Sparsity, Learning, and Endogenous Predictability in Asset Markets

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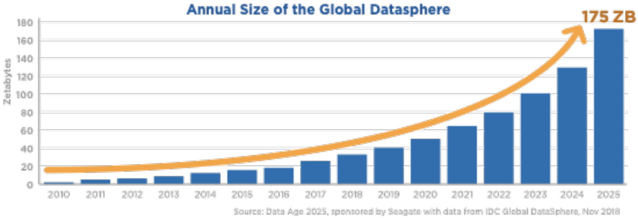
Information availability grows exponentially



Baseline Framework

- Standard **Lucas (1978)** economy with a continuum of risk–neutral agents.
- One risky asset paying stochastic dividends D_t.
- Dividends and consumption follow log–normal growth:

$$\frac{D_t}{D_{t-1}} = a\varepsilon_t^d, \quad \frac{C_t}{C_{t-1}} = a\varepsilon_t^c.$$

- Shocks:

$$\begin{pmatrix} \log \varepsilon_t^d \\ \log \varepsilon_t^c \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} -s_d^2/2 \\ -s_c^2/2 \end{pmatrix}, \begin{pmatrix} s_d^2 & \rho s_d s_c \\ \rho s_d s_c & s_c^2 \end{pmatrix} \right).$$

- Expected dividend growth: $\mathbb{E}_t D_{t+1} = aD_t$.

Equilibrium Pricing under RE

- No–arbitrage condition:

$$P_t = \delta \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} (P_{t+1} + D_{t+1}) \right].$$

- Under rational expectations:

$$P_t = \frac{\delta a^{1-\gamma} \rho_e}{1 - a^{1-\gamma} \rho_e} D_t, \quad \rho_e = e^{\gamma(1+\gamma) s_c^2/2 - \gamma \rho s_c s_d}.$$

- Price–dividend ratio constant.
- Gross return:

$$R_{t+1} = \frac{P_{t+1}}{P_t} = a \varepsilon^d_{t+1}.$$

- Log–return: $r_{t+1} = \log a + \log \varepsilon^d_{t+1}$.

Adaptive Learning Setup

- Agents are **boundedly rational**: they estimate expected returns via econometrics.
- Perceived Law of Motion (PLM):

$$r_{t+1} = x_t\beta + u_t, \quad x_t \sim \mathcal{N}(0, \sigma_x^2).$$

- Gross expected return:

$$\mathbb{E}_t(R_{t+1}) = \phi e^{x_t\beta}, \quad \phi = e^{\sigma_u^2/2}.$$

- Plug into pricing equation:

$$P_t = \frac{\delta a^{1-\gamma}\rho_e}{1 - \delta a^{-\gamma}\phi e^{x_t\beta}} D_t, \quad \kappa = \delta a^{-\gamma}\phi.$$

Actual Law of Motion (ALM)

- From price dynamics:

$$R_t = \frac{1 - \kappa e^{x_{t-1}\beta}}{1 - \kappa e^{x_t\beta}} \frac{D_t}{D_{t-1}}.$$

- Log form:

$$r_{t+1} = \log \varepsilon_{t+1} + \log(1 - \kappa e^{x_t\beta}) - \log(1 - \kappa e^{x_{t+1}\beta}).$$

- Nonlinear mapping between beliefs and realized returns $\Rightarrow$ PLM is misspecified.
- Agents update β recursively $\Rightarrow$ stochastic recursive system.

5. Cross-Moment Consistent Equilibrium

- Equilibrium condition:

$$T(\beta) - \beta = 0, \quad T(\beta) = \frac{\mathrm{Cov}(r_{t+1}, x_t)}{\mathrm{Var}(x_t)}.$$

- Approximation (Taylor around $x_t = 0$):

$$\log(1 - \kappa e^{x_t \beta}) \approx \log(1 - \kappa) - \frac{\kappa}{1 - \kappa} x_t \beta + \frac{\kappa}{2(1 - \kappa)^2} (x_t \beta)^2.$$

- Then:

$$T(\beta) \approx -\frac{\kappa}{1 - \kappa} \beta \quad \Rightarrow \quad \beta^* = 0.$$

- Stable if $\kappa < 1$.

Learning Algorithms

Recursive Least Squares (RLS):

$$\beta_t = \beta_{t-1} + t^{-1} S_{t-1}^{-1} x_{t-2}(r_{t-1} - x_{t-2}\beta_{t-1}), \quad S_t = S_{t-1} + t^{-1}(x_{t-2}^2 - S_{t-1}).$$

- Fits canonical stochastic recursive algorithm form.
- Stability $\Leftrightarrow$ ODE $\frac{d\beta}{d\tau} = T(\beta) - \beta$.

Constant-Gain (WLS):

$$b_t = b_{t-1} + \gamma M_t^{-1} x_t (y_t - x_t b_{t-1}),$$

- Gain $\gamma \in (0, 1)$ constant $\Rightarrow$ perpetual fluctuations.
- Stationary distribution:

$$\beta_t \sim \mathcal{N}\left(0,\ \gamma s_d^2 [2\sigma_x^2 (1 + \tfrac{\kappa}{1-\kappa})]^{-1}\right).$$

LASSO Learning and Implications

- Agents use LASSO estimator:

$$\beta_{LASSO} = \text{sign}(\beta_{OLS}) \max(0, |\beta_{OLS}| - \lambda).$$

- Mapping: $\beta_{OLS} = -\frac{\kappa}{1-\kappa}\beta_{LASSO}$.
- Only $\beta^* = 0$ stable for $\kappa < 1$. item Probability of nonzero estimate:

$$\mathbb{P}(|\beta_{OLS}| > \lambda) = 1 - [\Phi(\lambda/\sigma_{OLS}) - \Phi(-\lambda/\sigma_{OLS})].$$