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Using origami boxes to explore concepts of geometry and calculus

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The purpose of this classroom note is to provide an example of how a simple origami box can be used to explore important concepts of geometry and calculus. This article describes how an origami box can be folded, then it goes on to describe how its volume and surface area can be calculated. Finally, it describes how the box could be folded to maximize the surface area and the volume.

Keywords: origami; geometry; calculus; visual; concrete

The instruction of mathematics has changed dramatically over the last 40 years or so. Mathematics is no longer a subject that is lifeless, cold and austere. It is dynamic, colourful and accessible. It is something we all *understand*, *enjoy* and *appreciate*. Moreover, mathematics is to be made concrete and meaningful whenever possible [1]. According to D'Ambrosio, since mathematics is a cultural endeavour, it is ideal to teach mathematics as it manifests in the context of various cultures. Teachers must learn special instructional skills to accommodate different backgrounds and different learning strategies [2]. Appreciation of cultural diversity is not only important for the minority groups, but it is also important for the dominant ethnic group in any society because even the members of the dominant ethnic group will be working in an environment that is increasingly getting more and more diverse [3]. The teaching and

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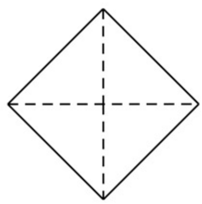
learning of mathematics can provide a powerful context for the appreciation of cultural diversity of our world.

For instance, a teacher can use pictures of beautiful North American quilt patterns to help her/his students explore and understand the meaning of transformation and symmetry. Another example could be the use of origami to teach important concepts of geometry or even calculus. Origami, or paper folding, creates a concrete context for exploring abstract ideas in mathematics. This not only provides opportunities for students to understand mathematics conceptually, but also creates a powerful platform for communication between teacher and students and among students. The purpose of this article is to introduce an origami project that is accessible to beginners and explore a few interesting mathematical properties of the origami model.

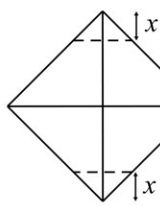
We will first use a square sheet of paper to construct a box. We will then find the internal surface area and the volume of the box. Finally, we will explore how one can fold this box to achieve the maximum possible volume, given the length of the square paper is fixed. In order to understand the mathematical components of the paper, one needs to have an understanding of elementary Euclidean geometry and calculus. Anyone with a bit of curiosity can follow the directions and make the box. No experience in origami is needed. Let us now follow the next few steps to construct the box. These steps are described in Figures 1–4.

Step 01



Start with a square sheet of paper, make sure that the plain side is facing up. Fold along the diagonals and form sharp creases, and then unfold.

Step 02



Remove the two triangular corners by cutting along the broken lines. These broken lines should be perpendicular to the vertical diagonal of the square sheet as shown above. The two triangular corners removed must be congruent and isosceles. Make sure x is about one-fourth of the diagonal of the original square sheet (precision is not necessary).

Figure 1. Construction of the box: Step 01–Step 02.

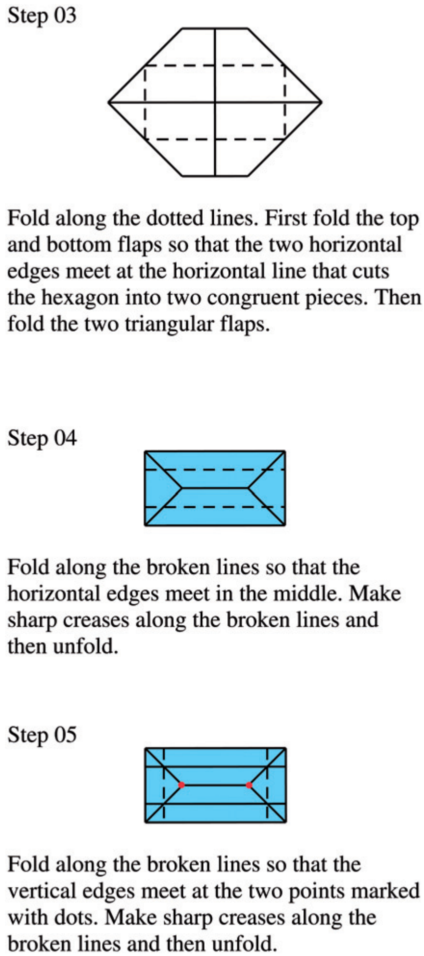


Figure 2. Construction of the box: Step 03–Step 05.

The fold in Step 10 can be understood better with the help of the photograph shown in Figure 5. In Step 02 we cut off two triangular corners. Traditional origami does not generally involve cutting. An essentially identical box can be made without cutting those triangular corners. We can just fold those triangular corners (instead of cutting them off), over the plain side of the square sheet in Step 02 and pretend that the triangular flaps are not there. If we follow the remaining steps, we will get a box that will essentially have the same properties as the one we have created by following the above steps. That is, the dimensions and the outward look of both boxes (one made from a square sheet with triangles cut off, and the other made from a similar square sheet with triangles folded) will be identical. In Step 02, the value of x, determines the dimensions of the box. Note when x is 0 the constructed box has a square base. Figure 6 shows two completed boxes, one with $x = 0$, the other with $x > 0$.

Now we have made a box with a rectangular base, see Figure 6. Let us now explore some of its properties. Suppose $2d$ is the length of the diagonal of the original

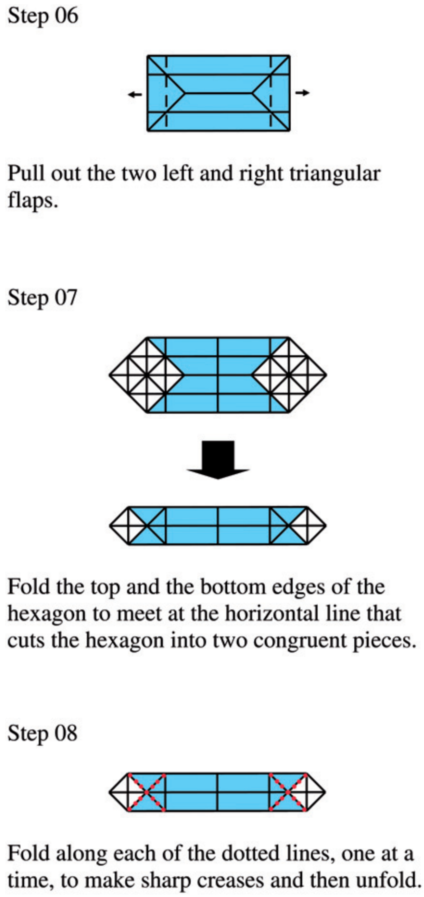


Figure 3. Construction of the box: Step 06–Step 08.

square sheet of paper that was used to make the rectangular box, and x is the height of each of the folded triangle. Note $x < d$. Suppose a, b, c represent the height, width and length of the rectangular box, respectively, see Figure 7.

By using elementary Euclidean geometry of similar triangles we can deduce the following.

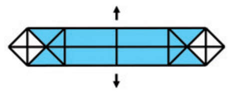
$$a = \frac{d - x}{4}, \tag{1}$$

$$b = \frac{d - x}{2}, \tag{2}$$

$$c = \frac{d + 3x}{2}. \tag{3}$$

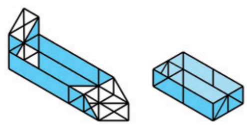
In order to express a, b and c in terms of d and x, one can fold and then unfold a box and analyse the creases thus created. Figure 8 shows the creases. Note the shaded triangles at the two corners are missing because they were cut off. The shaded dodecagon represents the parts of the sheet that *become* the four walls and the floor of the created box.

Step 09



Pull out the top and bottom flaps so that these trapezoidal flaps are perpendicular to the hexagonal base.

Step 10



Fold the left triangular flap up as shown above in the first picture, then fold it inside. Fold the right triangular flap (shown in the first picture) in a similar manner to form the box with a rectangular base.

Figure 4. Construction of the box: Step 09–Step 10.

The internal surface area of the box is $2ab+bc+2ac$. By using results (1), (2) and (3) we get the following.

$$2ab + bc + 2ac = \frac{3d^2 + 2dx - 5x^2}{4}.$$

Let

$$S(x) = \frac{3d^2 + 2dx - 5x^2}{4},$$

then

$$S'(x) = \frac{d - 5x}{2}.$$

$$S'(x) = 0 \Rightarrow x$$

$$S\left(\frac{d}{5}\right) = \frac{4d^2}{5}.$$

It turns out that the function $S(x)$ has its maximum at $x = \frac{d}{5}$, and the maximum value of the function is $\frac{4d^2}{5}$.

The graph in Figure 9 shows the relationship between the values of x and the internal surface of the box, $S(x)$, created with a sheet of square paper with a 20-unit long diagonal, that is, $d = 10$. Note the dot on the curve signifies the point where $S(x)$ reaches its maximum. The x- and y-coordinates of this point are 2 and 80, respectively.

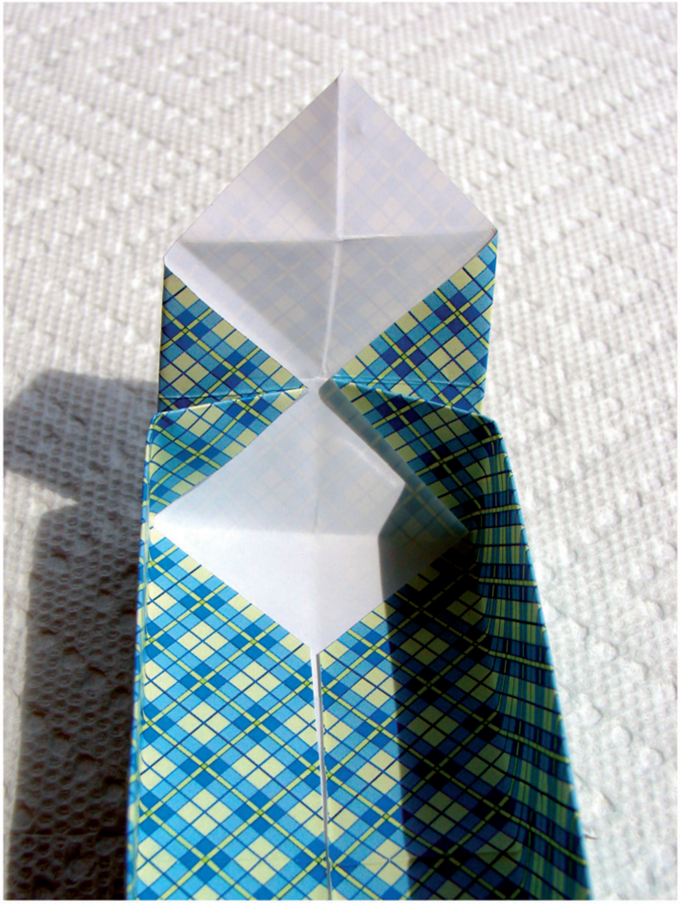


Figure 5. Photograph to understand Step 10.

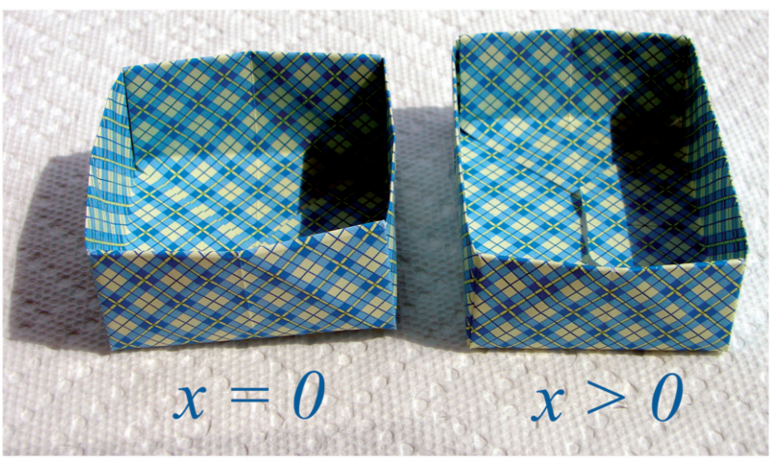


Figure 6. A photograph of two completed boxes.

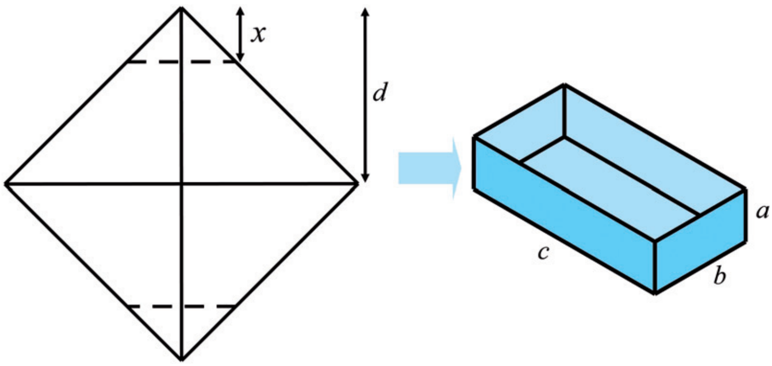


Figure 7. Measurements of the original square sheet and the dimensions of the constructed box.

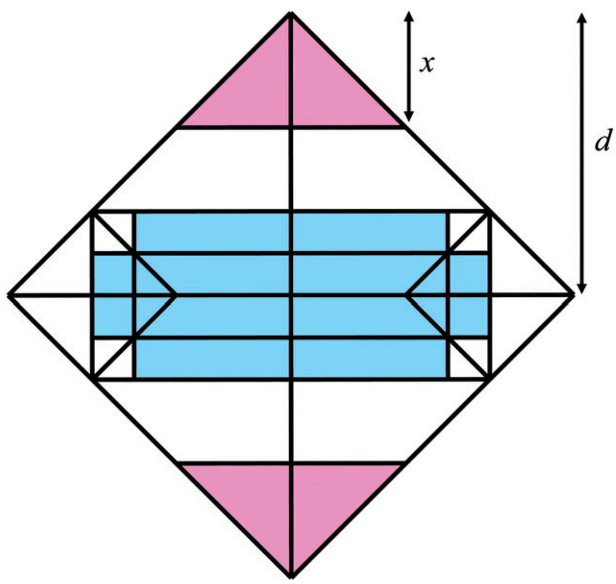


Figure 8. Parts of original sheet showing the creases created by the folds.

The volume of the box is abc. By using results (1), (2) and (3) we get the following.

$$abc = \frac{3x^3 - 5dx^2 + d^2x + d^3}{16}.$$

Let

$$V(x) = \frac{3x^3 - 5dx^2 + d^2x + d^3}{16}.$$

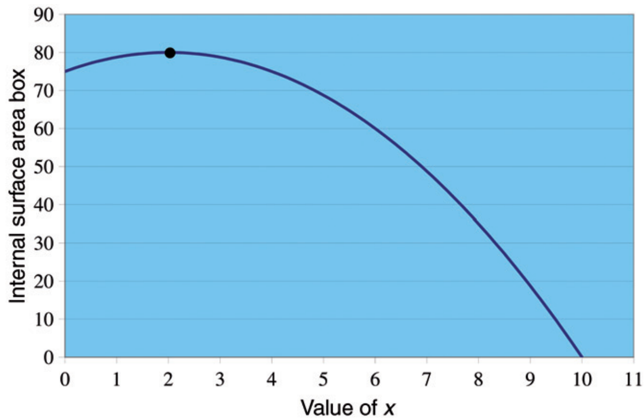


Figure 9. The graph of $S(x)$ plotted against the values of x.

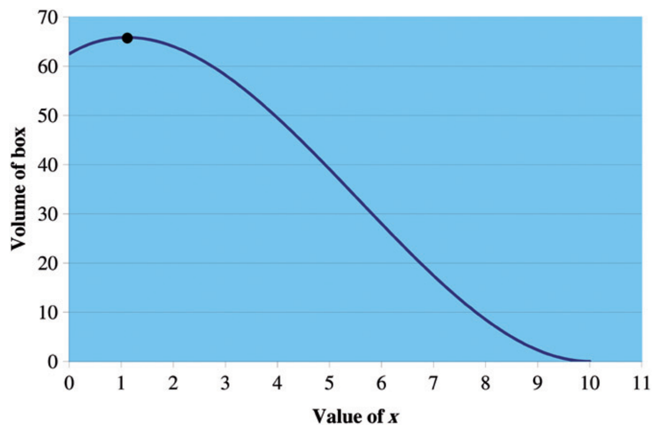


Figure 10. The graph of $V(x)$ against the values of x.

Let us now find out for what values of x, $V(x)$ is minimum or maximum.

Since the volume is a function of x, note d is a constant.

$$V(x) = \frac{3x^3 - 5dx^2 + d^2x + d^3}{16},$$
$$V'(x) = \frac{1}{16}(9x^2 - 10dx + d^2).$$

Note

$$V'(x) = 0$$
$$\Rightarrow 9x^2 - 10dx + d^2 = 0$$
$$\Rightarrow x = d, \frac{d}{9}.$$

Since $x < d$, we only consider $x = \frac{d}{9}$.

$$V\left(\frac{d}{9}\right) = \frac{16d^3}{243}.$$

It turns out that the function $V(x)$ has its maximum at $x = \frac{d}{9}$, and the maximum value of $V(x)$ is $\frac{16d^3}{243}$.

The graph in Figure 10 shows the relationship between the values of x and the volume of the box, $V(x)$, created with a sheet of square paper with a 20-unit long diagonal, that is, $d = 10$. Note the dot on the curve signifies the point where $V(x)$ reaches a maximum. The x- and y-coordinates of this point are 1.11 and 65.85, respectively.

It is interesting to note that the function $S(x)$ is increasing over the interval $(0, d/5)$. That is, over this interval, the larger the areas of the removed triangles, the larger the surface area of the box. Also note that the function $V(x)$ is increasing over the interval $(0, d/9)$. That is, over this interval, the larger the areas of the removed triangles, the larger the volume of the box. This seems counter intuitive because one may think that the larger the areas of the removed triangles, the smaller the surface area and the volume of the box.

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Exploring volumetrically indexed cups

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This article was inspired by a set of 12 cylindrical cups, which are volumetrically indexed; that is to say, the volume of cup n is equal to n

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