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Some remarks about flux time derivative

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Abstract Very often, in physical exercises, it is necessary to differentiate an integral with respect to a parameter. Generally we differentiate after the integral evaluation but, occasionally, it is desirable, from theoretical point of view, to interchange the order of differentiation and integration. In this letter, we examine in a simple way the derivative under the flux integral getting the result in a very simple manner that could be useful from didactic point of view.

Keywords Differentiation under the integral sign · Leibniz integral rule · Flux rule · Faraday's law

Mathematics Subject Classification 00A35 Methodology of mathematics, didactics · 97I40 Differential calculus · 97I50 Integral calculus · 00A79 Physics

1 Introduction

Everyone knows the Leibniz rule for differentiating an integral function. If it is required to differentiate with respect to t the function $F(t) = \int_{\alpha(t)}^{\beta(t)} f(x,t)dx$ where the functions $f(x,t)$ and $\frac{\partial f(x,t)}{\partial t}$ are both continuous and the functions $\alpha(t)$ and $\beta(t)$ are both continuous and both have continuous derivatives, then we have [1]

$$\frac{dF(t)}{dt} = \beta'(t)f(\beta(t),t) - \alpha'(t)f(\alpha(t),t) + \int_{\alpha(t)}^{\beta(t)} \frac{\partial f(x,t)}{\partial t}dx \tag{1}$$

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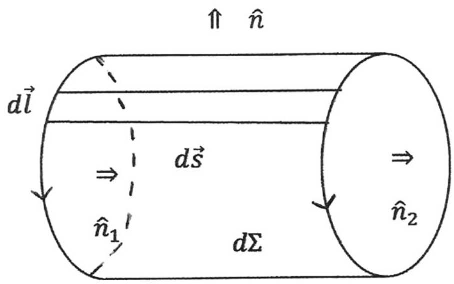
Fig. 1 Imaginary surface



This rule can be extended to 2-dimensional integrals $F(t) = \int_{\alpha(t)}^{\beta(t)} dx \int_{\gamma(t)}^{\delta(t)} f(x, y, t)dy$ obtaining for $\frac{dF}{dt}$ the following result

$$\beta'(t) \int_{\gamma(t)}^{\delta(t)} f(\beta, y, t)dy - \alpha'(t) \int_{\gamma(t)}^{\delta(t)} f(\alpha, y, t)dy + \int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial}{\partial t} \int_{\gamma(t)}^{\delta(t)} f(x, y, t)dy \right) dx \quad (2)$$

Moreover, if we have a flux integral $I = \int_S \vec{F}(r, t) \cdot d\vec{S}$, for time derivative $\frac{dI}{dt}$ we get

$$\int_S \left[\frac{\partial \vec{F}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{F} - \left(\vec{F} \cdot \vec{\nabla}\right) \vec{v} + \vec{F} \left(\vec{\nabla} \cdot \vec{v}\right) \right] \cdot d\vec{S} \quad (3)$$

where $\vec{v}$ is the velocity of motion of points of the surface. Equivalent relations are the following

$$\frac{dI}{dt} = \int_{S(t)} \left[\frac{\partial \vec{F}}{\partial t} + (\vec{\nabla} \cdot \vec{F})\vec{v} \right] \cdot d\vec{S} - \oint_l \left(\vec{v} \wedge \vec{F}\right) \cdot d\vec{l} \quad (4)$$

and

$$\frac{dI(t)}{dt} = \int_{S(t)} \left[\frac{\partial \vec{F}}{\partial t} + (\vec{\nabla} \cdot \vec{F})\vec{v} - \vec{\nabla} \wedge (\vec{v} \wedge \vec{F}) \right] \cdot d\vec{S}, \quad (5)$$

linked by

$$\vec{\nabla} \wedge (\vec{v} \wedge \vec{F}) = (\vec{\nabla} \cdot \vec{F} + \vec{F} \cdot \vec{\nabla})\vec{v} - (\vec{\nabla} \cdot \vec{v} + \vec{v} \cdot \vec{\nabla})\vec{F} \quad (6)$$

and by Kelvin–Stokes theorem. The three-dimensional generalization of the Leibniz integral rule is the Reynolds transport theorem

$$\frac{d}{dt} \int_{V(t)} F dV = \int_{V(t)} \frac{\partial}{\partial t} F dV + \int_{\partial V(t)} F \vec{v} \cdot d\vec{S} \quad (7)$$

The aim of this paper is to show a simple physicist's proof of a formula for the total derivative of flux across a moving and deforming surface that could be useful from didactic point of view (Fig. 1).

2 Time derivative of flux

Let us consider a closed line of any shape immersed in a vectorial field $\overrightarrow{F}$ constant in time. We suppose that the line moves and denote by $\overrightarrow{v}$ the velocity of its generic element $d\overrightarrow{l}$. We consider two time istants t_i and $t_f = t_i + dt$. In this time interval, $d\overrightarrow{l}$ moves of $d\overrightarrow{s} = \overrightarrow{v}\,dt$ and in this way it describes a small area $d\overrightarrow{S} = d\overrightarrow{l} \wedge d\overrightarrow{s} = d\overrightarrow{l} \wedge \overrightarrow{v}\,dt$. By considering the closed surface σ formed by $S(t_i)$, $S(t_f)$ and the lateral surface $d\Sigma$, thanks to the divergence theorem and using the usual convention for the normal, we can write

$$\Phi_\sigma = \Phi_f - \Phi_i + \Phi_{d\Sigma} = \int\limits_{A(t_f)} \overrightarrow{F} \cdot d\overrightarrow{A} - \int\limits_{A(t_i)} \overrightarrow{F} \cdot d\overrightarrow{A} + \int\limits_{d\Sigma} \overrightarrow{F} \cdot d\overrightarrow{S} = \int\limits_{V} \overrightarrow{\nabla} \cdot \overrightarrow{F}\,dV \quad (8)$$

Therefore

$$d\Phi = \Phi_f - \Phi_i = -\int\limits_{d\Sigma} \overrightarrow{F} \cdot d\overrightarrow{S} + \int\limits_{V} \overrightarrow{\nabla} \cdot \overrightarrow{F}\,dV \quad (9)$$

We have that $dV = d\overrightarrow{A} \cdot d\overrightarrow{s} = d\overrightarrow{A} \cdot \overrightarrow{v}\,dt$ and therefore

$$d\Phi = -\int\limits_{d\Sigma} \overrightarrow{F} \cdot d\overrightarrow{S} + \int\limits_{V} \overrightarrow{\nabla} \cdot \overrightarrow{F}\,dV = -\oint\limits_{l} \overrightarrow{F} \cdot (d\overrightarrow{l} \wedge \overrightarrow{v}\,dt) + \int\limits_{A} (\overrightarrow{\nabla} \cdot \overrightarrow{F})\overrightarrow{v} \cdot d\overrightarrow{A}\,dt \quad (10)$$

Remembering the vectorial identity

$$\overrightarrow{a} \cdot (\overrightarrow{b} \wedge \overrightarrow{c}) = (\overrightarrow{c} \wedge \overrightarrow{a}) \cdot \overrightarrow{b} \quad (11)$$

we obtain

$$d\Phi = -dt \oint\limits_{l} (\overrightarrow{v} \wedge \overrightarrow{F}) \cdot d\overrightarrow{l} + dt \int\limits_{A} (\overrightarrow{\nabla} \cdot \overrightarrow{F})\overrightarrow{v} \cdot d\overrightarrow{A} \quad (12)$$

Finally

$$\frac{d\Phi}{dt} = -\oint\limits_{l} (\overrightarrow{v} \wedge \overrightarrow{F}) \cdot d\overrightarrow{l} + \int\limits_{A} (\overrightarrow{\nabla} \cdot \overrightarrow{F})\overrightarrow{v} \cdot d\overrightarrow{A} \quad (13)$$

Now we consider a more general case of time dependent vector field. In this case we have

$$\frac{d\Phi}{dt} = \frac{\int_{A(t+dt)} \overrightarrow{F}(t+dt) \cdot d\overrightarrow{A} - \int_{A(t)} \overrightarrow{F}(t) \cdot d\overrightarrow{A}}{dt} \quad (14)$$

By using Taylor's theorem, $\overrightarrow{F}(t + dt) = \overrightarrow{F}(t) + \frac{\partial \overrightarrow{F}}{\partial t}dt$ and the second member of (14) becomes $\frac{\int_{A(t+dt)} (\overrightarrow{F}(t) + \frac{\partial \overrightarrow{F}}{\partial t}dt) \cdot d\overrightarrow{A} - \int_{A(t)} \overrightarrow{F}(t) \cdot d\overrightarrow{A}}{dt}$ finally getting

$$\frac{\int_{A(t+dt)} \overrightarrow{F}(t) \cdot d\overrightarrow{A} - \int_{A(t)} \overrightarrow{F}(t) \cdot d\overrightarrow{A}}{dt} + \int\limits_{A(t+dt)} \frac{\partial \overrightarrow{F}}{\partial t} \cdot d\overrightarrow{A} \quad (15)$$

The term $\frac{\int_{A(t+dt)} \overrightarrow{F}(t) \cdot d\overrightarrow{A} - \int_{A(t)} \overrightarrow{F}(t) \cdot d\overrightarrow{A}}{dt}$ is the variation of flux where $\overrightarrow{F}(t)$ is considered constant at the value it had at time t (as if it were "frozen" at the istant t). So variation is due only to the motion of the loop and it is equal to the second member of (13). Instead

$$\lim_{dt\to 0} \int\limits_{A(t+dt)} \frac{\partial \overrightarrow{F}}{\partial t} \cdot d\overrightarrow{A} = \int\limits_{A(t)} \frac{\partial \overrightarrow{F}}{\partial t} \cdot d\overrightarrow{A} \tag{16}$$

that is, it represents the variation due to the only change in time of $\overrightarrow{F}$. In conclusion we have the general case

$$\frac{d\Phi}{dt} = \int\limits_{A(t)} \left[\frac{\partial \overrightarrow{F}}{\partial t} + (\overrightarrow{\nabla} \cdot \overrightarrow{F})\overrightarrow{v} \right] \cdot d\overrightarrow{A} - \oint\limits_{l} (\overrightarrow{v} \wedge \overrightarrow{F}) \cdot d\overrightarrow{l} \tag{17}$$

3 Faraday–Lenz law

It is well know that Michael Faraday discovered the electromagnetic induction showing that [2–4]

$$EMF_{induction} = \oint\limits_{l} \overrightarrow{E} \cdot d\overrightarrow{l} = -\int\int\limits_{S} \frac{\partial \overrightarrow{B}}{\partial t} \cdot d\overrightarrow{S} \tag{18}$$

where EMF is the electromotive force, S is a surface that has a circuit l as its boundary [5–7]. Locally, thanks to the Kelvin–Stokes theorem, we get the third Maxwell equation [8–10]

$$\overrightarrow{\nabla} \wedge \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t} \tag{19}$$

Instead, if we have a moving wire, as a consequence of the Lorentz force, we can write [11–14]

$$EMF_{motional} = \oint [(\overrightarrow{v}_{drift} + \overrightarrow{v}) \wedge \overrightarrow{B}] \cdot d\overrightarrow{l} = \oint (\overrightarrow{v} \wedge \overrightarrow{B}) \cdot d\overrightarrow{l} \tag{20}$$

where v is the velocity of the boundary element. Moreover, *if the closed circuit is a loop of thin wire*, we obtain the so-called flux rule "The electromotive force in any closed circuit is equal to the negative of the time rate of change of the magnetic flux through the circuit". In fact the magnetic field is a divergence free vector field and we obtain [15–17]

$$EMF_{induction} + EMF_{motional} = -\int\int\limits_{S(t)} \frac{\partial \overrightarrow{B}}{\partial t} \cdot d\overrightarrow{S} + \oint\limits_{\partial S(t)} (\overrightarrow{v} \wedge \overrightarrow{B}) \cdot d\overrightarrow{l} = -\frac{d\Phi}{dt} \tag{21}$$

We note that, in this physical context, Stokes' theorem can not be applied to Eq. (21) and therefore the last equation is not equivalent to (5) because $\overrightarrow{v}$ is not a continuous field with continuous space derivatives.

4 Conclusion

The flux rule is described by

$$EMF = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int\limits_{S} \overrightarrow{B} \cdot d\overrightarrow{S} \tag{22}$$

where the integral is the magnetic flux. We emphasized that Eq. (21) is the only correct way to write down Faraday's law in the general case, but some other versions of Faraday's law, derived from mathematical point of view and often proposed in physics textbook, do not give the right result from physical point of view. Indeed we consider

$$-\frac{d\Phi}{dt} = -\int_{S(t)} \left[\frac{\partial \overrightarrow{B}}{\partial t} - \overrightarrow{\nabla} \wedge (\overrightarrow{v} \wedge \overrightarrow{B})\right] \cdot d\overrightarrow{S} \tag{23}$$

and

$$-\frac{d\Phi}{dt} = -\int_{S(t)} \left[\frac{\partial \overrightarrow{B}}{\partial t} + (\overrightarrow{v} \cdot \overrightarrow{\nabla})\overrightarrow{B}\right] \cdot d\overrightarrow{S} \tag{24}$$

If $\overrightarrow{B}$ is constant everywhere and the velocity has a constant value just on a part of the circuit and is zero elsewhere, Eqs. (23) and (24) erroneously predict a vanishing EMF. In fact Stokes' theorem can be applied only if the vector field $\overrightarrow{v} \wedge \overrightarrow{B}$ is a continuously differentiable vector function but $\overrightarrow{v}$ is not a continuous field with continuous space derivatives.

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