Chapter 13

Shape and Curvature

In this chapter, we study the relationship between the geometry of a regular surface $\mathcal{M}$ in 3-dimensional space $\mathbb{R}^3$ and the geometry of $\mathbb{R}^3$ itself. The basic tool is the *shape operator* defined in Section 13.1. The shape operator at a point $\mathbf{p}$ of $\mathcal{M}$ is a linear transformation S of the tangent space $\mathcal{M}_{\mathbf{p}}$ that measures how $\mathcal{M}$ bends in different directions. The shape operator can also be considered to be (minus) the differential of the Gauss map of $\mathcal{M}$ (Proposition 13.5), and its effect is illustrated using tangent vectors to coordinate curves on both the original surface and the sphere (Figure 13.1).

In Section 13.2, we define a variant of the shape operator called *normal curvature*. Given a tangent vector $\mathbf{v}_{\mathbf{p}}$ to a surface $\mathcal{M}$, the normal curvature $k(\mathbf{v}_{\mathbf{p}})$ is a real number that measures how $\mathcal{M}$ bends in the direction $\mathbf{v}_{\mathbf{p}}$. This number is easy to understand geometrically because it is the curvature of the *plane curve* formed by the intersection of $\mathcal{M}$ with the plane passing through $\mathbf{v}_{\mathbf{p}}$ meeting $\mathcal{M}$ perpendicularly.

Techniques for computing the shape operator and normal curvature are given in Section 13.3. The eigenvalues of the shape operator at $\mathbf{p} \in \mathcal{M}$ are studied at the end of that section. They turn out to be the maximum and minimum of the normal curvature at $\mathbf{p}$, the so-called *principal curvatures*, and the corresponding eigenvectors are orthogonal. The principal curvatures are graphed for the monkey saddle in Figure 13.6.

The most important curvature functions of a surface in $\mathbb{R}^3$ are the *Gaussian curvature* and the *mean curvature*, both defined in Section 13.4. This leads to a unified discussion of the the *first*, *second* and *third fundamental forms*. Two separate techniques for computing curvature from the parametric representation of a surface are described in Section 13.5, which includes calculations from Notebook 13 with reference to Monge patches and other examples.

Section 13.6 is devoted to a global curvature theorem, while we discuss how to compute the curvatures of nonparametric surfaces in Section 13.7.

13.1 The Shape Operator

We want to measure how a regular surface $\mathcal{M}$ bends in $\mathbb{R}^3$. A good way to do this is to estimate how the surface normal $\mathbf{U}$ changes from point to point. We use a linear operator called the shape operator to calculate the bending of $\mathcal{M}$. The shape operator came into wide use after its introduction in O'Neill's book [ON1]; however, it occurs much earlier, for example, implicitly in Blashke's classical book [Blas2][1], and explicitly in [BuBu][2].

The shape operator applied to a tangent vector $\mathbf{v_p}$ is the negative of the derivative of $\mathbf{U}$ in the direction $\mathbf{v_p}$:

Definition 13.1. *Let $\mathcal{M} \subset \mathbb{R}^3$ be a regular surface, and let $\mathbf{U}$ be a surface normal to $\mathcal{M}$ defined in a neighborhood of a point $\mathbf{p} \in \mathcal{M}$. For a tangent vector $\mathbf{v_p}$ to $\mathcal{M}$ at $\mathbf{p}$ we put*

$$S(\mathbf{v_p}) = -\boldsymbol{D}_{\mathbf{v}}\mathbf{U}.$$

*Then S is called the **shape operator**.*

Derivatives of vector fields were discussed in Section 9.5, and regular surfaces in Section 10.3. The precise definition of $\boldsymbol{D}_{\mathbf{v}}\mathbf{U}$ relies on Lemma 9.40, page 281, since $\mathbf{U}$ is not defined away from the surface $\mathcal{M}$.

It is easy to see that the shape operator of a plane is identically zero at all points of the plane. For a nonplanar surface the surface normal $\mathbf{U}$ will twist and turn from point to point, and S will be nonzero. At any point of an orientable regular surface there are two choices for the unit normal: $\mathbf{U}$ and $-\mathbf{U}$. The shape operator corresponding to $-\mathbf{U}$ is the negative of the shape operator corresponding to $\mathbf{U}$. If $\mathcal{M}$ is nonorientable, we have seen that a surface normal $\mathbf{U}$ cannot be defined continuously on all of $\mathcal{M}$. This does not matter in the present chapter, because all calculations involving $\mathbf{U}$ are local, so it suffices to perform the local calculations on an open subset $\mathcal{U}$ of $\mathcal{M}$ where $\mathbf{U}$ *is* defined continuously.

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[1] 
Wilhelm Johann Eugen Blaschke (1885–1962). Austrian-German mathematician. In 1919 he was appointed to a chair in Hamburg, where he built an important school of differential geometry.

[2] Around 1900 the Gibbs–Heaviside vector analysis notation (which one can read about in the interesting book [Crowe]) spread to differential geometry, although its use was controversial for another 30 years. Blashke's [Blas2] was one of the first differential geometry books to use vector analysis. The book coauthored by Burali-Forti and Burgatti [BuBu] contains an amusing attack on those resisting the new vector notation. In the multivolume works of Darboux [Darb1] and Bianchi [Bian] most formulas are written component by component. Compact vector notation, of course, is indispensable nowadays both for humans and for computers.

Recall (Lemma 10.26, page 299) that any point $\mathbf{q}$ in the domain of definition of a regular patch $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ has a neighborhood $\mathcal{U}_{\mathbf{q}}$ of $\mathbf{q}$ such that $\mathbf{x}(\mathcal{U}_{\mathbf{q}})$ is a regular surface. Therefore, the shape operator of a regular patch is also defined. Conversely, we can use patches on a regular surface $\mathcal{M} \subset \mathbb{R}^3$ to calculate the shape operator of $\mathcal{M}$.

The next lemmas establish some elementary properties.

Lemma 13.2. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ be a regular patch. Then*

$$S(\mathbf{x}_u) = -\mathbf{U}_u \qquad \text{and} \qquad S(\mathbf{x}_v) = -\mathbf{U}_v.$$

Proof. Fix v and define a curve $\boldsymbol{\alpha}$ by $\boldsymbol{\alpha}(u) = \mathbf{x}(u, v)$. Then by Lemma 9.40, page 281, we have

$$S\big(\mathbf{x}_u(u, v)\big) = S\big(\boldsymbol{\alpha}'(u)\big) = -\boldsymbol{D}_{\alpha'(u)}\mathbf{U} = -(\mathbf{U} \circ \boldsymbol{\alpha})'(u).$$

But $(\mathbf{U} \circ \boldsymbol{\alpha})'$ is just $\mathbf{U}_u$. Similarly, $S(\mathbf{x}_v) = -\mathbf{U}_v$. ∎

Lemma 13.3. *At each point $\mathbf{p}$ of a regular surface $\mathcal{M} \subset \mathbb{R}^3$, the shape operator is a linear map*

$$S\colon \mathcal{M}_{\mathbf{p}} \longrightarrow \mathcal{M}_{\mathbf{p}}.$$

Proof. That S is linear follows from the fact that $\boldsymbol{D}_{a\mathbf{v}+b\mathbf{w}} = a\boldsymbol{D}_{\mathbf{v}} + b\boldsymbol{D}_{\mathbf{w}}$. To prove that S maps $\mathcal{M}_{\mathbf{p}}$ into $\mathcal{M}_{\mathbf{p}}$ (instead of merely into $\mathbb{R}^3_{\mathbf{p}}$), we differentiate the equation $\mathbf{U} \cdot \mathbf{U} = 1$ and use (9.12) on page 276:

$$0 = \mathbf{v}_{\mathbf{p}}[\mathbf{U} \cdot \mathbf{U}] = 2(\boldsymbol{D}_{\mathbf{v}}\mathbf{U}) \cdot \mathbf{U}(\mathbf{p}) = -2S(\mathbf{v}_{\mathbf{p}}) \cdot \mathbf{U}(\mathbf{p}),$$

for any tangent vector $\mathbf{v}_{\mathbf{p}}$. Since $S(\mathbf{v}_{\mathbf{p}})$ is perpendicular to $\mathbf{U}(\mathbf{p})$, it must be tangent to $\mathcal{M}$; that is, $S(\mathbf{v}_{\mathbf{p}}) \in \mathcal{M}_{\mathbf{p}}$. ∎

Next, we find an important relation between the shape operator of a surface and the acceleration of a curve on the surface.

Lemma 13.4. *If $\boldsymbol{\alpha}$ is a curve on a regular surface $\mathcal{M} \subset \mathbb{R}^3$, then*

$$\boldsymbol{\alpha}'' \cdot \mathbf{U} = S(\boldsymbol{\alpha}') \cdot \boldsymbol{\alpha}'.$$

Proof. We restrict the vector field $\mathbf{U}$ to the curve $\boldsymbol{\alpha}$ and use Lemma 9.40. Since $\boldsymbol{\alpha}(t) \in \mathcal{M}$ for all t, the velocity $\boldsymbol{\alpha}'$ is always tangent to $\mathcal{M}$, so

$$\boldsymbol{\alpha}' \cdot \mathbf{U} = 0.$$

When we differentiate this equation and use Lemmas 13.2 and 13.3, we obtain

$$\boldsymbol{\alpha}'' \cdot \mathbf{U} = -\mathbf{U}' \cdot \boldsymbol{\alpha}' = S(\boldsymbol{\alpha}') \cdot \boldsymbol{\alpha}'. \quad ∎$$

Observe that $\boldsymbol{\alpha}'' \cdot \mathbf{U}$ can be interpreted geometrically as the *component of the acceleration of $\boldsymbol{\alpha}$ that is perpendicular to $\mathcal{M}$.*

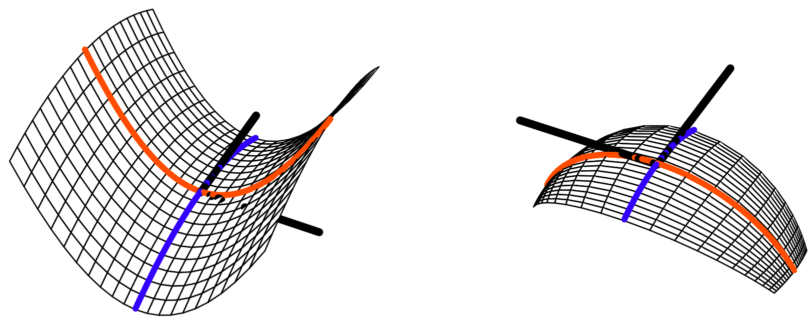

Figure 13.1: Coordinate curves and tangent vectors, together with (on the right) their Gauss images

Lemma 13.2 allows us to illustrate $-S$ by its effect on tangent vectors to coordinate curves to $\mathcal{M}$. These are mapped to the tangent vectors to the corresponding curves on the unit sphere defined by $\mathbf{U}$. This is expressed more invariantly by the next result that asserts that $-S$ is none other than the tangent map of the Gauss map.

Proposition 13.5. *Let $\mathcal{M}$ be a regular surface in $\mathbb{R}^3$ oriented by a unit normal vector field $\mathbf{U}$. View $\mathbf{U}$ as the Gauss map $\mathbf{U}\colon \mathcal{M} \to S^2(1)$, where $S^2(1)$ denotes the unit sphere in $\mathbb{R}^3$. If $\mathbf{v_p}$ is a tangent vector to $\mathcal{M}$ at $\mathbf{p} \in \mathcal{M}$, then $\mathbf{U}_*(\mathbf{v_p})$ is parallel to $-S(\mathbf{v_p}) \in \mathcal{M}_\mathbf{p}$.*

Proof. By Lemma 9.10, page 269, we have

$$\mathbf{U}_*(\mathbf{v_p}) = \big(\mathbf{v_p}[u_1],\ \mathbf{v_p}[u_2],\ \mathbf{v_p}[u_3]\big)_{\mathbf{U(p)}}.$$

On the other hand, Lemma 9.28, page 275, implies that

$$-S(\mathbf{v_p}) = \boldsymbol{D}_\mathbf{v}\mathbf{U} = \big(\mathbf{v_p}[u_1],\ \mathbf{v_p}[u_2],\ \mathbf{v_p}[u_3]\big)_{\mathbf{p}}.$$

Since the vectors $(\mathbf{v_p}[u_1], \mathbf{v_p}[u_2], \mathbf{v_p}[u_3])_{\mathbf{U(p)}}$ and $(\mathbf{v_p}[u_1], \mathbf{v_p}[u_2], \mathbf{v_p}[u_3])_{\mathbf{p}}$ are parallel, the lemma follows. ▮

We conclude this introductory section by noting a fundamental relationship between shape operators of surfaces and Euclidean motions of $\mathbb{R}^3$.

Theorem 13.6. *Let $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ be an orientation-preserving Euclidean motion, and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be oriented regular surfaces such that $F(\mathcal{M}_1) = \mathcal{M}_2$. Then*

(i) *the unit normals* $\mathbf{U}_1$ *and* $\mathbf{U}_2$ *of* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *can be chosen so that* $F_*(\mathbf{U}_1) = \mathbf{U}_2$;

(ii) *the shape operators* S_1 *and* S_2 *of the two surfaces* (*with the choice of unit normals given by* (i)) *are related by* $S_2 \circ F_* = F_* \circ S_1$.

Proof. Since F_* preserves lengths and inner products, it follows that $F_*(\mathbf{U}_1)$ is perpendicular to $\mathcal{M}_2$ and has unit length. Hence $F_*(\mathbf{U}_1) = \pm\mathbf{U}_2$; we choose the plus sign at all points of $\mathcal{M}_2$. Let $\mathbf{p} \in \mathcal{M}_1$ and $\mathbf{v_p} \in \mathcal{M}_{1\mathbf{p}}$, and let $\mathbf{w}_{F(\mathbf{p})} = F_*(\mathbf{v_p})$. A Euclidean motion is an affine transformation, so by Lemma 9.35, page 278, we have

$$\begin{aligned}(S_2 \circ F_*)(\mathbf{v_p}) = S_2(\mathbf{w}_{F(\mathbf{p})}) = -\boldsymbol{D}_{\mathbf{w}}\mathbf{U}_2 \;&=\; -\boldsymbol{D}_{\mathbf{w}}F_*(\mathbf{U}_1)\\ &=\; -F_*(\boldsymbol{D}_{\mathbf{v}}\mathbf{U}_1) = (F_* \circ S_1)(\mathbf{v_p}).\end{aligned}$$

Because $\mathbf{v_p}$ is arbitrary, we have $S_2 \circ F_* = F_* \circ S_1$. ▮

13.2 Normal Curvature

Although the shape operator does the job of measuring the bending of a surface in different directions, frequently it is useful to have a real-valued function, called the ***normal curvature***, which does the same thing. We shall define it in terms of the shape operator, though it is worth bearing in mind that normal curvature is explicitly a much older concept (see [Euler3], [Meu] and Corollary 13.20).

First, we need to make precise the notion of direction on a surface.

Definition 13.7. *A* ***direction*** ℓ *on a regular surface* $\mathcal{M}$ *is a* 1*-dimensional subspace of* (*that is, a line through the origin in*) *a tangent space to* $\mathcal{M}$.

A nonzero vector $\mathbf{v_p}$ in a tangent space $\mathcal{M}_\mathbf{p}$ determines a unique 1-dimensional subspace ℓ, so we can use the terminology 'the direction $\mathbf{v_p}$' to mean ℓ, provided the sign of $\mathbf{v_p}$ is irrelevant.

Definition 13.8. *Let* $\mathbf{u_p}$ *be a tangent vector to a regular surface* $\mathcal{M} \subset \mathbb{R}^3$ *with* $\|\mathbf{u_p}\| = 1$. *Then the* ***normal curvature of*** $\mathcal{M}$ ***in the direction*** $\mathbf{u_p}$ *is*

$$k(\mathbf{u_p}) = S(\mathbf{u_p}) \cdot \mathbf{u_p}.$$

More generally, if $\mathbf{v_p}$ *is any nonzero tangent vector to* $\mathcal{M}$ *at* $\mathbf{p}$, *we put*

$$k(\mathbf{v_p}) = \frac{S(\mathbf{v_p}) \cdot \mathbf{v_p}}{\|\mathbf{v_p}\|^2}. \tag{13.1}$$

If ℓ is a direction in a tangent space $\mathcal{M}_{\mathbf{p}}$ to a regular surface $\mathcal{M} \subset \mathbb{R}^3$, then $k(\mathbf{v_p})$ is easily seen to be the same for all nonzero tangent vectors $\mathbf{v_p}$ in ℓ (this is Exercise 9). Therefore, we call the common value of the normal curvature the ***normal curvature of the direction*** ℓ.

Let us single out two kinds of directions.

Definition 13.9. *Let ℓ be a direction in a tangent space $\mathcal{M}_{\mathbf{p}}$, where $\mathcal{M} \subset \mathbb{R}^3$ is a regular surface. If the normal curvature of ℓ is zero, we say that ℓ is an* ***asymptotic direction****. Similarly, if the normal curvature vanishes on a tangent vector $\mathbf{v_p}$ to $\mathcal{M}$, we say that $\mathbf{v_p}$ is an* ***asymptotic vector****. An* ***asymptotic curve*** *on $\mathcal{M}$ is a curve whose trace lies on $\mathcal{M}$ and whose tangent vector is everywhere asymptotic.*

Asymptotic curves will be studied in detail in Chapter 18.

Definition 13.10. *Let $\mathcal{M}$ be a regular surface in $\mathbb{R}^3$ and let $\mathbf{p} \in \mathcal{M}$. The maximum and minimum values of the normal curvature k of $\mathcal{M}$ at $\mathbf{p}$ are called the* ***principal curvatures*** *of $\mathcal{M}$ at $\mathbf{p}$ and are denoted by k_1 and k_2. Unit vectors $\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{M}_{\mathbf{p}}$ at which these extreme values occur are called* ***principal vectors****. The corresponding directions are called* ***principal directions****. A* ***principal curve*** *on $\mathcal{M}$ is a curve whose trace lies on $\mathcal{M}$ and whose tangent vector is everywhere principal.*

The principal curvatures measure the maximum and minimum bending of a regular surface $\mathcal{M}$ at each point $\mathbf{p} \in \mathcal{M}$. Principal curves will be studied again in Section 15.2, and in more detail in Chapter 19.

There is an important interpretation of normal curvature of a regular surface as the curvature of a space curve.

Lemma 13.11. (Meusnier) *Let $\mathbf{u_p}$ be a unit tangent vector to $\mathcal{M}$ at $\mathbf{p}$, and let $\boldsymbol{\beta}$ be a unit-speed curve in $\mathcal{M}$ with $\boldsymbol{\beta}(0) = \mathbf{p}$ and $\boldsymbol{\beta}'(0) = \mathbf{u_p}$. Then*

$$k(\mathbf{u_p}) = \boldsymbol{\kappa}[\boldsymbol{\beta}](0)\cos\theta, \tag{13.2}$$

where $\boldsymbol{\kappa}[\boldsymbol{\beta}](0)$ is the curvature of $\boldsymbol{\beta}$ at 0, and θ is the angle between the normal $\mathbf{N}(0)$ of $\boldsymbol{\beta}$ and the surface normal $\mathbf{U}(\mathbf{p})$. Thus all curves lying on a surface $\mathcal{M}$ and having the same tangent line at a given point $\mathbf{p} \in \mathcal{M}$ have the same normal curvature at $\mathbf{p}$.

Proof. Suppose that $\boldsymbol{\kappa}[\boldsymbol{\beta}](0) \neq 0$. By Lemma 13.4 and Theorem 7.10, page 197, we have

$$\begin{aligned} k(\mathbf{u_p}) = S(\mathbf{u_p}) \cdot \mathbf{u_p} &= \boldsymbol{\beta}''(0) \cdot \mathbf{U}(\mathbf{p}) \\ &= \boldsymbol{\kappa}[\boldsymbol{\beta}](0)\mathbf{N}(0) \cdot \mathbf{U}(\mathbf{p}) = \boldsymbol{\kappa}[\boldsymbol{\beta}](0)\cos\theta. \end{aligned} \tag{13.3}$$

In the exceptional case that $\boldsymbol{\kappa}[\boldsymbol{\beta}](0) = 0$, the normal $\mathbf{N}(0)$ is not defined, but we still have $k(\mathbf{u_p}) = 0$. ▮

To understand the meaning of normal curvature geometrically, we need to find curves on a surface to which we can apply Lemma 13.11.

Definition 13.12. *Let $\mathcal{M} \subset \mathbb{R}^3$ be a regular surface and $\mathbf{u_p}$ a unit tangent vector to $\mathcal{M}$. Denote by $\Pi\big(\mathbf{u_p}, \mathbf{U}(\mathbf{p})\big)$ the plane determined by $\mathbf{u_p}$ and the surface normal $\mathbf{U}(\mathbf{p})$. The **normal section** of $\mathcal{M}$ in the $\mathbf{u_p}$ direction is the intersection of $\Pi\big(\mathbf{u_p}, \mathbf{U}(\mathbf{p})\big)$ and $\mathcal{M}$.*

Corollary 13.13. *Let $\boldsymbol{\beta}$ be a unit-speed curve which lies in the intersection of a regular surface $\mathcal{M} \subset \mathbb{R}^3$ and one of its normal sections Π through $\mathbf{p} \in \mathcal{M}$. Assume that $\boldsymbol{\beta}(0) = \mathbf{p}$ and put $\mathbf{u_p} = \boldsymbol{\beta}'(0)$. Then the normal curvature $k(\mathbf{u_p})$ of $\mathcal{M}$ and the curvature of $\boldsymbol{\beta}$ are related by*

$$k(\mathbf{u_p}) = \pm\boldsymbol{\kappa}[\boldsymbol{\beta}](0). \tag{13.4}$$

Proof. We may assume that $\boldsymbol{\kappa}[\boldsymbol{\beta}](0) \neq 0$, for otherwise (13.4) is an obvious consequence of (13.2). Since $\boldsymbol{\beta}$ has unit speed, $\boldsymbol{\kappa}[\boldsymbol{\beta}](0)\mathbf{N}(0) = \boldsymbol{\beta}''(0)$ is perpendicular to $\boldsymbol{\beta}'(0)$. On the other hand, both $\mathbf{U}(\mathbf{p})$ and $\mathbf{N}(0)$ lie in Π, so the only possibility is $\mathbf{N}(0) = \pm\mathbf{U}(\mathbf{p})$. Hence $\cos\theta = \pm1$ in (13.2), and so we obtain (13.4). ∎

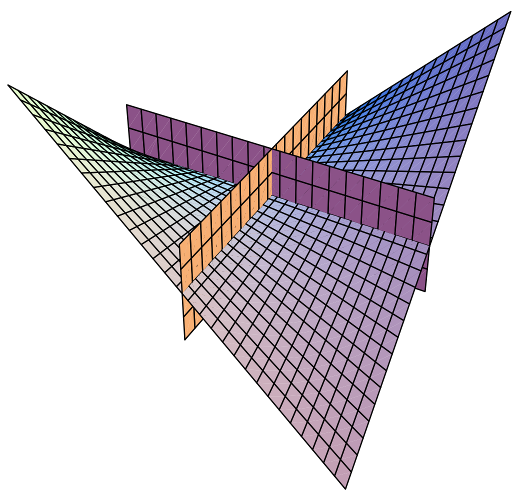

Figure 13.2: Normal sections to a paraboloid through asymptotic curves

Corollary 13.13 gives an excellent method for estimating normal curvature visually. For a regular surface $\mathcal{M} \subset \mathbb{R}^3$, suppose we want to know the normal curvature in various directions at $\mathbf{p} \in \mathcal{M}$. Each unit vector $\mathbf{u_p} \in \mathcal{M}_\mathbf{p}$, together with the surface normal $\mathbf{U}(\mathbf{p})$, determines a plane $\Pi\big(\mathbf{u_p}, \mathbf{U}(\mathbf{p})\big)$. Then the normal section in the direction $\mathbf{u_p}$ is the intersection of $\Pi\big(\mathbf{u_p}, \mathbf{U}(\mathbf{p})\big)$ and $\mathcal{M}$. This is a plane curve in $\Pi\big(\mathbf{u_p}, \mathbf{U}(\mathbf{p})\big)$ whose curvature is given by (13.4). There are three cases:

- If $k(\mathbf{u_p}) > 0$, then the normal section is bending in the same direction as $\mathbf{U}(\mathbf{p})$. Hence in the $\mathbf{u_p}$ direction $\mathcal{M}$ is bending toward $\mathbf{U}(\mathbf{p})$.
- If $k(\mathbf{u_p}) < 0$, then the normal section is bending in the opposite direction from $\mathbf{U}(\mathbf{p})$. Hence in the $\mathbf{u_p}$ direction $\mathcal{M}$ is bending away from $\mathbf{U}(\mathbf{p})$.
- If $k(\mathbf{u_p}) = 0$, then the curvature of the normal section vanishes at $\mathbf{p}$ so the normal to a curve in the normal section is undefined. It is impossible to conclude that there is no bending of $\mathcal{M}$ in the $\mathbf{u_p}$ direction since $\boldsymbol{\kappa}[\boldsymbol{\beta}]$ might vanish only at $\mathbf{p}$. But in some sense the bending is small.

As the unit tangent vector $\mathbf{u_p}$ turns, the surface may bend in different ways. A good example of this occurs at the center point of a hyperbolic paraboloid.

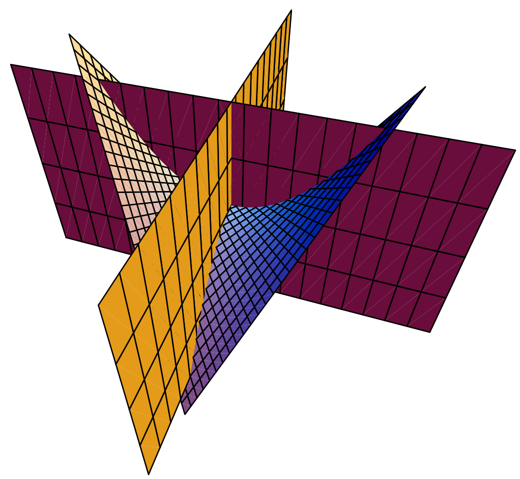

Figure 13.3: Normal sections to a paraboloid through principal curves

In Figure 13.2, both normal sections intersect the hyperbolic paraboloid in straight lines, and the normal curvature determined by each of these sections vanishes. These straight lines are in fact asymptotic curves, whereas the sections shown in Figure 13.3 intersect the surface in curves tangent to principal directions (recall Definitions 13.9 and 13.10). In the second case, the normal curvature determined by one section is positive, and that determined by the other is negative.

The normal sections at the center of a monkey saddle are similar to those of the hyperbolic paraboloid, but more complicated. In this case, there are three asymptotic directions passing through the center point $\mathbf{o}$ of the monkey saddle. It is this fact that forces S to vanish as a linear transformation of the tangent space $T_{\mathbf{o}}\mathcal{M}$ at the point $\mathbf{o}$ itself.

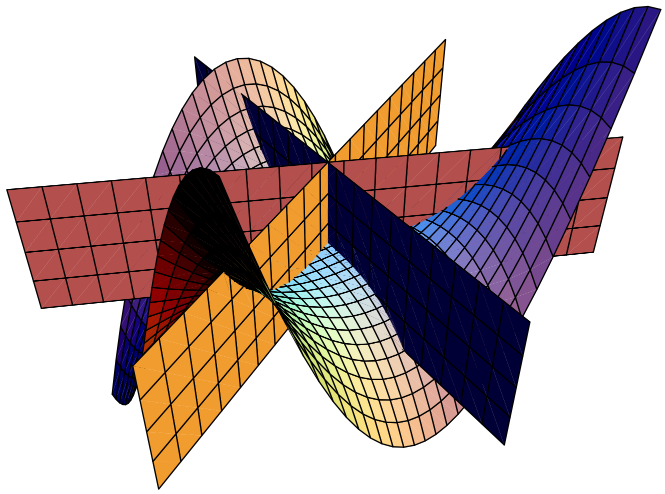

Figure 13.4: Normal sections to a monkey saddle

13.3 Calculation of the Shape Operator

Symmetric linear transformations are much easier to work with than general linear transformations. We shall exploit this in developing the theory of the shape operator, which fortunately falls into this category.

Lemma 13.14. The shape operator of a regular surface $\mathcal{M}$ is ***symmetric*** *or* ***self-adjoint****, meaning that*

$$S(\mathbf{v_p}) \cdot \mathbf{w_p} = \mathbf{v_p} \cdot S(\mathbf{w_p})$$

for all tangent vectors $\mathbf{v_p}, \mathbf{w_p}$ to $\mathcal{M}$.

Proof. Let $\mathbf{x}$ be an injective regular patch on $\mathcal{M}$. We differentiate the formula $\mathbf{U} \cdot \mathbf{x}_u = 0$ with respect to v and obtain

$$0 = \frac{\partial}{\partial v}(\mathbf{U} \cdot \mathbf{x}_u) = \mathbf{U}_v \cdot \mathbf{x}_u + \mathbf{U} \cdot \mathbf{x}_{uv}, \tag{13.5}$$

where $\mathbf{U}_v$ is the derivative of the vector field $v \mapsto \mathbf{U}(u,v)$ along any v-parameter curve. Since $\mathbf{U}_v = -S(\mathbf{x}_v)$, equation (13.5) becomes

$$S(\mathbf{x}_v) \cdot \mathbf{x}_u = \mathbf{U} \cdot \mathbf{x}_{uv}. \tag{13.6}$$

Similarly,

$$S(\mathbf{x}_u) \cdot \mathbf{x}_v = \mathbf{U} \cdot \mathbf{x}_{vu}. \tag{13.7}$$

From (13.6), (13.7) and the fact that $\mathbf{x}_{uv} = \mathbf{x}_{vu}$ we get

$$S(\mathbf{x}_u)\cdot\mathbf{x}_v = \mathbf{U}\cdot\mathbf{x}_{vu} = \mathbf{U}\cdot\mathbf{x}_{uv} = S(\mathbf{x}_v)\cdot\mathbf{x}_u. \tag{13.8}$$

The proof is completed by expressing $\mathbf{v_p}, \mathbf{w_p}$ in terms of $\mathbf{x}_u, \mathbf{x}_v$ and using linearity. ▮

Definition 13.15. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ be a regular patch. Then*

$$\begin{cases} e \;=\; -\mathbf{U}_u\cdot\mathbf{x}_u \;=\; \mathbf{U}\cdot\mathbf{x}_{uu}, \\ f \;=\; -\mathbf{U}_v\cdot\mathbf{x}_u \;=\; \mathbf{U}\cdot\mathbf{x}_{uv} \;=\; \mathbf{U}\cdot\mathbf{x}_{vu} \;=\; -\mathbf{U}_u\cdot\mathbf{x}_v, \\ g \;=\; -\mathbf{U}_v\cdot\mathbf{x}_v \;=\; \mathbf{U}\cdot\mathbf{x}_{vv}. \end{cases} \tag{13.9}$$

Classically, e, f, g are called the ***coefficients of the second fundamental form*** of $\mathbf{x}$.

In Section 12.1 we wrote the metric as

$$ds^2 = E\,du^2 + 2F\,du dv + G\,dv^2,$$

and E, F, G are called the ***coefficients of the first fundamental form*** of $\mathbf{x}$. The quantity

$$e\,du^2 + 2f\,du dv + g\,dv^2$$

has a more indirect interpretation, given on page 402. The notation e, f, g is that used in most classical differential geometry books, though many authors use L, M, N in their place. Incidentally, Gauss used the notation D, D', D'' for the respective quantities $e\sqrt{EG-F^2}$, $f\sqrt{EG-F^2}$, $g\sqrt{EG-F^2}$.

Theorem 13.16. (The Weingarten[3] equations) *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ be a regular patch. Then the shape operator S of $\mathbf{x}$ is given in terms of the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ by*

$$\begin{cases} -S(\mathbf{x}_u) \;=\; \mathbf{U}_u \;=\; \dfrac{f\,F - e\,G}{EG-F^2}\mathbf{x}_u + \dfrac{e\,F - f\,E}{EG-F^2}\mathbf{x}_v, \\[2ex] -S(\mathbf{x}_v) \;=\; \mathbf{U}_v \;=\; \dfrac{g\,F - f\,G}{EG-F^2}\mathbf{x}_u + \dfrac{f\,F - g\,E}{EG-F^2}\mathbf{x}_v. \end{cases} \tag{13.10}$$

Proof. Since $\mathbf{x}$ is regular, and $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent, we can write

$$\begin{cases} -S(\mathbf{x}_u) = \mathbf{U}_u = a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \\ -S(\mathbf{x}_v) = \mathbf{U}_v = a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v, \end{cases} \tag{13.11}$$

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[3]  Julius Weingarten (1836–1910). Professor at the Technische Universität in Berlin. A surface for which there is a definite functional relation between the principal curvatures is called a *Weingarten surface*.

for some functions $a_{11}, a_{21}, a_{12}, a_{22}$, which we need to compute. We take the scalar product of each of the equations in (13.11) with $\mathbf{x}_u$ and $\mathbf{x}_v$, and obtain

$$\begin{cases} -e = Ea_{11} + Fa_{21}, \\ -f = Fa_{11} + Ga_{21}, \\ -f = Ea_{12} + Fa_{22}, \\ -g = Fa_{12} + Ga_{22}. \end{cases} \tag{13.12}$$

Equations (13.12) can be written more concisely in terms of matrices:

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix};$$

hence

$$\begin{aligned} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \\ &= \frac{-1}{EG-F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \\ &= \frac{-1}{EG-F^2} \begin{pmatrix} Ge-Ff & Gf-Fg \\ -Fe+Ef & -Ff+Eg \end{pmatrix}. \end{aligned}$$

The result follows from (13.11). ∎

Although S is a symmetric linear operator, its matrix (a_{ij}) relative to $\{\mathbf{x}_u, \mathbf{x}_v\}$ need not be symmetric, because $\mathbf{x}_u$ and $\mathbf{x}_v$ are not in general perpendicular to one another.

There is also a way to express the normal curvature in terms of E, F, G and e, f, g:

Lemma 13.17. *Let $\mathcal{M} \subset \mathbb{R}^3$ be a regular surface and let $\mathbf{p} \in \mathcal{M}$. Let $\mathbf{x}$ be an injective regular patch on $\mathcal{M}$ with $\mathbf{p} = \mathbf{x}(u_0, v_0)$. Let $\mathbf{v_p} \in \mathcal{M}_\mathbf{p}$ and write*

$$\mathbf{v_p} = a\,\mathbf{x}_u(u_0, v_0) + b\,\mathbf{x}_v(u_0, v_0).$$

Then the normal curvature of $\mathcal{M}$ in the direction $\mathbf{v_p}$ is

$$k(\mathbf{v_p}) = \frac{e\,a^2 + 2f\,ab + g\,b^2}{E\,a^2 + 2F\,ab + G\,b^2}.$$

Proof. We have $\|\mathbf{v_p}\|^2 = \|a\mathbf{x}_u + b\mathbf{x}_v\|^2 = a^2E + 2abF + b^2G$, and

$$S(\mathbf{v_p}) \cdot \mathbf{v_p} = \big(a\,S(\mathbf{x}_u) + b\,S(\mathbf{x}_v)\big) \cdot (a\,\mathbf{x}_u + b\,\mathbf{x}_v) = a^2e + 2abf + b^2g.$$

The result follows from (13.1). ∎

Eigenvalues of the Shape Operator

We first recall an elementary version of the spectral theorem in linear algebra.

Lemma 13.18. *Let V be a real n-dimensional vector space with an inner product and let $A\colon V \to V$ be a linear transformation that is self-adjoint with respect to the inner product. Then the eigenvalues of A are real and A is diagonalizable: there is an orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of V such that*

$$A\mathbf{e}_j = \lambda_j \mathbf{e}_j \qquad \textit{for} > j = 1, \ldots, n.$$

Lemma 13.14 tells us that the shape operator S is a self-adjoint linear operator on each tangent space to a regular surface in $\mathbb{R}^3$, so the eigenvalues of S must be real. These eigenvalues are important geometric quantities associated with each regular surface in $\mathbb{R}^3$. Instead of proving Lemma 13.18 in its full generality, we prove it for the special case of the shape operator.

Lemma 13.19. *The eigenvalues of the shape operator S of a regular surface $\mathcal{M} \subset \mathbb{R}^3$ at $\mathbf{p} \in \mathcal{M}$ are precisely the principal curvatures k_1 and k_2 of $\mathcal{M}$ at $\mathbf{p}$. The corresponding unit eigenvectors are unit principal vectors, and vice versa. If $k_1 = k_2$, then S is scalar multiplication by their common value. Otherwise, the eigenvectors $\mathbf{e}_1, \mathbf{e}_2$ of S are perpendicular, and S is given by*

$$S\mathbf{e}_1 = k_1\mathbf{e}_1, \qquad S\mathbf{e}_2 = k_2\mathbf{e}_2. \tag{13.13}$$

Proof. Consider the normal curvature as a function $k\colon S^1_{\mathbf{p}} \to \mathbb{R}$, where $S^1_{\mathbf{p}}$ is the set of unit tangent vectors in the tangent space $\mathcal{M}_{\mathbf{p}}$. Since $S^1_{\mathbf{p}}$ is a circle, it is compact, and so k achieves its maximum at some unit vector, call it $\mathbf{e}_1 \in S^1_{\mathbf{p}}$. Choose $\mathbf{e}_2$ to be any vector in $S^1_{\mathbf{p}}$ perpendicular to $\mathbf{e}_1$, so $\{\mathbf{e}_1, \mathbf{e}_2\}$ is an orthonormal basis of $\mathcal{M}_{\mathbf{p}}$ and

$$\begin{cases} S\mathbf{e}_1 = (S\mathbf{e}_1 \cdot \mathbf{e}_1)\mathbf{e}_1 + (S\mathbf{e}_1 \cdot \mathbf{e}_2)\mathbf{e}_2, \\ S\mathbf{e}_2 = (S\mathbf{e}_2 \cdot \mathbf{e}_1)\mathbf{e}_1 + (S\mathbf{e}_2 \cdot \mathbf{e}_2)\mathbf{e}_2. \end{cases} \tag{13.14}$$

Define a function $\mathbf{u} = \mathbf{u}(\theta)$ by setting $\mathbf{u}(\theta) = \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta$, and write $k(\theta) = k\big(\mathbf{u}(\theta)\big)$. Then

$$k(\theta) = (S\mathbf{e}_1 \cdot \mathbf{e}_1)\cos^2\theta + 2(S\mathbf{e}_1 \cdot \mathbf{e}_2)\sin\theta\cos\theta + (S\mathbf{e}_2 \cdot \mathbf{e}_2)\sin^2\theta, \tag{13.15}$$

so that

$$\frac{d}{d\theta}k(\theta) = 2(S\mathbf{e}_2 \cdot \mathbf{e}_2 - S\mathbf{e}_1 \cdot \mathbf{e}_1)\sin\theta\cos\theta + 2(S\mathbf{e}_1 \cdot \mathbf{e}_2)(\cos^2\theta - \sin^2\theta).$$

In particular,

$$0 = \frac{dk}{d\theta}(0) = 2S\mathbf{e}_1 \cdot \mathbf{e}_2, \tag{13.16}$$

because $k(\theta)$ has a maximum at $\theta = 0$. Then (13.14) and (13.16) imply (13.13). From (13.13) it follows that both $\mathbf{e}_1$ and $\mathbf{e}_2$ are eigenvectors of S, and from (13.15) it follows that the principal curvatures of $\mathcal{M}$ at $\mathbf{p}$ are the eigenvalues of S. Hence the lemma follows. ▮

The principal curvatures determine the normal curvature completely:

Corollary 13.20. (Euler) *Let $k_1(\mathbf{p}), k_2(\mathbf{p})$ be the principal curvatures of a regular surface $\mathcal{M} \subset \mathbb{R}^3$ at $\mathbf{p} \in \mathcal{M}$, and let $\mathbf{e}_1, \mathbf{e}_2$ be the corresponding unit principal vectors. Let θ denote the oriented angle from $\mathbf{e}_1$ to $\mathbf{u_p}$, so that $\mathbf{u_p} = \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta$. Then the normal curvature $k(\mathbf{u_p})$ is given by*

$$k(\mathbf{u_p}) = k_1(\mathbf{p}) \cos^2\theta + k_2(\mathbf{p}) \sin^2\theta. \tag{13.17}$$

Proof. Since $S(\mathbf{e}_1) \cdot \mathbf{e}_2 = 0$, (13.15) reduces to (13.17). ▮

13.4 Gaussian and Mean Curvature

The notion of the curvature of a surface is a great deal more complicated than the notion of curvature of a curve. Let $\boldsymbol{\alpha}$ be a curve in $\mathbb{R}^3$, and let $\mathbf{p}$ be a point on the trace of $\boldsymbol{\alpha}$. The curvature of $\boldsymbol{\alpha}$ at $\mathbf{p}$ measures the rate at which $\boldsymbol{\alpha}$ leaves the tangent *line* to $\boldsymbol{\alpha}$ at $\mathbf{p}$. By analogy, the curvature of a surface $\mathcal{M} \subset \mathbb{R}^3$ at $\mathbf{p} \in \mathcal{M}$ should measure the rate at which $\mathcal{M}$ leaves the tangent *plane* to $\mathcal{M}$ at $\mathbf{p}$. But a difficulty arises for surfaces that was not present for curves: although a curve can separate from one of its tangent lines in only two directions, a surface separates from one of its tangent planes in infinitely many directions. In general, the rate of departure of a surface from one of its tangent planes depends on the direction.

There are several competing notions for the curvature of a surface in $\mathbb{R}^3$:

- the normal curvature k;
- the principal curvatures k_1, k_2;
- the mean curvature H;
- the Gaussian curvature K.

We defined normal curvature and the principal curvatures of a surface $\mathcal{M} \subset \mathbb{R}^3$ in Section 13.2. In the present section, we give the definitions of the Gaussian and mean curvatures; these are the most important functions in surface theory.

First, we recall some useful facts from linear algebra. If $S\colon V \to V$ is a linear transformation on a vector space V, we may define the ***determinant*** and ***trace*** of S, written $\det S$ and $\operatorname{tr} S$, merely as the determinant and trace of the matrix

A representing S with respect to any chosen basis. If P is an invertible matrix representing a change in basis then S is represented by $P^{-1}AP$ with respect to the new basis, but standard properties of the determinant and trace functions ensure that

$$\begin{aligned} \det(P^{-1}AP) &= \det A, \\ \operatorname{tr}(P^{-1}AP) &= \operatorname{tr} A, \end{aligned} \tag{13.18}$$

so that our definitions are independent of the choice of basis.

Definition 13.21. *Let $\mathcal{M}$ be a regular surface in $\mathbb{R}^3$. The* ***Gaussian curvature*** *K and* ***mean curvature*** *H of $\mathcal{M}$ are the functions $K, H\colon \mathcal{M} \to \mathbb{R}$ defined by*

$$K(\mathbf{p}) = \det\big(S(\mathbf{p})\big) \qquad \textit{and} \qquad H(\mathbf{p}) = \tfrac{1}{2}\operatorname{tr}\big(S(\mathbf{p})\big). \tag{13.19}$$

Note that although the shape operator S and the mean curvature H depend on the choice of unit normal $\mathbf{U}$, the Gaussian curvature K is independent of that choice. The name 'mean curvature' is due to Germain[4].

Definition 13.22. *A* ***minimal surface*** *in $\mathbb{R}^3$ is a regular surface for which the mean curvature vanishes identically. A regular surface is* ***flat*** *if and only if its Gaussian curvature vanishes identically.*

We shall see in Chapter 16 that surfaces of minimal area are indeed minimal in the sense of Definition 13.22.

The Gaussian curvature permits us to distinguish four kinds of points on a surface.

Definition 13.23. *Let $\mathbf{p}$ be a point on a regular surface $\mathcal{M} \subset \mathbb{R}^3$. We say that*

- $\mathbf{p}$ *is* ***elliptic*** *if $K(\mathbf{p}) > 0$ (equivalently, k_1 and k_2 have the same sign);*
- $\mathbf{p}$ *is* ***hyperbolic*** *if $K(\mathbf{p}) < 0$ (equivalently, k_1 and k_2 have opposite signs);*
- $\mathbf{p}$ *is* ***parabolic*** *if $K(\mathbf{p}) = 0$, but $S(\mathbf{p}) \neq 0$ (equivalently, exactly one of k_1 and k_2 is zero);*
- $\mathbf{p}$ *is* ***planar*** *if $K(\mathbf{p}) = 0$ and $S(\mathbf{p}) = 0$ (equivalently, $k_1 = k_2 = 0$).*

It is usually possible to glance at almost any surface and recognize which points are elliptic, hyperbolic, parabolic or planar. Consider, for example, the paraboloids shown in Figure 12.10 on page 375. Not surprisingly, all the points

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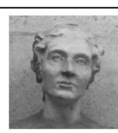
[4]  Sophie Germain (1776–1831). French mathematician, best known for her work on elasticity and Fermat's last theorem. Germain (under the pseudonym 'M. Blanc') corresponded with Gauss regarding her results in geometry and number theory.

on the left-hand elliptical paraboloid are elliptic, and all those on the right-hand hyperbolic paraboloid are hyperbolic. Calculations from the next section show that the monkey surface (13.25) has all its points hyperbolic except for its central point, $\mathbf{o} = (0, 0, 0)$, which is planar. This corresponds to the fact, illustrated in Figure 13.5, that its Gaussian curvature K both vanishes and achieves an absolute maximum at $\mathbf{o}$.

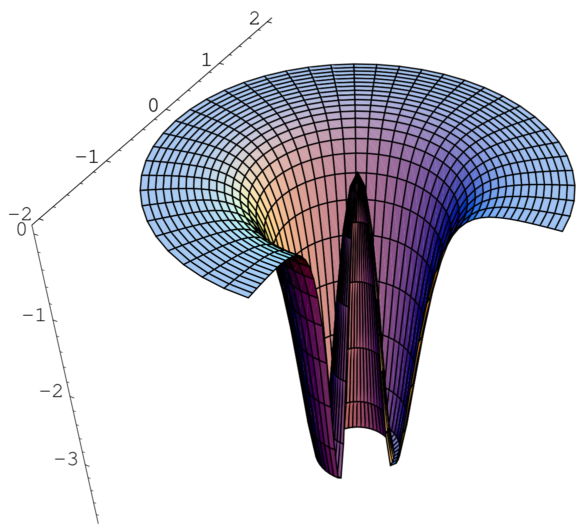


Figure 13.5: Gaussian curvature of the monkey saddle

There are two especially useful ways of choosing a basis of a tangent space to a surface in $\mathbb{R}^3$. Each gives rise to important formulas for the Gaussian and mean curvatures, which are presented in turn by the following proposition and subsequent theorem.

Proposition 13.24. *The Gaussian curvature and mean curvature of a regular surface $\mathcal{M} \subset \mathbb{R}^3$ are related to the principal curvatures by*

$$K = k_1 k_2 \qquad \textit{and} \qquad H = \tfrac{1}{2}(k_1 + k_2).$$

Proof. If we choose an orthonormal basis of eigenvectors of S for $\mathcal{M}_{\mathbf{p}}$, the matrix of S with respect to this basis is diagonal so that

$$K = \det\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} = k_1 k_2$$

$$H = \tfrac{1}{2}\operatorname{tr}\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} = \tfrac{1}{2}(k_1 + k_2). \quad \blacksquare$$

Let $\mathcal{M} \subset \mathbb{R}^3$ be a regular surface. The Gaussian and mean curvatures are functions $K, H\colon \mathcal{M} \to \mathbb{R}$; we have written $K(\mathbf{p})$ and $H(\mathbf{p})$ for their values at $\mathbf{p} \in \mathcal{M}$. We need a slightly different notation for a regular patch $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$. Strictly speaking, K and H are functions defined on $\mathbf{x}(\mathcal{U}) \to \mathbb{R}$. However, we follow conventional notation and abbreviate $K \circ \mathbf{x}$ to K and $H \circ \mathbf{x}$ to H. Thus $K(u, v)$ and $H(u, v)$ will denote the values of the Gaussian and mean curvatures at $\mathbf{x}(u, v)$.

Theorem 13.25. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ be a regular patch. The Gaussian curvature and mean curvature of $\mathbf{x}$ are given by the formulas*

$$K = \frac{eg - f^2}{EG - F^2}, \tag{13.20}$$

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)}, \tag{13.21}$$

where e, f, g are the coefficients of the second fundamental form relative to $\mathbf{x}$, and E, F, G are the coefficients of the first fundamental form.

Proof. This time we compute K and H using the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$, and the matrix

$$\frac{-1}{EG - F^2}\begin{pmatrix} Ge - Ff & Gf - Fg \\ -Fe + Ef & -Ff + Eg \end{pmatrix}$$

of Theorem 13.16. Taking the determinant and half the trace yields

$$K = \frac{(fF - eG)(fF - gE) - (eF - fE)(gF - fG)}{(EG - F^2)^2} = \frac{eg - f^2}{EG - F^2},$$

and

$$H = -\frac{(fF - eG) + (fF - gE)}{2(EG - F^2)} = \frac{eG - 2fF + gE}{2(EG - F^2)}. \blacksquare$$

The importance of Proposition 13.24 is theoretical, that of Theorem 13.25 more practical. Usually, one uses Theorem 13.25 to compute K and H, and afterwards Proposition 13.24 to find the principal curvatures. More explicitly,

Corollary 13.26. *The principal curvatures k_1, k_2 are the roots of the quadratic equation*

$$x^2 - 2Hx + K = 0.$$

Thus we can choose k_1, k_2 so that

$$k_1 = H + \sqrt{H^2 - K} \qquad \textit{and} \qquad k_2 = H - \sqrt{H^2 - K}. \tag{13.22}$$

Corollary 13.27. *Suppose that $\mathcal{M} \subset \mathbb{R}^3$ has negative Gaussian curvature K at $\mathbf{p}$. Then:*

(i) *there are exactly two asymptotic directions at $\mathbf{p}$, and they are bisected by the principal directions;*

(ii) *the two asymptotic directions at $\mathbf{p}$ are perpendicular if and only if the mean curvature H of $\mathcal{M}$ vanishes at $\mathbf{p}$.*

Proof. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be unit principal vectors corresponding to $k_1(\mathbf{p})$ and $k_2(\mathbf{p})$. Then $K(\mathbf{p}) = k_1(\mathbf{p})k_2(\mathbf{p}) < 0$ implies that $k_1(\mathbf{p})$ and $k_2(\mathbf{p})$ have opposite signs. Thus, there exists θ with $0 < \theta < \pi/2$ such that

$$\tan^2\theta = -\frac{k_1(\mathbf{p})}{k_2(\mathbf{p})}.$$

Put $\mathbf{u}_\mathbf{p}(\theta) = \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta$. Then (13.17) implies that $\mathbf{u}_\mathbf{p}(\theta)$ and $\mathbf{u}_\mathbf{p}(-\theta)$ are linearly independent asymptotic vectors at $\mathbf{p}$. The angle between $\mathbf{u}_\mathbf{p}(\theta)$ and $\mathbf{u}_\mathbf{p}(-\theta)$ is 2θ, and it is clear that $\mathbf{e}_1$ bisects the angle between $\mathbf{u}_\mathbf{p}(\theta)$ and $\mathbf{u}_\mathbf{p}(-\theta)$. This proves (i). For (ii) we observe that $H(\mathbf{p}) = 0$ if and only if θ equals $\pm\pi/4$, up to integral multiples of π. ∎

The Three Fundamental Forms

In classical differential geometry, there are frequent references to the 'second fundamental form' of a surface in $\mathbb{R}^3$, a notion that is essentially equivalent to the shape operator S. Such references can for example be found in the influential textbook [Eisen1] of Eisenhart[5].

Definition 13.28. *Let $\mathcal{M}$ be a regular surface in $\mathbb{R}^3$. The* ***second fundamental form*** *is the symmetric bilinear form $\mathbf{II}$ on a tangent space $\mathcal{M}_\mathbf{p}$ given by*

$$\mathbf{II}(\mathbf{v}_\mathbf{p}, \mathbf{w}_\mathbf{p}) = S(\mathbf{v}_\mathbf{p}) \cdot \mathbf{w}_\mathbf{p}$$

for $\mathbf{v}_\mathbf{p}, \mathbf{w}_\mathbf{p} \in \mathcal{M}_\mathbf{p}$.

Since there is a second fundamental form, there must be a ***first fundamental form***. It is nothing but the inner product between tangent vectors:

$$\mathbf{I}(\mathbf{v}_\mathbf{p}, \mathbf{w}_\mathbf{p}) = \mathbf{v}_\mathbf{p} \cdot \mathbf{w}_\mathbf{p}.$$

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5  Luther Pfahler Eisenhart (1876–1965). American differential geometer and dean at Princeton University.

Note that the first fundamental form **I** can in a sense defined whether or not the surface is in $\mathbb{R}^3$; this is the basis of theory to be discussed in Chapter 26.

The following lemma is an immediate consequence of the definitions.

Lemma 13.29. *Let* $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ *be a regular patch. Then*

$$\mathbf{I}(a\mathbf{x}_u + b\mathbf{x}_v,\ a\mathbf{x}_u + b\mathbf{x}_v) = E a^2 + 2F ab + G b^2,$$
$$\mathbf{II}(a\mathbf{x}_u + b\mathbf{x}_v,\ a\mathbf{x}_u + b\mathbf{x}_v) = e a^2 + 2f ab + g b^2.$$

The normal curvature is therefore given by

$$k(\mathbf{v_p}) = \frac{\mathbf{II}(\mathbf{v_p}, \mathbf{v_p})}{\mathbf{I}(\mathbf{v_p}, \mathbf{v_p})}$$

for any nonzero tangent vector $\mathbf{v_p}$.

This lemma explains why we call E, F, G the coefficients of the first fundamental form, and e, f, g the coefficients of the second fundamental form.

Finally, there is also a ***third fundamental form*** **III** for a surface in $\mathbb{R}^3$ given by

$$\mathbf{III}(\mathbf{v_p}, \mathbf{w_p}) = S(\mathbf{v_p}) \cdot S(\mathbf{w_p})$$

for $\mathbf{v_p}, \mathbf{w_p} \in \mathcal{M}_\mathbf{p}$. Note that **III**, in contrast to **II**, does not depend on the choice of surface normal **U**. The third fundamental form **III** contains no new information, since it is expressible in terms of **I** and **II**.

Lemma 13.30. *Let* $\mathcal{M} \subset \mathbb{R}^3$ *be a regular surface. Then the following relation holds between the first, second and third fundamental forms of* $\mathcal{M}$:

$$\mathbf{III} - 2H\,\mathbf{II} + K\,\mathbf{I} = 0, \tag{13.23}$$

where H *and* K *denote the mean curvature and Gaussian curvature of* $\mathcal{M}$.

Proof. Although (13.23) follows from Corollary 13.26 and the Cayley–Hamilton[6] Theorem (which states that a matrix satisfies its own characteristic polynomial), we prefer to give a direct proof.

First, note that the product $H\,\mathbf{II}$ is independent of the choice of surface normal **U**. Hence (13.23) makes sense whether or not $\mathcal{M}$ is orientable. To

6  Sir William Rowan Hamilton (1805–1865). Irish mathematician, best known for having been struck with the concept of quaternions as he crossed Brougham Bridge in Dublin (see Chapter 23), and for his work in dynamics.

prove it, we observe that since its left-hand side is a symmetric bilinear form, it suffices to show that for each $\mathbf{p} \in \mathcal{M}$ and some basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathcal{M}_{\mathbf{p}}$ we have

$$(\mathbf{III} - 2H\,\mathbf{II} + K\,\mathbf{I})(\mathbf{e}_i, \mathbf{e}_j) = 0, \tag{13.24}$$

for $i, j = 1, 2$. We choose $\mathbf{e}_1$ and $\mathbf{e}_2$ to be linearly independent principal vectors at $\mathbf{p}$. Then

$$(\mathbf{III} - 2H\,\mathbf{II} + K\,\mathbf{I})(\mathbf{e}_1, \mathbf{e}_2) = 0$$

because each term vanishes separately. Furthermore,

$$(\mathbf{III} - 2H\,\mathbf{II} + K\,\mathbf{I})(\mathbf{e}_i, \mathbf{e}_i) = k_i^2 - (k_1 + k_2)k_i + k_1 k_2 = 0$$

for $i = 1$ and 2, as required. ▮

13.5 More Curvature Calculations

In this section, we show how to compute by hand the Gaussian curvature K and the mean curvature H for a monkey saddle and a torus. Along the way we compute the coefficients of their first and second fundamental forms.

The Monkey Saddle

For the surface parametrized by

$$\mathbf{x}(u, v) = \mathsf{monkeysaddle}(u, v) = (u,\ v,\ u^3 - 3uv^2), \tag{13.25}$$

and described on page 304, we easily compute

$$\mathbf{x}_u(u, v) = (1, 0, 3u^2 - 3v^2), \qquad \mathbf{x}_v(u, v) = (0, 1, -6uv),$$

$$\mathbf{x}_{uu}(u, v) = (0, 0, 6u), \quad \mathbf{x}_{uv}(u, v) = (0, 0, -6v), \quad \mathbf{x}_{vv}(u, v) = (0, 0, -6u).$$

Therefore,

$$E = 1 + 9(u^2 - v^2)^2, \quad F = -18uv(u^2 - v^2), \quad G = 1 + 36u^2v^2,$$

and by inspection a unit surface normal is

$$\mathbf{U} = \frac{(-3u^2 + 3v^2,\ 6uv,\ 1)}{\sqrt{1 + 9u^4 + 18u^2v^2 + 9v^4}},$$

so that

$$\begin{aligned} e &= \mathbf{U} \cdot \mathbf{x}_{uu} = \frac{6u}{\sqrt{1 + 9u^4 + 18u^2v^2 + 9v^4}}, \\ f &= \mathbf{U} \cdot \mathbf{x}_{uv} = \frac{-6v}{\sqrt{1 + 9u^4 + 18u^2v^2 + 9v^4}}, \\ g &= \mathbf{U} \cdot \mathbf{x}_{vv} = \frac{-6u}{\sqrt{1 + 9u^4 + 18u^2v^2 + 9v^4}}. \end{aligned}$$

Theorem 13.25 yields

$$K = \frac{-36(u^2+v^2)}{(1+9u^4+18u^2v^2+9v^4)^2}, \quad H = \frac{-27u^5+54u^3v^2+81uv^4}{(1+9u^4+18u^2v^2+9v^4)^{3/2}}.$$

A glance at these expressions shows that $\mathbf{o} = (0,0,0)$ is a planar point of the monkey saddle and that every other point is hyperbolic. Furthermore, the Gaussian curvature of the monkey saddle is invariant under all rotations about the z-axis, even though the monkey saddle itself does *not* have this property.

The principal curvatures are determined by Corollary 13.26, and are easy to plot. Figure 13.6 shows graphically their singular nature at the point $\mathbf{o}$, which contrasts with the surface itself sandwiched in the middle.

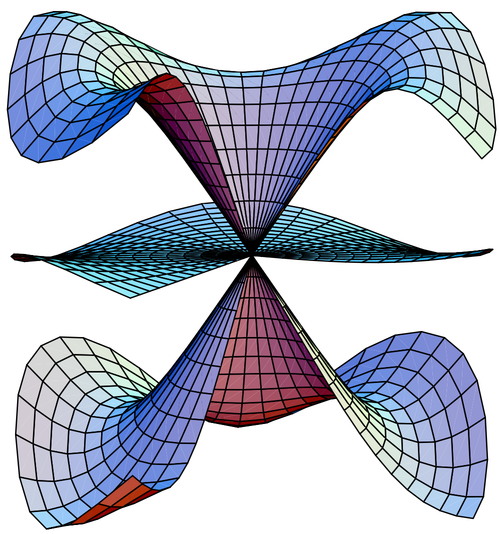

Figure 13.6: Principal curvatures of the monkey saddle

Alternative Formulas

Classically, the standard formulas for computing K and H for a patch $\mathbf{x}$ are (13.20) and (13.21). It is usually too tedious to compute K and H by hand in one step. Therefore, the functions E, F, G and e, f, g need to be computed before any of the curvature functions are calculated. The computation of E, F, G is straightforward: first one computes the first derivatives $\mathbf{x}_u$ and $\mathbf{x}_v$ and then the dot products $E = \mathbf{x}_u \cdot \mathbf{x}_u$, $F = \mathbf{x}_u \cdot \mathbf{x}_v$ and $G = \mathbf{x}_v \cdot \mathbf{x}_v$.

There are two methods for computing e, f, g. The direct approach using the definitions necessitates computing the surface normal via equation (10.11) on page 295, and then using the definition (13.9). The other method, explained in the next lemma, avoids computation of the surface normal; it uses instead the vector triple product which can be computed as a determinant.

Lemma 13.31. *Let* $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ *be a regular patch. Then*

$$e = \frac{[\mathbf{x}_{uu}\,\mathbf{x}_u\,\mathbf{x}_v]}{\sqrt{EG-F^2}}, \quad f = \frac{[\mathbf{x}_{uv}\,\mathbf{x}_u\,\mathbf{x}_v]}{\sqrt{EG-F^2}}, \quad g = \frac{[\mathbf{x}_{vv}\,\mathbf{x}_u\,\mathbf{x}_v]}{\sqrt{EG-F^2}}.$$

Proof. From (7.4) and (13.9) it follows that

$$e = \mathbf{x}_{uu} \cdot \mathbf{U} = \mathbf{x}_{uu} \cdot \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \frac{[\mathbf{x}_{uu}\,\mathbf{x}_u\,\mathbf{x}_v]}{\sqrt{EG-F^2}}.$$

The other formulas are proved similarly. ▮

On the other hand, at least theoretically, we can compute K and H for a regular patch $\mathbf{x}$ directly in terms of the first and second derivatives of $\mathbf{x}$. Here are the relevant formulas.

Corollary 13.32. *Let* $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ *be a regular patch. The Gaussian and mean curvatures of* $\mathcal{M}$ *are given by the formulas*

$$K = \frac{[\mathbf{x}_{uu}\,\mathbf{x}_u\,\mathbf{x}_v][\mathbf{x}_{vv}\,\mathbf{x}_u\,\mathbf{x}_v] - [\mathbf{x}_{uv}\,\mathbf{x}_u\,\mathbf{x}_v]^2}{\big(\|\mathbf{x}_u\|^2\|\mathbf{x}_v\|^2 - (\mathbf{x}_u \cdot \mathbf{x}_v)^2\big)^2}, \tag{13.26}$$

$$H = \frac{[\mathbf{x}_{uu}\,\mathbf{x}_u\,\mathbf{x}_v]\|\mathbf{x}_v\|^2 - 2[\mathbf{x}_{uv}\,\mathbf{x}_u\,\mathbf{x}_v](\mathbf{x}_u \cdot \mathbf{x}_v) + [\mathbf{x}_{vv}\,\mathbf{x}_u\,\mathbf{x}_v]\|\mathbf{x}_u\|^2}{2\big(\|\mathbf{x}_u\|^2\|\mathbf{x}_v\|^2 - (\mathbf{x}_u \cdot \mathbf{x}_v)^2\big)^{3/2}}. \tag{13.27}$$

Proof. The equations follow from (13.20), (13.21) and Lemma 13.31 when we write out e, f, g and E, F, G explicitly in terms of dot products. ▮

Equations 13.26 and 13.27 are used effectively in Notebook 13. They are usually too complicated for hand calculation, but we do use them below in the case of the torus.

It is neither enlightening nor useful to write out the formulas for the principal curvatures in terms of E, F, G, e, f, g in general. However, there is one special case when such formulas for k_1, k_2 are worth noting.

Corollary 13.33. *Let* $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ *be a regular patch for which* $f = F = 0$. *With respect to this patch, the principal curvatures are* e/E *and* g/G.

Proof. When $F = f = 0$, the Weingarten equations (13.10) reduce to

$$S(\mathbf{x}_u) = \frac{e}{E}\mathbf{x}_u \qquad \text{and} \qquad S(\mathbf{x}_v) = \frac{g}{G}\mathbf{x}_v.$$

By definition, e/E and g/G are the eigenvalues of the shape operator S. ▮

The Sphere

We compute the principal curvatures of the patch

$$\mathbf{x}(u,v) = \mathsf{sphere}[a](u,v) = \big(a\cos v\cos u,\ a\cos v\sin u,\ a\sin v\big) \tag{13.28}$$

of the sphere with center $\mathbf{o} = (0,0,0)$ and radius a. We find that

$$\begin{aligned}
\mathbf{x}_u(u,v) &= \big(-\,a\cos v\sin u,\ a\cos v\cos u,\ 0\big),\\
\mathbf{x}_v(u,v) &= \big(-\,a\sin v\cos u,\ -a\sin v\sin u,\ a\cos v\big),
\end{aligned}$$

and so

$$E = a^2\cos^2 v, \qquad F = 0, \qquad G = a^2.$$

Furthermore,

$$\begin{aligned}
\mathbf{x}_{uu}(u,v) &= \big(-\,a\cos v\cos u,\ -a\cos v\sin u,\ 0\big),\\
\mathbf{x}_{uv}(u,v) &= \big(\,a\sin v\sin u,\ -a\sin v\cos u,\ 0\big),\\
\mathbf{x}_{vv}(u,v) &= \big(-\,a\cos v\cos u,\ -a\cos v\sin u,\ -a\sin v\big) = -\mathbf{x}(u,v),
\end{aligned}$$

and Lemma 13.31 yields

$$e = \frac{\det\begin{pmatrix} -a\cos v\cos u & -a\cos v\sin u & 0\\ -a\cos v\sin u & a\cos v\cos u & 0\\ -a\sin v\cos u & -a\sin v\sin u & a\cos v\end{pmatrix}}{a^2\cos v} = -a\cos^2 v.$$

Changing just the first row of the determinant gives $f = 0$ and $g = -a$. Therefore,

$$K = a^{-2}, \qquad H = -a^{-1},$$

$$k_1 = -a^{-1} = k_2,$$

and the corresponding Weingarten matrix of the shape operator is

$$\begin{pmatrix} a^{-1} & 0\\ 0 & a^{-1}\end{pmatrix}.$$

The Circular Torus

We compute the Gaussian and mean curvatures of the patch

$$\mathbf{x}(u,v) = \mathsf{torus}[a,b,b](u,v) = \big((a+b\cos v)\cos u,\ (a+b\cos v)\sin u,\ b\sin v\big),$$

representing a torus with circular sections rather than the more general case on pages 210 and 305.

Setting $a = 0$ and then $b = a$ reduces the parametrization of the torus to that of the sphere (13.28). For this reason, the calculations are only slightly more involved than those above, though we now assume that $a > b > 0$ and jump to the results

$$e = -\cos v(a + b\cos v), \qquad f = 0, \qquad g = -b,$$

and

$$K = \frac{\cos v}{b(a + b\cos v)}, \qquad H = -\frac{a + 2b\cos v}{2b(a + b\cos v)},$$

$$k_1 = -\frac{\cos v}{a + b\cos v}, \qquad k_2 = -\frac{1}{b}.$$

It follows that the Gaussian curvature K of the torus vanishes along the curves given by $v = \pm\pi/2$. These are the two circles of contact, when the torus is held between two planes of glass. These circles consist exclusively of parabolic points, since the angle featuring in (13.2) on page 390 is $\pi/2$, and the normal curvature cannot change sign.

The set of hyperbolic points is $\{\mathbf{x}(u,v) \mid \frac{1}{2}\pi < v < \frac{3}{2}\pi\}$, and the set of elliptic points is $\{\mathbf{x}(u,v) \mid -\frac{1}{2}\pi < v < \frac{1}{2}\pi\}$. This situation can be illustrated using commands in Notebook 13 that produce different colors according to the sign of K.

The Astroidal Ellipsoid

If we modify the standard parametrization of the ellipsoid given on page 313 by replacing each coordinate by its cube, we obtain the ***astroidal ellipsoid***

$$\mathsf{astell}[a,b,c](u,v) = \big((a\cos u\cos v)^3,\ (b\sin u\cos v)^3,\ (c\sin v)^3\big).$$

Therefore, $\mathsf{astell}[a,a,a]$ has the nonparametric equation

$$x^{2/3} + y^{2/3} + z^{2/3} = a^2,$$

and is called the ***astroidal sphere***. Figure 13.7 depicts it touching an ordinary sphere.

Notebook 13 computes the Gaussian curvature of the astroidal sphere, which is given by

$$K = \frac{1024\sec^4 v}{9a^6\left(-18 + 2\cos 4u + \cos(4u-2v) + 14\cos 2v + \cos(4u+2v)\right)^2}.$$

Surprisingly, this function is continuous on the edges of the astroid, and is singular only at the vertices. This is confirmed by Figure 13.8.

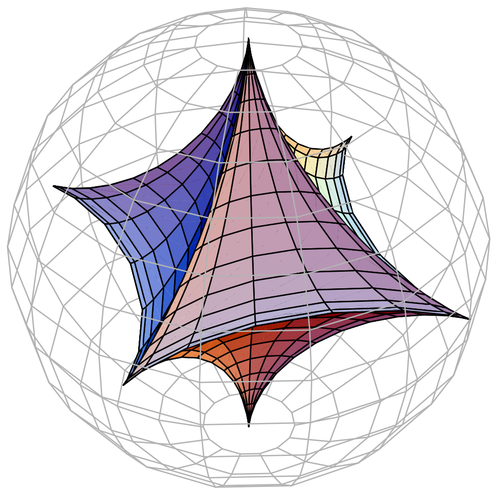

Figure 13.7: The astroidal sphere

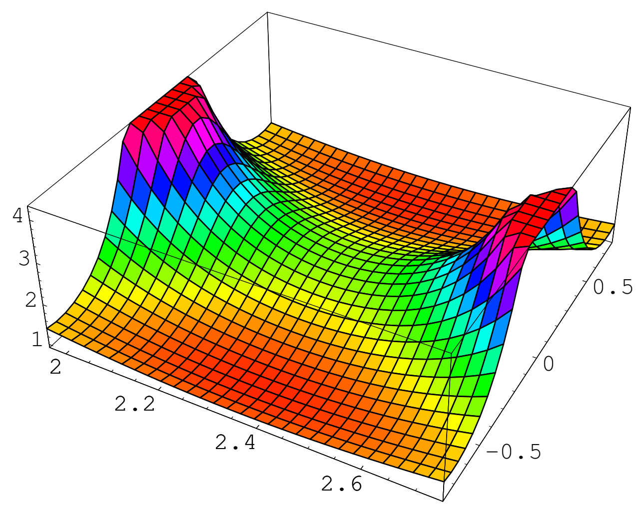

Figure 13.8: Gaussian curvature of the astroidal sphere

Monge Patches

It is not hard to compute directly the Gaussian and mean curvatures of graphs of functions of two variables; see, for example, [dC1, pages 162–163]. However, we can use computations from Notebook 13 to verify the following results:

Lemma 13.34. *For a Monge patch* $(u,v) \mapsto \big(u,\ v,\ h(u,v)\big)$ *we have*

$$E = 1 + h_u^2, \qquad F = h_u h_v, \qquad G = 1 + h_v^2,$$
$$e = \frac{h_{uu}}{(1 + h_u^2 + h_v^2)^{1/2}}, \qquad f = \frac{h_{uv}}{(1 + h_u^2 + h_v^2)^{1/2}}, \qquad g = \frac{h_{vv}}{(1 + h_u^2 + h_v^2)^{1/2}},$$
$$K = \frac{h_{uu}h_{vv} - h_{uv}^2}{(1 + h_u^2 + h_v^2)^2}, \qquad H = \frac{(1 + h_v^2)h_{uu} - 2h_u h_v h_{uv} + (1 + h_u^2)h_{vv}}{2(1 + h_u^2 + h_v^2)^{3/2}}.$$

See Exercise 18 for the detailed general computations.

Once the formulsa of Lemma (13.34) have been stored in Notebook 13, they can be applied to specific functions. For example, suppose we wish to determine the curvature of the graph of the function

$$p_{m,n}(u,v) = u^m v^n. \tag{13.29}$$

One quickly discovers that

$$K = \frac{m(1 - m - n)nu^{2m-2}v^{2n-2}}{\big(1 + m^2u^{2m-2}v^{2n-2} + n^2u^{2m}v^{2n-2}\big)^2}.$$

We see from this formula that the graph has nonpositive Gaussian curvature at all points, provided $n + m > 1$. For definiteness, let us illustrate what happens when $m = 2$ and $n = 4$.

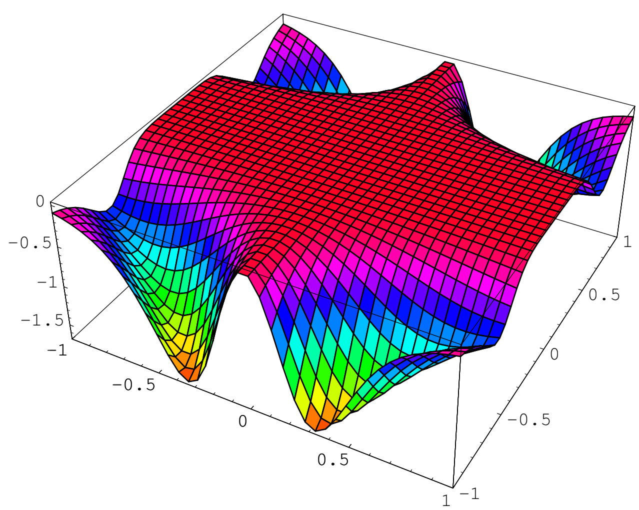

Figure 13.9: Gaussian curvature of the graph of $(u,v) \mapsto u^2v^4$

Figure 13.9 shows that the Gaussian curvature of $p_{2,4}$ is everywhere nonpositive. It is easier to see that all points are either hyperbolic or planar by plotting p in polar coordinates; this we do in Figure 13.10.

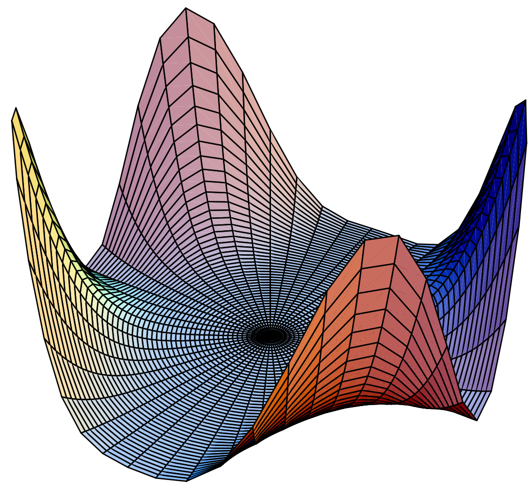

Figure 13.10: The graph of $p_{2,4}$ in polar coordinates

13.6 A Global Curvature Theorem

We recall the following fundamental fact about compact subsets of $\mathbb{R}^n$ (see page 373):

Lemma 13.35. *Let R be a compact subset of $\mathbb{R}^n$, and let $f\colon R \to \mathbb{R}$ be a continuous function. Then f assumes its maximum value at some point $\mathbf{p} \in R$.*

For a proof of this fundamental lemma, see [Buck, page 74].

Intuitively, it is reasonable that for each compact surface $\mathcal{M} \subset \mathbb{R}^3$, there is a point $\mathbf{p}_0 \in \mathcal{M}$ that is furthest from the origin, and at $\mathbf{p}_0$ the surface bends towards the origin. Thus it appears that the Gaussian curvature K of $\mathcal{M}$ is positive at $\mathbf{p}_0$. We now prove that this is indeed the case. The proof uses standard facts from calculus concerning a maximum of a differentiable function of one variable.

Theorem 13.36. *If $\mathcal{M}$ is a compact regular surface in $\mathbb{R}^3$, there is a point $\mathbf{p} \in \mathbb{R}^3$ at which the Gaussian curvature K is strictly positive.*

Proof. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(\mathbf{p}) = \|\mathbf{p}\|^2$. Then f is continuous (in fact, differentiable), since it can be expressed in terms of the natural coordinate functions of $\mathbb{R}^3$ as $f = u_1^2 + u_2^2 + u_3^2$. By Lemma 13.35, f assumes its maximum value at some point $\mathbf{p}_0 \in \mathcal{M}$. Let $\mathbf{v} \in \mathcal{M}_{\mathbf{p}}$ be a unit tangent vector, and choose a unit-speed curve $\boldsymbol{\alpha}\colon (a,b) \to \mathcal{M}$ such that $a < 0 < b$, $\boldsymbol{\alpha}(0) = \mathbf{p}_0$ and $\boldsymbol{\alpha}'(0) = \mathbf{v}$.

Since the function $g\colon (a,b) \to \mathbb{R}$ defined by $g = f \circ \boldsymbol{\alpha}$ has a maximum at 0, it follows that

$$g'(0) = 0 \qquad \text{and} \qquad g''(0) \leqslant 0.$$

But $g(t) = \boldsymbol{\alpha}(t) \cdot \boldsymbol{\alpha}(t)$, so that

$$0 = g'(0) = 2\boldsymbol{\alpha}'(0) \cdot \boldsymbol{\alpha}(0) = 2\mathbf{v} \cdot \mathbf{p}_0. \tag{13.30}$$

In (13.30), $\mathbf{v}$ can be an arbitrary unit tangent vector, and so $\mathbf{p}_0$ must be normal to $\mathcal{M}$ at $\mathbf{p}_0$. Clearly, $\mathbf{p}_0 \neq 0$, so that (13.30) implies that $\mathbf{p}_0/\|\mathbf{p}_0\|$ is a unit normal vector to $\mathcal{M}$ at $\mathbf{p}_0$. Furthermore,

$$0 \geqslant g''(0) = 2\boldsymbol{\alpha}''(0) \cdot \boldsymbol{\alpha}(0) + 2\boldsymbol{\alpha}'(0) \cdot \boldsymbol{\alpha}'(0) = 2\big(\boldsymbol{\alpha}''(0) \cdot \mathbf{p}_0 + 1\big),$$

so that $\boldsymbol{\alpha}''(0) \cdot \mathbf{p}_0 \leqslant -1$, or

$$k(\mathbf{v}) = \boldsymbol{\alpha}''(0) \cdot \frac{\mathbf{p}_0}{\|\mathbf{p}_0\|} \leqslant -\frac{1}{\|\mathbf{p}_0\|}, \tag{13.31}$$

where $k(\mathbf{v})$ is the normal curvature determined by the tangent vector $\mathbf{v}$ and the unit normal vector $\mathbf{p}_0/\|\mathbf{p}_0\|$. In particular, the principal curvatures of $\mathcal{M}$ at $\mathbf{p}_0$ (with respect to $\mathbf{p}_0/\|\mathbf{p}_0\|$) satisfy

$$k_1(\mathbf{p}_0), k_2(\mathbf{p}_0) \leqslant -\frac{1}{\|\mathbf{p}_0\|}.$$

This implies that the Gaussian curvature of $\mathcal{M}$ at $\mathbf{p}_0$ satisfies

$$K(\mathbf{p}_0) = k_1(\mathbf{p}_0)k_2(\mathbf{p}_0) \geqslant \frac{1}{\|\mathbf{p}_0\|^2} > 0. \blacksquare$$

Noncompact surfaces of positive Gaussian curvature exist (see Exercise 15). On the other hand, for surfaces of negative curvature, we have the following result (see Exercise 16):

Corollary 13.37. Any surface in $\mathbb{R}^3$ whose Gaussian curvature is everywhere nonpositive must be noncompact.

13.7 Nonparametrically Defined Surfaces

So far we have discussed computing the curvature of a surface from its parametric representation. In this section we show how in some cases the curvature can be computed from the nonparametric form of a surface.

Lemma 13.38. Let $\mathbf{p}$ be a point on a regular surface $\mathcal{M} \subset \mathbb{R}^3$, and let $\mathbf{v_p}$ and $\mathbf{w_p}$ be tangent vectors to $\mathcal{M}$ at $\mathbf{p}$. Then the Gaussian and mean curvatures of $\mathcal{M}$ at $\mathbf{p}$ are related to the shape operator by the formulas

$$S(\mathbf{v_p}) \times S(\mathbf{w_p}) \;=\; K(\mathbf{p})\,\mathbf{v_p} \times \mathbf{w_p}, \tag{13.32}$$

$$S(\mathbf{v_p}) \times \mathbf{w_p} + \mathbf{v_p} \times S(\mathbf{w_p}) \;=\; 2H(\mathbf{p})\,\mathbf{v_p} \times \mathbf{w_p}. \tag{13.33}$$

Proof. First, assume that $\mathbf{v_p}$ and $\mathbf{w_p}$ are linearly independent. Then we can write

$$S(\mathbf{v_p}) = a\mathbf{v_p} + b\mathbf{w_p} \qquad \text{and} \qquad S(\mathbf{w_p}) = c\mathbf{v_p} + d\mathbf{w_p},$$

so that

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

is the matrix of S with respect to $\mathbf{v_p}$ and $\mathbf{w_p}$. It follows from (13.19), page 398, that

$$\begin{aligned} S(\mathbf{v_p}) \times \mathbf{w_p} + \mathbf{v_p} \times S(\mathbf{w_p}) &= (a\mathbf{v_p} + b\mathbf{w_p}) \times \mathbf{w_p} + \mathbf{v_p} \times (c\mathbf{v_p} + d\mathbf{w_p}) \\ &= (a+d)\,\mathbf{v_p} \times \mathbf{w_p} \\ &= (\operatorname{tr} S(\mathbf{p}))\,\mathbf{v_p} \times \mathbf{w_p} = 2H(\mathbf{p})\,\mathbf{v_p} \times \mathbf{w_p}, \end{aligned}$$

proving (13.33) in the case that $\mathbf{v_p}$ and $\mathbf{w_p}$ are linearly independent.

If $\mathbf{v_p}$ and $\mathbf{w_p}$ are linearly dependent, they are still the limits of linearly independent tangent vectors. Since both sides of (13.33) are continuous in $\mathbf{v_p}$ and $\mathbf{w_p}$, we get (13.33) in the general case. equation (13.32) is proved by the same method (see Exercise 22). ∎

Theorem 13.39. Let $\mathbf{Z}$ *be a nonvanishing vector field on a regular surface* $\mathcal{M} \subset \mathbb{R}^3$ *which is everywhere perpendicular to* $\mathcal{M}$*. Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be vector fields tangent to* $\mathcal{M}$ *such that* $\mathbf{V} \times \mathbf{W} = \mathbf{Z}$*. Then*

$$K = \frac{[\mathbf{Z}\ \boldsymbol{D}_\mathbf{V}\mathbf{Z}\ \boldsymbol{D}_\mathbf{W}\mathbf{Z}]}{\|\mathbf{Z}\|^4}, \tag{13.34}$$

$$H = \frac{[\mathbf{Z}\ \mathbf{W}\ \boldsymbol{D}_\mathbf{V}\mathbf{Z}] - [\mathbf{Z}\ \mathbf{V}\ \boldsymbol{D}_\mathbf{W}\mathbf{Z}]}{2\|\mathbf{Z}\|^3}. \tag{13.35}$$

Proof. Let $\mathbf{U} = \mathbf{Z}/\|\mathbf{Z}\|$; then (9.5), page 267, implies that

$$\boldsymbol{D}_\mathbf{V}\mathbf{U} = \frac{\boldsymbol{D}_\mathbf{V}\mathbf{Z}}{\|\mathbf{Z}\|} + \mathbf{V}\left[\frac{1}{\|\mathbf{Z}\|}\right]\mathbf{Z}.$$

Therefore,

$$S(\mathbf{V}) = -\boldsymbol{D}_\mathbf{V}\mathbf{U} = \frac{-\boldsymbol{D}_\mathbf{V}\mathbf{Z}}{\|\mathbf{Z}\|} + \mathbf{N_V},$$

where $\mathbf{N_V}$ is a vector field normal to $\mathcal{M}$. By Lemma 13.38 we have

$$\begin{aligned} K\,\mathbf{V} \times \mathbf{W} &= S(\mathbf{V}) \times S(\mathbf{W}) \\ &= \left(\frac{-\boldsymbol{D}_\mathbf{V}\mathbf{Z}}{\|\mathbf{Z}\|} + \mathbf{N_V}\right) \times \left(\frac{-\boldsymbol{D}_\mathbf{W}\mathbf{Z}}{\|\mathbf{Z}\|} + \mathbf{N_W}\right). \end{aligned} \tag{13.36}$$

Since $\mathbf{N_V}$ and $\mathbf{N_W}$ are linearly dependent, it follows from (13.36) that

$$K\mathbf{Z} = \frac{\boldsymbol{D}_{\mathbf{V}}\mathbf{Z} \times \boldsymbol{D}_{\mathbf{W}}\mathbf{Z}}{\|\mathbf{Z}\|^2} + \text{some vector field tangent to } \mathcal{M}.$$

Taking the scalar product of both sides with $\mathbf{Z}$ yields (13.34). Equation (13.35) is proved in a similar fashion. ▮

In order to make use of Theorem 13.39, we need an important function that measures the distance from the origin of each tangent plane to a surface.

Definition 13.40. *Let $\mathcal{M}$ be an oriented regular surface in $\mathbb{R}^3$ with surface normal $\mathbf{U}$. Then the* ***support function*** *of $\mathcal{M}$ is the function $h\colon \mathcal{M} \to \mathbb{R}$ given by*

$$h(\mathbf{p}) = \mathbf{p} \cdot \mathbf{U}(\mathbf{p}).$$

Geometrically, $h(\mathbf{p})$ is the distance from the origin to the tangent space $\mathcal{M}_{\mathbf{p}}$.

Corollary 13.41. *Let $\mathcal{M}$ be the surface*

$$\{\, (u_1, u_2, u_3) \in \mathbb{R}^3 \mid f_1 u_1^k + f_2 u_2^k + f_3 u_3^k = 1 \,\},$$

where f_1, f_2, f_3 are constants, not all zero, and k is a nonzero real number. Then the support function, Gaussian curvature and mean curvature of $\mathcal{M}$ are given by

$$h = \frac{1}{\sqrt{f_1^2 u_1^{2k-2} + f_2^2 u_2^{2k-2} + f_3^2 u_3^{2k-2}}}, \tag{13.37}$$

$$K = \frac{(k-1)^2 f_1 f_2 f_3 (u_1 u_2 u_3)^{k-2}}{\left(\displaystyle\sum_{i=1}^{3} f_i^2 u_i^{2k-2}\right)^2} = h^4 (k-1)^2 f_1 f_2 f_3 (u_1 u_2 u_3)^{k-2}, \tag{13.38}$$

$$\begin{aligned} H = {} & \frac{-k+1}{2\left(\displaystyle\sum_{i=1}^{3} f_i^2 u_i^{2k-2}\right)^{\frac{3}{2}}} \Big(f_1 f_2 (u_1 u_2)^{k-2} (f_1 u_1^k + f_2 u_2^k) \\ & + f_2 f_3 (u_2 u_3)^{k-2} (f_2 u_2^k + f_3 u_3^k) + f_3 f_1 (u_3 u_1)^{k-2} (f_3 u_3^k + f_1 u_1^k) \Big). \end{aligned} \tag{13.39}$$

Proof. Let $g(u_1, u_2, u_3) = f_1 u_1^k + f_2 u_2^k + f_3 u_3^k - 1$, so that

$$\mathcal{M} = \{\, \mathbf{p} \in \mathbb{R}^3 \mid g(\mathbf{p}) = 0 \,\}.$$

Then $\mathbf{Z} = \operatorname{grad} g$ is a nonvanishing vector field that is everywhere perpendicular to $\mathcal{M}$. Explicitly,

$$\mathbf{Z} = k \sum_{i=1}^{3} f_i u_i^{k-1} \mathbf{U}_i,$$

u_1, u_2, u_3 being the natural coordinate functions of $\mathbb{R}^3$, and $\mathbf{U}_i = \partial/\partial u_i$ (see Definition 9.20). The vector field $\mathbf{X} = \sum u_i \mathbf{U}_i$ satisfies

$$\mathbf{X} \cdot \mathbf{Z} = k \sum f_i u_i^k, \tag{13.40}$$

in which the summations continue to be over $i = 1, 2, 3$. The support function of $\mathcal{M}$ is given by

$$h = \mathbf{X} \cdot \frac{\mathbf{Z}}{\|\mathbf{Z}\|} = \frac{\sum f_i u_i^k}{\sqrt{\sum f_i^2 u_i^{2k-2}}}.$$

Since $\sum f_i u_i^k$ equals 1 on $\mathcal{M}$, we get (13.37).

Next, let

$$\mathbf{V} = \sum v_i \mathbf{U}_i \quad \text{and} \quad \mathbf{W} = \sum w_i \mathbf{U}_i$$

be vector fields on $\mathbb{R}^3$. Since f_1, f_2, f_3 are constants, we have

$$\boldsymbol{D}_{\mathbf{V}}\mathbf{Z} = k \sum \mathbf{V}[f_i u_i^{k-1}]\mathbf{U}_i = k(k-1) \sum f_i v_i u_i^{k-2} \mathbf{U}_i,$$

and similarly for $\mathbf{W}$. Therefore, the triple product $[\mathbf{Z}\ \boldsymbol{D}_{\mathbf{V}}\mathbf{Z}\ \boldsymbol{D}_{\mathbf{W}}\mathbf{Z}]$ equals

$$\det\begin{pmatrix} k f_1 u_1^{k-1} & k f_2 u_2^{k-1} & k f_3 u_3^{k-1} \\ k(k-1) f_1 v_1 u_1^{k-2} & k(k-1) f_2 v_2 u_2^{k-2} & k(k-1) f_3 v_3 u_3^{k-2} \\ k(k-1) f_1 w_1 u_1^{k-2} & k(k-1) f_2 w_2 u_2^{k-2} & k(k-1) f_3 w_3 u_3^{k-2} \end{pmatrix}$$

$$= k^3 (k-1)^2 f_1 f_2 f_3 u_1^{k-2} u_2^{k-2} u_3^{k-2} [\mathbf{X}\ \mathbf{V}\ \mathbf{W}].$$

Now we choose $\mathbf{V}$ and $\mathbf{W}$ so that they are tangent to $\mathcal{M}$ and $\mathbf{V} \times \mathbf{W} = \mathbf{Z}$. Using (13.34), we obtain

$$K = \frac{k^3 (k-1)^2 f_1 f_2 f_3 u_1^{k-2} u_2^{k-2} u_3^{k-2} \mathbf{X} \cdot \mathbf{Z}}{\|\mathbf{Z}\|^4} = \frac{(k-1)^2 f_1 f_2 f_3 u_1^{k-2} u_2^{k-2} u_3^{k-2}}{\left(\displaystyle\sum_{i=1}^{3} f_i^2 u_i^{2k-2}\right)^2}.$$

This proves (13.38). The proof of (13.39) is similar. ▮

Computations in Notebook 13 yield three special cases of Corollary 13.41.

Corollary 13.42. *The support function and Gaussian curvature of*

(i) *the ellipsoid* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$,

(ii) *the hyperboloid of one sheet* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$,

(iii) *the hyperboloid of two sheets* $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

are given in each case by

$$h = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{-1/2} \qquad \text{and} \qquad K = \pm\frac{h^4}{a^2b^2c^2},$$

where the minus sign only applies in (ii).

We shall return to these quadrics in Section 19.6, where we describe geometrically useful parametrizations of them. We conclude the present section by considering a ***superquadric***, which is a surface of the form

$$f_1x^k + f_2y^k + f_3z^k = 1,$$

where k is different from 2. We do the special case $k = 2/3$, which is an astroidal ellipsoid (see page 407).

Corollary 13.43. *The Gaussian curvature of the superquadric*

$$f_1x^{2/3} + f_2y^{2/3} + f_3z^{2/3} = 1$$

is given by

$$K = \frac{f_1f_2f_3}{9\left(f_3^2(xy)^{2/3} + f_2^2(xz)^{2/3} + f_1^2(yz)^{2/3}\right)^2}.$$

13.8 Exercises

M **1.** Plot the graph of $p_{2,3}$, defined by (13.29), and its Gaussian curvature.

M **2.** Plot the following surfaces and describe in each case (without additional calculation) the set of elliptic, hyperbolic, parabolic and planar points:

(a) a sphere,

(b) an ellipsoid,

(c) an elliptic paraboloid,

(d) a hyperbolic paraboloid,

(e) a hyperboloid of one sheet,

(f) a hyperboloid of two sheets.

This exercise continues on page 450.

3. By studying the plots of the surfaces listed in Exercise 2, describe the general shape of the image of their Gauss maps.

4. Compute by hand the coefficients of the first fundamental form, those of the second fundamental form, the unit normal, the mean curvature and the principal curvatures of the following surfaces:

(a) the elliptical torus,

(b) the helicoid,

(c) Enneper's minimal surface.

Refer to the exercises on page 377.

5. Compute the first fundamental form, the second fundamental form, the unit normal, the Gaussian curvature, the mean curvature and the principal curvatures of the patch

$$\mathbf{x}(u,v) = \big(u^2 + v,\ v^2 + u,\ uv\big).$$

6. The ***translation surface*** determined by curves $\boldsymbol{\alpha}, \boldsymbol{\gamma} \colon (a,b) \to \mathbb{R}^3$ is the patch

$$(u,v) \mapsto \boldsymbol{\alpha}(u) + \boldsymbol{\gamma}(v).$$

It is the surface formed by moving $\boldsymbol{\alpha}$ parallel to itself in such a way that a point of the curve moves along $\boldsymbol{\gamma}$. Show that $f = 0$ for a translation surface.

7. Explain the difference between the translation surface formed by a circle and a lemniscate lying in perpendicular planes, and the twisted surface formed from a lemniscate according to Section 11.6.

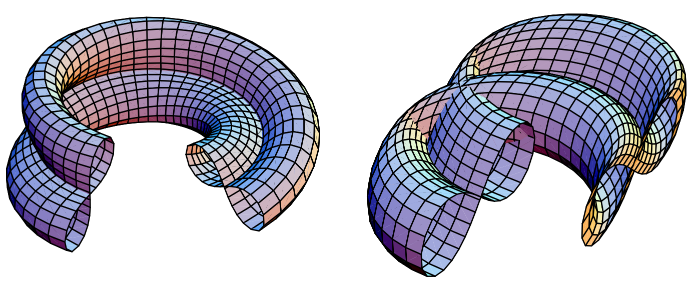

Figure 13.11: Translation and twisted surface formed by a lemniscate

8. Compute by hand the first fundamental form, the second fundamental form, the unit normal, the Gaussian curvature, the mean curvature and the principal curvatures of the patch

$$\mathbf{x}_n(u,v) = \big(u^n,\ v^n,\ uv\big).$$

For $n = 3$, see the picture on page 291.

9. Prove the statement immediately following Definition 13.8.

10. Show that the first fundamental form of the Gauss map of a patch $\mathbf{x}$ coincides with the third fundamental form of $\mathbf{x}$.

11. Show that an orientation-preserving Euclidean motion $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ leaves unchanged both principal curvatures and principal vectors.

12. Show that the ***Bohemian dome*** defined by

$$\mathsf{bohdom}[a,b,c,d](u,v) = (a\cos u,\ b\sin u + c\cos v,\ d\sin v)$$

is the translation surface of two ellipses (see Exercise 6). Compute by hand the Gaussian curvature, the mean curvature and the principal curvatures of $\mathsf{bohdom}[a,b,b,a]$.

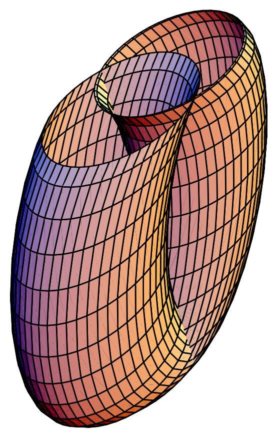

Figure 13.12: The Bohemian dome $\mathsf{bohdom}[1,2,2,3]$

13. Show that the mean curvature $H(\mathbf{p})$ of a surface $\mathcal{M} \subset \mathbb{R}^3$ at $\mathbf{p} \in \mathcal{M}$ is given by

$$H(\mathbf{p}) = \frac{1}{\pi}\int_0^{\pi} k(\theta)\,d\theta,$$

where $k(\theta)$ denotes the normal curvature, as in the proof of Lemma 13.19.

14. (Continuation) Let n be an integer larger than 2 and for $0 \leqslant i \leqslant n-1$, put $\theta_i = \psi + 2\pi i/n$, where ψ is some angle. Show that

$$H(\mathbf{p}) = \frac{1}{n}\sum_{i=0}^{n-1} k(\theta_i).$$

15. Give examples of a noncompact surface

(a) whose Gaussian curvature is negative,

(b) whose Gaussian curvature is identically zero,

(c) whose Gaussian curvature is positive,

(d) containing elliptic, hyperbolic, parabolic and planar points.

16. Prove Corollary 13.37 under the assumption that K is everywhere negative.

17. Show that there are no compact minimal surfaces in $\mathbb{R}^3$.

M **18.** Find the coefficients the first, second and third fundamental forms to the following surfaces:

(a) the elliptical torus defined on page 377,

(b) the helicoid defined on page 377,

(c) Enneper's minimal surface defined on page 378,

(d) a Monge patch defined on page 302.

19. Determine the principal curvatures of the Whitney umbrella. Its mean curvature is shown in Figure 13.13.

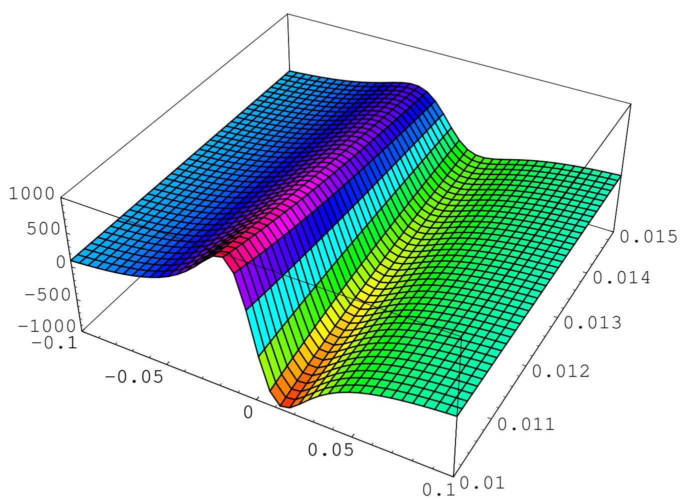

Figure 13.13: Gaussian curvature of the Whitney umbrella

M **20.** Find the formulas for the coefficients e, f, g of the second fundamental forms of the following surfaces parametrized in Chapter 11: the Möbius strip, the Klein bottle, Steiner's Roman surface, the cross cap.

M **21.** Find the formulas for the Gaussian and mean curvatures of the surfaces of the preceding exercise.

22. Prove equation (13.32).

23. Prove

Lemma 13.44. *Let $\mathcal{M} \subset \mathbb{R}^3$ be a surface, and suppose that $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ and $\mathbf{y}\colon \mathcal{V} \to \mathcal{M}$ are coherently oriented patches on $\mathcal{M}$ with $\mathbf{x}(\mathcal{U}) \cap \mathbf{y}(\mathcal{V})$ nonempty. Let $\mathbf{x}^{-1} \circ \mathbf{y} = (\bar{u}, \bar{v})\colon \mathcal{U} \cap \mathcal{V} \to \mathcal{U} \cap \mathcal{V}$ be the associated change of coordinates, so that*

$$\mathbf{y}(u, v) = \mathbf{x}\big(\bar{u}(u, v), \bar{v}(u, v)\big).$$

Let $e_{\mathbf{x}}, f_{\mathbf{x}}, g_{\mathbf{x}}$ denote the coefficients of the second fundamental form of $\mathbf{x}$, and let $e_{\mathbf{y}}, f_{\mathbf{y}}, g_{\mathbf{y}}$ denote the coefficients of the second fundamental form of $\mathbf{y}$. Then

$$(13.41) \qquad \begin{cases} e_{\mathbf{y}} = e_{\mathbf{x}} \left(\dfrac{\partial \bar{u}}{\partial u}\right)^2 + 2 f_{\mathbf{x}} \dfrac{\partial \bar{u}}{\partial u} \dfrac{\partial \bar{v}}{\partial u} + g_{\mathbf{x}} \left(\dfrac{\partial \bar{v}}{\partial u}\right)^2, \\[2ex] f_{\mathbf{y}} = e_{\mathbf{x}} \dfrac{\partial \bar{u}}{\partial u} \dfrac{\partial \bar{u}}{\partial v} + f_{\mathbf{x}} \left(\dfrac{\partial \bar{u}}{\partial u} \dfrac{\partial \bar{v}}{\partial v} + \dfrac{\partial \bar{u}}{\partial v} \dfrac{\partial \bar{v}}{\partial u}\right) + g_{\mathbf{x}} \dfrac{\partial \bar{v}}{\partial u} \dfrac{\partial \bar{v}}{\partial v}, \\[2ex] g_{\mathbf{y}} = e_{\mathbf{x}} \left(\dfrac{\partial \bar{u}}{\partial v}\right)^2 + 2 f_{\mathbf{x}} \dfrac{\partial \bar{u}}{\partial v} \dfrac{\partial \bar{v}}{\partial v} + g_{\mathbf{x}} \left(\dfrac{\partial \bar{v}}{\partial v}\right)^2. \end{cases}$$

24. Prove equation (13.39) of Corollary 13.41.