mid C=1)P(C=1) + P(R=1 \mid C=0)P(C=0)}$$

$$\overline{bel}(x_t) = P(x_t \mid u_{1:t}, z_{1:t-1})$$

$$= \int P(x_t \mid x_{t-1}, u_{1:t}, \cancel{z_{1:t-1}})P(x_{t-1} \mid u_{1:t}, z_{1:t-1})dx_{t-1}$$

$$= \int P(x_t \mid x_{t-1}, u_{1:t})bel(x_{t-1})dx_{t-1}$$

Bayes filter

- Prediction

$$\boxed{P(x_t \mid x_{t-1}, u_t, z_{t-1})} = \int \underbrace{P(x_t \mid x_{t-1}, u_t)}_{\text{Known (motion model)}} \boxed{P(x_{t-1} \mid z_{t-1})} dx_{t-1}$$

- Correction

$$\boxed{P(x_t \mid z_t)} = \eta \bullet \underbrace{P(z_t \mid x_t)}_{\text{Known (sensor model)}} \boxed{P(x_t \mid x_{t-1}, u_t, z_{t-1})}$$

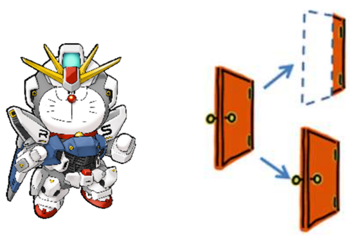
constant

Since the robot motion (u_t) and the door state (X_t) is not relevant

$\Rightarrow u_t$ can be skipped

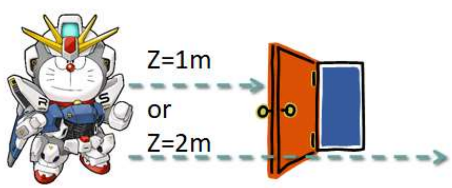
Bayes filter

$$\boxed{P(x_t \mid x_{t-1}, z_{t-1})} = \int \underbrace{P(x_t \mid x_{t-1})}_{\text{Known}} \boxed{P(x_{t-1} \mid z_{t-1})} dx_{t-1}$$



$$\boxed{P(x_t \mid z_t)} = \eta \bullet \underbrace{P(z_t \mid x_t)}_{\text{Known}} \boxed{P(x_t \mid x_{t-1}, z_{t-1})}$$

constant



Bayes filter

- EX:

$$P(Z_{t_0}=2\mid x_{t_0}=o)=0.6 \qquad P(Z_{t_0}=2\mid x_{t_0}=c)=0.3$$

$$P(x_{t_0}=o)=P(x_{t_0}=c)=0.5\ \text{(by experiments)}$$

$$\left(Z_{t_0},Z_{t_1},Z_{t_2}\right)=\{2,2,1\}$$

$$P(x_t=o\mid x_{t-1}=o)=P(x_t=c\mid x_{t-1}=c)=0.8$$

$$P(x_t=o\mid x_{t-1}=c)=P(x_t=c\mid x_{t-1}=o)=0.2$$

$$\text{When t}=t_0,t_1,t_2\text{，}P\left(x_t\mid Z_t\right)=?$$

$$\text{When } t=t_0, P\left(x_t\mid Z_t=2\right)=?$$

Bayes filter

t 0, Z=2m

- EX:

$$P(z_2 \mid open) = 0.6 \qquad P(z_2 \mid close) = 0.3$$

$$P(open) = P(close) = 0.5$$

$$\left(Z_{t_0}, Z_{t_1}, Z_{t_2}\right) = \{2,2,1\}$$

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z \mid open)\, p(open) + P(z \mid close)\, p(close)}$$

$$P(open \mid z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.66$$

t 1, Z=2m

Bayes filter

Given:

$$P(z_2 \mid x_t = o) = 0.6 \qquad P(z_2 \mid x_t = c) = 0.3$$

$$\boxed{\begin{aligned} P(x_t = o \mid x_{t-1} = o) = P(x_t = c \mid x_{t-1} = c) = 0.8 \\ P(x_t = o \mid x_{t-1} = c) = P(x_t = c \mid x_{t-1} = o) = 0.2 \end{aligned}}$$

Transition matrix

$$P(x_t \mid x_{t-1}, z_{t-1}) = \int P(x_t \mid x_{t-1}) P(x_{t-1} \mid z_{t-1}) dx_{t-1}$$

$$\begin{cases} P(x_t = o \mid x_{t-1}, z_{t-1}) = \underbrace{P(x_t = o \mid x_{t-1} = o)}_{0.8} \underbrace{P(x_{t-1} = o \mid z_{t-1})}_{0.66} + \underbrace{P(x_t = o \mid x_{t-1} = c)}_{0.2} \underbrace{P(x_{t-1} = c \mid z_{t-1})}_{0.33} = 0.6 \\ P(x_t = c \mid x_{t-1}, z_{t-1}) = \underbrace{P(x_t = c \mid x_{t-1} = o)}_{0.2} \underbrace{P(x_{t-1} = o \mid z_{t-1})}_{0.66} + \underbrace{P(x_t = c \mid x_{t-1} = c)}_{0.8} \underbrace{P(x_{t-1} = c \mid z_{t-1})}_{0.33} = 0.4 \end{cases}$$

$$P(x_t \mid z_t) = \eta \bullet P(z_t \mid x_t) P(x_t \mid x_{t-1}, z_{t-1})$$

$$\begin{cases} P(x_t = o \mid z_t) = \eta \bullet \underbrace{P(z_t \mid x_t = o)}_{0.6} \underbrace{P(x_t = o \mid x_{t-1}, z_{t-1})}_{0.6} = 0.75 \\ P(x_t = c \mid z_t) = \eta \bullet \underbrace{P(z_t \mid x_t = c)}_{0.3} \underbrace{P(x_t = c \mid x_{t-1}, z_{t-1})}_{0.4} = 0.25 \end{cases}$$

$$(0.48\eta = 1, \eta = 2.08)$$

t 2, Z=1m

Bayes filter

$$P(z_2 \mid x_t = o) = 0.6 \qquad P(z_2 \mid x_t = c) = 0.3$$

$$P(x_t = o \mid x_{t-1} = o) = P(x_t = c \mid x_{t-1} = c) = 0.8$$

$$P(x_t = o \mid x_{t-1} = c) = P(x_t = c \mid x_{t-1} = o) = 0.2$$

$$P(x_t \mid x_{t-1}, z_{t-1}) = \int P(x_t \mid x_{t-1}) P(x_{t-1} \mid z_{t-1}) dx_{t-1}$$

$$\begin{cases} P(x_t = o \mid x_{t-1}, z_{t-1}) = \underbrace{P(x_t = o \mid x_{t-1} = o)}_{0.8}\underbrace{P(x_{t-1} = o \mid z_{t-1})}_{0.75} + \underbrace{P(x_t = o \mid x_{t-1} = c)}_{0.2}\underbrace{P(x_{t-1} = c \mid z_{t-1})}_{0.25} = 0.65 \\ P(x_t = c \mid x_{t-1}, z_{t-1}) = \underbrace{P(x_t = c \mid x_{t-1} = o)}_{0.2}\underbrace{P(x_{t-1} = o \mid z_{t-1})}_{0.75} + \underbrace{P(x_t = c \mid x_{t-1} = c)}_{0.8}\underbrace{P(x_{t-1} = c \mid z_{t-1})}_{0.25} = 0.35 \end{cases}$$

$$P(x_t \mid z_t) = \eta \bullet P(z_t \mid x_t) P(x_t \mid x_{t-1}, z_{t-1})$$

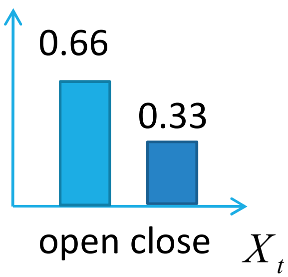
$$\begin{cases} P(x_t = o \mid z_t) = \eta \bullet \underbrace{P(z_t \mid x_t = o)}_{0.4}\underbrace{P(x_t = o \mid x_{t-1}, z_{t-1})}_{0.65} = 0.51 \\ P(x_t = c \mid z_t) = \eta \bullet \underbrace{P(z_t \mid x_t = c)}_{0.7}\underbrace{P(x_t = c \mid x_{t-1}, z_{t-1})}_{0.35} = 0.49 \end{cases}$$

$$(0.505\eta = 1, \eta = 1.98)$$

Bayes filter

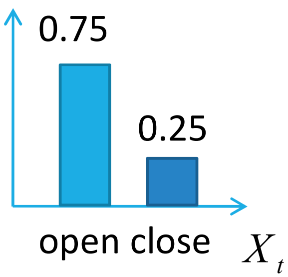
$$\begin{cases} P(x_t \mid x_{t-1}, z_{t-1}) = \int P(x_t \mid x_{t-1}) P(x_{t-1} \mid z_{t-1}) dx_{t-1} \\ P(x_t \mid z_t) = \eta \bullet P(z_t \mid x_t) P(x_t \mid x_{t-1}, z_{t-1}) \end{cases}$$

$P(X_t \mid Z_t = 2)$



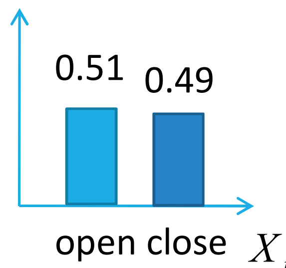
$t0$

$P(X_t \mid Z_t = 2)$



$t1$

$P(X_t \mid Z_t = 1)$



$t2$

Bayes filter

- EX3-2:

$$\frac{p(open\,|\,Z)}{p(close\,|\,Z)} > \frac{(R_{CC} - R_{OC})}{(R_{OO} - R_{CO})} \Rightarrow Decision : Move!$$

$$\frac{p(open\,|\,Z)}{p(close\,|\,Z)} < \frac{(R_{CC} - R_{OC})}{(R_{OO} - R_{CO})} \Rightarrow Decision : Stop!$$

$$R_{OC} = -20(\text{hit})$$

$$R_{CC} = +0\ (\text{wait})$$

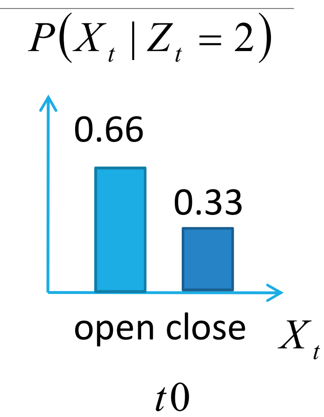
$$R_{CO} = \text{-}2\ \ (\text{waste power})$$

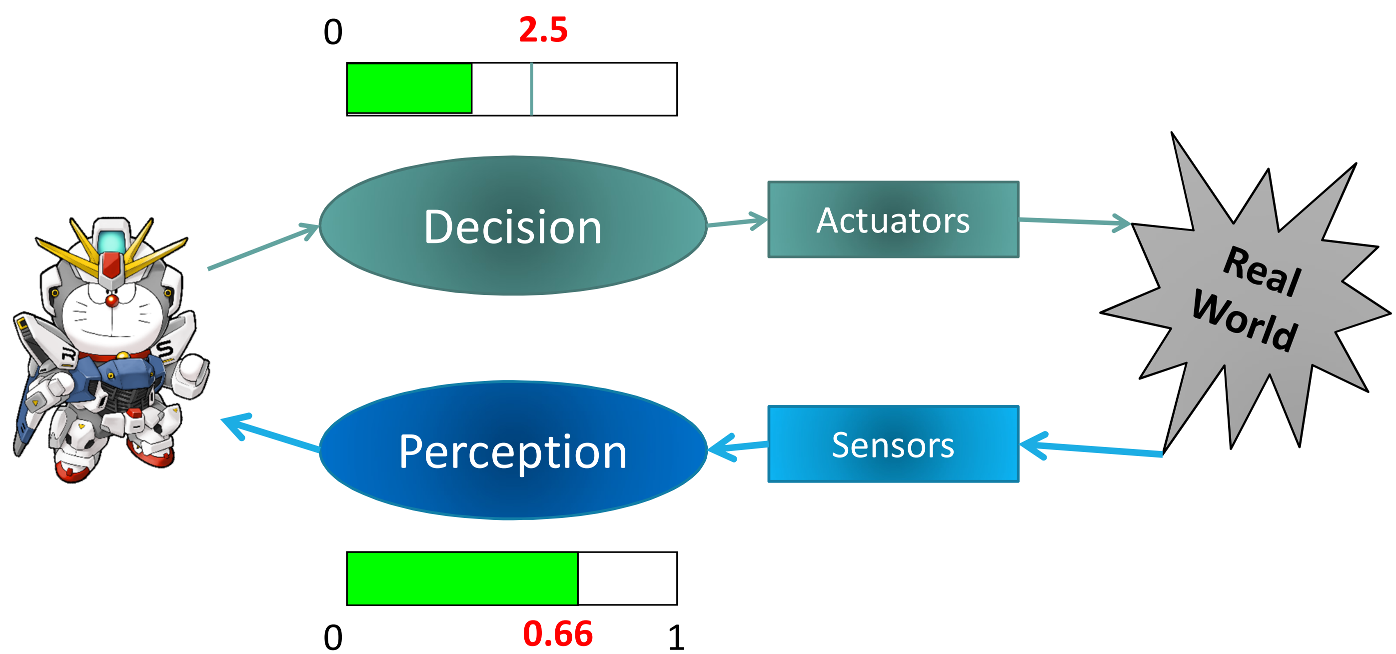
$$R_{OO} = +6\ (\text{charging})$$

Bayes filter

t 0, Z=2m

$$\frac{p(X_{t0}=O\mid Z_{t0}=2)}{p(X_{t0}=C\mid Z_{t0}=2)}=\frac{0.66}{0.33}=2<2.5\,(Stop!)$$



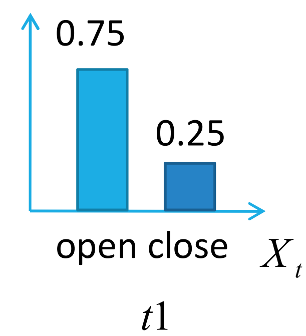


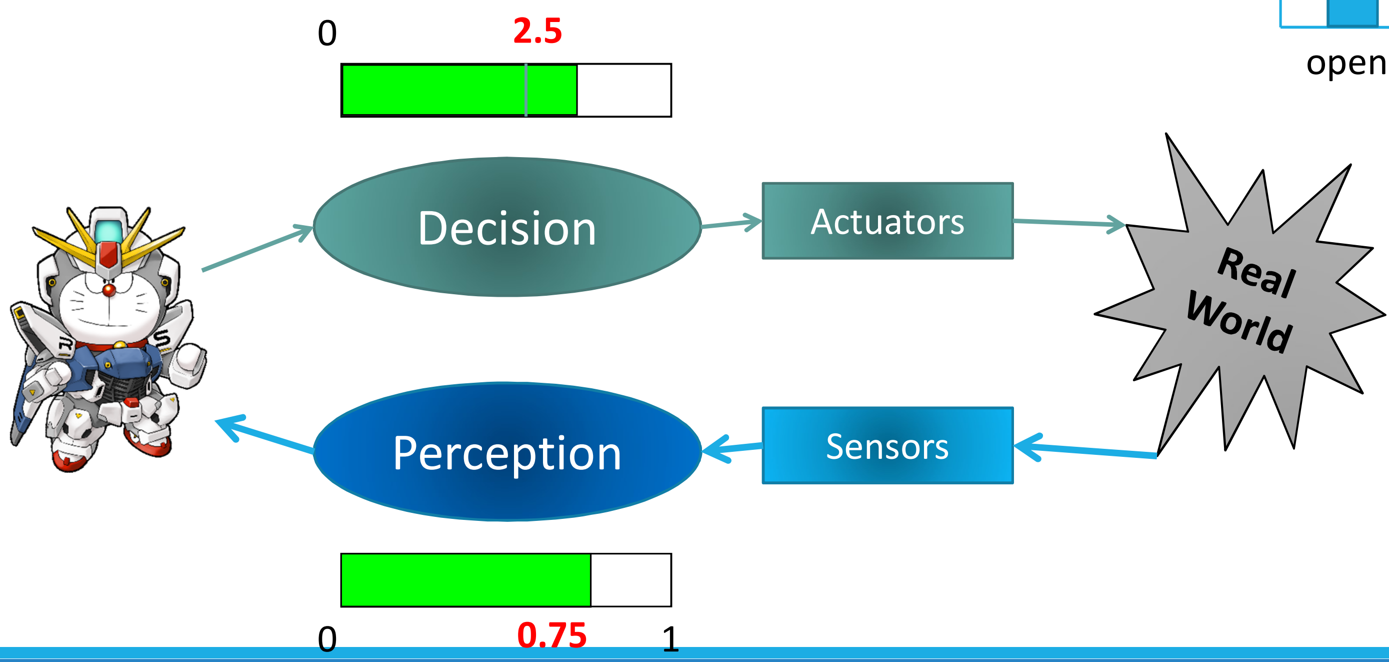
Bayes filter

t 1, Z=2m

$$\frac{p(X_{t1}=O \mid Z_{t1}=2)}{p(X_{t1}=C \mid Z_{t1}=2)} = \frac{0.75}{0.25} = 3 > 2.5\ (Move!)$$

$P(X_t \mid Z_t = 2)$

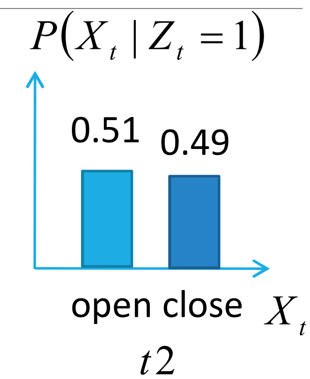


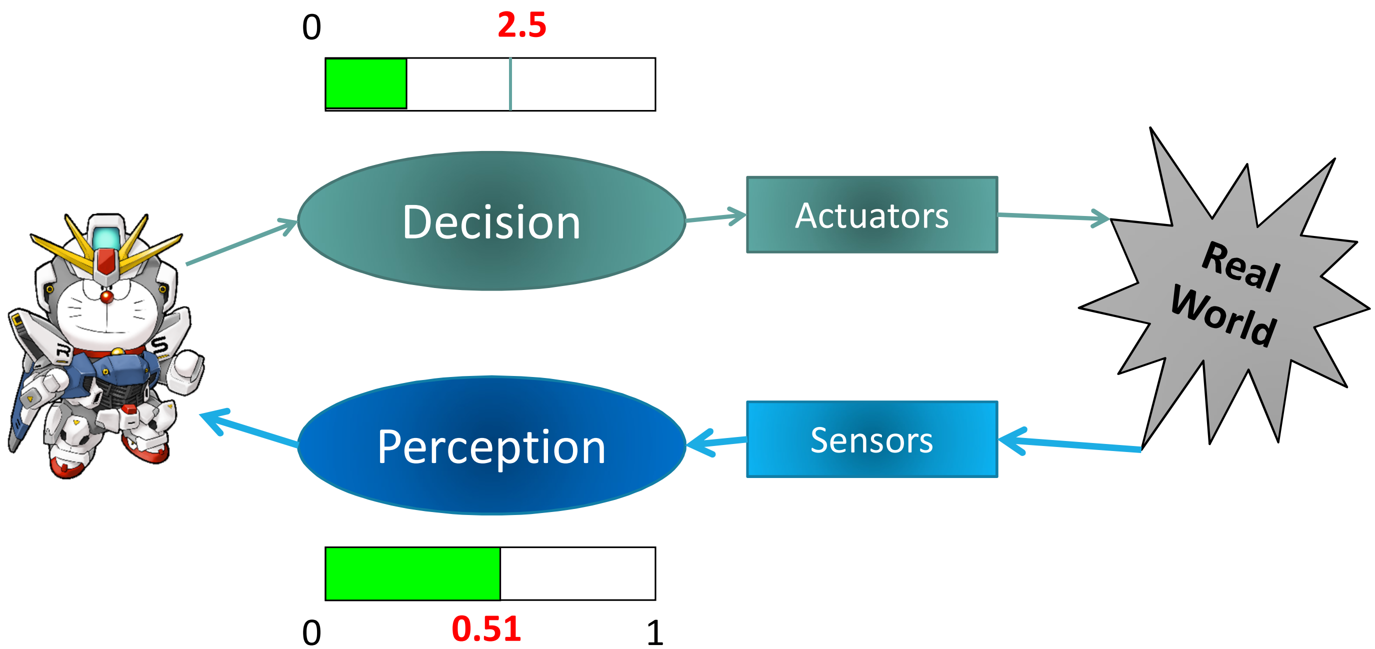


Bayes filter

t 2, Z=1m

$$\frac{p(X_{t2}=O \mid Z_{t2}=1)}{p(X_{t2}=C \mid Z_{t2}=1)} = \frac{0.51}{0.49} = 1.04 < 2.5\ (Stop!)$$

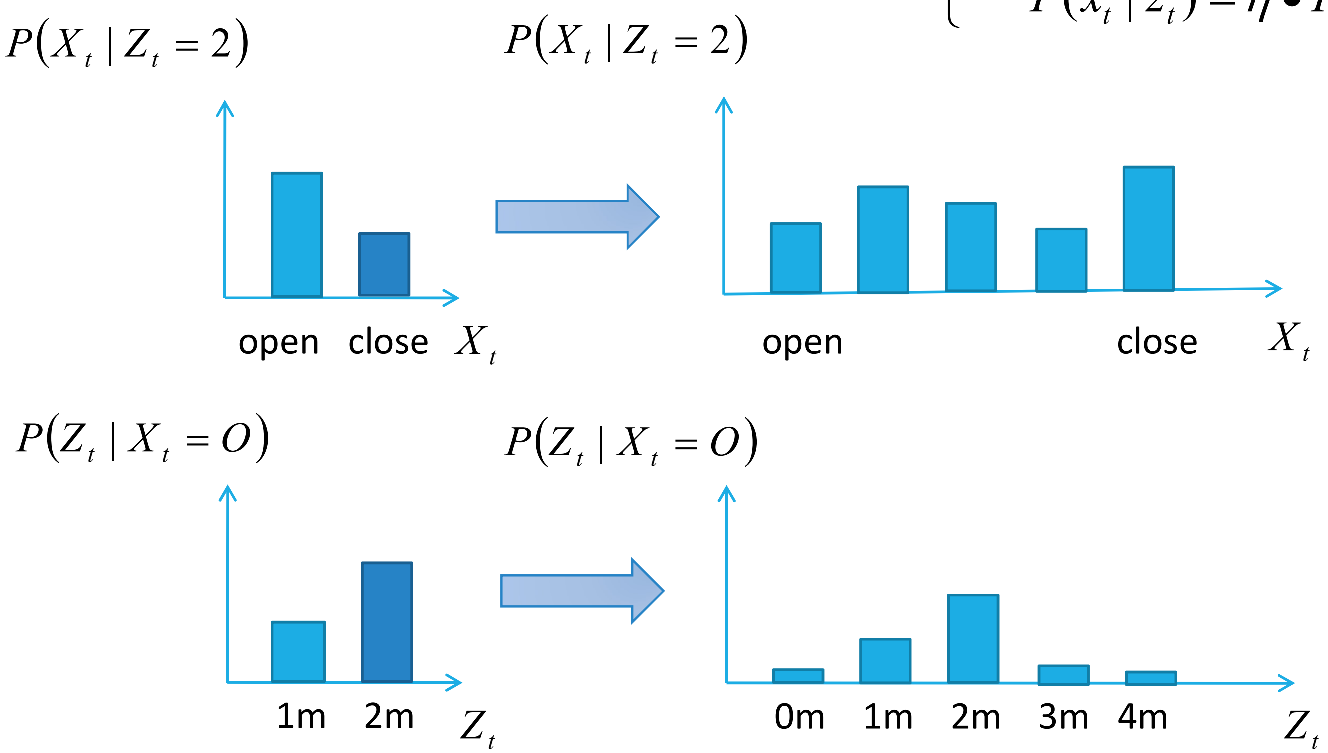




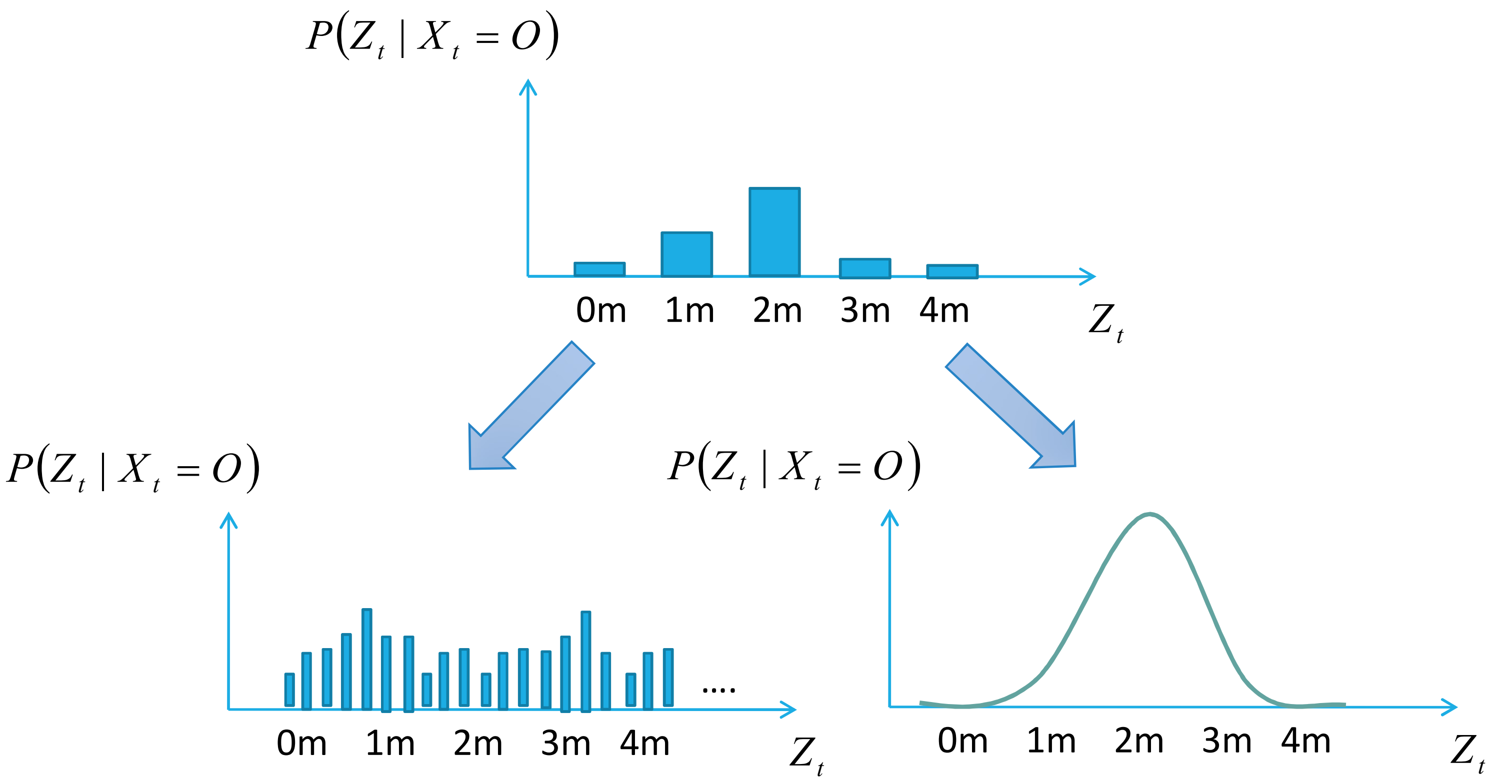
Two kinds of Bayes filter

- Probability density function (pdf)

$$\begin{cases} P(x_t \mid x_{t-1}, z_{t-1}) = \int P(x_t \mid x_{t-1}) P(x_{t-1} \mid z_{t-1}) dx_{t-1} \\ P(x_t \mid z_t) = \eta \bullet P(z_t \mid x_t) P(x_t \mid x_{t-1}, z_{t-1}) \end{cases}$$



Two kinds of Bayes filter



Discrete pdf

Continuous pdf

Two kinds of Bayes filter

- Prediction

$$P(x_t \mid x_{t-1}, u_t, z_{t-1}) = \int P(x_t \mid x_{t-1}, u_t) P(x_{t-1} \mid z_{t-1}) dx_{t-1}$$

- Correction

$$P(x_t \mid z_t) = \eta \bullet P(z_t \mid x_t) P(x_t \mid x_{t-1}, u_t, z_{t-1})$$

- Computational complexity?

Two kinds of Bayes filter

Assume there are n states and m possible measurements,
What's the computational complexity of Bayes filter?

$$\text{Prediction}: \underbrace{P(x_t \mid x_{t-1}, u_t, z_{t-1})}_{\text{n by 1 matrix}} = \sum_{x_{t-1}} \underbrace{P(x_t \mid x_{t-1}, u_t)}_{\text{n by n matrix}} \underbrace{P(x_{t-1} \mid z_{t-1})}_{\text{n by 1 matrix}}$$

$$\text{Correction}: \underbrace{P(x_t \mid z_t)}_{\text{n by 1 matrix}} = \eta \bullet \underbrace{P(z_t \mid x_t)}_{\text{n vector}} \underbrace{P(x_t \mid x_{t-1}, u_t, z_{t-1})}_{\text{n by 1 matrix}}$$

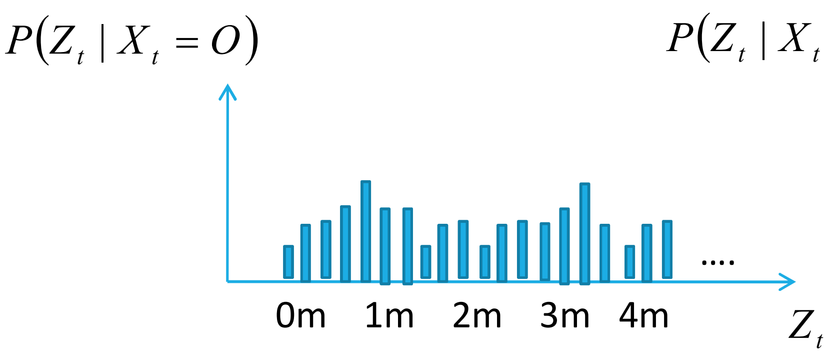
Two kinds of Bayes filter

- Prediction

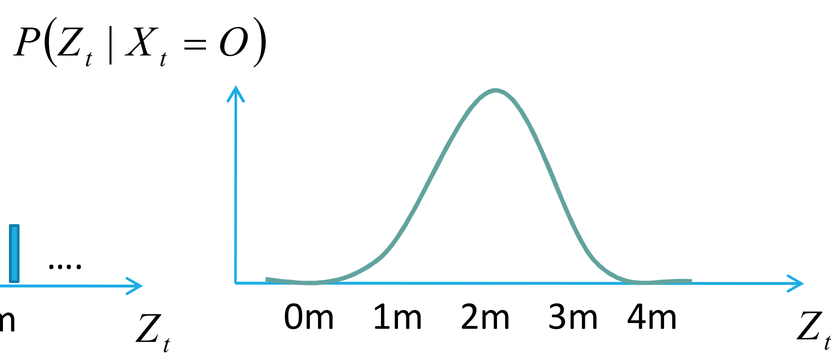
$$P(x_t \mid x_{t-1}, u_t, z_{t-1}) = \int P(x_t \mid x_{t-1}, u_t) P(x_{t-1} \mid z_{t-1}) dx_{t-1}$$

- Correction

$$P(x_t \mid z_t) = \eta \bullet P(z_t \mid x_t) P(x_t \mid x_{t-1}, u_t, z_{t-1})$$



Particle Filter 194X~



Kalman Filter 1960~

Conclusion

Perception

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Decision

$$\frac{P(X = open \mid Z)}{P(X = close \mid Z)} > \frac{(R_{CC} - R_{OC})}{(R_{OO} - R_{CO})}$$

Feedback

$$\begin{cases} P(x_t \mid x_{t-1}, u_t, z_{t-1}) = \int P(x_t \mid x_{t-1}, u_t) P(x_{t-1} \mid z_{t-1}) dx_{t-1} \\ P(x_t \mid z_t) = \eta \bullet P(z_t \mid x_t) P(x_t \mid x_{t-1}, u_t, z_{t-1}) \end{cases}$$

Particle filter

Kalman filter

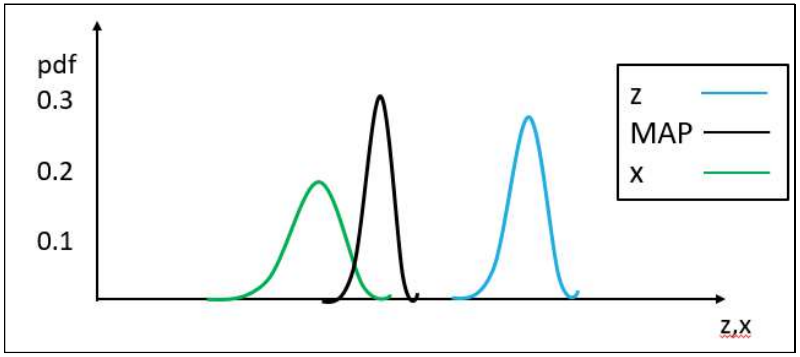
Conclusion

Feedback

$$\begin{cases} P(x_t \mid x_{t-1}, u_t, z_{t-1}) = \int P(x_t \mid x_{t-1}, u_t) P(x_{t-1} \mid z_{t-1}) dx_{t-1} \\ P(x_t \mid z_t) = \eta \bullet P(z_t \mid x_t) P(x_t \mid x_{t-1}, u_t, z_{t-1}) \end{cases}$$

Bayes filter (recursive)

Kalman filter



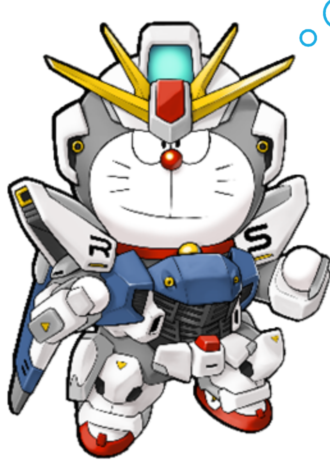
Gaussian
MMSE

$$MMSE \begin{cases} \mu_{xx|z} = \mu_x + \Sigma_{xz} \Sigma_{zz}^{-1} (\mathbf{z} - \mu_z) \\ \Sigma_{xx|z} = \Sigma_{xx} - \Sigma_{xz} \Sigma_{zz}^{-1} \Sigma_{zx} \end{cases}$$

Gaussian Processing

L10

Q&A



$$P(A \mid B,C) = \frac{P(B,C \mid A)P(A)}{P(B,\mathrm{C})}$$

Appendix A

$$P(x_t \mid u_{1:t}, z_{1:t}) = \frac{P(u_{1:t}, z_{1:t} \mid x_t)P(x_t)}{P(u_{1:t}, z_{1:t})} = \frac{P(u_{1:t}, z_t, z_{1:t-1} \mid x_t)P(x_t)}{P(u_{1:t}, z_t, z_{1:t-1})}$$

$$= \frac{P(z_t \mid z_{1:t-1}, u_{1:t}, x_t)\boxed{P(z_{1:t-1}, u_{1:t} \mid x_t)}P(x_t)}{P(z_t \mid z_{1:t-1}, u_{1:t})P(z_{1:t-1}, u_{1:t})}$$

$$P(A,C \mid B) = P(A \mid B,C)P(C \mid B)$$

$$= \frac{P(z_t \mid z_{1:t-1}, u_{1:t}, x_t)\boxed{P(z_{1:t-1}, u_{1:t}, x_t)}P(x_t)}{P(z_t \mid z_{1:t-1}, u_{1:t})P(z_{1:t-1}, u_{1:t})\boxed{P(x_t)}}$$

$$P(A \mid C) = P(A,C)/\boxed{P(C)}$$

$$= \frac{P(z_t \mid z_{1:t-1}, u_{1:t}, x_t)P(x_t \mid z_{1:t-1}, u_{1:t})P(z_{1:t-1}, u_{1:t})}{P(z_t \mid z_{1:t-1}, u_{1:t})P(z_{1:t-1}, u_{1:t})}$$

$$P(A,C) = P(A \mid C)\boxed{P(C)}$$

$$= \frac{P(z_t \mid u_{1:t}, z_{1:t-1}, x_t)P(x_t \mid u_{1:t}, z_{1:t-1})}{P(z_t \mid u_{1:t}, z_{1:t-1})}$$

$$= bel(x_t) = \eta \bullet P(z_t \mid x_t)\overline{bel}(x_t)$$

Appendix A

- Total probability

$$P(B) = \int P(B, A)dA$$

$$P(A \mid B) = \int P(A \mid B, C)P(C)dC$$

$$\overline{bel}(x_t) = P(x_t \mid u_{1:t}, z_{1:t-1})$$

$$= \int P(x_t \mid x_{t-1}, u_{1:t}, \cancel{z_{1:t-1}})P(x_{t-1} \mid u_{1:t}, z_{1:t-1})dx_{t-1}$$

$$= \int P(x_t \mid x_{t-1}, u_{1:t})bel(x_{t-1})dx_{t-1}$$

$$P(B) = \sum_{x=1}^{x=n} P(B \cap A_{x=i}) = \sum_{x=1}^{x=n} P(B \mid A_i)P(A_i)$$

Example:

$$\frac{P(R=1 \mid C=1)P(C=1)}{P(R=1)} = \frac{P(R=1 \mid C=1)P(C=1)}{P(R=1 \mid C=1)P(C=1) + P(R=1 \mid C=0)P(C=0)}$$

$$\overline{bel}(x_t) = \int P(x_t \mid x_{t-1}, u_{1:t})bel(x_{t-1})dx_{t-1}$$

$$bel(x_t) = \eta \bullet P(z_t \mid x_t)\overline{bel}(x_t)$$

Appendix A

- Dynamic Bayesian Network (DBN) & PDF
  - Prediction: $\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}$
  - Correction: $bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$

$$
\begin{cases}
P(x_t \mid x_{t-1}, u_t, z_{t-1}) = \int P(x_t \mid x_{t-1}, u_t) P(x_{t-1} \mid z_{t-1}) dx_{t-1} \\
P(x_t \mid z_t) = \eta \bullet P(z_t \mid x_t) P(x_t \mid x_{t-1}, u_t, z_{t-1})
\end{cases}
$$