SUBJECT: ____________ **LECTURE:** ____________ **DATE:** ___/___/___

IMPORTANT (EQUATIONS, LAWS, ETC.)

NOTES

The nine-point circle and the incircle of the triangle are mutually tangent

Recall:

$$S = \sqrt{p(p-a)(p-b)(p-c)}$$

$$S = rp \rightarrow r = \frac{S}{p}$$

$$abc = 4RS \rightarrow RS = \frac{abc}{4}$$

a) $a^2 + b^2 + c^2 = 2p^2 - 2r^2 - 8Rr$

(s: semiperimeter, r: radius of inscribed circle, R: radius of circumscribed circle)

Proof: $2p^2 - 2r^2 - 8Rr = 2p^2 - 2 \cdot \frac{S^2}{p^2} - 2\,\frac{abc}{p}$

$= 2p^2 - 2r^2 - 8Rr$

$= 2p^2 - \frac{2\cancel{p}(p-a)(p-b)(p-c)}{p^{\cancel{2}}} - \frac{2abc}{p}$

$= \frac{2p^3 - 2(p-c)(p^2 - pb - pa + ab) - 2abc}{p}$

$= \frac{\cancel{2p^3} - 2(\cancel{p^3} - p^2b - p^2a + abp - cp^2 + pbc + acp - \cancel{abc}) - \cancel{2abc}}{p}$

$= 2(pb + pa + pc - ab - ac - bc)$

$= (a+b+c)b + (a+b+c)a + (a+b+c)c - 2ab - 2ac - 2bc$

$= \cancel{ab} + b^2 + \cancel{bc} + a^2 + \cancel{ab} + \cancel{ac} + \cancel{ac} + \cancel{bc} + c^2 - \cancel{2ab} - \cancel{2ac} - \cancel{2bc}$

$= a^2 + b^2 + c^2$

b) Connection between G - centroid and M - arbitrary point

$$3MG^2 = MA^2 + MB^2 + MC^2 - \frac{1}{3}(AB^2 + AC^2 + BC^2)$$

Proof: $3\vec{MG} = \vec{MA} + \vec{MB} + \vec{MC}$

$9MG^2 = MA^2 + MB^2 + MC^2 + 2\vec{MA}\cdot\vec{MB} + 2\vec{MB}\,\vec{MC} + 2\vec{MC}\,\vec{MA}$

$(\vec{MB} - \vec{MA})^2 = MA^2 + MB^2 - 2\vec{MA}\,MB$

$9MG^2 = MA^2 + MB^2 + MC^2 + (MA^2 + MB^2 - AB^2) + (MB^2 + MC^2 - BC^2) + (MC^2 + MA^2 - AC^2)$

$\vec{MA} \cdot \vec{MB} = MA^2 + MB^2 - AB^2$

$9MG^2 = 3(MA^2 + MB^2 + MC^2) - (AB^2 + AC^2 + BC^2)$

$3MG^2 = MA^2 + MB^2 + MC^2 - \frac{1}{3}(AB^2 + AC^2 + BC^2)$

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c) Distance from I - incenter to G - centroid

$3MG^2 = MA^2 + MB^2 + MC^2 - \frac{1}{3}(AB^2 + AC^2 + BC^2)$

$I \equiv M \Rightarrow 3IG^2 = IA^2 + IB^2 + IC^2 - \frac{1}{3}(AB^2 + AC^2 + BC^2)$

Draw $IK \perp AB \Rightarrow AK = p - a$

$\Delta AKI \ (\angle K = 90^\circ) \rightarrow IA^2 = IK^2 + KA^2$

$= (p-a)^2 + r^2$

Similarly, $IB^2 = (p-b)^2 + r^2$

$IC^2 = (p-c)^2 + r^2$

Therefore, $3IG^2 = (p-a)^2 + (p-b)^2 + (p-c)^2 + r - \frac{1}{3}(a^2 + b^2 + c^2)$

$3IG^2 = 3r^2 + p^2 + \frac{2}{3}(a^2 + b^2 + c^2)$

From a) $a^2 + b^2 + c^2 = 2p^2 - 2r^2 - 8Rr$

$\rightarrow 3IG^2 = 3r^2 - p^2 + \frac{2}{3}(2p^2 - 2r^2 - 8Rr)$

$\Leftrightarrow 9IG^2 = 9r^2 - 3p^2 + 4p^2 - 4r^2 - 16Rr$

$\Leftrightarrow 9IG^2 = 5r^2 - p^2 - 16Rr$

$\Leftrightarrow IG^2 = \frac{5r^2 + p^2 - 16Rr}{9}$

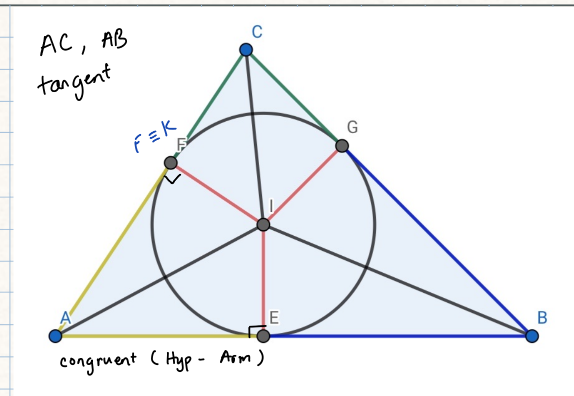
Recall: $\vec{OH} = 3\vec{OG} = 6\vec{GE}$

$OI^2 = OG^2 + GI^2 - 2OG.GI.\cos OGI$ (1)

$EI^2 = EG^2 + GI^2 - 2EG.GI.\cos OGI$ (2)

$2 \times (2) + (1)$

$\Leftrightarrow 2EI^2 + OI^2 = OG^2 + 2EG^2 + 3GI^2$



$AK = p - a$

$p = AK + FC + BG$

$p = AK + CG + BG$

$p = AK + BC$

$AK = p - a$

$3IG^2 = 3r^2 + p^2 - 2ap + a^2 + p^2 - 2bp + b^2 + p^2 - 2cp + c^2 - \frac{1}{3}(a^2 + b^2 + c^2)$

$= 3r^2 + 3p^2 - 2ap - 2bp - 2cp + \frac{2}{3}(a^2 + b^2 + c^2)$

$= 3r^2 + 3p^2 - 2p(a + b + c) + \frac{2}{3}(a^2 + b^2 + c^2)$

$= 3r^2 - p^2 + \frac{2}{3}(a^2 + b^2 + c^2)$

Euler's theorem: $\vec{OG} = 2\vec{GE}$

$\vec{OI}^2 = R^2 - 2Rr$

E - nine-point center
O - circumcenter
G - centroid
I - incenter
H - orthocenter

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$(\Leftarrow)\quad EI^2 = \frac{1}{2}\left(OG^2 + 2EG^2 + 3GI^2 - OI^2\right)$

$(\Leftarrow)\quad EI^2 = \frac{1}{2}\left(\frac{3}{2}OG^2 + 3GI^2 - OI^2\right)$

$OI^2 = R^2 - 2Rr$

$GI^2 = \dfrac{5r^2 + p^2 - 16Rr}{9}$

$OG^2 = \frac{1}{9}\left(9R^2 + 2r^2 - 2p^2 + 8Rr\right)$

$\frac{3}{2}OG^2 = \frac{3}{2}R^2 + \frac{1}{3}r^2 - \frac{1}{3}p^2 + \frac{4}{3}Rr$

$3GI^2 = \frac{5}{3}r^2 + \frac{3}{9}p^2 - \frac{16}{3}Rr$

$-OI^2 = -R^2 + 2Rr$

$(\Leftarrow)\quad EI^2 = \frac{1}{2}\left(\frac{1}{2}R^2 + 2r^2 - 2Rr\right)$

$(\Leftarrow)\quad EI^2 = \frac{1}{4}(R - 2r)^2$

$(\Leftarrow)\quad EI = \left|\frac{R}{2} - r\right|$

Radius of the 9-point circle is $\frac{R}{2}$

Radius of inscribed circle is r

$\Rightarrow$ I and E tangent

$\vec{OG} = 2\vec{EG}$

$OG^2 = 4EG^2$

$\frac{OG^2}{2} = 2EG^2$

$3MG^2 = MA^2 + MB^2 + MC^2 - \frac{1}{3}\left(AB^2 + AC^2 + BC^2\right)$ (from b))

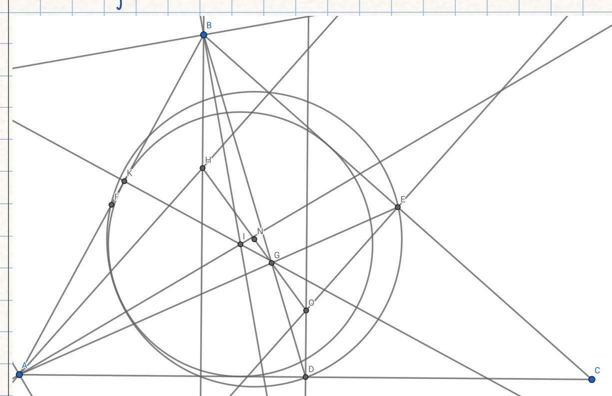
$O \equiv M \Rightarrow$

$3OG^2 = OA^2 + OB^2 + OC^2 - \frac{1}{3}\left(a^2 + b^2 + c^2\right)$

$3OG^2 = 3R^2 - \frac{1}{3}\left(2p^2 - 2r^2 + 8Rr\right)$

$OG^2 = R^2 - \frac{1}{9}\left(2p^2 - 2r^2 + 8Rr\right)$

$OG^2 = \frac{1}{9}\left(R^2 - 2p^2 - 2r^2 + 8Rr\right)$



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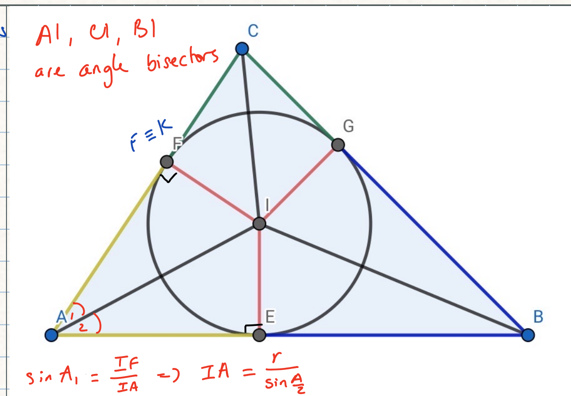
IMPORTANT (EQUATIONS, LAWS, ETC.)

NOTES

The nine-point circle is tangent externally to the 3 excircles

$IA = \dfrac{r}{\sin\frac{A}{2}}$, $IB = \dfrac{r}{\sin\frac{B}{2}}$, $IC = \dfrac{r}{\sin\frac{C}{2}}$

Similarly, $AI_1 = \dfrac{r_1}{\sin\frac{A}{2}}$, $BI_1 = \dfrac{r_1}{\cos\frac{B}{2}}$, $CI_1 = \dfrac{r_1}{\cos\frac{C}{2}}$



If E be the center of 9-point circle of ΔABC and M be an arbitrary point then:

$$4EM^2 = \frac{R^2}{\Delta}\left[(\sin 2B + \sin 2C)AM^2 + (\sin 2A + \sin 2C)BM^2 + (\sin 2A + \sin 2B)CM^2\right] - (a^2+b^2+c^2-5R^2)$$

$\sin A + \sin B = 2\sin(A+B)\cos(A-B)$, $\sin 2A = 2\sin A\cos A$

$\to \sin 2B + \sin 2C = 2\sin(B+C)\cos(B-C)$

$= 2\sin(\pi - A)\cos(B-C)$

$= 2\sin A\cos(B-C) = \sin\frac{A}{2}\cos\frac{A}{2}\cos(B-C)$

let $I_1 \equiv M$

$$4EI_1^2 = \frac{R^2}{\Delta}\left[(\sin 2B + \sin 2C)I_1A^2 + (\sin 2A + \sin 2C)I_1B^2 + (\sin 2A + \sin 2B)I_1C^2\right] - (a^2+b^2+c^2-5R^2)$$

$$= \frac{R^2}{\Delta}\left[\sin\frac{A}{2}\cos\frac{A}{2}\cos(B-C)\frac{r_1^2}{\sin^2\frac{A}{2}} + \sin\frac{B}{2}\cos\frac{B}{2}\cos(A-C)\frac{r_1^2}{\sin^2\frac{B}{2}} + \sin\frac{C}{2}\cos\frac{C}{2}\cos(B-C)\frac{r_1^2}{\sin^2\frac{C}{2}}\right] - (a^2+b^2+c^2-5R^2)$$

$$= \frac{R^2 r_1^2}{\Delta}\left[\cot\frac{A}{2}\cos(B-C) + \tan\frac{B}{2}\cos(C-A) + \tan\frac{C}{2}\cos(A-B)\right] - (a^2+b^2+c^2-5R^2)$$

$\left(\dfrac{\cos}{\sin} = \cot , \dfrac{\sin}{\cos} = \tan\right)$

$$\cot\frac{A}{2}\cos(B-C) + \tan\frac{B}{2}\cos(C-A) + \tan\frac{C}{2}\cos(A-B) = \frac{2\Delta}{r_1^2R^2}(2r_1^2 - r^2 + p^2 - 4Rr + 2Rr_1 - 2R^2)$$

$$= \frac{\cancel{R^2}\cancel{r_1^2}}{\cancel{\Delta}}\left[\frac{2\cancel{\Delta}}{\cancel{r_1^2}\cancel{R^2}}(2r_1^2 - r^2 + p^2 - 4Rr + 2Rr_1 - 2R^2)\right] - (\underbrace{2p^2 - 2r^2 - 8Rr}_{\text{proved in a}} - 5R^2)$$

$$4I_1E^2 = 4r_1^2 - \cancel{2r^2} + \cancel{2p^2} - \cancel{8Rr} + 4Rr_1 - 4R^2 - \cancel{2p^2} + \cancel{2r^2} + \cancel{8Rr} + 5R^2$$

$$4I_1E^2 = 4r_1^2 + 4Rr_1 + R^2$$

$$I_1E^2 = r_1^2 + Rr_1 + \frac{R^2}{4}$$

$$I_1E^2 = \left(r_1 + \frac{R}{2}\right)^2$$

$I_1E = \left|r_1 + \dfrac{R}{2}\right|$ with r_1 : radius of excircle I_1 $\Rightarrow I_1, E$ tangent.

$\dfrac{R}{2}$: radius of 9-point circle

Similarly we can prove

$I_2E = \dfrac{R}{2} + r_2$, $I_3E = \dfrac{R}{2} + r_3$