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Automated short proof generation for projective geometric theorems with Cayley and bracket algebras II. Conic geometry

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Abstract

In this paper we study plane conic geometry, particularly different representations of geometric constructions and relations in plane conic geometry, with Cayley and bracket algebras. We propose three powerful simplification techniques for bracket computation involving conic points, and an algorithm for rational Cayley factorization in conic geometry. The factorization algorithm is not a general one, but works for all the examples tried so far. We establish a series of elimination rules for various geometric constructions based on the idea of bracket-oriented representation and elimination, and an algorithm for optimal representation of the conclusion in theorem proving. These techniques can be used in any applications involving brackets and conics. In theorem proving, our algorithm based on these techniques can produce extremely short proofs for difficult theorems in conic geometry.

Keywords: Bracket algebra; Cayley algebra; Automated theorem proving; Projective conic geometry

1. Introduction

Because of its nonlinear nature, projective conic geometry is more complicated than incidence geometry. Maybe this is the reason why this geometry is not as well studied as incidence geometry with Cayley and bracket algebras (Barnabei et al., 1985; Bokowski and Sturmfels, 1989; Doubilet et al., 1974). For a basic geometric relation like six points on a conic, it is well known that this can be represented by a degree-4 bracket binomial

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equality, which is the bracket representation of Pascal's conic theorem (Richter-Gebert, 1995, 1996; Sturmfels, 1991, 1993; Sturmfels and Whiteley, 1991). It is less well known that there are 15 such equalities representing the same relation, and it is unknown how to employ all these equalities efficiently in bracket computation involving conic points, and how to select one of the representations for this type of conclusion in theorem proving.

Readable theorem proving in conic geometry is challenging because of its difficulty, at the same time it is fascinating because of the beauty of the short proofs for difficult theorems (Crapo and Richter-Gebert, 1994; Gao and Wang, 2000; Havel, 1991; Hestenes and Ziegler, 1991; Li and Wu, 2000; Wang, 2001). The first necessary work is exploring conic geometry with Cayley and bracket algebras, particularly the representations of geometric constructions like free conic points, intersections of lines and conics, intersections of conics, conjugates, poles, polars and tangents, conics determined by lines and tangents, and the relations among different representations. This is the content of Section 2 of this paper. It turns out that the Cayley–bracket-algebra version of conic geometry has two prominent features: multiple representations and high degrees. The latter feature can be illustrated by the fourth intersection of two conics, whose representation is a linear combination of three vectors, and each coefficient is a four-termed bracket polynomial of degree 12.

This paper is a continuation of Li and Wu (2001). In that paper we studied the Cayley expansions of some typical Cayley expressions and established a series of theorems on factored and shortest expansions. In bracket computation, these conclusions are used to obtain factored/shortest result for each bracket. In this paper, the bracket-oriented manipulations are further extended to include bracket-oriented representation, to overcome the difficulty of multiple representations in conic geometry.

The idea of bracket-oriented representation is embodied in a series of elimination rules for various geometric constructions, which is the content of Section 3. The main algorithm there is the conic points selection algorithm, which to a given construction, selects a suitable sequence of representative conic points for each bracket or wedge product containing the construction. Combining the representations with eliminations and expansions, we form a key idea to overcome the difficulty of multiple representations, eliminations and expansions—"**breefs**" (Li and Wu, 2001), an abbreviation of **b**racket-oriented **r**epresentation, **e**limination and **e**xpansion for **f**actored and **s**hortest results.

However, there is an exception. As mentioned before, the six-point-on-conic construction is a kind of infrastructure. Its algebraic representations are needed whenever there is a bracket polynomial with at least six conic points. The fundamentality and universality of this structure suggests that it is better for us to treat its algebraic representations as computation rules, just like we use the collinear structure to evaluate brackets in incidence geometry (Li and Wu, 2001). Based on this idea, we develop three powerful techniques for bracket simplification in conic geometry, which are comparable to the three simplification techniques for general bracket computation in Li and Wu (2001). This is the content of Section 4.

Then it comes to the problem of choosing a suitable representation for the conclusion in theorem proving. While an unlucky representation can make the proving very complicated, a lucky one often leads to an extremely short proof. In Section 5, we propose an algorithm on optimal representation of a very typical conclusion with multiple representations—the six-point-on-conic conclusion.

In conic computation based on Cayley and bracket algebras, a common phenomenon is that a large number of common rational factors may be produced, most of which are brackets. For example, in Example 7.8 of Section 7 there are 52 common bracket factors. They can contribute tremendously to simplifying the computation. Because of this, it is greatly desired to produce more factors from bracket polynomials. Section 6 is devoted to this need. There will be rational bracket factors together with wedge products in the factorization, so we call it *rational Cayley factorization*. The Cayley factorization techniques developed in Li and Wu (2001) play an important role there, besides the three bracket simplification techniques in conic geometry.

Putting together the algorithms for representations, eliminations, expansions, simplifications and factorizations, in Section 7 we form an algorithm for theorem proving in conic geometry. The algorithm is designed to produce short and readable algebraic proofs, which is in general a challenging task. Another drive of the research is to produce new techniques for symbolic computation with Cayley and bracket algebras, which can be immediately applied to other areas where these algebras are needed.

Because of this, the theorem proving algorithm is not a simple one. It is not designed to perform the verification of the conclusion superfast. As an algorithm, it is good enough to produce so many two-termed proofs for so many difficult theorems, which amounts to 80% of the 40 nontrivial theorems tested in conic geometry. And it is efficient enough to produce generally two-termed proofs for geometric theorems involving free conic points, poles and tangents. In Section 7 there is a collection of typical examples and their proofs by the algorithm.

2. Conic geometry with Cayley and bracket algebras

We first study the representations of geometric constructions and relations in conic geometry with Cayley and bracket algebras.

2.1. Conics determined by five points

There are three kinds of projective conics. The first is *double-line conic*, which is composed of a line and itself. The second is *line-conic*, which is composed of two different lines. The third is *nondegenerate conic*, which does not contain any line. In this paper we consider only the latter two conics. The numbers field is always assumed to be complex[1]. The intersection of the two lines in a line-conic is the *double point* of the conic.

According to Pascal's theorem, six points $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ are on the same conic, called *conconic*, if and only if the intersections $\mathbf{12} \cap \mathbf{56}$, $\mathbf{13} \cap \mathbf{45}$, $\mathbf{24} \cap \mathbf{36}$ are collinear. Expanding $[(\mathbf{12} \wedge \mathbf{56})(\mathbf{13} \wedge \mathbf{45})(\mathbf{24} \wedge \mathbf{36})] = 0$, we get

$$\mathrm{conic}(\mathbf{123456}) = [\mathbf{135}][\mathbf{245}][\mathbf{126}][\mathbf{346}] - [\mathbf{125}][\mathbf{345}][\mathbf{136}][\mathbf{246}] = 0. \tag{2.1}$$

[1] It must be pointed out that all the representations and computations in this paper do not depend on the numbers field as long as its characteristic is not 2.

Proposition 2.1. *For any six points* $\mathbf{1}, \ldots, \mathbf{6}$ *in the plane, the expression conic*$(\mathbf{123456})$ *is antisymmetric with respect to the six points. Furthermore, for any point* $\mathbf{6}'$ *in the plane,*

$$\begin{aligned}&[\mathbf{126'}][\mathbf{346'}]\mathrm{conic}(\mathbf{123456}) + [\mathbf{125}][\mathbf{345}]\mathrm{conic}(\mathbf{123466'})\\ &\quad = [\mathbf{126}][\mathbf{346}]\mathrm{conic}(\mathbf{123456'}).\end{aligned} \tag{2.2}$$

Proof. Obviously, conic$(\mathbf{123456})$ is antisymmetric with respect to any of the three pairs $\mathbf{14}$, $\mathbf{23}$, $\mathbf{56}$. We only need to prove the antisymmetry with respect to any of the two pairs $\mathbf{12}$, $\mathbf{15}$. This can be verified by contractions:

$$\begin{aligned}\mathrm{conic}(\mathbf{123456}) + \mathrm{conic}(\mathbf{213456}) &= [\mathbf{126}][\mathbf{346}]([\mathbf{135}][\mathbf{245}] - [\mathbf{235}][\mathbf{145}])\\ &\quad -[\mathbf{125}][\mathbf{345}]([\mathbf{136}][\mathbf{246}] - [\mathbf{236}][\mathbf{146}])\\ &\overset{\text{contract}}{=} 0,\\ \mathrm{conic}(\mathbf{123456}) + \mathrm{conic}(\mathbf{523416}) &= [\mathbf{135}][\mathbf{346}]([\mathbf{245}][\mathbf{126}] + [\mathbf{124}][\mathbf{256}])\\ &\quad -[\mathbf{125}][\mathbf{246}]([\mathbf{345}][\mathbf{136}] + [\mathbf{134}][\mathbf{356}])\\ &\overset{\text{contract}}{=} 0. \quad \Box\end{aligned}$$

Corollary 2.2. *If* $\mathbf{1}, \ldots, \mathbf{5}, \mathbf{1}', \ldots, \mathbf{5}'$ *are conconic, then for any point* $\mathbf{A}$ *in the plane,*

$$\begin{aligned}\frac{\mathrm{conic}(\mathbf{A12345})}{\mathrm{conic}(\mathbf{A12345'})} &= \frac{[\mathbf{125}][\mathbf{345}]}{[\mathbf{125'}][\mathbf{345'}]},\\ \frac{\mathrm{conic}(\mathbf{A12345})}{\mathrm{conic}(\mathbf{A1234'5'})} &= \frac{[\mathbf{124}][\mathbf{125}][\mathbf{345}]}{[\mathbf{124'}][\mathbf{125'}][\mathbf{345'}]},\\ \frac{\mathrm{conic}(\mathbf{A12345})}{\mathrm{conic}(\mathbf{A123'4'5'})} &= \frac{[\mathbf{123}][\mathbf{124}][\mathbf{125}][\mathbf{345}]}{[\mathbf{123'}][\mathbf{124'}][\mathbf{125'}][\mathbf{3'4'5'}]}.\end{aligned} \tag{2.3}$$

Obviously the list (2.3) can continue. From it we derive the following important concept.

Definition 2.1. Let $C(\mathcal{S})$ be a Cayley or bracket expression of points $\mathcal{S} = \mathbf{1}, \ldots, \mathbf{i}$ on a conic, and assume that C is either symmetric or antisymmetric. The *transformation rules* of C with respect to $\mathcal{S}$ are the properties that for any conic points $\mathbf{1}'$, $\mathbf{2}'$, $\mathbf{3}'$,

$$\frac{C(\mathbf{1}, \mathbf{2}, \ldots, \mathbf{i})}{C(\mathbf{1}', \mathbf{2}, \ldots, \mathbf{i})} = \frac{[\mathbf{1k}_1\mathbf{k}_2][\mathbf{1k}_3\mathbf{k}_4]}{[\mathbf{1'k}_1\mathbf{k}_2][\mathbf{1'k}_3\mathbf{k}_4]}, \tag{2.4}$$

where $\mathbf{k}_1, \ldots, \mathbf{k}_4$ are any four elements in $\mathcal{S}$ different from $\mathbf{1}$, $\mathbf{1}'$; and

$$\frac{C(\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \mathbf{i})}{C(\mathbf{1}', \mathbf{2}', \mathbf{3}, \ldots, \mathbf{i})} = \frac{[\mathbf{12k}_3][\mathbf{1k}_1\mathbf{k}_2][\mathbf{2k}_1\mathbf{k}_2]}{[\mathbf{1'2'k}_3][\mathbf{1'k}_1\mathbf{k}_2][\mathbf{2'k}_1\mathbf{k}_2]}, \tag{2.5}$$

where $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ are any three elements in $\mathcal{S}$ different from $\mathbf{1}$, $\mathbf{2}$, $\mathbf{1}'$, $\mathbf{2}'$; and

$$\frac{C(\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \ldots, \mathbf{i})}{C(\mathbf{1}', \mathbf{2}', \mathbf{3}', \mathbf{4}, \ldots, \mathbf{i})} = \frac{[\mathbf{123}][\mathbf{1k}_1\mathbf{k}_2][\mathbf{2k}_1\mathbf{k}_2][\mathbf{3k}_1\mathbf{k}_2]}{[\mathbf{1'2'3'}][\mathbf{1'k}_1\mathbf{k}_2][\mathbf{2'k}_1\mathbf{k}_2][\mathbf{3'k}_1\mathbf{k}_2]}, \tag{2.6}$$

where $\mathbf{k}_1$, $\mathbf{k}_2$ are any two elements in $\mathcal{S}$ different from $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$, $\mathbf{1}'$, $\mathbf{2}'$, $\mathbf{3}'$; and so on. The ratios in the formulae are called *transformation coefficients*.

Assume that **1**, **2**, **3**, **4**, **5** are five distinct points in which no four are collinear. They determine a unique conic, denoted by **12345**. The juxtaposition here does not denote the outer product of vectors, although it is antisymmetric with respect to the elements.

Proposition 2.3. *Any point* **X** *on conic* **12345** *satisfies*

$$\begin{aligned}\mathrm{conic}_{\mathbf{123,45}}(\mathbf{X}) &= [\mathbf{145}][\mathbf{234}][\mathbf{235}][\mathbf{X12}][\mathbf{X13}] - [\mathbf{134}][\mathbf{135}][\mathbf{245}][\mathbf{X12}][\mathbf{X23}]\\ &\quad +[\mathbf{124}][\mathbf{125}][\mathbf{345}][\mathbf{X13}][\mathbf{X23}]\\ &= 0.\end{aligned} \tag{2.7}$$

If **1**, **2**, **3** are not collinear, a point **X** in the plane is on conic **12345** if and only if (2.7) holds. The expression $\mathrm{conic}_{\mathbf{123,45}}(\mathbf{X})$ is symmetric with respect to **1**, **2**, **3** but antisymmetric with respect to **4**, **5**. It satisfies the transformation rules with respect to **4**, **5**.

Proof. Applying the following Grassmann–Plücker (GP) relations to (2.7),

$$[\mathbf{134}][\mathbf{X23}] = [\mathbf{123}][\mathbf{X34}] + [\mathbf{234}][\mathbf{X13}],$$
$$[\mathbf{124}][\mathbf{X23}] = [\mathbf{123}][\mathbf{X24}] + [\mathbf{234}][\mathbf{X12}],$$

we get

$$\begin{aligned}\mathrm{conic}_{\mathbf{123,45}}(\mathbf{X}) \;&=\; [\mathbf{123}]([\mathbf{125}][\mathbf{345}][\mathbf{13X}][\mathbf{24X}] - [\mathbf{135}][\mathbf{245}][\mathbf{12X}][\mathbf{34X}])\\ &\quad +[\mathbf{X12}][\mathbf{X13}][\mathbf{234}]([\mathbf{145}][\mathbf{235}] + [\mathbf{125}][\mathbf{345}] - [\mathbf{135}][\mathbf{245}])\\ &\overset{\text{contract}}{=} -[\mathbf{123}]\,\mathrm{conic}(\mathbf{12345X}).\end{aligned}$$ □

2.2. Polars and tangents

Recall that for four distinct collinear points **A**, **B**, **C**, **D**, the pair **C**, **D** are *conjugate* with respect to **A**, **B**, or equivalently, the pair **A**, **B** are *conjugate* with respect to **C**, **D**, if the cross ratio $(\mathbf{AB}; \mathbf{CD}) = -1$. A degenerate case is $\mathbf{A} = \mathbf{B}$. Then the pair **A**, **A** are *conjugate* with respect to **A** and any point **D** in the plane.

Two distinct points **A**, **B** are said to be *conjugate* with respect to a conic, if either they are conjugate with respect to the points **C**, **D** in which **AB** meets the conic, or line **AB** is part of the conic. A point is *conjugate* to itself with respect to a conic if it is on the conic.

As is well known, if **A** is not a double point of a conic, then the conjugates of **A** with respect to the conic form a line, called the *polar* of **A**. In particular, if **A** is on the conic, its polar is the *tangent* at **A**. Dually, the points on a line l which is not part of a conic, have a unique common conjugate with respect to the conic, called the *pole* of l. When l is tangent to the conic, its pole is the *point of tangency*.

Proposition 2.4. *Two points* **A**, **B** *are conjugate with respect to conic* **12345** *if and only if*

$$\begin{aligned}\mathrm{conjugate}_{\mathbf{12345}}(\mathbf{A},\mathbf{B}) &= [\mathbf{135}][\mathbf{245}]([\mathbf{12A}][\mathbf{34B}] + [\mathbf{12B}][\mathbf{34A}])\\ &\quad -[\mathbf{125}][\mathbf{345}]([\mathbf{13A}][\mathbf{24B}] + [\mathbf{13B}][\mathbf{24A}])\\ &= 0.\end{aligned} \tag{2.8}$$

When **A** *is not a double point of the conic, its polar is*

$$\begin{aligned}\mathrm{polar}_{\mathbf{A}}(\mathbf{12345}) &= [\mathbf{135}][\mathbf{245}]([\mathbf{12A}]\,\mathbf{34} + [\mathbf{34A}]\,\mathbf{12})\\ &\quad -[\mathbf{125}][\mathbf{345}]([\mathbf{13A}]\,\mathbf{24} + [\mathbf{24A}]\,\mathbf{13}).\end{aligned} \tag{2.9}$$

The expressions $\mathrm{conjugate}_{\mathbf{12345}}(\mathbf{A},\mathbf{B})$ *and* $\mathrm{polar}_{\mathbf{A}}(\mathbf{12345})$ *are antisymmetric with respect to* **1**, **2**, **3**, **4**, **5**, *and follow the transformation rules with respect to the five points.*

Proof. If $\mathbf{A}=\mathbf{B}$, then $\mathrm{conjugate}_{\mathbf{12345}}(\mathbf{A},\mathbf{A}) = 2\,\mathrm{conic}(\mathbf{12345A})$. If $\mathbf{A}\neq\mathbf{B}$ and $\mathbf{A}$ is not a conic point, let $\mathbf{X}$ be an intersection of line $\mathbf{AB}$ with the conic. Let $\lambda = -[\mathbf{BX}]/[\mathbf{AX}]$. Substituting $\mathbf{X}=\mathbf{B}+\lambda\mathbf{A}$ into $\mathrm{conic}(\mathbf{12345X})$ and expanding the result, we get

$$\begin{aligned}\mathrm{conic}(\mathbf{12345X}) &= [\mathbf{135}][\mathbf{245}]([\mathbf{12B}]+\lambda[\mathbf{12A}])([\mathbf{34B}]+\lambda[\mathbf{34A}])\\ &\quad -[\mathbf{125}][\mathbf{345}]([\mathbf{13B}]+\lambda[\mathbf{13A}])([\mathbf{24B}]+\lambda[\mathbf{24A}])\\ &= \mathrm{conic}(\mathbf{12345B})+\lambda\,\mathrm{conjugate}_{\mathbf{12345}}(\mathbf{A},\mathbf{B})\\ &\quad +\lambda^2\,\mathrm{conic}(\mathbf{12345A}).\end{aligned}\tag{2.10}$$

Let $\mathbf{C}$, $\mathbf{D}$ be the points of intersection of $\mathbf{AB}$ with the conic. They are conjugate with respect to $\mathbf{A}$, $\mathbf{B}$, i.e. $\frac{[\mathbf{BC}]}{[\mathbf{AC}]}+\frac{[\mathbf{BD}]}{[\mathbf{AD}]}=0$, if and only if the two roots of (2.10) have zero sum, i.e. the linear part of the equation is zero, which is just (2.8).

If $\mathbf{A}\neq\mathbf{B}$ and $\mathbf{B}$ is not a conic point, we still have (2.8). If $\mathbf{A}\neq\mathbf{B}$ and both are conic points, they are conjugate with respect to the conic if and only if line $\mathbf{AB}$ is part of the conic, i.e. if and only if (2.10) holds for any scalar λ. Since the quadratic and constant parts of the equation are already zero, the linear part must also be zero. □

There are other representations of $\mathrm{polar}_{\mathbf{A}}(\mathbf{12345})$. When $[\mathbf{123}]\neq 0$, substituting $\mathbf{4}=[\mathbf{234}]\mathbf{1}-[\mathbf{134}]\mathbf{2}+[\mathbf{124}]\mathbf{3}$ into (2.9) and using contractions, we get the following two forms:

$$\begin{aligned}\mathrm{polar}_{\mathbf{A}}(\mathbf{12345}) &= ([\mathbf{124}][\mathbf{135}][\mathbf{235}][\mathbf{34A}]-[\mathbf{125}][\mathbf{134}][\mathbf{234}][\mathbf{35A}])\mathbf{12}\\ &\quad -([\mathbf{134}][\mathbf{125}][\mathbf{235}][\mathbf{24A}]-[\mathbf{135}][\mathbf{124}][\mathbf{234}][\mathbf{25A}])\mathbf{13}\\ &\quad +([\mathbf{234}][\mathbf{125}][\mathbf{135}][\mathbf{14A}]-[\mathbf{235}][\mathbf{124}][\mathbf{134}][\mathbf{15A}])\mathbf{23}\\ &= ([\mathbf{134}][\mathbf{135}][\mathbf{23A}][\mathbf{245}]-[\mathbf{234}][\mathbf{235}][\mathbf{13A}][\mathbf{145}])\mathbf{12}\\ &\quad -([\mathbf{124}][\mathbf{125}][\mathbf{23A}][\mathbf{345}]-[\mathbf{234}][\mathbf{235}][\mathbf{12A}][\mathbf{145}])\mathbf{13}\\ &\quad +([\mathbf{134}][\mathbf{135}][\mathbf{12A}][\mathbf{245}]-[\mathbf{124}][\mathbf{125}][\mathbf{13A}][\mathbf{345}])\mathbf{23}.\end{aligned}\tag{2.11}$$

Proposition 2.5. *The tangent of conic* **12345** *at point* **5**, *which is not a double point of the conic, is*

$$\frac{\mathbf{12}}{[\mathbf{125}]}+\frac{\mathbf{34}}{[\mathbf{345}]}-\frac{\mathbf{13}}{[\mathbf{135}]}-\frac{\mathbf{24}}{[\mathbf{245}]}.\tag{2.12}$$

It can also be written in the following form:

$$\mathrm{tangent}_{\mathbf{5},\mathbf{1234}} = [\mathbf{134}][\mathbf{235}][\mathbf{245}]\,\mathbf{15}-[\mathbf{135}][\mathbf{145}][\mathbf{234}]\,\mathbf{25}.\tag{2.13}$$

The expression $\mathrm{tangent}_{\mathbf{5},\mathbf{1234}}$ *is antisymmetric with respect to* **1**, **2**, **3**, **4** *and satisfies the transformation rules with respect to the four points.*

2.3. Poles

The pole of line $\mathbf{AB}$ with respect to a conic is the intersection of the polars of the two points. Computing the wedge product of the polars directly leads to a complicated result. In this paper, instead, we construct poles by tangents.

Let $\mathbf{1}$, $\mathbf{2}$ be the intersection of the line and conic $\mathbf{12345}$. Then

$$\begin{aligned}\text{tangent}_{\mathbf{1},\mathbf{2345}} \wedge \text{tangent}_{\mathbf{2},\mathbf{1345}} &= ([\mathbf{134}][\mathbf{135}][\mathbf{245}]\mathbf{12} - [\mathbf{124}][\mathbf{125}][\mathbf{345}]\mathbf{13}) \\ &\quad \wedge([\mathbf{145}][\mathbf{234}][\mathbf{235}]\mathbf{12} + [\mathbf{124}][\mathbf{125}][\mathbf{345}]\mathbf{23}) \\ &= [\mathbf{123}][\mathbf{124}][\mathbf{125}][\mathbf{345}]\,([\mathbf{145}][\mathbf{234}][\mathbf{235}]\mathbf{1} \\ &\quad +[\mathbf{134}][\mathbf{135}][\mathbf{245}]\mathbf{2} - [\mathbf{124}][\mathbf{125}][\mathbf{345}]\mathbf{3}). \end{aligned} \tag{2.14}$$

Proposition 2.6. *If line* $\mathbf{12}$ *is not tangent to conic* $\mathbf{12345}$*, then the pole of* $\mathbf{12}$ *with respect to the conic is*

$$\text{pole}_{\mathbf{12},\mathbf{345}} = [\mathbf{145}][\mathbf{234}][\mathbf{235}]\mathbf{1} + [\mathbf{134}][\mathbf{135}][\mathbf{245}]\mathbf{2} - [\mathbf{124}][\mathbf{125}][\mathbf{345}]\mathbf{3}. \tag{2.15}$$

It is symmetric with respect to $\mathbf{1}$, $\mathbf{2}$*, antisymmetric with respect to* $\mathbf{3}$, $\mathbf{4}$, $\mathbf{5}$*, and follows the transformation rules with respect to* $\mathbf{3}$, $\mathbf{4}$, $\mathbf{5}$.

Proposition 2.7. *Let* $\mathbf{1}, \ldots, \mathbf{5}, \mathbf{2}', \ldots, \mathbf{5}'$ *be distinct conic points. Then*

$$\text{tangent}_{\mathbf{1},\mathbf{2345}} \wedge \text{tangent}_{\mathbf{2},\mathbf{13'4'5'}} = [\mathbf{123'}][\mathbf{124'}][\mathbf{125'}][\mathbf{3'4'5'}]\,\text{pole}_{\mathbf{12},\mathbf{345}} \tag{2.16}$$

$$\text{pole}_{\mathbf{12},\mathbf{345}} \vee \text{pole}_{\mathbf{13},\mathbf{24'5'}} = 2\,[\mathbf{14'5'}][\mathbf{234'}][\mathbf{235'}]\,\text{tangent}_{\mathbf{1},\mathbf{2345}} \tag{2.17}$$

$$[\text{pole}_{\mathbf{45},\mathbf{123}}\,\text{tangent}_{\mathbf{1},\mathbf{2'3'4'5}}] = -2\,[\mathbf{124}][\mathbf{12'5}][\mathbf{134}][\mathbf{13'5}][\mathbf{14'5}][\mathbf{2'3'4'}][\mathbf{235}]. \tag{2.18}$$

Proof. (2.16) is the result of (2.14) and the transformation rule of $\text{tangent}_{\mathbf{2},\mathbf{1345}}$ with respect to $\mathbf{3}$, $\mathbf{4}$, $\mathbf{5}$. When $\mathbf{4} = \mathbf{4}'$ and $\mathbf{5} = \mathbf{5}'$, (2.17) is straightforward from (2.15), with

$$\text{tangent}_{\mathbf{1},\mathbf{2345}} = -[\mathbf{245}][\mathbf{134}][\mathbf{135}]\mathbf{12} + [\mathbf{345}][\mathbf{124}][\mathbf{125}]\mathbf{13}. \tag{2.19}$$

By the transformation rule of $\text{pole}_{\mathbf{13},\mathbf{245}}$ with respect to $\mathbf{4}$, $\mathbf{5}$,

$$\begin{aligned}\text{pole}_{\mathbf{12},\mathbf{345}} \vee \text{pole}_{\mathbf{13},\mathbf{24'5'}} &= \frac{[\mathbf{14'5'}][\mathbf{234'}][\mathbf{235'}]}{[\mathbf{145}][\mathbf{234}][\mathbf{235}]}\text{pole}_{\mathbf{12},\mathbf{345}} \vee \text{pole}_{\mathbf{13},\mathbf{245}} \\ &= 2\,[\mathbf{14'5'}][\mathbf{234'}][\mathbf{235'}]\,\text{tangent}_{\mathbf{1},\mathbf{2345}}.\end{aligned}$$

To prove (2.18), first assume that $\mathbf{2}' = \mathbf{2}$, $\mathbf{3}' = \mathbf{3}$ and $\mathbf{4}' = \mathbf{4}$. Then

$$\begin{aligned}[\text{pole}_{\mathbf{45},\mathbf{123}}\,\text{tangent}_{\mathbf{1},\mathbf{2345}}] &= [\mathbf{124}][\mathbf{125}][\mathbf{135}]\{[\mathbf{134}][\mathbf{235}]([\mathbf{124}][\mathbf{345}] \\ &\quad -[\mathbf{134}][\mathbf{245}]) + [\mathbf{134}][\mathbf{234}]([\mathbf{125}][\mathbf{345}] \\ &\quad -[\mathbf{135}][\mathbf{245}])\} \\ &\overset{\text{contract}}{=} -2\,[\mathbf{124}][\mathbf{125}][\mathbf{134}][\mathbf{135}][\mathbf{145}][\mathbf{234}][\mathbf{235}].\end{aligned}$$

By the transformation rule of $\text{tangent}_{\mathbf{1},\mathbf{2345}}$ with respect to $\mathbf{2}$, $\mathbf{3}$, $\mathbf{4}$, we get (2.18). $\square$

Corollary 2.8. *For distinct conic points* $\mathbf{1}, \ldots, \mathbf{5}, \mathbf{4}', \mathbf{5}', \mathbf{1}'', \ldots, \mathbf{5}''$,

$$\begin{aligned}&[\text{pole}_{\mathbf{12},\mathbf{34''5''}}\ \text{pole}_{\mathbf{13},\mathbf{24'5'}}\ \text{pole}_{\mathbf{45},\mathbf{1''23}}] \\ &\quad = -4\,[\mathbf{1''24}][\mathbf{125}][\mathbf{134}][\mathbf{1''35}][\mathbf{14'5'}][\mathbf{14''5''}][\mathbf{234'}][\mathbf{234''}][\mathbf{235'}][\mathbf{235''}], \\ &[\text{pole}_{\mathbf{12},\mathbf{34''5''}}\ \text{pole}_{\mathbf{13},\mathbf{245'}}\ \text{pole}_{\mathbf{45},\mathbf{12''3''}}] \\ &\quad = -4\,[\mathbf{124}][\mathbf{12''5}][\mathbf{134}][\mathbf{13''5}][\mathbf{14''5''}][\mathbf{145'}][\mathbf{2''3''4}][\mathbf{234''}][\mathbf{235''}][\mathbf{235'}].\end{aligned} \tag{2.20}$$

Proposition 2.9. (1) *For distinct conic points* $\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{2}', \ldots, \mathbf{5}'$,

$$[\mathbf{1}\,\mathrm{pole}_{\mathbf{12},\mathbf{3'45}}\,\mathrm{pole}_{\mathbf{3'4'},\mathbf{12'5'}}]$$
$$= 2\,[\mathbf{123'}][\mathbf{12'4'}][\mathbf{13'4}][\mathbf{13'5}][\mathbf{14'5'}][\mathbf{2'3'5'}][\mathbf{245}]. \qquad (2.21)$$

(2)

$$\frac{[\mathbf{1}\,\mathrm{pole}_{\mathbf{12},\mathbf{345}}\,\mathrm{pole}_{\mathbf{34},\mathbf{125}}]}{[\mathbf{2}\,\mathrm{pole}_{\mathbf{12},\mathbf{345}}\,\mathrm{pole}_{\mathbf{34},\mathbf{125}}]} = \frac{[\mathbf{134}]}{[\mathbf{234}]},$$
$$\frac{[\mathbf{1}\,\mathrm{pole}_{\mathbf{12},\mathbf{345}}\,\mathrm{pole}_{\mathbf{34},\mathbf{125}}]}{[\mathbf{3}\,\mathrm{pole}_{\mathbf{12},\mathbf{345}}\,\mathrm{pole}_{\mathbf{34},\mathbf{125}}]} = -\frac{[\mathbf{124}]}{[\mathbf{234}]}. \qquad (2.22)$$

Proof. For (2.21), first remove all the primes. Then

$$\begin{aligned}[\mathbf{1}\,\mathrm{pole}_{\mathbf{12},\mathbf{345}}\,\mathrm{pole}_{\mathbf{34},\mathbf{125}}] &= [\mathbf{123}][\mathbf{124}][\mathbf{134}][\mathbf{135}][\mathbf{245}]([\mathbf{145}][\mathbf{235}] \\ &\quad +[\mathbf{135}][\mathbf{245}] - [\mathbf{125}][\mathbf{345}]) \\ &\overset{\text{contract}}{=} 2\,[\mathbf{123}][\mathbf{124}][\mathbf{134}][\mathbf{135}][\mathbf{145}][\mathbf{235}][\mathbf{245}]. \qquad (2.23)\end{aligned}$$

(2.21) can be obtained from (2.23) as follows: first replace $\mathbf{3}$, $\mathbf{4}$ by $\mathbf{3'}$, $\mathbf{4'}$, then change $\mathbf{4'}$ in $\mathrm{pole}_{\mathbf{12},\mathbf{3'4'5}}$ to $\mathbf{4}$, and change $\mathbf{2}$, $\mathbf{5}$ in $\mathrm{pole}_{\mathbf{3'4'},\mathbf{125}}$ to $\mathbf{2'}$, $\mathbf{5'}$. The transformation coefficient from $[\mathbf{1}\,\mathrm{pole}_{\mathbf{12},\mathbf{3'45}}\,\mathrm{pole}_{\mathbf{3'4'},\mathbf{125}}]$ to $[\mathbf{1}\,\mathrm{pole}_{\mathbf{12},\mathbf{3'45}}\,\mathrm{pole}_{\mathbf{3'4'},\mathbf{12'5'}}]$ is $\frac{[\mathbf{13'4}][\mathbf{245}]}{[\mathbf{13'4'}][\mathbf{24'5}]}\,\frac{[\mathbf{12'4'}][\mathbf{14'5'}][\mathbf{2'3'5'}]}{[\mathbf{124'}][\mathbf{14'5}][\mathbf{23'5}]}$. (2.22) is a direct corollary of (2.23). □

2.4. Intersections

Proposition 2.10. *When line* $\mathbf{AB}$ *is not part of conic* $\mathbf{A1234}$*, the second intersection of the line and the conic, denoted by* $\mathbf{X} = \mathbf{AB} \cap \mathbf{A1234}$*, is*

$$\mathbf{X}_{\mathbf{AB},\mathbf{1234}} = [\mathbf{134}][\mathbf{24A}][\mathbf{3AB}]\,\mathbf{12} \wedge \mathbf{AB} - [\mathbf{124}][\mathbf{34A}][\mathbf{2AB}]\,\mathbf{13} \wedge \mathbf{AB}. \qquad (2.24)$$

It is antisymmetric with respect to $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$, $\mathbf{4}$ *and satisfies the transformation rules with respect to the four points.*

Proof. Expanding (2.24) by separating $\mathbf{A}$, $\mathbf{B}$ in the wedge products, we get

$$\begin{aligned}\mathbf{X}_{\mathbf{AB},\mathbf{1234}} &= ([\mathbf{12B}][\mathbf{134}][\mathbf{24A}][\mathbf{3AB}] - [\mathbf{124}][\mathbf{13B}][\mathbf{2AB}][\mathbf{34A}])\mathbf{A} \\ &\quad +([\mathbf{124}][\mathbf{13A}][\mathbf{2AB}][\mathbf{34A}] - [\mathbf{12A}][\mathbf{134}][\mathbf{24A}][\mathbf{3AB}])\mathbf{B} \\ &= c\mathbf{A} + t\mathbf{B}. \qquad (2.25)\end{aligned}$$

The two coefficients are $c = \mathrm{conic}(\mathbf{123AB4})$ and $t = [\mathbf{B}\,\mathrm{tangent}_{\mathbf{A},\mathbf{2314}}]$. Since line $\mathbf{AB}$ is not part of the conic, c and t cannot be both zero. Substituting (2.25) into $\mathrm{conic}(\mathbf{4321AX}) = [\mathbf{12X}][\mathbf{13A}][\mathbf{24A}][\mathbf{34X}] - [\mathbf{12A}][\mathbf{13X}][\mathbf{24X}][\mathbf{34A}]$, we get

$$\begin{aligned}\mathrm{conic}(\mathbf{4321AX}_{\mathbf{AB},\mathbf{1234}}) &= ct\{[\mathbf{12A}][\mathbf{24A}]([\mathbf{13A}][\mathbf{34B}] - [\mathbf{13B}][\mathbf{34A}]) \\ &\quad +[\mathbf{13A}][\mathbf{34A}]([\mathbf{12B}][\mathbf{24A}] - [\mathbf{12A}][\mathbf{24B}])\} \\ &\quad +t^2([\mathbf{12B}][\mathbf{13A}][\mathbf{24A}][\mathbf{34B}] \\ &\quad -[\mathbf{12A}][\mathbf{13B}][\mathbf{24B}][\mathbf{34A}]) \\ &\overset{\text{contract}}{=} ct([\mathbf{12A}][\mathbf{134}][\mathbf{24A}][\mathbf{3AB}]\end{aligned}$$

$$
\begin{aligned}
& -[\mathbf{124}][\mathbf{13A}][\mathbf{2AB}][\mathbf{34A}]) + ct^2 \\
= \quad & 0.
\end{aligned}
$$

So $\mathbf{X}$ is a point on both line $\mathbf{AB}$ and conic $\mathbf{A1234}$.

The expression $\mathbf{X_{AB,1234}}$ is obviously antisymmetric with respect to $\mathbf{2}$, $\mathbf{3}$. Its antisymmetry with respect to $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{1}$, $\mathbf{4}$ can be proved by contractions. $\square$

Corollary 2.11. *Let* $\mathbf{1}, \mathbf{2}, \mathbf{A}$ *be not collinear. Let* $\mathbf{1}' = \mathbf{12345} \cap \mathbf{1A}$ *and* $\mathbf{2}' = \mathbf{12345} \cap \mathbf{2A}$. *Then*

$$\mathrm{polar}_{\mathbf{A}}(\mathbf{12345}) = \mathbf{12}'_{\mathbf{2A},\mathbf{1345}} - \mathbf{21}'_{\mathbf{1A},\mathbf{2345}}. \tag{2.26}$$

Proposition 2.12. *For two distinct conics* $\mathbf{12345}$ *and* $\mathbf{1234'5'}$, *if lines* $\mathbf{12}$, $\mathbf{13}$, $\mathbf{23}$ *are not part of the conics, then the fourth intersection of the two conics, denoted by* $\mathbf{X} = \mathbf{12345} \cap \mathbf{1234'5'}$, *is*

$$\mathbf{X} = \lambda_{\mathbf{2}}\lambda_{\mathbf{3}}\, \mathbf{1} + \lambda_{\mathbf{1}}\lambda_{\mathbf{3}}\, \mathbf{2} + \lambda_{\mathbf{1}}\lambda_{\mathbf{2}}\, \mathbf{3}, \tag{2.27}$$

where for any permutation $\mathbf{i}, \mathbf{j}, \mathbf{k}$ *of* $\mathbf{1}, \mathbf{2}, \mathbf{3}$,

$$
\begin{aligned}
\lambda_{\mathbf{i}} &= [\mathbf{i}\, \mathrm{pole}_{\mathbf{jk},\mathbf{i45}}\, \mathrm{pole}_{\mathbf{jk},\mathbf{i4'5'}}] = [\mathbf{i}\, \mathrm{pole}_{\mathbf{ij},\mathbf{k45}}\, \mathrm{pole}_{\mathbf{ik},\mathbf{j4'5'}}] \\
&= -[\mathbf{i}\, \mathrm{pole}_{\mathbf{ij},\mathbf{k45}}\, \mathrm{pole}_{\mathbf{ij},\mathbf{k4'5'}}].
\end{aligned} \tag{2.28}
$$

Proof. First, by the hypotheses, $\mathbf{1}, \mathbf{2}, \mathbf{3}$ are not collinear, and the two conics do not have a line in common. So they have four points at the intersection. According to (2.7),

$$
\begin{aligned}
\mathrm{conic}_{\mathbf{123},\mathbf{45}}(\mathbf{X}) &= [\mathbf{145}][\mathbf{234}][\mathbf{235}][\mathbf{X12}][\mathbf{X13}] - [\mathbf{134}][\mathbf{135}][\mathbf{245}][\mathbf{X12}][\mathbf{X23}] \\
&\quad + [\mathbf{124}][\mathbf{125}][\mathbf{345}][\mathbf{X13}][\mathbf{X23}] \\
&= 0, \\
\mathrm{conic}_{\mathbf{123},\mathbf{4'5'}}(\mathbf{X}) &= [\mathbf{14'5'}][\mathbf{234'}][\mathbf{235'}][\mathbf{X12}][\mathbf{X13}] - [\mathbf{134'}][\mathbf{135'}][\mathbf{24'5'}][\mathbf{X12}][\mathbf{X23}] \\
&\quad + [\mathbf{124'}][\mathbf{125'}][\mathbf{34'5'}][\mathbf{X13}][\mathbf{X23}] \\
&= 0.
\end{aligned}
$$

Let $\lambda'_1 = [\mathbf{X12}][\mathbf{X13}]$, $\lambda'_2 = -[\mathbf{X12}][\mathbf{X23}]$, $\lambda'_3 = [\mathbf{X13}][\mathbf{X23}]$. Then vector $(\lambda'_1, \lambda'_2, \lambda'_3)^T$ is parallel to vector

$$\mathbf{V} = \begin{pmatrix} [\mathbf{145}] & [\mathbf{234}] & [\mathbf{235}] \\ [\mathbf{134}] & [\mathbf{135}] & [\mathbf{245}] \\ [\mathbf{124}] & [\mathbf{125}] & [\mathbf{345}] \end{pmatrix} \times \begin{pmatrix} [\mathbf{14'5'}] & [\mathbf{234'}] & [\mathbf{235'}] \\ [\mathbf{134'}] & [\mathbf{135'}] & [\mathbf{24'5'}] \\ [\mathbf{124'}] & [\mathbf{125'}] & [\mathbf{34'5'}] \end{pmatrix}. \tag{2.29}$$

By (2.15), it is easy to verify the equality of the three forms of $\lambda_{\mathbf{i}}$ in (2.28), and the equality $\mathbf{V} = (\lambda_{\mathbf{1}}, \lambda_{\mathbf{2}}, \lambda_{\mathbf{3}})^T/[\mathbf{123}]$. On the other hand, by Cramer's rule $[\mathbf{123}]\mathbf{X} = [\mathbf{X23}]\mathbf{1} - [\mathbf{X13}]\mathbf{2} + [\mathbf{X12}]\mathbf{3}$, the vector formed by the coordinates of $\mathbf{X}$ with respect to the basis $\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$ is parallel to vector $(\lambda'_2\lambda'_3, \lambda'_1\lambda'_3, \lambda'_1\lambda'_2)^T$, and is also parallel to vector $(\lambda_{\mathbf{2}}\lambda_{\mathbf{3}}, \lambda_{\mathbf{1}}\lambda_{\mathbf{3}}, \lambda_{\mathbf{1}}\lambda_{\mathbf{2}})^T$.

We only need to show that $\mathbf{V} \neq 0$. Since $[\mathbf{123}] \neq 0$, a point $\mathbf{Y}$ is on conic $\mathbf{12345}$ if and only if $\mathrm{conic}_{\mathbf{123},\mathbf{45}}(\mathbf{Y}) = 0$. So $\mathbf{V} = 0$ if and only if the two conics are identical. $\square$

2.5. Conics determined by points and tangents

Proposition 2.13. *Given five points* $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$, *such that* (1) $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ *are distinct and noncollinear,* (2) $\mathbf{4}, \mathbf{5}$ *are distinct,* (3) *either points* $\mathbf{1}, \mathbf{2}, \mathbf{3}$ *are not on line* $\mathbf{45}$, *and point* $\mathbf{4}$ *is not on any of the lines* $\mathbf{12}, \mathbf{13}, \mathbf{23}$, *or only one of the points* $\mathbf{1}, \mathbf{2}, \mathbf{3}$ *is on line* $\mathbf{45}$, *then there exists a unique conic passing through points* $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ *and tangent to line* $\mathbf{45}$, *denoted by conic* $(\mathbf{1234}, \mathbf{45})$. *A point* $\mathbf{X}$ *is on the conic if and only if*

$$\mathrm{conic}(\mathbf{X1234}, \mathbf{45}) = [\mathbf{134}][\mathbf{245}][\mathbf{14X}][\mathbf{23X}] - [\mathbf{234}][\mathbf{145}][\mathbf{13X}][\mathbf{24X}] = 0. \quad (2.30)$$

The expression $\mathrm{conic}(\mathbf{X1234}, \mathbf{45})$ is antisymmetric with respect to $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{X}$.

Proof. If $[\mathbf{145}] = 0$, then $[\mathbf{134}]$, $[\mathbf{245}]$ are nonzero. (2.30) is the equation of the line-conic $\mathbf{14}, \mathbf{23}$. If $[\mathbf{245}] = 0$, then (2.30) is the equation of the line-conic $\mathbf{13}, \mathbf{24}$. If $[\mathbf{345}] = 0$, by contractions

$$[\mathbf{134}][\mathbf{245}] - [\mathbf{234}][\mathbf{145}] = 0, \quad [\mathbf{14X}][\mathbf{23X}] - [\mathbf{13X}][\mathbf{24X}] = -[\mathbf{12X}][\mathbf{34X}],$$

(2.30) becomes $-[\mathbf{134}][\mathbf{245}][\mathbf{12X}][\mathbf{34X}]$, which is the equation of the line-conic $\mathbf{12}, \mathbf{34}$. Below we assume that points $\mathbf{1}, \mathbf{2}, \mathbf{3}$ are not on line $\mathbf{45}$.

By the hypotheses, $[\mathbf{124}]$, $[\mathbf{134}]$, $[\mathbf{234}]$ are all nonzero. If $[\mathbf{123}] = 0$, by contractions

$$[\mathbf{134}][\mathbf{23X}] - [\mathbf{234}][\mathbf{13X}] = 0, \quad [\mathbf{245}][\mathbf{14X}] - [\mathbf{145}][\mathbf{24X}] = -[\mathbf{124}][\mathbf{45X}],$$

(2.30) becomes $-[\mathbf{124}][\mathbf{134}][\mathbf{23X}][\mathbf{45X}]$, which is the equation of the line-conic $\mathbf{23}, \mathbf{45}$. So we further assume that $[\mathbf{123}] \neq 0$. Then no three of the four points $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ are collinear. Let $\mathbf{X}$ be any point in the plane distinct from the four points, then no four of the five points $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{X}$ are collinear, so they determine a unique conic $\mathbf{1234X}$. Obviously none of the four points $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ can be a double point of the conic, and the tangent at $\mathbf{4}$ exists. Comparing (2.30) with (2.13), we find that (2.30) is exactly the equation of the tangent at $\mathbf{4}$ of conic $\mathbf{1234X}$, with $\mathbf{5}$ as the indeterminate point. In other words, (2.30) holds if and only if $\mathbf{45}$ is tangent to the conic $\mathbf{1234X}$.

The antisymmetry of $\mathrm{conic}(\mathbf{X1234}, \mathbf{45})$ with respect to $\mathbf{1}, \mathbf{2}$ is obvious. The antisymmetry with respect to $\mathbf{1}, \mathbf{3}$ and $\mathbf{1}, \mathbf{X}$ can be proved by contractions. □

Corollary 2.14. (1) *Any point* $\mathbf{X}$ *on conic* $(\mathbf{1234}, \mathbf{45})$ *satisfies*

$$\begin{aligned}&[\mathbf{234}]^2[\mathbf{145}][\mathbf{12X}][\mathbf{13X}] - [\mathbf{134}]^2[\mathbf{245}][\mathbf{12X}][\mathbf{23X}]\\ &\quad + [\mathbf{124}]^2[\mathbf{345}][\mathbf{13X}][\mathbf{23X}] = 0.\end{aligned} \quad (2.31)$$

If $[\mathbf{123}] \neq 0$, *a point* $\mathbf{X}$ *in the plane is on the conic if and only if* (2.31) *holds.*

(2) *If* $\mathbf{1}', \mathbf{2}', \mathbf{3}'$ *are points on conic* $(\mathbf{1234}, \mathbf{45})$, *then for any point* $\mathbf{X}$ *in the plane,*

$$\begin{aligned}
\frac{\mathrm{conic}(\mathbf{X1234}, \mathbf{45})}{\mathrm{conic}(\mathbf{X1'234}, \mathbf{45})} &= \frac{[\mathbf{124}][\mathbf{134}]}{[\mathbf{1'24}][\mathbf{1'34}]} = \frac{[\mathbf{123}][\mathbf{145}]}{[\mathbf{1'23}][\mathbf{1'45}]},\\
\frac{\mathrm{conic}(\mathbf{X1234}, \mathbf{45})}{\mathrm{conic}(\mathbf{X1'2'34}, \mathbf{45})} &= \frac{[\mathbf{124}][\mathbf{134}][\mathbf{234}]}{[\mathbf{1'2'4}][\mathbf{1'34}][\mathbf{2'34}]} = \frac{[\mathbf{123}][\mathbf{145}][\mathbf{245}]}{[\mathbf{1'2'3}][\mathbf{1'45}][\mathbf{2'45}]},\\
\frac{\mathrm{conic}(\mathbf{X1234}, \mathbf{45})}{\mathrm{conic}(\mathbf{X1'2'3'4}, \mathbf{45})} &= \frac{[\mathbf{124}][\mathbf{134}][\mathbf{234}]}{[\mathbf{1'2'4}][\mathbf{1'3'4}][\mathbf{2'3'4}]} = \frac{[\mathbf{123}][\mathbf{145}][\mathbf{245}][\mathbf{345}]}{[\mathbf{1'2'3'}][\mathbf{1'45}][\mathbf{2'45}][\mathbf{3'45}]}.
\end{aligned} \quad (2.32)$$

(3) *If line* $\mathbf{4'5'}$ *is tangent to conic* $(\mathbf{1234}, \mathbf{45})$ *at point* $\mathbf{4'}$, *then for any point* $\mathbf{X}$ *in the plane, any conic point* $\mathbf{1}$ *which is not a double point,*

$$\frac{\mathrm{conic}(\mathbf{X1234}, \mathbf{45})}{\mathrm{conic}(\mathbf{X1234'}, \mathbf{4'5'})} = \frac{[\mathbf{145}][\mathbf{234}]^2}{[\mathbf{14'5'}][\mathbf{234'}]^2}. \tag{2.33}$$

Proposition 2.15. (1) *Let* $\mathbf{A}, \mathbf{B}$ *be two points in the plane. They are conjugate with respect to conic* $(\mathbf{1234}, \mathbf{45})$ *if and only if*

$$\begin{aligned} \mathrm{conjugate}_{\mathbf{1234},\mathbf{45}}(\mathbf{A}, \mathbf{B}) &= [\mathbf{134}][\mathbf{245}]([\mathbf{23A}][\mathbf{14B}] + [\mathbf{14A}][\mathbf{23B}]) \\ &\quad -[\mathbf{234}][\mathbf{145}]([\mathbf{24A}][\mathbf{13B}] + [\mathbf{13A}][\mathbf{24B}]) \\ &= 0. \end{aligned} \tag{2.34}$$

The polar of point $\mathbf{A} \neq \mathbf{23} \cap \mathbf{45}$ *with respect to conic* $(\mathbf{1234}, \mathbf{45})$ *is*

$$\begin{aligned} \mathrm{polar}_{\mathbf{A}}(\mathbf{1234}, \mathbf{45}) &= [\mathbf{134}][\mathbf{245}]([\mathbf{23A}]\mathbf{14} + [\mathbf{14A}]\mathbf{23}) \\ &\quad -[\mathbf{234}][\mathbf{145}]([\mathbf{24A}]\mathbf{13} + [\mathbf{13A}]\mathbf{24}). \end{aligned} \tag{2.35}$$

Both expressions are antisymmetric with respect to $\mathbf{1}, \mathbf{2}, \mathbf{3}$, *and satisfy the same transformation rules with respect to* $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$ *as* $\mathrm{conic}(\mathbf{X1234}, \mathbf{45})$.

(2) *The tangent at point* $\mathbf{1} \neq \mathbf{23} \cap \mathbf{45}$ *of the conic is*

$$\begin{aligned} \mathrm{tangent}_{\mathbf{1}}(\mathbf{1234}, \mathbf{45}) &= [\mathbf{134}]^2[\mathbf{245}]\mathbf{12} - [\mathbf{124}]^2[\mathbf{345}]\mathbf{13} \\ &= [\mathbf{134}][\mathbf{145}][\mathbf{234}]\mathbf{12} - [\mathbf{123}][\mathbf{124}][\mathbf{345}]\mathbf{14} \\ &= [\mathbf{124}][\mathbf{145}][\mathbf{234}]\mathbf{13} - [\mathbf{123}][\mathbf{134}][\mathbf{245}]\mathbf{14}. \end{aligned} \tag{2.36}$$

It is antisymmetric with respect to $\mathbf{2}, \mathbf{3}$, *and satisfies the same transformation rules with respect to* $\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$ *as* $\mathrm{conic}(\mathbf{X1234}, \mathbf{45})$.

Proposition 2.16. *Given five points* $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$ *such that* (1) $\mathbf{1}, \mathbf{2}, \mathbf{3}$ *are not collinear,* (2) $\mathbf{24}, \mathbf{35}$ *are lines and are distinct,* (3) *either* $\mathbf{1}$ *is on one of the lines* $\mathbf{24}, \mathbf{35}$, *or* $\mathbf{1}, \mathbf{3}$ *are not on line* $\mathbf{24}$ *and* $\mathbf{1}, \mathbf{2}$ *are not on line* $\mathbf{35}$, *then there exists a unique conic passing through* $\mathbf{1}, \mathbf{2}, \mathbf{3}$ *and tangent to lines* $\mathbf{24}, \mathbf{35}$, *denoted by conic* $(\mathbf{123}, \mathbf{24}, \mathbf{35})$. *A point* $\mathbf{X}$ *is on the conic if and only if* $\mathrm{conic}(\mathbf{X123}, \mathbf{24}, \mathbf{35}) = 0$, *where*

$$\begin{aligned} \mathrm{conic}(\mathbf{X123}, \mathbf{24}, \mathbf{35}) &= [\mathbf{123}]^2[\mathbf{24X}][\mathbf{35X}] - [\mathbf{124}][\mathbf{135}][\mathbf{23X}]^2 \\ &= [\mathbf{12X}][\mathbf{13X}][\mathbf{234}][\mathbf{235}] - [\mathbf{12X}][\mathbf{23X}][\mathbf{135}][\mathbf{234}] \\ &\quad -[\mathbf{13X}][\mathbf{23X}][\mathbf{124}][\mathbf{235}]. \end{aligned} \tag{2.37}$$

Proof. The equality of the two forms in (2.37) is the result of Cramer's rule $[\mathbf{123}]\mathbf{X} = [\mathbf{12X}]\mathbf{3} - [\mathbf{13X}]\mathbf{2} + [\mathbf{23X}]\mathbf{1}$ and the inequality $[\mathbf{123}] \neq 0$. In the following, by (2.37) we mean the first form.

If $[\mathbf{124}]$ or $[\mathbf{135}]$ is zero, then (2.37) represents the line-conic $\mathbf{24}, \mathbf{35}$. If $[\mathbf{234}] = 0$ or $[\mathbf{235}] = 0$, then $[\mathbf{135}] = 0$. Below we assume that $[\mathbf{124}], [\mathbf{135}], [\mathbf{234}], [\mathbf{235}]$ are nonzero. For any point $\mathbf{X}$ not on lines $\mathbf{12}$ and $\mathbf{23}$, we have (1) $\mathbf{3}, \mathbf{1}, \mathbf{X}, \mathbf{2}$ are distinct and noncollinear, (2) $\mathbf{2}, \mathbf{4}$ are distinct, (3) either $\mathbf{X}$ is on line $\mathbf{24}$, or $\mathbf{X}$ is not on any of the lines $\mathbf{12}, \mathbf{23}, \mathbf{24}$. So there exists a unique conic $(\mathbf{31X2}, \mathbf{24})$. Since $\mathbf{3}$ is not on line $\mathbf{24}$, it cannot be a double point of the conic, and the tangent at **3** exists, whose equation is just (2.37), with **5** as the indeterminate point. □

Corollary 2.17. *If point* $\mathbf{1}'$ *is on conic* $(\mathbf{123}, \mathbf{24}, \mathbf{35})$*, then for any point* $\mathbf{X}$ *in the plane,*

$$\frac{\mathrm{conic}(\mathbf{X123}, \mathbf{24}, \mathbf{35})}{\mathrm{conic}(\mathbf{X1'23}, \mathbf{24}, \mathbf{35})} = \frac{[\mathbf{123}]^2}{[\mathbf{1'23}]^2}. \tag{2.38}$$

If lines $\mathbf{2'4'}, \mathbf{3'5'}$ *are tangent to the conic at points* $\mathbf{2'}, \mathbf{3'}$*, then*

$$\begin{aligned} \frac{\mathrm{conic}(\mathbf{X123}, \mathbf{24}, \mathbf{35})}{\mathrm{conic}(\mathbf{X123'}, \mathbf{24}, \mathbf{3'5'})} &= \frac{[\mathbf{123}]^2[\mathbf{235}]}{[\mathbf{123'}]^2[\mathbf{23'5'}]} = \frac{[\mathbf{234}]^2[\mathbf{135}]}{[\mathbf{23'4}]^2[\mathbf{13'5'}]}, \\ \frac{\mathrm{conic}(\mathbf{X123}, \mathbf{24}, \mathbf{35})}{\mathrm{conic}(\mathbf{X12'3'}, \mathbf{2'4'}, \mathbf{3'5'})} &= \frac{[\mathbf{123}]^2[\mathbf{23'4}][\mathbf{235}]}{[\mathbf{12'3'}]^2[\mathbf{2'3'4'}][\mathbf{23'5'}]}. \end{aligned} \tag{2.39}$$

Proposition 2.18. (1) *Let* $\mathbf{A}, \mathbf{B}$ *be two points in the plane. They are conjugate with respect to conic* $(\mathbf{123}, \mathbf{24}, \mathbf{35})$ *if and only if*

$$\begin{aligned} \mathrm{conjugate}_{\mathbf{123},\mathbf{24},\mathbf{35}}(\mathbf{A}, \mathbf{B}) &= [\mathbf{123}]^2([\mathbf{35A}][\mathbf{24B}] + [\mathbf{24A}][\mathbf{35B}]) \\ &\quad -2\,[\mathbf{124}][\mathbf{135}][\mathbf{23A}][\mathbf{23B}] \\ &= 0. \end{aligned} \tag{2.40}$$

The polar of point $\mathbf{A} \neq \mathbf{24} \cap \mathbf{35}$ *with respect to conic* $(\mathbf{123}, \mathbf{24}, \mathbf{35})$ *is*

$$\mathrm{polar}_{\mathbf{A}}(\mathbf{123}, \mathbf{24}, \mathbf{35}) = [\mathbf{123}]^2([\mathbf{35A}]\mathbf{24} + [\mathbf{24A}]\mathbf{35}) - 2\,[\mathbf{124}][\mathbf{135}][\mathbf{23A}]\mathbf{23}. \tag{2.41}$$

They satisfy the same transformation rules with respect to $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$ *as* $\mathrm{conic}(\mathbf{X123}, \mathbf{24}, \mathbf{35})$.

(2) *The tangent at* $\mathbf{1} \neq \mathbf{24} \cap \mathbf{35}$ *is*

$$\mathrm{tangent}_{\mathbf{1}}(\mathbf{123}, \mathbf{24}, \mathbf{35}) = [\mathbf{135}][\mathbf{234}]\mathbf{12} + [\mathbf{124}][\mathbf{235}]\mathbf{13}. \tag{2.42}$$

In particular, when $\mathbf{4} = \mathbf{5}$ *(the pole of* $\mathbf{23}$*), the tangent is*

$$\mathrm{tangent}_{\mathbf{1},\mathbf{4}}(\mathbf{123}, \mathbf{24}, \mathbf{34}) = [\mathbf{134}]\mathbf{12} + [\mathbf{124}]\mathbf{13}. \tag{2.43}$$

Corollary 2.19. *If lines* $\mathbf{2'4'}, \mathbf{3'5'}$ *are tangent to conic* $(\mathbf{123}, \mathbf{24}, \mathbf{35})$ *at points* $\mathbf{2'}, \mathbf{3'}$ *respectively, then*

$$\begin{aligned} \frac{\mathrm{tangent}_{\mathbf{1}}(\mathbf{123}, \mathbf{24}, \mathbf{35})}{\mathrm{tangent}_{\mathbf{1}}(\mathbf{123'}, \mathbf{24}, \mathbf{3'5'})} &= \frac{[\mathbf{123}][\mathbf{235}]}{[\mathbf{123'}][\mathbf{23'5'}]}, \\ \frac{\mathrm{tangent}_{\mathbf{1}}(\mathbf{123}, \mathbf{24}, \mathbf{35})}{\mathrm{tangent}_{\mathbf{1}}(\mathbf{12'3'}, \mathbf{2'4'}, \mathbf{3'5'})} &= \frac{[\mathbf{123}][\mathbf{23'4}][\mathbf{235}]}{[\mathbf{12'3'}][\mathbf{2'3'4'}][\mathbf{23'5'}]}. \end{aligned} \tag{2.44}$$

If $\mathbf{4} = \mathbf{5}, \mathbf{4'} = \mathbf{5'}$*, then*

$$\begin{aligned} \frac{\mathrm{tangent}_{\mathbf{1},\mathbf{4}}(\mathbf{123}, \mathbf{24}, \mathbf{34})}{\mathrm{tangent}_{\mathbf{1},\mathbf{4'}}(\mathbf{123'}, \mathbf{24'}, \mathbf{3'4'})} &= \frac{[\mathbf{123}][\mathbf{124}]}{[\mathbf{123'}][\mathbf{124'}]} = \frac{[\mathbf{123}][\mathbf{23'4}]}{[\mathbf{123'}][\mathbf{23'4'}]}, \\ \frac{\mathrm{tangent}_{\mathbf{1},\mathbf{4}}(\mathbf{123}, \mathbf{24}, \mathbf{34})}{\mathrm{tangent}_{\mathbf{1},\mathbf{4'}}(\mathbf{12'3'}, \mathbf{2'4'}, \mathbf{3'4'})} &= \frac{[\mathbf{123}][\mathbf{23'4}]}{[\mathbf{12'3'}][\mathbf{23'4'}]}. \end{aligned} \tag{2.45}$$

3. Elimination rules

3.1. Geometric constructions

Summarizing what we have studied so far, we get the following list of typical geometric constructions. Each construction is associated with a set of *given nondegeneracy conditions* (Buchberger, 1988; Chou, 1988; Mourrain, 1991; Wu, 1994, 2000, 2001).

Construction 1. **X** is a free point in the plane: no nondegeneracy condition.

Construction 2. **X** is a semifree point on line **12**: $\exists$**12**.

Construction 3. **X** is the conjugate of point **3** on line **12**: $\exists$**12**.

Construction 4. **X** is the intersection of two lines **12**, **34**: $\exists$**12**, $\exists$**34**, and **1**, **2**, **3**, **4** are not collinear.

Construction 5. **X** is a free point on conic (a) **12345**, (b) (**1234**, **45**), (c) (**123**, **24**, **35**). The nondegeneracy conditions of (b), (c) are already given in Propositions 2.13 and 2.16. They are denoted by $\exists$(**1234**, **45**) and $\exists$(**1234**, **24**, **35**) respectively. The nondegeneracy conditions of (a) are that the five points are distinct, and no four of them are collinear; denoted by $\exists$**12345**.

Construction 6. Conic **12** . . . **i**, where the number of elements $i \geq 6$:
The construction means that among the i points, five of them are free points in the plane determining a conic, and the others are free points on the conic. The i points are called *free conic points*. The nondegeneracy condition is that there exist five points $\mathbf{j}_1, \ldots, \mathbf{j}_5$ in $\{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{i}\}$ such that $\exists \mathbf{j}_1 \ldots \mathbf{j}_5$, denoted by $\exists$**12** . . . **i**.

Construction 7. l is the polar (including tangent) of point **A** with respect to conic (a) **12345**, (b) (**1234**, **45**), (c) (**123**, **24**, **35**).
The nondegeneracy conditions are denoted by $\exists \text{polar}_{\mathbf{A}}$. They are (a) $\exists$**12345**, (b) $\exists$(**1234**, **45**), (c) $\exists$(**123**, **24**, **35**), and in each case, **A** is not a double point of the conic.

Construction 8. **X** is a semifree point on the tangent at point **1** of a conic: $\exists \text{polar}_{\mathbf{1}}$.

Construction 9. **X** is the intersection of line **AB** and the polar of point **1** with respect to a conic: $\exists$**AB**, $\exists \text{polar}_{\mathbf{1}}$, either **A** or **B** is not conjugate to **1** with respect to the conic.

Construction 10. **X** is the pole of line **AB** with respect to a conic: $\exists$**AB**, the conic exists, **AB** is not part of the conic; denoted by $\exists \text{pole}_{\mathbf{AB}}$.

Construction 11. **X** is the second intersection of line **AB** and conic **A1234**: $\exists \text{pole}_{\mathbf{AB}}$.

Construction 12. **X** is the fourth intersection of two conics **12345** and **1234′5′**: the 10 points are not conconic, $\exists$**123**, $\exists \text{pole}_{\mathbf{12}}(\mathbf{12345})$, $\exists \text{pole}_{\mathbf{12}}(\mathbf{1234'5'})$, $\exists \text{pole}_{\mathbf{13}}(\mathbf{12345})$, $\exists \text{pole}_{\mathbf{13}}(\mathbf{1234'5'})$, $\exists \text{pole}_{\mathbf{23}}(\mathbf{12345})$, $\exists \text{pole}_{\mathbf{23}}(\mathbf{1234'5'})$.

The first four constructions in the list belong to incidence geometry. In Li and Wu (2001), a *free point* by default indicates a free point in the plane, and points by the second to the fourth constructions are called *incidence points*. In a geometric construction, the elements (points and lines) directly involved in the construction are the *parents*, and the constructed element is the *child*. By the parent–child partial order, for two comparable elements, one is an *ascendant* and the other is a *descendent*. The same terminology will also be used in this paper.

The following is an important concept in the selection of representations.

Definition 3.1. Let $\mathbf{X} = \mathbf{X}(\mathbf{1}, \ldots, \mathbf{i})$ be a geometric construction. A point $\mathbf{j}$ in $\mathbf{1}, \ldots, \mathbf{i}$ is called an *essential point* of the construction if when $\mathbf{X}$ is represented by a linear combination of vectors, $\mathbf{j}$ is one of the vectors, or when $\mathbf{X}$ is represented by a wedge product, $\mathbf{j}$ occurs in the wedge product. If there is no such point, then $\mathbf{X}$ itself is the *essential point* of the construction.

For the above 12 constructions, the corresponding essential points are

(1)$\mathbf{X}$	(2)$\mathbf{1}, \mathbf{2}$	(3)$\mathbf{1}, \mathbf{2}$	(4)$\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$	(5)$\mathbf{X}$	(6)$\mathbf{1}, \ldots, \mathbf{i}$
(7)$\mathbf{A}$	(8)$\mathbf{1}$	(9)$\mathbf{A}, \mathbf{B}, \mathbf{1}$	(10)$\mathbf{A}, \mathbf{B}$	(11)$\mathbf{A}, \mathbf{B}$	(12)$\mathbf{1}, \mathbf{2}, \mathbf{3}$.

3.2. Free conic points and intersections

Elimination rule 1. Let $\mathbf{X}$ be a free conic point. To eliminate $\mathbf{X}$ from a bracket expression $p(\mathbf{X})$,

(1) eliminate from $p(\mathbf{X})$ *all points* other than free conic points.

(2) Order the bracket mates of $\mathbf{X}$ in $p(\mathbf{X})$ by their numbers of occurrences in the brackets containing $\mathbf{X}$. Let $\mathbf{1}, \mathbf{2}, \mathbf{3}$ be the first three bracket mates with maximal occurrences. Substitute into $p(\mathbf{X})$ the following Cramer's rule:

$$\mathbf{X} = \frac{1}{[\mathbf{123}]}([\mathbf{X12}]\mathbf{3} - [\mathbf{X13}]\mathbf{2} + [\mathbf{X23}]\mathbf{1}). \tag{3.1}$$

Set $[\mathbf{X12}]$, $[\mathbf{X13}]$, $[\mathbf{X23}]$ to be *mute* Li and Wu (2001), i.e. they only satisfy the relation $\mathbf{B2}$ of brackets: $[\mathbf{A}_{i_1}\mathbf{A}_{i_2}\mathbf{A}_{i_3}] = \mathrm{sign}(\sigma)[\mathbf{A}_{i_{\sigma(1)}}\mathbf{A}_{i_{\sigma(2)}}\mathbf{A}_{i_{\sigma(3)}}]$ for any permutation σ of 1, 2, 3.

(3) If there are at least two other free conic points in $p(\mathbf{X})$, let $\mathbf{4}, \mathbf{5}$ be two of them. Use the following version of $\mathrm{conic}_{\mathbf{123},\mathbf{45}}(\mathbf{X}) = 0$ to eliminate $[\mathbf{X23}]$ from $p(\mathbf{X})$:

$$[\mathbf{X23}] = \frac{[\mathbf{145}][\mathbf{234}][\mathbf{235}][\mathbf{X12}][\mathbf{X13}]}{[\mathbf{134}][\mathbf{135}][\mathbf{245}][\mathbf{X12}] - [\mathbf{124}][\mathbf{125}][\mathbf{345}][\mathbf{X13}]}. \tag{3.2}$$

An application of Elimination rule 1 can be found in the remark of Example 7.4.

For the fourth intersection of two conics, the representation generally has 12 terms, in which the bracket coefficients have degree 12. Some simplifications must be carried out before eliminating the intersection.

Elimination rule 2. Let $\mathbf{X}$ be the fourth intersection of conics $\mathbf{12345}$ and $\mathbf{1234'5'}$. To eliminate $\mathbf{X}$ from a bracket expression $p(\mathbf{X})$,

(1) apply Cramer's rule $\mathbf{X} = [\mathbf{X23}]\mathbf{1} - [\mathbf{X13}]\mathbf{2} + [\mathbf{X12}]\mathbf{3}$ to $p(\mathbf{X})$.

(2) Compute

$$\begin{aligned} &\mu_1 = [\mathbf{145}][\mathbf{234}][\mathbf{235}], && \mu_2 = [\mathbf{134}][\mathbf{135}][\mathbf{245}], && \mu_3 = [\mathbf{124}][\mathbf{125}][\mathbf{345}], \\ &\mu_1' = [\mathbf{14'5'}][\mathbf{234'}][\mathbf{235'}], && \mu_2' = [\mathbf{134'}][\mathbf{135'}][\mathbf{24'5'}], && \mu_3' = [\mathbf{124'}][\mathbf{125'}][\mathbf{34'5'}] \end{aligned}$$

by eliminating all their incidence points.

(3) Compute $\lambda_1 = \mu_2\mu_3' - \mu_3\mu_2'$, $\lambda_2 = \mu_3\mu_1' - \mu_1\mu_3'$, $\lambda_3 = \mu_1\mu_2' - \mu_2\mu_1'$. Substitute $[\mathbf{X23}] = \lambda_2\lambda_3$, $[\mathbf{X13}] = -\lambda_1\lambda_3$, $[\mathbf{X12}] = \lambda_1\lambda_2$ into $p(\mathbf{X})$ after removing their common factors.

Before introducing the elimination rule of the second intersection of a line and a conic, let us look at an example.

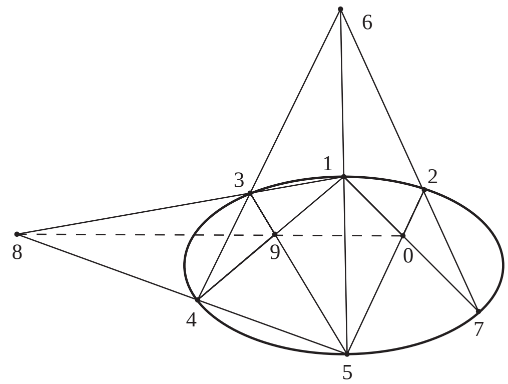

Fig. 1. Example 3.1.

Example 3.1. (A Reformulation of Theorem 5.51 in O'Hara and Ward, 1936, p. 112)
Free points: **1**, **2**, **3**, **4**, **5**.

Intersections: $\mathbf{6} = \mathbf{34} \cap \mathbf{15}$, $\mathbf{7} = \mathbf{12345} \cap \mathbf{26}$, $\mathbf{8} = \mathbf{13} \cap \mathbf{45}$, $\mathbf{9} = \mathbf{14} \cap \mathbf{35}$, $\mathbf{0} = \mathbf{25} \cap \mathbf{17}$.

Conclusion: **8**, **9**, **0** are collinear.

Proof.

Rules		$[\mathbf{890}]$
$[\mathbf{890}] = (\mathbf{13}\wedge\mathbf{45}\wedge\mathbf{25})(\mathbf{14}\wedge\mathbf{35}\wedge\mathbf{17}) - (\mathbf{13}\wedge\mathbf{45}\wedge\mathbf{17})(\mathbf{14}\wedge\mathbf{35}\wedge\mathbf{25}) = [\mathbf{245}][\mathbf{135}]^2[\mathbf{147}]-[\mathbf{137}][\mathbf{145}]^2[\mathbf{235}]$	$\overset{\mathbf{8},\mathbf{9},\mathbf{0}}{=}$	$[\mathbf{135}]^2[\mathbf{147}][\mathbf{245}] - [\mathbf{137}][\mathbf{145}]^2[\mathbf{235}]$
$\mathbf{7}_{\mathbf{26},\mathbf{1345}} = -[\mathbf{145}][\mathbf{235}][\mathbf{246}]\mathbf{13}\wedge\mathbf{26} + [\mathbf{135}][\mathbf{236}][\mathbf{245}]\mathbf{14}\wedge\mathbf{26}$	$\overset{\mathbf{7}}{=}$	$\underbrace{[\mathbf{126}][\mathbf{134}][\mathbf{135}][\mathbf{145}][\mathbf{235}][\mathbf{245}]}([\mathbf{135}][\mathbf{246}] - [\mathbf{145}][\mathbf{236}])$
	$\overset{\text{contract}}{=}$	$0.$

Additional nondegeneracy condition: none. The common bracket factors in each step are underbraced and are removed at the end of the step. □

In the above proof, the initial Cayley expansion produces a 2-term result in which **7** occurs in brackets [**137**] and [**147**]. Since **1** occurs in both brackets, **3** and **4** each occur in a bracket, the unique best representation for **7** is $\mathbf{7}_{\mathbf{26},\mathbf{1345}}$. This example suggests the following elimination method:

Elimination rule 3. Let **X** be the second intersection of line **AB** with a conic passing through **A**. To eliminate **X** from a bracket expression $p(\mathbf{X})$,

(1) use the following conic points selection algorithm to find for each bracket $q(\mathbf{X})$ containing **X** a sequence s_q. Substitute the first four elements of s_q into **1**, **2**, **3**, **4** of (2.24).

(2) Fix any $q(\mathbf{X})$, substitute the corresponding representation of **X** into it. For any other bracket containing **X**, substitute the corresponding representation of **X** and multiply the result by the corresponding transformation coefficient.

Algorithm. Conic points selection.

Input: (1) A construction $x = x(\mathcal{P})$ related to a conic, (2) $\mathcal{C}$, the set of conic points constructed before x, (3) $p(x)$, a Cayley expression where x occurs either in brackets or in wedge products.

Output: A set of pairs $(q(x), s_q)$, where the q's are the brackets or wedge products in $p(x)$ containing x, and s_q is a sequence of elements from $\mathcal{C}$.

Step 1. Let $\mathcal{C}_x$ be the elements in $\mathcal{C}$ which are not essential to x. For each bracket (or wedge product) $q(x)$ in $p(x)$ containing x, do the following. (1) For every bracket (or wedge product) mate y of x, find all its essential points $\mathcal{E}_y$ in $\mathcal{C}_x$. Set the *essential weight* of each element in $\mathcal{E}_y$ to be $(\#(\mathcal{E}_y))^{-1}$, where $\#(\mathcal{E}_y)$ is the number of elements in $\mathcal{E}_y$. (2) Let the union of all the $\mathcal{E}_y$'s be $\mathcal{E}_q$. Compute the sum of the essential weights for each element in $\mathcal{E}_q$. Order the elements by their essential weights, denote the descending sequence by the same symbol $\mathcal{E}_q$.

Step 2. If there is only one $q(x)$ then return $(q(x), \mathcal{E}_q, \mathcal{C}_x - \mathcal{E}_q)$. Else, let $\mathcal{E}$ be the union of all the $\mathcal{E}_q$'s. Compute the sum of the essential weights for each element in $\mathcal{E}$. Order the elements by their essential weights, denote the descending sequence by the same symbol $\mathcal{E}$.

Step 3. For all the q's, return $(q(x), \mathcal{E}_q, \mathcal{E} - \mathcal{E}_q, \mathcal{C}_x - \mathcal{E})$.

3.3. Polars, tangents and poles

According to (2.9), (2.35) and (2.41), for conic **12345**, the polar of point **A** has three forms:

$$\begin{aligned}
&\text{polar}_{\mathbf{A}}(\mathbf{12345})\\
&\quad = [\mathbf{135}][\mathbf{245}]([\mathbf{12A}]\,\mathbf{34} + [\mathbf{34A}]\,\mathbf{12}) - [\mathbf{125}][\mathbf{345}]([\mathbf{13A}]\,\mathbf{24} + [\mathbf{24A}]\,\mathbf{13})\\
&\quad = [\mathbf{145}][\mathbf{235}]([\mathbf{13A}]\,\mathbf{24} + [\mathbf{24A}]\,\mathbf{13}) - [\mathbf{135}][\mathbf{245}]([\mathbf{14A}]\,\mathbf{23} + [\mathbf{23A}]\,\mathbf{14})\\
&\quad = [\mathbf{145}][\mathbf{235}]([\mathbf{12A}]\,\mathbf{34} + [\mathbf{34A}]\,\mathbf{12}) - [\mathbf{125}][\mathbf{345}]([\mathbf{14A}]\,\mathbf{23} + [\mathbf{23A}]\,\mathbf{14}).
\end{aligned}\tag{3.3}$$

For conic $(\mathbf{1234}, \mathbf{45})$, the polar also has three forms:

$$\begin{aligned}
&\text{polar}_{\mathbf{A}}(\mathbf{1234}, \mathbf{45})\\
&\quad = [\mathbf{124}][\mathbf{345}]([\mathbf{13A}]\mathbf{24} + [\mathbf{24A}]\mathbf{13}) - [\mathbf{134}][\mathbf{245}]([\mathbf{34A}]\mathbf{12} + [\mathbf{12A}]\mathbf{34})\\
&\quad = [\mathbf{134}][\mathbf{245}]([\mathbf{23A}]\mathbf{14} + [\mathbf{14A}]\mathbf{23}) - [\mathbf{234}][\mathbf{145}]([\mathbf{24A}]\mathbf{13} + [\mathbf{13A}]\mathbf{24})\\
&\quad = [\mathbf{124}][\mathbf{345}]([\mathbf{23A}]\mathbf{14} + [\mathbf{14A}]\mathbf{23}) - [\mathbf{234}][\mathbf{145}]([\mathbf{34A}]\mathbf{12} + [\mathbf{12A}]\mathbf{34}).
\end{aligned}\tag{3.4}$$

For conic $(\mathbf{123}, \mathbf{24}, \mathbf{35})$, the polar has the unique form (2.41).

As to tangents, for conic **12345**, the tangent at point **1** has a representation (2.19), and there are six such representations by points **2**, **3**, **4**, **5**. For conic $(\mathbf{1234}, \mathbf{45})$, the tangent has three forms in the representation (2.36). For conic $(\mathbf{123}, \mathbf{24}, \mathbf{35})$, the tangent has the unique form (2.42), and for conic $(\mathbf{123}, \mathbf{24}, \mathbf{34})$, the tangent has the unique form (2.43). As to poles, by (2.15), the pole of **12** with respect to conic **12345** has three representations.

When there are more conic points and tangents, the number of representations grows quickly. In designing elimination rules, the key is to find in every related bracket or wedge product a suitable representation of the polar, tangent or pole.

Elimination rule 4. Let l be the polar of point **A**, or the tangent at point **1**. To eliminate l from a Cayley expression $p(l)$,

(1) If l is a polar, the conic is **12345** and has only the five points constructed before l, then substitute the three forms of (3.3) into $p(l)$ and select the shortest result.

If the conic is conic(**1234**, **45**), has only four points and one tangent constructed before l, then substitute the three forms of (3.4) if l is a polar, or (2.36) if l is a tangent, into $p(l)$ and select the shortest result.

If the conic is conic(**123**, **24**, **35**), has only three points and two tangents constructed before l, then substitute (2.41) or (2.42) or (2.43) into $p(l)$.

(2) For other cases, first use the conic points selection algorithm to find a sequence s_q for each bracket or wedge product $q(l)$ containing l. Then

Case 1. If the conic is constructed by points, substitute the first five elements of s_q into $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$ of (3.3), or substitute the first four elements of s_q into $\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$ of (2.19).

Case 2. If the conic is constructed by points and a tangent, then find in s_q the first element with a tangent constructed before l, denote it by **4** and denote the tangent by **45**. Substitute the first three elements in the remaining sequence into $\mathbf{1}, \mathbf{2}, \mathbf{3}$ of (3.4), or substitute the first two elements in the remaining sequence into $\mathbf{2}, \mathbf{3}$ of (2.36), together with $\mathbf{4}, \mathbf{5}$.

Case 3. If the conic is constructed by tangents and a point, then find in s_q the first two elements with tangents constructed before l, denote them by $\mathbf{2}, \mathbf{3}$, and denote the tangents by $\mathbf{24}, \mathbf{35}$. Substitute the first element in the remaining sequence into **1** of (2.41), together with $\mathbf{24}, \mathbf{35}$, or substitute $\mathbf{24}, \mathbf{35}$ into (2.42), or substitute $\mathbf{24}, \mathbf{34}$ into (2.43).

Finally, fix any $q(l)$, substitute the corresponding representation of l into it, and if there are three forms, select the shortest result. For any other $q(l)$, substitute the corresponding representation, select the shortest result, and multiply the result by the corresponding transformation coefficient.

Elimination rule 5. Let $\mathbf{X}$ be a free point on the tangent l at point **1** of conic **12345**. To eliminate $\mathbf{X}$ from a bracket expression $p(\mathbf{X})$, first use the conic points selection algorithm to find *one representation* of l (by the return of $\mathcal{E}$). Let it be $l = \mathbf{1} \vee \mathbf{A}$, where $\mathbf{A}$ is a vector expression. Substitute $\mathbf{X} = [\mathbf{1X}]\mathbf{A} - [\mathbf{AX}]\mathbf{1}$ into $p(\mathbf{X})$, set $[\mathbf{1X}], [\mathbf{AX}]$ to be mute.

Elimination rule 6. Let l be the polar/tangent of point $\mathbf{A}$. Let $\mathbf{X}$ be the intersection of l with l', which is either a polar/tangent or a line connecting two points. To eliminate $\mathbf{X}$ from a Cayley expression $p(\mathbf{X})$,

Case 1. If $l \wedge l'$ has a factored expansion, then substitute it into $p(\mathbf{X})$.

Case 2. For other cases, (1) replace each bracket $[\mathbf{X1'2'}]$ in $p(\mathbf{X})$ by $l \wedge l' \wedge \mathbf{1'2'}$. (2) Replace each wedge product $\mathbf{XY} \wedge \mathbf{1'2'} \wedge \mathbf{1''2''}$ by a factored/shortest expansion of $-[\mathbf{Y}(l \wedge l')(\mathbf{1'2'} \wedge \mathbf{1''2''})]$. (3) Replace each expression $[\mathbf{X}(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})]$ by a factored/shortest expansion of $[(l \wedge l')(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})]$. (4) For other Cayley expressions in $p(\mathbf{X})$, first expand them into bracket polynomials, then eliminate $\mathbf{X}$ from the brackets.

Elimination rule 7. Let $\mathbf{X}$ be the pole of line **12** with respect to a conic constructed by points. To eliminate $\mathbf{X}$ from a bracket expression $p(\mathbf{X})$,

(1) For each bracket $q(\mathbf{X})$ containing $\mathbf{X}$, if it has more than one pole, the poles should be eliminated in a batch. Formulae (2.17), (2.18), (2.20) and (2.21) may be used in the elimination.

(2) For each $q(\mathbf{X})$, use the conic points selection algorithm to find a sequence s_q. Substitute the first three elements of s_q into **3**, **4**, **5** of (2.15).

(3) Fix any $q(\mathbf{X})$, substitute the corresponding representation of $\mathbf{X}$ into it. For any other bracket containing $\mathbf{X}$, substitute the corresponding representation and multiply the result by the corresponding transformation coefficient.

4. Simplification techniques in conic computation

In this section we develop three simplification techniques in conic computation, based on the relations (2.1) and (2.7) among free conic points.

4.1. Conic transformation

Definition 4.1. Let p be a bracket polynomial which is neither contractible nor factorable in the polynomial ring of brackets. For any six conic points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{X}, \mathbf{Y}$ in p, if the transformation

$$[\mathbf{XAB}][\mathbf{XCD}][\mathbf{YAC}][\mathbf{YBD}] = [\mathbf{XAC}][\mathbf{XBD}][\mathbf{YAB}][\mathbf{YCD}] \tag{4.1}$$

either reduces the number of terms of p, or makes it factorable in the polynomial ring of brackets, or makes it contractible, the transformation is called a *conic transformation*.

The following are basic properties of the transformation (4.1):

(1) Any three brackets on the left side determine a unique transformation.
(2) There are only two transformations involving $[\mathbf{XAB}][\mathbf{XCD}]$ and $\mathbf{Y}$: the two brackets containing $\mathbf{Y}$ are either $[\mathbf{YAC}][\mathbf{YBD}]$ or $[\mathbf{YAD}][\mathbf{YBC}]$.
(3) For a conic point $\mathbf{X}$, if there is a point $\mathbf{Z}$ which occurs in every bracket containing $\mathbf{X}$, then there is no transformation involving $\mathbf{X}$.

The following is the criterion for (4.1) to be a conic transformation.

Proposition 4.1. *Let p be a bracket polynomial which is neither contractible nor factorable in the polynomial ring of brackets. Let T be the term of p containing the left side of* (4.1)*. Then* (4.1) *is a conic transformation if and only if one of the following conditions is satisfied:*

(1) *A bracket on the right side of* (4.1) *is in every term of p other than T.*
(2) *T after the transformation becomes a like term of another term in p.*
(3) *T after the transformation has only two brackets different from some other terms of p, and all the different brackets form a contractible degree-2 polynomial.*

In particular, if p has only two terms, then (4.1) is a conic transformation if and only if a bracket on the right side of (4.1) is in the other term of p.

Algorithm. Conic transformation.

Input: A bracket polynomial p of degree at least 4 and involving at least six conic points. Assume that p is already factored in the polynomial ring of brackets.

Output: A bracket polynomial q.

Procedure: Move the factors of p with degree less than 4 to q.

While p is not empty, do the following for each factor f of p, for each term T of f:

Step 1. Let $\mathcal{C}$ be the conic points in T. If $\#(\mathcal{C}) < 6$ then move f to q.

Step 2. Let p' be the square-free bracket factors of T formed by points in $\mathcal{C}$. Count the *degree* of each point in p', which is the number of occurrences of the point in p'. Denote by $\mathcal{C}'$ the points with degree at least 2.
If $\#(\mathcal{C}') < 6$, then move f to q, else if $\mathcal{C}' \neq \mathcal{C}$, set $\mathcal{C} = \mathcal{C}'$, go back to the beginning of this step.

Step 3. Let $\mathbf{X}$ be a point in $\mathcal{C}$ with the lowest degree in p'. Let $b(\mathbf{X})$ be the brackets of p' containing $\mathbf{X}$, and let $\bar{b}(\mathbf{X})$ be the brackets of p' without $\mathbf{X}$. Find from $b(\mathbf{X})$ the bracket pairs $[\mathbf{XAB}][\mathbf{XCD}]$ such that $\{\mathbf{A},\mathbf{B}\}\cap\{\mathbf{C},\mathbf{D}\}$ is empty. If there is no such pair, go to Step 5.

Step 4. For each pair $[\mathbf{XAB}][\mathbf{XCD}]$, set

$$\mathcal{R}_{\mathbf{C}} = \{\mathbf{Y} \in \mathcal{C} - \{\mathbf{X},\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\} \mid [\mathbf{YAC}][\mathbf{YBD}] \in \bar{b}(\mathbf{X})\},$$
$$\mathcal{R}_{\mathbf{D}} = \{\mathbf{Y} \in \mathcal{C} - \{\mathbf{X},\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\} \mid [\mathbf{YAD}][\mathbf{YBC}] \in \bar{b}(\mathbf{X})\}.$$

(1) For each point $\mathbf{Y}$ in $\mathcal{R}_{\mathbf{C}}$, let m be the smallest power of $[\mathbf{XAB}][\mathbf{XCD}][\mathbf{YAC}][\mathbf{YBD}]$ in T. If

$$([\mathbf{XAB}][\mathbf{XCD}][\mathbf{YAC}][\mathbf{YBD}])^m = ([\mathbf{XAC}][\mathbf{XBD}][\mathbf{YAB}][\mathbf{YCD}])^m$$

is a conic transformation, go to Step 6.
(2) For each point $\mathbf{Y}$ in $\mathcal{R}_{\mathbf{D}}$, let m be the smallest power of $[\mathbf{XAB}][\mathbf{XCD}][\mathbf{YAD}][\mathbf{YBC}]$ in p. If

$$([\mathbf{XAB}][\mathbf{XCD}][\mathbf{YAD}][\mathbf{YBC}])^m = ([\mathbf{XAD}][\mathbf{XBC}][\mathbf{YAB}][\mathbf{YCD}])^m$$

is a conic transformation, go to Step 6.

Step 5. If $\#(\mathcal{C}) > 6$, delete $\mathbf{X}$ from $\mathcal{C}$, go back to Step 2. Else, skip to the next term of f, and if f has no more terms, move f to q.

Step 6. Perform the conic transformation, contract and factor the result, replace f by the factors, and go back to the beginning of the Procedure.

Example 4.1 (From Example 7.2 in Section 7). Let $\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6}$ be conic points. Let

$$\begin{aligned} p = {} & [\mathbf{124}]^3[\mathbf{135}]^2[\mathbf{136}][\mathbf{256}][\mathbf{346}] - [\mathbf{124}]^2[\mathbf{125}][\mathbf{134}]^2[\mathbf{136}][\mathbf{256}][\mathbf{356}] \\ & + [\mathbf{124}][\mathbf{126}]^2[\mathbf{134}][\mathbf{135}]^2[\mathbf{245}][\mathbf{346}] - [\mathbf{125}][\mathbf{126}]^2[\mathbf{134}]^3[\mathbf{245}][\mathbf{356}]. \end{aligned}$$

The first, the third and the last terms of p each have one conic transformation:

$$\begin{aligned} &[\mathbf{124}][\mathbf{135}][\mathbf{256}][\mathbf{346}] = [\mathbf{125}][\mathbf{134}][\mathbf{246}][\mathbf{356}], \\ &[\mathbf{126}][\mathbf{135}][\mathbf{245}][\mathbf{346}] = [\mathbf{125}][\mathbf{136}][\mathbf{246}][\mathbf{345}], \\ &[\mathbf{126}][\mathbf{134}][\mathbf{245}][\mathbf{356}] = [\mathbf{124}][\mathbf{136}][\mathbf{256}][\mathbf{345}]. \end{aligned} \tag{4.2}$$

The second term has no conic transformation, as according to the algorithm, among the conic points $\mathcal{C} = \mathbf{1}^4, \mathbf{2}^3, \mathbf{3}^3, \mathbf{4}^2, \mathbf{5}^3, \mathbf{6}^3$, for $\mathbf{X} = \mathbf{4}$, point $\mathbf{1}$ occurs in both brackets of $b(\mathbf{X}) = [\mathbf{124}][\mathbf{134}]$.

The first conic transformation in (4.2) produces a common factor $[\mathbf{134}]$ for p. After its removal we get

$$p = [\mathbf{124}]^2[\mathbf{125}][\mathbf{135}][\mathbf{136}][\mathbf{246}][\mathbf{356}] - [\mathbf{124}]^2[\mathbf{125}][\mathbf{134}][\mathbf{136}][\mathbf{256}][\mathbf{356}] \\ + [\mathbf{124}][\mathbf{126}]^2[\mathbf{135}]^2[\mathbf{245}][\mathbf{346}] - [\mathbf{125}][\mathbf{126}]^2[\mathbf{134}]^2[\mathbf{245}][\mathbf{356}].$$

The second conic transformation produces a common factor $[\mathbf{125}]$ for p, and after its removal we get

$$p = [\mathbf{124}]^2[\mathbf{135}][\mathbf{136}][\mathbf{246}][\mathbf{356}] - [\mathbf{124}]^2[\mathbf{134}][\mathbf{136}][\mathbf{256}][\mathbf{356}] \\ + [\mathbf{124}][\mathbf{126}][\mathbf{135}][\mathbf{136}][\mathbf{246}][\mathbf{345}] - [\mathbf{126}]^2[\mathbf{134}]^2[\mathbf{245}][\mathbf{356}].$$

The last conic transformation produces two common factors $[\mathbf{124}][\mathbf{136}]$. Finally, with all common factors retrieved, we have

$$p = \underbrace{[\mathbf{124}][\mathbf{125}][\mathbf{134}][\mathbf{136}]}\{[\mathbf{124}][\mathbf{135}][\mathbf{246}][\mathbf{356}] - [\mathbf{124}][\mathbf{134}][\mathbf{256}][\mathbf{356}] \\ + [\mathbf{126}][\mathbf{135}][\mathbf{246}][\mathbf{345}] - [\mathbf{126}][\mathbf{134}][\mathbf{256}][\mathbf{345}]\}. \tag{4.3}$$

4.2. Pseudoconic transformation

Definition 4.2. Let p be a bracket polynomial which is neither contractible nor factorable in the polynomial ring of brackets, nor conic transformable. For any six conic points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{X}, \mathbf{Y}$ in p, if by the transformation

$$[\mathbf{XAB}][\mathbf{XCD}][\mathbf{YAC}] = \frac{[\mathbf{XAC}][\mathbf{XBD}][\mathbf{YAB}][\mathbf{YCD}]}{[\mathbf{YBD}]} \tag{4.4}$$

and the removal of common rational factors, either the degree of p is decreased, or p becomes contractible, then the transformation is called a *pseudoconic transformation*.

Proposition 4.2. *Let p be a bracket polynomial which is neither contractible nor factorable in the polynomial ring of brackets, nor conic transformable. Then*

1. *The transformation* (4.4) *cannot reduce the number of terms in p.*
2. *Let T be the term of p containing the left side of* (4.4)*. Then* (4.4) *is a pseudoconic transformation if and only if one of the following conditions is satisfied:*
   (1) *Two brackets in the numerator of the right side of* (4.4) *are in every term different from T.*
   (2) *The numerator of T after the transformation has only two brackets different from some other terms of p multiplied by* $[\mathbf{YBD}]$*, and all the different brackets form a contractible degree-*2 *polynomial.*
3. *If p is a degree-*3 *binomial, then it has no pseudoconic transformation.*
4. *When p is degree-*3 *and has at least three terms, then* (4.4) *is a pseudoconic transformation if and only if one of the following terms is in p, where λ is the coefficient of the term in p containing the left side of* (4.4):

(1) $-\lambda[\mathbf{XAB}][\mathbf{XAC}][\mathbf{YCD}]$,
(2) $-\lambda[\mathbf{XAC}][\mathbf{XCD}][\mathbf{YAB}]$,
(3) $-\lambda[\mathbf{XAC}][\mathbf{XBD}][\mathbf{YAC}]$.

Proof. 1. If p has a term which when multiplied by $[\mathbf{YBD}]$ becomes a like term of the numerator of the new T, then $[\mathbf{YBD}]$ has to be in the numerator of the new T, contradicting the hypothesis that p has no conic transformation.

2. Obvious.

3. Let U be the other term of p. Then (4.4) is a pseudoconic transformation if and only if $U[\mathbf{YBD}]$ and $[\mathbf{XAC}][\mathbf{XBD}][\mathbf{YAB}][\mathbf{YCD}]$ have two common bracket factors. So two of the latter four brackets must be in U, contradicting the hypothesis that p is not factorable.

4. Let T be the term containing the left side of (4.4). Let U be a term of p which when multiplied by $[\mathbf{YBD}]$ has two brackets in $[\mathbf{XAC}][\mathbf{XBD}][\mathbf{YAB}][\mathbf{YCD}]$. Since p is homogeneous, the third bracket in U is unique, and there are only the three cases in the proposition up to coefficient. For the same reason, no two brackets of $[\mathbf{XAC}][\mathbf{XBD}][\mathbf{YAB}][\mathbf{YCD}]$ can be common to every term different from T. So one term, for example $U[\mathbf{YBD}]$ must form a GP transformable pair with $\lambda[\mathbf{XAC}][\mathbf{XBD}][\mathbf{YAB}][\mathbf{YCD}]$. The negative signs in the three cases come from this requirement. □

Algorithm. Pseudoconic transformation.

Input: A bracket polynomial p of degree at least 3 and involving at least six conic points. Assume that p is factored in the polynomial ring of brackets, whose factors do not have conic transformations.

Output: A rational bracket polynomial q.

Procedure: Move to q the factors of p with degree less than 3, and the degree-3 binomial factors. While p is not empty, do the following for each factor f of p, for each term T of f:

Step 1. Let $\mathcal{C}$ be the conic points in T. If $\#(\mathcal{C}) < 6$ then move f to q.

Step 2. Let p' be the square-free bracket factors of T formed by points in $\mathcal{C}$. Count the degree of each point in p'. Denote by $\mathcal{C}'$ the points with degree at least 2. If $\#(\mathcal{C}') < 3$ then move f to q.

Step 3. Start from a point in $\mathcal{C}'$ with the lowest degree in p', say $\mathbf{X}$, do the following: (1) Let $b(\mathbf{X})$ be the brackets of p' containing $\mathbf{X}$, and let $\bar{b}(\mathbf{X})$ be the brackets of p' without $\mathbf{X}$.

(2) Find from $b(\mathbf{X})$ the bracket pairs $[\mathbf{XAB}][\mathbf{XCD}]$ such that $\{\mathbf{A}, \mathbf{B}\} \cap \{\mathbf{C}, \mathbf{D}\}$ is empty. If there is no such pair, skip to the point in $\mathcal{C}'$, and if $\mathcal{C}'$ has no more points, move f to q.

(3) For each pair $[\mathbf{XAB}][\mathbf{XCD}]$, let $\mathcal{R}$ be the brackets in $\bar{b}(\mathbf{X})$ whose elements are one of the pairs $\mathbf{AC}, \mathbf{BD}, \mathbf{AD}, \mathbf{BC}$ and a point other than $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$. For each bracket in $\mathcal{R}$, for example $[\mathbf{YAC}]$, let m be the smallest power of $[\mathbf{XAB}][\mathbf{XCD}][\mathbf{YAC}]$ in T. If

$$([\mathbf{XAB}][\mathbf{XCD}][\mathbf{YAC}])^m = ([\mathbf{XAC}][\mathbf{XBD}][\mathbf{YAB}][\mathbf{YCD}]/[\mathbf{YBD}])^m$$

is a pseudoconic transformation, perform it, contract and factor the result, put it in q and delete f from p, go back to the beginning of the Procedure.

Step 4. If T is the last term of f, move f to q.

Example 4.2. Consider $\mathrm{conic}_{\mathbf{123},\,\mathbf{45}}(\mathbf{X})$ in (2.7), where $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{X}$ are conic points:

$$\begin{aligned} p = {} & [\mathbf{145}][\mathbf{234}][\mathbf{235}][\mathbf{X12}][\mathbf{X13}] - [\mathbf{134}][\mathbf{135}][\mathbf{245}][\mathbf{X12}][\mathbf{X23}] \\ & + [\mathbf{124}][\mathbf{125}][\mathbf{345}][\mathbf{X13}][\mathbf{X23}]. \end{aligned}$$

Let T be the first term of p. The conic points with their degrees are $\mathcal{C} = \mathcal{C}' = \mathbf{1}^3, \mathbf{2}^3, \mathbf{3}^3, \mathbf{4}^2, \mathbf{5}^2, \mathbf{X}^2$. For point $\mathbf{X}$, since $b(\mathbf{X}) = [\mathbf{X12}][\mathbf{X13}]$, there is no transformation. For point $\mathbf{5}$, we have $\bar{b}(\mathbf{5}) = [\mathbf{145}][\mathbf{235}]$ and $\mathcal{R} = [\mathbf{X12}][\mathbf{X13}]$. The transformation

$$[\mathbf{145}][\mathbf{235}][\mathbf{X12}] = [\mathbf{125}][\mathbf{345}][\mathbf{14X}][\mathbf{23X}]/[\mathbf{X34}]$$

changes p to

$$\begin{aligned} & [\mathbf{125}][\mathbf{234}][\mathbf{345}][\mathbf{X13}][\mathbf{X14}] - [\mathbf{134}][\mathbf{135}][\mathbf{245}][\mathbf{X12}][\mathbf{X34}] \\ & \quad + [\mathbf{124}][\mathbf{125}][\mathbf{345}][\mathbf{X13}][\mathbf{X34}] \end{aligned}$$

multiplied by $[\mathbf{X23}]/[\mathbf{X34}]$. Then a contraction between the first and the third terms yields

$$[\mathbf{134}]([\mathbf{125}][\mathbf{345}][\mathbf{X13}][\mathbf{X24}] - [\mathbf{135}][\mathbf{245}][\mathbf{X12}][\mathbf{X34}]).$$

A conic transformation changes it to zero.

4.3. Conic contraction

The degree-5 polynomial $\mathrm{conic}_{\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4,\mathbf{A}_5\mathbf{A}_6}(\mathbf{A}_1) = 0$ can be written in the following form:

$$\begin{aligned} & [\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3][\mathbf{A}_2\mathbf{A}_4\mathbf{A}_5][\mathbf{A}_2\mathbf{A}_4\mathbf{A}_6][\mathbf{A}_3\mathbf{A}_5\mathbf{A}_6] - [\mathbf{A}_1\mathbf{A}_2\mathbf{A}_4][\mathbf{A}_2\mathbf{A}_3\mathbf{A}_5][\mathbf{A}_2\mathbf{A}_3\mathbf{A}_6][\mathbf{A}_4\mathbf{A}_5\mathbf{A}_6] \\ & \quad = \frac{[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3][\mathbf{A}_1\mathbf{A}_2\mathbf{A}_4][\mathbf{A}_2\mathbf{A}_5\mathbf{A}_6][\mathbf{A}_3\mathbf{A}_4\mathbf{A}_5][\mathbf{A}_3\mathbf{A}_4\mathbf{A}_6]}{[\mathbf{A}_1\mathbf{A}_3\mathbf{A}_4]}. \end{aligned} \tag{4.5}$$

The equality can be easily established by any pseudoconic transformation on the left side followed by a contraction. On the other hand, as shown in Example 4.2, the equality $\mathrm{conic}_{\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4,\mathbf{A}_5\mathbf{A}_6}(\mathbf{A}_1) = 0$ can only be established by a pseudoconic transformation, a contraction and a conic transformation. So it is better that we use (4.5) directly in bracket simplification. This is the idea of *conic contraction*.

Algorithm. Conic contraction.

Input: A bracket polynomial p of degree at least 4 and involving at least six conic points. Assume that p is already factored in the polynomial ring of brackets.

Output: A rational bracket polynomial q.

Procedure: Move the factors of p with degree less than 4 to q.

While p is not empty, do the following for each factor f of p, for each pair of terms $t_1 + t_2$ of f:

Step 1. If any of the following conditions is not satisfied, skip to the next pair of terms, and if f has no more pairs of terms, move f to q.
(1) Each term has four brackets, all of which are square-free.
(2) The coefficients are ± 1.
(3) t_1 has six points, all of which are from the same conic.
(4) The points are $\mathbf{A}_1$ to $\mathbf{A}_6$, with degrees 1, 3, 2, 2, 2, 2 respectively.
(5) The pair $\mathbf{A}_1\mathbf{A}_2$ occurs once in each term. The corresponding brackets are denoted by $[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3]$, $[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_4]$ respectively.
(6) $t_1 = \epsilon[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3][\mathbf{A}_2\mathbf{A}_4\mathbf{A}_5][\mathbf{A}_2\mathbf{A}_4\mathbf{A}_6][\mathbf{A}_3\mathbf{A}_5\mathbf{A}_6]$, and
$t_2 = -\epsilon[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_4][\mathbf{A}_2\mathbf{A}_3\mathbf{A}_5][\mathbf{A}_2\mathbf{A}_3\mathbf{A}_6][\mathbf{A}_4\mathbf{A}_5\mathbf{A}_6]$, where $\epsilon = \pm 1$.

Step 2. Substitute

$$t_1 + t_2 \\ = \epsilon[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3][\mathbf{A}_1\mathbf{A}_2\mathbf{A}_4][\mathbf{A}_2\mathbf{A}_5\mathbf{A}_6][\mathbf{A}_3\mathbf{A}_4\mathbf{A}_5][\mathbf{A}_3\mathbf{A}_4\mathbf{A}_6]/[\mathbf{A}_1\mathbf{A}_3\mathbf{A}_4]$$

into f, contract and factor the result, put it in q and remove f from p.

Example 4.3 (From Example 6.2 in Section 6). Let $\mathbf{1}, \ldots, \mathbf{6}$ be conic points. Let

$$p = [\mathbf{126}][\mathbf{234}][\mathbf{245}][\mathbf{356}] - [\mathbf{124}][\mathbf{236}][\mathbf{256}][\mathbf{345}].$$

The degrees of the points are $\mathbf{1}^1, \mathbf{2}^3, \mathbf{3}^2, \mathbf{4}^2, \mathbf{5}^2, \mathbf{6}^2$. The two brackets $[\mathbf{126}]$ and $[\mathbf{124}]$ each occur in a term of p. So $\mathbf{A}_1 = \mathbf{1}$, $\mathbf{A}_2 = \mathbf{2}$, $\mathbf{A}_3 = \mathbf{6}$, $\mathbf{A}_4 = \mathbf{4}$, $\mathbf{A}_5 = \mathbf{5}$, $\mathbf{A}_6 = \mathbf{3}$, and p matches the pattern. So

$$p = -[\mathbf{124}][\mathbf{126}][\mathbf{235}][\mathbf{346}][\mathbf{456}]/[\mathbf{146}].$$

Putting together the three techniques, we form an algorithm for simplifying bracket computation in conic geometry:

Algorithm. Conic combination.

Input: A bracket polynomial p involving at least six conic points. Assume that p is already factored in the polynomial ring of brackets.
Output: p after conic combination, or p itself.
Procedure. For every factor f of p, do the following until it no longer changes:

(1) Do conic transformations.
(2) Do conic contraction.
(3) Do pseudoconic transformation.

5. Conclusion representation

There are four typical conclusions in conic geometry:

(1) points $\mathbf{1}, \mathbf{2}, \mathbf{3}$ are collinear,
(2) lines $\mathbf{12}, \mathbf{1}'\mathbf{2}', \mathbf{1}''\mathbf{2}''$ concur,
(3) points $\mathbf{A}, \mathbf{B}$ are conjugate with respect to a conic,
(4) six points $\mathbf{1}, \ldots, \mathbf{6}$ are on a conic (*conconic*).

The first two conclusions have unique representations. In this section we study the latter two conclusions and their optimal representations.

5.1. *Representation of the conic-conjugacy conclusion*

This type of conclusion generally has multiple representations. One reason is that for a fixed set of representative conic points and tangents, (2.8) and (2.34) both have three different forms. The other reason is that there may be different sets of representative points and tangents available. Considering the situation that the representations are three to four terms in general, the first guideline of optimal representation is to replace the conclusion with a simpler one.

In the following cases, the conclusion can be replaced by simpler ones:

(1) A point of intersection of line **AB** with the conic is given: construct the other point of intersection, prove the conjugacy on line **AB**.

(2) A point of tangency **C** of the tangent passing through **A** (or **B**) is given: construct the other intersection of line **BC** (or **AC**) with the conic, prove the tangency of the line connecting the point and **A** (or **B**).

The second guideline is to make the zero terms in the representation maximal. By (2.8) and (2.34), this is only possible when some brackets involving **A**, **B** are zero. By (2.40), since the case in which **A** or **B** is on a tangent is already ruled out, only the brackets [**23A**] and [**23B**] can be zero.

Cue 1. Select lines passing through two conic points and either **A** or **B**. Make the conic points on the lines occur in the brackets containing **A** or **B** in the representation.

The third guideline is to use the essential conic points of **A**, **B** in the representation, so that in later manipulations the opportunity to obtain factored or short results is increased.

Cue 2. For **A**, **B**, find respectively their essential conic points and assign the essential weights, which are generally one over the number of such points. Order the points by adding up their essential weights. Use the points with bigger essential weights in the representation. The following is an example of applying the two cues.

Example 5.1 (From Example 7.9 in Section 7). Let **7**, **9** be points outside conic (**123**, **24**, **34**). Let **8** be the intersection of line **23** and the tangent at **1**. Let $\mathbf{A} = \mathbf{23} \cap \mathbf{47}$ and $\mathbf{B} = \mathbf{12} \cap \mathbf{79}$. Represent the conclusion that **A**, **B** are conjugate with respect to the conic.

The conic has three points **1**, **2**, **3** and three corresponding tangents **18**, **24**, **34**. The essential conic points of **A**, **B** with their essential weights are $\mathbf{2}^1$; $\mathbf{1}^{\frac{1}{2}}, \mathbf{3}^{\frac{1}{2}}$. So there are two optimal representations of the conic: (**321**, **24**, **18**) and (**123**, **24**, **34**). The proofs based on these representations are in Section 7.

We can certainly design an algorithm to make the representation automatic. However, in our experiments, the finding of a good representation following the two cues is very easy, so there is no need to do so.

5.2. *Representation of the conconic conclusion*

The conclusion that points **1**, **2**, **3**, **4**, **5**, **6** are conconic has 15 different representations: without using GP relations, the expression

$$\mathrm{conic}(\mathbf{123456}) = [\mathbf{135}][\mathbf{245}][\mathbf{126}][\mathbf{346}] - [\mathbf{125}][\mathbf{345}][\mathbf{136}][\mathbf{246}]$$

is antisymmetric with respect to each of the pairs **14**, **23**, **56**, and is symmetric with respect to the three pairs, so the number of representations is $C_6^2 \times C_4^2/3! = 15$. They are listed as follows:

123456 **123546** **123645** **124356** **124536** **124635**
125346 **125436** **125634** **126345** **126435** **126534**
134256 **135246** **136245**.

Without computing the brackets any representation is just as good as any other one. The six points form $C_6^3 = 20$ brackets. To compute them generally means to eliminate all their incidence points by Cayley expansions. An obvious criterion for a good representation is that the common factors of the two terms from bracket computation have the maximal degree, called *maximal discarded degree*.

We introduce a fast representation algorithm based on a simplified version of the above criterion: we only compute the *factored expansions* of the p_I to q_{III} typed Cayley expressions (Li and Wu, 2001) obtained by eliminating all incidence points in a bracket, assuming that there is no incidence constraint among different points in the expressions. The corresponding brackets formed by the six points are called *special brackets*. In our experiments, the criterion works very well without any exception.

The following is a list of formulae. Among the notations, $\mathbf{A_{ij}} = \lambda_\mathbf{j}\mathbf{i} + \lambda_\mathbf{i}\mathbf{j}$ and $\mathbf{B_{kl}} = \mu_\mathbf{l}\mathbf{k} + \mu_\mathbf{k}\mathbf{l}$ are points on lines **ij** and **kl** respectively, where the λ's and μ's are polynomials.

Formulae on factored expansions of special brackets
Double line:

$$\begin{array}{lll}
\mathbf{1'2'}: & [\mathbf{1}(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1'2'} \wedge \mathbf{3''4''})] & = [\mathbf{11'2'}]\mathbf{1'2'} \wedge \mathbf{3'4'} \wedge \mathbf{3''4''}, \\
\mathbf{12}: & [(\mathbf{12} \wedge \mathbf{34})(\mathbf{12} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})] & = (\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{3'4'})(\mathbf{12} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''}), \\
\mathbf{1'2'}: & [\mathbf{1}(\mathbf{1'2'} \wedge \mathbf{3'4'})\mathbf{A_{1'2'}}] & = [\mathbf{11'2'}][\mathbf{3'4'A_{1'2'}}], \\
\mathbf{5'6'}: & [(\mathbf{12} \wedge \mathbf{34})\mathbf{A_{5'6'}B_{5'6'}}] & = (\lambda_{\mathbf{6'}}\mu_{\mathbf{5'}} - \lambda_{\mathbf{5'}}\mu_{\mathbf{6'}})\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{5'6'}, \\
\mathbf{12}: & [(\mathbf{12} \wedge \mathbf{34})\mathbf{A_{12}B_{5''6''}}] & = [\mathbf{12B_{5''6''}}][\mathbf{34A_{12}}], \\
\mathbf{12}: & [(\mathbf{12} \wedge \mathbf{34})(\mathbf{12} \wedge \mathbf{3'4'})\mathbf{A_{5''6''}}] & = [\mathbf{12A_{5''6''}}]\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{3'4'}, \\
\mathbf{12}: & [(\mathbf{12} \wedge \mathbf{34})(\mathbf{1'2'} \wedge \mathbf{3'4'})\mathbf{A_{12}}] & = [\mathbf{34A_{12}}]\mathbf{12} \wedge \mathbf{1'2'} \wedge \mathbf{3'4'}.
\end{array}$$

Recursion of **1**:

$$\begin{array}{ll}
\mathbf{12} \wedge \mathbf{12'} \wedge \mathbf{1''2''} & = [\mathbf{122'}][\mathbf{11''2''}], \\
[\mathbf{1}(\mathbf{12'} \wedge \mathbf{3'4'})\mathbf{A_{5''6''}}] & = [\mathbf{13'4'}][\mathbf{12'A_{5''6''}}], \\
[\mathbf{1}(\mathbf{1'2'} \wedge \mathbf{3'4'})\mathbf{A_{16''}}] & = -\lambda_\mathbf{1}\mathbf{16''} \wedge \mathbf{1'2'} \wedge \mathbf{3'4'}, \\
[\mathbf{1}(\mathbf{12'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})] & = [\mathbf{13'4'}]\mathbf{12'} \wedge \mathbf{1''2''} \wedge \mathbf{3''4''}.
\end{array}$$

Complete quadrilateral **1234**:

$$[(\mathbf{12} \wedge \mathbf{34})(\mathbf{13} \wedge \mathbf{24})(\mathbf{14} \wedge \mathbf{23})] = -2\,[\mathbf{123}][\mathbf{124}][\mathbf{134}][\mathbf{234}].$$

Triangle pair (**122'**, **344'**):

$$[(\mathbf{12} \wedge \mathbf{34})\,(\mathbf{12'} \wedge \mathbf{34'})\,(\mathbf{22'} \wedge \mathbf{44'})] = -[\mathbf{122'}][\mathbf{344'}]\mathbf{13} \wedge \mathbf{24} \wedge \mathbf{2'4'}.$$

Quadrilateral (**1234**, **14**):

$$[(\mathbf{12} \wedge \mathbf{34})\,(\mathbf{13} \wedge \mathbf{24})\,(\mathbf{14} \wedge \mathbf{3''4''})] = -[\mathbf{124}][\mathbf{134}]([\mathbf{123}][\mathbf{43''4''}]$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad + [\mathbf{13''4''}][\mathbf{234}]),\\
&[(\mathbf{12}\wedge\mathbf{34})(\mathbf{13}\wedge\mathbf{24})\mathbf{A_{14}}] \qquad\quad = [\mathbf{124}][\mathbf{134}](\lambda_{\mathbf{4}}[\mathbf{123}]-\lambda_{\mathbf{1}}[\mathbf{234}]).
\end{aligned}
$$

Triangle $\mathbf{122'}$:

$$
\begin{aligned}
{[(\mathbf{12}\wedge\mathbf{34})\,(\mathbf{12'}\wedge\mathbf{3'4'})\,(\mathbf{22'}\wedge\mathbf{3''4''})]} &= [\mathbf{122'}]([\mathbf{134}][\mathbf{23''4''}][\mathbf{2'3'4'}]\\
&\quad -[\mathbf{13'4'}][\mathbf{234}][\mathbf{2'3''4''}]),\\
{[(\mathbf{12}\wedge\mathbf{34})\mathbf{A_{12'}B_{22'}}]} &= [\mathbf{122'}](\lambda_{\mathbf{1}}\mu_{\mathbf{2'}}[\mathbf{234}]-\lambda_{\mathbf{2'}}\mu_{\mathbf{2}}[\mathbf{134}]),\\
{[(\mathbf{12}\wedge\mathbf{34})(\mathbf{12'}\wedge\mathbf{3'4'})\mathbf{A_{22'}}]} &= [\mathbf{122'}](\lambda_{\mathbf{2'}}[\mathbf{13'4'}][\mathbf{234}]\\
&\quad +\lambda_{\mathbf{2}}[\mathbf{134}][\mathbf{2'3'4'}]).
\end{aligned}
$$

Except for the first triangle pattern (type p_{IV}), the results of the factored expansions are unique in the sense that if there is any other factored result q, then (1) any bracket factor in q is either in the corresponding formula, or from a monomial expansion of a wedge product in the formula, (2) any 2-termed factor in q is either in the corresponding formula, or has a higher degree than any factor in the formula.

In the exceptional case, by the three distributive expansions of $p_{IV} = [(\mathbf{12}\wedge\mathbf{34})\,(\mathbf{12'}\wedge\mathbf{3'4'})\,(\mathbf{22'}\wedge\mathbf{3''4''})]$, we have

$$
\begin{aligned}
p_{IV} &= [\mathbf{122'}]([\mathbf{23''4''}]\mathbf{12'}\wedge\mathbf{34}\wedge\mathbf{3'4'}-[\mathbf{13'4'}]\mathbf{22'}\wedge\mathbf{34}\wedge\mathbf{3''4''})\\
&= [\mathbf{122'}]([\mathbf{2'3''4''}]\mathbf{12}\wedge\mathbf{34}\wedge\mathbf{3'4'}-[\mathbf{134}]\mathbf{22'}\wedge\mathbf{3'4'}\wedge\mathbf{3''4''})\\
&= [\mathbf{122'}]([\mathbf{23''4''}]\mathbf{12'}\wedge\mathbf{34}\wedge\mathbf{3'4'}-[\mathbf{13'4'}]\mathbf{22'}\wedge\mathbf{34}\wedge\mathbf{3''4''}). \qquad (5.1)
\end{aligned}
$$

Any expansion in (5.1) leads to a 2-termed result when both of its wedge products have monomial expansions.

Algorithm. Fast representation of the conconic conclusion.

Input: A list of constructions of six points $\mathbf{1},\ldots,\mathbf{6}$.
Output: A sequence of permutations of the six points.
Step 1. Let $\mathcal{B}$ be the set of 20 brackets formed by the six points. Find and compute all the special brackets.
Step 2. If there are two p_{IV}-typed triangle brackets having different degree-3 binomial factors, say p_1 and p_2, compare them by contracting $p_1 \pm p_2$. If they are equal then unify the factors. This step is necessary because the expansion results of such brackets may not be unique.
Step 3. Let $\mathcal{C}$ be the set of 15 representations of the conclusion. For every element $c\in\mathcal{C}$, substitute the results of the special brackets. Collect common factors from the two terms, and sum up their degrees by assuming that they are expanded into brackets. The sum is called the *discarded degree*.
Step 4. Find the elements in $\mathcal{C}$ with the maximal discarded degree. Order them by the number of special brackets and output the descending sequence.

Below we illustrate the algorithm with an example.

Example 5.2 (See Chou et al., 1994, Example 6.395 for a circle). If points $\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6}$ are on a conic, then $\mathbf{12}\cap\mathbf{34}$, $\mathbf{13}\cap\mathbf{24}$, $\mathbf{14}\cap\mathbf{23}$, $\mathbf{34}\cap\mathbf{56}$, $\mathbf{35}\cap\mathbf{46}$, $\mathbf{45}\cap\mathbf{36}$ are conconic.

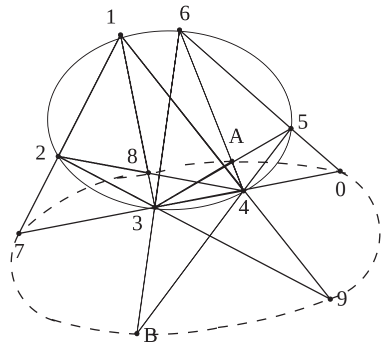

Fig. 2. Example 5.2.

Free conic points: **1**, **2**, **3**, **4**, **5**, **6**.
Intersections:

$$\mathbf{7} = \mathbf{12} \cap \mathbf{34}, \quad \mathbf{8} = \mathbf{13} \cap \mathbf{24}, \quad \mathbf{9} = \mathbf{14} \cap \mathbf{23},$$
$$\mathbf{0} = \mathbf{34} \cap \mathbf{56}, \quad \mathbf{A} = \mathbf{35} \cap \mathbf{46}, \quad \mathbf{B} = \mathbf{36} \cap \mathbf{45}.$$

Conclusion: **7**, **8**, **9**, **0**, **A**, **B** are on a conic. **Steps 1–2.** The following are special brackets:

1 line: **70** on line **34**. There are four associated brackets:

$$\begin{aligned}
[\mathbf{780}] &= -[\mathbf{134}][\mathbf{234}]\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56}, & [\mathbf{790}] &= [\mathbf{134}][\mathbf{234}]\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56},\\
[\mathbf{70A}] &= [\mathbf{345}][\mathbf{346}]\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56}, & [\mathbf{70B}] &= [\mathbf{345}][\mathbf{346}]\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56}.
\end{aligned}$$

2 complete quadrilaterals: [**789**] of **1234**, and [**0AB**] of **3456**.

$$[\mathbf{789}] = -2[\mathbf{123}][\mathbf{124}][\mathbf{134}][\mathbf{234}], \qquad [\mathbf{0AB}] = -2[\mathbf{345}][\mathbf{346}][\mathbf{356}][\mathbf{456}].$$

2 quadrilaterals: [**890**] of (**1234**, **34**), and [**7AB**] of (**3456**, **34**).

$$\begin{aligned}
[\mathbf{7AB}] &= [\mathbf{345}][\mathbf{346}]([\mathbf{123}][\mathbf{456}] + [\mathbf{124}][\mathbf{356}]),\\
[\mathbf{890}] &= [\mathbf{134}][\mathbf{234}]([\mathbf{123}][\mathbf{456}] + [\mathbf{124}][\mathbf{356}]).
\end{aligned}$$

Steps 3–4. The 15 conclusions with their discarded degrees are

$$\begin{array}{llllll}
\mathbf{7890AB}^{(0)} & \mathbf{789A0B}^{(2)} & \mathbf{789B0A}^{(2)} & \mathbf{7809AB}^{(2)} & \overset{(6)}{\underbrace{\mathbf{780A9B}}} & \overset{(6)}{\underbrace{\mathbf{780B9A}}}\\
\overset{(6)}{\underbrace{\mathbf{78A90B}}} & \overset{(6)}{\underbrace{\mathbf{78A09B}}} & \overset{(6)}{\underbrace{\mathbf{78AB90}}} & \overset{(6)}{\underbrace{\mathbf{78B90A}}} & \overset{(6)}{\underbrace{\mathbf{78B09A}}} & \overset{(6)}{\underbrace{\mathbf{78BA90}}}\\
\mathbf{7908AB}^{(2)} & \overset{(6)}{\underbrace{\mathbf{79A80B}}} & \overset{(6)}{\underbrace{\mathbf{79B80A}}}. & & &
\end{array}$$

10 representations have maximal discarded degree 6.

Let us see how the proof goes on when choosing one of the 10 representations, say $\mathbf{78A09B}^{(6)}$. A very nice property of the 12 nonspecial brackets is that their 2-termed expansion results are all unique.

Proof.

$$
\begin{array}{cll}
\text{Rules} & & \mathrm{conic}(\mathbf{78A09B}) \\
 & = & [\mathbf{789}][\mathbf{7AB}][\mathbf{80B}][\mathbf{90A}] \\
 & & -[\mathbf{78B}][\mathbf{79A}][\mathbf{890}][\mathbf{0AB}] \\
\boxed{\begin{array}{l}
[\mathbf{789}]=-2\,[\mathbf{123}][\mathbf{124}][\mathbf{134}][\mathbf{234}] \\
[\mathbf{0AB}]=-2\,[\mathbf{345}][\mathbf{346}][\mathbf{356}][\mathbf{456}] \\
[\mathbf{7AB}]=[\mathbf{345}][\mathbf{346}]([\mathbf{123}][\mathbf{456}]+[\mathbf{124}][\mathbf{356}]) \\
[\mathbf{890}]=[\mathbf{134}][\mathbf{234}]([\mathbf{123}][\mathbf{456}]+[\mathbf{124}][\mathbf{356}]) \\
[\mathbf{80B}]=[\mathbf{134}][\mathbf{245}][\mathbf{346}][\mathbf{356}]-[\mathbf{136}][\mathbf{234}][\mathbf{345}][\mathbf{456}] \\
[\mathbf{90A}]=[\mathbf{134}][\mathbf{235}][\mathbf{346}][\mathbf{456}]-[\mathbf{146}][\mathbf{234}][\mathbf{345}][\mathbf{356}] \\
[\mathbf{78B}]=[\mathbf{123}][\mathbf{134}][\mathbf{245}][\mathbf{346}]-[\mathbf{124}][\mathbf{136}][\mathbf{234}][\mathbf{345}] \\
[\mathbf{79A}]=[\mathbf{124}][\mathbf{134}][\mathbf{235}][\mathbf{346}]-[\mathbf{123}][\mathbf{146}][\mathbf{234}][\mathbf{345}]
\end{array}}
 & \overset{\mathbf{7,8,9,0,A,B}}{=} & \begin{array}{l}
2\;\underbrace{[\mathbf{134}]^2[\mathbf{234}]^2[\mathbf{345}]^2[\mathbf{346}]^2} \\
\underbrace{([\mathbf{123}][\mathbf{456}]+[\mathbf{124}][\mathbf{356}])} \\
([\mathbf{123}][\mathbf{124}][\mathbf{146}][\mathbf{245}][\mathbf{356}]^2 \\
+[\mathbf{123}][\mathbf{124}][\mathbf{136}][\mathbf{235}][\mathbf{456}]^2 \\
-[\mathbf{123}]^2[\mathbf{146}][\mathbf{245}][\mathbf{356}][\mathbf{456}] \\
-[\mathbf{124}]^2[\mathbf{136}][\mathbf{235}][\mathbf{356}][\mathbf{456}])
\end{array} \\
 & = & \begin{array}{l}
\underbrace{([\mathbf{124}][\mathbf{356}]-[\mathbf{123}][\mathbf{456}])} \\
([\mathbf{123}][\mathbf{146}][\mathbf{245}][\mathbf{356}] \\
-[\mathbf{124}][\mathbf{136}][\mathbf{235}][\mathbf{456}])
\end{array} \\
 & \overset{\mathrm{conic}}{=} & 0.
\end{array}
$$

The next to the last step is a factorization in the polynomial ring of brackets. □

Additional nondegeneracy condition: none.

6. Rational Cayley factorization

By now we already have representations of the hypotheses and the conclusion, the elimination and simplification techniques, so we are ready for theorem proving. In this section we show that there is one more necessary technique, without which the proving is often not only difficult, but also fragile in that it is extremely sensitive to Cayley expansions.

The technique is called *rational Cayley factorization*, different from the Cayley factorization of White (1975, 1991). It is an integration of the Cayley factorization techniques developed in Li and Wu (2001) and the conic combination technique, and can significantly simplify bracket computation in conic geometry. For maximal factorization of a bracket polynomial involving free conic points, bracket monomials must be allowed to occur in the denominator, which is the feature of this factorization.

6.1. Bracket unification

We first introduce a small but very useful algorithm, called *bracket unification*. The purpose is to produce more common bracket factors before merging two polynomials.

Algorithm: Bracket unification.

Input: Two bracket polynomials p_1, p_2. Assume that they have no common factors and are factored in the polynomial ring of brackets.

Output: p_1, p_2 after bracket unification.

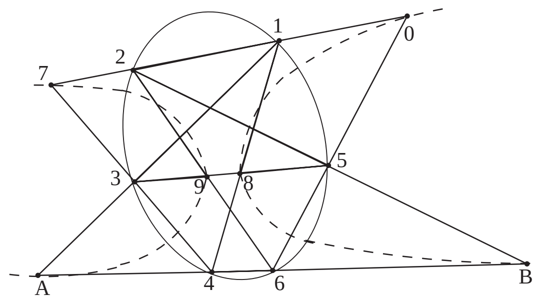

Fig. 3. Example 6.1.

Procedure. Let b_1, b_2 be the bracket factors of p_1, p_2, let c_1, c_2 be the polynomial factors. Let $d = b_1 + \lambda b_2$, where λ is an indeterminate. Set $b = 1$.Do the following to d until it no longer changes, then output $p_1 = bd_1c_1$ and $p_2 = bd_2c_2$, where $d = d_1 + \lambda d_2$.

(1) Do conic transformations. If d becomes factored, move the bracket factors to b.

(2) Do pseudoconic transformation. If d becomes factored, move the rational bracket factors to b.

Example 6.1 (See Chou et al., 1994, Example 6.397 for a circle). If points **1**, **2**, **3**, **4**, **5**, **6** are on a conic, then the intersections $\mathbf{12} \cap \mathbf{34}$, $\mathbf{14} \cap \mathbf{35}$, $\mathbf{35} \cap \mathbf{26}$, $\mathbf{12} \cap \mathbf{56}$, $\mathbf{13} \cap \mathbf{46}$, $\mathbf{25} \cap \mathbf{46}$ are on a conic. Free conic points: **1**, **2**, **3**, **4**, **5**, **6**.

Intersections:

$$\mathbf{7} = \mathbf{12} \cap \mathbf{34}, \quad \mathbf{8} = \mathbf{14} \cap \mathbf{35}, \quad \mathbf{9} = \mathbf{26} \cap \mathbf{35},$$
$$\mathbf{0} = \mathbf{12} \cap \mathbf{56}, \quad \mathbf{A} = \mathbf{13} \cap \mathbf{46}, \quad \mathbf{B} = \mathbf{25} \cap \mathbf{46}.$$

Conclusion: **7**, **8**, **9**, **0**, **A**, **B** are conconic.

The following are special brackets:

3 lines: **70** on line **12**, **89** on line **35**, and **AB** on line **46**. There are 12 associated brackets:

$$\begin{aligned}
[\mathbf{780}] &= -[\mathbf{124}][\mathbf{135}]\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56}, & [\mathbf{790}] &= -[\mathbf{126}][\mathbf{235}]\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56},\\
[\mathbf{70A}] &= [\mathbf{123}][\mathbf{146}]\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56}, & [\mathbf{70B}] &= [\mathbf{125}][\mathbf{246}]\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56},\\
[\mathbf{789}] &= [\mathbf{123}][\mathbf{345}]\mathbf{14} \wedge \mathbf{26} \wedge \mathbf{35}, & [\mathbf{890}] &= -[\mathbf{125}][\mathbf{356}]\mathbf{14} \wedge \mathbf{26} \wedge \mathbf{35},\\
[\mathbf{89A}] &= -[\mathbf{135}][\mathbf{346}]\mathbf{14} \wedge \mathbf{26} \wedge \mathbf{35}, & [\mathbf{89B}] &= [\mathbf{235}][\mathbf{456}]\mathbf{14} \wedge \mathbf{26} \wedge \mathbf{35},\\
[\mathbf{7AB}] &= [\mathbf{124}][\mathbf{346}]\mathbf{13} \wedge \mathbf{25} \wedge \mathbf{46}, & [\mathbf{8AB}] &= [\mathbf{146}][\mathbf{345}]\mathbf{13} \wedge \mathbf{25} \wedge \mathbf{46},\\
[\mathbf{9AB}] &= -[\mathbf{246}][\mathbf{356}]\mathbf{13} \wedge \mathbf{25} \wedge \mathbf{46}, & [\mathbf{0AB}] &= -[\mathbf{126}][\mathbf{456}]\mathbf{13} \wedge \mathbf{25} \wedge \mathbf{46}.
\end{aligned}$$

2 triangles: [**78A**] of **134**, and [**90B**] of **256**.

$$[\mathbf{78A}] = -[\mathbf{134}]([\mathbf{123}][\mathbf{146}][\mathbf{345}] + [\mathbf{124}][\mathbf{135}][\mathbf{346}]),$$
$$[\mathbf{90B}] = -[\mathbf{256}]([\mathbf{125}][\mathbf{246}][\mathbf{356}] + [\mathbf{126}][\mathbf{235}][\mathbf{456}]).$$

The 15 representations with their discarded degrees are

$$\begin{array}{llllll}
\mathbf{7890AB}^{(0)} & \mathbf{789A0B}^{(4)} & \mathbf{789B0A}^{(4)} & \mathbf{7809AB}^{(4)} & \overset{(6)}{\underbrace{\mathbf{780A9B}}} & \overset{(6)}{\underbrace{\mathbf{780B9A}}} \\
\overset{(6)}{\underbrace{\mathbf{78A90B}}} & \mathbf{78A09B}^{(4)} & \overset{(6)}{\underbrace{\mathbf{78AB90}}} & \overset{(6)}{\underbrace{\mathbf{78B90A}}} & \mathbf{78B09A}^{(4)} & \overset{(6)}{\underbrace{\mathbf{78BA90}}} \\
\mathbf{7908AB}^{(4)} & \overset{(6)}{\underbrace{\mathbf{79A80B}}} & \overset{(6)}{\underbrace{\mathbf{79B80A}}}\,. & & &
\end{array}$$

There are eight representations with maximal discarded degree 6, two of which have all their brackets as special ones: $\mathbf{78B90A}^{(6)}$, $\mathbf{780B9A}^{(6)}$. Choosing any of the two representations, say the first one, we get

$$\begin{aligned}
&\text{conic}(\mathbf{78B90A}) \\
&\quad = [\mathbf{780}][\mathbf{7AB}][\mathbf{89A}][\mathbf{90B}] - [\mathbf{78A}][\mathbf{70B}][\mathbf{890}][\mathbf{9AB}] \\
&\quad = \underbrace{(\mathbf{12} \wedge \mathbf{34} \wedge \mathbf{56})(\mathbf{14} \wedge \mathbf{26} \wedge \mathbf{35})(\mathbf{13} \wedge \mathbf{25} \wedge \mathbf{46})} \\
&\qquad \{[\mathbf{125}]^2[\mathbf{134}][\mathbf{246}]^2[\mathbf{356}]^2([\mathbf{123}][\mathbf{146}][\mathbf{345}] + [\mathbf{124}][\mathbf{135}][\mathbf{346}]) \\
&\qquad - [\mathbf{124}]^2[\mathbf{135}]^2[\mathbf{256}][\mathbf{346}]^2([\mathbf{125}][\mathbf{246}][\mathbf{356}] + [\mathbf{126}][\mathbf{235}][\mathbf{456}])\}.
\end{aligned}$$

Now let us see how the bracket unification works. For $b_1 = [\mathbf{125}]^2[\mathbf{134}][\mathbf{246}]^2[\mathbf{356}]^2$ and $b_2 = [\mathbf{124}]^2[\mathbf{135}]^2[\mathbf{256}][\mathbf{346}]^2$,

$$\begin{array}{lcl}
\text{Transformations} & & b_1 + \lambda b_2 \\
\boxed{[\mathbf{124}][\mathbf{135}][\mathbf{256}][\mathbf{346}]=[\mathbf{125}][\mathbf{134}][\mathbf{246}][\mathbf{356}]} & \overset{\text{conic}}{=} & \underbrace{[\mathbf{125}][\mathbf{134}][\mathbf{246}][\mathbf{356}]} \\
 & & ([\mathbf{125}][\mathbf{246}][\mathbf{356}] + \lambda[\mathbf{124}][\mathbf{135}][\mathbf{346}]) \\
\boxed{[\mathbf{124}][\mathbf{135}][\mathbf{346}]=[\mathbf{125}][\mathbf{134}][\mathbf{246}][\mathbf{356}]/[\mathbf{256}]} & \overset{\text{pseudo}}{=} & \underbrace{([\mathbf{125}][\mathbf{246}][\mathbf{356}]/[\mathbf{256}])}([\mathbf{256}] + \lambda[\mathbf{134}]).
\end{array}$$

So essentially b_1 and b_2 are simplified to $[\mathbf{256}]$ and $[\mathbf{134}]$ respectively. Substituting the results and removing common factors, we get

$$\begin{aligned}
\text{conic}(\mathbf{78B90A}) \quad &= [\mathbf{123}][\mathbf{146}][\mathbf{256}][\mathbf{345}] + [\mathbf{124}][\mathbf{135}][\mathbf{256}][\mathbf{346}] \\
&\quad -[\mathbf{125}][\mathbf{134}][\mathbf{246}][\mathbf{356}] - [\mathbf{126}][\mathbf{134}][\mathbf{235}][\mathbf{456}] \\
&\overset{\text{conic}}{=} 0.
\end{aligned} \tag{6.1}$$

Additional nondegeneracy condition: $\exists \mathbf{256}$.

6.2. Conic combination and Cayley factorization

In proving theorems with a conconic conclusion, generally it is not difficult to find binomial expansions for nonspecial brackets, what is difficult is that when there are several binomial results for the same bracket, they can form a huge number of combinations. It is a common phenomenon that the successive manipulations work well for one particular combination of expansions, but not for any other one. Thus, the proving is very fragile.

Example 6.2. In Example 6.1, instead of choosing a representation with maximal discarded degree, we choose one with discarded degree 4, for example

$$\mathrm{conic}(\mathbf{78B09A}) = [\mathbf{78A}][\mathbf{79B}][\mathbf{890}][\mathbf{0AB}] - [\mathbf{789}][\mathbf{7AB}][\mathbf{80A}][\mathbf{90B}]. \tag{6.2}$$

The two nonspecial brackets [**79B**], [**80A**] in (6.2) each have six binomial expansions:

$$\begin{aligned}
[\mathbf{79B}] &= [\mathbf{124}][\mathbf{235}][\mathbf{236}][\mathbf{456}] - [\mathbf{123}][\mathbf{245}][\mathbf{246}][\mathbf{356}]\\
&= [\mathbf{125}][\mathbf{234}][\mathbf{236}][\mathbf{456}] - [\mathbf{123}][\mathbf{245}][\mathbf{256}][\mathbf{346}]\\
&= [\mathbf{125}][\mathbf{234}][\mathbf{246}][\mathbf{356}] - [\mathbf{124}][\mathbf{235}][\mathbf{256}][\mathbf{346}]\\
&= [\mathbf{126}][\mathbf{234}][\mathbf{245}][\mathbf{356}] - [\mathbf{124}][\mathbf{236}][\mathbf{256}][\mathbf{345}]\\
&= [\mathbf{126}][\mathbf{234}][\mathbf{235}][\mathbf{456}] - [\mathbf{123}][\mathbf{246}][\mathbf{256}][\mathbf{345}]\\
&= [\mathbf{126}][\mathbf{235}][\mathbf{245}][\mathbf{346}] - [\mathbf{125}][\mathbf{236}][\mathbf{246}][\mathbf{345}]\\
[\mathbf{80A}] &= [\mathbf{125}][\mathbf{136}][\mathbf{146}][\mathbf{345}] - [\mathbf{126}][\mathbf{135}][\mathbf{145}][\mathbf{346}]\\
&= [\mathbf{123}][\mathbf{146}][\mathbf{156}][\mathbf{345}] - [\mathbf{126}][\mathbf{134}][\mathbf{135}][\mathbf{456}]\\
&= [\mathbf{123}][\mathbf{145}][\mathbf{156}][\mathbf{346}] - [\mathbf{125}][\mathbf{134}][\mathbf{136}][\mathbf{456}]\\
&= [\mathbf{124}][\mathbf{136}][\mathbf{156}][\mathbf{345}] - [\mathbf{126}][\mathbf{134}][\mathbf{145}][\mathbf{356}]\\
&= [\mathbf{124}][\mathbf{135}][\mathbf{156}][\mathbf{346}] - [\mathbf{125}][\mathbf{134}][\mathbf{146}][\mathbf{356}]\\
&= [\mathbf{123}][\mathbf{145}][\mathbf{146}][\mathbf{356}] - [\mathbf{124}][\mathbf{135}][\mathbf{136}][\mathbf{456}].
\end{aligned} \tag{6.3}$$

The conic combination works well for the first expansions of the two brackets, but not for any other combination. As a result, the proof based on the first expansions goes on smoothly as in the previous proof, while proofs based on other expansions are very difficult to finish.

How to overcome the difficulty of finding the unique suitable combination of expansions of nonspecial brackets? In this example, a very nice property of the nonspecial brackets is that all their binomial expansion results can be conic contracted to rational monomials.

By means of conic contraction, ANY of the 15 *representations is just as good as any other one*. For the representation (6.2) and the expansions (6.3), the conic contractions give

$$\begin{aligned}
[\mathbf{79B}] &= -[\mathbf{123}][\mathbf{124}][\mathbf{256}][\mathbf{345}][\mathbf{346}]/[\mathbf{134}]\\
&= -[\mathbf{123}][\mathbf{125}][\mathbf{246}][\mathbf{345}][\mathbf{356}]/[\mathbf{135}]\\
&= -[\mathbf{124}][\mathbf{125}][\mathbf{236}][\mathbf{345}][\mathbf{456}]/[\mathbf{145}]\\
&= -[\mathbf{124}][\mathbf{126}][\mathbf{235}][\mathbf{346}][\mathbf{456}]/[\mathbf{146}]\\
&= -[\mathbf{123}][\mathbf{126}][\mathbf{245}][\mathbf{346}][\mathbf{356}]/[\mathbf{136}]\\
&= -[\mathbf{125}][\mathbf{126}][\mathbf{234}][\mathbf{356}][\mathbf{456}]/[\mathbf{156}]\\
[\mathbf{80A}] &= -[\mathbf{125}][\mathbf{126}][\mathbf{134}][\mathbf{356}][\mathbf{456}]/[\mathbf{256}]\\
&= -[\mathbf{123}][\mathbf{126}][\mathbf{145}][\mathbf{346}][\mathbf{356}]/[\mathbf{236}]\\
&= -[\mathbf{123}][\mathbf{125}][\mathbf{146}][\mathbf{345}][\mathbf{356}]/[\mathbf{235}]\\
&= -[\mathbf{124}][\mathbf{126}][\mathbf{135}][\mathbf{346}][\mathbf{456}]/[\mathbf{246}]\\
&= -[\mathbf{124}][\mathbf{125}][\mathbf{136}][\mathbf{345}][\mathbf{456}]/[\mathbf{245}]\\
&= -[\mathbf{123}][\mathbf{124}][\mathbf{156}][\mathbf{345}][\mathbf{346}]/[\mathbf{234}].
\end{aligned} \tag{6.4}$$

If we choose in (6.4) the first result for each bracket, then without bracket unification we get (6.1) directly by substitution and removal of common factors. The proof has no additional nondegeneracy condition. The combination is the best.

If we choose the worst combination, which is the last result for each bracket, then

$$\begin{aligned}\mathrm{conic}(\mathbf{78B09A}) = \frac{1}{\underbrace{[\mathbf{156}][\mathbf{234}]}}&\{[\mathbf{125}]^2[\mathbf{126}]^2[\mathbf{134}][\mathbf{234}]^2[\mathbf{356}]^2[\mathbf{456}]^2\\&\times([\mathbf{123}][\mathbf{146}][\mathbf{345}] + [\mathbf{124}][\mathbf{135}][\mathbf{346}])\\&-[\mathbf{123}]^2[\mathbf{124}]^2[\mathbf{156}]^2[\mathbf{256}][\mathbf{345}]^2[\mathbf{346}]^2\\&\times([\mathbf{125}][\mathbf{246}][\mathbf{356}] + [\mathbf{126}][\mathbf{235}][\mathbf{456}])\}.\end{aligned}$$

For $b_1 = [\mathbf{125}]^2[\mathbf{126}]^2[\mathbf{134}][\mathbf{234}]^2[\mathbf{356}]^2[\mathbf{456}]^2$, $b_2 = [\mathbf{123}]^2[\mathbf{124}]^2[\mathbf{156}]^2[\mathbf{256}][\mathbf{345}]^2[\mathbf{346}]^2$, the bracket unification is as follows:

Transformations		$b_1 + \lambda b_2$
$\boxed{([\mathbf{125}][\mathbf{234}][\mathbf{356}])^2 = ([\mathbf{124}][\mathbf{156}][\mathbf{235}][\mathbf{346}]/[\mathbf{146}])^2}$	$\overset{\text{pseudo}}{=}$	$\underbrace{([\mathbf{124}]^2[\mathbf{156}]^2[\mathbf{346}]^2/[\mathbf{146}]^2)}$ $([\mathbf{126}]^2[\mathbf{134}][\mathbf{235}]^2[\mathbf{456}]^2 + \lambda[\mathbf{123}]^2[\mathbf{146}]^2[\mathbf{256}][\mathbf{345}]^2)$
$\boxed{[\mathbf{126}][\mathbf{134}][\mathbf{235}][\mathbf{456}] = [\mathbf{123}][\mathbf{146}][\mathbf{256}][\mathbf{345}]}$	$\overset{\text{conic}}{=}$	$\underbrace{[\mathbf{123}][\mathbf{146}][\mathbf{256}][\mathbf{345}]}([\mathbf{126}][\mathbf{235}][\mathbf{456}] + \lambda[\mathbf{123}][\mathbf{146}][\mathbf{345}])$
$\boxed{[\mathbf{126}][\mathbf{235}][\mathbf{456}] = [\mathbf{123}][\mathbf{146}][\mathbf{256}][\mathbf{345}]/[\mathbf{134}]}$	$\overset{\text{pseudo}}{=}$	$\underbrace{([\mathbf{123}][\mathbf{146}][\mathbf{345}]/[\mathbf{134}])}([\mathbf{256}] + \lambda[\mathbf{134}])$

We get the same brackets [**256**] and [**134**] as in the best combination. The difference is the additional nondegeneracy conditions: ∃**134**, ∃**146**, ∃**156**, ∃**234**.

This example suggests the application of conic combination immediately after the expansion of nonspecial brackets. For more complicated problems, for instance Example 7.2 in Section 7, conic combination alone is not sufficient to make the proving robust, and must be followed by Cayley factorization.

In theorem proving, there is the need to factor a polynomial composed of brackets and wedge products of type p_I to maximal extent, with brackets allowed in the denominator. Such a polynomial generally occurs after bracket-wise eliminations and expansions, and is a linear combination of some multiplications of polynomials. Owning to their invariant inheritance from the eliminated brackets, the polynomial components are generally much easier to be factored, but not so after expanding their multiplications. The following factorization algorithm is based on this experience.

Algorithm: Rational Cayley factorization.

Input: A polynomial p composed of brackets and wedge products of type p_I, and involving at least six conic points.

Output: q, a rational polynomial of brackets and wedge products of type p_I.

Procedure. Let p be an i-termed polynomial, whose terms are multiplications of polynomial components.

Step 1. For each polynomial component of p, do conic combination, followed by Cayley combination.

Step 2. Move to q the rational factors common to the terms of p.

Step 3. If $i = 2$, do bracket unification to p. Move to q the rational factors common to the terms of p.

Step 4. Expand p, do conic combination and Cayley combination. Return $q = pq$.

7. Automated theorem proving

Similar to incidence geometry, the first manipulation to the conclusion of a theorem in conic geometry is *initial batch elimination*.

Algorithm: Initial batch elimination.
Input: A Cayley expression *conc*, and a construction sequence of elements (points, polars and tangents).
Output: *conc* after some eliminations and expansions, and the procedure to obtain it.
Procedure: Let $\mathcal{E}$ be the elements in *conc* which are neither free points nor free conic points, and which have no descendents in *conc*.

(1) If *conc* is not composed of brackets and wedge products of type p_I, then expand it into bracket polynomials.
(2) In each related bracket or wedge product of *conc*, eliminate points in $\mathcal{E}$ at the same time by Cayley expansion and the elimination rules. If this is impossible for some wedge products, then expand the wedge products into bracket polynomials before the batch elimination; if this is impossible for some brackets, then eliminate the maximal number of points in $\mathcal{E}$ from the brackets, and continue to eliminate the rest of the points from the results.
(3) Contract and remove common factors of *conc*.

Below we present a theorem proving algorithm which integrates all the previous techniques. The algorithm is implemented with Maple V.4, and has been tested by 40 nontrivial problems, most of which are difficult theorems (selected from Brannan et al., 1998, Hodge and Pedoe, 1953, Kadison and Kromann, 1996, Pedoe, 1963 etc.). Eight theorems cannot be given two-termed proofs, while all the others can, which include nearly all the theorems we encountered on free conic points, tangents and poles related to tangents. For the theorems without two-termed proofs, most of which are on intersections and more general poles and polars, generally we can still find very short and interesting proofs.

Algorithm: Short proof generation in conic geometry.
Input: A sequence of elements (points, polars and tangents) together with their constructions; a conclusion which is either conconic or of the form $conc = 0$, where *conc* is a Cayley expression.
Output: (1) Representation of the conconic conclusion;
(2) eliminations and the corresponding elimination rules;
(3) Cayley expansions;
(4) (strong, level) contractions;
(5) (pseudo)conic transformations, conic contractions;
(6) Cayley factorizations;
(7) removal of common factors;
(8) additional nondegeneracy conditions.

Step 1. [Registration] Collect points, lines, conics, polars, tangents.

(1) A line is composed of at least three points.
(2) A point is composed of the name and the construction.
(3) A conic is composed of the construction and all its points and tangents.

(4) A polar/tangent is composed of the conic and the pole/point of tangency.

Step 2. [Conclusion representation and initial batch elimination]

(1) If the conclusion is conconic then find a representation $conc = 0$.
(2) Do initial batch elimination to $conc$.

Step 3. [Elimination] Start from the last element x of $conc$ in the construction sequence, do the following:

(1) If $conc = 0$ then go to Step 6, else if $conc$ has only free points and free conic points, go to Step 4.
(2) Eliminate x from $conc$. Then do contraction and remove common factors.

Step 4. [Rational Cayley factorization] If $conc = 0$ then go to Step 6, else do rational Cayley factorization to $conc$. Remove common factors.

Step 5. [Complete elimination] If there are wedge products in $conc$, then expand them into bracket polynomials and contract the result.

While $conc \neq 0$ do the following. At the end of each step, carry out contraction, conic combination and remove common factors.

(1) Do level contraction.
(2) Do strong contraction.
(3) Eliminate the last point of $conc$ in the construction sequence.

Step 6. [Additional nondegeneracy conditions] There are two resources:

(1) the denominators which are produced by the transformation rules of different representations, Cramer's rules, conic contractions and pseudoconic transformations, and which are not cancelled after substitutions;
(2) the given nondegeneracy conditions of different representations which are not included in the original geometric constructions.

Remark. The geometric constructions include both nonlinear types (e.g. free conic points) and reducible types (e.g. intersections). The algorithm is complete by the point-by-point elimination in Step 5. However, no theorem in our experiments needs to go through any elimination of free point or free conic point. All theorems except one finish by Step 4. The only exception is Example 7.4, whose proof finishes after a level contraction, a strong contraction and two conic combinations without eliminating any free (conic) point.

7.1. Almost incidence geometry

If the constructions of a geometric problem involve only free points, free conic points and incidence points, we say it is a problem of *almost incidence geometry*. Such problems are among the simplest in conic geometry, and our algorithm can generally produce 2-termed proofs for them.

Example 7.1. (Nine-point Conic Theorem, See O'Hara and Ward, 1936, p. 135, Theorem 6.32) Let **1234** be a quadrilateral, and let **7**, **8**, **9** be the three intersections **12** ∩ **34**, **13** ∩ **24** and **23** ∩ **14**. Any line l intersects with the six sides of **1234** at points **5**, **6**, **0**, **A**, **B**, **C**

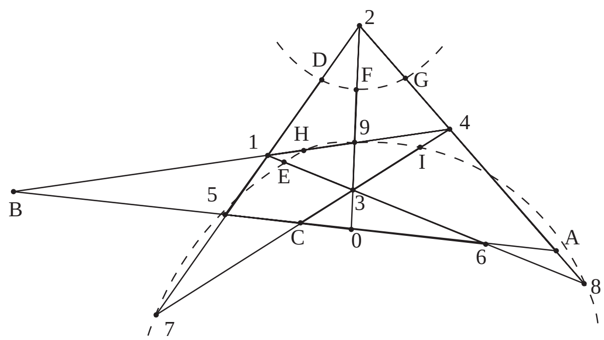

Fig. 4. Example 7.1.

respectively. Then points **7**, **8**, **9** and the six harmonic conjugates of **5**, **6**, **0**, **A**, **B**, **C** with respect to the vertices of **1234** are on a conic.

Free points: **1**, **2**, **3**, **4**.

Semifree points: **5** on **12**, **6** on **13**.

Intersections:

$$\mathbf{7} = \mathbf{12} \cap \mathbf{34}, \quad \mathbf{8} = \mathbf{13} \cap \mathbf{24}, \quad \mathbf{9} = \mathbf{23} \cap \mathbf{14}, \quad \mathbf{0} = \mathbf{23} \cap \mathbf{56},$$
$$\mathbf{A} = \mathbf{24} \cap \mathbf{56}, \quad \mathbf{B} = \mathbf{14} \cap \mathbf{56}, \quad \mathbf{C} = \mathbf{34} \cap \mathbf{56}.$$

Conjugates:

$$\mathbf{D} = \text{conjugate}_{\mathbf{12}}(\mathbf{5}), \quad \mathbf{E} = \text{conjugate}_{\mathbf{13}}(\mathbf{6}), \quad \mathbf{F} = \text{conjugate}_{\mathbf{23}}(\mathbf{0}),$$
$$\mathbf{G} = \text{conjugate}_{\mathbf{24}}(\mathbf{A}), \quad \mathbf{H} = \text{conjugate}_{\mathbf{14}}(\mathbf{B}), \quad \mathbf{I} = \text{conjugate}_{\mathbf{34}}(\mathbf{C}).$$

Conclusion: **7**, **8**, **9**, **D**, **E**, **F**, **G**, **H**, **I** are conconic.

Analysis

If we can prove that **7**, **8**, **9**, **D**, **F**, **I** are conconic, then by symmetry, we have the following 6-tuples of conconic points: {**7**, **8**, **9**, **D**, **G**, **I**}, {**7**, **8**, **9**, **D**, **E**, **I**}, {**7**, **8**, **9**, **D**, **H**, **I**}. Under the additional nondegeneracy condition ∃**789DI**, the nine points are on the same conic. The six points **7**, **8**, **9**, **D**, **F**, **I** have the following special brackets, according to

$$\mathbf{D} = [\mathbf{25}]\mathbf{1} + [\mathbf{15}]\mathbf{2}, \qquad \mathbf{F} = [\mathbf{30}]\mathbf{2} + [\mathbf{20}]\mathbf{3}, \qquad \mathbf{I} = [\mathbf{4C}]\mathbf{3} + [\mathbf{3C}]\mathbf{4}.$$

3 lines: **7D** of line **12**, **9F** of line **23**, and **7I** of line **34**. There are 11 associated brackets:

$$\begin{aligned}
[\mathbf{78D}] &= [\mathbf{123}][\mathbf{124}]([\mathbf{15}][\mathbf{234}] + [\mathbf{25}][\mathbf{134}]) \\
[\mathbf{79D}] &= [\mathbf{123}][\mathbf{124}]([\mathbf{15}][\mathbf{234}] + [\mathbf{25}][\mathbf{134}]) \\
[\mathbf{7DF}] &= -[\mathbf{20}][\mathbf{123}]([\mathbf{15}][\mathbf{234}] + [\mathbf{25}][\mathbf{134}]) \\
[\mathbf{7DI}] &= -([\mathbf{3C}][\mathbf{124}] + [\mathbf{4C}][\mathbf{123}])([\mathbf{15}][\mathbf{234}] + [\mathbf{25}][\mathbf{134}]) \\
[\mathbf{79F}] &= -[\mathbf{123}][\mathbf{234}]([\mathbf{20}][\mathbf{134}] + [\mathbf{30}][\mathbf{124}]) \\
[\mathbf{89F}] &= [\mathbf{123}][\mathbf{234}]([\mathbf{20}][\mathbf{134}] + [\mathbf{30}][\mathbf{124}]) \\
[\mathbf{9DF}] &= -[\mathbf{25}][\mathbf{123}]([\mathbf{20}][\mathbf{134}] + [\mathbf{30}][\mathbf{124}])
\end{aligned}$$

$$\begin{aligned}
[\mathbf{9FI}] &= [\mathbf{3C}][\mathbf{234}]([\mathbf{20}][\mathbf{134}] + [\mathbf{30}][\mathbf{124}])\\
[\mathbf{78I}] &= -[\mathbf{134}][\mathbf{234}]([\mathbf{3C}][\mathbf{124}] + [\mathbf{4C}][\mathbf{123}])\\
[\mathbf{79I}] &= -[\mathbf{134}][\mathbf{234}]([\mathbf{3C}][\mathbf{124}] + [\mathbf{4C}][\mathbf{123}])\\
[\mathbf{7FI}] &= -[\mathbf{30}][\mathbf{234}]([\mathbf{3C}][\mathbf{124}] + [\mathbf{4C}][\mathbf{123}]).
\end{aligned}$$

1 complete quadrilateral: [**789**] of **1234**.

$$[\mathbf{789}] = -2\,[\mathbf{123}][\mathbf{124}][\mathbf{134}][\mathbf{234}].$$

3 quadrilaterals: [**78F**] of (**1234**, **23**), [**89D**] of (**1234**, **12**), and [**89I**] of (**1234**, **34**).

$$\begin{aligned}
[\mathbf{78F}] &= [\mathbf{123}][\mathbf{234}]([\mathbf{30}][\mathbf{124}] - [\mathbf{20}][\mathbf{134}])\\
[\mathbf{89D}] &= [\mathbf{123}][\mathbf{124}]([\mathbf{15}][\mathbf{234}] - [\mathbf{25}][\mathbf{134}])\\
[\mathbf{89I}] &= [\mathbf{134}][\mathbf{234}]([\mathbf{4C}][\mathbf{123}] - [\mathbf{3C}][\mathbf{124}]).
\end{aligned}$$

2 triangles: [**8DF**] of **123**, and [**8FI**] of **234**.

$$\begin{aligned}
[\mathbf{8DF}] &= [\mathbf{123}]([\mathbf{25}][\mathbf{30}][\mathbf{124}] + [\mathbf{15}][\mathbf{20}][\mathbf{234}])\\
[\mathbf{8FI}] &= [\mathbf{234}]([\mathbf{20}][\mathbf{3C}][\mathbf{134}] + [\mathbf{30}][\mathbf{4C}][\mathbf{123}]).
\end{aligned}$$

The conclusion can be represented by conic(**78F9DI**).

Proof.

$$\begin{array}{ccl}
\text{Rules} & & \mathrm{conic}(\mathbf{78F9DI})\\
 & = & \begin{array}{l}\quad[\mathbf{78D}][\mathbf{7FI}][\mathbf{89I}][\mathbf{9DF}]\\ -[\mathbf{78I}][\mathbf{7DF}][\mathbf{89D}][\mathbf{9FI}]\end{array}\\
 & & \underbrace{[\mathbf{123}]^2[\mathbf{124}][\mathbf{134}][\mathbf{234}]^2}\underbrace{([\mathbf{15}][\mathbf{234}] + [\mathbf{25}][\mathbf{134}])}\\
 & \overset{\mathbf{7,8,9,D,F,I}}{=} & \underbrace{([\mathbf{20}][\mathbf{134}] + [\mathbf{30}][\mathbf{124}])}\underbrace{([\mathbf{3C}][\mathbf{124}] + [\mathbf{4C}][\mathbf{123}])}\\
 & & \{[\mathbf{25}][\mathbf{30}][\mathbf{4C}][\mathbf{123}] - [\mathbf{25}][\mathbf{30}][\mathbf{3C}][\mathbf{124}]\\
 & & +[\mathbf{25}][\mathbf{20}][\mathbf{3C}][\mathbf{134}] - [\mathbf{15}][\mathbf{20}][\mathbf{3C}][\mathbf{234}]\}\\
\boxed{\begin{array}{ll}[\mathbf{3C}]=[\mathbf{34}][\mathbf{356}], & [\mathbf{4C}]=[\mathbf{34}][\mathbf{456}]\\ [\mathbf{20}]=[\mathbf{23}][\mathbf{256}], & [\mathbf{30}]=[\mathbf{23}][\mathbf{356}]\end{array}} & \overset{\mathbf{0,C}}{=} & \underbrace{[\mathbf{23}][\mathbf{34}][\mathbf{356}]}([\mathbf{25}][\mathbf{123}][\mathbf{456}] - [\mathbf{25}][\mathbf{124}][\mathbf{356}]\\
 & & \qquad\qquad +[\mathbf{25}][\mathbf{134}][\mathbf{256}] - [\mathbf{15}][\mathbf{234}][\mathbf{256}])\\
 & \overset{\text{contract}}{=} & \underbrace{[\mathbf{234}]}([\mathbf{25}][\mathbf{156}] - [\mathbf{15}][\mathbf{256}])\\
 & \overset{\text{contract}}{=} & 0.
\end{array}$$

Additional nondegeneracy condition: ∃**789DI**.

Example 7.2. [Steiner's Theorem, See Chou et al., 1994, Example 6.393 for a circle] Let points **1**, **2**, **3**, **4**, **5**, **6** be on a conic, and let **7**, **8**, **9**, **0**, **A**, **B** be the intersections **12** ∩ **35**, **13**∩**45**, **14**∩**25**, **13**∩**26**, **12**∩**46**, **14**∩**36** respectively, then lines **7B**, **8A**, **90** are concurrent.

Free conic points: **1**, **2**, **3**, **4**, **5**, **6**.
Intersections:

$$\begin{array}{lll}
\mathbf{7} = \mathbf{12} \cap \mathbf{35}, & \mathbf{8} = \mathbf{13} \cap \mathbf{45}, & \mathbf{9} = \mathbf{14} \cap \mathbf{25},\\
\mathbf{0} = \mathbf{13} \cap \mathbf{26}, & \mathbf{A} = \mathbf{12} \cap \mathbf{46}, & \mathbf{B} = \mathbf{14} \cap \mathbf{36}.
\end{array}$$

Conclusion: **7B**, **8A**, **90** are concurrent.

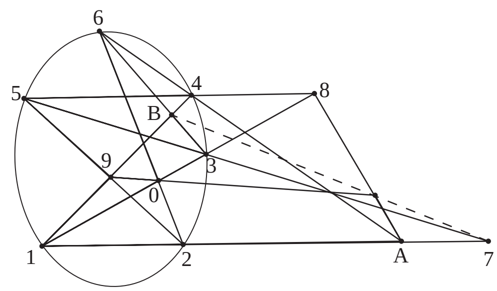

Fig. 5. Example 7.2.

Proof.

$$\begin{array}{ccl}
\text{Rules} & & \mathbf{7B}\wedge\mathbf{90}\wedge\mathbf{8A} \\
 & \overset{\text{expand}}{=} & [\mathbf{7AB}][\mathbf{890}]-[\mathbf{78B}][\mathbf{90A}] \\
\boxed{\begin{array}{l}[\mathbf{7AB}]=[\mathbf{124}][\mathbf{125}]\mathbf{12}\wedge\mathbf{35}\wedge\mathbf{46}\\ [\mathbf{890}]=[\mathbf{125}][\mathbf{134}]\mathbf{13}\wedge\mathbf{26}\wedge\mathbf{45}\\ [\mathbf{78B}]=([\mathbf{125}][\mathbf{134}][\mathbf{356}]/[\mathbf{256}])\mathbf{13}\wedge\mathbf{26}\wedge\mathbf{45}\\ [\mathbf{90A}]=([\mathbf{126}][\mathbf{134}][\mathbf{245}]/[\mathbf{345}])\mathbf{12}\wedge\mathbf{35}\wedge\mathbf{46}\end{array}} & \overset{\mathbf{7},\mathbf{8},\mathbf{9},\mathbf{0},\mathbf{A},\mathbf{B}}{=} & \begin{array}{l}\underbrace{([\mathbf{125}][\mathbf{134}]/[\mathbf{256}][\mathbf{345}])}\\ \underbrace{(\mathbf{12}\wedge\mathbf{35}\wedge\mathbf{46})(\mathbf{13}\wedge\mathbf{26}\wedge\mathbf{45})}\\ ([\mathbf{124}][\mathbf{125}][\mathbf{256}][\mathbf{345}]\\ -[\mathbf{126}][\mathbf{134}][\mathbf{245}][\mathbf{356}])\end{array} \\
 & \overset{\text{conic}}{=} & 0.
\end{array}$$

Procedure of computing brackets [**78B**], [**90A**]:

$$\begin{array}{ccl}
\text{Transformations} & & [\mathbf{78B}] \\
 & \overset{\text{expand}}{=} & [\mathbf{124}][\mathbf{135}]^2[\mathbf{346}]-[\mathbf{125}][\mathbf{134}]^2[\mathbf{356}] \\
\boxed{\begin{array}{l}[\mathbf{124}][\mathbf{135}][\mathbf{346}]\\ =[\mathbf{125}][\mathbf{134}][\mathbf{246}][\mathbf{356}]/[\mathbf{256}]\end{array}} & \overset{\text{pseudo}}{=} & ([\mathbf{125}][\mathbf{134}][\mathbf{356}]/[\mathbf{256}])([\mathbf{135}][\mathbf{246}]-[\mathbf{134}][\mathbf{256}]) \\
 & \overset{\text{factor}}{=} & ([\mathbf{125}][\mathbf{134}][\mathbf{356}]/[\mathbf{256}])\mathbf{13}\wedge\mathbf{26}\wedge\mathbf{45}
\end{array}$$

$$\begin{array}{ccl}
\text{Transformations} & & [\mathbf{90A}] \\
 & \overset{\text{expand}}{=} & [\mathbf{124}]^2[\mathbf{136}][\mathbf{256}]+[\mathbf{126}]^2[\mathbf{134}][\mathbf{245}] \\
\boxed{\begin{array}{l}[\mathbf{124}][\mathbf{136}][\mathbf{256}]\\ =[\mathbf{126}][\mathbf{134}][\mathbf{245}][\mathbf{356}]/[\mathbf{345}]\end{array}} & \overset{\text{pseudo}}{=} & ([\mathbf{126}][\mathbf{134}][\mathbf{245}]/[\mathbf{345}])([\mathbf{124}][\mathbf{356}]+[\mathbf{126}][\mathbf{345}]) \\
 & \overset{\text{factor}}{=} & ([\mathbf{126}][\mathbf{134}][\mathbf{245}]/[\mathbf{345}])\mathbf{12}\wedge\mathbf{35}\wedge\mathbf{46}
\end{array}$$

Additional nondegeneracy conditions: $\exists\mathbf{256}$, $\exists\mathbf{345}$.

Remark. By symmetry, the conclusion $\mathbf{7B}\wedge\mathbf{90}\wedge\mathbf{8A}=0$ can be expanded in any of the three ways. In the expansion used in the proof, brackets [**78B**], [**90A**] each have three binomial expansions:

$$\begin{aligned}
[\mathbf{78B}] &= [\mathbf{124}][\mathbf{135}]^2[\mathbf{346}]-[\mathbf{125}][\mathbf{134}]^2[\mathbf{356}] \\
&= [\mathbf{123}][\mathbf{134}][\mathbf{145}][\mathbf{356}]+[\mathbf{124}][\mathbf{135}][\mathbf{136}][\mathbf{345}] \qquad (7.1)\\
&= [\mathbf{123}][\mathbf{135}][\mathbf{145}][\mathbf{346}]+[\mathbf{125}][\mathbf{134}][\mathbf{136}][\mathbf{345}]
\end{aligned}$$

$$\begin{aligned}[\mathbf{90A}] &= [\mathbf{124}]^2[\mathbf{136}][\mathbf{256}] + [\mathbf{126}]^2[\mathbf{134}][\mathbf{245}] \\ &= [\mathbf{123}][\mathbf{124}][\mathbf{146}][\mathbf{256}] + [\mathbf{125}][\mathbf{126}][\mathbf{134}][\mathbf{246}] \\ &= [\mathbf{124}][\mathbf{125}][\mathbf{136}][\mathbf{246}] - [\mathbf{123}][\mathbf{126}][\mathbf{146}][\mathbf{245}].\end{aligned}$$

If the conic combination is not carried out immediately after the expansions, then rational Cayley factorization works well for the first expansions of the two brackets:

$$\begin{aligned}[\mathbf{78B}][\mathbf{90A}] = {} & [\mathbf{124}]^3[\mathbf{135}]^2[\mathbf{136}][\mathbf{256}][\mathbf{346}] - [\mathbf{124}]^2[\mathbf{125}][\mathbf{134}]^2[\mathbf{136}][\mathbf{256}][\mathbf{356}] \\ & + [\mathbf{124}][\mathbf{126}]^2[\mathbf{134}][\mathbf{135}]^2[\mathbf{245}][\mathbf{346}] - [\mathbf{125}][\mathbf{126}]^2[\mathbf{134}]^3[\mathbf{245}][\mathbf{356}].\end{aligned}$$

The details of the conic combination have been provided in Example 4.1 of Section 4. The result after removing bracket factors is

$$\begin{aligned}& [\mathbf{124}][\mathbf{135}][\mathbf{246}][\mathbf{356}] - [\mathbf{124}][\mathbf{134}][\mathbf{256}][\mathbf{356}] \\ & \quad + [\mathbf{126}][\mathbf{135}][\mathbf{246}][\mathbf{345}] - [\mathbf{126}][\mathbf{134}][\mathbf{256}][\mathbf{345}].\end{aligned}$$

A Cayley combination then changes it to $(\mathbf{12} \wedge \mathbf{35} \wedge \mathbf{46})(\mathbf{13} \wedge \mathbf{26} \wedge \mathbf{45})$, which is identical to the result from the term $[\mathbf{7AB}][\mathbf{890}]$. No additional nondegeneracy condition occurs.

However, the latter proof is too fragile in that the proving based on any other combination of the expansions of $[\mathbf{78B}], [\mathbf{90A}]$ is very difficult to finish. *The conic combination before breaking up the parentheses is indispensable*.

Let us see how rational Cayley factorization makes the proving robust. In (7.1), each expansion has two pseudoconic transformations. All together there are 12 different results from the conic combinations:

$$\begin{aligned}[\mathbf{78B}] &= [\mathbf{125}][\mathbf{134}][\mathbf{356}]([\mathbf{135}][\mathbf{246}] - [\mathbf{134}][\mathbf{256}])/[\mathbf{256}] \\ &= [\mathbf{124}][\mathbf{135}][\mathbf{346}]([\mathbf{135}][\mathbf{246}] - [\mathbf{134}][\mathbf{256}])/[\mathbf{246}] \\ &= [\mathbf{124}][\mathbf{135}]([\mathbf{134}][\mathbf{236}][\mathbf{456}] + [\mathbf{136}][\mathbf{246}][\mathbf{345}])/[\mathbf{246}] \\ &= [\mathbf{134}][\mathbf{356}]([\mathbf{123}][\mathbf{145}][\mathbf{256}] + [\mathbf{126}][\mathbf{135}][\mathbf{245}])/[\mathbf{256}] \\ &= [\mathbf{125}][\mathbf{134}]([\mathbf{136}][\mathbf{256}][\mathbf{345}] + [\mathbf{135}][\mathbf{236}][\mathbf{456}])/[\mathbf{256}] \\ &= [\mathbf{135}][\mathbf{346}]([\mathbf{126}][\mathbf{134}][\mathbf{245}] + [\mathbf{123}][\mathbf{145}][\mathbf{246}])/[\mathbf{246}] \\ [\mathbf{90A}] &= [\mathbf{126}][\mathbf{134}][\mathbf{245}]([\mathbf{124}][\mathbf{356}] + [\mathbf{126}][\mathbf{345}])/[\mathbf{345}] \\ &= [\mathbf{124}][\mathbf{136}][\mathbf{256}]([\mathbf{124}][\mathbf{356}] + [\mathbf{126}][\mathbf{345}])/[\mathbf{356}] \\ &= [\mathbf{126}][\mathbf{134}]([\mathbf{124}][\mathbf{235}][\mathbf{456}] + [\mathbf{125}][\mathbf{246}][\mathbf{345}])/[\mathbf{345}] \\ &= [\mathbf{124}][\mathbf{256}]([\mathbf{126}][\mathbf{135}][\mathbf{346}] + [\mathbf{123}][\mathbf{146}][\mathbf{356}])/[\mathbf{356}] \\ &= [\mathbf{126}][\mathbf{245}]([\mathbf{124}][\mathbf{135}][\mathbf{346}] - [\mathbf{123}][\mathbf{146}][\mathbf{345}])/[\mathbf{345}] \\ &= [\mathbf{124}][\mathbf{136}]([\mathbf{125}][\mathbf{246}][\mathbf{356}] - [\mathbf{126}][\mathbf{235}][\mathbf{456}])/[\mathbf{356}].\end{aligned}$$

After degree-2 and degree-3 Cayley factorizations, there are only four different results:

$$\begin{aligned}[\mathbf{78B}] &= ([\mathbf{125}][\mathbf{134}][\mathbf{356}]/[\mathbf{256}])\mathbf{13} \wedge \mathbf{26} \wedge \mathbf{45} \\ &= ([\mathbf{124}][\mathbf{135}][\mathbf{346}]/[\mathbf{246}])\mathbf{13} \wedge \mathbf{26} \wedge \mathbf{45} \\ [\mathbf{90A}] &= ([\mathbf{126}][\mathbf{134}][\mathbf{245}]/[\mathbf{345}])\mathbf{12} \wedge \mathbf{35} \wedge \mathbf{46} \\ &= ([\mathbf{124}][\mathbf{136}][\mathbf{256}]/[\mathbf{356}])\mathbf{12} \wedge \mathbf{35} \wedge \mathbf{46}.\end{aligned}$$

Thus, proofs based on different combinations of the expansions are much the same.

7.2. Intersections

Example 7.3 (See Bix, 1998, p. 107, Theorem 6.10). Let K and G be two conics through four points $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$. Let $\mathbf{5}$ and $\mathbf{7}$ be two points of K that do not lie on G, and are such that

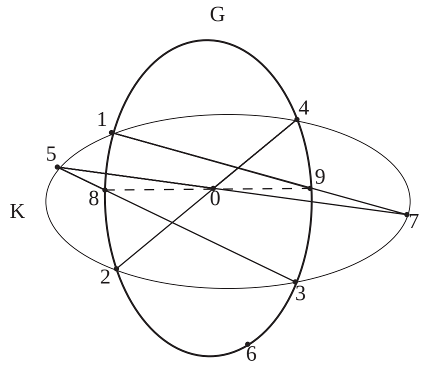

Fig. 6. Example 7.3.

5 does not lie on the tangent to G at **3**, and **7** does not lie on the tangent to G at **1**. Then **35** intersects G at a point **8** other than **3**, and **71** intersects G at a point **9** other than **1**, and **24** intersects **57** at a point **0** collinear with **8**, **9**.

Free conic points: **1**, **2**, **3**, **4**, **5**, **7**.
Free point: **6**.
Intersections: $\mathbf{8} = \mathbf{35} \cap \mathbf{12346}$, $\mathbf{9} = \mathbf{17} \cap \mathbf{12346}$, $\mathbf{0} = \mathbf{24} \cap \mathbf{57}$.
Conclusion: **8**, **9**, **0** are collinear.

Proof.

$$
\begin{aligned}
& [\mathbf{890}] \\
\overset{\mathbf{8,9,0}}{=} \; & [\mathbf{126}][\mathbf{146}][\mathbf{147}][\mathbf{236}][\mathbf{345}][\mathbf{346}]\,[(\mathbf{12}\wedge\mathbf{35})\,(\mathbf{23}\wedge\mathbf{17})\,(\mathbf{24}\wedge\mathbf{57})] \\
& -[\mathbf{126}]^2[\mathbf{147}][\mathbf{235}][\mathbf{346}]^2\,[(\mathbf{14}\wedge\mathbf{35})\,(\mathbf{23}\wedge\mathbf{17})\,(\mathbf{24}\wedge\mathbf{57})] \\
& -[\mathbf{127}][\mathbf{146}]^2[\mathbf{236}]^2[\mathbf{345}]\,[(\mathbf{12}\wedge\mathbf{35})\,(\mathbf{34}\wedge\mathbf{17})\,(\mathbf{24}\wedge\mathbf{57})] \\
& +[\mathbf{126}][\mathbf{127}][\mathbf{146}][\mathbf{235}][\mathbf{236}][\mathbf{346}]\,[(\mathbf{14}\wedge\mathbf{35})\,(\mathbf{34}\wedge\mathbf{17})\,(\mathbf{24}\wedge\mathbf{57})] \\
\overset{\text{expand}}{=} \; & -[\mathbf{126}][\mathbf{146}][\mathbf{147}][\mathbf{236}][\mathbf{345}][\mathbf{346}]([\mathbf{127}][\mathbf{135}][\mathbf{234}][\mathbf{257}] \\
& +[\mathbf{123}][\mathbf{157}][\mathbf{235}][\mathbf{247}]) - [\mathbf{126}]^2[\mathbf{147}][\mathbf{235}][\mathbf{346}]^2([\mathbf{123}][\mathbf{157}][\mathbf{247}][\mathbf{345}] \\
& -[\mathbf{127}][\mathbf{135}][\mathbf{234}][\mathbf{457}]) - [\mathbf{127}][\mathbf{146}]^2[\mathbf{236}]^2[\mathbf{345}]([\mathbf{135}][\mathbf{147}][\mathbf{234}][\mathbf{257}] \\
& -[\mathbf{134}][\mathbf{157}][\mathbf{235}][\mathbf{247}]) + [\mathbf{126}][\mathbf{127}][\mathbf{146}][\mathbf{235}][\mathbf{236}][\mathbf{346}] \\
& \times([\mathbf{135}][\mathbf{147}][\mathbf{234}][\mathbf{457}] + [\mathbf{134}][\mathbf{157}][\mathbf{247}][\mathbf{345}]) \\
\overset{\text{combine}}{=} \; & \underbrace{([\mathbf{126}][\mathbf{146}][\mathbf{157}][\mathbf{236}][\mathbf{245}][\mathbf{247}][\mathbf{346}][\mathbf{357}]/[\mathbf{257}][\mathbf{457}])} \\
& \times([\mathbf{127}][\mathbf{134}][\mathbf{235}][\mathbf{457}] - [\mathbf{123}][\mathbf{147}][\mathbf{257}][\mathbf{345}]) \\
\overset{\text{conic}}{=} \; & 0.
\end{aligned}
$$

The following are representations of **8**, **9**:

$$
\begin{aligned}
\mathbf{8} = \mathbf{8}_{\mathbf{35},\mathbf{1246}} &= [\mathbf{146}][\mathbf{236}][\mathbf{345}]\,\mathbf{12}\wedge\mathbf{35} - [\mathbf{126}][\mathbf{235}][\mathbf{346}]\,\mathbf{14}\wedge\mathbf{35}, \\
\mathbf{9} = \mathbf{9}_{\mathbf{17},\mathbf{3246}} &= [\mathbf{126}][\mathbf{147}][\mathbf{346}]\,\mathbf{23}\wedge\mathbf{17} - [\mathbf{127}][\mathbf{146}][\mathbf{236}]\,\mathbf{34}\wedge\mathbf{17}.
\end{aligned}
$$

The conic combination in the next to the last step contains two conic contractions (CC) and two conic transformations:

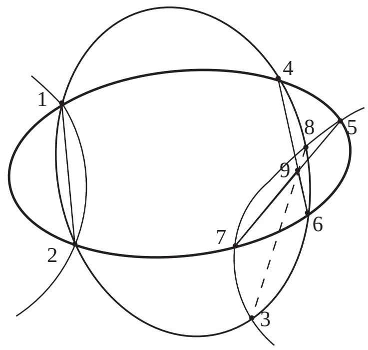

Fig. 7. Example 7.4.

$$\begin{aligned}
&[\mathbf{127}][\mathbf{135}][\mathbf{234}][\mathbf{257}]+[\mathbf{123}][\mathbf{157}][\mathbf{235}][\mathbf{247}] \overset{\text{CC}}{=} [\mathbf{123}][\mathbf{157}][\mathbf{245}][\mathbf{247}][\mathbf{357}]/[\mathbf{457}],\\
&[\mathbf{123}][\mathbf{157}][\mathbf{247}][\mathbf{345}]-[\mathbf{127}][\mathbf{135}][\mathbf{234}][\mathbf{457}] \overset{\text{conic}}{=} 0,\\
&[\mathbf{135}][\mathbf{147}][\mathbf{234}][\mathbf{257}]-[\mathbf{134}][\mathbf{157}][\mathbf{235}][\mathbf{247}] \overset{\text{conic}}{=} 0,\\
&[\mathbf{135}][\mathbf{147}][\mathbf{234}][\mathbf{457}]+[\mathbf{134}][\mathbf{157}][\mathbf{247}][\mathbf{345}] \overset{\text{CC}}{=} [\mathbf{134}][\mathbf{157}][\mathbf{245}][\mathbf{247}][\mathbf{357}]/[\mathbf{257}].
\end{aligned}$$

Additional nondegeneracy conditions: $\exists \mathbf{257}$, $\exists \mathbf{457}$.

Example 7.4 (See Graustein, 1930, p. 296, Theorem 2). If three conics have a common chord, and the three conics are taken in pairs and the common chord of each pair which is opposite to the given common chord is drawn, the three resulting lines are concurrent.

Free conic points: **1**, **2**, **4**, **5**, **6**, **7**.
Free point: **3**.
Intersections: $\mathbf{8} = \mathbf{12346} \cap \mathbf{12357}$, $\mathbf{9} = \mathbf{46} \cap \mathbf{57}$.
Conclusion: **3**, **8**, **9** are collinear.

Proof.

Rules

$$\boxed{\mathbf{8}=[\mathbf{238}]\mathbf{1}-[\mathbf{138}]\mathbf{2}+[\mathbf{128}]\mathbf{3}}$$

$$\boxed{\begin{aligned}
&[\mathbf{138}]=[\mathbf{125}][\mathbf{127}][\mathbf{134}][\mathbf{136}][\mathbf{246}][\mathbf{357}]\\
&\qquad -[\mathbf{124}][\mathbf{126}][\mathbf{135}][\mathbf{137}][\mathbf{257}][\mathbf{346}]\\
&[\mathbf{238}]=-[\mathbf{124}][\mathbf{126}][\mathbf{157}][\mathbf{235}][\mathbf{237}][\mathbf{346}]\\
&\qquad +[\mathbf{125}][\mathbf{127}][\mathbf{146}][\mathbf{234}][\mathbf{236}][\mathbf{357}]\\
&\mathbf{23}\wedge\mathbf{46}\wedge\mathbf{57}=[\mathbf{234}][\mathbf{567}]+[\mathbf{236}][\mathbf{457}]\\
&\mathbf{13}\wedge\mathbf{46}\wedge\mathbf{57}=[\mathbf{134}][\mathbf{567}]+[\mathbf{136}][\mathbf{457}]
\end{aligned}}$$

$$\begin{aligned}
&[\mathbf{389}]\\
\overset{\mathbf{9}}{=}\ &\mathbf{38}\wedge\mathbf{46}\wedge\mathbf{57}\\
\overset{\text{Cramer}}{=}\ &[\mathbf{138}]\mathbf{23}\wedge\mathbf{46}\wedge\mathbf{57}-[\mathbf{238}]\mathbf{13}\wedge\mathbf{46}\wedge\mathbf{57}\\
\overset{\mathbf{8}}{=}\ &\ [\mathbf{125}][\mathbf{127}][\mathbf{134}][\mathbf{136}][\mathbf{234}][\mathbf{246}][\mathbf{357}][\mathbf{567}]\\
&-[\mathbf{124}][\mathbf{126}][\mathbf{135}][\mathbf{137}][\mathbf{234}][\mathbf{257}][\mathbf{346}][\mathbf{567}]\\
&+[\mathbf{125}][\mathbf{127}][\mathbf{134}][\mathbf{136}][\mathbf{236}][\mathbf{246}][\mathbf{357}][\mathbf{457}]\\
&-[\mathbf{124}][\mathbf{126}][\mathbf{135}][\mathbf{137}][\mathbf{236}][\mathbf{257}][\mathbf{346}][\mathbf{457}]\\
&+[\mathbf{124}][\mathbf{126}][\mathbf{134}][\mathbf{157}][\mathbf{235}][\mathbf{237}][\mathbf{346}][\mathbf{567}]\\
&-[\mathbf{125}][\mathbf{127}][\mathbf{134}][\mathbf{146}][\mathbf{234}][\mathbf{236}][\mathbf{357}][\mathbf{567}]\\
&+[\mathbf{124}][\mathbf{126}][\mathbf{136}][\mathbf{157}][\mathbf{235}][\mathbf{237}][\mathbf{346}][\mathbf{457}]\\
&-[\mathbf{125}][\mathbf{127}][\mathbf{136}][\mathbf{146}][\mathbf{234}][\mathbf{236}][\mathbf{357}][\mathbf{457}]
\end{aligned}$$

$$\boxed{\begin{array}{l}[\mathbf{136}][\mathbf{246}]-[\mathbf{146}][\mathbf{236}]=[\mathbf{126}][\mathbf{346}]\\ [\mathbf{134}][\mathbf{246}]-[\mathbf{146}][\mathbf{234}]=[\mathbf{124}][\mathbf{346}]\end{array}}$$

$$\overset{\text{contract}}{=} \underbrace{[\mathbf{346}]}\{[\mathbf{124}][\mathbf{125}][\mathbf{127}][\mathbf{136}][\mathbf{236}][\mathbf{357}][\mathbf{457}] \\ +[\mathbf{125}][\mathbf{126}][\mathbf{127}][\mathbf{134}][\mathbf{234}][\mathbf{357}][\mathbf{567}] \\ -[\mathbf{124}][\mathbf{126}][\mathbf{135}][\mathbf{137}][\mathbf{234}][\mathbf{257}][\mathbf{567}] \\ -[\mathbf{124}][\mathbf{126}][\mathbf{135}][\mathbf{137}][\mathbf{236}][\mathbf{257}][\mathbf{457}] \\ +[\mathbf{124}][\mathbf{126}][\mathbf{134}][\mathbf{157}][\mathbf{235}][\mathbf{237}][\mathbf{567}] \\ +[\mathbf{124}][\mathbf{126}][\mathbf{136}][\mathbf{157}][\mathbf{235}][\mathbf{237}][\mathbf{457}]\}$$

$$\boxed{\begin{array}{l}[\mathbf{234}][\mathbf{567}]+[\mathbf{236}][\mathbf{457}]\\ \qquad =[\mathbf{235}][\mathbf{467}]+[\mathbf{237}][\mathbf{456}]\\ [\mathbf{134}][\mathbf{567}]+[\mathbf{136}][\mathbf{457}]\\ \qquad =[\mathbf{135}][\mathbf{467}]+[\mathbf{137}][\mathbf{456}]\\ -[\mathbf{137}][\mathbf{257}]+[\mathbf{157}][\mathbf{237}]=-[\mathbf{127}][\mathbf{357}]\\ -[\mathbf{135}][\mathbf{257}]+[\mathbf{157}][\mathbf{235}]=-[\mathbf{125}][\mathbf{357}]\end{array}}$$

$$\overset{\text{level}}{=} \underbrace{[\mathbf{357}]}\{[\mathbf{124}][\mathbf{125}][\mathbf{127}][\mathbf{136}][\mathbf{236}][\mathbf{457}] \\ +[\mathbf{125}][\mathbf{126}][\mathbf{127}][\mathbf{134}][\mathbf{234}][\mathbf{567}] \\ -[\mathbf{124}][\mathbf{125}][\mathbf{126}][\mathbf{137}][\mathbf{237}][\mathbf{456}] \\ -[\mathbf{124}][\mathbf{126}][\mathbf{127}][\mathbf{135}][\mathbf{235}][\mathbf{467}]\}$$

$$\boxed{\begin{array}{l}[\mathbf{134}][\mathbf{567}]=[\mathbf{135}][\mathbf{467}]\\ \qquad -[\mathbf{136}][\mathbf{457}]+[\mathbf{137}][\mathbf{456}]\\ [\mathbf{125}][\mathbf{234}]-[\mathbf{124}][\mathbf{235}]=-[\mathbf{123}][\mathbf{245}]\\ [\mathbf{124}][\mathbf{236}]-[\mathbf{126}][\mathbf{234}]=[\mathbf{123}][\mathbf{246}]\\ [\mathbf{127}][\mathbf{234}]-[\mathbf{124}][\mathbf{237}]=-[\mathbf{123}][\mathbf{247}]\end{array}}$$

$$\overset{\text{strong}}{=} \underbrace{[\mathbf{123}]}\{-[\mathbf{126}][\mathbf{127}][\mathbf{135}][\mathbf{245}][\mathbf{467}] \\ +[\mathbf{125}][\mathbf{127}][\mathbf{136}][\mathbf{246}][\mathbf{457}] \\ -[\mathbf{125}][\mathbf{126}][\mathbf{137}][\mathbf{247}][\mathbf{456}]\}$$

$$\boxed{\begin{array}{l}[\mathbf{126}][\mathbf{245}][\mathbf{467}]\\ =[\mathbf{125}][\mathbf{167}][\mathbf{246}][\mathbf{457}]/[\mathbf{157}]\end{array}}$$

$$\overset{\text{pseudo}}{=} \underbrace{([\mathbf{125}]/[\mathbf{157}])}\{-[\mathbf{127}][\mathbf{135}][\mathbf{167}][\mathbf{246}][\mathbf{457}] \\ +[\mathbf{127}][\mathbf{136}][\mathbf{157}][\mathbf{246}][\mathbf{457}] \\ -[\mathbf{126}][\mathbf{137}][\mathbf{157}][\mathbf{247}][\mathbf{456}]\}$$

$$\boxed{-[\mathbf{135}][\mathbf{167}]+[\mathbf{136}][\mathbf{157}]=[\mathbf{137}][\mathbf{156}]}$$

$$\overset{\text{contract}}{=} \underbrace{[\mathbf{137}]}([\mathbf{127}][\mathbf{156}][\mathbf{246}][\mathbf{457}] \\ -[\mathbf{126}][\mathbf{157}][\mathbf{247}][\mathbf{456}])$$

$$\overset{\text{conic}}{=} 0.$$

Procedure of deriving the elimination rules of **8**:

$$\begin{aligned}
\mu_1 &= [\mathbf{146}][\mathbf{234}][\mathbf{236}]\\
\mu_2 &= [\mathbf{134}][\mathbf{136}][\mathbf{246}]\\
\mu_3 &= [\mathbf{124}][\mathbf{126}][\mathbf{346}]\\
\mu_1' &= [\mathbf{157}][\mathbf{235}][\mathbf{237}]\\
\mu_2' &= [\mathbf{135}][\mathbf{137}][\mathbf{257}]\\
\mu_3' &= [\mathbf{125}][\mathbf{127}][\mathbf{357}]\\
\lambda_1 &= [\mathbf{125}][\mathbf{127}][\mathbf{134}][\mathbf{136}][\mathbf{246}][\mathbf{357}] - [\mathbf{124}][\mathbf{126}][\mathbf{135}][\mathbf{137}][\mathbf{257}][\mathbf{346}]\\
\lambda_2 &= [\mathbf{124}][\mathbf{126}][\mathbf{157}][\mathbf{235}][\mathbf{237}][\mathbf{346}] - [\mathbf{125}][\mathbf{127}][\mathbf{146}][\mathbf{234}][\mathbf{236}][\mathbf{357}]\\
\lambda_3 &= [\mathbf{135}][\mathbf{137}][\mathbf{146}][\mathbf{234}][\mathbf{236}][\mathbf{257}] - [\mathbf{134}][\mathbf{136}][\mathbf{157}][\mathbf{235}][\mathbf{237}][\mathbf{246}]
\end{aligned}$$

$$[\mathbf{238}] = \underbrace{\lambda_3}\lambda_2$$

$$[\mathbf{138}] = -\underbrace{\lambda_3}\lambda_1.$$

Additional nondegeneracy condition: $\exists$**157**.

Remark. (1) After the elimination of **8** and the successive contraction, there is no conic combination before the degree of **3** is reduced to one. The reason is that **3** takes too many brackets, which often makes the number of conic points less than 6 after removing

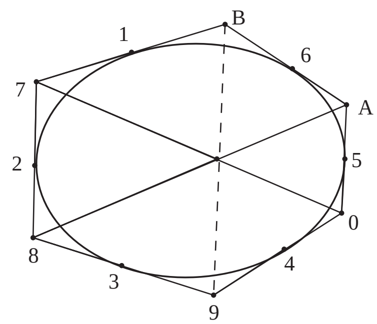

Fig. 8. Example 7.5.

the brackets containing **3**. The level contraction and strong contraction each produce a common bracket containing **3**, and thus reduce the degree of **3** by two. Only after these simplifications can the conic combination occur, which gets rid of the last degree of **3**.

(2) After the strong contraction, if instead of doing conic combination, we directly eliminate point **7** by Cramer's rule $[\mathbf{156}]\mathbf{7} = [\mathbf{567}]\mathbf{1} - [\mathbf{167}]\mathbf{5} + [\mathbf{157}]\mathbf{6}$, then we obtain a bracket polynomial of 10 terms and degree 7, which can be changed into $[\mathbf{123}][\mathbf{156}]\,\mathrm{conic}_{\mathbf{156},\mathbf{42}}(\mathbf{7})$ after five contractions. If we eliminate **3** instead of **7**, then by Cramer's rule $[\mathbf{156}]\mathbf{3} = [\mathbf{356}]\mathbf{1} + [\mathbf{136}]\mathbf{5} - [\mathbf{135}]\mathbf{6}$, we obtain a bracket polynomial of four terms and degree 6, which is changed to zero after two conic transformations. Obviously the latter elimination is simpler.

7.3. *Tangency and polarity*

Example 7.5. (Brianchon's Theorem, the Dual of Pascal's Theorem, See Berger, 1987) If a conic can be drawn to touch all the sides of a given hexagon, then the lines joining the pairs of opposite vertices of the hexagon are concurrent. Free conic points: **1**, **2**, **3**, **4**, **5**, **6**.

Poles:

$$\begin{aligned}
&\mathbf{7} = \mathrm{pole}_{\mathbf{12}}(\mathbf{123456}), \quad &&\mathbf{8} = \mathrm{pole}_{\mathbf{23}}(\mathbf{123456}), \quad &&\mathbf{9} = \mathrm{pole}_{\mathbf{34}}(\mathbf{123456}),\\
&\mathbf{0} = \mathrm{pole}_{\mathbf{45}}(\mathbf{123456}), \quad &&\mathbf{A} = \mathrm{pole}_{\mathbf{56}}(\mathbf{123456}), \quad &&\mathbf{B} = \mathrm{pole}_{\mathbf{61}}(\mathbf{123456}).
\end{aligned}$$

Conclusion: **9B**, **8A**, **70** are concurrent.

Proof.

$$\begin{aligned}
& \mathbf{9B} \wedge \mathbf{8A} \wedge \mathbf{70}\\
\overset{\text{expand}}{=}\ & [\mathbf{780}][\mathbf{9AB}] - [\mathbf{70A}][\mathbf{89B}]\\
\overset{\mathbf{7},\mathbf{8},\mathbf{9},\mathbf{0},\mathbf{A},\mathbf{B}}{=}\ & [\mathbf{7}_{\mathbf{21},\mathbf{345}}\,\mathbf{8}_{\mathbf{23},\mathbf{145}}\,\mathbf{0}_{\mathbf{45},\mathbf{213}}][\mathbf{9}_{\mathbf{34},\mathbf{651}}\,\mathbf{A}_{\mathbf{65},\mathbf{134}}\,\mathbf{B}_{\mathbf{61},\mathbf{534}}] - [\mathbf{7}_{\mathbf{12},\mathbf{546}}\,\mathbf{0}_{\mathbf{54},\mathbf{612}}\,\mathbf{A}_{\mathbf{56},\mathbf{412}}]\\
& \times [\mathbf{8}_{\mathbf{32},\mathbf{461}}\,\mathbf{9}_{\mathbf{34},\mathbf{261}}\,\mathbf{B}_{\mathbf{61},\mathbf{324}}]\,\frac{\mathbf{7}_{\mathbf{21},\mathbf{345}}\,\mathbf{8}_{\mathbf{23},\mathbf{145}}\,\mathbf{9}_{\mathbf{34},\mathbf{651}}\,\mathbf{0}_{\mathbf{45},\mathbf{213}}\,\mathbf{A}_{\mathbf{65},\mathbf{134}}\,\mathbf{B}_{\mathbf{61},\mathbf{534}}}{\mathbf{7}_{\mathbf{12},\mathbf{546}}\,\mathbf{8}_{\mathbf{32},\mathbf{461}}\,\mathbf{9}_{\mathbf{34},\mathbf{261}}\,\mathbf{0}_{\mathbf{54},\mathbf{612}}\,\mathbf{A}_{\mathbf{56},\mathbf{412}},\,\mathbf{B}_{\mathbf{61},\mathbf{324}}}\\
=\ & 16\ \underbrace{([\mathbf{124}][\mathbf{125}][\mathbf{134}]^2[\mathbf{136}][\mathbf{145}]^2[\mathbf{146}][\mathbf{235}][\mathbf{245}][\mathbf{346}][\mathbf{356}]/[\mathbf{126}]^4[\mathbf{234}]^2[\mathbf{456}]^2)}\\
& \{[\mathbf{126}]^4[\mathbf{135}]^4[\mathbf{234}]^3[\mathbf{245}][\mathbf{346}][\mathbf{456}]^3 - [\mathbf{123}]^3[\mathbf{125}][\mathbf{136}][\mathbf{156}]^3[\mathbf{246}]^4[\mathbf{345}]^4\}\\
\overset{\text{conic}}{=}\ & \underbrace{[\mathbf{123}]^3[\mathbf{156}]^3[\mathbf{246}]^3[\mathbf{345}]^3}\{[\mathbf{126}][\mathbf{135}][\mathbf{245}][\mathbf{346}] - [\mathbf{125}][\mathbf{136}][\mathbf{246}][\mathbf{345}]\}\\
\overset{\text{conic}}{=}\ & 0.
\end{aligned}$$

The elimination rules of **7**, **8**, **9**, **0**, **A**, **B** are from formula (2.20):

$$\begin{aligned}
[\mathbf{7}_{\mathbf{21},\mathbf{345}}\,\mathbf{8}_{\mathbf{23},\mathbf{145}}\,\mathbf{0}_{\mathbf{45},\mathbf{213}}] &= -4\,[\mathbf{124}][\mathbf{125}][\mathbf{134}]^2[\mathbf{135}]^2[\mathbf{234}][\mathbf{235}][\mathbf{245}]^2\\
[\mathbf{9}_{\mathbf{34},\mathbf{651}}\,\mathbf{A}_{\mathbf{65},\mathbf{134}}\,\mathbf{B}_{\mathbf{61},\mathbf{534}}] &= -4\,[\mathbf{135}]^2[\mathbf{136}][\mathbf{145}]^2[\mathbf{146}][\mathbf{346}]^2[\mathbf{356}][\mathbf{456}]\\
[\mathbf{7}_{\mathbf{12},\mathbf{546}}\,\mathbf{0}_{\mathbf{54},\mathbf{612}}\,\mathbf{A}_{\mathbf{56},\mathbf{412}}] &= -4\,[\mathbf{125}]^2[\mathbf{145}][\mathbf{146}]^2[\mathbf{156}][\mathbf{245}][\mathbf{246}]^2[\mathbf{256}]\\
[\mathbf{8}_{\mathbf{32},\mathbf{461}}\,\mathbf{9}_{\mathbf{34},\mathbf{261}}\,\mathbf{B}_{\mathbf{61},\mathbf{324}}] &= -4\,[\mathbf{123}][\mathbf{124}]^2[\mathbf{134}][\mathbf{136}]^2[\mathbf{236}][\mathbf{246}]^2[\mathbf{346}].
\end{aligned}$$

The conic transformation in the next to the last step is

$$([\mathbf{126}][\mathbf{135}][\mathbf{234}][\mathbf{456}])^3 = ([\mathbf{123}][\mathbf{156}][\mathbf{246}][\mathbf{345}])^3.$$

Additional nondegeneracy conditions: $\exists$**12345**, $\exists$**12346**, $\exists$**12456**, $\exists$**13456**, $\exists$**126**, $\exists$**234**, $\exists$**456**.

Remark. By symmetry, the conclusion $\mathbf{9B} \wedge \mathbf{8A} \wedge \mathbf{70} = 0$ can be expanded in any of the three ways. In the expansion used in the proof, two different representations of each point are used to eliminate the six points. If we reduce the changes in representation, we can reduce the number of additional nondegeneracy conditions. Notice that formula (2.20) allows a lot of freedom in choosing representative points. If we choose the following representations,

$$\begin{aligned}
[\mathbf{7}_{\mathbf{21},\mathbf{345}}\,\mathbf{8}_{\mathbf{23},\mathbf{145}}\,\mathbf{0}_{\mathbf{45},\mathbf{261}}] &= 4\,[\mathbf{124}][\mathbf{125}][\mathbf{134}]^2[\mathbf{135}][\mathbf{156}][\mathbf{235}][\mathbf{245}]^2[\mathbf{246}]\\
[\mathbf{9}_{\mathbf{34},\mathbf{612}}\,\mathbf{A}_{\mathbf{65},\mathbf{134}}\,\mathbf{B}_{\mathbf{61},\mathbf{534}}] &= 4\,[\mathbf{123}][\mathbf{135}][\mathbf{136}][\mathbf{145}]^2[\mathbf{146}][\mathbf{246}][\mathbf{346}]^2[\mathbf{356}]\\
[\mathbf{7}_{\mathbf{12},\mathbf{534}}\,\mathbf{0}_{\mathbf{54},\mathbf{612}}\,\mathbf{A}_{\mathbf{56},\mathbf{412}}] &= 4\,[\mathbf{125}]^2[\mathbf{135}][\mathbf{145}][\mathbf{146}]^2[\mathbf{234}][\mathbf{245}][\mathbf{246}][\mathbf{256}]\\
[\mathbf{8}_{\mathbf{32},\mathbf{461}}\,\mathbf{9}_{\mathbf{34},\mathbf{261}}\,\mathbf{B}_{\mathbf{61},\mathbf{345}}] &= 4\,[\mathbf{124}]^2[\mathbf{134}][\mathbf{135}][\mathbf{136}]^2[\mathbf{236}][\mathbf{246}][\mathbf{346}][\mathbf{456}],
\end{aligned}$$

then

$$\begin{aligned}
[\mathbf{9BC}] &= [\mathbf{7}_{\mathbf{21},\mathbf{345}}\,\mathbf{8}_{\mathbf{23},\mathbf{145}}\,\mathbf{0}_{\mathbf{45},\mathbf{261}}][\mathbf{9}_{\mathbf{34},\mathbf{612}}\,\mathbf{A}_{\mathbf{65},\mathbf{134}}\,\mathbf{B}_{\mathbf{61},\mathbf{534}}]\\
&\quad -[\mathbf{7}_{\mathbf{12},\mathbf{534}}\,\mathbf{0}_{\mathbf{54},\mathbf{612}}\,\mathbf{A}_{\mathbf{56},\mathbf{412}}][\mathbf{8}_{\mathbf{32},\mathbf{461}}\,\mathbf{9}_{\mathbf{34},\mathbf{261}}\,\mathbf{B}_{\mathbf{61},\mathbf{345}}]\frac{\mathbf{8}_{\mathbf{23},\mathbf{145}}\,\mathbf{A}_{\mathbf{65},\mathbf{134}}}{\mathbf{8}_{\mathbf{32},\mathbf{461}}\,\mathbf{A}_{\mathbf{56},\mathbf{412}}}\\
&= 16\,\underbrace{[\mathbf{124}][\mathbf{125}][\mathbf{134}]^2[\mathbf{135}]^2[\mathbf{136}][\mathbf{145}]^2[\mathbf{146}][\mathbf{235}][\mathbf{245}][\mathbf{246}]^2}\\
&\qquad \underbrace{[\mathbf{346}][\mathbf{356}]}\{[\mathbf{123}][\mathbf{156}][\mathbf{245}][\mathbf{346}] - [\mathbf{125}][\mathbf{136}][\mathbf{234}][\mathbf{456}]\}\\
&\overset{\text{conic}}{=} 0.
\end{aligned}$$

Additional nondegeneracy conditions: $\exists$**12345**, $\exists$**12346**, $\exists$**12456**, $\exists$**13456**.

Example 7.6 (See Semple and Kneebone, 1952, p. 126, Exercise 11). Let there be a conic touching the three sides **90**, **49**, **40** of a triangle **490** at points **1**, **2**, **3** respectively. Show that the three points $\mathbf{12} \cap \mathbf{40}$, $\mathbf{13} \cap \mathbf{49}$, $\mathbf{23} \cap \mathbf{90}$ lie on a line. If the lines joining **4**, **9**, **0** to any point **7** of this line meet **23**, **12**, **13** at points **A**, **B**, **C** respectively, prove that triangle **ABC** is self-polar relative to the conic.

Free points: **1**, **2**, **3**, **4**.
Intersections and semifree points:

$$\begin{array}{lll}
\mathbf{5} = \mathbf{12} \cap \mathbf{34}, & \mathbf{6} = \mathbf{13} \cap \mathbf{24}, & \mathbf{7} \text{ on } \mathbf{56},\\
\mathbf{8} = \mathbf{23} \cap \text{tangent}_{\mathbf{1}}(\mathbf{123}, \mathbf{24}, \mathbf{34}), & \mathbf{9} = \mathbf{24} \cap \mathbf{18}, & \mathbf{0} = \mathbf{34} \cap \mathbf{18},\\
\mathbf{A} = \mathbf{23} \cap \mathbf{47}, & \mathbf{B} = \mathbf{12} \cap \mathbf{79}, & \mathbf{C} = \mathbf{13} \cap \mathbf{70}.
\end{array}$$

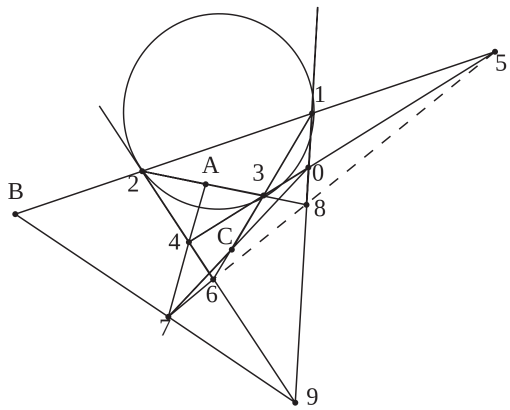

Fig. 9. Example 7.6.

Conclusion: (1) **5**, **6**, **8** are collinear; (2) any of the pairs (**A**, **B**), (**B**, **C**) and (**A**, **C**) are conjugate with respect to conic (**123**, **24**, **34**).

Proof. (1)

Rules	[**568**]
$\mathbf{8}=[\mathbf{134}]\mathbf{23}\wedge\mathbf{12}+[\mathbf{124}]\mathbf{23}\wedge\mathbf{13}$ $=-\underbrace{[\mathbf{123}]}([\mathbf{134}]\mathbf{2}+[\mathbf{124}]\mathbf{3})$, $[(\mathbf{12}\wedge\mathbf{34})(\mathbf{13}\wedge\mathbf{24})\mathbf{8}]$ $=[\mathbf{123}][\mathbf{234}]([\mathbf{134}][\mathbf{124}]-[\mathbf{124}][\mathbf{134}])$	$\overset{\mathbf{5},\,\underline{\mathbf{6}},\,\mathbf{8}}{=}\ 0.$

Additional nondegeneracy condition: none.

(2) By symmetry, it suffices to prove that **A** and **B** are conjugate. The following proof is based on the conic representation (**123**, **24**, **34**):

Rules	$\text{conjugate}_{\mathbf{123},\mathbf{24},\mathbf{34}}(\mathbf{A},\mathbf{B})$
	$=\ \underbrace{[\mathbf{123}]}^2([\mathbf{34A}][\mathbf{24B}]+[\mathbf{24A}][\mathbf{34B}])$
$[\mathbf{24A}]=-[\mathbf{234}][\mathbf{247}]$, $[\mathbf{24B}]=-[\mathbf{124}][\mathbf{279}]$ $[\mathbf{34A}]=-[\mathbf{234}][\mathbf{347}]$, $[\mathbf{34B}]=-\mathbf{12}\wedge\mathbf{34}\wedge\mathbf{79}$	$\overset{\mathbf{A},\,\mathbf{B}}{=}\ \underbrace{[\mathbf{234}]}([\mathbf{124}][\mathbf{279}][\mathbf{347}]+[\mathbf{247}]\mathbf{12}\wedge\mathbf{34}\wedge\mathbf{79})$
$[\mathbf{279}]=[\mathbf{128}][\mathbf{247}]$, $\mathbf{12}\wedge\mathbf{34}\wedge\mathbf{79}$ $=[\mathbf{127}]\mathbf{24}\wedge\mathbf{18}\wedge\mathbf{34}-[\mathbf{347}]\mathbf{24}\wedge\mathbf{18}\wedge\mathbf{12}$ $=[\mathbf{127}][\mathbf{148}][\mathbf{234}]+[\mathbf{347}][\mathbf{124}][\mathbf{128}]$	$\overset{\mathbf{9}}{=}\ \underbrace{[\mathbf{247}]}(2\,[\mathbf{124}][\mathbf{128}][\mathbf{347}]+[\mathbf{127}][\mathbf{148}][\mathbf{234}])$
$\mathbf{8}=[\mathbf{134}]\mathbf{2}+[\mathbf{124}]\mathbf{3}$	$\overset{\mathbf{8}}{=}\ 2\,\underbrace{[\mathbf{124}]}([\mathbf{123}][\mathbf{124}][\mathbf{347}]-[\mathbf{127}][\mathbf{134}][\mathbf{234}])$
$\mathbf{7}=[\mathbf{57}]\mathbf{6}-[\mathbf{67}]\mathbf{5}$	$\overset{\mathbf{7}}{=}\ \underbrace{[\mathbf{57}]}([\mathbf{123}][\mathbf{124}][\mathbf{346}]-[\mathbf{126}][\mathbf{134}][\mathbf{234}])$
$[\mathbf{126}]=[\mathbf{123}][\mathbf{124}]$, $[\mathbf{346}]=[\mathbf{134}][\mathbf{234}]$	$\overset{\mathbf{6}}{=}\ 0.$

Additional nondegeneracy condition: none.

The following proof is based on the conic representation (**312**, **18**, **24**):

Rules		
		$\mathrm{conjugate}_{\mathbf{312},\mathbf{18},\mathbf{24}}(\mathbf{A},\mathbf{B})$
	$=$	$\underbrace{[\mathbf{123}]^2}([\mathbf{24A}][\mathbf{18B}]+[\mathbf{18A}][\mathbf{24B}])$
$[\mathbf{18A}]=[\mathbf{123}][\mathbf{478}]$, $[\mathbf{18B}]=-[\mathbf{129}][\mathbf{178}]$ $[\mathbf{24A}]=-[\mathbf{234}][\mathbf{247}]$, $[\mathbf{24B}]=[\mathbf{129}][\mathbf{247}]$	$\stackrel{\mathbf{A},\,\mathbf{B}}{=}$	$\underbrace{[\mathbf{129}][\mathbf{247}]}([\mathbf{123}][\mathbf{478}]+[\mathbf{178}][\mathbf{234}])$
$\mathbf{8}=[\mathbf{134}]\mathbf{2}+[\mathbf{124}]\mathbf{3}$	$\stackrel{\mathbf{8}}{=}$	$[\mathbf{123}][\mathbf{134}][\mathbf{247}]+[\mathbf{123}][\mathbf{124}][\mathbf{347}]$ $-[\mathbf{127}][\mathbf{134}][\mathbf{234}]-[\mathbf{124}][\mathbf{137}][\mathbf{234}]$
$\mathbf{7}=[\mathbf{57}]\mathbf{6}-[\mathbf{67}]\mathbf{5}$	$\stackrel{\mathbf{7}}{=}$	$-[\mathbf{67}][\mathbf{123}][\mathbf{134}][\mathbf{245}]+[\mathbf{57}][\mathbf{123}][\mathbf{124}][\mathbf{346}]$ $-[\mathbf{57}][\mathbf{126}][\mathbf{134}][\mathbf{234}]+[\mathbf{67}][\mathbf{124}][\mathbf{135}][\mathbf{234}]$
$[\mathbf{245}]=-[\mathbf{124}][\mathbf{234}]$, $[\mathbf{346}]=[\mathbf{134}][\mathbf{234}]$ $[\mathbf{135}]=-[\mathbf{123}][\mathbf{134}]$, $[\mathbf{126}]=[\mathbf{123}][\mathbf{124}]$	$\stackrel{\mathbf{5},\,\mathbf{6}}{=}$	$0.$

Additional nondegeneracy condition: $\exists$(**312**, **18**, **24**).

8. Conclusion

In the two papers, we have established the Cayley expansion theory in Cayley and bracket algebras, particularly the classification of factored and binomial expansions of some typical Cayley expressions into bracket polynimials. The results can lead to significant simplifications in bracket computation. Based on the expansion theory, we set up a group of formulae and algorithms for Cayley factorization, and use them in theorem proving. We propose three important techniques for bracket simplification: contraction, level contraction and strong contraction. For conic computation, we propose three additional simplification techniques: conic transformation, pseudoconic transformation and conic contraction, and an algorithm for rational Cayley factorization.

We study conic geometry with Cayley and bracket algebras, establish some concise representations and their transformation rules. To overcome the difficulty of multiple representations and eliminations in theorem proving, we design a set of elimination rules for both incidence geometry and conic geometry, an algorithm for conic points selection, and an algorithm for optimal representation of the conconic conclusion. The central idea is bracket-oriented representation, elimination and expansion for factored and shortest results (**breefs**). We use these algorithms in theorem proving to generate extremely short proofs. Among more than 70 theorems tested by the algorithms, nearly all theorems in incidence geometry have two-termed proofs. In conic geometry, the overwhelming majority of the theorems can be given two-termed proofs. For those without such proofs, generally very short and nice proofs can be found.

Finally, all the representations, simplifications, expansions and elimination techniques are valid for any numbers field whose characteristic is not 2.

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