

Chapter 15

Surfaces of Revolution and Constant Curvature

Surfaces of revolution form the most easily recognized class of surfaces. We know that ellipsoids and hyperboloids are surfaces of revolution provided that two of their axes are equal; this is evident from the figures on pages 312 and 314. Similarly for tori and elliptical paraboloids. Indeed, many objects from everyday life such as cans, table glasses, and furniture legs are surfaces of revolution. The process of lathing wood produces surfaces of revolution by its very nature.

We begin an analytic study of surfaces of revolution by defining *standard parametrizations* whose coordinate curves are parallels and meridians. Then, in order to provide a fresh example, we define the catenoid, an important minimal surface. We compare it with a hyperboloid of revolution, which (only) at first sight it resembles. We also display a surface obtained by rotating a plane curve with assigned curvature (Figure 15.3).

We point out in Section 15.2 that parallels and meridians are principal curves, being tangent to the directions determined by the principal curvatures. We pursue this topic analytically in Section 15.3, in which formulas are given for the Gaussian and mean curvatures of a surface of revolution. All the accompanying computations are carried out in Notebook 15. More curvature formulas are given in Section 15.4 for the generalized helicoids, which constitute a class of surfaces encompassing both helicoids and surfaces of revolution.

While the concept of a surface of revolution, like that of a ruled surface, may be understood in terms of elementary geometry, some of the most interesting surfaces of revolution are those of *constant Gaussian curvature*. We develop this theme in the second half of the chapter by exhibiting surfaces of revolution with constant positive and negative Gaussian curvature in Sections 15.5 and 15.6 respectively. Understanding their equations requires at least cursory knowledge

of the *elliptic integrals* of Legendre¹, which we mention.

Other than the sphere, there are two types of surfaces of revolution with constant positive curvature, one shaped like an American football, and the other like a barrel. There are also three types in the negative case, the most famous of which is the pseudosphere, the surface of revolution obtained by rotating a tractrix. Their equations will be studied further in Chapter 21.

In Section 15.7, we take a glimpse at more exotic surfaces with constant Gaussian curvature. We define a flat generalized helicoid, thereby extending a discussion of flat ruled surfaces in Section 14.2. Then we illustrate the surfaces of Kuen and Dini, whose definitions will be justified mathematically in Sections 21.7 and 21.8.

15.1 Surfaces of Revolution

A surface of revolution is formed by revolving a plane curve about a line in \mathbb{R}^3 . More precisely:

Definition 15.1. Let Π be a plane in \mathbb{R}^3 , let ℓ be a line in Π , and let \mathcal{C} be a point set in Π . When \mathcal{C} is rotated in \mathbb{R}^3 about ℓ , the resulting point set \mathcal{M} is called the **surface of revolution** generated by \mathcal{C} , which is called the **profile curve**. The line ℓ is called the **axis of revolution** of \mathcal{M} .

For convenience, we shall choose Π to be the xz -plane and ℓ to be the z -axis. We shall assume that the point set \mathcal{C} has a parametrization $\alpha: (a, b) \rightarrow \mathcal{C}$ that is differentiable. Write $\alpha = (\varphi, \psi)$.

Definition 15.2. The patch $\text{surfrev}[\alpha]: (0, 2\pi) \times (a, b) \rightarrow \mathbb{R}^3$ defined by

$$(15.1) \quad \text{surfrev}[\alpha](u, v) = (\varphi(v) \cos u, \varphi(v) \sin u, \psi(v)).$$

is called the **standard parametrization** of the surface of revolution \mathcal{M} .

The patch is of necessity defined with the angle u lying in the open interval $(0, 2\pi)$, which means that the standard parametrization must be combined with an analogous patch in order to cover the surface. One usually assumes $\varphi(v) > 0$ in (15.1) to ensure that the profile curve does not cross the axis of revolution;

¹



Adrien Marie Legendre (1752–1833). French mathematician, who made numerous contributions to number theory and the theory of elliptic functions. In 1782, Legendre determined the attractive force for certain solids of revolution by introducing an infinite series of polynomials that are now called Legendre polynomials. In his 3-volume work *Traité des fonctions elliptiques* (1825, 1826, 1830) Legendre founded the theory of elliptic integrals.

however, we do not make this assumption initially. In particular, the curvature formulas that we derive hold in the general case.

A meridian on the earth is a great circle that passes through the north and south poles. A parallel is a circle on the earth parallel to the equator. These notions extend to an arbitrary surface of revolution.

Definition 15.3. Let \mathcal{C} be a point set in a plane $\Pi \subset \mathbb{R}^3$, and let $\mathcal{M}[\mathcal{C}]$ be the surface of revolution in \mathbb{R}^3 generated by revolving \mathcal{C} about a line $\ell \subset \Pi$. A **meridian** on $\mathcal{M}[\mathcal{C}]$ is the intersection of $\mathcal{M}[\mathcal{C}]$ with a plane containing the axis of the surface of revolution ℓ . A **parallel** on $\mathcal{M}[\mathcal{C}]$ is the intersection of $\mathcal{M}[\mathcal{C}]$ with a plane orthogonal to the axis of the surface of revolution.

For a surface parametrized by (15.1), the parallels

$$(15.2) \quad u \mapsto \text{surfref}[\alpha](u, v_0) = (\varphi(v_0) \cos u, \varphi(v_0) \sin u, \psi(v_0)),$$

and the meridians

$$(15.3) \quad v \mapsto \text{surfref}[\alpha](u_0, v) = (\varphi(v) \cos u_0, \varphi(v) \sin u_0, \psi(v))$$

are coordinate curves.

Many of the computer-generated plots of surfaces of revolution display (polygonal approximations of) the parallels and meridians by default. This phenomenon is evident in Figure 15.1, which shows a surface of revolution flanked by its parallels (on the left) and meridians (on the right).

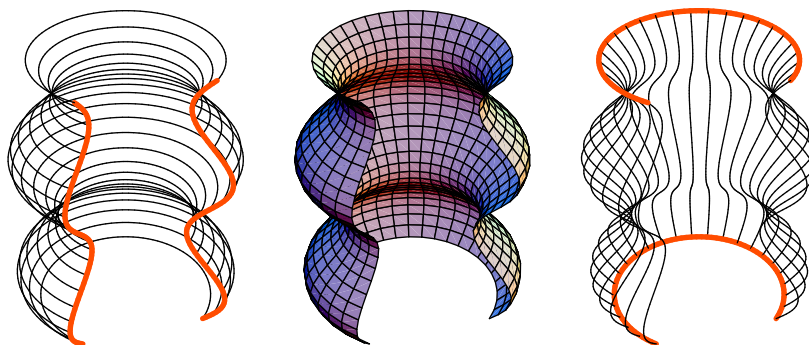


Figure 15.1: Parallels and meridians on a surface of revolution generated by the curve $t \mapsto (2 + \frac{1}{2} \sin 2t, t)$

The quantities $\varphi(v_0)$ and $\psi(v_0)$ in (15.2) have geometric interpretations: $|\varphi(v_0)|$ represents the radius of the parallel, whereas $\psi(v_0)$ can be interpreted as the distance (measured positively or negatively) of the center of the same parallel from the origin.

The Catenoid

We have already discussed various basic surfaces of revolution. Next, we consider the **catenoid**, the surface of revolution generated by a catenary. The catenoid has a standard parametrization

$$\text{catenoid}[c](u, v) = \left(c \cos u \cosh \frac{v}{c}, c \sin u \cosh \frac{v}{c}, v \right).$$

It is easy to compute the principal curvatures of the catenoid, with the result that

$$k_1 = -k_2 = \frac{1}{c \left(\cosh \frac{v}{c} \right)^2}.$$

The Gaussian curvature is therefore

$$K = \frac{-1}{c^2 \left(\cosh \frac{v}{c} \right)^4},$$

whereas the mean curvature H vanishes. It follows that the catenoid is a minimal surface, a concept defined on page 398, and we shall prove in Section 16.3 that any surface of revolution which is also a minimal surface is contained in a catenoid or a plane.

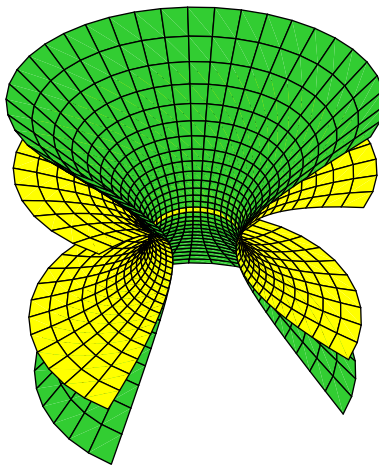


Figure 15.2: Hyperboloid of revolution and catenoid

The shape of a small portion of the catenoid around the equator is similar to that of the hyperboloid of revolution around the equator, but large portions are quite different, as are their Gauss maps. We know that the image of the Gauss map of the whole hyperboloid omits disks around the north and south

poles (see Figure 11.2 on page 336 and the discussion there). By contrast, the image of the Gauss map of the whole catenoid omits only the north and south poles. This is demonstrated in Notebook 15. There is a well-known relationship between the catenoid and the helicoid, discussed in Section 16.4.

Revolving Curves with Prescribed Curvature

One can also consider the surfaces formed by revolving curves with specified curvature. This provides a means of rendering ‘solid’ many of the examples of curves studied in Chapter 6. An example is illustrated in Figure 15.3; others are given in Notebook 15.

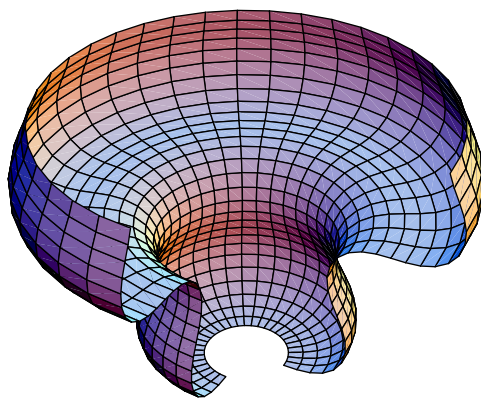


Figure 15.3: Surface of revolution whose meridian has $\kappa_2(s) = \sin s$

15.2 Principal Curves

In this section, we take the first step in the study of curves with particular properties that lie on surfaces. Later, in Chapter 18, we shall study other classes, namely asymptotic curves and geodesics.

For simplicity, we shall deal only with orientable surfaces. For such a surface \mathcal{M} , we choose a globally-defined surface unit normal \mathbf{U} . Recall

Definition 15.4. A curve α on a regular surface $\mathcal{M} \subset \mathbb{R}^3$ is called a **principal curve** if and only if the velocity α' always points in a principal direction. Thus,

$$S(\alpha') = k_i \alpha',$$

where S denotes the shape operator of \mathcal{M} with respect to \mathbf{U} , and k_i ($i = 1$ or 2) is a principal curvature of \mathcal{M} .

A useful characterization of a principal vector is provided by

Lemma 15.5. *A nonzero tangent vector \mathbf{v}_p to a regular surface $\mathcal{M} \subset \mathbb{R}^3$ is principal if and only if*

$$S(\mathbf{v}_p) \times \mathbf{v}_p = \mathbf{0}.$$

Hence a curve α on \mathcal{M} is a principal curve if and only if $S(\alpha') \times \alpha' = \mathbf{0}$.

Proof. If $S(\mathbf{v}_p) = k_i \mathbf{v}_p$, then $S(\mathbf{v}_p) \times \mathbf{v}_p = k_i \mathbf{v}_p \times \mathbf{v}_p = \mathbf{0}$. Conversely, if $S(\mathbf{v}_p) \times \mathbf{v}_p = \mathbf{0}$, then $S(\mathbf{v}_p)$ and \mathbf{v}_p are linearly dependent. ■

More often than not, principal curves can be found by geometrical considerations. For example, we shall construct them in Chapter 19 using triply orthogonal families of surfaces. Another example is the next theorem, due to Terquem² and Joachimsthal³ [Joach1], which provides a simple but useful criterion for the intersection of two surfaces to be a principal curve on both.

Theorem 15.6. *Let α be a curve whose trace lies in the intersection of regular surfaces $\mathcal{M}_1, \mathcal{M}_2 \subset \mathbb{R}^3$. Denote by \mathbf{U}_i the unit surface normal to \mathcal{M}_i , $i = 1, 2$. Suppose that along α the surfaces $\mathcal{M}_1, \mathcal{M}_2$ meet at a constant angle; that is, $\mathbf{U}_1 \cdot \mathbf{U}_2$ is constant along α . Then α is a principal curve in \mathcal{M}_1 if and only if it is a principal curve in \mathcal{M}_2 .*

Proof. Along the curve of intersection we have

$$(15.4) \quad 0 = \frac{d}{dt}(\mathbf{U}_1 \cdot \mathbf{U}_2) = \left(\frac{d}{dt}\mathbf{U}_1\right) \cdot \mathbf{U}_2 + \mathbf{U}_1 \cdot \left(\frac{d}{dt}\mathbf{U}_2\right).$$

Suppose that the curve of intersection α is a principal curve in \mathcal{M}_1 . Then

$$(15.5) \quad \frac{d}{dt}\mathbf{U}_1 = -k_1\alpha',$$

where k_1 is a principal curvature on \mathcal{M}_1 . But α' is also orthogonal to \mathbf{U}_2 , so we conclude from (15.4) and (15.5) that

$$(15.6) \quad \mathbf{U}_1 \cdot \left(\frac{d}{dt}\mathbf{U}_2\right) = 0.$$

Since $d\mathbf{U}_2/dt$ is also perpendicular to \mathbf{U}_2 , it follows from (15.6) that

$$\frac{d}{dt}\mathbf{U}_2 = -k_2\alpha'$$

for some k_2 . In other words, α is a principal curve in \mathcal{M}_2 . ■

²Olry Terquem (1782–1862). French mathematician and religious correspondent. Known for his study about the nine point circle of a given triangle, and also for suggesting some radical reforms aimed at improving the standing of the Jewish community in France.

³Ferdinand Joachimsthal (1818–1861). German mathematician, student of Kummer and professor at Halle and Breslau. Joachimsthal was a great teacher; his book [Joach2] was one of the first to explain the results of the Monge school and Gauss.

As an important application of Theorem 15.6, we find principal curves on a surface of revolution.

Theorem 15.7. *Suppose the surface of revolution $\mathcal{M}[\alpha]$ generated by a plane curve α is a regular surface. Then the meridians and parallels on $\mathcal{M}[\alpha]$ are principal curves.*

Proof. Each meridian is sliced from $\mathcal{M}[\alpha]$ by a plane Π_m containing the axis of rotation of $\mathcal{M}[\alpha]$. For $\mathbf{p} \in \mathcal{M}[\alpha] \cap \Pi_m$ it is clear that the surface normal $\mathbf{U}(\mathbf{p})$ of $\mathcal{M}[\alpha]$ lies in Π_m . Hence $\mathbf{U}(\mathbf{p})$ and the unit surface normal of Π_m are orthogonal. Therefore, Theorem 15.6 implies that the meridians are principal curves of $\mathcal{M}[\alpha]$.

Next, let Π_p be a plane orthogonal to the axis of $\mathcal{M}[\alpha]$. By rotational symmetry, the unit surface normal \mathbf{U} of $\mathcal{M}[\alpha]$ makes a constant angle with the unit surface normal of Π_p . Again, Theorem 15.6 implies that the parallels are principal curves. ■

Principal curves give rise to an important class of patches for which curvature computations are especially simple.

Definition 15.8. A **principal patch** is a patch $\mathbf{x}: \mathcal{U} \rightarrow \mathbb{R}^3$ for which the curves

$$u \mapsto \mathbf{x}(u, v) \quad \text{and} \quad v \mapsto \mathbf{x}(u, v)$$

are principal curves.

It now follows that the standard parametrization (15.1) of a surface of revolution is a principal patch. This fact will be derived independently in the next section, using

Lemma 15.9. *Let $\mathbf{x}: \mathcal{U} \rightarrow \mathbb{R}^3$ be a patch.*

- (i) *If $F = 0 = f$ at all points of \mathcal{U} , then \mathbf{x} is a principal patch.*
- (ii) *If \mathbf{x} is a principal patch with distinct principal curvatures, then $F = 0 = f$ on \mathcal{U} .*

Proof. If $F = 0 = f$, the Weingarten equations (Theorem 13.16, page 394) imply that \mathbf{x}_u and \mathbf{x}_v are eigenvectors of the shape operator. Conversely, if \mathbf{x} is principal, then both \mathbf{x}_u and \mathbf{x}_v are eigenvectors of the shape operator. If in addition the principal curvatures are distinct, Lemma 13.19 (page 396) implies that $F = 0 = f$. ■

In the case of the plane or sphere, the principal curvatures are equal and every point of the surface is said to be **umbilic**. Moreover, there are infinitely many principal curves through each point of the plane or sphere. This situation will be discussed further in Chapter 19.

15.3 Curvature of a Surface of Revolution

First, we compute the coefficients of the first and second fundamental forms, and also the unit surface normal for a general surface of revolution.

Lemma 15.10. *Let \mathcal{M} be a surface of revolution with profile curve $\alpha = (\varphi, \psi)$. Let $\mathbf{x} = \text{surfref}[\alpha]$ be the standard parametrization (15.1) of \mathcal{M} . Then*

$$(15.7) \quad E = \varphi'^2, \quad F = 0, \quad G = \varphi'^2 + \psi'^2.$$

Thus \mathbf{x} is regular wherever φ and $\varphi'^2 + \psi'^2$ are nonzero. When this is the case,

$$(15.8) \quad e = \frac{-|\varphi|\psi'}{\sqrt{\varphi'^2 + \psi'^2}}, \quad f = 0, \quad g = \frac{(\text{sign } \varphi)(\varphi''\psi' - \varphi'\psi'')}{\sqrt{\varphi'^2 + \psi'^2}},$$

and the unit surface normal is

$$(15.9) \quad \mathbf{U}(u, v) = \frac{\text{sign } \varphi}{\sqrt{\varphi'^2 + \psi'^2}}(\psi' \cos u, \psi' \sin u, -\varphi').$$

Proof. From (15.1) it follows that the first partial derivatives of \mathbf{x} are given by

$$(15.10) \quad \begin{cases} \mathbf{x}_u = (-\varphi(v) \sin u, \varphi(v) \cos u, 0), \\ \mathbf{x}_v = (\varphi'(v) \cos u, \varphi'(v) \sin u, \psi'(v)). \end{cases}$$

Then (15.7) is immediate from (15.10) and the definitions of E, F, G . The unit normal

$$\mathbf{U} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

is easily computed.

Next, it is necessary to write down the second partial derivatives of \mathbf{x} , and use Lemma 13.31 on page 405 to find e, f, g . We omit the calculations, which are easily checked. ■

Recalling Lemma 15.9, we obtain another proof of the following result.

Corollary 15.11. *The standard parametrization (15.1) of a surface of revolution is a principal patch.*

At this point, one could compute the Gaussian curvature K by means of Theorem 13.25. However, we prefer to compute the principal curvatures first. We know that the coordinate curves (15.2) and (15.3) are principal curves. Hence the principal curvatures for a surface of revolution have special meaning, so we denote them by k_p, k_m instead of the usual k_1, k_2 . To be specific, k_p is the curvature of the parallel (15.2) and k_m is the curvature of the meridian (15.3) (the reverse of alphabetical order!).

Theorem 15.12. *The principal curvatures of a surface of revolution parametrized by (15.1) are given by*

$$(15.11) \quad \begin{cases} k_p = \frac{e}{E} = \frac{-\psi'}{|\varphi| \sqrt{\varphi'^2 + \psi'^2}}, \\ k_m = \frac{g}{G} = \frac{(\text{sign } \varphi)(\varphi''\psi' - \varphi'\psi'')}{(\varphi'^2 + \psi'^2)^{3/2}}. \end{cases}$$

The Gaussian curvature is given by

$$(15.12) \quad K = \frac{-\psi'^2\varphi'' + \varphi'\psi'\psi''}{\varphi(\varphi'^2 + \psi'^2)^2},$$

and the mean curvature by

$$(15.13) \quad H = \frac{\varphi(\varphi''\psi' - \varphi'\psi'') - \psi'(\varphi'^2 + \psi'^2)}{2|\varphi|(\varphi'^2 + \psi'^2)^{3/2}}.$$

Proof. Since $F = f = 0$, it follows that

$$\left\{ \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|}, \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|} \right\}$$

forms an orthonormal basis which diagonalizes the shape operator S wherever \mathbf{x} is regular. Hence by Corollary 13.33 on page 405,

$$(15.14) \quad S(\mathbf{x}_u) = \frac{e}{E}\mathbf{x}_u \quad \text{and} \quad S(\mathbf{x}_v) = \frac{g}{G}\mathbf{x}_v.$$

Thus, $k_m = g/G$, $k_p = e/E$, and (15.11) follows from (15.7) and (15.8). Finally, (15.12) and (15.13) follow from the relations $K = k_p k_m$, $H = \frac{1}{2}(k_p + k_m)$. ■

We know from the geometric description of the normal curvature given in Section 13.2 that k_m at $\mathbf{p} \in \mathcal{M}$ coincides, up to sign, with the curvature κ of the meridian through \mathbf{p} . The second equation of (15.11) confirms this fact.

Corollary 15.13. *For a surface of revolution, the functions*

$$K, H, k_p, k_m, E, F, G, e, f, g$$

are all constant along parallels.

Proof. All these functions are expressible in terms of φ and ψ and their derivatives. But φ and ψ do not depend on the angle u . ■

We may in theory choose a profile curve to have unit speed. In this case, the formulas we have derived so far for a surface of revolution simplify considerably.

Corollary 15.14. Let \mathbf{x} be the standard parametrization (15.1) of a surface of revolution in \mathbb{R}^3 whose profile curve $\alpha = (\varphi, \psi)$ has unit speed. Then

$$\begin{aligned} E &= \varphi^2, & F &= 0, & G &= 1, \\ e &= -|\varphi|\psi', & f &= 0, & g &= (\text{sign } \varphi)(\varphi''\psi' - \varphi'\psi''), \\ k_p &= \frac{-\psi'}{|\varphi|}, & k_m &= (\text{sign } \varphi)(\varphi''\psi' - \varphi'\psi''), \\ 2H &= (\text{sign } \varphi)(\varphi''\psi' - \varphi'\psi'') - \frac{\psi'}{|\varphi|}, & K &= \frac{-\varphi''}{\varphi}. \end{aligned}$$

In particular, the signed curvature $\kappa_2[\alpha]$ equals $\pm k_m$.

Proof. Since $1 = \|\alpha'\| = \sqrt{\varphi'^2 + \psi'^2}$, all formulas, except for the last, are immediate from (15.7), (15.8), (15.11). That for K follows from (15.12) and the equality $\varphi'\varphi'' + \psi'\psi'' = 0$. The last statement follows from Newton's equation

$$\kappa_2[\alpha] = \frac{-\psi'\varphi'' + \varphi'\psi''}{(\varphi'^2 + \psi'^2)^{3/2}}$$

on page 15. ■

15.4 Generalized Helicoids

Both helicoids and surfaces of revolution are examples of the class of surfaces known as generalized helicoids, which were first studied by Minding⁴ in 1839 (see [Mind]).

Definition 15.15. Let Π be a plane in \mathbb{R}^3 , ℓ be a line in Π , and \mathcal{C} be a point set in Π . Suppose that \mathcal{C} is rotated in \mathbb{R}^3 about ℓ and simultaneously displaced parallel to ℓ so that the speed of displacement is proportional to the speed of rotation. Then the resulting point set \mathcal{M} is called the **generalized helicoid** generated by \mathcal{C} , which is called the **profile curve** of \mathcal{M} . The line ℓ is called the **axis** of \mathcal{M} . The ratio of the speed of displacement to the speed of rotation is called the **slant** of the generalized helicoid \mathcal{M} .

The Euclidean motion (consisting of a simultaneous translation and rotation) used in this definition is called a **screw motion**. Clearly, a surface of revolution

4



Ernst Ferdinand Adolf Minding (1806–1885). German professor, later dean of the faculty, at the University of Dorpat (now Tartu) in Estonia. Minding was a self-taught mathematician. While a school teacher, he studied for his doctorate, which was awarded by Halle for a thesis on approximating the values of double integrals. In 1864, Minding became a Russian citizen and in the same year was elected to the St. Petersburg Academy. In the 1830s Minding was one of the first mathematicians to use Gauss's approach to the differential geometry of surfaces.

is a generalized helicoid of slant 0, and a generalized helicoid reduces to an ordinary helicoid when the profile curve is $t \mapsto (bt, 0)$.

Darboux ([Darb2, Volume 1, page 128]) put it very well when he observed that a surface of revolution has an important kinematic property: when it rotates about its axis, it glides over itself. In more modern language, we would say that a surface of revolution is the orbit of a curve in a plane Π by a 1-parameter group of rotations of \mathbb{R}^3 about a line in Π . Generalized helicoids enjoy a similar property: when a generalized helicoid is subjected to a screw motion, it glides over itself. Thus, a generalized helicoid is the orbit of a curve in a plane Π by a 1-parameter group of screw motions of \mathbb{R}^3 about a line in Π .

Just as we did for surfaces of revolution on page 462, we choose Π to be the xz -plane and ℓ to be the z -axis. We shall assume that the point set \mathcal{C} has a parametrization $\alpha: (a, b) \rightarrow \mathcal{C}$ which is differentiable, and we write $\alpha = (\varphi, \psi)$. This allows us to give the alternative

Definition 15.16. *Let α be a plane curve and write $\alpha(t) = (\varphi(t), \psi(t))$. Then the generalized helicoid with profile curve α and slant c is the surface in \mathbb{R}^3 parametrized by*

$$\text{genhel}[c, \alpha](u, v) = (\varphi(v) \cos u, \varphi(v) \sin u, cu + \psi(v)).$$

To summarize, a generalized helicoid is generated by a plane curve, which is rotated about a fixed axis and at the same time translated in the direction of the axis with speed a constant multiple of the speed of rotation.

It is also possible to define the notion of meridian for a generalized helicoid.

Definition 15.17. *Let \mathcal{C} be a point set in a plane $\Pi \subset \mathbb{R}^3$, and let $\mathcal{M}[\mathcal{C}, c]$ be the generalized helicoid in \mathbb{R}^3 of slant c generated by revolving \mathcal{C} about a line $\ell \subset \Pi$ and simultaneously displacing it parallel to ℓ . A **meridian** on $\mathcal{M}[\mathcal{C}, c]$ is the intersection of $\mathcal{M}[\mathcal{C}, c]$ with a plane containing the axis ℓ .*

The generalization to generalized helicoids of the notion of parallel given on page 463 is more complicated. For a surface of revolution a parallel is a circle; for a generalized helicoid the corresponding curve is a helical curve, which we call a **parallel helical curve**. The v -parameter curves are the meridians and the u -parameter curves are the parallel helical curves. In contrast to the situation with a surface of revolution, the meridians and parallel helical curves will not be perpendicular to one another for a generalized helicoid of nonzero slope. In Theorem 15.25, page 483, we shall show that a generalized helicoid with the property that its meridians are principal curves must either be a surface of revolution or Dini's surface illustrated in Figure 15.13.

Generalized helicoids can be considered to be 'twisted' surfaces of revolution, though this is unrelated to the construction on page 348. For example, Figure 15.4 is evidently a twisted version of Figure 15.1; it has slant $1/2$.

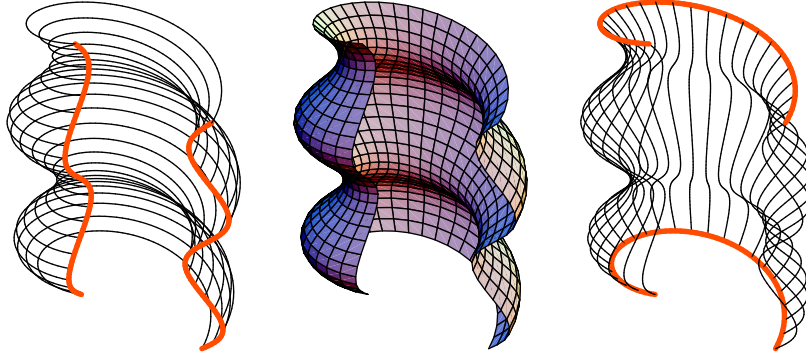


Figure 15.4: A generalized helicoid together with its parallel helical curves and meridians

The proof of the following lemma is an easy extension of the proof of Lemma 15.10, and the calculations are carried out in Notebook 15.

Lemma 15.18. *Let \mathbf{x} be a generalized helicoid in \mathbb{R}^3 whose profile curve is $\alpha = (\varphi, \psi)$. Then*

$$E = \varphi^2 + c^2, \quad F = c\psi', \quad G = \varphi'^2 + \psi'^2.$$

Therefore, \mathbf{x} is regular wherever

$$EG - F^2 = \varphi^2(\varphi'^2 + \psi'^2) + c^2\varphi'^2$$

is nonzero. In this case, denoting its square root by D , we have

$$e = \frac{-\varphi^2\psi'}{D}, \quad f = \frac{c\varphi'^2}{D}, \quad g = \frac{\varphi(\varphi''\psi' - \varphi'\psi'')}{D}.$$

Furthermore

$$(15.15) \quad K = \frac{-c^2\varphi'^4 + \varphi^3\psi'(-\psi'\varphi'' + \varphi'\psi'')}{D^4},$$

and

$$H = \frac{-2c^2\varphi'^2\psi' - \varphi^2\psi'(\varphi'^2 + \psi'^2) + \varphi(c^2 + \varphi^2)(\psi'\varphi'' - \varphi'\psi'')}{2D^3}.$$

We know that the sphere $S^2(a)$ is a surface of revolution parametrized by

$$(15.16) \quad \text{sphere}[a](u, v) = (a \cos v \cos u, a \cos v \sin u, a \sin v);$$

it has constant positive Gaussian curvature $K = 1/a^2$. We call the twisted version of the sphere the **corkscrew surface** or **twisted sphere**. Explicitly, it is given by

$$\text{twisphere}[a, b](u, v) = (a \cos u \cos v, a \sin u \cos v, a \sin v + b u).$$

In Notebook 15, the Gaussian curvature of the twisted sphere is found to be

$$\frac{4a^2 \cos^4 v - 4b^2 \sin^4 v}{(a^2 + b^2 + (a^2 - b^2) \cos 2v)^2},$$

and simplifies to $(\cos 2v)/a^2$ if $b = a$. This formula is used to color the left-hand side of Figure 15.5.

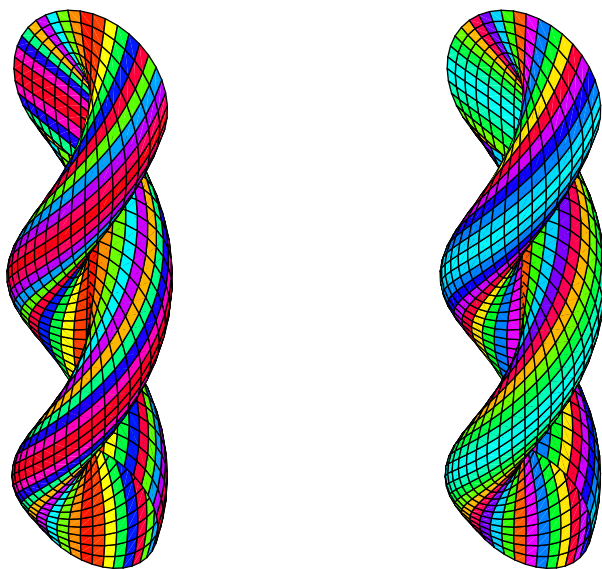


Figure 15.5: Corkscrew surfaces colored by Gaussian and mean curvature

15.5 Surfaces of Constant Positive Curvature

The sphere is not the only surface in \mathbb{R}^3 that has constant curvature. We can take a spherical cap made up of some thin inelastic material, for example, half of a ping-pong ball. Then the spherical cap can be bent without stretching into many different shapes. Since there is no stretching involved and the Gaussian curvature is an isometric invariant (see Theorem 17.5 on page 536), a bent cap also has constant positive curvature. On the other hand, it is clear intuitively that a whole ping-pong ball cannot be bent. Indeed, Liebmann's Theorem (Theorem 19.13, page 603) states that the sphere $S^2(a)$ is rigid in this sense.

To find other surfaces of revolution in \mathbb{R}^3 that have constant positive curvature, we proceed backwards: we assume that we are given a surface of revolution \mathcal{M} with constant positive curvature and seek restrictions on a parametrization \mathbf{x} of \mathcal{M} . First, we determine the profile curves of a surface of constant positive Gaussian curvature.

Theorem 15.19. *Let \mathcal{M} be a surface of revolution whose Gaussian curvature is a positive constant $1/a^2$, where $a > 0$. Then \mathcal{M} is part of a surface parametrized by a patch \mathbf{x} of the form*

$$\mathbf{x}(u, v) = (\varphi(v) \cos u, \varphi(v) \sin u, \psi(v)),$$

where

$$(15.17) \quad \begin{cases} \varphi(v) = b \cos \frac{v}{a}, \\ \psi(v) = \int_0^{v/a} \sqrt{a^2 - b^2 \sin^2 t} \, dt, \end{cases}$$

for some constant $b > 0$. The parameter v has one of the following ranges:

$$\text{if } b = a, \quad \text{then } -\frac{\pi a}{2} \leq v \leq \frac{\pi a}{2};$$

$$\text{if } b < a, \quad \text{then } -\infty \leq v \leq \infty;$$

$$\text{if } b > a, \quad \text{then } -a \arcsin \frac{a}{b} \leq v \leq a \arcsin \frac{a}{b}.$$

The patch \mathbf{x} is regular at (u, v) if and only if $\varphi(v) \neq 0$, that is, $v \neq (n + \frac{1}{2})\pi a$.

Proof. Assume without loss of generality that $\mathbf{x} = \text{surfrev}[\boldsymbol{\alpha}]$ is given by (15.1), and that the profile curve $\boldsymbol{\alpha} = (\varphi, \psi)$ has unit speed, so that $\varphi'^2 + \psi'^2 = 1$. If \mathcal{M} has constant positive curvature $1/a^2$, then Corollary 15.14, page 470, implies that φ satisfies the differential equation

$$(15.18) \quad \varphi'' + \frac{1}{a^2} \varphi = 0,$$

whose general solution is $\varphi(v) = b \cos((v/a) + c)$. Without loss of generality, we can assume that $c = 0$; this amounts to translating the profile curve along the axis of revolution so that the profile curve is farthest from the axis of revolution when $v = 0$. Also, by taking a mirror image if necessary, we may assume that $b > 0$. Thus we get the first equation of (15.17).

We can assume that $\psi'(v) \geq 0$ for all v ; otherwise, replace v by $-v$. Then $\varphi'^2 + \psi'^2 = 1$ implies that

$$(15.19) \quad a\psi'(v) = \sqrt{a^2 - b^2 \sin^2 \frac{v}{a}},$$

and when we integrate this equation from 0 to v , and then rescale the variable, we get the second equation of (15.17).

In order that $\psi(v)$ be well defined, it is necessary that the quantity under the square root in (15.19) be nonnegative. For $b < a$ this quantity is always positive, so $\psi(v)$ is defined for all v . When $b = a$ the profile curve is a part of a circle; in that case the requirement that $-\pi/2 \leq v/a \leq \pi/2$ ensures that the profile curve does not overlap itself and is thus a semicircle. If $b > a$ then (15.19) is defined if and only if $-a/b \leq \sin(v/a) \leq a/b$. ■

When $b = a$, the profile curve is

$$v \mapsto \left(a \cos \frac{v}{a}, a \sin \frac{v}{a} \right), \quad -\frac{\pi}{2} \leq \frac{v}{a} \leq \frac{\pi}{2},$$

which when revolved about the z -axis yields a sphere $S^2(a)$ of radius a . If $b < a$, the profile curve (φ, ψ) makes a shallower arc and first crosses the z -axis when

$$z = \pm \int_0^{\pi/2} \sqrt{a^2 - b^2 \sin^2 t} \, dt.$$

Its length between these two points is πa , which is the same as the length of a semicircle of radius a . But the profile curve can be continued indefinitely, so as to weave its way up and down the z -axis, instead of closing up (a hint of this behavior can be seen in Figure 15.6).

When $b > a$ the profile curve is only defined on the interval

$$-a \arcsin \frac{a}{b} \leq v \leq a \arcsin \frac{a}{b},$$

and the resulting surface of revolution resembles a barrel when b is moderately larger than a . The closer the ratio a/b is to zero, the larger the hole in the middle. The outside surface in Figure 15.7 illustrates the case $a/b = 3/4$.

In conclusion,

Corollary 15.20. *Let $S(a, b)$ be the surface of revolution whose profile curve is $\alpha = (\varphi, \psi)$ with φ, ψ given by (15.17).*

- (i) $S(a, a)$ is an ordinary sphere of radius a .
- (ii) (*Spindle type*) If $0 < b < a$, then $S(a, b)$ is a surface of revolution that resembles an infinite string of beads, each of which is shaped like a football with its vertices on the axis of revolution.
- (iii) (*Bulge type*) If $0 < a < b$, then $S(a, b)$ is barrel-shaped and does not meet the axis of revolution.

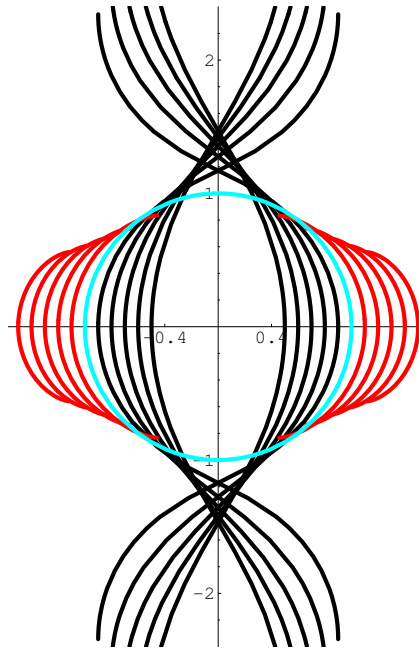


Figure 15.6: Profile curves with $b \leq a$ and $b \geq a$

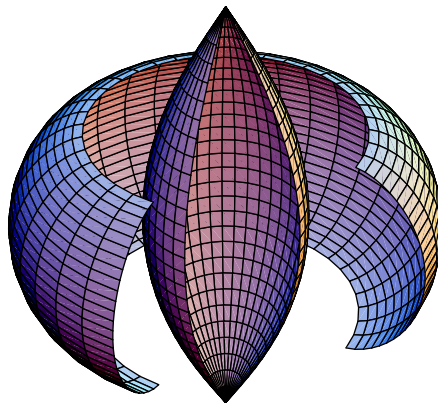


Figure 15.7: The surfaces $\mathcal{S}(a, b)$ with $a/b = 2$ and $3/4$

Elliptic Integrals

We shall make a slight detour into the complicated subject of elliptic functions and integrals, which will be useful when we study surfaces of revolution of constant negative curvature.

Definition 15.21. The *elliptic integral of the second kind* is defined by

$$E(\phi | m) = \int_0^\phi \sqrt{1 - m \sin^2 \theta} d\theta,$$

whereas the *complete elliptic integral of the second kind* is

$$E\left(\frac{\pi}{2} | m\right) = \int_0^{\pi/2} \sqrt{1 - m \sin^2 \theta} d\theta.$$

The motivation for this definition comes from the formula for the arc length of an ellipse, given in Exercise 9. But we have already made implicit use of it, for in (15.17) we may write

$$\psi(v) = a E\left(\frac{v}{a} \middle| \frac{b^2}{a^2}\right),$$

after having brought a factor of a outside the square root.

Notice that $E(\phi | 1) = \sin \phi$; thus $E(\phi | m)$ can be considered to be a generalization of the sine function. The corresponding generalization of the hyperbolic sine function turns out to be $-i E(i\phi | -m)$ because

$$(15.20) \quad -i E(i\phi | -m) = \int_0^\phi \sqrt{1 - m \sinh^2 \theta} d\theta,$$

as can be checked by changing variables in the integral and using the identity $\sinh ix = i \sin x$. We shall need (15.20) in the next section.

15.6 Surfaces of Constant Negative Curvature

The determination of the surfaces of revolution of constant negative curvature proceeds along the same lines as that in Section 15.5. However, the resulting surfaces are quite different in appearance.

Theorem 15.22. Let \mathcal{M} be a surface of revolution whose Gaussian curvature is a negative constant $-1/a^2$. Then \mathcal{M} is part of a surface parametrized by a patch \mathbf{x} such that

$$\mathbf{x}(u, v) = (\varphi(v) \cos u, \varphi(v) \sin u, \psi(v)),$$

where the profile curve $\alpha = (\varphi, \psi)$ is one of the following types:

(i) (Pseudosphere)

$$(15.21) \quad \alpha(v) = \begin{cases} \left(a e^{-v/a}, \int_0^v \sqrt{1 - e^{-2t/a}} dt \right) & \text{for } 0 \leq v < \infty, \\ \left(a e^{v/a}, \int_0^v \sqrt{1 - e^{2t/a}} dt \right) & \text{for } -\infty < v \leq 0. \end{cases}$$

(ii) (*Hyperboloid type*)

$$(15.22) \quad \begin{cases} \varphi(v) = b \cosh \frac{v}{a}, \\ \psi(v) = \int_0^{v/a} \sqrt{a^2 - b^2 \sinh^2 t} dt = -ia E\left(\frac{iv}{a} \mid \frac{-b^2}{a^2}\right), \end{cases}$$

for some constant $b > 0$, and v satisfies $-a \operatorname{arcsinh} \frac{a}{b} \leq v \leq a \operatorname{arcsinh} \frac{a}{b}$.

(iii) (*Conic type*)

$$(15.23) \quad \begin{cases} \varphi(v) = b \sinh \frac{v}{a}, \\ \psi(v) = \int_0^{v/a} \sqrt{a^2 - b^2 \cosh^2 t} dt \\ \quad = -i \sqrt{a^2 - b^2} E\left(\frac{iv}{a} \mid \frac{-b^2}{a^2 - b^2}\right), \end{cases}$$

for some constant b with $0 < b \leq a$, and v must satisfy

$$-a \operatorname{arcsinh} \frac{\sqrt{a^2 - b^2}}{b} \leq v \leq a \operatorname{arcsinh} \frac{\sqrt{a^2 - b^2}}{b}.$$

Proof. Without loss of generality, $a > 0$. The general solution of the negative analogue $\varphi'' - \varphi/a^2 = 0$ of (15.18) is

$$(15.24) \quad \varphi(v) = A e^{v/a} + B e^{-v/a}.$$

Case 1. First, suppose that A is zero in (15.24). We can assume that $B > 0$ by reflecting the profile curve about the y -axis if necessary, so that (φ, ψ) becomes $(-\varphi, \psi)$. Moreover, substituting $v \rightsquigarrow v + a \log B - a \log a$, we can assume that $B = a$ and $\varphi(v) = a e^{-v/a}$. Since

$$0 \leq \psi'(v)^2 = 1 - \varphi'(v)^2 = 1 - e^{-2v/a},$$

we must have $v \geq 0$. Thus we get the first alternative in (15.21). Similarly, $B = 0$ leads to the second alternative.

Next, suppose that A and B are both different from zero in (15.24). Using the change of variables

$$v \rightsquigarrow v + \frac{a}{2} \log \left| \frac{B}{A} \right|$$

if necessary, we may assume that $|A| = |B|$.

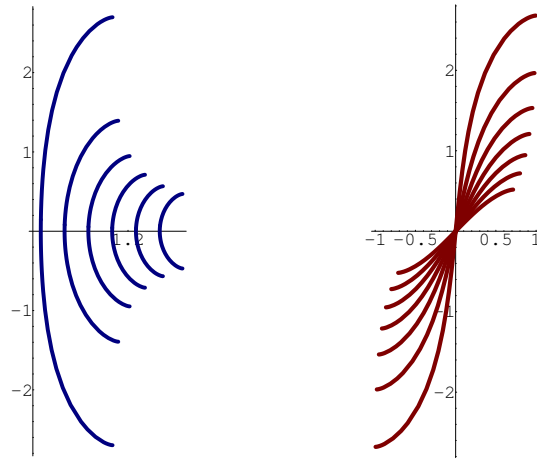


Figure 15.8: Profile curves for constant negative curvature

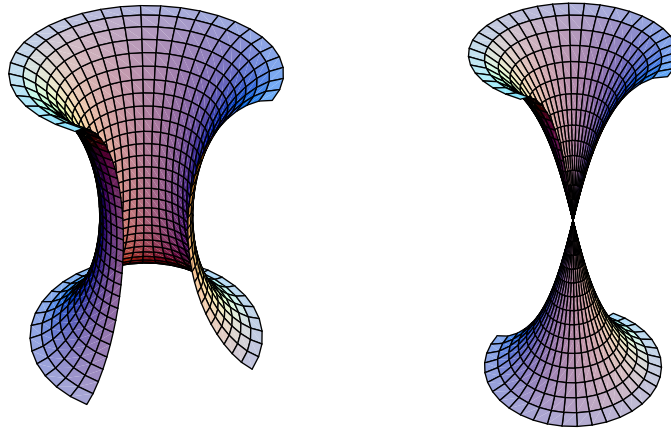


Figure 15.9: Surfaces of hyperboloid and conic type

Case 2. When $A = B$, we can (using a mirror image of the profile curve if necessary) assume that $A > 0$. Then

$$\varphi(v) = A(e^{v/a} + e^{-v/a}) = 2A \cosh \frac{v}{a}.$$

We put $b = 2A$ and obtain (15.22).

Case 3. If $A = -B$, we can (changing v to $-v$ if necessary) assume that $A > 0$. Thus (15.24) becomes

$$\varphi(v) = A(e^{v/a} - e^{-v/a}) = 2A \sinh \frac{v}{a}.$$

We put $b = 2A$ and obtain (15.23). ■

Corollary 15.23. *The surface of revolution whose profile curve is given by (15.21) is a **pseudosphere** or **tractoid**; that is, the surface of revolution of a tractrix.*

Proof. This is an immediate consequence of Lemma 2.1, page 52, and part (i) of Theorem 15.22. ■

In the light of the previous corollary, equation (2.16) gives rise to the parametrization

$$\text{pseudosphere}[a](u, v) = a \left(\cos u \sin v, \sin u \sin v, \cos v + \log \left(\tan \frac{v}{2} \right) \right).$$

In Notebook 15, we verify that this surface has constant curvature $-1/a^2$.

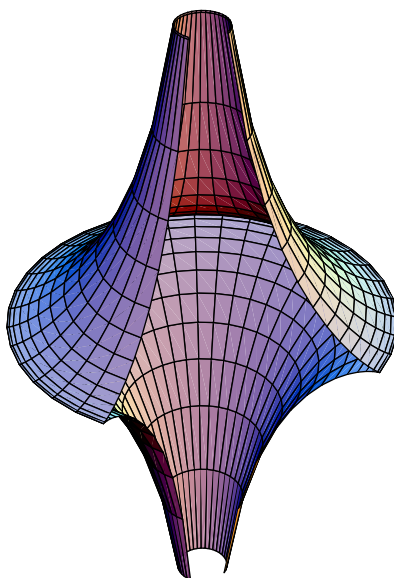


Figure 15.10: Pseudosphere

15.7 More Examples of Constant Curvature

In this final section, we determine the *flat* generalized helicoids and introduce the surfaces of Dini and Kuen, which will be studied in greater detail by transform methods in Chapter 21.

Flat Generalized Helicoids

Recall the parametrization of a generalized helicoid given in Definition 15.16.

Theorem 15.24. *A generalized helicoid is flat if and only if its profile curve can be parametrized as $\alpha(t) = (t, \psi(t))$, where*

$$(15.25) \quad \pm\psi(t) = t\sqrt{a^2 - \frac{c^2}{t^2}} + c \arcsin \frac{c}{at}.$$

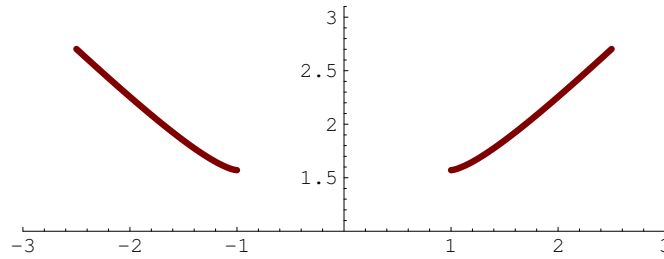


Figure 15.11: Zero-curvature profile curve

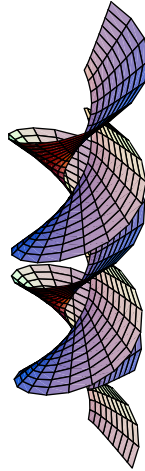


Figure 15.12: Zero-curvature generalized helicoid

Proof. Equation (15.15), page 472, implies that a generalized helicoid is flat if and only if its profile curve $\alpha = (\varphi, \psi)$ satisfies the differential equation

$$(15.26) \quad \varphi^3(\varphi'\psi'\psi'' - \psi'^2\varphi'') - c^2\varphi'^4 = 0.$$

The profile curve can be parametrized so that $\varphi(t) = t$; then (15.26) reduces to

$$t^3\psi'\psi'' - c^2 = 0.$$

Thus,

$$\frac{d}{dt}(\psi')^2 = \frac{2c^2}{t^3},$$

and

$$(15.27) \quad \psi'(t) = \pm \sqrt{a^2 - \frac{c^2}{t^2}},$$

where a^2 is a constant of integration, and $a > 0$. Then (15.27) can be integrated by computer to give (15.25), apart from an irrelevant constant of integration. ■

In Notebook 15, we verify that the generalized helicoid generated by

$$t \mapsto (t, \psi(t)),$$

with ψ defined in (15.25), does indeed have zero Gaussian curvature.

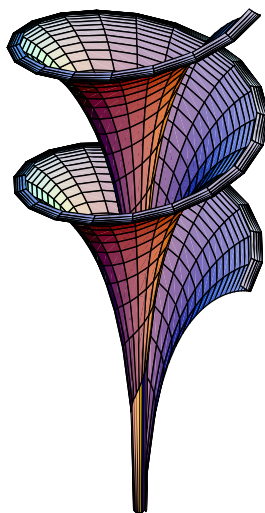


Figure 15.13: Dini's surface

Dini's Surface

The twisted pseudosphere is associated with the name of Dini⁵. The explicit parametrization of Dini's surface is obtained using by applying the **genhel** construction on page 471 to a tractrix, and is therefore

$$\text{dini}[a, b](u, v) = \left(a \cos u \sin v, a \sin u \sin v, a \left(\cos v + \log \left(\tan \frac{v}{2} \right) \right) + cu \right).$$

5



Ulisse Dini (1845–1918). Italian mathematician who worked mainly in Pisa. He made fundamental contributions to surface theory and real analysis. His statue can be found off the Piazza dei Cavalieri, not far from the leaning tower.

In contrast to the corkscrew surface, Dini's has constant curvature. Figure 15.13 shows clearly that it is a deformation by twisting of the pseudosphere.

The following theorem is taken from volume 1 page 353 of Luigi Bianchi's classical text⁶, *Lezioni di Geometria Differenziale* [Bian].

Theorem 15.25. *Let \mathcal{M} be a generalized helicoid with the property that the meridians are principal curves. Then \mathcal{M} is part of Dini's surface.*

Proof. Without loss of generality, we can suppose that the profile curve of the generalized helicoid is of the form $\alpha(t) = (t, \psi(t))$, so that

$$\text{genhel}[c, \alpha](u, v) = (v \cos u, v \sin u, cu + \psi(v)).$$

The unit normal \mathbf{U} to $\text{genhel}[c, \alpha]$ is given by

$$\mathbf{U}(u, v) = \frac{(-c \sin u + v\psi'(v) \cos u, c \cos u + v\psi'(v) \sin u, -v)}{\sqrt{c^2 + v^2 + v^2\psi'(v)^2}},$$

a formula extracted from Notebook 15. The meridian $v \mapsto \text{genhel}[c, \alpha](u, v)$ is the intersection of the generalized helicoid with a plane Π through its axis. The unit normal to this plane is given by

$$\mathbf{V}(u, v) = (-\sin u, \cos u, 0).$$

Now suppose that α is a principal curve on the generalized helicoid. Since α is automatically a principal curve on Π , Theorem 15.26 (which is Exercise 4) implies that the vector fields \mathbf{U} and \mathbf{V} meet at a constant angle σ along α . Thus

$$(15.28) \quad \cos \sigma = \mathbf{U} \cdot \mathbf{V} = \frac{c}{\sqrt{c^2 + v^2 + v^2\psi'(v)^2}}.$$

Then (15.28) implies that ψ satisfies the differential equation

$$v^2(1 + \psi'(v)^2) = c^2 \tan^2 \sigma.$$

But this is equivalent to the differential equation (2.17) on page 51, so α is a tractrix. Hence the generalized helicoid is a surface of Dini. ■

6



Luigi Bianchi (1856–1928). Italian mathematician who worked mainly in Pisa. Although he is most remembered for the ‘Bianchi identities’, he also made fundamental contributions to surface theory.

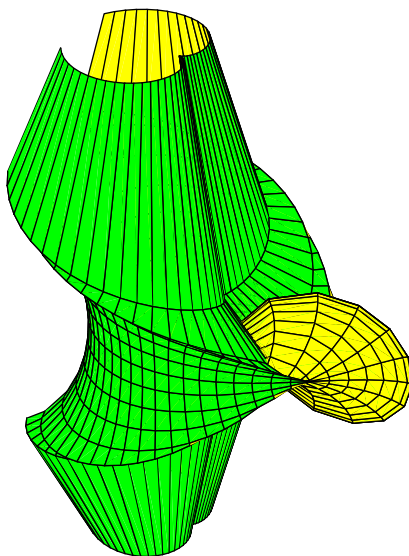


Figure 15.14: Kuen's surface

Kuen's Surface

A more complicated surface of constant negative curvature is that of Kuen⁷ [Kuen]. It can be parametrized by

$$\left(\frac{(\cos u + u \sin u) \sin v}{1 + u^2 \sin^2 v}, \frac{(\sin u - u \cos u) \sin v}{1 + u^2 \sin^2 v}, \frac{1}{2} \log\left(\tan \frac{v}{2}\right) + \frac{\cos v}{1 + u^2 \sin^2 v} \right).$$

A computation in Notebook 15 shows that this surface has Gaussian curvature

$$K = -4.$$

It also turns out that, for this parametrization, the ‘mixed’ coefficients F and f of both the first and second fundamental forms vanish identically. As a consequence, the u and v parameter curves are principal; these are the curves visible in Figure 15.14. We shall explain this fact and see exactly how the equation for Kuen's surface arises in Section 21.7.

An extraordinary plaster model was made of this surface; plate 86 of [Fischer] is a photo of this plaster model. Reckziegel has given an excellent description of Kuen's surface (see pages 30–41 of the Commentaries to [Fischer]).

⁷Th. Kuen used Bianchi's parametrization to make a plaster model of his surface (see [Kuen])

15.8 Exercises

- M 1.** Show that the surface of revolution generated by the graph of a function $h: \mathbb{R} \rightarrow \mathbb{R}$ can be parametrized by

$$\mathbf{x}(u, v) = (v \cos u, v \sin u, h(v)), \quad 0 < u < 2\pi.$$

Determine the associated functions $E, F, G, e, f, g, k_p, k_m, K$, in terms of the function h .

- 2.** Find the formulas for the Gaussian, mean and principal curvatures of a surface of revolution with profile curve $\alpha(t) = (t, \psi(t))$.

- M 3.** Plot the surfaces of revolution corresponding to each of the following curves: cycloid, cissoid, logarithmic spiral, lemniscate, cardioid, astroid, deltoid, nephroid, triangle.

- 4.** Prove the following converse to Theorem 15.6 that was used to prove Theorem 15.25.

Theorem 15.26. *Let α be a curve which lies on the intersection of regular surfaces $\mathcal{M}_1, \mathcal{M}_2 \subset \mathbb{R}^3$. Suppose that α is a principal curve in \mathcal{M}_1 and also in \mathcal{M}_2 . Then the normals to \mathcal{M}_1 and \mathcal{M}_2 meet at a constant angle along α .*

- 5.** With reference to Lemma 15.9, show that the mapping $\mathbf{x}: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by

$$\mathbf{x}(u, v) = (u + v, u, 0)$$

is a principal patch for which $F \neq 0$.

- M 6.** Compute the Gaussian curvature and mean curvature of the generalized helicoid of a catenary. Plot the generalized helicoid of `catenary`[1] with slant $1/2$.

- 7.** Show that a surface of revolution whose Gaussian curvature is zero is a part of a plane, circular cone or circular cylinder. These are therefore the only *flat* surfaces of revolution.

- M 8.** Plot the elliptic functions

$$\phi \mapsto E\left(\phi \middle| \frac{n}{5}\right) \quad \text{and} \quad \phi \mapsto -i E\left(i\phi \middle| -\frac{n}{5}\right)$$

for $n = 0, 1, 2, 3, 4, 5$.

9. Find a formula for the length of the ellipse $(x/a)^2 + (y/b)^2 = 1$. in terms of the elliptic integral E.
10. A surface of revolution is formed by moving a curve γ in a plane Π_1 about a circle in a plane Π_2 , where Π_1 and Π_2 are perpendicular. There is a more general construction in which the circle is replaced by an arbitrary curve α in Π_2 :

$$\text{gensurfrev}[\alpha, \gamma](u, v) = (\alpha_1(u)\gamma_1(v), \alpha_2(u)\gamma_2(v), \alpha_3(u)\gamma_3(v)).$$

Determine when a generalized surface of revolution is a principal patch.

- M 11. Compute the Gaussian curvature K and the mean curvatures H of the generalized surface of revolution formed by moving an eight curve along an eight curve. This is represented in Figure 15.15 left), while the function K is plotted on the right.

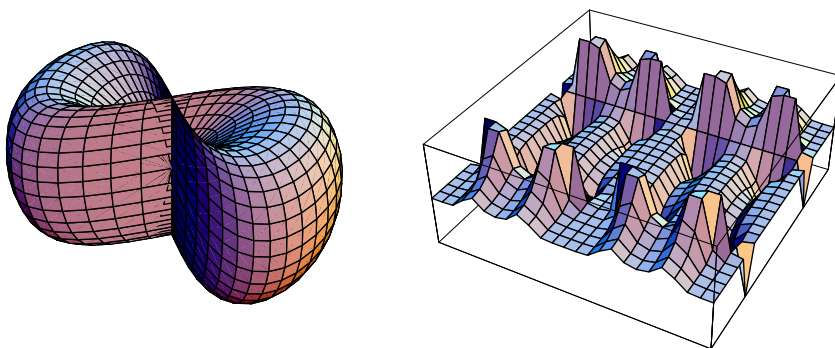


Figure 15.15: A generalized surface of revolution and its Gaussian curvature

- M 12. Find the principal curvatures and the Gauss map of a pseudosphere.
- M 13. **Sievert's surface** is defined by $\mathbf{x} = (x_1, x_2, x_3)$, where

$$\begin{cases} x_1 = \frac{2\sqrt{2+2\sin^2 u} \sin v}{2 - \sin^2 v \cos^2 u} \cos\left(-\frac{u}{\sqrt{2}} + \arctan(\sqrt{2} \tan u)\right) \\ x_2 = \frac{2\sqrt{2+2\sin^2 u} \sin v}{2 - \sin^2 v \cos^2 u} \sin\left(-\frac{u}{\sqrt{2}} + \arctan(\sqrt{2} \tan u)\right), \\ x_3 = \log\left(\tan \frac{v}{2}\right) + \frac{4 \cos v}{2 - \sin^2 v \cos^2 u}. \end{cases}$$

Show that it has constant positive Gaussian curvature (see [Siev] and [Fischer, Commentary, page 38]). Investigate the more general patch, $\text{sievert}[a]$, depending upon a parameter a (equal to 1 above) defined in Notebook 15. Figure 15.16 illustrates the case $a = 4/5$.

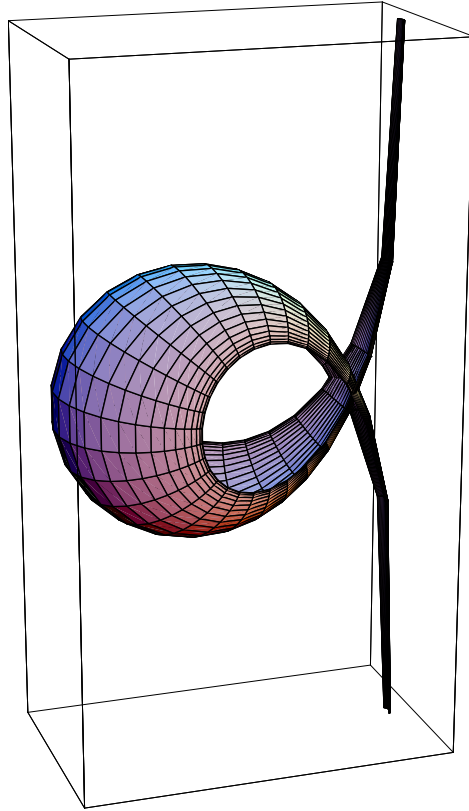


Figure 15.16: The surface $\text{sievert}[0.8]$

14. Suppose that \mathbf{x} is a generalized helicoid in \mathbb{R}^3 with slant c whose profile curve $\alpha = (\varphi, \psi)$ has unit speed. Simplify the equations of Lemma 15.18 that apply in this case.
15. Fill in the details of the proof of Theorem 15.25.