ESAPPM 446 Final Project

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March 2022

1 Introduction

When I was picking a 2D PDE, I wanted to choose something that was related to finance, as that is ultimately the field to which I wish to apply my applied math modeling skills. There are many partial differential equations in the field of finance, and the most famous of which is probably the Black-Scholes equation; however, that equation is not 2D. So I researched different ways that financial researchers using PDEs to model the market, and found that the Feynman-Kac method is a very common technique.

The Feynman-Kac formula solves certain differential equations by simulating random paths of a stochastic process, and essentially transforms a PDE of n dimensions into a conditional expectation of an Ito process. However, as I read more into the Feynman-Kac process and its use cases in Ito calculus, it became apparent that I do not know how to model stochastic processes and use stochastic random variables to set up conditional expectations that equate to the generalized PDE. So in this project I will be mostly analyzing solutions to PDEs that can result from the generalized PDE from the Feynman-Kac equation.

The equation that we are setting up is the general solution that the Feynman–Kac formula attempts to solve using expected value, and it is

$$\partial_t u + \mu \nabla u + \frac{1}{2}\sigma^2 \mathrm{H}_u u - Vu = f$$

Where u is a 2D function of x and y, and we will make u periodic in x with boundary conditions $u(\pm y) = 0$. ∇ is the gradient of u, and H_u is the Hessian matrix of u. In the Feynman-Kac PDE transformation, the Hessian is used, however, as we will see in alternative equations of this form, many times the Laplacian is used

$$\partial_t u + \mu \nabla u + \frac{1}{2}\sigma^2 \nabla^2 u - Vu = f$$

The solution of this equation can be written as a conditional expectation

$$u = E^Q[\int_t^T e^{-\int_t^r V(X_\tau,\tau} d\tau) f(X_r, r) dr + e^{-\int_t^T V(X_\tau,\tau) d\tau} \psi(X_T) | X_T = x]$$

where X is an Ito process driven by the equation

$$dX = \mu(X,t)dt + \sigma(X,t)dW^Q$$

2 Setup and Derivation

$$\partial_t u + \mu \nabla u + \frac{1}{2}\sigma^2 \mathrm{H}_u u - Vu = f$$

We separate the general PDE into two variables u and u_y, we let

$$\partial_y u = u_y \to \partial_y u - u_y + \tau_1 = 0$$

Where τ_1 is a variable needed to constrain the boundary conditions. The above equation can then be split into a system of equations

$$\partial_y u - u_y + \tau_1 = 0 \tag{1}$$

$$\partial_t u + \mu \partial_x u + \mu u_y + \frac{1}{2}\sigma^2 \partial_x^2 u + \frac{1}{2}\sigma^2 \partial_y u_y + \frac{1}{2}\sigma^2 \partial_x u_y + \frac{1}{2}\sigma^2 \partial_y \partial_x u - Vu + \tau_2 = f \tag{2}$$

We discretize the variables u, u_y, τ_1, τ_2 by

$$u = \sum_{n,m} a_{n,m} \cos(nx) T_m(y) - \sum_{n,m} b_{n,m} \sin(nx) T_m(y)$$

$$u_y = \sum_{n,m} c_{n,m} \cos(nx) T_m(y) - \sum_{n,m} d_{n,m} \sin(nx) T_m(y)$$

$$\partial_x u = -n \sum_{n,m} b_{n,m} \cos(nx) T_m(y) - n \sum_{n,m} a_{n,m} \sin(nx) T_m(y)$$

$$\partial_x^2 u = -n^2 \sum_{n,m} a_{n,m} \cos(nx) T_m(y) + n^2 \sum_{n,m} b_{n,m} \sin(nx) T_m(y) = -n^2 u$$

$$\tau_1 = \sum_n \tau_{1,n} \cos(nx) - \sum_n \tilde{\tau}_{1,n} \sin(nx)$$

$$\tau_2 = \sum_n \tau_{2,n} \cos(nx) - \sum_n \tilde{\tau}_{2,n} \sin(nx)$$

Plugging these discretized equations back into equations (1) and (2) and projecting out,

$$Da_{n,m} - Cc_{n,m} + \tau_{1,n} = 0$$

$$Db_{n,m} - Cd_{n,m} + \tilde{\tau}_{1,n} = 0$$

$$\begin{aligned}\partial_t C a_{n,m} - \mu C n b_{n,m} + \mu C c_{n,m} - \frac{1}{2}\sigma^2 n^2 C a_{n,m} + \frac{1}{2}\sigma^2 D c_{n,m} - \frac{1}{2}\sigma^2 C n d_{n,m} \\ - \frac{1}{2}\sigma^2 D n b_{n,m} - CV a_{n,m} + \tau_{2,n} = Cf\end{aligned}$$

$$
\partial_t C b_{n,m} + \mu C n a_{n,m} + \mu C d_{n,m} - \frac{1}{2}\sigma^2 n^2 C b_{n,m} + \frac{1}{2}\sigma^2 D d_{n,m} + \frac{1}{2}\sigma^2 C n c_{n,m} \\ + \frac{1}{2}\sigma^2 D n a_{n,m} - C V b_{n,m} + \tilde{\tau}_{2,n} = C f
$$

The vector of variables is

$$
X_n = [a_{n,m}, b_{n,m}, c_{n,m}, d_{n,m}, \tau_{1,n}, \tilde{\tau}_{1,n}, \tau_{2,n}, \tilde{\tau}_{2,n}]^T
$$

and the RHS is represented by

$$
F = [0, 0, Cf, Cf, 0, 0, 0, 0]^T
$$

The resulting M and L matrices are then

$$
M = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
L = \begin{bmatrix}
D & 0 & -C & 0 & \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} & 0 & 0 & 0 \\
0 & D & 0 & -C & 0 & \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} & 0 & 0 \\
-\frac{1}{2}\sigma^2 n^2 C - CV & -\mu C n - \frac{1}{2}\sigma^2 D n & \mu C + \frac{1}{2}\sigma^2 D & -\frac{1}{2}\sigma^2 C n & 0 & 0 & \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} & 0 \\
\mu C n + \frac{1}{2}\sigma^2 D n & -\frac{1}{2}\sigma^2 n^2 C - CV & \frac{1}{2}\sigma^2 C n & \mu C + \frac{1}{2}\sigma^2 D & 0 & 0 & 0 & \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \\
(1, -1, 1, -1, \ldots) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1, 1, 1, 1, \ldots) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (1, -1, 1, -1, \ldots) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (1, 1, 1, 1, \ldots) & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

If we are only concerning ourselves with the Laplacian of u, then the following L matrix should be used

$$
L = \begin{bmatrix}
D & 0 & -C & 0 & \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix} & 0 & 0 & 0 \\
0 & D & 0 & -C & 0 & \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix} & 0 & 0 \\
-\frac{1}{2}\sigma^2 n^2 C - CV & -\mu C n & \mu C + \frac{1}{2}\sigma^2 D & 0 & 0 & 0 & \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix} & 0 \\
\mu C n & -\frac{1}{2}\sigma^2 n^2 C - CV & 0 & \mu C + \frac{1}{2}\sigma^2 D & 0 & 0 & 0 & \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix} \\
(1,-1,1,-1,\ldots) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1,1,1,1,\ldots) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (1,-1,1,-1,\ldots) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (1,1,1,1,\ldots) & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

3 Example Simulations and Verification

From the equation

$$\partial_t u + \mu \nabla u + \frac{1}{2}\sigma^2 \nabla^2 u - Vu = f$$

we can turn this into the non-homogeneous heat equation or the reaction-diffusion equation by setting $\mu = V = 0$ and $\frac{1}{2}\sigma^2 < 0$. If we set the RHS f to be $u(1-u^2)$, aka the Rayleigh-Bernard convection, we get the following evolution.

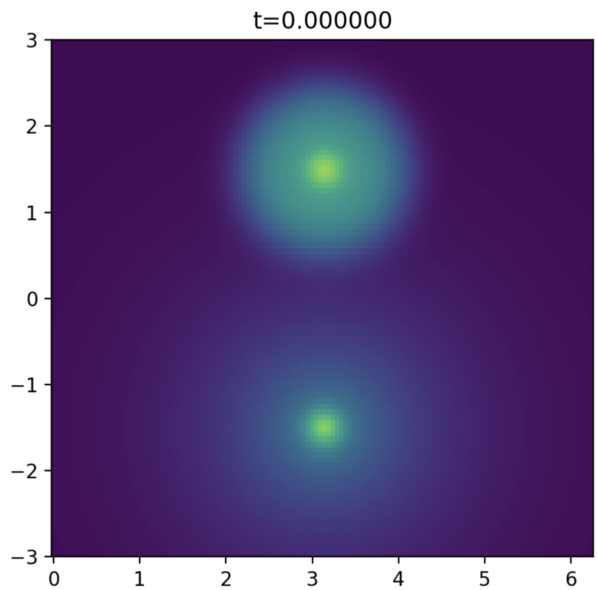

Figure 1: T = 0

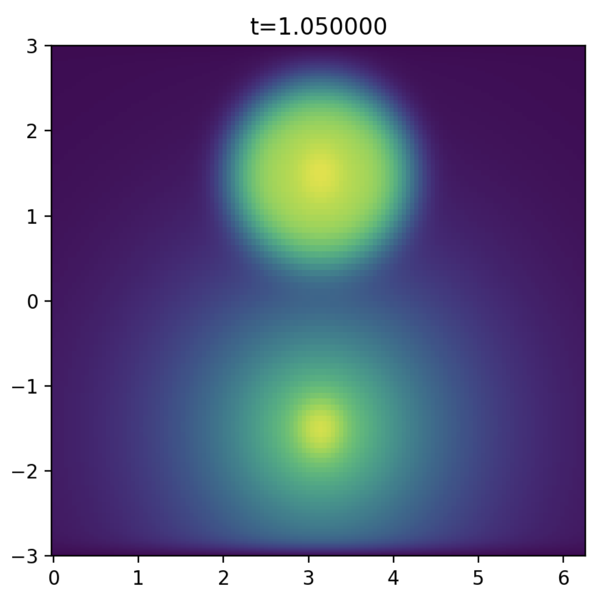

Figure 2: T = 1

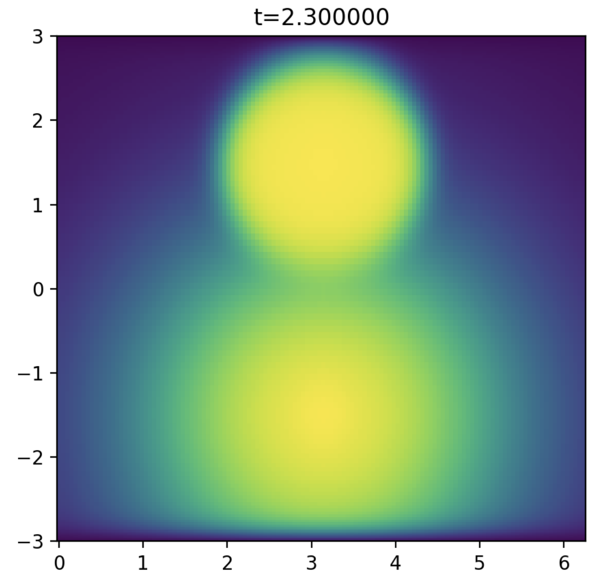

Figure 3: T = 2.3

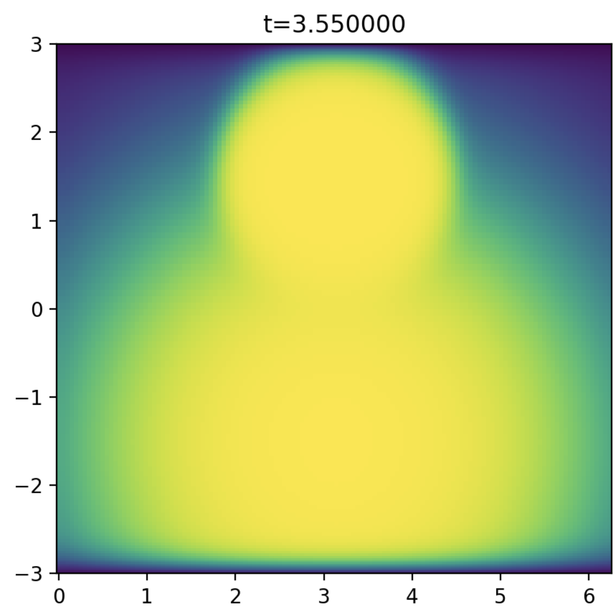

Figure 4: T = 3.5

This models a reaction being diffused through the 2D domain, with initial condition characterized by

$$u = 0.3e^{-20r_1^2} + 0.5e^{-r_1^4} + 0.3e^{-20r_2^2} + 0.5e^{-r_2}$$
$$r_1 = \sqrt{(x-\pi)^2 + (y-1.5)^2}$$
$$r_2 = \sqrt{(x-\pi)^2 + (y+1.5)^2}$$

We can also model the **Fokker–Planck equation**, by altering some of the variables above. The Fokker–Planck equation is characterized by

$$\partial_t u = -p\nabla u + D\mathrm{H}_u u$$

where again H_u is the Hessian of u. The Fokker–Planck equation models the time evolution of the probability density function of the velocity of a particle under drag forces and random forces, such as in Brownian motions. We can attempt to set up the same RHS equation and use the L matrix with the Hessian matrix to see the evolution. We set p to be 0.01, and D to be 0.001.

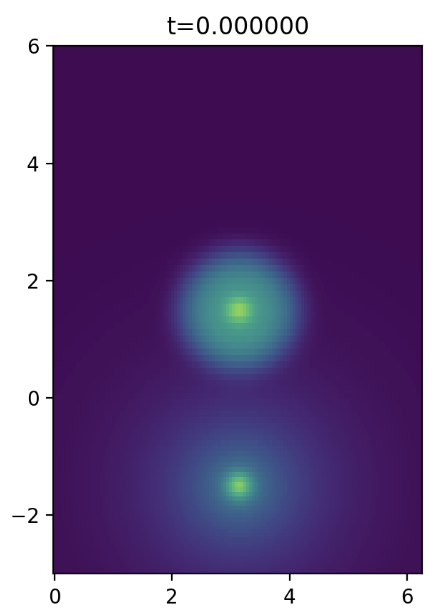

Figure 5: T = 0

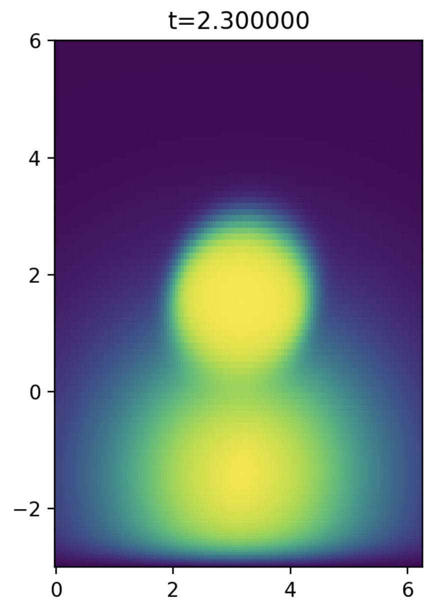

Figure 6: T = 2.3

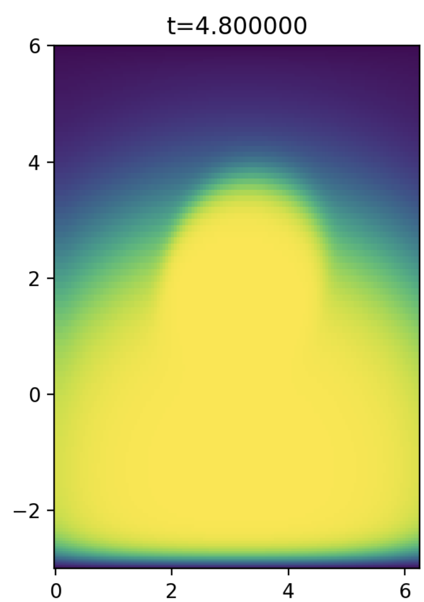

Figure 7: T = 4.8

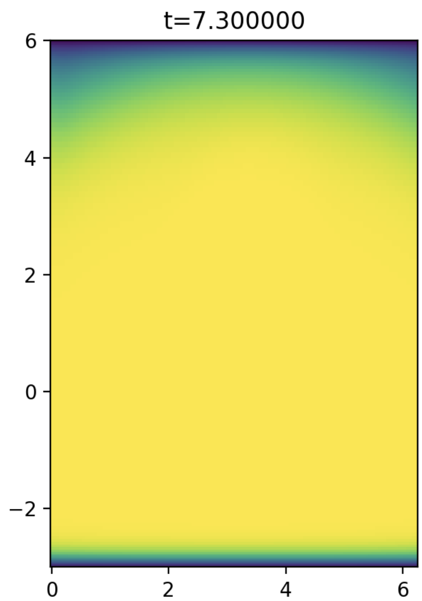

Figure 8: T = 7.3

In this simulation, we see that since there is a positive velocity term (drift term) as well as a diffusive term, the two initial injects are diffusing slowly in the $+x$ and $+y$ direction. In one dimension, the equation is supposed to exhibit the following behavior

Figure 9: Initial State

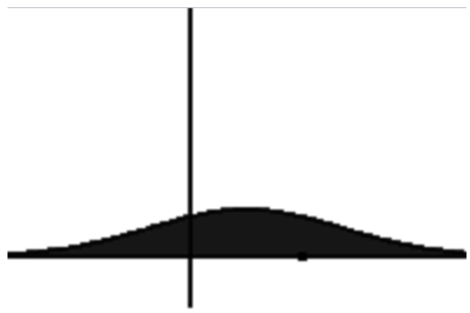

Figure 10: Final State

And my simulation of this phenomenon did exactly that; the injection widened/diffused, and the center moved.

4 Conclusion

In this project I used the spectral method to create numerical solutions to a general differential equation that, although initially used to model stochastic random variables primarily used in financial applications, can be altered to model the heat equation, diffusion equations, and probability density functions. The different equations used either the Laplacian or the Hessian matrix of 2D variable u, but in either case my simulations conformed to theoretical solutions and illustrated the time evolution of these partial differential equations.