MACHINE LEARNING

REINFORCEMENT LEARNING

AGENDA

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05 Learning MDP Models

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AI

INTRODUCTION

REINFORCEMENT LEARNING

INTRODUCTION

There are many **problems**, where it is **very difficult** to **provide** the "**right**" **answers** to the algorithm, as we have been doing for supervised learning.

For example, if we have just built a **four-legged robot** and are **trying** to **program** it to **walk**, then initially we **have no idea** what the "**correct**" **actions** to take are to make it walk **(credit assignment problem).**

In **reinforcement learning**, we will **provide algorithms** only a **reward function**, to **indicate** when the **learning agent** is **doing well** or **poorly**.

It will be the learning algorithm's job to figure out how to choose actions over time to maximize rewards

REINFORCEMENT LEARNING

I N T R O D U C T I O N

Many of the **applications** of **reinforcement learning include**:

- **Autonomous helicopter flight.**
- **Robot Legged Locomotion.**
- **Cell-phone network routing.**
- **Marketing strategy selection.**
- **Factory control.**
- **Efficient web indexing.**
- **Traffic light control.**
- **Chemical reaction optimization.**
- **Personalized recommendations.**
- **Bidding and Advertisement.**

MARKOV DECISION PROCESSES

MARKOV DECISION PROCESSES

A **Markov decision process** is a **tuple**:

$$(\boldsymbol{S}, \boldsymbol{A}, \{\boldsymbol{P}_{\boldsymbol{sa}}\}, \boldsymbol{\gamma}, \boldsymbol{R})$$

- $\boldsymbol{S}$**:** the set of **states** (i.e. all positions and orientations of a helicopter).
- $\boldsymbol{A}$: the set of **actions** (i.e. all directions in which you can push the helicopter's sticks).
- $\boldsymbol{P}_{\boldsymbol{sa}}$: state **transition probability distributions**. For **each state** $\boldsymbol{s} \in \boldsymbol{S}$ and action $\boldsymbol{a} \in \boldsymbol{A}$**,** $\boldsymbol{P}_{\boldsymbol{sa}}$ is a **distribution** over the **state space.** Thus, $\boldsymbol{P}_{\boldsymbol{sa}}$ **gives** the **distribution** over what **states** the **algorithm** will **transition,** if it takes **action** $\boldsymbol{a}$ in **state** $\boldsymbol{s}$.

$$\sum_{\boldsymbol{s'}} \boldsymbol{P}_{\boldsymbol{sa}}(\boldsymbol{s'}) = \mathbf{1}$$

REINFORCEMENT LEARNING

MARKOV DECISION PROCESSES

A **Markov decision process** is a **tuple**:

$$(\boldsymbol{S}, \boldsymbol{A}, \{\boldsymbol{P}_{\boldsymbol{sa}}\}, \boldsymbol{\gamma}, \boldsymbol{R})$$

- $\boldsymbol{\gamma} \in [\mathbf{0}, \mathbf{1})$: is called the **discount factor**.
- $\boldsymbol{R}: \boldsymbol{S}\ \boldsymbol{X}\ \boldsymbol{A} \rightarrow \mathbb{R}$: the **reward function**. (It can be written only as a function of $\boldsymbol{S}$, $\boldsymbol{R}: \boldsymbol{S} \rightarrow \mathbb{R}$).

REINFORCEMENT LEARNING

MARKOV DECISION PROCESSES

EXAMPLE:

We will **define** a **robot navigation task**, in which you **have** a **robot** in a **grid environment** where **all gray cells** are **obstacles**.

The **reward** is **defined by** the +1 and the **punishment** as -1.

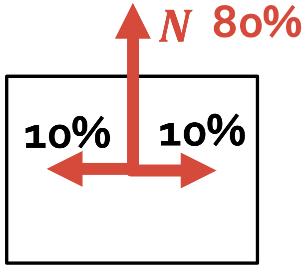
The **robot can move** in **all cells except** for the **gray cells.**

11 states

			+1
			-1

STOCHASTIC ACTIONS

$$\boldsymbol{A} = \{\boldsymbol{N}, \boldsymbol{W}, \boldsymbol{S}, \boldsymbol{E}\}$$



REINFORCEMENT LEARNING

MARKOV DECISION PROCESSES

EXAMPLE:

The **robot** can **transit from** one **state** to another **state** in **single steps.**

When the **robot reaches (4,3) or (4,2)** the **world ends**.

$(1,3)$	$(2,3)$	$(3,3)$	$(4,3)$ **+1**
$(1,2)$	$(2,2)$	$(3,2)$ 80%	$(4,2)$ **-1**
$(1,1)$	$(2,1)$	10% 10% $(3,1)$	$(4,1)$

Transitions

$$P_{s_t a_t}(s')$$

$$P_{(3,1)N}\big((3,2)\big) = 0.8$$

$$P_{(3,1)N}\big((4,1)\big) = 0.1$$

$$P_{(3,1)N}\big((2,1)\big) = 0.1$$

$$P_{(3,1)N}\big((3,3)\big) = 0$$

$$\vdots$$

Rewards

$$R(s)$$

$$R\big((4,3)\big) = +1$$

$$R\big((4,2)\big) = -1$$

$$R(s) = -0.2$$

(fuel consumption)

$$\vdots$$

REINFORCEMENT LEARNING

MARKOV DECISION PROCESSES

The dynamics of a **Markov decision process** is as follows:

1. The **algorithm starts** in a **state** $\boldsymbol{s_0}$ and **choose** an **action** $\boldsymbol{a_0} \in \boldsymbol{A}$.

2. There is a **random transition from state** $\boldsymbol{s_0}$ to **state** $\boldsymbol{s_1}$ **drawn according** to $\boldsymbol{s_1 \sim P_{s_0 a_0}}$.

3. The **process repeats indefinitely** where there is a **random selection** to **traverse** from state $\boldsymbol{s_t}$ to state $\boldsymbol{s_{t+1}}$, which is **drawn from** $\boldsymbol{s_{t+1} \sim P_{s_t a_t}}$.

$$\boldsymbol{s_0 \underset{a_0}{\rightarrow} s_1 \underset{a_1}{\rightarrow} s_2 \underset{a_2}{\rightarrow} \ldots}$$

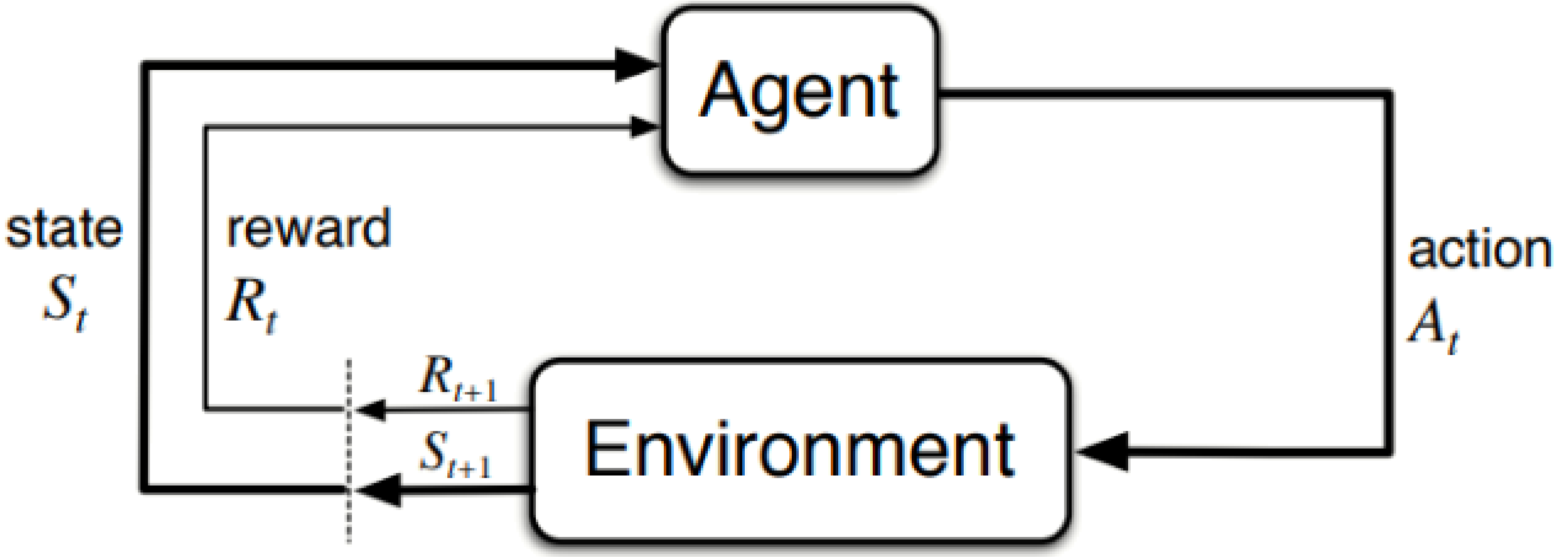
4. The **total payoff** is be given by:

$$\boldsymbol{R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots}$$

(Rewards diminish as times goes by).

REINFORCEMENT LEARNING

MARKOV DECISION PROCESSES



REINFORCEMENT LEARNING

MARKOV DECISION PROCESSES

The **objective** in **reinforcement learning** is to **choose actions over time** to **maximize** the **expected value** of the **total payoff**:

$$E\left[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots\right]$$

The **reward** at **timestep** t is **discounted** by a **factor** of γ^t where $\gamma \in [0,1)$ is the **discount factor**.

Therefore, to **make** this **expectation large**, we would like to:

- **Obtain positive rewards** as **soon** as **possible**.
- **Postpone negative rewards,** as **long as possible**.

POLICY AND VALUE FUNCTION

REINFORCEMENT LEARNING

POLICY AND VALUE FUNCTION

POLICY

A **policy** is **any function** $\pi : S \rightarrow A$ that **maps from states** to **actions (for every state, what action** is **recommended** to **take** in **that state)**.

EXAMPLE: (**Optimal Policy**)

$(1,3)$ →	$(2,3)$ →	$(3,3)$ →	+1
$(1,2)$ ↑	$(2,2)$	$(3,2)$ ↑	-1
$(1,1)$ ↑	$(2,1)$ ←	$(3,1)$ ←	$(4,1)$ ←

MDPs are very good in finding the best tradeoffs between actions (policies).

REINFORCEMENT LEARNING

POLICY AND VALUE FUNCTION

VALUE FUNCTION:

For **any policy** $\boldsymbol{\pi}$, we **define** the **value function** $\boldsymbol{V^{\pi}}$: $\boldsymbol{S} \rightarrow \mathbb{R}$ as the **expected total payoff** upon **starting** in **state** $\boldsymbol{s}$ and **taking actions according** to the **policy** $\boldsymbol{\pi}$**.**

$$\boldsymbol{V^{\pi}(s) = E\left[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots | s_0 = s, \pi\right]}$$

As a **side note**, this **notation** is **technically incorrect** because we **cannot condition** on $\boldsymbol{\pi}$, which **does not represent** a **random variable**.

REINFORCEMENT LEARNING

POLICY AND VALUE FUNCTION

EXAMPLE: Value Function and Policy

Very bad policy π

$(1,3)$ →	$(2,3)$ →	$(3,3)$ →	+1
$(1,2)$ ↓	$(2,2)$	$(3,2)$ →	-1
$(1,1)$ →	$(2,1)$ →	$(3,1)$ ↑	$(4,1)$ ↑

Value Function $V^{\pi}(s)$

$V^{\pi}((1,3))$ 0.52	$V^{\pi}((2,3))$ 0.73	$V^{\pi}((3,3))$ 0.77	+1
$V^{\pi}((1,2))$ -0.90		$V^{\pi}((3,2))$ -0.82	-1
$V^{\pi}((1,1))$ -0.88	$V^{\pi}((2,1))$ -0.87	$V^{\pi}((3,1))$ -0.85	$V^{\pi}((4,1))$ -1

$$V^{\pi}(s) = E\left[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots \mid s_0 = s, \pi\right]$$

Better to start at the top

REINFORCEMENT LEARNING

POLICY AND VALUE FUNCTION

BELLMAN EQUATIONS:

We can **represent** the **value function** in the **following way**:

$$V^{\pi}(s) = E\left[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots \mid s_0 = s, \pi\right]$$

$$V^{\pi}(s) = E[R(s_0) + \gamma \underbrace{(R(s_1) + \gamma R(s_2) + \cdots)}_{\substack{\text{Expected sum of future} \\ \text{discounted rewards.} \\ V^{\pi}(s_1)}} | s_0 = s, \pi]$$

Immediate reward
for starting in state s

Therefore, we can **write** the **equation** as a **recursive form** of **itself** ($s_0 \rightarrow s$ and $s_0 \rightarrow s'$) **using** the **definition** of **expected value** $E[X] = \sum x_i p_i$ (s' is a **random variable**):

$$V^{\pi}(s) = R(s) + \gamma \sum_{s' \in S} P_{s\pi(s)}(s')\, V^{\pi}(s')$$

REINFORCEMENT LEARNING

POLICY AND VALUE FUNCTION

BELLMAN EQUATIONS:

Given a **fixed policy** $\boldsymbol{\pi}$**,** its **value function** $\boldsymbol{V^{\pi}}$ **satisfies Bellman equations**:

$$\boldsymbol{V^{\pi}(s) = R(s) + \gamma \sum_{s' \in S} P_{s\pi(s)}(s')\, V^{\pi}(s')}$$

The **first term** is the **immediate reward** for **starting** in **state** $\boldsymbol{s}$**.**

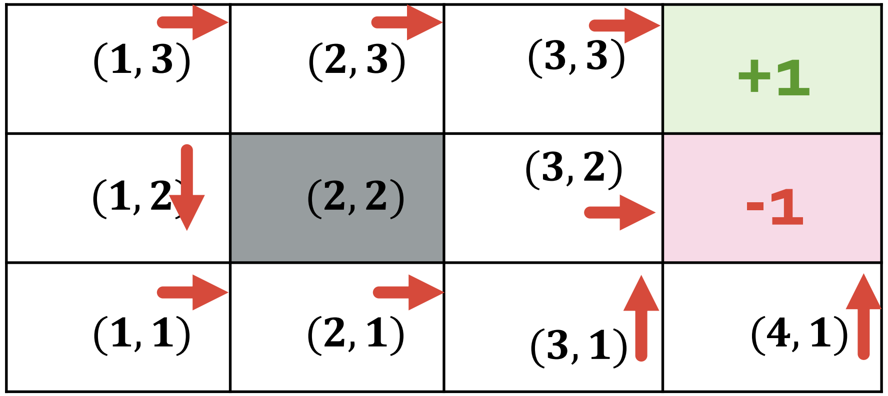
The **second term** gives the **expected sum** of **discounted rewards** obtained after the **first step** in the **MDP.**

REINFORCEMENT LEARNING

POLICY AND VALUE FUNCTION

EXAMPLE: Bellman Equation

Let us **assume** that we **start** at **state** (**3,1**) and the **policy** $\boldsymbol{\pi}$ **indicates** to take a **step** in the **north direction** $\boldsymbol{\pi}\big((3,1)\big) = \boldsymbol{N}$.



The **Bellman Equation** is:

$$\boldsymbol{V^{\pi}(s) = R(s) + \gamma \sum_{s' \in S} P_{s\pi(s)}(s')\, V^{\pi}(s')}$$

$$\boldsymbol{V^{\pi}\big((3,1)\big) = R\big((3,1)\big) + \gamma\big[0.8V^{\pi}\big((3,2)\big) + 0.1V^{\pi}\big((4,1)\big) + 0.1V^{\pi}\big((2,1)\big)\big]}$$

You will have **11 variables** and **11 equations**, **each** one **corresponding** to **each state**.

REINFORCEMENT LEARNING

POLICY AND VALUE FUNCTION

OPTIMAL VALUE FUNCTION:

The **optimal value function** will be the **best possible expected sum** of **discounted rewards** that can be attained **using any policy**.

$$\boldsymbol{V^*(s) = \max_{\pi} V^{\pi}(s)}$$

The **Bellman representation** will be:

Expected future rewards if we take the BEST action $\boldsymbol{a}$ in state $\boldsymbol{s}$.

$$\boldsymbol{V^*(s) = R(s) + \boxed{\max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s')\, V^*(s')}}$$

The **first term** above is the **immediate reward**. The **second term** is the **maximum over all actions $\boldsymbol{a}$** of the **expected future sum** of **discounted rewards** we'll get **upon after action $\boldsymbol{a}$.**

REINFORCEMENT LEARNING

POLICY AND VALUE FUNCTION

OPTIMAL POLICY:

The **optimal policy** π^* indicates the **best action** a to **take** in **state** s, which **maximizes** the **expected future rewards**.

$$\pi^*(s) = \arg\max_{a \in A} \sum_{s' \in S} P_{sa}(s')\, V^*(s')$$

This **derives** in the **following expression**:

$$V^*(s) = V^{\pi^*}(s) \geq V^{\pi}(s)$$

In conclusion, π^* is the **optimal policy for all states** s.

We can use the same policy π^* no matter what the initial state of our MDP is.

REINFORCEMENT LEARNING

POLICY AND VALUE FUNCTION

OPTIMAL POLICY:

To **find** the **optimal policy** $\boldsymbol{\pi}^*(\boldsymbol{s})$, we would **need** to **obtain** the **best value function** $\boldsymbol{V}^*(\boldsymbol{s}')$ and **substitute it** in the **equation** we have just obtained:

$$\boldsymbol{\pi}^*(\boldsymbol{s}) = \underset{\mathbf{a} \in \mathbf{A}}{\mathbf{arg\,max}} \sum_{\boldsymbol{s'} \in \boldsymbol{S}} \boldsymbol{P}_{\boldsymbol{sa}}(\boldsymbol{s}')\,\boldsymbol{V}^*(\boldsymbol{s}')$$

The **problem** is **finding** $\boldsymbol{V}^*(\boldsymbol{s}')$ which corresponds to:

$$\boldsymbol{V}^*(\boldsymbol{s}) = \max_{\boldsymbol{\pi}} \boldsymbol{V}^{\boldsymbol{\pi}}(\boldsymbol{s})$$

The **bottleneck** is that you would **need** to **solve** an **exponentially large number** of **systems** of **equations because** there is an **exponential number** of **policies** $\boldsymbol{\pi}$.

VALUE ITERATION AND POLICY ITERATION

REINFORCEMENT LEARNING

VALUE ITERATION AND POLICY ITERATION

VALUE ITERATION:

1. **For each state** s, **initialize** $V(s) := 0 \ \forall s$.

2. **Repeat until convergence**{

   For every state, **update** $V(s) := R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s') V(s')$.

   }

3. **Substitute** $V^*(s)$ in $\pi^*(s) = \arg\max_{a \in A} \sum_{s' \in S} P_{sa}(s') V^*(s')$.

The **algorithm repeatedly tries** to **update** the **estimated value function** using **Bellman Equations.** By **implementing this** you **ensure** that the **value function converges** to the **optimum**.

$$V(s) \rightarrow V^*(s)$$

REINFORCEMENT LEARNING

VALUE ITERATION AND POLICY ITERATION

VALUE ITERATION:

There are **two possible ways** of **performing** the **updates** in the **inner loop**.

$$\boldsymbol{V}\,(\boldsymbol{s}) := \boldsymbol{R}(\boldsymbol{s}) + \max_{\boldsymbol{a}\in\boldsymbol{A}} \boldsymbol{\gamma} \sum_{\boldsymbol{s}'\in\boldsymbol{S}} \boldsymbol{P}_{\boldsymbol{sa}}(\boldsymbol{s}')\,\boldsymbol{V}(\boldsymbol{s}').$$

1. **Synchronous update** (current estimate → new estimate):
   - **Compute** the **new values** of $V(s)$.
   - **Overwrite** the **old values with** the **new values**.

2. **Asynchronous updates**
   - **Loop over** the **states** (**predefined order**).
   - **Update** the **values** of $V(s)$ **one** at a **time**.

VALUE ITERATION AND POLICY ITERATION

EXAMPLE: Value Iteration

Running value iteration on our **previous example**, we obtain the **following results** for $V^*(s)$:

$\boxed{V^*(s)}$

$V^\pi((1,3))$ $\mathbf{0.86}$	$V^\pi((2,3))$ $\mathbf{0.90}$	$V^\pi((3,3))$ $\mathbf{0.93}$	**+1**
$V^\pi((1,2))$ $\mathbf{0.82}$		$V^\pi((3,2))$ $\mathbf{0.69}$	**-1**
$V^\pi((1,1))$ $\mathbf{0.78}$	$V^\pi((2,1))$ $\mathbf{0.75}$	$V^\pi((3,1))$ $\mathbf{0.71}$	$V^\pi((4,1))$ $\mathbf{0.49}$

REINFORCEMENT LEARNING

VALUE ITERATION AND POLICY ITERATION

EXAMPLE: Value Iteration

Let us see how we **obtain $\pi^*(s)$** for a **single state:** $(3,1)$

$$W = \sum_{s' \in S} P_{sa}(s')V(s') = (0.8 * 0.75) + (0.1 * 0.69) + (0.1 * 0.71) = 0.78$$

The action west maximizes our future rewards if we start at state (3,1).

$$N = \sum_{s' \in S} P_{sa}(s')V(s') = (0.8 * 0.69) + (0.1 * 0.75) + (0.1 * 0.49) = 0.67$$

$$S = \sum_{s' \in S} P_{sa}(s')V(s') = (0.8 * 0.71) + (0.1 * 0.75) + (0.1 * 0.49) = 0.69$$

$$E = \sum_{s' \in S} P_{sa}(s')V(s') = (0.8 * 0.49) + (0.1 * 0.69) + (0.1 * 0.71) = 0.53$$

$\boxed{V^*(s)}$

$V^\pi((1,3))$ 0.86	$V^\pi((2,3))$ 0.90	$V^\pi((3,3))$ 0.93	+1
$V^\pi((1,2))$ 0.82		$V^\pi((3,2))$ 0.69	-1
$V^\pi((1,1))$ 0.78	$V^\pi((2,1))$ 0.75	$V^\pi((3,1))$ 0.71	$V^\pi((4,1))$ 0.49

EXAMPLE: Value Iteration

Applying the **previous procedure** to **all states,** we **obtain** our **optimal policy** (**best actions** $\boldsymbol{a} = \boldsymbol{\pi}(\boldsymbol{s})$ to take **at every state** $\boldsymbol{s}$**).**

$\boxed{\boldsymbol{\pi^*(s)}}$

$(\mathbf{1,3})$ →	$(\mathbf{2,3})$ →	$(\mathbf{3,3})$ →	**+1**
$(\mathbf{1,2})$ ↑	$(\mathbf{2,2})$	$(\mathbf{3,2})$ ↑	**-1**
$(\mathbf{1,1})$ ↑	$(\mathbf{2,1})$ ←	$(\mathbf{3,1})$ ←	$(\mathbf{4,1})$ ←

VALUE ITERATION AND POLICY ITERATION

POLICY ITERATION:

1. **Initialize π randomly.**
2. **Repeat until convergence**{

   a) Let $V := V^{\pi}$ **(Solve Bellman equations using policy π).**

   b) For every state, **compute** $\pi(s) := R(s) \underset{a \in A}{\operatorname{argmax}} \sum_{s' \in S} P_{sa}(s') V(s')$.

}

The **policy π found** in **step (b)** is also **called** the **policy** that is **greedy** with **respect** to V.
By **implementing this** you **ensure** that the **value function converges** to the **optimum**.

$$V(s) \rightarrow V^{*}(s)$$
$$\pi(s) \rightarrow \pi^{*}(s)$$

REINFORCEMENT LEARNING

VALUE ITERATION AND POLICY ITERATION

POLICY ITERATION VS VALUE ITERATION:

POLICY ITERATION	VALUE ITERATION
For **small MDPs** it is **very fast.**	For **small MDPs** it is **slower.**
For **MDPs** with **large state spaces** it is **computationally expensive** (large matrices)	For **MDPs** with **large state spaces** it is has **less computational expense**.

Value iteration seems to be **used more often** than policy iteration.

LEARNING MDP MODELS

REINFORCEMENT LEARNING

LEARNING MDP MODELS

In **many realistic problems**, we are **not given** state **transition probabilities** and **rewards explicitly**. Therefore, we need to **estimate them from data.**

More generally, in a Markov Decision Process $(\boldsymbol{S}, \boldsymbol{A}, \{\boldsymbol{P}_{sa}\}, \boldsymbol{\gamma}, \boldsymbol{R})$ the **states $\boldsymbol{S}$,** the **actions $\boldsymbol{A}$** and the **discount factors $\boldsymbol{\gamma}$** are **almost** always **known.** The **discount factor** is **chosen based** on **how much tradeoff** do you want **between current** and **future rewards**.

We can **easily derive** the **maximum likelihood estimates** for the **state transition probabilities**:

$$\boldsymbol{P}_{sa}(\boldsymbol{s}') = \frac{\#\ \boldsymbol{times\ we\ took\ action\ a\ in\ state\ s\ and\ got\ to\ s'}}{\#\ \boldsymbol{times\ we\ took\ action\ a\ in\ state\ s}}$$

If there is the **case** that you have **never taken action $\boldsymbol{a}$ in state $\boldsymbol{s}$** (0/0) we can **compute** the **uniform distribution over all states** :

$$\boldsymbol{P}_{sa}(\boldsymbol{s}') = \frac{\mathbf{1}}{|\boldsymbol{S}|}$$

REINFORCEMENT LEARNING

LEARNING MDP MODELS

Putting together model learning and **value iteration**, here is **one possible algorithm** for **learning** in an **MDP** with **unknown state transition probabilities**:

1. **Initialize π randomly.**

2. **Repeat** {

   a) Execute π in the **MDP** for some **number** of **trials**

   b) Using the **accumulated experience** in the **MDP**, **update** our **estimates** for P_{sa} (and R, **if applicable**).

   c) Apply value iteration with the **estimated state transition probabilities** and **rewards** to **get** a **new estimated value function V** (**use** this **update** to **initialize V** in **next iteration**).

   (d) Update π to be the **greedy policy** with **respect** to V.

}