Chapter 2

Famous Plane Curves

Plane curves have been a subject of much interest beginning with the Greeks. Both physical and geometric problems frequently lead to curves other than ellipses, parabolas and hyperbolas. The literature on plane curves is extensive. Diocles studied the *cissoid* in connection with the classic problem of doubling the cube. Newton[1] gave a classification of cubic curves (see [Newton] or [BrKn]). Mathematicians from Fermat to Cayley often had curves named after them. In this chapter, we illustrate a number of historically-interesting plane curves.

Cycloids are discussed in Section 2.1, *lemniscates of Bernoulli* in Section 2.2 and *cardioids* in Section 2.3. Then in Section 2.4 we derive the differential equation for a *catenary*, a curve that at first sight resembles the parabola. We shall present a geometrical account of the *cissoid of Diocles* in Section 2.5, and an analysis of the *tractrix* in Section 2.6. Section 2.7 is devoted to an illustration of *clothoids*, though the significance of these curves will become

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Sir Isaac Newton (1642–1727).English mathematician, physicist, and astronomer. Newton's contributions to mathematics encompass not only his fundamental work on calculus and his discovery of the binomial theorem for negative and fractional exponents, but also substantial work in algebra, number theory, classical and analytic geometry, methods of computation and approximation, and probability. His classification of cubic curves was published as an appendix to his book on optics; his work in analytic geometry included the introduction of polar coordinates.

As Lucasian professor at Cambridge, Newton was required to lecture once a week, but his lectures were so abstruse that he frequently had no auditors. Twice elected as Member of Parliament for the University, Newton was appointed warden of the mint; his knighthood was awarded primarily for his services in that role. In *Philosophiæ Naturalis Principia Mathematica*, Newton set forth fundamental mathematical principles and then applied them to the development of a world system. This is the basis of the Newtonian physics that determined how the universe was perceived until the twentieth century work of Einstein.

clearer in Chapter 5. Finally, *pursuit curves* are discussed in Section 2.8.

There are many books on plane curves. Four excellent classical books are those of Cesàro [Ces], Gomes Teixeira[2] [Gomes], Loria[3] [Loria1], and Wieleitner [Wiel2]. In addition, Struik's book [Stru2] contains much useful information, both theoretical and historical. Modern books on curves include [Arg], [BrGi], [Law], [Lock], [Sav], [Shikin], [vonSeg], [Yates] and [Zwi].

2.1 Cycloids

The general cycloid is defined by

$$\mathsf{cycloid}[a,b](t) = \big(at - b\sin t,\ a - b\cos t\big).$$

Taking $a = b$ gives $\mathsf{cycloid}[a,a]$, which describes the locus of points traced by a point on a circle of radius a which rolls without slipping along a straight line.

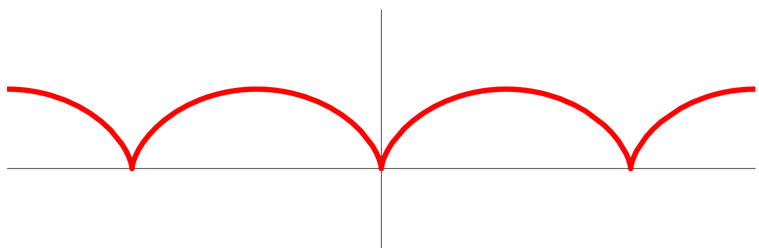

Figure 2.1: The cycloid $t \mapsto (t - \sin t,\ 1 - \cos t)$

In Figure 2.2, we plot the graph of the curvature $\boldsymbol{\kappa 2}(t)$ of $\mathsf{cycloid}[1,1](t)$ over the range $0 \leqslant t \leqslant 2\pi$. Note that the horizontal axis represents the variable t, and not the x-coordinate $t - \sin t$ of Figure 2.1. A more faithful representation of the curvature function is obtained by expressing the parameter t as a function of x. In Figure 2.3, the curvature of a given point $(x(t), y(t))$ of $\mathsf{cycloid}[1, \frac{1}{2}]$ is represented by the point $(x(t), \boldsymbol{\kappa 2}(t))$ on the graph vertically above or below it.

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2  Francisco Gomes Teixeira (1851–1933). A leading Portuguese mathematician of the last half of the 19^{th} century. There is a statue of him in Porto.

3  Gino Loria (1862–1954). Italian mathematician, professor at the University of Genoa. In addition to his books on curve theory, [Loria1] and [Loria2], Loria wrote several books on the history of mathematics.

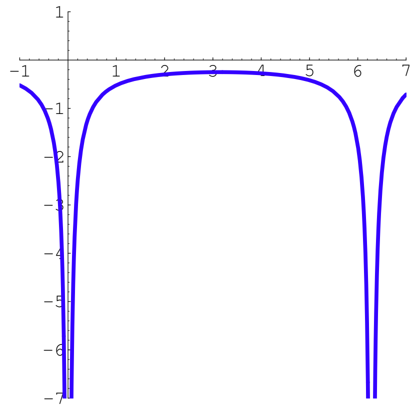

Figure 2.2: Curvature of cycloid$[1,1]$

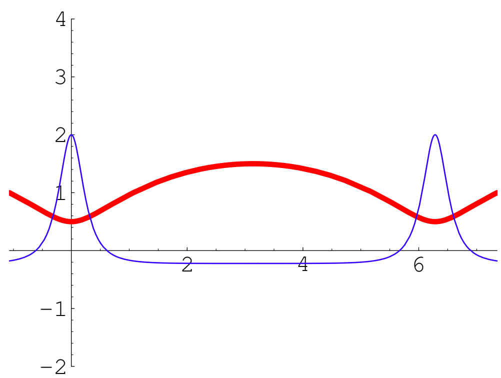

Figure 2.3: cycloid$[1,\frac{1}{2}]$ together with its curvature

Consider now the case in which a and b are not necessarily equal. The curve cycloid$[a,b]$ is that traced by a point on a circle of radius b when a concentric circle of radius a rolls without slipping along a straight line. The cycloid is ***prolate*** if $a<b$ and ***curtate*** if $a>b$.

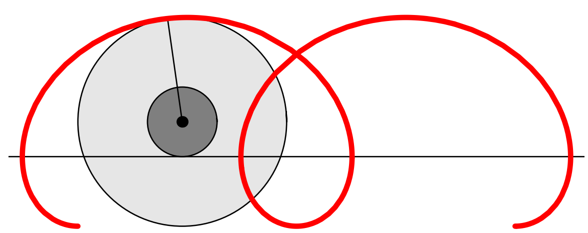

Figure 2.4: The prolate cycloid $t \mapsto (t-3\sin t,\ 1-3\cos t)$

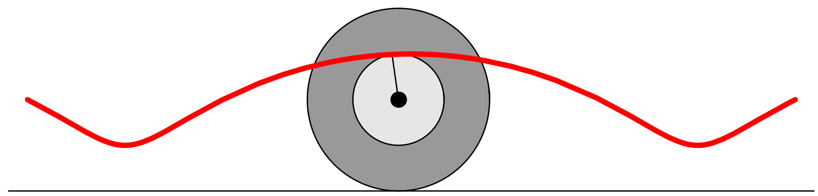

Figure 2.5: The curtate cycloid $t \mapsto (2t - \sin t,\ 2 - \cos t)$

The final figure in this section demonstrates the fact that the tangent and normal to the cycloid always intersect the vertical diameter of the generating circle on the circle itself. A discussion of this and other properties of cycloids can be found in [Lem, Chapter 4] and [Wagon, Chapter 2].

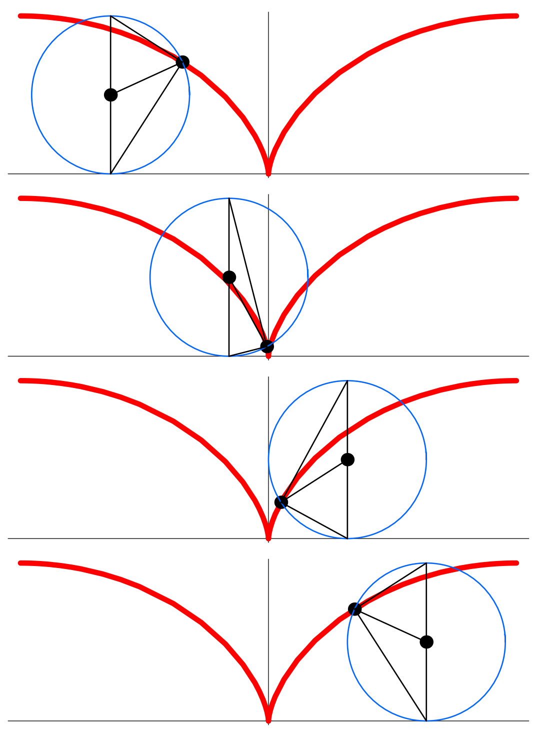

Figure 2.6: Properties of tangent and normal

Instructions for animating Figures 2.4–2.6 are given in Notebook 2.

2.2 Lemniscates of Bernoulli

Each curve in the family

$$\text{lemniscate}[a](t) = \left(\frac{a\cos t}{1+\sin^2 t}, \frac{a\sin t\cos t}{1+\sin^2 t}\right), \tag{2.1}$$

is called a ***lemniscate of Bernoulli***[4]. Like an ellipse, a lemniscate has foci F_1 and F_2, but the lemniscate is the locus $\mathscr{L}$ of points P for which the *product* of distances from F_1 and F_2 is a certain constant f^2. More precisely,

$$\mathscr{L} = \left\{\, (x,y) \mid \text{distance}\big((x,y),F_1\big)\,\text{distance}\big((x,y),F_2\big) = f^2 \,\right\},$$

where $\text{distance}(F_1,F_2) = 2f$. This choice ensures that the midpoint of the segment connecting F_1 with F_2 lies on the curve $\mathscr{L}$.

Let us derive (2.1) from the focal property. Let the foci be $(\pm f, 0)$ and let $\mathscr{L}$ be a set of points containing $(0,0)$ such that the product of the distances from $F_1 = (-f,0)$ and $F_2 = (f,0)$ is the same for all $P \in \mathscr{L}$. Write $P=(x,y)$. Then the condition that P lie on $\mathscr{L}$ is

$$\big((x-f)^2+y^2\big)\big((x+f)^2+y^2\big) = f^4, \tag{2.2}$$

or equivalently

$$(x^2+y^2)^2 = 2f^2(x^2-y^2). \tag{2.3}$$

The substitutions $y = x\sin t$, $f = a/\sqrt{2}$ transform (2.3) into (2.1).

Figure 2.7 displays four lemniscates. Starting from the largest, each successively smaller one passes through the foci of the previous one. Figure 2.8 plots the curvature of one of them as a function of the x-coordinate.

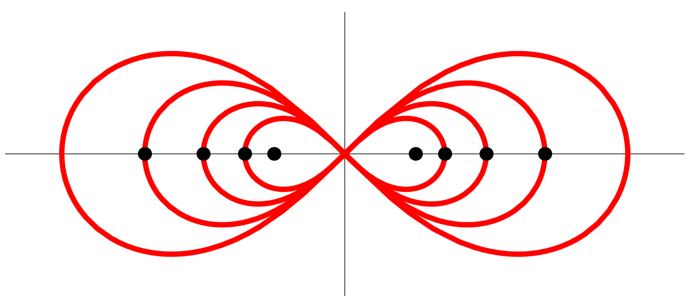

Figure 2.7: A family of lemniscates

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4  Jakob Bernoulli (1654–1705). Jakob and his brother Johann were the first of a Swiss mathematical dynasty. The work of the Bernoullis was instrumental in establishing the dominance of Leibniz's methods of exposition. Jakob Bernoulli laid basic foundations for the development of the calculus of variations, as well as working in probability, astronomy and mathematical physics. In 1694 Bernoulli studied the lemniscate named after him.

Although a lemniscate of Bernoulli resembles the figure eight curve parametrized in the simper way by

$$\mathsf{eight}[a](t) = \big(\sin t,\ \sin t\cos t\big), \tag{2.4}$$

a comparison of the graphs of their respective curvatures shows the difference between the two curves: the curvature of a lemniscate has only one maximum and one minimum in the range $0 \leqslant t < 2\pi$, whereas (2.4) has three maxima, three minima, and two inflection points.

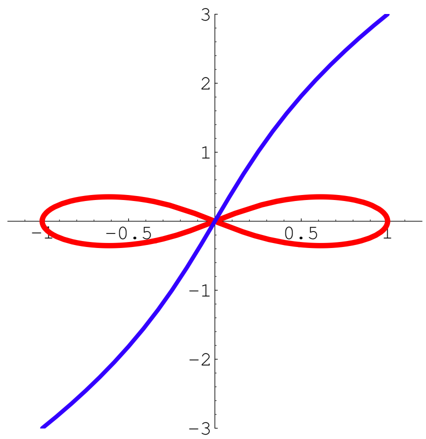

Figure 2.8: Part of a lemniscate and its curvature

In Section 6.1, we shall define the ***total signed curvature*** by integrating $\boldsymbol{\kappa 2}$ through a full turn. It is zero for both (2.1) and (2.4); for the former, this follows because the curvature graphed in Figure 2.8 is an odd function.

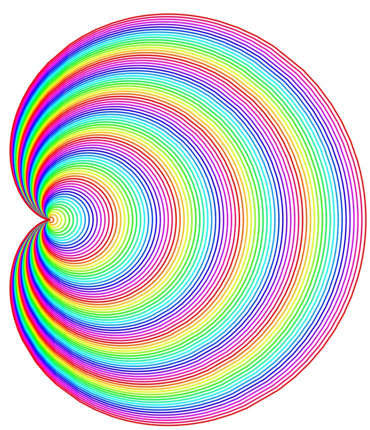

Figure 2.9: A family of cardioids

2.3 Cardioids

A ***cardioid*** is the locus of points traced out by a point on a circle of radius a which rolls without slipping on another circle of the same radius a. Its parametric equation is

$$\mathsf{cardioid}[a](t) = \big(2a(1+\cos t)\cos t,\ 2a(1+\cos t)\sin t\big).$$

The curvature of the cardioid can be simplified by hand to get

$$\boldsymbol{\kappa 2}[\mathsf{cardioid}[a]](t) = \frac{3}{8|a\cos(t/2)|}.$$

The result is illustrated below using the same method as in Figures 2.3 and 2.8. The curvature consists of two branches which meet at one of two points where the value of $\boldsymbol{\kappa 2}$ coincides with the cardioid's y-coordinate.

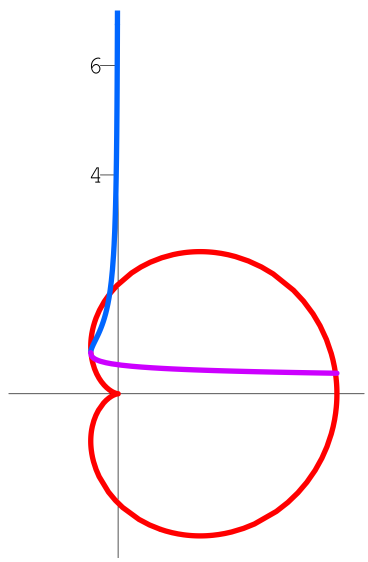

Figure 2.10: A cardioid and its curvature

2.4 The Catenary

In 1691, Jakob Bernoulli gave a solution to the problem of finding the curve assumed by a flexible inextensible cord hung freely from two fixed points; Leibniz has called such a curve a ***catenary*** (which stems from the Latin word *catena*, meaning chain). The solution is based on the differential equation

$$\frac{d^2y}{dx^2} = \frac{1}{a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}. \tag{2.5}$$

To derive (2.5), we consider a portion $\overline{\mathbf{pq}}$ of the cable between the lowest point $\mathbf{p}$ and an arbitrary point $\mathbf{q}$. Three forces act on the cable: the weight of the portion $\overline{\mathbf{pq}}$, as well as the tensions $\mathbf{T}$ and $\mathbf{U}$ at $\mathbf{p}$ and $\mathbf{q}$. If w is the linear density and s is the length of $\overline{\mathbf{pq}}$, then the weight of the portion $\overline{\mathbf{pq}}$ is ws.

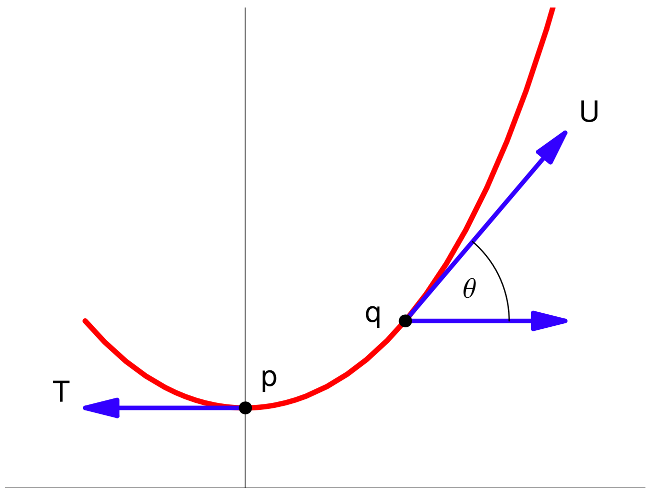

Figure 2.11: Definition of a catenary

Let $|\mathbf{T}|$ and $|\mathbf{U}|$ denote the magnitudes of the forces $\mathbf{T}$ and $\mathbf{U}$, and write

$$\mathbf{U} = \big(|\mathbf{U}|\cos\theta,\ |\mathbf{U}|\sin\theta\big),$$

with θ the angle shown in Figure 2.11. Because of equilibrium we have

$$|\mathbf{T}| = |\mathbf{U}|\cos\theta \qquad \text{and} \qquad w\,s = |\mathbf{U}|\sin\theta. \tag{2.6}$$

Let $\mathbf{q} = (x, y)$, where x and y are functions of s. From (2.6) we obtain

$$\frac{dy}{dx} = \tan\theta = \frac{w\,s}{|\mathbf{T}|}. \tag{2.7}$$

Since the length of $\overline{\mathbf{pq}}$ is

$$s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx,$$

the fundamental theorem of calculus tells us that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{2.8}$$

When we differentiate (2.7) and use (2.8), we get (2.5) with $a = \omega/|T|$.

Although at first glance the catenary looks like a parabola, it is in fact the graph of the hyperbolic cosine. A solution of (2.5) is given by

$$y = a \cosh \frac{x}{a}. \tag{2.9}$$

The next figure compares a catenary and a parabola having the same curvature at $x = 0$. The reader may need to refer to Notebook 2 to see which is which.

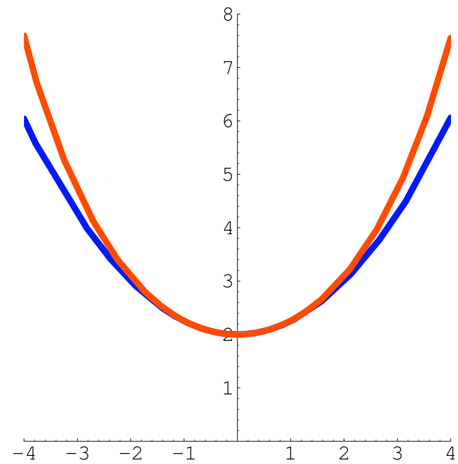

Figure 2.12: Catenary and parabola

We rotate the graph of (2.9) to define

$$\text{catenary}[a](t) = \left(a \cosh \frac{t}{a},\ t \right), \tag{2.10}$$

where without loss of generality, we assume that $a > 0$. A catenary is one of the few curves for which it is easy to compute the arc length function. Indeed,

$$|\text{catenary}[a]'(t)| = \cosh \frac{t}{a},$$

and it follows that a unit-speed parametrization of a catenary is given by

$$s \mapsto a \left(\sqrt{1 + \frac{s^2}{a^2}},\ \operatorname{arcsinh} \frac{s}{a} \right).$$

In Chapter 15, we shall use the catenary to construct an important minimal surface called the ***catenoid***.

2.5 The Cissoid of Diocles

The ***cissoid of Diocles*** is the curve defined nonparametrically by

$$x^3 + x y^2 - 2 a y^2 = 0. \tag{2.11}$$

To find a parametrization of the cissoid, we substitute $y = xt$ in (2.11) and obtain

$$x = \frac{2at^2}{1+t^2}, \qquad y = \frac{2at^3}{1+t^2}.$$

Thus we define

$$\mathsf{cissoid}[a](t) = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right). \tag{2.12}$$

The Greeks used the cissoid of Diocles to try to find solutions to the problems of doubling a cube and trisecting an angle. For more historical information and the definitions used by the Greeks and Newton see [BrKn, pages 9–12], [Gomes, volume 1, pages 1–25] and [Lock, pages 130–133]. Cissoid means 'ivy-shaped'. Observe that $\mathsf{cissoid}[a]'(0) = 0$ so that cissoid is not regular at 0. In fact, a cissoid has a cusp at 0, as can be seen in Figure 2.14.

The definition of the cissoid used by the Greeks and by Newton can best be explained by considering a generalization of the cissoid. Let $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ be two curves and A a fixed point. Draw a line ℓ through A cutting $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ at points Q and R respectively, and find a point P on ℓ such that the distance from A to P equals the distance from Q to R. The locus of such points P is called the ***cissoid of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ with respect to*** A.

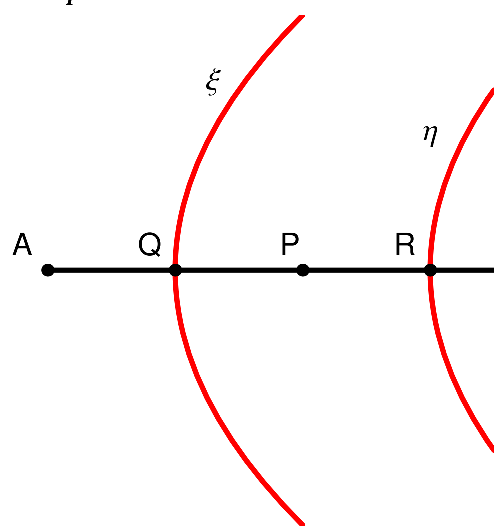

Figure 2.13: The cissoid of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ with respect to the point A

Then $\mathsf{cissoid}[a]$ is precisely the cissoid of a circle of radius a and one of its tangent lines with respect to the point diametrically opposite to the tangent line, as in Figure 2.14. Let us derive (2.11). Consider a circle of radius a centered at $(a, 0)$. Let (x, y) be the coordinates of a point P on the cissoid. Then $2a = \mathsf{distance}(A, S)$, so by the Pythagorean theorem we have

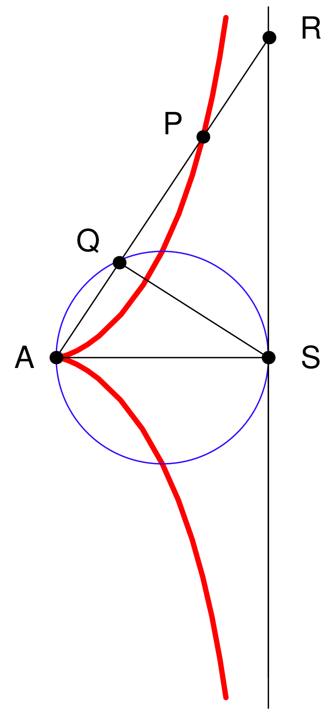

Figure 2.14: AR moves so that $\mathsf{distance}(A,P) = \mathsf{distance}(Q,R)$

$$\begin{aligned} \mathsf{distance}(Q,S)^2 &= \mathsf{distance}(A,S)^2 - \mathsf{distance}(A,Q)^2 \\ &= 4a^2 - \big(\mathsf{distance}(A,R) - \mathsf{distance}(Q,R)\big)^2. \end{aligned} \tag{2.13}$$

The definition of the cissoid says that

$$\mathsf{distance}(Q,R) = \mathsf{distance}(A,P) = \sqrt{x^2+y^2}.$$

By similar triangles, $\mathsf{distance}(A,R)/(2a) = \mathsf{distance}(A,P)/x$. Therefore, (2.13) becomes

$$\mathsf{distance}(Q,S)^2 = 4a^2 - \left(\frac{2a}{x} - 1\right)^2 (x^2+y^2). \tag{2.14}$$

On the other hand,

$$\begin{aligned} \mathsf{distance}(Q,S)^2 &= \mathsf{distance}(R,S)^2 - \mathsf{distance}(Q,R)^2 \\ &= \mathsf{distance}(R,S)^2 - \mathsf{distance}(A,P)^2 \\ &= \left(\frac{2a\,y}{x}\right)^2 - (x^2+y^2), \end{aligned} \tag{2.15}$$

since by similar triangles, $\mathsf{distance}(R,S)/(2a) = y/x$. Then (2.11) follows easily by equating the right-hand sides of (2.14) and (2.15).

From Notebook 2, the curvature of the cissoid is given by

$$\boldsymbol{\kappa 2}[\mathsf{cissoid}[a]](t) = \frac{3}{a|t|(4+t^2)^{3/2}},$$

and is everywhere strictly positive.

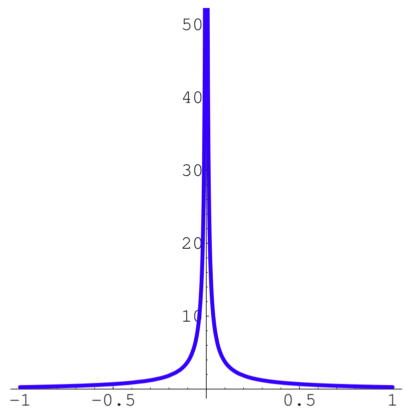

Figure 2.15: Curvature of the cissoid

2.6 The Tractrix

A ***tractrix*** is a curve $\boldsymbol{\alpha}$ passing through the point $A = (a, 0)$ on the horizontal axis with the property that the length of the segment of the tangent line from any point on the curve to the vertical axis is constant, as shown in Figure 2.16.

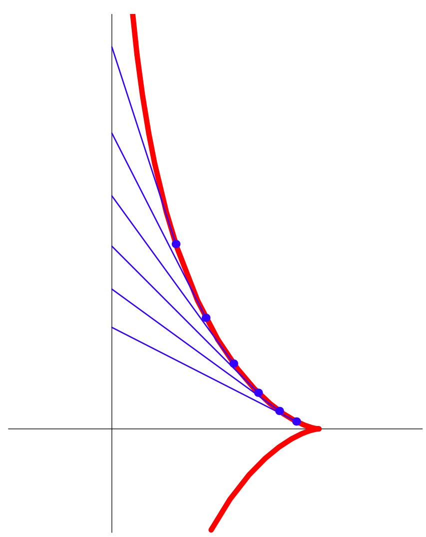

Figure 2.16: Tangent segments have equal lengths

The German word for tractrix is the more descriptive *Hundekurve*. It is the path that an obstinate dog takes when his master walks along a north-south path. One way to parametrize the curve is by means of

$$\text{tractrix}[a](t) = a\Big(\sin t,\ \cos t + \log\big(\tan\frac{t}{2}\big)\Big), \tag{2.16}$$

It approaches the vertical axis asymptotically as $t \to 0$ or $t \to \pi$, and has a cusp at $t = \pi/2$.

To find the differential equation of a tractrix, write $\text{tractrix}[a](t) = (x(t), y(t))$. Then dy/dx is the slope of the curve, and the differential equation is therefore

$$\frac{dy}{dx} = -\frac{\sqrt{a^2 - x^2}}{x}. \tag{2.17}$$

It can be checked (with the help of (2.19) overleaf) that the differential equation is satisfied by the components $x(t) = a\sin t$ and $y(t) = a\big(\cos t + \log(\tan(t/2))\big)$ of (2.16), and that both sides of (2.17) equal $\cot t$.

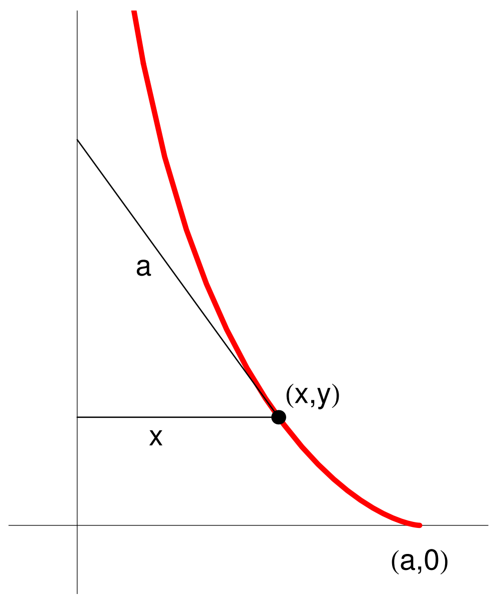

Figure 2.17: Finding the differential equation

The curvature of the tractrix is

$$\boldsymbol{\kappa 2}[\text{tractrix}[1]](t) = -|\tan t\,|.$$

In particular, it is everywhere negative and approaches $-\infty$ at the cusp. Figure 2.18 graphs $\boldsymbol{\kappa 2}$ as a function of the x-coordinate $\sin t$, so that the curvature at a given point on the tractrix is found by referring to the point on the curvature graph vertically below.

For future use let us record

Lemma 2.1. *A unit-speed parametrization of the tractrix is given by*

$$\boldsymbol{\alpha}(s) = \begin{cases} \left(ae^{-s/a},\ \displaystyle\int_0^s \sqrt{1-e^{-2t/a}}\,dt\right) & \text{for } 0 \leqslant s < \infty, \\ \left(ae^{s/a},\ \displaystyle\int_0^s \sqrt{1-e^{2t/a}}\,dt\right) & \text{for } -\infty < s \leqslant 0. \end{cases} \tag{2.18}$$

Note that

$$\int_0^s \sqrt{1-e^{-2t/a}}\,dt = a\,\operatorname{arctanh}\left(\sqrt{1-e^{-2s/a}}\right) - a\sqrt{1-e^{-2s/a}}.$$

Proof. First, we compute

$$\mathsf{tractrix}[a]'(\phi) = a\left(\cos\phi,\ -\sin\phi + \frac{1}{\sin\phi}\right). \tag{2.19}$$

Define $\phi(s) = \pi - \arcsin(e^{-s/a})$ for $s \geqslant 0$. Then $\sin\phi(s) = e^{-s/a}$; furthermore, $\pi/2 \leqslant \phi(s) < \pi$ for $s \geqslant 0$, so that $\cos\phi(s) = -\sqrt{1-e^{-2s/a}}$. Hence

$$\phi'(s) = \frac{e^{-s/a}}{a\sqrt{1-e^{-2s/a}}} = -\frac{\sin\phi(s)}{a\cos\phi(s)}. \tag{2.20}$$

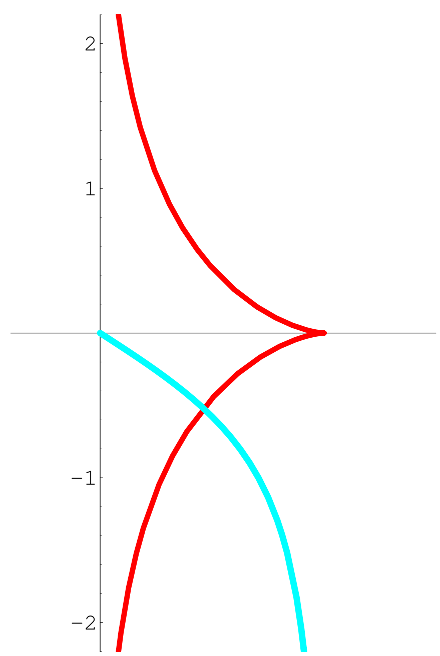

Figure 2.18: A tractrix and its curvature

Therefore, if we define a curve $\boldsymbol{\beta}$ by $\boldsymbol{\beta}(s) = \mathsf{tractrix}[a]\big(\phi(s)\big)$, it follows from (2.19) and (2.20) that

$$\begin{aligned}
\boldsymbol{\beta}'(s) &= \mathsf{tractrix}[a]'\big(\phi(s)\big)\phi'(s) \\
&= a\left(\cos\phi(s), -\sin\phi(s) + \frac{1}{\sin\phi(s)}\right)\left(-\frac{\sin\phi(s)}{a\cos\phi(s)}\right) \\
&= \big(-\sin\phi(s), -\cos\phi(s)\big) = \left(-e^{-s/a}, \sqrt{1-e^{-2s/a}}\right) = \boldsymbol{\alpha}'(s).
\end{aligned}$$

Also, $\boldsymbol{\beta}(0) = (a, 0) = \boldsymbol{\alpha}(0)$. Thus $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ coincide for $0 \leqslant s < \infty$, so that $\boldsymbol{\alpha}$ is a reparametrization of a tractrix in that range. The proof that $\boldsymbol{\alpha}$ is a reparametrization of $\mathsf{tractrix}[a]$ for $-\infty < s \leqslant 0$ is similar. Finally, an easy calculation shows that $\boldsymbol{\alpha}$ has unit speed. ▮

2.7 Clothoids

One of the most elegant of all plane curves is the ***clothoid*** or ***spiral of Cornu***[5]. We give a generalization of the clothoid by defining

$$\mathsf{clothoid}[n, a](t) = a\left(\int_0^t \sin\left(\frac{u^{n+1}}{n+1}\right)du,\ \int_0^t \cos\left(\frac{u^{n+1}}{n+1}\right)du\right).$$

Clothoids are important curves used in freeway and railroad construction (see [Higg] and [Roth]). For example, a clothoid is needed to make the gradual transition from a highway, which has zero curvature, to the midpoint of a freeway exit, which has nonzero curvature. A clothoid is clearly preferable to a path consisting of straight lines and circles, for which the curvature is discontinuous.

The standard clothoid is $\mathsf{clothoid}[1, a]$, represented by the larger one of two plotted in Figure 2.19. The quantities

$$\int_0^t \sin(\pi u^2/2)\,du \qquad \text{and} \qquad \int_0^t \cos(\pi u^2/2)\,du,$$

are called ***Fresnel***[6] ***integrals***; $\mathsf{clothoid}[1, a]$ is expressible in terms of them. Since

$$\int_0^{\pm\infty} \sin(u^2/2)du = \int_0^{\pm\infty} \cos(u^2/2)du = \pm\frac{\sqrt{\pi}}{2}$$

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[5] Marie Alfred Cornu (1841–1902). French scientist, who studied the clothoid in connection with diffraction. The clothoid was also known to Euler and Jakob Bernoulli. See [Gomes, volume 2, page 102–107] and [Law, page 190].

[6]  Augustin Jean Fresnel (1788–1827). French physicist, one of the founders of the wave theory of light.

(as is easily checked by computer), the ends of the $\mathsf{clothoid}[1,a]$ curl around the points $\pm\frac{1}{2}a\sqrt{\pi}(1,1)$. The first clothoid is symmetric with respect to the origin, but the second one (smaller in Figure 2.19) is symmetric with respect to the horizontal axis. The odd clothoids have shapes similar to $\mathsf{clothoid}[1,a]$, while the even clothoids have shapes similar to $\mathsf{clothoid}[2,a]$.

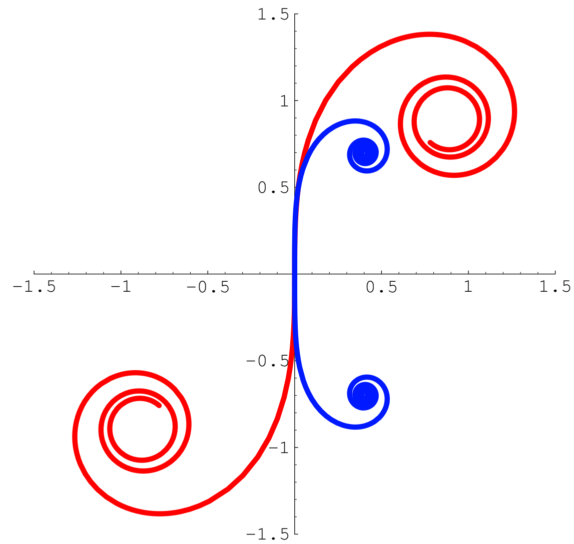

Figure 2.19: $\mathsf{clothoid}[1,1]$ and $\mathsf{clothoid}[2,\frac{1}{2}]$

Although the definition of $\mathsf{clothoid}[n,a]$ is quite complicated, its curvature is simple:

$$\boldsymbol{\kappa 2}[\mathsf{clothoid}[n,a]](t) = -\frac{t^n}{a}. \tag{2.21}$$

In Chapter 5, we shall show how to define the clothoid as a numerical solution to a differential equation arising from (2.21).

2.8 Pursuit Curves

The problem of pursuit probably originated with Leonardo da Vinci. It is to find the curve by which a vessel moves while pursuing another vessel, supposing that the speeds of the two vessels are always in the same ratio. Let us formulate this problem mathematically.

Definition 2.2. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be plane curves parametrized on an interval $[a,b]$. We say that ***$\boldsymbol{\alpha}$ is a pursuit curve of $\boldsymbol{\beta}$*** *provided that*

(i) *the velocity vector $\boldsymbol{\alpha}'(t)$ points towards the point $\boldsymbol{\beta}(t)$ for $a < t < b$; that is, $\boldsymbol{\alpha}'(t)$ is a multiple of $\boldsymbol{\alpha}(t) - \boldsymbol{\beta}(t)$;*

(ii) *the speeds of* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *are related by* $\|\boldsymbol{\alpha}'\| = k\|\boldsymbol{\beta}'\|$, *where* k *is a positive constant. We call* k *the* ***speed ratio***.

A ***capture point*** *is a point* $\mathbf{p}$ *for which* $\mathbf{p} = \boldsymbol{\alpha}(t_1) = \boldsymbol{\beta}(t_1)$ *for some* t_1.

In Figure 2.20, $\boldsymbol{\alpha}$ is the curve of the pursuer and $\boldsymbol{\beta}$ the curve of the pursued.

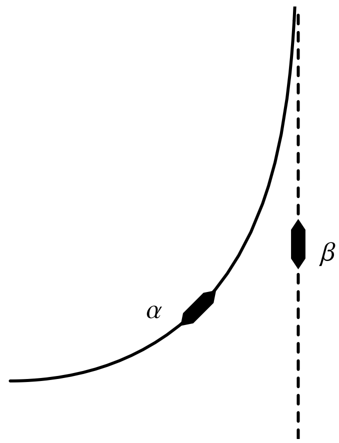

Figure 2.20: A pursuit curve

When the speed ratio k is larger than 1, the pursuer travels faster than the pursued. Although this would usually be the case in a physical situation, it is not a necessary assumption for the mathematical analysis of the problem.

We derive differential equations for pursuit curves in terms of coordinates.

Lemma 2.3. *Write* $\boldsymbol{\alpha} = (x, y)$ *and* $\boldsymbol{\beta} = (f, g)$, *and assume that* $\boldsymbol{\alpha}$ *is a pursuit curve of* $\boldsymbol{\beta}$. *Then*

$$x'^2 + y'^2 = k^2(f'^2 + g'^2) \tag{2.22}$$

and

$$x'(y - g) - y'(x - f) = 0. \tag{2.23}$$

Proof. Equation (2.22) is the same as $\|\boldsymbol{\alpha}'\| = k\|\boldsymbol{\beta}'\|$. To prove (2.23), we observe that $\boldsymbol{\alpha}(t) - \boldsymbol{\beta}(t) = \big(x(t) - f(t), y(t) - g(t)\big)$ and $\boldsymbol{\alpha}'(t) = \big(x'(t), y'(t)\big)$. Note that the vector $J\big(\boldsymbol{\alpha}(t) - \boldsymbol{\beta}(t)\big) = \big(-y(t) + g(t), x(t) - f(t)\big)$ is perpendicular to $\boldsymbol{\alpha}(t) - \boldsymbol{\beta}(t)$. The condition that $\boldsymbol{\alpha}'(t)$ is a multiple of $\boldsymbol{\alpha}(t) - \boldsymbol{\beta}(t)$ is conveniently expressed by saying that $\boldsymbol{\alpha}'(t)$ is perpendicular to $J\big(\boldsymbol{\alpha}(t) - \boldsymbol{\beta}(t)\big)$, which is equivalent to (2.23). ∎

Next, we specialize to the case when the curve of the pursued is a straight line. Assume that the curve $\boldsymbol{\beta}$ of the pursued is a vertical straight line passing through the point $(a, 0)$, and that the speed ratio k is larger than 1. We want to find the curve $\boldsymbol{\alpha}$ of the pursuer, assuming the initial conditions $\boldsymbol{\alpha}(0) = (0, 0)$ and $\boldsymbol{\alpha}'(0) = (1, 0)$.

We can parametrize $\boldsymbol{\beta}$ as

$$\boldsymbol{\beta}(t) = \bigl(a,\ g(t)\bigr).$$

Furthermore, the curve $\boldsymbol{\alpha}$ of the pursuer can be parametrized as

$$\boldsymbol{\alpha}(t) = \bigl(t,\ y(t)\bigr).$$

The condition (2.22) becomes

$$1 + y'^2 = k^2 g'^2, \tag{2.24}$$

and (2.23) reduces to

$$(y - g) - y'(t - a) = 0.$$

Differentiation with respect to t yields

$$-y''(t - a) = g'. \tag{2.25}$$

From (2.24) and (2.25) we get

$$1 + y'^2 = k^2(a - t)^2 y''^2. \tag{2.26}$$

Let $p = y'$; then (2.26) can be rewritten as

$$\frac{k\,dp}{\sqrt{1+p^2}} = \frac{dt}{a-t}.$$

This separable first-order equation has the solution

$$\operatorname{arcsinh} p = -\frac{1}{k}\log\left(\frac{a-t}{a}\right), \tag{2.27}$$

when we make use of the initial condition $y'(0) = 0$. Then (2.27) can be rewritten as

$$\begin{aligned} y' \;=\; p \;=\; \sinh\bigl(\operatorname{arcsinh} p\bigr) \;&=\; \tfrac{1}{2}\left(e^{\operatorname{arcsinh} p} - e^{-\operatorname{arcsinh} p}\right) \\ &=\; \frac{1}{2}\left(\left(\frac{a-t}{a}\right)^{-1/k} - \left(\frac{a-t}{a}\right)^{1/k}\right). \end{aligned}$$

Integrating, with the initial condition $y(0) = 0$, yields

$$y = \frac{a\,k}{k^2 - 1} + \frac{1}{2}\left(\frac{a\,k}{k+1}\left(\frac{a-t}{a}\right)^{1+1/k} - \frac{a\,k}{k-1}\left(\frac{a-t}{a}\right)^{1-1/k}\right). \tag{2.28}$$

The curve of the pursuer is then $\boldsymbol{\alpha}(t) = (t,\ y(t))$, where y is given by (2.28). Since $\boldsymbol{\alpha}(t_1) = \boldsymbol{\beta}(t_1)$ if and only if $t_1 = a$, the capture point is

$$\mathbf{p} = \left(a,\ \frac{a\,k}{k^2 - 1}\right). \tag{2.29}$$

The graph below depicts the case when $a = 1$ and k has the values $2, 3, 4, 5$. As the speed ratio k becomes smaller and smaller, the capture point goes higher and higher.

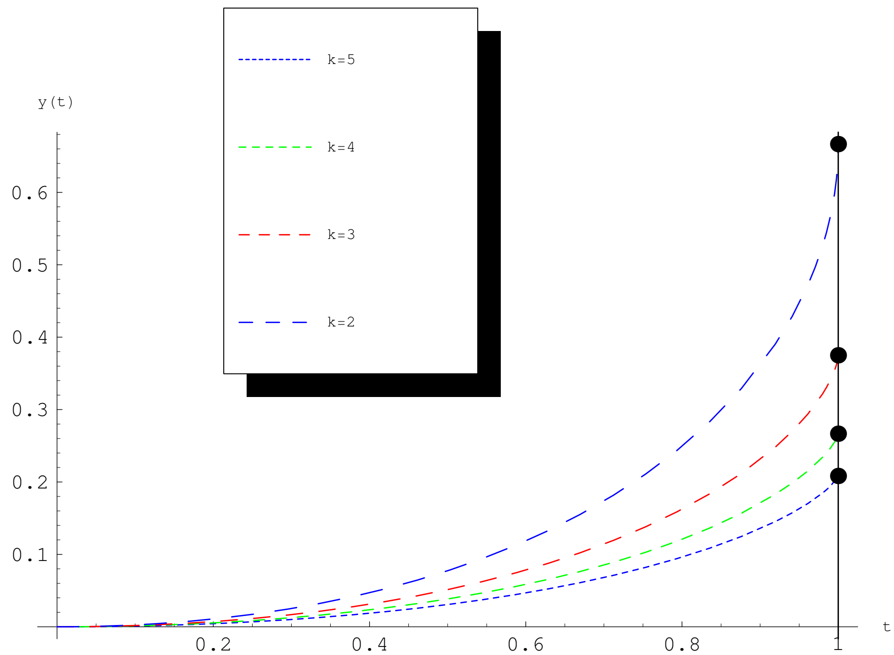

Figure 2.21: The case in which the pursued moves in a straight line

2.9 Exercises

M **1.** Graph the curvatures of the cycloids illustrated in Figures 2.4 and 2.5. Find the formula for the curvature $\boldsymbol{\kappa}\mathbf{2}$ of the general cycloid $\mathsf{cycloid}[a, b]$. Then define and draw ordinary, prolate and curtate cycloids together with the defining circle such as those on page 42.

M **2.** A ***deltoid*** is defined by

$$\mathsf{deltoid}[a](t) = \big(2a\cos t(1 + \cos t) - a,\ 2a\sin t(1 - \cos t)\big).$$

The curve is so named because it resembles a Greek capital delta. It is a particular case of a curve called ***hypocycloid*** (see Exercise 13 of Chapter 6). Plot as one graph the deltoids $\mathsf{deltoid}[a]$ for $a = 1, 2, 3, 4$. Graph the curvature of the first deltoid.

M **3.** The ***Lissajous***[7] or ***Bowditch curve***[8] is defined by

$$\mathsf{lissajous}[n, d, a, b](t) = \big(a\sin(n\,t + d),\ b\sin t\big).$$

Draw several of these curves and plot their curvatures. (One is shown in Figure 11.19 on page 349.)

M **4.** The ***limaçon***, sometimes called ***Pascal's snail***, named after Étienne Pascal, father of Blaise Pascal[9], is a generalization of the cardioid. It is defined by

$$\mathsf{limacon}[a, b](t) = (2a\cos t + b)\big(\cos t,\ \sin t\big).$$

Find the formula for the curvature of the limaçon, and plot several of them.

5. Consider a circle with center $C = (0, a)$ and radius a. Let ℓ be the line tangent to the circle at $(0, 2a)$. A line from the origin $O = (0, 0)$ intersecting ℓ at a point A intersects the circle at a point Q. Let x be the first coordinate of A and y the second coordinate of Q, and put $P = (x, y)$. As A varies along ℓ the point P traces out a curve called ***versiera***, in Italian and misnamed in English as the ***witch of Agnesi***[10]. Verify that a parametrization of the Agnesi versiera is

$$\mathsf{agnesi}[a](t) = \big(2a\tan t,\ 2a\cos^2 t\big)\,.$$

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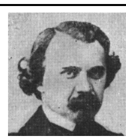
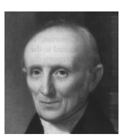
7 

Jules Antoine Lissajous (1822–1880). French physicist, who studied similar curves in 1857 in connection with his optical method for studying vibrations.

8 

Nathaniel Bowditch curve (1773–1838). American mathematician and astronomer. His *New American Practical Navigator*, written in 1802, was highly successful. Bowditch also translated and annotated *Laplace's Mécanique Céleste.* His study of pendulums in 1815 included the figures named after him. Preferring his post as president of the Essex Fire and Marine Insurance Company from 1804 to 1823, Bowditch refused chairs of mathematics at several universities.

9 

Blaise Pascal (1623–1662). French mathematician, philosopher and inventor. Pascal was an early investigator in projective geometry and invented the first mechanical device for performing addition and subtraction.

10 

Maria Gaetana Agnesi (1718–1799). Professor at the University of Bologna. She was the first woman to occupy a chair of mathematics. Her widely used calculus book *Instituzioni Analitiche* was translated into French and English.

M **6.** Define the curve

$$\mathsf{tschirnhausen}[n,a](t) = \left(a\frac{\cos t}{(\cos(t/3))^n},\ a\frac{\sin t}{(\cos(t/3))^n}\right).$$

When $n = 1$, this curve is attributed to Tschirnhausen[11]. Find the formula for the curvature of $\mathsf{tschirnhausen}[n,a][t]$ and make a simultaneous plot of the curves for $1 \leqslant n \leqslant 8$.

7. In the special case that the speed ratio is 1, show that the equation for the pursuit curve is

$$y(t) = \frac{a}{4}\left(\left(\frac{a-t}{a}\right)^2 - 1 - 2\log\left(\frac{a-t}{a}\right)\right),$$

and that the pursuer never catches the pursued.

M **8.** Equation (2.28) defines the function

$$y(t) = \frac{ak}{k^2-1} + \frac{k(a-t)^{1+1/k}}{2a^{1/k}(1+k)} - \frac{a^{1/k}k(a-t)^{1-1/k}}{2(k-1)}.$$

Plot a pursuit curve with $a = 1$ and $k = 1.2$.

[11]Ehrenfried Walter Tschirnhausen (1651–1708). German mathematician, who tried to solve equations of any degree by removing all terms except the first and last. He contributed to the rediscovery of the process for making hard-paste porcelain. Sometimes the name is written *von Tschirnhaus.*