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Cutting Against the Grain: Volumes of Solids of Revolution via Cross-Sections Parallel to the Rotation Axis

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Among the first applications of the definite integral students meet in a calculus class is the calculation of volumes of solids of revolution. This is really just a special case of the general approach of slicing a solid and integrating the cross-sectional area in which the slices are circles centered along the axis of rotation. If the solid is obtained by rotating the graph of $y = f(x)$ for $a \leq x \leq b$ around the x-axis, then the resulting formula for the volume, dutifully memorized by students is

$$V = \pi \int_a^b (f(x))^2 \, dx.$$

But what if we slice the solid another way, via planes parallel to the xy-plane rather than perpendicular to it?

An excellent example arises via the graph of $y = 1/x$ for $x \geq 1$. The resulting surface, often called Gabriel's horn (or Torricelli's trumpet) is full of surprises; see Figure 1. The region in the plane that is rotated about the x-axis has infinite area, yet the solid of revolution has finite volume π [**5**, p. 574]. Even more counterintuitive is the fact that the surface area of the horn is infinite [**3**]. We can explain the finite volume by noting that the cross-sections are circles of radius $1/x$ and, as x increases, the areas of these slices approach zero rapidly enough to ensure an arbitrarily small volume for a sufficiently chosen tail. In particular, each slice has a finite area, quadratic in $1/x$, and it is easy to believe that they sum up to a finite number.

If instead we slice the horn parallel to the xy-plane, then we get slices of arbitrarily large area close to the infinite-area slice in the xy-plane. Yet somehow these add up to a finite volume. This seems odd, so we will see what insights come from working out these calculations.

Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/ucmj.
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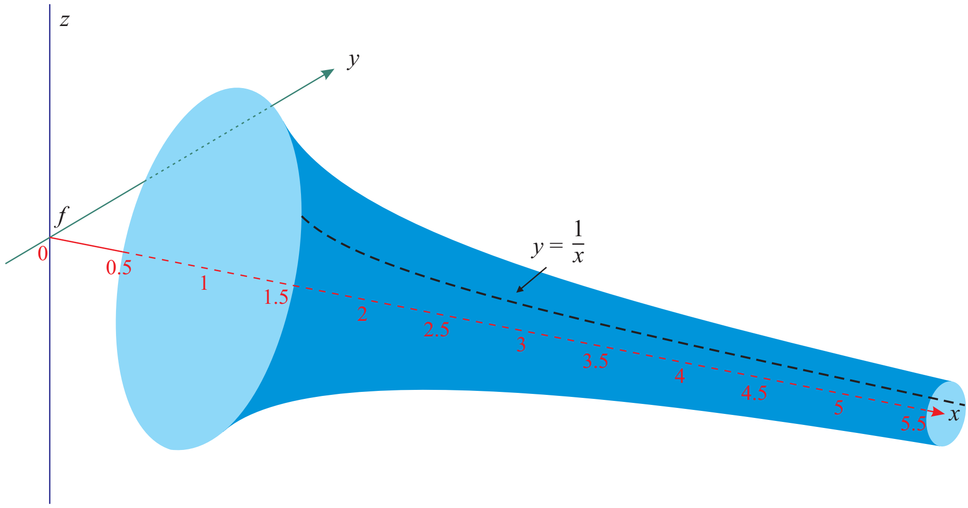

Figure 1. Gabriel's horn.

Solids of revolution

For our unconventional slicing, it is convenient to parameterize the surface. Assume we have a curve $y = f(x)$ for $a \le x \le b$ in the xy-plane and let R denote the region bounded by the curve and the x-axis. For simplicity, we assume that $f(x)$ is monotonic on the interval $[a, b]$. Rotate R about the x-axis to generate a solid S. The surface of S has a parametrization $(x, f(x)\cos\theta, f(x)\sin\theta)$ for $a \le x \le b$ and $0 \le \theta \le 2\pi$.

Suppose we slice the solid via the plane $z = c \ge 0$. What is the area $A(c)$ of the resulting cross-section of S? It is a region bounded by some curve, so its area can be computed as an integral, provided we can determine the equation of the curve. The parametrization of the surface of S allows us to do this: Equating z-coordinates gives $f(x)\sin\theta = c$. Therefore, $\sin^2\theta = c^2/f(x)^2$ and using the fundamental identity $\cos^2\theta + \sin^2\theta = 1$ gives

$$y^2 = (f(x))^2\cos^2\theta = (f(x))^2(1 - \sin^2\theta) = (f(x))^2 - c^2.$$

Now the region contained by this curve is symmetric about the x-axis, so we can compute the area

$$A(c) = 2\int \sqrt{(f(x))^2 - c^2}\,dx$$

where the limits of integration need to be determined as functions of the slice parameter c. Again using symmetry, we integrate the cross-sectional area to determine the volume of S,

$$V = 2\int_0^d A(c)\,dc$$

where $d = \max\{f(x) \mid a \le x \le b\}$, the upper bound on the possible slices. Note that our requirement for a monotonic f means that d is either $f(a)$ or $f(b)$ depending on whether f is decreasing or increasing.

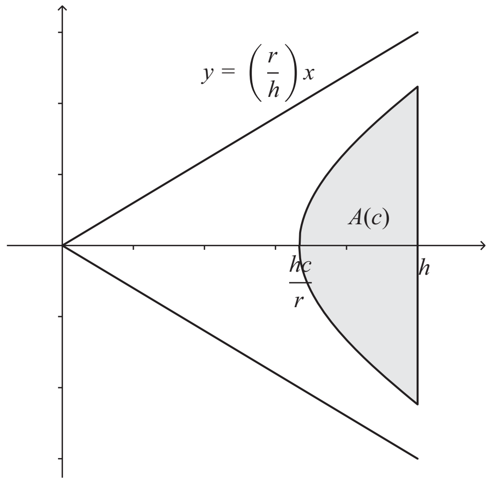

Figure 2. The hyperbolic cross-section of the cone.

The volume of a cone

First, we use this technique on a familiar example: Consider the right circular cone S with the base radius r and height h. This can be obtained by rotating $y = (r/h)x$ for $0 \le x \le h$ about the x-axis. The familiar disc method uses the circular cross-sections perpendicular to the x-axis. As shown in Figure 2, slicing S parallel to the xy-plane yields a different conic section—a hyperbola.

The calculations of the previous section tell us that the equation of this hyperbola is $y^2 = (r/h)^2x^2 - c^2$ and it intersects the positive x-axis at $x = hc/r$. Therefore,

$$A(c) = \frac{2}{h}\int_{hc/r}^{h} \sqrt{r^2x^2 - c^2h^2}\, dx.$$

We can evaluate this by the trigonometric substitution $rx = ch \sec t$ which, after some algebraic manipulation using the identity $\sec^2 t - 1 = \tan^2 t$, brings us to a constant times the integral of $\sec^3 t - \sec t$, a standard example. A routine if tedious calculation produces

$$A(c) = h\sqrt{r^2 - c^2} - \frac{c^2h}{r}\left(\ln(r + \sqrt{r^2 - c^2}) + \ln c\right).$$

Note that as $c \to 0^+$, we get the area of the triangular cross-section lying in the xy-plane having base $2r$ and height h (an application of L'Hôpital's rule helps with the limit of the second summand).

Now to find the volume of the cone, we integrate this function of c:

$$\begin{aligned} V &= 2\int_0^r \left[h\sqrt{r^2 - c^2} - \frac{c^2h}{r}\left(\ln(r + \sqrt{r^2 - c^2}) + \ln c\right)\right] dc \\ &= \frac{\pi r^2 h}{2} + \frac{2h}{r}\left(\frac{r^3}{3}\ln r - \frac{r^3}{9}\right) - \frac{2h}{r}\left(\frac{r^3}{3}\ln r + \frac{r^3}{3}\left[\frac{\pi}{4} - \frac{1}{3}\right]\right) \\ &= \frac{\pi r^2 h}{3}. \end{aligned}$$

Here are some notes on the solution. The first integrand gives the area of a quarter-circle of radius r and the rest of the integral can be managed via integration by parts. The integral of $c^2 \ln c$ is routine. The integral of $c^2 \ln(r + \sqrt{r^2 - c^2})$ is trickier; after performing integration by parts once, we are reduced to computing

$$\int \frac{c^4}{(r + \sqrt{r^2 - c^2})\sqrt{r^2 - c^2}}\, dc.$$

This may be calculated by a standard trigonometric substitution $c = r \sin t$, followed by some algebraic manipulations involving trigonometric identities for powers of $\cos t$. Substituting back to the original variable c and evaluating at the endpoints gives the familiar volume result.

An improper integral

Before addressing Gabriel's horn, consider the horn-shaped object obtained by rotating the graph of $y = e^{-x}$ for $x \geq 1$ about the x-axis. Its volume is easily calculated as

$$V = \pi \int_1^\infty e^{-2x}\, dx = \frac{\pi}{2e^2}.$$

With our unconventional cross-sections, cutting via the plane $z = c$ and using the integral derived before gives

$$A(c) = 2 \int_1^{-\ln c} \sqrt{e^{-2x} - c^2}\, dx.$$

This integral is not too difficult: First, substitute $u = e^{-x}$ to transform it into

$$-\int \frac{\sqrt{u^2 - c^2}}{u}\, du$$

and then make the substitution $u = c \sec t$ to reduce to the integral

$$-c \int \tan^2 t\, dt.$$

Replacing the integrand by $\sec^2 t - 1$ allows us to see

$$\begin{aligned} A(c) &= -2 \left(\sqrt{e^{-2x} - c^2} + c \arctan\left(\frac{c}{\sqrt{e^{-2x} - c^2}} \right) \right) \Bigg|_1^{-\ln c} \\ &= -\pi c + 2\sqrt{e^{-2} - c^2} + 2c \arctan\left(\frac{c}{\sqrt{e^{-2} - c^2}} \right). \end{aligned}$$

Note that as $c \to 0^+$, the cross-sections all have finite volume approaching $2/e$. To compute the volume of the solid we now must integrate this function of c. The first summand has a simple antiderivative and the second may be interpreted as the area of a portion of a circle. The third's antiderivative may be found via integration by parts

($dv = c\,dc$), which reduces it to the integral of $c^2(c^2 - r^2)/(r^2 - c^2)^{3/2}$, amenable to trigonometric substitution. So we see the volume of the solid is

$$
\begin{aligned}
V &= 2\int_0^{1/e} A(c)\,dc \\
&= 2\left[\left(-\frac{\pi c^2}{2}\right)\Bigg|_0^{1/e} + 2\left(\frac{\pi}{4e^2}\right)\right] + 4\int_0^{1/e} c\arctan\left(\frac{c}{\sqrt{e^{-2} - c^2}}\right) dc \\
&= \left[c\sqrt{e^{-2} - c^2} - (e^{-2} - 2c^2)\arctan\left(\frac{c}{\sqrt{e^{-2} - c^2}}\right)\right]_0^{1/e} \\
&= \frac{\pi}{2e^2}.
\end{aligned}
$$

Gabriel's horn

Now to our primary example, the solid S obtained by revolving about the x-axis the region R, bounded by $y = 1/x$ and the x-axis for $x \geq 1$. The surprising results are that the area of R is infinite, but Gabriel's horn S has finite volume, namely π. Let us try slicing this solid with planes parallel to the xy-plane.

The bounding surface has parametrization

$$\left(x, \frac{1}{x}\cos\theta, \frac{1}{x}\sin\theta\right)$$

for $x \geq 1$ and $0 \leq \theta \leq 2\pi$. Following the ideas of the previous sections, slicing this surface by the plane $z = c$ for $0 < c \leq 1$ gives the bounding curve $y^2 = 1/x^2 - c^2$. Note that this curve intersects the x-axis at $x = 1/c$ (see Figure 3). The area of this slice is then

$$A(c) = 2\int_1^{1/c} \frac{\sqrt{1 - c^2x^2}}{x}\,dx.$$

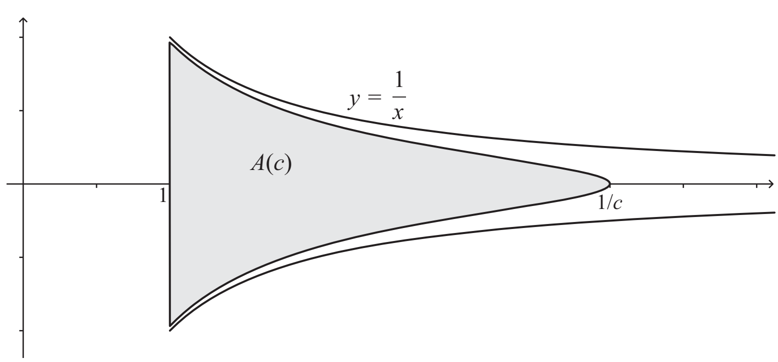

Figure 3. The horizontal cross-section of Gabriel's horn.

The substitution $cx = \sin t$ reduces this to the integral of $\csc t - \sin t$ and thus

$$
\begin{aligned}
A(c) &= 2\left(-\ln\left|\frac{1}{cx} + \frac{\sqrt{1-c^2x^2}}{cx}\right| + \sqrt{1-c^2x^2}\right)\Bigg|_1^{1/c} \\
&= 2\ln\left(\frac{1+\sqrt{1-c^2}}{c}\right) - 2\sqrt{1-c^2}.
\end{aligned}
$$

Observe that $\lim_{c\to 0^+} A(c) = \infty$, as we expect since the slice of S in the xy-plane has infinite area. However, the areas $A(c)$ tend to infinity at the rate of $\ln u$ as $u \to \infty$, a rather slow rate (e.g., $\ln 10^{100}$ is only about 230).

We are now able to calculate the volume of Gabriel's horn.

$$
\begin{aligned}
V &= 2\int_0^1 A(c)\,dc \\
&= 4\int_0^1 \left(-\ln c + \ln(1+\sqrt{1-c^2}) - \sqrt{1-c^2}\right) dc \\
&= 4\left[\,(-c\ln c + c)\Big|_0^1 - \frac{\pi}{4} + \int_0^1 \ln(1+\sqrt{1-c^2})\,dc\right] \\
&= 4\left[1 - \frac{\pi}{4} + \left(c\ln(1+\sqrt{1-c^2}) - \arcsin(\sqrt{1-c^2}) + c\right)\Big|_0^1\right] \\
&= \pi.
\end{aligned}
$$

In that computation, the antiderivative of $\ln(1+\sqrt{1-c^2})$ comes from integration by parts, reducing it to

$$
\int \frac{c^2}{(1-c^2)+\sqrt{1-c^2}}\,dc,
$$

then the substitution $c = \sin t$ leaves the integral of $1 - \cos t$.

This calculation is perhaps more surprising than the standard method for finding the volume of Gabriel's horn. In the familiar computation, it is easy to explain that if one lets x get large enough, the discs of radius $1/x$ at the tail of the horn have very small area and are thus inconsequential in the limit. Here, however, we have slices whose areas are becoming arbitrarily large as we get closer to the xy-plane, albeit very slowly. So slowly, in fact, that when we stack them up they fill only a finite volume. The heuristic for why this happens is that when c is close to 0, the area $A(c)$ is on the order of $\ln(1/c)$ and so the volume of the cylinder of height c on this slice would have volume on the order of $c\ln(1/c)$. This is a reasonable approximation to the slab of the solid near zero, and since c goes to 0 much faster than $\ln(1/c)$ goes to ∞, we see that this slab has vanishing volume in the limit.

To be more precise, if we set some threshold D for c near 0, then the volume of the band of slices between $-D$ and D is

$$
4\left(\frac{\pi}{2} - \arcsin(\sqrt{1-D^2}) + 2D - \frac{D}{2}\sqrt{1-D^2} + D\ln\left(\frac{1+\sqrt{1-D^2}}{D}\right) - \arcsin D\right).
$$

It is easy to see that this goes to 0 as $D \to 0^+$.

This example presents a good opportunity to discuss the rate of growth of the natural logarithm function. While $\ln u$ does become arbitrarily large, it is certainly in no hurry to get there. When viewed as cross-sectional areas of an object with finite volume, it becomes clear just how slowly the logarithm function grows.

Suggestions for further study

Gabriel's horn is a popular object; further information about the history of the calculation of its volume may be found in [1]. See [4] for an example of a bounded surface with finite volume and infinite surface area. And [2] considers higher dimensional generalizations of solids of revolution.

Readers are encouraged to try slicing their own examples of solids of revolution in the method described here. One interesting surface is the paraboloid obtained by rotating the graph of $y = \sqrt{x}$ about the x-axis: The resulting cross-sections are easily seen to be parabolas, so it is possible to compute their areas without calculus (Archimedes gave a formula). You might also try $y = \sqrt[4]{x}$ for $0 \le x \le 16$, and $y = \sqrt{\ln x}$ for $1 \le x \le 2$.

Note that applying the technique to $\sqrt{\ln x}$ will involve an error function. Indeed, coming up with examples with a manageable integral for computing the cross-sectional area is difficult precisely because we end up integrating a square root. For example, innocuous functions such as $x^{3/2}$, x^2, or $\sin x$ lead to cross-sections whose areas involve integrals of functions with no simple antiderivative. In such cases, it may still be an interesting exercise to estimate $A(c)$ as a function of c via Simpson's method, and then use that to estimate the volume of the solid.

Another question to consider is what happens when the function $f(x)$ is not monotone. In that case, some slices may have several disjoint components. This complicates matters, of course, but it provides substantive food for thought. Since improper integrals of nonmonotone functions are often poorly behaved (that is, convergence may be difficult to prove), here is an example of a bounded domain to try: $f(x) = |x - 2|$ for $0 \le x \le 3$. The surface obtained by rotating this graph around the x-axis is the union of two cones joined at their vertices and it has a total volume of 3π.

Summary. Rather than slicing solids of revolution perpendicular to the axis of rotation, we consider what happens when we cut the surface with planes parallel to the axis. Using this approach to explore the solid known as Gabriel's horn, which has finite volume and infinite surface area, yields insights into the growth rate of the logarithm function.

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