Chapter 12

Metrics on Surfaces

In this chapter, we begin the study of the geometry of surfaces from the point of view of distance and area. When mathematicians began to study surfaces at the end of the eighteenth century, they did so in terms of infinitesimal distance and area. We explain these notions intuitively in Sections 12.1 and 12.4, but from a modern standpoint.

We first define the coefficients E, F, G of the first fundamental form, and investigate their transformation under a change of coordinates (Lemma 12.4). Lengths and areas are defined by integrals involving the quantities E, F, G.

The concept of an *isometry* between surfaces is defined and illustrated by means of a circular cone in Section 12.2. The extension of this concept to *conformal maps* is introduced in Section 12.3. The latter also includes a discussion of the *distance function* defined by a metric on a surface.

In Section 12.5, we compute metrics and areas for a selection of simple surfaces, as an application of the theory.

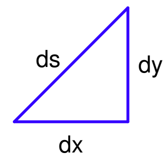
12.1 The Intuitive Idea of Distance

So far we have not discussed how to measure distances on a surface. One of the key facts about distance in Euclidean space $\mathbb{R}^n$ is that the Pythagorean Theorem holds. This means that if $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ are points in $\mathbb{R}^n$, then the distance s from $\mathbf{p}$ to $\mathbf{q}$ is given by

$$s^2 = (p_1 - q_1)^2 + \cdots + (p_n - q_n)^2. \tag{12.1}$$

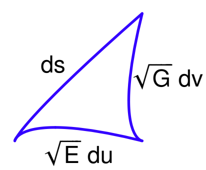
How is this notion different for a surface? Because a general surface is curved, distance on it is not the same as in Euclidean space; in particular, (12.1) is in general false however we interpret the coordinates. To describe how to measure distance on a surface, we need the mathematically imprecise notion of infinitesimal. The infinitesimal version of (12.1) for $n = 2$ is

$$ds^2 = dx^2 + dy^2. \tag{12.2}$$



We can think of dx and dy as small quantities in the x and y directions. The formula (12.2) is valid for $\mathbb{R}^2$. For a surface, or more precisely a patch, the corresponding equation is

$$ds^2 = E\,du^2 + 2F\,dudv + G\,dv^2. \tag{12.3}$$



This is the classical notation[1] for a metric on a surface. We may consider (12.3) as an infinitesimal warped version of the Pythagorean Theorem.

Among the many ways to define metrics on surfaces, one is especially simple and important. Let $\mathcal{M}$ be a regular surface in $\mathbb{R}^n$. The scalar or dot product of $\mathbb{R}^n$ gives rise to a scalar product on $\mathcal{M}$ by restriction. If $\mathbf{v_p}$ and $\mathbf{w_p}$ are tangent vectors to $\mathcal{M}$ at $\mathbf{p} \in \mathcal{M}$, we can take the scalar product $\mathbf{v_p} \cdot \mathbf{w_p}$ because $\mathbf{v_p}$ and $\mathbf{w_p}$ are also tangent vectors to $\mathbb{R}^n$ at $\mathbf{p}$.

In this chapter we deal only with local properties of distance, so without loss of generality, we can assume that $\mathcal{M}$ is the image of an injective regular patch. However, the following definition makes sense for any patch.

Definition 12.1. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ be a patch. Define functions $E, F, G\colon \mathcal{U} \to \mathbb{R}$ by*

$$E = \|\mathbf{x}_u\|^2, \qquad F = \mathbf{x}_u \cdot \mathbf{x}_v, \qquad G = \|\mathbf{x}_v\|^2.$$

Then $ds^2 = E\,du^2 + 2F\,dudv + G\,dv^2$ is the ***Riemannian metric*** *or* ***first fundamental form*** *of the patch $\mathbf{x}$. Furthermore, E, F, G are called the* ***coefficients of the first fundamental form*** *of $\mathbf{x}$.*

The equivalence of the formal definition of E, F, G with the infinitesimal version (12.3) can be better understood if we consider curves on a patch.

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[1] This notation (including the choice of letters E, F and G) was already in use in the early part of the 19th century; it can be found, for example, in the works of Gauss (see [Gauss2] and [Dom]). Gauss had the idea to study properties of a surface that are independent of the way the surface sits in space, that is, properties of a surface that are expressible in terms of E, F and G alone. One such property is Gauss's *Theorema Egregium*, which we shall prove in Section 17.2.

Lemma 12.2. *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$ be a curve that lies on a regular injective patch $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$, and fix c with $a < c < b$. The arc length function s of $\boldsymbol{\alpha}$ starting at $\boldsymbol{\alpha}(c)$ is given by*

$$s(t) = \int_c^t \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}\,dt. \tag{12.4}$$

Proof. Write $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$ for $a < t < b$. By the definition of the arc length function and Corollary 10.15 on page 294, we have

$$\begin{aligned} s(t) &= \int_c^t \|\boldsymbol{\alpha}'(t)\|dt = \int_c^t \big\|u'(t)\mathbf{x}_u\big(u(t),v(t)\big) + v'(t)\mathbf{x}_v\big(u(t),v(t)\big)\big\|\,dt \\ &= \int_c^t \sqrt{u'(t)^2\big\|\mathbf{x}_u\big\|^2 + 2u'(t)v'(t)\mathbf{x}_u\cdot\mathbf{x}_v + v'(t)^2\big\|\mathbf{x}_v\big\|^2}\,dt \\ &= \int_c^t \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}\,dt. \quad \blacksquare \end{aligned}$$

It follows from (12.4) that

$$\frac{ds}{dt} = \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}. \tag{12.5}$$

Now we can make sense of (12.3). We square both sides of (12.5) and multiply through by dt^2. Although strictly speaking multiplication by the infinitesimal dt^2 is not permitted, at least formally we obtain (12.3).

Notice that the right-hand side of (12.3) does not involve the parameter t, except insofar as u and v depend on t. We may think of ds as the ***infinitesimal arc length***, because it gives the arc length function when integrated over any curve. Geometrically, ds can be interpreted as the infinitesimal distance from a point $\mathbf{x}(u,v)$ to a point $\mathbf{x}(u+du, v+dv)$ measured along the surface. Indeed, to first order,

$$\boldsymbol{\alpha}(t+dt) \;\approx\; \boldsymbol{\alpha}(t) + \boldsymbol{\alpha}'(t)dt \;=\; \boldsymbol{\alpha}(t) + \mathbf{x}_u u'(t)dt + \mathbf{x}_v v'(t)dt,$$

so that

$$\begin{aligned} \big\|\boldsymbol{\alpha}(t+dt) - \boldsymbol{\alpha}(t)\big\| &\approx \big\|\mathbf{x}_u u'(t) + \mathbf{x}_v v'(t)\big\|\,dt \\ &= \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\,dt \;=\; ds. \end{aligned}$$

A standard notion from calculus of several variables is that of the *differential* of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$; it is given by

$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv.$$

More generally, if $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ is a regular injective patch and $f\colon \mathcal{M} \to \mathbb{R}$ is a differentiable function, we put

$$df = \frac{\partial(f \circ \mathbf{x})}{\partial u} du + \frac{\partial(f \circ \mathbf{x})}{\partial v} dv.$$

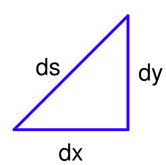
We call df the *differential* of f. The differentials of the functions $\mathbf{x}(u,v) \mapsto u$ and $\mathbf{x}(u,v) \mapsto v$ are denoted by du and dv. In spite of its appearance, ds will hardly ever be the differential of a function on a surface, since ds^2 represents a *nondegenerate* quadratic form. But formally, $d\mathbf{x} = \mathbf{x}_u du + \mathbf{x}_v dv$, so that

$$\begin{aligned} d\mathbf{x} \cdot d\mathbf{x} &= \big(\mathbf{x}_u du + \mathbf{x}_v dv\big) \cdot \big(\mathbf{x}_u du + \mathbf{x}_v dv\big) \\ &= \|\mathbf{x}_u\|^2 du^2 + 2\mathbf{x}_u \cdot \mathbf{x}_v du dv + \|\mathbf{x}_v\|^2 dv^2 \\ &= E\, du^2 + 2F\, du dv + G\, dv^2 \\ &= ds^2. \end{aligned}$$

Equation (12.2) represents the square of the infinitesimal distance on $\mathbb{R}^2$ written in terms of the Cartesian coordinates x and y. There is a different, but equally familiar, expression for ds^2 in polar coordinates.

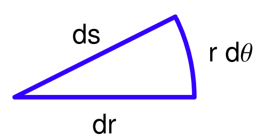
Lemma 12.3. *The metric ds^2 on $\mathbb{R}^2$, given in Cartesian coordinates as*

$$ds^2 = dx^2 + dy^2,$$



in polar coordinates becomes

$$ds^2 = dr^2 + r^2 d\theta^2.$$



Proof. We have the standard change of variable formulas from rectangular to polar coordinates:

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

Therefore,

$$\begin{cases} dx = -r\sin\theta\, d\theta + \cos\theta\, dr, \\ dy = r\cos\theta\, d\theta + \sin\theta\, dr. \end{cases} \tag{12.6}$$

Hence

$$\begin{aligned} dx^2 + dy^2 &= (-r\sin\theta\, d\theta + \cos\theta\, dr)^2 + (r\cos\theta\, d\theta + \sin\theta\, dr)^2 \\ &= dr^2 + r^2 d\theta^2. \quad ▮ \end{aligned}$$

We need to know how the expression for a metric changes under a change of coordinates.

Lemma 12.4. *Let $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ and $\mathbf{y}\colon \mathcal{V} \to \mathcal{M}$ be patches on a regular surface $\mathcal{M}$ with $\mathbf{x}(\mathcal{U}) \cap \mathbf{y}(\mathcal{V})$ nonempty. Let $\mathbf{x}^{-1} \circ \mathbf{y} = (\bar{u}, \bar{v})\colon \mathcal{U} \cap \mathcal{V} \to \mathcal{U} \cap \mathcal{V}$ be the associated change of coordinates, so that*

$$\mathbf{y}(u,v) = \mathbf{x}\big(\bar{u}(u,v), \bar{v}(u,v)\big).$$

Suppose that $\mathcal{M}$ has a metric and denote the induced metrics on $\mathbf{x}$ and $\mathbf{y}$ by

$$ds_{\mathbf{x}}^2 = E_{\mathbf{x}} d\bar{u}^2 + 2F_{\mathbf{x}} d\bar{u} d\bar{v} + G_{\mathbf{x}} d\bar{v}^2 \quad \textit{and} \quad ds_{\mathbf{y}}^2 = E_{\mathbf{y}} du^2 + 2F_{\mathbf{y}} du dv + G_{\mathbf{y}} dv^2.$$

Then

$$\left\{ \begin{aligned} E_{\mathbf{y}} &= E_{\mathbf{x}} \left(\frac{\partial \bar{u}}{\partial u}\right)^2 + 2F_{\mathbf{x}} \frac{\partial \bar{u}}{\partial u}\frac{\partial \bar{v}}{\partial u} + G_{\mathbf{x}} \left(\frac{\partial \bar{v}}{\partial u}\right)^2, \\ F_{\mathbf{y}} &= E_{\mathbf{x}} \frac{\partial \bar{u}}{\partial u}\frac{\partial \bar{u}}{\partial v} + F_{\mathbf{x}} \left(\frac{\partial \bar{u}}{\partial u}\frac{\partial \bar{v}}{\partial v} + \frac{\partial \bar{u}}{\partial v}\frac{\partial \bar{v}}{\partial u}\right) + G_{\mathbf{x}} \frac{\partial \bar{v}}{\partial u}\frac{\partial \bar{v}}{\partial v}, \\ G_{\mathbf{y}} &= E_{\mathbf{x}} \left(\frac{\partial \bar{u}}{\partial v}\right)^2 + 2F_{\mathbf{x}} \frac{\partial \bar{u}}{\partial v}\frac{\partial \bar{v}}{\partial v} + G_{\mathbf{x}} \left(\frac{\partial \bar{v}}{\partial v}\right)^2. \end{aligned} \right. \tag{12.7}$$

Proof. We use Lemma 10.31 on page 300, to compute

$$\begin{aligned} E_{\mathbf{y}} = \mathbf{y}_u \cdot \mathbf{y}_u &= \left(\frac{\partial \bar{u}}{\partial u}\mathbf{x}_{\bar{u}} + \frac{\partial \bar{v}}{\partial u}\mathbf{x}_{\bar{v}}\right) \cdot \left(\frac{\partial \bar{u}}{\partial u}\mathbf{x}_{\bar{u}} + \frac{\partial \bar{v}}{\partial u}\mathbf{x}_{\bar{v}}\right) \\ &= \left(\frac{\partial \bar{u}}{\partial u}\right)^2 \mathbf{x}_{\bar{u}} \cdot \mathbf{x}_{\bar{u}} + 2\left(\frac{\partial \bar{u}}{\partial u}\right)\left(\frac{\partial \bar{v}}{\partial v}\right) \mathbf{x}_{\bar{u}} \cdot \mathbf{x}_{\bar{v}} + \left(\frac{\partial \bar{v}}{\partial v}\right)^2 \mathbf{x}_{\bar{v}} \cdot \mathbf{x}_{\bar{v}} \\ &= \left(\frac{\partial \bar{u}}{\partial u}\right)^2 E_{\mathbf{x}} + 2\frac{\partial \bar{u}}{\partial u}\frac{\partial \bar{v}}{\partial u} F_{\mathbf{x}} + \left(\frac{\partial \bar{v}}{\partial u}\right)^2 G_{\mathbf{x}}. \end{aligned}$$

The other equations are proved similarly. ▮

12.2 Isometries between Surfaces

In Section 10.5, we defined the notion of a mapping F between surfaces in $\mathbb{R}^n$. Even if F is a local diffeomorphism (see page 309), $F(\mathcal{M})$ can be quite different from $\mathcal{M}$. Good examples are the maps romanmap and crosscapmap of Section 11.5 that map a sphere onto Steiner's Roman surface and a cross cap. Differentiable maps that preserve infinitesimal distances, on the other hand, distort much less. The search for such maps leads to the following definition.

Definition 12.5. *Let $\mathcal{M}_1, \mathcal{M}_2$ be regular surfaces in $\mathbb{R}^n$. A map $\Phi\colon \mathcal{M}_1 \to \mathcal{M}_2$ is called a* ***local isometry*** *provided its tangent map satisfies*

$$\|\Phi_*(\mathbf{v_p})\| = \|\mathbf{v_p}\| \tag{12.8}$$

for all tangent vectors $\mathbf{v_p}$ to $\mathcal{M}_1$. An ***isometry*** *is a surface mapping which is simultaneously a local isometry and a diffeomorphism.*

Lemma 12.6. *A local isometry is a local diffeomorphism.*

Proof. It is easy to see that (12.8) implies that each tangent map of a local isometry Φ is injective. Then the inverse function theorem implies that Φ is a local diffeomorphism. ▮

Every isometry of $\mathbb{R}^n$ which maps a regular surface $\mathcal{M}_1$ onto a regular surface $\mathcal{M}_2$ obviously restricts to an isometry between $\mathcal{M}_1$ and $\mathcal{M}_2$. When we study minimal surfaces in Chapter 16, we shall see that there are isometries between surfaces that do not however arise in this fashion.

Let $\mathbf{v_p}, \mathbf{w_p}$ be tangent vectors to $\mathcal{M}_1$. Suppose that (12.8) holds. Since Φ_* is linear,

$$\begin{aligned}
\Phi_*(\mathbf{v_p}) \cdot \Phi_*(\mathbf{w_p}) &= \frac{1}{2}\Big(\|\Phi_*(\mathbf{v_p}+\mathbf{w_p})\|^2 - \|\Phi_*(\mathbf{v_p})\|^2 - \|\Phi_*(\mathbf{w_p})\|^2\Big)\\
&= \frac{1}{2}\Big(\|(\mathbf{v_p}+\mathbf{w_p})\|^2 - \|\mathbf{v_p}\|^2 - \|\mathbf{w_p}\|^2\Big)\\
&= \mathbf{v_p} \cdot \mathbf{w_p}.
\end{aligned}$$

It follows that (12.8) is equivalent to

$$\Phi_*(\mathbf{v_p}) \cdot \Phi_*(\mathbf{w_p}) = \mathbf{v_p} \cdot \mathbf{w_p}. \tag{12.9}$$

Next, we show that a surface mapping is an isometry if and only if it preserves Riemannian metrics.

Lemma 12.7. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ be a regular injective patch and let $\mathbf{y}\colon \mathcal{U} \to \mathbb{R}^n$ be any patch. Let*

$$ds_{\mathbf{x}}^2 = E_{\mathbf{x}}du^2 + 2F_{\mathbf{x}}dudv + G_{\mathbf{x}}dv^2 \quad \textit{and} \quad ds_{\mathbf{y}}^2 = E_{\mathbf{y}}du^2 + 2F_{\mathbf{y}}dudv + G_{\mathbf{y}}dv^2$$

denote the induced Riemannian metrics on $\mathbf{x}$ and $\mathbf{y}$. Then the map

$$\Phi = \mathbf{y} \circ \mathbf{x}^{-1}\colon \mathbf{x}(\mathcal{U}) \longrightarrow \mathbf{y}(\mathcal{U})$$

is a local isometry if and only if

$$ds_{\mathbf{x}}^2 = ds_{\mathbf{y}}^2. \tag{12.10}$$

Proof. First, note that

$$\Phi_*(\mathbf{x}_u) = \big((\mathbf{y}\circ\mathbf{x}^{-1})_*\circ\mathbf{x}_*\big)\left(\frac{\partial}{\partial u}\right) = \mathbf{y}_*\left(\frac{\partial}{\partial u}\right) = \mathbf{y}_u;$$

similarly, $\Phi_*(\mathbf{x}_v) = \mathbf{y}_v$. If Φ is a local isometry, then

$$E_{\mathbf{y}} = \|\mathbf{y}_u\|^2 = \|\Phi_*(\mathbf{x}_u)\|^2 = \|\mathbf{x}_u\|^2 = E_{\mathbf{x}}.$$

In the same way, $F_{\mathbf{y}} = F_{\mathbf{x}}$ and $G_{\mathbf{y}} = G_{\mathbf{x}}$. Thus (12.10) holds.

To prove the converse, consider a curve $\boldsymbol{\alpha}$ of the form

$$\boldsymbol{\alpha}(t) = \mathbf{x}\big(u(t), v(t)\big);$$

then $(\Phi\circ\boldsymbol{\alpha})(t) = \mathbf{y}(u(t), v(t))$. From Corollary 10.15 we know that

$$\boldsymbol{\alpha}' = u'\mathbf{x}_u + v'\mathbf{x}_v \qquad\text{and}\qquad (\Phi\circ\boldsymbol{\alpha})' = u'\mathbf{y}_u + v'\mathbf{y}_v.$$

Hence

$$\text{(12.11)}\qquad \begin{cases} \|\boldsymbol{\alpha}'\|^2 \;=\; E_{\mathbf{x}}u'^2 + 2F_{\mathbf{x}}u'v' + G_{\mathbf{x}}v'^2, \\ \|(\Phi\circ\boldsymbol{\alpha})'\|^2 \;=\; E_{\mathbf{y}}u'^2 + 2F_{\mathbf{y}}u'v' + G_{\mathbf{y}}v'^2. \end{cases}$$

Then (12.10) and (12.11) imply that

$$\text{(12.12)}\qquad \|\boldsymbol{\alpha}'(t)\| = \|(\Phi\circ\boldsymbol{\alpha})'(t)\|.$$

Since every tangent vector to $\mathbf{x}(\mathcal{U})$ can be represented as $\boldsymbol{\alpha}'(0)$ for some curve $\boldsymbol{\alpha}$, it follows that Φ is an isometry. ∎

Corollary 12.8. Let $\Phi\colon \mathcal{M}_1 \to \mathcal{M}_2$ be a surface mapping. Given a patch $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}_1$, set $\mathbf{y} = \Phi\circ\mathbf{x}$. Then Φ is a local isometry if and only if for each regular injective patch $\mathbf{x}$ on $\mathcal{M}_1$ we have $ds^2_{\mathbf{x}} = ds^2_{\mathbf{y}}$.

We shall illustrate these results with a vivid example by constructing a patch on a circular cone in terms of Euclidean coordinates (u, v) in the plane. We begin with the region in the plane surrounded by a circular arc AB of radius 1 subtending an angle AOB of α radians, shown in Figure 12.4.

We shall attach the edge OA to OB, in the fashion of the previous chapter, but make this process explicit by identifying the result with a circular (half) cone in 3-dimensional space. The top circle of the cone has circumference α (this being the length of the existing arc AB) and thus radius $\alpha/2\pi$. It follows that if β denotes the angle (between the axis and a generator) of the cone, then

$$\sin\beta = \frac{\alpha}{2\pi}.$$

For example, if $\alpha = \pi$, the initial region is a semicircular and the cone has angle $\beta = \pi/6$ or 30°.

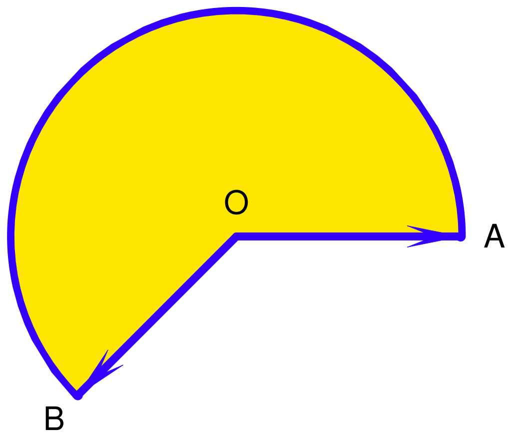


Figure 12.4: A planar region

The planar region may be realized as a surface in $\mathbb{R}^3$ by merely mapping it into the xy-plane. Using polar coordinates, it is then parametrized by the patch

$$\mathbf{x}(r,\theta) = (r\cos\theta,\ r\sin\theta,\ 0)$$

defined for $0 \leqslant r \leqslant 1$ and $0 \leqslant \theta \leqslant \alpha$. We can parametrize the resulting conical surface by the patch by realizing that a radial line in the preceding plot becomes a generator of the cone. A distance r along this generator contributes a horizontal distance of $r\sin\beta$ and a vertical distance $r\cos\beta$. The angle about the z-axis has to run through a full turn, so a patch of the conical surface is

$$\mathbf{y}(r,\theta) = \Big(r\cos\frac{2\pi\theta}{\alpha}\sin\beta,\ r\sin\frac{2\pi\theta}{\alpha}\sin\beta,\ r\cos\beta\Big), \tag{12.13}$$

for $0 \leqslant r \leqslant 1$ and $0 \leqslant \theta \leqslant \alpha$. The final result is close to Figure 12.5, right.

Let $\mathcal{V}$ denote the open interior of the circular region in the xy-plane. The above procedure determines a mapping $\Phi\colon \mathcal{V} \to \mathbb{R}^3$ for which

$$\mathbf{y}(u,v) = \Phi\big(\mathbf{x}(u,v)\big),$$

in accord with Corollary 12.8. Indeed, Φ represents the 'rigid' folding of a piece of paper representing Figure 12.1 into the cone, and is therefore an isometry. This is confirmed by a computation of the respective first fundamental forms. The metric $ds^2_{\mathbf{x}}$ is given in polar coordinates (r,θ) by Lemma 12.3. But an easy computation of E, F, G for $\mathbf{y}$ (carried out in Notebook 12) shows that $ds^2_{\mathbf{y}}$ has the identical form. This means that the same norms and angles are induced on the corresponding tangent vectors on the surface of the cone.

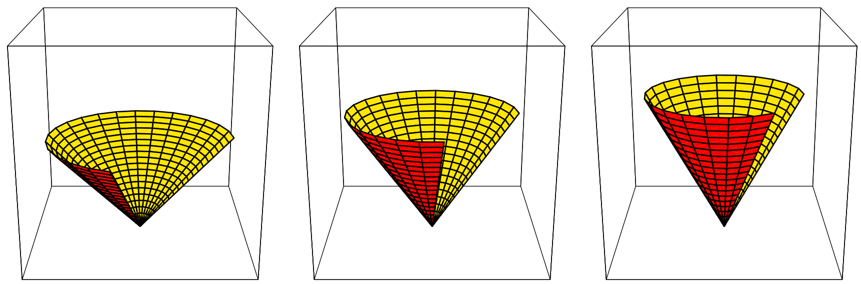


Figure 12.5: Isometries between conical surfaces

The following patch interpolates between $\mathbf{x}$ and $\mathbf{y}$ by representing an intermediate conical surface

$$\mathbf{x}_t(r,\theta) = \left(r\cos\frac{\theta}{\lambda_t}\sin\beta_t,\ r\sin\frac{\theta}{\lambda_t}\sin\beta_t,\ r\cos\beta_t\right),$$

where

$$\lambda_t = 1 - t + \frac{\alpha t}{2\pi}, \qquad \beta_t = \arcsin\lambda_t.$$

For each fixed t with $0 \leqslant t \leqslant 1$, this process determines an isometry from the planar region to the intermediate surface, for example Figure 12.5, middle. In Notebook 12, it is used to create an animation of surfaces passing from $\mathbf{x}$ to the closed cone $\mathbf{y}$, representing the act of folding the piece of paper.

12.3 Distance and Conformal Maps

A Riemannian metric determines a function which measures distances on a regular surface.

Definition 12.9. *Let $\mathcal{M} \subset \mathbb{R}^n$ be a regular surface, and let $\mathbf{p},\mathbf{q} \in \mathcal{M}$. Then the* ***intrinsic distance*** $\boldsymbol{\rho}(\mathbf{p},\mathbf{q})$ *is the greatest lower bound of the lengths of all piecewise-differentiable curves $\boldsymbol{\alpha}$ that lie entirely on $\mathcal{M}$ and connect $\mathbf{p}$ to $\mathbf{q}$. We call $\boldsymbol{\rho}$ the* ***distance function*** *of $\mathcal{M}$.*

In general, the intrinsic distance $\boldsymbol{\rho}(\mathbf{p},\mathbf{q})$ will be greater than the Euclidean distance $\|\mathbf{p}-\mathbf{q}\|$, since the surface will not contain the straight line joining $\mathbf{p}$ to $\mathbf{q}$. This fact is evident in Figure 12.6.

An isometry also preserves the intrinsic distance:

Lemma 12.10. *Let $\Phi\colon \mathcal{M}_1 \to \mathcal{M}_2$ be an isometry. Then Φ identifies the intrinsic distances $\boldsymbol{\rho}_1,\boldsymbol{\rho}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ in the sense that*

$$\boldsymbol{\rho}_2\big(\Phi(\mathbf{p}),\Phi(\mathbf{q})\big) = \boldsymbol{\rho}_1(\mathbf{p},\mathbf{q}), \tag{12.14}$$

for $\mathbf{p},\mathbf{q} \in \mathcal{M}_1$.

Proof. Let $\boldsymbol{\alpha}\colon (c,d) \to \mathcal{M}$ be a piecewise-differentiable curve with $\boldsymbol{\alpha}(a) = \mathbf{p}$ and $\boldsymbol{\alpha}(b) = \mathbf{q}$, where $c < a < b < d$. From the definition of local isometry, it follows that

$$\mathsf{length}[\boldsymbol{\alpha}] = \int_a^b \|\boldsymbol{\alpha}'(t)\|\,dt = \int_a^b \|(\Phi \circ \boldsymbol{\alpha})'(t)\|\,dt = \mathsf{length}[\Phi \circ \boldsymbol{\alpha}].$$

Since Φ is a diffeomorphism, there is a one-to-one correspondence between the piecewise-differentiable curves on $\mathcal{M}_1$ connecting $\mathbf{p}$ to $\mathbf{q}$ and the piecewise-differentiable curves on $\mathcal{M}_2$ connecting $\Phi(\mathbf{p})$ to $\Phi(\mathbf{q})$. Since corresponding curves have equal lengths, we obtain (12.14). ▮

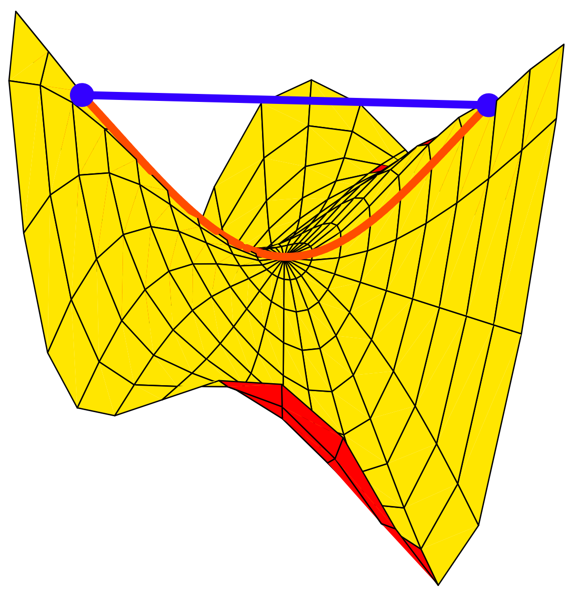


Figure 12.6: Distance on a surface

Next, let us consider a generalization of the notion of local isometry.

Definition 12.11. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be regular surfaces in $\mathbb{R}^n$. Then a map $\Phi\colon \mathcal{M}_1 \to \mathcal{M}_2$ is called a **conformal map** provided there is a differentiable everywhere positive function $\lambda\colon \mathcal{M}_1 \to \mathbb{R}$ such that*

$$\|\Phi_*(\mathbf{v_p})\| = \lambda(\mathbf{p})\|\mathbf{v_p}\| \tag{12.15}$$

*for all $\mathbf{p} \in \mathcal{M}_1$ and all tangent vectors $\mathbf{v_p}$ to $\mathcal{M}_1$ at $\mathbf{p}$. We call λ the **scale factor**. A **conformal diffeomorphism** is a surface mapping which is simultaneously a conformal map and a diffeomorphism.*

Since the tangent map Φ_* of an isometry preserves inner products, it also preserves both lengths and angles. The tangent map of a conformal diffeomorphism in general will change the *lengths* of tangent vectors, but we do have

Lemma 12.12. *A conformal map $\Phi\colon \mathcal{M}_1 \to \mathcal{M}_2$ preserves the angles between nonzero tangent vectors.*

Proof. The proof that (12.9) is equivalent to (12.8) can be easily modified to prove that (12.15) is equivalent to

$$\Phi_*(\mathbf{v_p}) \cdot \Phi_*(\mathbf{w_p}) = \lambda(\mathbf{p})^2\, \mathbf{v_p} \cdot \mathbf{w_p} \tag{12.16}$$

for all nonzero tangent vectors $\mathbf{v_p}$ and $\mathbf{w_p}$ to $\mathcal{M}_1$. Then (12.16) implies that

$$\frac{\Phi_*(\mathbf{v_p}) \cdot \Phi_*(\mathbf{w_p})}{\|\Phi_*(\mathbf{v_p})\|\,\|\Phi_*(\mathbf{w_p})\|} = \frac{\mathbf{v_p} \cdot \mathbf{w_p}}{\|\mathbf{v_p}\|\,\|\mathbf{w_p}\|}.$$

From the definition of angle on page 3, it follows that Φ_* preserves angles between tangent vectors. ∎

Here is an important example of a conformal map, which was in fact introduced in Section 8.6.

Definition 12.13. *Let $S^2(1)$ denote the unit sphere in $\mathbb{R}^3$. The* ***stereographic map*** *$\boldsymbol{\Upsilon}\colon \mathbb{R}^2 \to S^2(1)$ is defined by*

$$\boldsymbol{\Upsilon}(p_1, p_2) = \frac{(2p_1,\ 2p_2,\ p_1^2 + p_2^2 - 1)}{p_1^2 + p_2^2 + 1}.$$

We abbreviate this definition to

$$\boldsymbol{\Upsilon}(\mathbf{p}) = \frac{(2\mathbf{p};\ \|\mathbf{p}\|^2 - 1)}{1 + \|\mathbf{p}\|^2}, \tag{12.17}$$

for $\mathbf{p} \in \mathbb{R}^2$ (the semicolon reminds us that the numerator is a vector, not an inner product). It is easy to see that $\boldsymbol{\Upsilon}$ is differentiable, and that $\|\boldsymbol{\Upsilon}(\mathbf{p})\| = 1$. Moreover,

Lemma 12.14. *$\boldsymbol{\Upsilon}$ is a conformal map.*

Proof. Let $\boldsymbol{\alpha}\colon (a, b) \to \mathbb{R}^2$ be a curve. It follows from (12.17) that the image of $\boldsymbol{\alpha}$ by $\boldsymbol{\Upsilon}$ is the curve

$$\boldsymbol{\Upsilon} \circ \boldsymbol{\alpha} = \frac{(2\boldsymbol{\alpha};\ \|\boldsymbol{\alpha}\|^2 - 1)}{1 + \|\boldsymbol{\alpha}\|^2},$$

and this equation can be rewritten as

$$(1 + \|\boldsymbol{\alpha}\|^2)(\boldsymbol{\Upsilon} \circ \boldsymbol{\alpha}) = (2\boldsymbol{\alpha};\ \|\boldsymbol{\alpha}\|^2 - 1). \tag{12.18}$$

Differentiating (12.18), we obtain

$$2(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}')(\boldsymbol{\Upsilon}\circ\boldsymbol{\alpha}) + (1+\|\boldsymbol{\alpha}\|^2)(\boldsymbol{\Upsilon}\circ\boldsymbol{\alpha})' = 2(\boldsymbol{\alpha}';\ \boldsymbol{\alpha}\cdot\boldsymbol{\alpha}'),$$

and taking norms,

$$\left\|2(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}')(\boldsymbol{\Upsilon}\circ\boldsymbol{\alpha}) + (1+\|\boldsymbol{\alpha}\|^2)(\boldsymbol{\Upsilon}\circ\boldsymbol{\alpha})'\right\|^2 = 4\|(\boldsymbol{\alpha}';\ \boldsymbol{\alpha}\cdot\boldsymbol{\alpha}')\|^2. \tag{12.19}$$

Since $\boldsymbol{\Upsilon}(\mathbb{R}^2)\subset S^2(1)$, we have

$$\|\boldsymbol{\Upsilon}\circ\boldsymbol{\alpha}\|^2 = 1 \qquad \text{and} \qquad (\boldsymbol{\Upsilon}\circ\boldsymbol{\alpha})\cdot(\boldsymbol{\Upsilon}\circ\boldsymbol{\alpha})' = 0.$$

It therefore follows from (12.19) that

$$4(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}')^2 + (1+\|\boldsymbol{\alpha}\|^2)^2\left\|(\boldsymbol{\Upsilon}\circ\boldsymbol{\alpha})'\right\|^2 = 4\|\boldsymbol{\alpha}'\|^2 + 4(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}')^2,$$

whence

$$\left\|(\boldsymbol{\Upsilon}\circ\boldsymbol{\alpha})'\right\| = \frac{2\|\boldsymbol{\alpha}'\|}{1+\|\boldsymbol{\alpha}\|^2}.$$

We conclude that

$$\left\|\boldsymbol{\Upsilon}_*(\mathbf{v_p})\right\| = \frac{2\|\mathbf{v_p}\|}{1+\|\mathbf{p}\|^2}. \tag{12.20}$$

for all $\mathbf{p}$ and all tangent vectors $\mathbf{v_p}$ to $\mathbb{R}^2$ at $\mathbf{p}$. Hence $\boldsymbol{\Upsilon}$ is conformal with scale factor $\lambda(\mathbf{p}) = 2/(1+\|\mathbf{p}\|^2)$. ∎

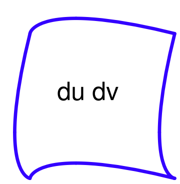
12.4 The Intuitive Idea of Area

In $\mathbb{R}^n$, the infinitesimal hypercube bounded by $dx_1,\ldots,dx_n$ has as its volume the product

$$dV = dx_1 dx_2\cdots dx_n.$$

We call dV the ***infinitesimal volume element*** of $\mathbb{R}^n$. The corresponding concept for a surface with metric (12.3), page 362, is the ***element of area*** dA, given by

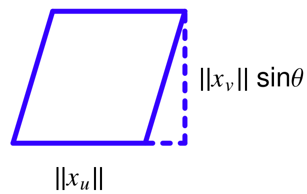
$$dA = \sqrt{EG-F^2}\,dudv \tag{12.21}$$



For a patch $\mathbf{x}\colon\mathcal{U}\to\mathbb{R}^3$, it is easy to see why this is the appropriate definition. By (7.2) on page 193, we have

$$\sqrt{EG-F^2} = \|\mathbf{x}_u\times\mathbf{x}_v\| = \|\mathbf{x}_u\|\,\|\mathbf{x}_v\|\sin\theta, \tag{12.22}$$

where θ is the oriented angle from the vector $\mathbf{x}_u$ to the vector $\mathbf{x}_v$. On the other hand, the quantity $\|\mathbf{x}_u\|\,\|\mathbf{x}_v\|\sin\theta$ represents the area of the parallelogram with sides $\mathbf{x}_u, \mathbf{x}_v$ and angle θ between the sides:



It is important to realize that the expression '$dudv$' occurring in (12.21) has a very different meaning to that in the middle term $2F dudv$ of (12.3). The latter arises from the *symmetric* product $\mathbf{x}_u \cdot \mathbf{x}_v$, while equation (12.22) shows that dA is determined by an *antisymmetric* product which logically should be written $du \times dv$ or, as is customary, $du \wedge dv$, satisfying

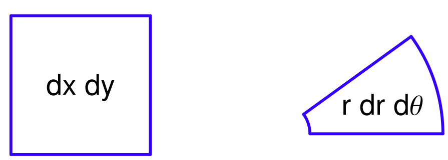
$$du \wedge dv = -dv \wedge du. \tag{12.23}$$

In fact, (12.23) is an example of a ***differential form***, an object which is subject to transformation rules to reflect (12.25) below and the associated change of variable formula in double integrals. To keep the presentation simple, we shall avoid this notation, though the theory can be developed using the approach of Section 24.6.

Let us compute the element of area for the usual metric on the plane, but in polar coordinates $u = r > 0$ and $v = \theta$. Lemma 12.3 tells us that in this case, $E = 1$, $F = 0$ and $G = r^2$. Thus,

Lemma 12.15. *The metric ds^2 on $\mathbb{R}^2$ given by* (12.2) *has as its element of area*

$$dA = dxdy = r\,dr d\theta.$$



Equation (12.21) motivates the following definition of the area of a closed subset of the trace of a patch. Recall that a subset S of $\mathbb{R}^n$ is ***bounded***, provided there exists a number M such that $\|\mathbf{p}\| \leqslant M$ for all $\mathbf{p} \in S$. A ***compact*** subset of $\mathbb{R}^n$ is a subset which is closed and bounded.

Definition 12.16. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ be an injective regular patch, and let $\mathcal{R}$ be a compact subset of $\mathbf{x}(\mathcal{U})$. Then the* ***area*** *of $\mathcal{R}$ is*

$$\mathsf{area}(\mathcal{R}) = \iint_{\mathbf{x}^{-1}(\mathcal{R})} \sqrt{EG - F^2}\, du dv. \tag{12.24}$$

That this definition is geometric is a consequence of

Lemma 12.17. *The definition of area is independent of the choice of patch.*

Proof. Let $\mathbf{x}\colon \mathcal{V} \to \mathbb{R}^n$ and $\mathbf{y}\colon \mathcal{W} \to \mathbb{R}^n$ be injective regular patches, and assume that $\mathcal{R} \subseteq \mathbf{x}(\mathcal{V}) \cap \mathbf{y}(\mathcal{W})$. A long but straightforward computation using Lemma 12.4 (and its notation) shows that

$$\sqrt{E_{\mathbf{y}}G_{\mathbf{y}} - F_{\mathbf{y}}^2} \;=\; \sqrt{E_{\mathbf{x}}G_{\mathbf{x}} - F_{\mathbf{x}}^2}\left|\frac{\partial \bar{u}}{\partial u}\frac{\partial \bar{v}}{\partial v} - \frac{\partial \bar{u}}{\partial v}\frac{\partial \bar{v}}{\partial u}\right|. \tag{12.25}$$

But the last factor is the determinant of the Jacobian matrix of $\mathbf{x}^{-1} \circ \mathbf{y}$. By the change of variables formulas for multiple integrals, we have

$$\begin{aligned}\iint_{\mathbf{y}^{-1}(\mathcal{R})} \sqrt{E_{\mathbf{y}}G_{\mathbf{y}} - F_{\mathbf{y}}^2}\, du dv \;&=\; \iint_{\mathbf{y}^{-1}(\mathcal{R})} \sqrt{E_{\mathbf{x}}G_{\mathbf{x}} - F_{\mathbf{x}}^2}\left|\frac{\partial \bar{u}}{\partial u}\frac{\partial \bar{v}}{\partial v} - \frac{\partial \bar{u}}{\partial v}\frac{\partial \bar{v}}{\partial u}\right| du dv\\ &= \iint_{\mathbf{x}^{-1}(\mathcal{R})} \sqrt{E_{\mathbf{x}}G_{\mathbf{x}} - F_{\mathbf{x}}^2}\, d\bar{u}\, d\bar{v}. \quad \blacksquare\end{aligned}$$

Surface area cannot, in general, be computed by taking the limit of the area of approximating polyhedra. For example, see [Krey1, pages 115–117]. Thus the analog for surfaces of Theorem 1.14, page 10, is false.

12.5 Examples of Metrics

The Sphere

The components of the metric and the infinitesimal area of a sphere $S^2(a)$ of radius a are easily computed. For the standard parametrization on page 288, we obtain from Notebook 12 that

$$E = a^2 \cos^2 v, \qquad F = 0, \qquad G = a^2.$$

Hence the Riemannian metric of $S^2(a)$ can be written as

$$ds^2 = a^2\big(\cos^2 v\, du^2 + dv^2\big). \tag{12.26}$$

The element of area is

$$dA = \sqrt{EG - F^2}\, du dv = a^2 \cos v\, du dv.$$

If we note that, except for one missing meridian, $\mathsf{sphere}[a]$ covers the sphere exactly once when $0 < u < 2\pi$ and $-\pi/2 < v < \pi/2$, we can compute the total area of a sphere by computer. As expected, the result is $4a\pi$.

Paraboloids

The two types of paraboloid are captured by the single definition

$$\mathsf{paraboloid}[a,b](u,v) = (u,\ v,\ au^2 + bv^2),$$

copied from page 303, where both a and b are nonzero. If these two parameters have the same sign, we obtain the ***elliptical paraboloid***, and otherwise the ***hyperbolic paraboloid***; both types are visible in Figure 12.10.

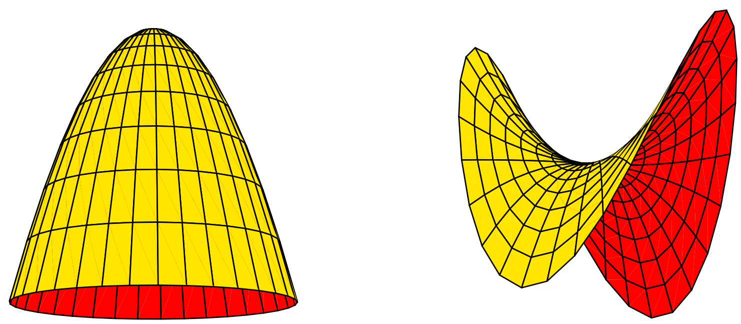


Figure 12.10: Elliptical and hyperbolic paraboloids

The metric can be computed as a function of a, b, and the result is

$$ds^2 = (1 + 4a^2u^2)du^2 + 8ab\,uv\,du\,dv + (1 + 4b^2v^2)dv^2.$$

Only the sign of $F = 4ab\,uv$ changes in passing from 'elliptical' to 'hyperbolic', as E and G are invariant. The element of area

$$dA = \sqrt{1 + 4a^2u^2 + 4b^2v^2}\,du\,dv$$

is identical in the two cases.

Cylinders

Perhaps an even simpler example is provided by the two cylinders illustrated in Figure 12.11. Whilst the first is a conventional circular cylinder, the leaning one on the right is formed by continuously translating an ellipse in a constant direction. Indeed, by a 'cylinder', we understand the surface parametrized by

$$\mathsf{cylinder}[\mathbf{d},\,\boldsymbol{\gamma}](u,v) = \boldsymbol{\gamma}(u) + v\,\mathbf{d},$$

where $\mathbf{d}$ is some fixed nonzero vector (representing the direction of the axis), and $\boldsymbol{\gamma}$ is a fixed curve. Our two cylinders are determined by the choices

$$\begin{aligned} \mathbf{d} &= (0,0,1), \qquad & \boldsymbol{\gamma}(u) &= (\cos u,\ \sin u,\ 0),\\ \mathbf{d} &= (0,1,1), \qquad & \boldsymbol{\gamma}(u) &= (\sin u,\ \tfrac{1}{4}\cos u,\ 0). \end{aligned}$$

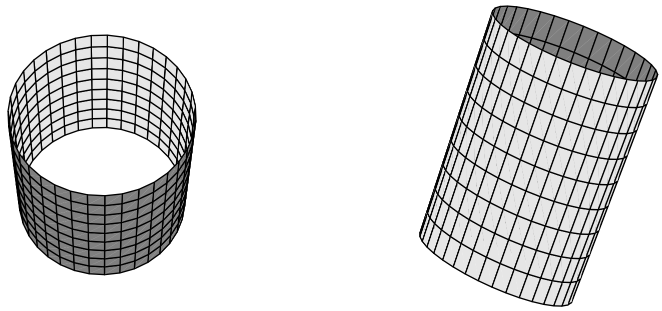


Figure 12.11: Cylinders

Whilst the circular cylinder has the Pythagorean metric $ds^2 = du^2 + dv^2$ (as in (12.2) apart from the change of variables), the elliptical cylinder has

$$ds^2 = (1 - \tfrac{15}{16}\sin^2 u)du^2 - \tfrac{1}{2}\sin u\, du\, dv + 2\, dv^2.$$

As we shall see later, this reflects the fact that no piece of the second cylinder can be deformed into the plane without distorting distance.

The Helicoid

The geometric definition of the *helicoid* is the surface generated by a line ℓ attached orthogonally to an axis m such that ℓ moves along m and also rotates, both at constant speed. The effect depends whether or not ℓ extends to both sides of m (see Figure 12.12); any point of ℓ not on the axis describes a circular helix.

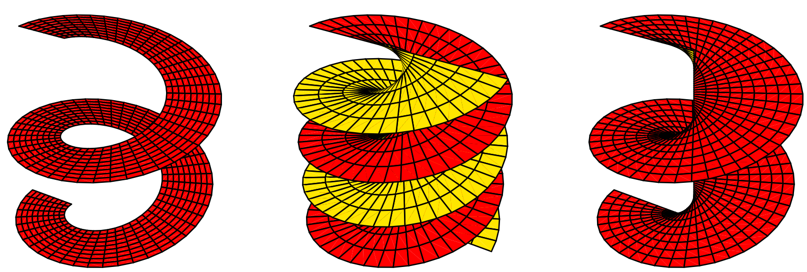


Figure 12.12: Helicoids

The helicoid surface is (in Exercise 3) explicitly parametrized by

(12.27) $$\mathsf{helicoid}[a,b](u,v) = \big(a\,v\cos u,\ a\,v\sin u,\ b\,u\big).$$

The Monkey Saddle

We can use Notebook 12 to find the metric and infinitesimal area of the monkey saddle defined on page 304. The Riemannian metric of the monkey saddle is given by

$$ds^2 = \big(1 + (3u^2 - 3v^2)^2\big)du^2 - 36uv(u^2 - v^2)du\,dv + \big(1 + 36u^2v^2\big)dv^2,$$

and a computation gives

$$dA = \sqrt{1 + 9u^4 + 18u^2v^2 + 9v^4}\,du\,dv.$$

Numerical integration can be used to find the area of (for instance) the square-like portion

$$S = \{\mathsf{monkeysaddle}(p, q) \mid -1 < p, q < 1\},$$

which is approximately 2.33.

Twisted Surfaces

Recall Definition 11.15 from the previous chapter.

Lemma 12.18. *Suppose* $\mathbf{x}$ *is a twisted surface in* $\mathbb{R}^3$ *whose profile curve is* $\boldsymbol{\alpha} = (\varphi, \psi)$. *Then*

$$\begin{aligned} E \;&=\; a^2 + 2a\varphi(v)\cos bu + \varphi(v)^2\cos^2 bu + \psi(v)^2\sin^2 bu \\ &\qquad + b^2(\varphi(v)^2 + \psi(v)^2) - 2a\psi(v)\sin bu - \varphi(v)\psi(v)\sin(2bu), \\ F \;&=\; b(-\psi\varphi' + \varphi\psi'), \\ G \;&=\; \varphi'^2 + \psi'^2. \end{aligned}$$

This can be proved computationally (see Exercise 4).

12.6 Exercises

1. Find the metric, infinitesimal area, and total area of a circular torus defined by

$$\mathsf{torus}[a, b](u, v) = \big((a + b\cos v)\cos u,\ (a + b\cos v)\sin u,\ b\sin v\big).$$

2. Fill in the details of the proof of Lemma 12.12.

M **3.** Find the metric and infinitesimal area of the helicoid parametrized by (12.27). Compute the area of the region

$$\big\{\,\mathsf{helicoid}[a, b](p, q) \mid 0 < p < 2\pi,\ c < q < d\,\big\},$$

and plot the helicoid with $a = b = 1$.

M **4.** Prove Lemma 12.18 using the program metric from Notebook 12.

5. Describe an explicit isometry between an open rectangle in the plane and a circular cylinder, in analogy to (12.13) and the analysis carried out in Section 12.2 for the circular cone.

M **6.** Determine the metric and infinitesimal area of ***Enneper's minimal surface***[2], defined by

$$\mathsf{enneper}(u,v) = \left(u - \frac{u^3}{3} + u\,v^2,\ -v + \frac{v^3}{3} - v\,u^2,\ u^2 - v^2\right).$$

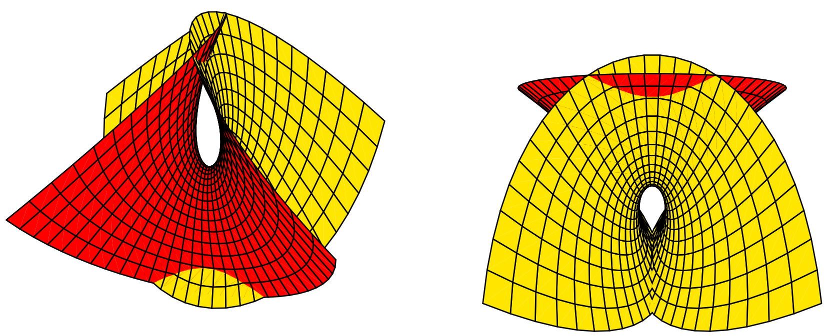


Figure 12.13: Two views of Enneper's surface

Plot Enneper's surface with viewpoints different from the ones shown in Figure 12.13. Finally, compute the area of the image under enneper of the set $[-1,1]\times[-1,1]$. (See also page 509 and [Enn1].)

M **7.** Find formulas for the components E, F, G of the metrics of the following surfaces parametrized in Chapter 10: the circular paraboloid, the eight surface and the Whitney umbrella.

M **8.** Find E, F, G for the following surfaces parametrized in Chapter 11: the Möbius strip, the Klein bottle, Steiner's Roman surface, the cross cap and the pseudo cross cap.

[2] Alfred Enneper (1830–1885). Professor at the University of Göttingen. Enneper also studied minimal surfaces and surfaces of constant negative curvature.