Chapter 8

Construction of Space Curves

The analog of the Fundamental Theorem of Plane Curves (Theorem 5.13) is the *Fundamental Theorem of Space Curves*. The uniqueness part of this theorem, proved in Section 8.1, states that two curves with the same torsion and positive curvature differ only by a Euclidean motion of $\mathbb{R}^3$. The torsion of a space curve is not defined at a point where the curvature is zero, so it is not unreasonable to require as a hypothesis that the curvature never vanish. But this requirement turns out to be essential, in the light of a counter-example that we present (see Figure 8.1).

In Section 8.2, we prove the existence part, namely that space curves exist with prescribed torsion and positive curvature. We then describe some of the examples with specified curvature and torsion that were generated by the associated computer programs in Notebook 8. Closed curves can be obtained by imposing constant curvature and periodic torsion (see Figure 8.14 on page 253).

The notion of contact between two curves or between a curve and a surface is defined in Section 8.3. This part of the chapter is also relevant to the theory of plane curves, although it leads eventually to the definition of the *evolute* of a space curve. The evolute is traced out by the centers of spheres having third order contact with the given space curve at each of its points.

Curves that lie on spheres are characterized in Section 8.4, and they furnish examples for the rest of the chapter. Curves of constant slope are generalizations of a helix, and such a curve $\boldsymbol{\gamma}$ can be constructed so as to project to a given plane curve $\boldsymbol{\beta}$. We see in Section 8.5 that the assumption that $\boldsymbol{\gamma}$ lie on a sphere requires $\boldsymbol{\beta}$ to be an epicycloid. Loxodromes on spheres are defined, discussed and plotted in Section 8.6.

8.1 The Fundamental Theorem of Space Curves

The curvature and torsion of a space curve determine the curve in very much the same way as the signed curvature $\boldsymbol{\kappa 2}$ determines a plane curve. First, we establish the invariance of curvature and torsion under Euclidean motions of $\mathbb{R}^3$. For that we need a fact about the vector triple product, defined in Section 7.1.

Lemma 8.1. *Let $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map and $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. Then*

$$[(A\mathbf{a})\,(A\mathbf{b})\,(A\mathbf{c})] = \det(A)[\mathbf{a}\,\mathbf{b}\,\mathbf{c}].$$

Proof. If we treat $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as column vectors then we may regard A as a matrix premultiplying the vectors. Let $(\mathbf{a}|\mathbf{b}|\mathbf{c})$ denote the 3×3 matrix whose columns are $\mathbf{a}, \mathbf{b}, \mathbf{c}$. With this notation,

$$\begin{aligned}[(A\mathbf{a})\,(A\mathbf{b})\,(A\mathbf{c})] = A\mathbf{a} \cdot (A\mathbf{b} \times A\mathbf{c}) &= \det(A\mathbf{a}|A\mathbf{b}|A\mathbf{c}) \\ &= \det\left(A(\mathbf{a}|\mathbf{b}|\mathbf{c})\right) = \det(A)[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]. \quad \blacksquare\end{aligned}$$

Now we can determine the effect of a Euclidean motion on arc length, curvature and torsion.

Theorem 8.2. *The curvature and the absolute value of the torsion are invariant under Euclidean motions of $\mathbb{R}^3$, though the torsion changes sign under an orientation-reversing Euclidean motion.*

Proof. Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^3$ be a curve, and let $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a Euclidean motion. Denote by A the linear part of F, so that for all $\mathbf{p} \in \mathbb{R}^3$ we have $F(\mathbf{p}) = A\mathbf{p} + F(0)$. We define a curve $\boldsymbol{\gamma}\colon (a,b) \to \mathbb{R}^3$ by $\boldsymbol{\gamma} = F \circ \boldsymbol{\alpha}$; then for $a < t < b$ we have

$$\boldsymbol{\gamma}(t) = A\boldsymbol{\alpha}(t) + F(0).$$

Hence $\boldsymbol{\gamma}'(t) = A\boldsymbol{\alpha}'(t)$, $\boldsymbol{\gamma}''(t) = A\boldsymbol{\alpha}''(t)$ and $\boldsymbol{\gamma}'''(t) = A\boldsymbol{\alpha}'''(t)$. Let s_α and s_γ denote the arc length functions with respect to $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$. Since A is an orthogonal transformation, we have

$$s_\gamma'(t) = \left\|\boldsymbol{\gamma}'(t)\right\| = \left\|A\boldsymbol{\alpha}'(t)\right\| = \left\|\boldsymbol{\alpha}'(t)\right\| = s_\alpha'(t). \tag{8.1}$$

We compute the curvature of $\boldsymbol{\gamma}$, making use of the Lagrange identity (7.2) on page 193:

$$\begin{aligned}\boldsymbol{\kappa}[\boldsymbol{\gamma}](t) &= \frac{\left\|\boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t)\right\|}{\left\|\boldsymbol{\gamma}'(t)\right\|^3} = \frac{\left\|A\boldsymbol{\alpha}'(t) \times A\boldsymbol{\alpha}''(t)\right\|}{\left\|A\boldsymbol{\alpha}'(t)\right\|^3} \\ &= \frac{\sqrt{\left\|A\boldsymbol{\alpha}'(t)\right\|^2\left\|A\boldsymbol{\alpha}''(t)\right\|^2 - \left(A\boldsymbol{\alpha}'(t) \cdot A\boldsymbol{\alpha}''(t)\right)^2}}{\left\|A\boldsymbol{\alpha}'(t)\right\|^3}\end{aligned}$$

$$= \frac{\sqrt{\|\boldsymbol{\alpha}'(t)\|^2\|\boldsymbol{\alpha}''(t)\|^2 - \big(\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}''(t)\big)^2}}{\|\boldsymbol{\alpha}'(t)\|^3} = \frac{\|\boldsymbol{\alpha}'(t)\times\boldsymbol{\alpha}''(t)\|}{\|\boldsymbol{\alpha}'(t)\|^3} = \boldsymbol{\kappa}[\boldsymbol{\alpha}](t).$$

Similarly, Lemma 8.1 and the fact that $\det(A) = \pm 1$ implies that

$$\begin{aligned}\boldsymbol{\tau}[\boldsymbol{\gamma}](t) &= \frac{[\boldsymbol{\gamma}'(t)\,\boldsymbol{\gamma}''(t)\,\boldsymbol{\gamma}'''(t)]}{\|\boldsymbol{\gamma}'(t)\times\boldsymbol{\gamma}''(t)\|^2} = \frac{[(A\boldsymbol{\alpha}'(t))(A\boldsymbol{\alpha}''(t))(A\boldsymbol{\alpha}'''(t))]}{\|A\boldsymbol{\alpha}'(t)\times A\boldsymbol{\alpha}''(t)\|^2}\\ &= \frac{\det(A)[\boldsymbol{\alpha}'(t)\,\boldsymbol{\alpha}''(t)\,\boldsymbol{\alpha}'''(t)]}{\|\boldsymbol{\alpha}'(t)\times\boldsymbol{\alpha}''(t)\|^2} = \det(A)\boldsymbol{\tau}[\boldsymbol{\alpha}](t) = \pm\boldsymbol{\tau}[\boldsymbol{\alpha}](t).\ \blacksquare\end{aligned}$$

Inherent in the above proof, and a consequence of (8.1), is the fact that the arc length of a curve is itself invariant under Euclidean motions.

In Theorem 5.13, page 136, we showed that a plane curve is determined up to a Euclidean motion of the plane by its signed curvature. This leads us to the conjecture that a space curve is determined up to a Euclidean motion of $\mathbb{R}^3$ by its curvature and torsion. We prove that this is true, with one additional (but essential) assumption.

Theorem 8.3. (Fundamental Theorem, Uniqueness) *Let $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ be unit-speed curves in $\mathbb{R}^3$ defined on the same interval (a, b), and assume they have the same torsion and the same positive curvature. Then there is a Euclidean motion F of $\mathbb{R}^3$ that maps $\boldsymbol{\alpha}$ onto $\boldsymbol{\gamma}$.*

Proof. Since both $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ have nonzero curvature, both of the Frenet frames $\{\mathbf{T}_\alpha, \mathbf{N}_\alpha, \mathbf{B}_\alpha\}$ and $\{\mathbf{T}_\gamma, \mathbf{N}_\gamma, \mathbf{B}_\gamma\}$ are well-defined. Fix s_0 with $a < s_0 < b$. There exists a translation of $\mathbb{R}^3$ taking $\boldsymbol{\alpha}(s_0)$ into $\boldsymbol{\gamma}(s_0)$. Choose a rotation A that maps $\boldsymbol{\alpha}'(s_0)$ onto $\boldsymbol{\gamma}'(s_0)$. By rotating again in the plane perpendicular to this vector, we may assume that A also maps $\mathbf{N}_\alpha(s_0)$ onto $\mathbf{N}_\gamma(s_0)$. Lemma 8.1 then guarantees that A maps $\mathbf{B}_\alpha(s_0)$ onto $\mathbf{B}_\gamma(s_0)$. Thus there exists a Euclidean motion F of $\mathbb{R}^3$ such that

$$F\big(\boldsymbol{\alpha}(s_0)\big) = \boldsymbol{\gamma}(s_0)$$

$$A\big(\mathbf{T}_\alpha(s_0)\big) = \mathbf{T}_\gamma(s_0), \quad A\big(\mathbf{N}_\alpha(s_0)\big) = \mathbf{N}_\gamma(s_0), \quad A\big(\mathbf{B}_\alpha(s_0)\big) = \mathbf{B}_\gamma(s_0).$$

To show that $F\circ\boldsymbol{\alpha}$ coincides with $\boldsymbol{\gamma}$, we define a real-valued function f by

$$f(s) = \|(A\circ\mathbf{T}_\alpha)(s) - \mathbf{T}_\gamma(s)\|^2 + \|(A\circ\mathbf{N}_\alpha)(s) - \mathbf{N}_\gamma(s)\|^2 + \|(A\circ\mathbf{B}_\alpha)(s) - \mathbf{B}_\gamma(s)\|^2$$

for $a < s < b$. A computation similar to that of (5.5) on page 137 (using the assumptions that $\boldsymbol{\kappa}[\boldsymbol{\alpha}] = \boldsymbol{\kappa}[\boldsymbol{\gamma}]$ and $\boldsymbol{\tau}[\boldsymbol{\alpha}] = \boldsymbol{\tau}[\boldsymbol{\gamma}]$, see Exercise 1) shows that the derivative of f is 0. Since $f(s_0) = 0$, we conclude that $f(s) = 0$ for all s. Hence $(F\circ\boldsymbol{\alpha})'(s) = (A\circ\mathbf{T}_\alpha)(s) = \boldsymbol{\gamma}'(s)$ for all s, and so there exists $\mathbf{q}\in\mathbb{R}^3$ such that

$$(F\circ\boldsymbol{\alpha})(s) = \boldsymbol{\gamma}(s) + \mathbf{q}$$

for all s. By the choice of F we have $\mathbf{q} = 0$, and F maps $\boldsymbol{\alpha}$ onto $\boldsymbol{\gamma}$. $\blacksquare$

We now discuss what can go wrong when the assumption of nowhere-zero curvature is dropped. The first thing to note is that it is not possible to distinguish a unit normal vector $\mathbf{N}(t_0)$ at a point where $\boldsymbol{\kappa}(t_0) = 0$, and consequently $\boldsymbol{\tau}(t_0)$ is undefined. Nonetheless, if $\boldsymbol{\gamma}\colon (a, b) \to \mathbb{R}^3$ is a regular curve whose torsion vanishes everywhere except for a single t_0 with $a < t_0 < b$, it is reasonable to say that $\boldsymbol{\gamma}$ has zero torsion. With this convention, we shall explain that the assumption in Theorem 8.3 that $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ have nonzero curvature is essential.

There are indeed two space curves that have the same curvature and torsion for which it is impossible to find a Euclidean motion mapping one onto the other. For example, consider curves $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ defined by

$$\boldsymbol{\alpha}(t) = \begin{cases} 0, & \text{if } t = 0, \\ (t,\ 0,\ 5e^{-1/t^2}), & \text{if } t \neq 0, \end{cases}$$

and

$$\boldsymbol{\gamma}(t) = \begin{cases} (t,\ 5e^{-1/t^2},\ 0) & \text{if } t < 0, \\ 0, & \text{if } t = 0, \\ (t,\ 0,\ 5e^{-1/t^2}), & \text{if } t > 0. \end{cases}$$

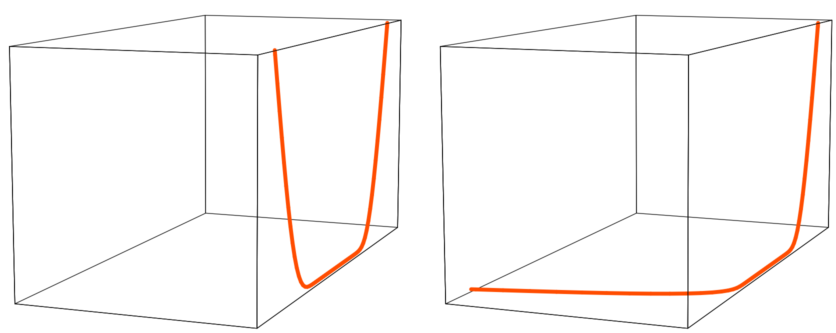

Figure 8.1: The 'very flat' curves $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$

The function $f(t) = e^{-1/t^2}$ (with $f(0) = 0$) is infinitely differentiable, and $f^{(n)}(0) = 0$ for all $n \geqslant 0$. It follows that $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ are regular, infinitely differentiable curves, which have identical curvature functions vanishing at $t = 0$. Furthermore, their torsion vanishes for both $t > 0$ and $t < 0$ since within these parameter ranges the curves are planar. Any Euclidean motion mapping $\boldsymbol{\alpha}$ onto $\boldsymbol{\gamma}$ would have to be the identity on one portion of $\mathbb{R}^3$ and a rotation on another portion of $\mathbb{R}^3$, which is impossible. Despite the nonexistence of an isometry between them, $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ have the same curvature, and the same torsion function, wherever the latter can legitimately be defined.

8.2 Assigned Curvature and Torsion

Now we turn to the question of the existence of curves with prescribed curvature and torsion.

Theorem 8.4. (Fundamental Theorem of Space Curves, Existence) *Suppose that $\boldsymbol{\kappa}\colon (a,b) \to \mathbb{R}$ and $\boldsymbol{\tau}\colon (a,b) \to \mathbb{R}$ are differentiable functions with $\boldsymbol{\kappa} > 0$. Then there exists a unit-speed curve $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^3$ whose curvature and torsion are $\boldsymbol{\kappa}$ and $\boldsymbol{\tau}$. For $a < s_0 < b$ the value $\boldsymbol{\beta}(s_0)$ can be prescribed arbitrarily. Also, the values of $\mathbf{T}(s_0)$ and $\mathbf{N}(s_0)$ can be prescribed subject to the conditions that $\|\mathbf{T}(s_0)\| = \|\mathbf{N}(s_0)\| = 1$ and $\mathbf{T}(s_0) \cdot \mathbf{N}(s_0) = 0$.*

Proof. Consider the following system of 12 differential equations

$$\left\{\begin{array}{rcl} x_i'(s) &=& t_i(s), \\ t_i'(s) &=& \boldsymbol{\kappa}(s) n_i(s), \\ n_i'(s) &=& -\boldsymbol{\kappa}(s) t_i(s) + \boldsymbol{\tau}(s) b_i(s), \\ b_i'(s) &=& -\boldsymbol{\tau}(s) n_i(s), \end{array}\right. \tag{8.2}$$

all for $1 \leqslant i \leqslant 3$, together with the initial conditions

$$\left\{\begin{array}{rcl} x_i(s_0) &=& p_i, \\ t_i(s_0) &=& q_i, \\ n_i(s_0) &=& r_i, \\ b_1(s_0) &=& q_2 r_3 - q_3 r_2, \\ b_2(s_0) &=& q_3 r_1 - q_1 r_3, \\ b_3(s_0) &=& q_1 r_2 - q_2 r_1. \end{array}\right. \tag{8.3}$$

In (8.3) we require that

$$\sum_{i=1}^{3} q_i^2 = 1 = \sum_{i=1}^{3} r_i^2 \quad \text{and} \quad \sum_{i=1}^{3} q_i r_i = 0.$$

From the theory of systems of differential equations, we know that this linear system has a unique solution. If we put $\boldsymbol{\beta} = (x_1, x_2, x_3)$ and

$$\mathbf{T} = (t_1, t_2, t_3), \qquad \mathbf{N} = (n_1, n_2, n_3), \qquad \mathbf{B} = (b_1, b_2, b_3),$$

we see that (8.2) can be compressed into the equations $\boldsymbol{\beta}' = \mathbf{T}$ and

$$\left\{\begin{array}{rcl} \mathbf{T}' &=& \boldsymbol{\kappa}\,\mathbf{N}, \\ \mathbf{N}' &=& -\boldsymbol{\kappa}\,\mathbf{T} + \boldsymbol{\tau}\,\mathbf{B}, \\ \mathbf{B}' &=& -\boldsymbol{\tau}\,\mathbf{N}. \end{array}\right. \tag{8.4}$$

Comparison with the Frenet formulas (7.12) shows that $\boldsymbol{\kappa}$ is the curvature of the curve $\boldsymbol{\beta}$, and $\boldsymbol{\tau}$ is the torsion. ∎

In practice, Theorem 8.4 is implemented by a program in Notebook 8 which solves the system of differential equations numerically, and then plots the resulting space curve. The following initial examples of its use are designed to investigate cases in which one of $\boldsymbol{\kappa}, \boldsymbol{\tau}$ is constant.

If $\boldsymbol{\kappa} = 0$ then the definition of $\boldsymbol{\tau}$ is irrelevant, as a straight line results. If $\boldsymbol{\tau} = 0$ then the curve is planar. If both $\boldsymbol{\kappa}, \boldsymbol{\tau}$ are constant, then the curve is a helix, and this will be oriented according to the assignment of the initial Frenet frame $\{\mathbf{T}(s_0), \mathbf{N}(s_0), \mathbf{B}(s_0)\}$.

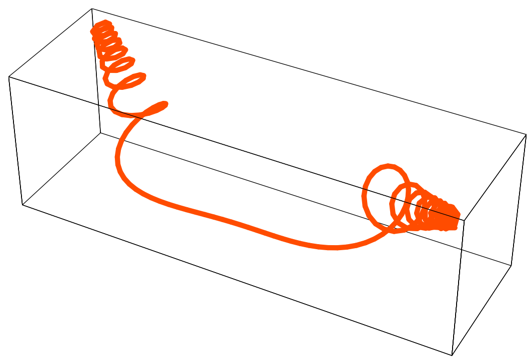

Figure 8.2: Curve with $\boldsymbol{\kappa}(s) = s$ and $\boldsymbol{\tau}(s) = 1$

Less familiar examples are formed by taking at least one of $\boldsymbol{\kappa}, \boldsymbol{\tau}$ to be the identity function $s \mapsto s$, or $s \mapsto cs$ with $c > 0$ a constant. Figure 8.2 is a 3-dimensional analogue of a clothoid, though a closer resemblance with a helix over a clothoid is obtained by setting $\boldsymbol{\kappa}(s) = |s|$ and $\boldsymbol{\tau}(s) = 1$. Figure 8.3 shows a 'double corkscrew' characterized by both $\boldsymbol{\kappa}, \boldsymbol{\tau}$ being linear.

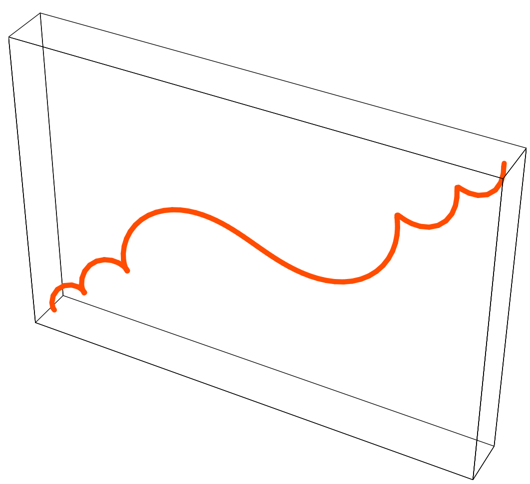

Figure 8.3: A curve with $\boldsymbol{\kappa}(s) = s$ and $\boldsymbol{\tau}(s) = s$

Figure 8.4 displays a straightforward example of a space curve with $\boldsymbol{\kappa}$ constant. It starts off at the origin resembling a semicircle, but becomes less planar as $|\boldsymbol{\tau}|$ increases. There are many other types of curves with constant curvature in space, characterized by different assignments of the torsion $\boldsymbol{\tau}(s)$ as a function of arc length.

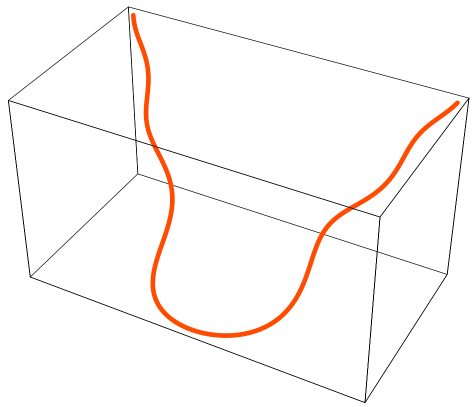

Figure 8.4: A curve with $\boldsymbol{\kappa}(s) = 1$ and $\boldsymbol{\tau}(s) = s$

Figure 8.5 consists of frames of an animation detailing the evolution of a space curve with constant curvature and periodic torsion proportional to $\sin s$. It begins with a circle traversed twice, which is then broken open and eventually joined up to become a simple closed curve admitting one point with $\boldsymbol{\tau}(s) = 1$. A similarly-formed closed curve is shown in Figure 8.14.

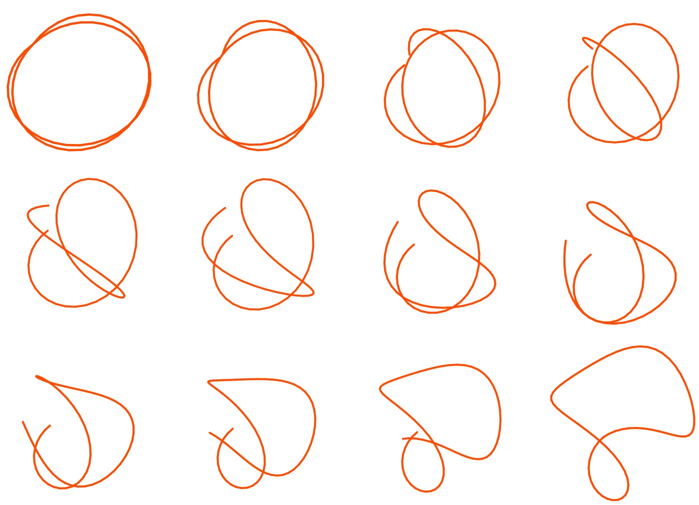

Figure 8.5: Curves with $\boldsymbol{\kappa}(s) = 1$ and $\boldsymbol{\tau}(s) = \frac{1}{12} n \sin s$, $n = 1, \ldots 12$

8.3 Contact

In Section 4.1, we defined the evolute of a plane curve as the locus of its centers of curvature. In this section we show how to define the evolute of a space curve. This requires that we generalize the notion of center of curvature to space curves. For this purpose, we need the notion of *contact*. The latter is also important for plane curves, so we discuss it in a generality that will apply to both cases.

We begin by extending Definition 3.1.

Definition 8.5. *An **implicitly-defined hypersurface** in $\mathbb{R}^n$ is the set of zeros of a differentiable function $F\colon \mathbb{R}^n \to \mathbb{R}$. We denote the set of zeros by $F^{-1}(0)$.*

Clearly, an implicitly-defined hypersurface in $\mathbb{R}^2$ is a curve, and an implicitly-defined hypersurface in $\mathbb{R}^3$ is a surface. These are the main cases that we need.

The simplest hypersurfaces are hyperplanes and hyperspheres. A ***hyperplane*** in $\mathbb{R}^n$ consists of the points represented by the set

$$\{\, \mathbf{p} \in \mathbb{R}^n \mid (\mathbf{p} - \mathbf{u}) \cdot \mathbf{v} = 0 \,\} \tag{8.5}$$

of vectors whose difference from a fixed one $\mathbf{u}$ is always perpendicular to another fixed vector $\mathbf{v}$ (normal to the plane). A hyperplane in $\mathbb{R}^2$ is a line (see the analogous Lemma 6.23), and a hyperplane in $\mathbb{R}^3$ is a plane.

A ***hypersphere*** in $\mathbb{R}^n$ of radius r centered at $\mathbf{q} \in \mathbb{R}^n$ is the set of points

$$\{\, \mathbf{p} \in \mathbb{R}^n \mid \|\mathbf{p} - \mathbf{q}\| = r \,\} \tag{8.6}$$

at a distance r from $\mathbf{q}$. In particular, a hypersphere in $\mathbb{R}^2$ is a circle and a hypersphere in $\mathbb{R}^3$ is a sphere. We leave it to the reader to find differentiable functions whose zero sets are precisely (8.5) and (8.6); recall (3.1) on page 75.

Among all lines passing through a point $\mathbf{p}$ on a curve $\boldsymbol{\alpha}$, the tangent line affords the best approximation, though other curves through $\mathbf{p}$ may approximate the curve even more closely. The mathematics that makes this idea precise arises from

Definition 8.6. *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$ be a regular curve, and let $F\colon \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. We say that a parametrically-defined curve $\boldsymbol{\alpha}$ and an implicitly-defined hypersurface $F^{-1}(0)$ have **contact of order** k at $\boldsymbol{\alpha}(t_0)$ provided*

$$(F \circ \boldsymbol{\alpha})(t_0) = (F \circ \boldsymbol{\alpha})'(t_0) = \cdots = (F \circ \boldsymbol{\alpha})^{(k)}(t_0) = 0, \tag{8.7}$$

but

$$(F \circ \boldsymbol{\alpha})^{(k+1)}(t_0) \neq 0. \tag{8.8}$$

To check that contact is a geometric concept, we prove:

Lemma 8.7. *The definition of contact between* $\boldsymbol{\alpha}$ *and* $F^{-1}(0)$ *is independent of the parametrization of* $\boldsymbol{\alpha}$.

Proof. Let $\boldsymbol{\gamma}\colon (c,d) \to \mathbb{R}^n$ be a reparametrization of $\boldsymbol{\alpha}$; then there exists a differentiable function $h\colon (c,d) \to (a,b)$ such that $\boldsymbol{\gamma}(u) = \boldsymbol{\alpha}(h(u))$ for $c < u < d$. Let u_0 be such that $h(u_0) = h_0$. Then

$$\begin{aligned}
(F \circ \boldsymbol{\gamma})(u_0) &= (F \circ \boldsymbol{\alpha})(h_0) = 0,\\
(F \circ \boldsymbol{\gamma})'(u_0) &= (F \circ \boldsymbol{\alpha})'(h_0)h'(u_0) = 0,\\
(F \circ \boldsymbol{\gamma})''(u_0) &= (F \circ \boldsymbol{\alpha})''(h_0)h'(u_0)^2 + (F \circ \boldsymbol{\alpha})'(h_0)h''(u_0) = 0,\\
(F \circ \boldsymbol{\gamma})'''(u_0) &= (F \circ \boldsymbol{\alpha})'''(h_0)h'(u_0)^3 + 3(F \circ \boldsymbol{\alpha})''(h_0)h'(u_0)h''(u_0)\\
&\qquad\qquad\qquad + (F \circ \boldsymbol{\alpha})'(h_0)h'''(u_0) = 0,
\end{aligned}$$

and so forth. Such Leibniz formulas were computed in Notebook 1. In any case, we see that (8.7) implies that

$$(F \circ \boldsymbol{\gamma})(u_0) = (F \circ \boldsymbol{\gamma})'(u_0) = \cdots = (F \circ \boldsymbol{\gamma})^{(k)}(u_0) = 0,$$

and (8.8) implies that

$$(F \circ \boldsymbol{\gamma})^{(k+1)}(u_0) \neq 0. \quad \blacksquare$$

For example, the x-axis is the set of zeros of the function $F\colon \mathbb{R}^2 \to \mathbb{R}$ given by $F(x,y) = y$. If n is a positive integer, the ***generalized parabola*** $t \mapsto (t,\, t^n)$ has contact of order $n-1$ with this horizontal line at $(0,0)$. A number of these curves are plotted in Figure 8.6.

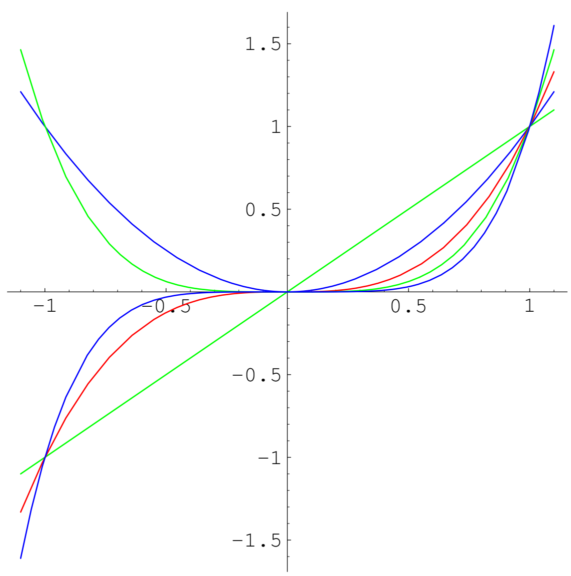

Figure 8.6: Generalized parabolas

Next, we discuss contact between hyperplanes and curves in $\mathbb{R}^n$.

Lemma 8.8. *Let $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^n$ be a unit-speed curve, let $a < s_0 < b$ and let $\mathbf{v} \in \mathbb{R}^n$. Then the hyperplane $\{\, \mathbf{q} \mid (\mathbf{q} - \boldsymbol{\beta}(s_0)) \cdot \mathbf{v} = 0 \,\}$ has at least order 1 contact with $\boldsymbol{\beta}$ at $\boldsymbol{\beta}(s_0)$ if and only if $\mathbf{v}$ is perpendicular to $\boldsymbol{\beta}'(s_0)$.*

Proof. The hyperplane in question is the zero set of the function $F\colon \mathbb{R}^n \to \mathbb{R}$ given by $F(\mathbf{q}) = (\mathbf{q} - \boldsymbol{\beta}(s_0)) \cdot \mathbf{v}$, so we define $f(s) = (\boldsymbol{\beta}(s) - \boldsymbol{\beta}(s_0)) \cdot \mathbf{v}$. Then

$$f'(s) = \boldsymbol{\beta}'(s) \cdot \mathbf{v}.$$

Hence $f'(s_0) = 0$ if and only if $\boldsymbol{\beta}'(s_0) \cdot \mathbf{v} = 0$. ▮

Thus at each point on a regular curve, there are hyperplanes with at least order 1 contact with the curve. The tangent line to a *plane* curve is such a hypersurface, even though it only has dimension 1. As we have just seen, higher order contact with a hyperplane is possible for special curves; in this regard, see Exercise 11 and Figure 7.3.

We turn to contact between hyperspheres and curves. Intuitively, it should be the case that a hypersphere can be chosen to have higher order contact with a curve at a given point than is possible with a hyperplane. We first consider plane curves.

Lemma 8.9. *Let $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^2$ be a unit-speed curve, and choose s_0 such that $a < s_0 < b$. Let $\mathbf{v} \in \mathbb{R}^2$ be distinct from $\boldsymbol{\beta}(s_0)$ so that $r = \|\boldsymbol{\beta}(s_0) - \mathbf{v}\| > 0$. Let $\mathscr{C} = \{\, \mathbf{q} \in \mathbb{R}^2 \mid \|\mathbf{q} - \mathbf{v}\| = r \,\}$.*

(i) *The circle $\mathscr{C}$ has at least order 1 contact with $\boldsymbol{\beta}$ at $\boldsymbol{\beta}(s_0)$ if and only if its center $\mathbf{v}$ lies on the normal line to $\boldsymbol{\beta}$ at $\boldsymbol{\beta}(s_0)$.*

(ii) *Suppose that $\boldsymbol{\kappa 2}(s_0) \neq 0$. Then $\mathscr{C}$ has at least order 2 contact with $\boldsymbol{\beta}$ at $\boldsymbol{\beta}(s_0)$ if and only if*

$$\mathbf{v} = \boldsymbol{\beta}(s_0) + \frac{1}{\boldsymbol{\kappa 2}(s_0)} J\mathbf{T}(s_0).$$

When this is the case, the radius r of $\mathscr{C}$ equals $1/|\boldsymbol{\kappa 2}(s_0)|$.

Proof. Define $g(s) = \|\boldsymbol{\beta}(s) - \mathbf{v}\|^2$. Then $g' = 2\boldsymbol{\beta}' \cdot (\boldsymbol{\beta} - \mathbf{v}) = 2\mathbf{T} \cdot (\boldsymbol{\beta} - \mathbf{v})$, so that $g'(s_0) = 0$ if and only if $\boldsymbol{\beta}'(s_0) \cdot (\boldsymbol{\beta}(s_0) - \mathbf{v}) = 0$. This proves (i).

It follows from (i) that there is a number λ such that $\boldsymbol{\beta}(s_0) - \mathbf{v} = \lambda J\mathbf{T}(s_0)$. Also, we have

$$g'' = 2\big(\mathbf{T} \cdot \mathbf{T} + \mathbf{T}' \cdot (\boldsymbol{\beta} - \mathbf{v})\big) = 2\big(1 + \boldsymbol{\kappa 2}\, J\mathbf{T} \cdot (\boldsymbol{\beta} - \mathbf{v})\big);$$

hence

$$g''(s_0) = 2\big(1 + \boldsymbol{\kappa 2}(s_0)\lambda\big).$$

Thus $g''(s_0) = 0$ if and only if $\lambda = -1/\boldsymbol{\kappa 2}(s_0)$, proving (ii). ▮

Part (ii) of Lemma 8.9 yields a characterization of the osculating circle defined in Section 4.4:

Corollary 8.10. *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$ be a regular plane curve, and let $a < t_0 < b$ be such that $\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t_0) \neq 0$. Then the osculating circle of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(t_0)$ is the unique circle which has at least order 2 contact with $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(t_0)$.*

Next, we consider contact between space curves and spheres. In many cases contact of order 3 is possible.

Theorem 8.11. *Let $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^3$ be a unit-speed curve, and choose s_0 such that $a < s_0 < b$. Let $\mathbf{v} \in \mathbb{R}^3$ be distinct from $\boldsymbol{\beta}(s_0)$ so that $r = \|\boldsymbol{\beta}(s_0) - \mathbf{v}\| > 0$. We abbreviate $\boldsymbol{\kappa}[\boldsymbol{\beta}]$ and $\boldsymbol{\tau}[\boldsymbol{\beta}]$ to $\boldsymbol{\kappa}$ and $\boldsymbol{\tau}$, and let $\mathscr{S} = \{\, \mathbf{q} \in \mathbb{R}^3 \mid \|\mathbf{q}-\mathbf{v}\| = r \,\}$.*

(i) *The sphere $\mathscr{S}$ has at least order 1 contact with $\boldsymbol{\beta}$ at $\boldsymbol{\beta}(s_0)$ if and only if its center $\mathbf{v}$ lies on a line perpendicular to $\boldsymbol{\beta}'(s_0)$ at $\boldsymbol{\beta}(s_0)$.*

(ii) *Suppose that $\boldsymbol{\kappa}(s_0) \neq 0$. Then $\mathscr{S}$ has at least order 2 contact with $\boldsymbol{\beta}$ at $\boldsymbol{\beta}(s_0)$ if and only if*

$$\mathbf{v} - \boldsymbol{\beta}(s_0) = \frac{1}{\boldsymbol{\kappa}(s_0)}\mathbf{N}(s_0) - b\,\mathbf{B}(s_0)$$

for some number b.

(iii) *Suppose that $\boldsymbol{\kappa}(s_0) \neq 0$ and $\boldsymbol{\tau}(s_0) \neq 0$. Then $\mathscr{S}$ has at least order 3 contact with $\boldsymbol{\beta}$ at s_0 if and only if*

$$\mathbf{v} - \boldsymbol{\beta}(s_0) = \frac{1}{\boldsymbol{\kappa}(s_0)}\mathbf{N}(s_0) - \frac{\boldsymbol{\kappa}'(s_0)}{\boldsymbol{\kappa}(s_0)^2\boldsymbol{\tau}(s_0)}\mathbf{B}(s_0).$$

Proof. Define $g(s) = \big\|\boldsymbol{\beta}(s) - \mathbf{v}\big\|^2$. We calculate as in part (i) of Lemma 8.9:

$$g' = 2\boldsymbol{\beta}' \cdot (\boldsymbol{\beta} - \mathbf{v}) = 2\mathbf{T} \cdot (\boldsymbol{\beta} - \mathbf{v}),$$

so that $g'(s_0) = 0$ if and only if $\boldsymbol{\beta}'(s_0) \cdot \big(\boldsymbol{\beta}(s_0) - \mathbf{v}\big) = 0$, proving (i). It follows from (i) that there are numbers a and b such that

$$\boldsymbol{\beta}(s_0) - \mathbf{v} = a\,\mathbf{N}(s_0) + b\,\mathbf{B}(s_0).$$

Also, computing as in part (ii) of Lemma 8.9, we find that

$$\tfrac{1}{2}g'' = \mathbf{T} \cdot \mathbf{T} + \mathbf{T}' \cdot (\boldsymbol{\beta} - \mathbf{v}) = 1 + \boldsymbol{\kappa}\mathbf{N} \cdot (\boldsymbol{\beta} - \mathbf{v}). \tag{8.9}$$

Hence

$$\tfrac{1}{2}g''(s_0) = 1 + \boldsymbol{\kappa}(s_0)\mathbf{N}(s_0) \cdot \big(a\,\mathbf{N}(s_0) + b\,\mathbf{B}(s_0)\big) = 1 + \boldsymbol{\kappa}(s_0)a.$$

Thus $g''(s_0) = 0$ if and only if $a = -1/\boldsymbol{\kappa}(s_0)$, proving (ii).

For (iii) we use (8.9) to compute

$$\tfrac{1}{2}g''' = \big(\boldsymbol{\kappa}'\mathbf{N} + \boldsymbol{\kappa}\mathbf{N}'\big)\boldsymbol{\cdot}(\boldsymbol{\beta} - \mathbf{v}) + \boldsymbol{\kappa}\mathbf{N}\boldsymbol{\cdot}\mathbf{T} = \big(\boldsymbol{\kappa}'\mathbf{N} - \boldsymbol{\kappa}^2\mathbf{T} + \boldsymbol{\kappa}\boldsymbol{\tau}\mathbf{B}\big)\boldsymbol{\cdot}(\boldsymbol{\beta} - \mathbf{v}).$$

In particular, $g'''(s_0)/2$ equals

$$\begin{aligned}\big(\boldsymbol{\kappa}'(s_0)\mathbf{N}(s_0) - \boldsymbol{\kappa}(s_0)^2\mathbf{T}(s_0) + \boldsymbol{\kappa}(s_0)\boldsymbol{\tau}(s_0)\mathbf{B}(s_0)\big)\boldsymbol{\cdot}&\left(\frac{\mathbf{N}(s_0)}{\boldsymbol{\kappa}(s_0)} + b\mathbf{B}(s_0)\right)\\ &= -\frac{\boldsymbol{\kappa}'(s_0)}{\boldsymbol{\kappa}(s_0)} + \boldsymbol{\kappa}(s_0)\boldsymbol{\tau}(s_0)b.\end{aligned}$$

Thus $g'''(s_0) = 0$ if and only if $b = \boldsymbol{\kappa}'(s_0)/\big(\boldsymbol{\kappa}(s_0)^2\,\boldsymbol{\tau}(s_0)\big)$. ▮

The space curve analog of an osculating circle is an osculating sphere, first considered by Fuss[1] in his paper [Fuss]. It is defined precisely as follows:

Definition 8.12. *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^3$ be a regular space curve, and let $a < t_0 < b$ be such that $\boldsymbol{\kappa}[\boldsymbol{\alpha}](t_0) \neq 0$ and $\boldsymbol{\tau}[\boldsymbol{\alpha}](t_0) \neq 0$. Then the* ***osculating sphere*** *of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(t_0)$ is the unique sphere which has at least order 3 contact with $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(t_0)$.*

Corollary 8.13. *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^3$ be a regular space curve whose curvature and torsion do not vanish at t_0. Then the osculating sphere at $\boldsymbol{\alpha}(t_0)$ is a sphere of radius*

$$\sqrt{\frac{1}{\boldsymbol{\kappa}[\boldsymbol{\alpha}](t_0)^2} + \frac{\boldsymbol{\kappa}[\boldsymbol{\alpha}]'(t_0)^2}{\boldsymbol{\tau}[\boldsymbol{\alpha}](t_0)^2\boldsymbol{\kappa}[\boldsymbol{\alpha}](t_0)^4}} \tag{8.10}$$

and center

$$\mathbf{v} = \boldsymbol{\alpha}(t_0) + \frac{1}{\boldsymbol{\kappa}[\boldsymbol{\alpha}](t_0)}\mathbf{N}(t_0) - \frac{\boldsymbol{\kappa}[\boldsymbol{\alpha}]'(t_0)}{\big\|\boldsymbol{\alpha}'(t_0)\big\|\boldsymbol{\kappa}[\boldsymbol{\alpha}](t_0)^2\boldsymbol{\tau}[\boldsymbol{\alpha}](t_0)}\mathbf{B}(t_0). \tag{8.11}$$

Proof. Let $\boldsymbol{\beta}$ be a unit-speed reparametrization of $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}(t) = \boldsymbol{\beta}(s(t))$; then by the chain rule

$$\boldsymbol{\kappa}[\boldsymbol{\beta}]'(s(t))s'(t) = \boldsymbol{\kappa}[\boldsymbol{\alpha}]'(t).$$

From this fact and part (iii) of Theorem 8.11 we get (8.11). Since the radius of the osculating sphere at $\boldsymbol{\alpha}(t_0)$ is $\|\mathbf{v} - \boldsymbol{\alpha}(t_0)\|$, equation (8.10) follows from (8.11). ▮

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[1]Nicolaus Fuss (1755–1826). Born in Basel, Fuss went to St. Petersburg, where he became Euler's secretary. He edited Euler's works and wrote papers on insurance and many textbooks. From 1800 to his death he was permanent secretary of the St. Petersburg Academy.

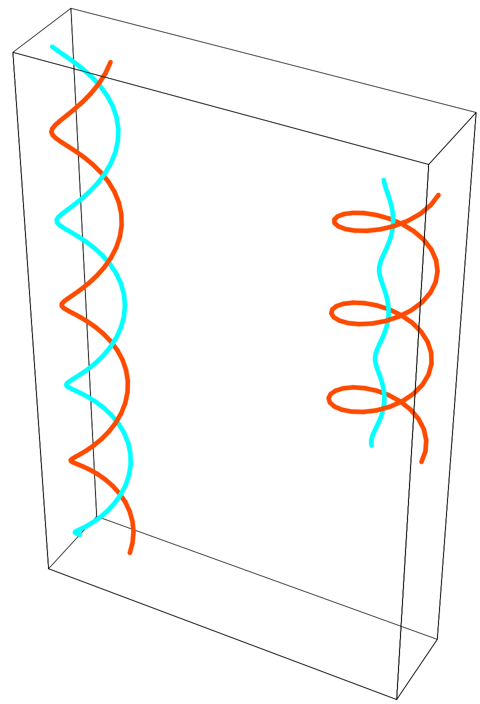

Figure 8.7: Dual pairs helix$[1,1]$, helix$[-1,1]$ (left) and helix$[\frac{3}{2},\frac{1}{2}]$, helix$[-\frac{1}{6},\frac{1}{2}]$ (right)

In Section 4.1, we defined the evolute of a plane curve to be the locus of the centers of the osculating circles to the curve. It is natural to define the evolute of a space curve to be the locus of the centers of the osculating spheres.

Definition 8.14. The ***evolute*** *of a regular space curve* $\boldsymbol{\alpha}$ *is the curve given by*

$$\mathsf{evolute3d}[\boldsymbol{\alpha}](t) = \boldsymbol{\alpha}(t) + \frac{1}{\boldsymbol{\kappa}[\boldsymbol{\alpha}](t)}\mathbf{N}(t) - \frac{\boldsymbol{\kappa}[\boldsymbol{\alpha}]'(t)}{\|\boldsymbol{\alpha}'(t)\|\boldsymbol{\kappa}[\boldsymbol{\alpha}](t)^2\boldsymbol{\tau}[\boldsymbol{\alpha}](t)}\mathbf{B}(t).$$

Consider for example $\boldsymbol{\alpha} = \mathsf{helix}[a,b]$. From (7.16), we have $\boldsymbol{\kappa}[\boldsymbol{\alpha}]' = 0$. Using (7.17), with t in place of $s/\sqrt{a^2+b^2}$, yields

$$\mathsf{evolute3d}[\boldsymbol{\alpha}](t) = \left(-\frac{b^2}{a}\cos t,\ -\frac{b^2}{a}\sin t,\ bt\right).$$

This is merely helix$[-b^2/a,\ b]$ (it is now convenient to allow the radial parametrer to be negative). The original helix can be recovered by applying taking the evolute again, so the two helices are 'dual' to each other; see Figure 8.7. As a second exmple, the evolute of part of a twisted cubic is shown in Figure 8.8.

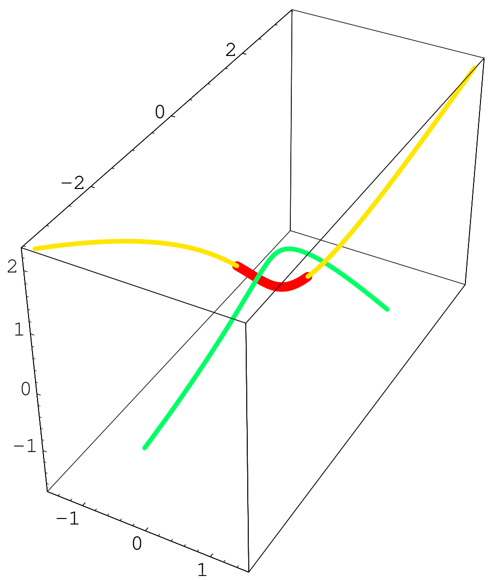

Figure 8.8: The evolute of a segment (thickened underneath) of the twisted cubic $t \mapsto (t, t^3, t^2)$

8.4 Space Curves that Lie on a Sphere

If a space curve (such as Viviani's curve) lies on a sphere, then it has contact of order n with that sphere for any positive integer n. In this section we show that certain relations between the curvature and torsion of a space curve are necessary and sufficient for the space curve to lie on a sphere.

Theorem 8.15. *Let $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^3$ be a unit-speed curve, with curvature $\boldsymbol{\kappa}$ and torsion $\boldsymbol{\tau}$. Suppose that $\boldsymbol{\beta}$ lies on the sphere of radius $c > 0$ centered at $\mathbf{q} \in \mathbb{R}^3$. Then*:

(i) $\boldsymbol{\kappa} \geqslant 1/c$;

(ii) $\boldsymbol{\kappa}$ *and* $\boldsymbol{\tau}$ *are related by*

$$\boldsymbol{\tau}^2\left(c^2 - \frac{1}{\boldsymbol{\kappa}^2}\right) = \left(\frac{\boldsymbol{\kappa}'}{\boldsymbol{\kappa}^2}\right)^2; \qquad (8.12)$$

(iii) $\boldsymbol{\kappa}'(t_0) = 0$ *implies* $\boldsymbol{\tau}(t_0) = 0$ *or* $\boldsymbol{\kappa}(t_0) = 1/c$;

(iv) $\dfrac{\boldsymbol{\tau}}{\boldsymbol{\kappa}} = \left(\dfrac{\boldsymbol{\kappa}'}{\boldsymbol{\tau}\boldsymbol{\kappa}^2}\right)'$;

(v) $\boldsymbol{\kappa}'(t_0) = \boldsymbol{\kappa}''(t_0) = 0$ *implies* $\boldsymbol{\tau}(t_0) = 0$;

(vi) $\boldsymbol{\kappa} \equiv 1/a$ *if and only if* $\boldsymbol{\beta}$ *is a circle of radius* a*, necessarily contained in some plane.*

Proof. Differentiation of the equation $\|\boldsymbol{\beta} - \mathbf{q}\|^2 = c^2$ yields $(\boldsymbol{\beta} - \mathbf{q}) \cdot \mathbf{T} = 0$. Another differentiation results in

$$1 + (\boldsymbol{\beta} - \mathbf{q}) \cdot \mathbf{T}' = 0.$$

Hence $\mathbf{T}'$ is nonzero, which implies the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is well-defined on $\boldsymbol{\beta}$. Thus

$$1 + (\boldsymbol{\beta} - \mathbf{q}) \cdot (\boldsymbol{\kappa}\mathbf{N}) = 0. \tag{8.13}$$

The Cauchy–Schwarz inequality implies that

$$1 = |(\boldsymbol{\beta} - \mathbf{q}) \cdot (\boldsymbol{\kappa}\mathbf{N})| \leqslant \boldsymbol{\kappa}\|\boldsymbol{\beta} - \mathbf{q}\|\,\|\mathbf{N}\| = \boldsymbol{\kappa} c,$$

proving (i).

Differentiating (8.13), we obtain

$$\begin{aligned} 0 &= \mathbf{T} \cdot (\boldsymbol{\kappa}\mathbf{N}) + (\boldsymbol{\beta} - \mathbf{q}) \cdot \big(\boldsymbol{\kappa}'\mathbf{N} + \boldsymbol{\kappa}(-\boldsymbol{\kappa}\mathbf{T} + \boldsymbol{\tau}\mathbf{B})\big) \\ &= \boldsymbol{\kappa}'(\boldsymbol{\beta} - \mathbf{q}) \cdot \mathbf{N} + \boldsymbol{\kappa}\boldsymbol{\tau}(\boldsymbol{\beta} - \mathbf{q}) \cdot \mathbf{B} \\ &= -\frac{\boldsymbol{\kappa}'}{\boldsymbol{\kappa}} + \boldsymbol{\kappa}\boldsymbol{\tau}(\boldsymbol{\beta} - \mathbf{q}) \cdot \mathbf{B}. \end{aligned} \tag{8.14}$$

We may now write

$$\begin{aligned} \boldsymbol{\tau}(\boldsymbol{\beta} - \mathbf{q}) &= \big(\boldsymbol{\tau}(\boldsymbol{\beta} - \mathbf{q}) \cdot \mathbf{T}\big)\mathbf{T} + \big(\boldsymbol{\tau}(\boldsymbol{\beta} - \mathbf{q}) \cdot \mathbf{N}\big)\mathbf{N} \\ &\qquad + \big(\boldsymbol{\tau}(\boldsymbol{\beta} - \mathbf{q}) \cdot \mathbf{B}\big)\mathbf{B} \\ &= -\frac{\boldsymbol{\tau}}{\boldsymbol{\kappa}}\mathbf{N} + \frac{\boldsymbol{\kappa}'}{\boldsymbol{\kappa}^2}\mathbf{B}, \end{aligned} \tag{8.15}$$

the last line from (8.13) and (8.14). Taking norms,

$$\boldsymbol{\tau}^2 c^2 = \boldsymbol{\tau}^2\|\boldsymbol{\beta} - \mathbf{q}\|^2 = \left(\frac{\boldsymbol{\tau}}{\boldsymbol{\kappa}}\right)^2 + \left(\frac{\boldsymbol{\kappa}'}{\boldsymbol{\kappa}^2}\right)^2,$$

from which we obtain (ii). The latter implies (iii).

To prove (iv), we first rearrange (8.14) as

$$\boldsymbol{\tau}(\boldsymbol{\beta} - \mathbf{q}) \cdot \mathbf{B} = \frac{\boldsymbol{\kappa}'}{\boldsymbol{\kappa}^2}. \tag{8.16}$$

Differentiation of (8.16) together with (8.13) yields

$$\begin{aligned} \left(\frac{\boldsymbol{\kappa}'}{\boldsymbol{\kappa}^2}\right)' &= \boldsymbol{\tau}'(\boldsymbol{\beta} - \mathbf{q}) \cdot \mathbf{B} + \boldsymbol{\tau}\big(\mathbf{T} \cdot \mathbf{B} + (\boldsymbol{\beta} - \mathbf{q}) \cdot (-\boldsymbol{\tau}\mathbf{N})\big) \\ &= \frac{\boldsymbol{\kappa}'\boldsymbol{\tau}'}{\boldsymbol{\kappa}^2\boldsymbol{\tau}} + \frac{\boldsymbol{\tau}^2}{\boldsymbol{\kappa}}. \end{aligned} \tag{8.17}$$

This implies (iv). Furthermore, (iv) implies (v).

Finally, if $\boldsymbol{\kappa} \equiv 1/a$, then (v) implies that $\boldsymbol{\tau} \equiv 0$. Hence by Lemma 7.11 we see that $\boldsymbol{\alpha}$ is a plane curve of constant curvature. From Theorem 1.22, page 16 it follows that $\boldsymbol{\alpha}$ is a circle. ▮

It is interesting to check that for Viviani's curve (which lies on a sphere of radius $2a$), $\boldsymbol{\tau}$ and $\boldsymbol{\kappa}'$ have the same zeros. This is already suggested by Figure 7.7. Instead, Figure 8.9 shows the curve

$$\boldsymbol{\alpha}(t) = \left(\tfrac{3}{4}\cos t,\ \tfrac{1}{2}\sin t,\ \sqrt{1 - \tfrac{9}{16}\cos^2 t - \tfrac{1}{4}\sin^2 t} \right)$$

lying on a sphere of radius 1, projecting to an ellipse in the xy-plane. Figure 8.10 plots $\boldsymbol{\kappa}[\boldsymbol{\alpha}]'$ (with maximum value in excess of 4) and $\boldsymbol{\tau}[\boldsymbol{\alpha}]$.

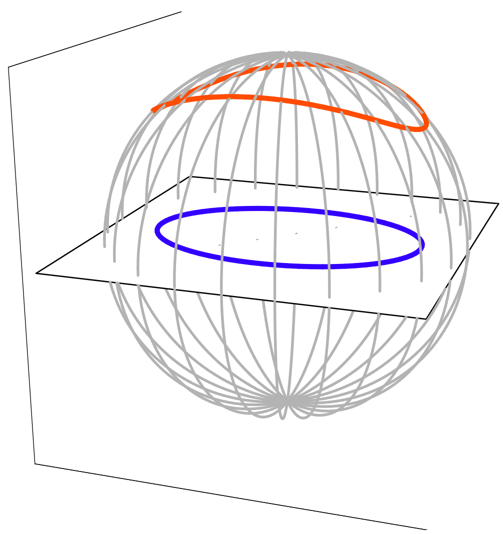

Figure 8.9: The elliptical space curve $\boldsymbol{\alpha}$

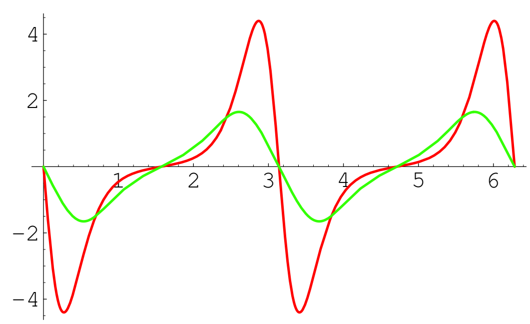

Figure 8.10: $\boldsymbol{\kappa}[\boldsymbol{\alpha}]'$ and $\boldsymbol{\tau}[\boldsymbol{\alpha}]$

Notice that equation (8.12) also holds for a helix of slope b over a circle of radius a where $ca = a^2 + b^2$, but that conditions (iv) and (v) fail. Indeed, a helix with nonzero torsion lies on no sphere, and we cannot therefore expect to characterize spherical curves by (8.12) alone. Nevertheless, we have the following result, which uses the technique of Theorem 1.22.

Theorem 8.16. *Let $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^3$ be a unit-speed curve whose curvature $\boldsymbol{\kappa}$ and torsion $\boldsymbol{\tau}$ satisfy* (8.12). *Assume that $\boldsymbol{\kappa}$, $\boldsymbol{\tau}$ and $\boldsymbol{\kappa}'$ vanish only at isolated points. Then $\boldsymbol{\beta}$ lies on a sphere of radius c centered at some $\mathbf{q} \in \mathbb{R}^3$.*

Proof. Let

$$\boldsymbol{\gamma} = \boldsymbol{\beta} + \frac{1}{\boldsymbol{\kappa}}\mathbf{N} - \left(\frac{\boldsymbol{\kappa}'}{\boldsymbol{\tau}\boldsymbol{\kappa}^2}\right)\mathbf{B}.$$

The Frenet formulas (7.10) imply that

$$\begin{aligned}\boldsymbol{\gamma}' &= \mathbf{T} - \frac{\boldsymbol{\kappa}'}{\boldsymbol{\kappa}^2}\mathbf{N} + \frac{1}{\boldsymbol{\kappa}}(-\boldsymbol{\kappa}\mathbf{T} + \boldsymbol{\tau}\mathbf{B}) - \left(\frac{\boldsymbol{\kappa}'}{\boldsymbol{\tau}\boldsymbol{\kappa}^2}\right)'\mathbf{B} + \left(\frac{\boldsymbol{\kappa}'}{\boldsymbol{\tau}\boldsymbol{\kappa}^2}\right)\boldsymbol{\tau}\mathbf{N} \\ &= \left(\frac{\boldsymbol{\tau}}{\boldsymbol{\kappa}} - \left(\frac{\boldsymbol{\kappa}'}{\boldsymbol{\tau}\boldsymbol{\kappa}^2}\right)'\right)\mathbf{B}.\end{aligned} \tag{8.18}$$

Differentiation of (8.12) yields

$$\left(\frac{\boldsymbol{\kappa}'}{\boldsymbol{\tau}\boldsymbol{\kappa}^2}\right)\left(\frac{\boldsymbol{\kappa}'}{\boldsymbol{\tau}\boldsymbol{\kappa}^2}\right)' = \frac{\boldsymbol{\kappa}'}{\boldsymbol{\kappa}^3}. \tag{8.19}$$

From (8.18) and (8.19), we get

$$\left(\frac{\boldsymbol{\kappa}'}{\boldsymbol{\tau}\boldsymbol{\kappa}^2}\right)\boldsymbol{\gamma}' = \left(\frac{\boldsymbol{\tau}}{\boldsymbol{\kappa}}\frac{\boldsymbol{\kappa}'}{\boldsymbol{\tau}\boldsymbol{\kappa}^2} - \frac{\boldsymbol{\kappa}'}{\boldsymbol{\kappa}^3}\right)\mathbf{B} = 0$$

or $\boldsymbol{\kappa}'\boldsymbol{\gamma}' = 0$. Thus $\boldsymbol{\gamma}'(t) = 0$ whenever t is such that $\boldsymbol{\kappa}'(t) \neq 0$. Since $\boldsymbol{\gamma}'$ is continuous, $\boldsymbol{\gamma}'(t) = 0$ for all t. Hence there exists $\mathbf{q} \in \mathbb{R}^3$ such that $\boldsymbol{\gamma}(t) = \mathbf{q}$ for all t. In other words,

$$\boldsymbol{\beta}(t) - \mathbf{q} = -\frac{1}{\boldsymbol{\kappa}(t)}\mathbf{N}(t) + \frac{\boldsymbol{\kappa}'(t)}{\boldsymbol{\tau}(t)\boldsymbol{\kappa}(t)^2}\mathbf{B}(t) \tag{8.20}$$

for all t. Then (8.12) implies that

$$\|\boldsymbol{\beta}(t) - \mathbf{q}\|^2 = c^2.$$

Hence $\boldsymbol{\beta}$ lies on a sphere of radius c centered at $\mathbf{q}$. ▮

The conclusion of Theorem 8.16 can be expressed by saying that the osculating spheres of $\boldsymbol{\beta}$ all coincide.

8.5 Curves of Constant Slope

Since a helix has constant curvature $\boldsymbol{\kappa}$ and torsion $\boldsymbol{\tau}$, the ratio $\boldsymbol{\tau}/\boldsymbol{\kappa}$ is constant. Are there other curves with this property? In order to answer this question, we first show that the constancy of $\boldsymbol{\tau}/\boldsymbol{\kappa}$ for a curve is equivalent to the constancy of the angle between the curve and a fixed vector. More precisely:

Definition 8.17. *A curve $\boldsymbol{\gamma}\colon (a,b) \to \mathbb{R}^3$ is said to have* ***constant slope*** *with respect to a unit vector $\mathbf{u} \in \mathbb{R}^3$, provided the angle ϕ between $\mathbf{u}$ and the unit tangent vector $\mathbf{T}$ of $\boldsymbol{\gamma}$ is constant. The analytical condition is*

$$\mathbf{T} \cdot \mathbf{u} = \cos\phi,$$

and we call $\cot\phi$ the ***slope*** *of $\boldsymbol{\gamma}$.*

With this convention, if the trace of $\boldsymbol{\gamma}$ is a straight line lying in the yz-plane $x = 0$ and $\mathbf{u} = (0,0,1)$, then the slope of $\boldsymbol{\gamma}$ with respect to $\mathbf{u}$ equals its gradient dz/dy.

Lemma 8.18. *If a curve $\boldsymbol{\gamma}\colon (a,b) \to \mathbb{R}^3$ has nonzero curvature $\boldsymbol{\kappa}$ and constant slope with respect to some unit vector $\mathbf{u}$, then $\boldsymbol{\tau}/\boldsymbol{\kappa}$ has the constant value $\pm\cot\phi$.*

Proof. Without loss of generality, $\boldsymbol{\gamma}$ has unit speed. We differentiate $\mathbf{T} \cdot \mathbf{u} = \cos\phi$, obtaining $(\boldsymbol{\kappa}\mathbf{N}) \cdot \mathbf{u} = 0$. Therefore, $\mathbf{u}$ is perpendicular to $\mathbf{N}$, and so we can write

$$\mathbf{u} = \mathbf{T}\cos\phi \pm \mathbf{B}\sin\phi.$$

When we differentiate this equation, we obtain

$$0 = \mathbf{N}(\boldsymbol{\kappa}\cos\phi \mp \boldsymbol{\tau}\sin\phi),$$

and the lemma follows. ∎

The converse of Lemma 8.18 is true.

Lemma 8.19. *Let $\boldsymbol{\gamma}\colon (a,b) \to \mathbb{R}^3$ be a curve for which the ratio $\boldsymbol{\tau}/\boldsymbol{\kappa}$ is constant. Write $\boldsymbol{\tau}/\boldsymbol{\kappa} = \cot\phi$. Then $\boldsymbol{\gamma}$ has constant slope $\cot\phi$ with respect to some unit vector $\mathbf{u} \in \mathbb{R}^3$.*

Proof. Without loss of generality, $\boldsymbol{\gamma}$ has unit speed. We have

$$0 = \mathbf{N}(\boldsymbol{\kappa}\cos\phi - \boldsymbol{\tau}\sin\phi) = \frac{d}{ds}(\mathbf{T}\cos\phi + \mathbf{B}\sin\phi).$$

It follows that there exists a constant unit vector $\mathbf{u} \in \mathbb{R}^3$ such that

$$\mathbf{u} = \mathbf{T}\cos\phi + \mathbf{B}\sin\phi.$$

In particular, $\mathbf{u} \cdot \mathbf{T}$ is constant. ∎

Returning to the helix, it is easily verified that (7.15) has constant slope b/a with respect to $(0,0,1)$ (Exercise 3). Next, we show that any unit-speed plane curve gives rise to a constant-slope space curve in the same way that a circle gives rise to a helix.

Lemma 8.20. *Let $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^2$ be a unit-speed plane curve, and write $\boldsymbol{\beta}(s) = \big(b_1(s), b_2(s)\big)$. Define a space curve $\boldsymbol{\gamma}$ by*

$$\boldsymbol{\gamma}(s) = \big(b_1(s),\ b_2(s),\ s\cos\psi\big),$$

where ψ is constant. Then $\boldsymbol{\gamma}$ has constant speed $\sqrt{1+\cos^2\psi}$ and constant slope $\cos\psi$ with respect to $(0,0,1)$.

Proof. The speed of $\boldsymbol{\gamma}$ is the norm of $\boldsymbol{\gamma}'$ and is obviously $\sqrt{1+\cos^2\psi}$. Using this fact, the angle ϕ between $\boldsymbol{\gamma}'(t)$ and $(0,0,1)$ is determined by

$$\cos\phi = \frac{\boldsymbol{\gamma}'(t)\cdot(0,0,1)}{\|\boldsymbol{\gamma}'(t)\|} = \frac{\cos\psi}{\sqrt{1+\cos^2\psi}},$$

whence

$$\sec^2\phi = 1 + \frac{1}{\cos^2\psi},$$

and $\cot\phi = \cos\psi$ as stated. ∎

The reverse construction also works: a space curve of constant slope projects to a plane curve in a natural way.

Lemma 8.21. *Let $\boldsymbol{\gamma}\colon (a,b) \to \mathbb{R}^3$ be a unit-speed curve that has constant slope $\cot\phi$ with respect to a unit vector $\mathbf{u} \in \mathbb{R}^3$. Assume that $\cot\phi \neq 1$. Let $\boldsymbol{\beta}$ be the projection of $\boldsymbol{\gamma}$ on a plane perpendicular to $\mathbf{u}$; that is,*

$$\boldsymbol{\beta}(s) = \boldsymbol{\gamma}(s) - (\boldsymbol{\gamma}(s)\cdot\mathbf{u})\mathbf{u}$$

for $a < s < b$. Then $\boldsymbol{\beta}$ has constant speed $|\sin\phi|$; furthermore, the curvature of $\boldsymbol{\gamma}$ and the signed curvature of $\boldsymbol{\beta}$ are related by

$$\boldsymbol{\kappa 2}[\boldsymbol{\beta}] = \pm\boldsymbol{\kappa}[\boldsymbol{\gamma}]\csc^2\phi. \tag{8.21}$$

Proof. Let v denote the speed of $\boldsymbol{\beta}$, and let $\mathbf{T}_\beta$ and $\mathbf{T}_\gamma$ denote the unit tangent vectors of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. We have

$$v\mathbf{T}_\beta = \boldsymbol{\beta}'(s) = \mathbf{T}_\gamma - (\mathbf{T}_\gamma\cdot\mathbf{u})\mathbf{u} = \mathbf{T}_\gamma - (\cos\phi)\mathbf{u}.$$

Thus

$$1 = \|\mathbf{T}_\gamma\|^2 = v^2 + \cos^2\phi,$$

proving that $v = |\sin\phi|$. Also, we have $J\mathbf{T}_\beta = \pm\mathbf{N}_\gamma$. Hence

$$\boldsymbol{\kappa}[\gamma]\mathbf{N}_\gamma = \mathbf{T}'_\gamma = v\mathbf{T}'_\beta = v^2\boldsymbol{\kappa 2}[\beta]J\mathbf{T}_\beta = \pm(\sin^2\phi)\boldsymbol{\kappa 2}[\boldsymbol{\beta}]\mathbf{N}_\gamma.$$

Since $\sin\phi \neq 0$, we get (8.21). ∎

To find interesting curves of constant slope other than a helix or a plane curve, we look for curves of constant slope that lie on some sphere. Figure 8.11 represents a typical example of such a curve (viewed from above and from the sides). First, we determine what relations must exist between the curvature and torsion of such curves.

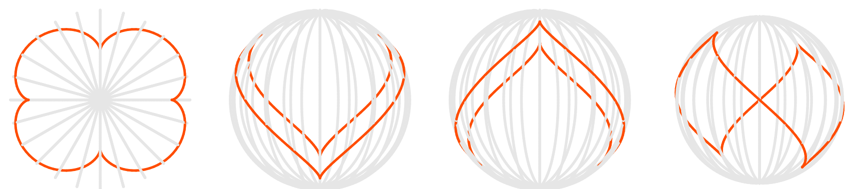

Figure 8.11: Four views of a curve on a sphere with constant slope

Lemma 8.22. *Let $\boldsymbol{\gamma}\colon (a,b) \to \mathbb{R}^3$ be a unit-speed curve that has constant slope* $\cot\phi$ *with respect to a unit vector* $\mathbf{u} \in \mathbb{R}^3$*, where* $0 < \phi < \pi/2$*. Assume also that* $\boldsymbol{\gamma}$ *lies on a sphere of radius* $c > 0$*.*

(i) *The curvature and torsion of* $\boldsymbol{\gamma}$ *are given by*

$$\boldsymbol{\kappa}[\boldsymbol{\gamma}](s)^2 = \frac{1}{c^2 - s^2\cot^2\phi} \quad \text{and} \quad \boldsymbol{\tau}[\boldsymbol{\gamma}](s)^2 = \frac{1}{c^2\tan^2\phi - s^2}. \tag{8.22}$$

(ii) *Let* $\boldsymbol{\beta}$ *be the projection of* $\boldsymbol{\gamma}$ *on a plane perpendicular to* $\mathbf{u}$*. Then the signed curvature of* $\boldsymbol{\beta}$ *satisfies*

$$\boldsymbol{\kappa}\mathbf{2}[\boldsymbol{\beta}](s_1)^2 = \frac{1}{c^2\sin^4\phi - s_1{}^2\cos^2\phi}, \tag{8.23}$$

where $s_1 = s\sin\phi$ *is the arc length function of* $\boldsymbol{\beta}$*.*

(iii) *Let* $c = a + 2b$ *and* $\cos\phi = a/(a+2b)$*. Then the natural equation* (8.23) *of* $\boldsymbol{\beta}$ *is the same as that of* epicycloid$[a,b]$ *defined on page* 144*.*

Proof. Since $\boldsymbol{\tau} = \boldsymbol{\kappa}\cot\phi$, from (8.12) we get

$$\left(\frac{\boldsymbol{\kappa}'}{\boldsymbol{\kappa}^2}\right)^2 = \boldsymbol{\tau}^2\left(c^2 - \frac{1}{\boldsymbol{\kappa}^2}\right) = (\boldsymbol{\kappa}^2c^2 - 1)\cot^2\phi,$$

or

$$\frac{\boldsymbol{\kappa}'}{\boldsymbol{\kappa}^2\sqrt{\boldsymbol{\kappa}^2c^2 - 1}} = \pm\cot\phi. \tag{8.24}$$

Integrating (8.24), we obtain

$$\frac{1}{\boldsymbol{\kappa}}\sqrt{\boldsymbol{\kappa}^2c^2 - 1} = \pm s\cot\phi. \tag{8.25}$$

When we solve (8.25) for $\boldsymbol{\kappa}$, we obtain (8.22), proving (i). Then (ii) follows from (8.21) and (i).

For (iii), we first compute

$$\sin^4\phi = \left(1 - \frac{a^2}{(a+2b)^2}\right)^2 = \frac{16b^2(a+b)^2}{(a+2b)^4}.$$

Hence (8.23) becomes

$$\frac{1}{\boldsymbol{\kappa 2}[\boldsymbol{\beta}](s_1)^2} = \frac{16b^2(a+b)^2}{(a+2b)^2} - \frac{a^2 s_1^2}{(a+2b)^2}. \tag{8.26}$$

Then (8.26) is the same as (5.13) with s replaced by s_1. ▮

Lemma 8.22 suggests the definition of a new kind of space curve.

Definition 8.23. *Write* epicycloid$[a,b](t) = \big(x(t), y(t)\big)$. *The* ***spherical helix*** *of parameters a, b is defined by*

$$\mathsf{sphericalhelix}[a,b](t) = \left(x(t),\ y(t),\ 2\sqrt{ab+b^2}\cos\frac{at}{2b}\right).$$

From Lemma 8.22 we have immediately

Lemma 8.24. *The space curve* sphericalhelix$[a,b]$ *has constant slope with respect to* $(0,0,1)$ *and lies on a sphere of radius* $a+2b$ *centered at the origin.*

Figure 8.12 illustrates two the following two examples. On the unit sphere $S^2(1)$, the curves of constant slope that project to a nephroid (see Exercise 6 on page 86) and a cardioid are respectively

$$\mathsf{sphericalhelix}[\tfrac{1}{2}, \tfrac{1}{4}] \qquad \text{and} \qquad \mathsf{sphericalhelix}[\tfrac{1}{3}, \tfrac{1}{3}].$$

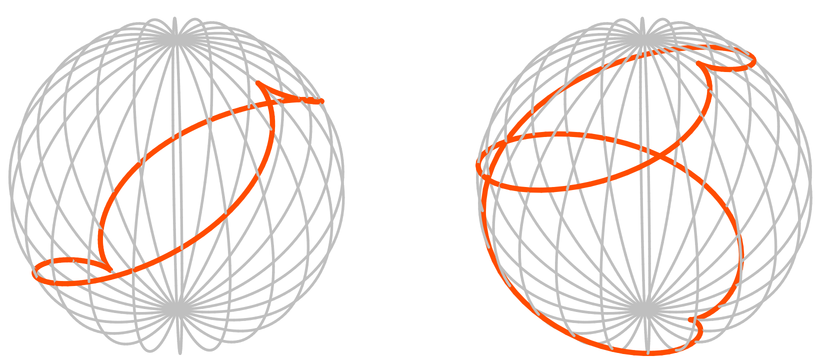

Figure 8.12: Spherical nephroid (left) and cardioid (right)

8.6 Loxodromes on Spheres

A ***meridian*** on a sphere is a great circle that passes through the north and south poles. A ***parallel*** on a sphere is a circle parallel to the equator. More generally:

Definition 8.25. *A **spherical loxodrome** or **rhumb line** is a curve on a sphere which meets each meridian of the sphere at the same angle, which we call the **pitch** of the loxodrome.*

A loxodrome has the north and south poles as asymptotic points. In the early years of terrestrial navigation, many sailors thought that a loxodrome was the same as a great circle, but Nuñez[2] distinguished the two.

To explain how to obtain the parametrization of a spherical loxodrome, we need the the so-called ***stereographic map*** $\boldsymbol{\Upsilon}\colon \mathbb{R}^2 \to S^2(1)$, defined by

$$\boldsymbol{\Upsilon}(p_1,\, p_2) = \frac{(2p_1,\, 2p_2,\, p_1^2 + p_2^2 - 1)}{p_1^2 + p_2^2 + 1}.$$

In fact, $\boldsymbol{\Upsilon}$ is the inverse of stereographic projection stereo, that we shall define in Chapter 22 using complex numbers (see page 730). In the meantime, the reader may check that the straight line joining (p_1, p_2) and the 'north pole' $\mathbf{n} = (0, 0, 1)$ intersects the sphere in $\boldsymbol{\Upsilon}(p_1, p_2)$. It is easy to see that $\boldsymbol{\Upsilon}$ is differentiable and that $\|\boldsymbol{\Upsilon}(\mathbf{p})\| = 1$ for all $\mathbf{p} \in \mathbb{R}^2$.

Lemma 8.26. *The map $\boldsymbol{\Upsilon}$ has the following properties*:

(i) $\boldsymbol{\Upsilon}$ *maps circles in the plane onto circles on the sphere.*

(ii) $\boldsymbol{\Upsilon}$ *maps straight lines in the plane onto circles on the sphere that pass through* $(0, 0, 1)$.

(iii) $\boldsymbol{\Upsilon}$ *maps straight lines through the origin onto meridians.*

(iv) $\boldsymbol{\Upsilon}$ *preserves angles.*

Proof. Any circle or line in the plane is given implicitly by an equation of the form

$$a(x^2 + y^2) + bx + cy + d = 0. \tag{8.27}$$

Let $W = 1 + x^2 + y^2$, $X = 2x/W$, $Y = 2y/W$ and $Z = 1 - 2/W$. Under $\boldsymbol{\Upsilon}$, the equation (8.27) is transformed into

$$bX + cY + (a - d)Z + (d + a) = 0, \tag{8.28}$$

[2] Pedro Nuñez Salaciense (1502–1578). The first great Portuguese mathematician. Professor at Coimbra.

which is the equation of a plane in $\mathbb{R}^3$. This plane meets the sphere $S^2(1)$ in a great circle. This proves (i).

In the case of a straight line in the plane, $a = 0$ in (8.27). From (8.28), we see that $(0, 0, 1)$ is on the image curve, proving (ii). If in addition the straight line passes through the origin, then $a = d = 0$ in (8.27). Then from (8.28) we see that the plane containing the image curve also contains the Z-axis. Hence the curve is a meridian, proving (iii).

Statement (iv) will be proved in Lemma 12.14 on page 371. ▮

Lemma 8.27. *A spherical loxodrome is the image under a stereographic map of a logarithmic spiral.*

Proof. Lemma 1.29, page 24, implies that a logarithmic spiral meets every radial line from the origin at the same angle. The stereographic map $\boldsymbol{\Upsilon}$ maps each of these radial lines into a meridian of the sphere. Since $\boldsymbol{\Upsilon}$ preserves angles, it must map each logarithmic spiral onto a spherical loxodrome. ▮

This leads us to the definition

$$\text{sphericalloxodrome}[a,b](t) = \left(\frac{2a\,e^{bt}\cos t}{1+a^2e^{2bt}}, \frac{2a\,e^{bt}\sin t}{1+a^2e^{2bt}}, \frac{a^2e^{2bt}-1}{1+a^2e^{2bt}}\right),$$

a particular case of which can be seen in Figure 8.13.

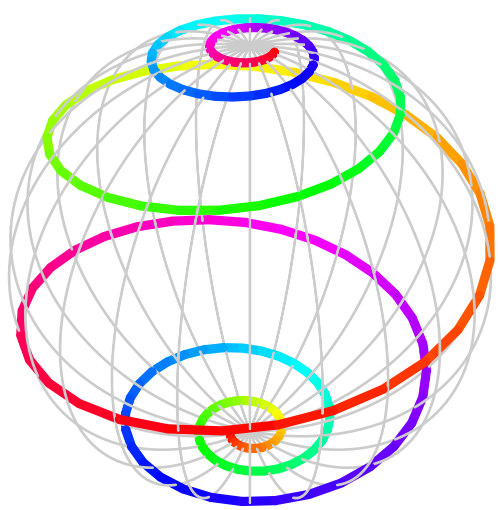

Figure 8.13: A spherical loxodrome of pitch 0.15

8.7 Exercises

1. Fill in the details of the proof of Theorem 8.3.

2. What is the evolute of Viviani's curve? No calculation is necessary!

3. Show that a helix has constant slope with respect to $(0,0,1)$.

4. Show that any curve in the xy-plane has zero slope with respect to $(0,0,1)$.

M **5.** Consider a space curve $\boldsymbol{\gamma}$ that projects to a curve $\boldsymbol{\alpha}$ in the xy-plane so that the third component of $\boldsymbol{\gamma}$ equals $\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]$, as explained in Section 7.5. Plot the result when $\boldsymbol{\gamma}$ is a figure eight, cardioid, astroid, cycloid.

M **6.** Another twisted generalization of the plane curve $\boldsymbol{\alpha}$ is the space curve

$$\mathsf{writhe}[\boldsymbol{\alpha}](t) = \big(a_1(t),\ a_2(t)\cos t,\ a_2(t)\sin t\big)$$

where $\boldsymbol{\alpha} = (a_1, a_2)$. Compute the curvature and torsion of $\mathsf{writhe}[\mathsf{circle}[a]]$. Verify that $\mathsf{writhe}[\mathsf{circle}[a]]$ and $\mathsf{viviani}[\frac{a}{2}](2t+\pi)$ have the same curvature and torsion. Conclude that there is a Euclidean motion that takes one curve into the other. Then plot the curves.

M **7.** Compute the curvature and torsion of $\mathsf{writhe}[\mathsf{lemniscate}[1]]$, and plot the curve.

8. Let $\boldsymbol{\theta}\colon (a,b) \to \mathbb{R}$ be an arbitrary differentiable function and define a curve $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^3$ by

$$\boldsymbol{\alpha}(t) = \left(\int_a^t \cos\boldsymbol{\theta}(u)\,du,\ \int_a^t \sin\boldsymbol{\theta}(u)\,du,\ ct\right),$$

where c is a constant. Compute the curvature and torsion of $\boldsymbol{\alpha}$ and show that $\boldsymbol{\alpha}$ has constant slope.

M **9.** Compute the curvature and torsion of $\mathsf{sphericalhelix}[a, 2b]$. Check that the ratio of the curvature to torsion is constant and that $\mathsf{sphericalhelix}[a, 2b]$ is a curve on a sphere. Plot several spherical helices.

10. A plane curve is defined by

$$\mathsf{teardrop}[a,b,c,d,n](t) = \big(a\sin t,\ d(b\sin t + c\sin 2t)^n\big).$$

Find the order of contact between $\mathsf{teardrop}[8,1,2,1,n]$ and the x-axis at the origin $(0,0)$.

11. Let $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^n$ be a unit-speed curve, let $a < s_0 < b$ and let $\mathbf{v} \in \mathbb{R}^n$. Then the hyperplane $\{\, \mathbf{q} \mid (\mathbf{q}-\boldsymbol{\beta}(s_0)) \cdot \mathbf{v} = 0 \,\}$ has at least order 2 contact with $\boldsymbol{\beta}$ at $\boldsymbol{\beta}(s_0)$ if and only if $\mathbf{v}$ is perpendicular to $\boldsymbol{\beta}'(s_0)$ and the curvature of $\boldsymbol{\beta}$ vanishes at the point of contact $\boldsymbol{\beta}(s_0)$.

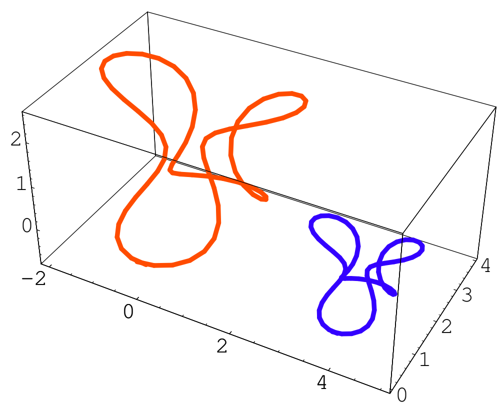

Figure 8.14: Space curves with $\boldsymbol{\kappa}(s) = 1$, $\boldsymbol{\tau}(s) = \sin\dfrac{s}{2}$ (large) and $\boldsymbol{\kappa}(s) = 2$, $\boldsymbol{\tau}(s) = 2\sin s$ (small)

12. Suppose that a space curve $\mathscr{C}$ has arc length s, curvature $\boldsymbol{\kappa}(s) > 0$ and torsion $\boldsymbol{\tau}(s)$. Let $c > 0$ be a constant, and set

$$\tilde{\boldsymbol{\kappa}}(s) = c\boldsymbol{\kappa}(cs), \quad \tilde{\boldsymbol{\tau}}(s) = c\boldsymbol{\tau}(cs).$$

Prove that a space curve with arc length s, curvature $\tilde{\boldsymbol{\kappa}}(s)$ and torsion $\tilde{\boldsymbol{\tau}}(s)$ is a rescaled version of $\mathscr{C}$. An example of this phenomenon can be seen in Figure 8.14.