# Chapter 7

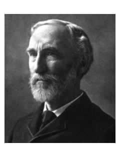
# Curves in Space

In previous chapters, we have seen that the curvature $\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]$ of a plane curve $\boldsymbol{\alpha}$ measures the failure of $\boldsymbol{\alpha}$ to be a straight line. In the present chapter, we define a similar curvature $\boldsymbol{\kappa}[\boldsymbol{\alpha}]$ for a curve in $\mathbb{R}^n$; it measures the failure of the curve to be a straight line in space. The function $\boldsymbol{\kappa}[\boldsymbol{\alpha}]$ reduces to the absolute value of $\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]$ when $n = 2$. For curves in $\mathbb{R}^3$, we can also measure the failure of the curve to lie in a plane by means of another function called the *torsion*, and denoted $\boldsymbol{\tau}[\boldsymbol{\alpha}]$.

We shall need the Gibbs[1] vector cross product to study curves in $\mathbb{R}^3$, just as we needed a complex structure to study curves in $\mathbb{R}^2$. We recall the definition and some of the properties of the vector cross product on $\mathbb{R}^3$ in Section 7.1. For completeness, and its use in Chapter 22, we take the opportunity to record analogous definitions for the complex vector spaces $\mathbb{C}^n$ and $\mathbb{C}^3$.

Curvature and torsion are defined in Section 7.2. We shall also define three orthogonal unit vector fields $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ along a space curve, that constitute the *Frenet frame field* of the curve. In Chapters 1 – 5, we made frequent use of the frame field $\{\mathbf{T}, J\mathbf{T}\}$ to study the geometry of a plane curve. The Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ plays the same role for space curves, but the algebra is somewhat different.

The twisting and turning of the Frenet frame field can be measured by curvature and torsion. In fact, the Frenet formulas use curvature and torsion to express the derivatives of the three vector fields of the Frenet frame field in terms

1 

Josiah Willard Gibbs (1839–1903). American physicist. When he tried to make use of Hamilton's quaternions (see Chapter 23), he found it more useful to split quaternion multiplication into a scalar part and a vector part, and to regard them as separate multiplications. Although this caused great consternation among some of Hamilton's followers, the formalism of Gibbs eventually prevailed; engineering and physics books using it started to appear in the early 1900s. Gibbs also played a large role in reviving Grassman's vector calculus.

of the vector fields themselves. We establish the Frenet formulas for unit-speed space curves in Section 7.2, and for arbitrary-speed space curves in Section 7.4. The simplest space curves are discussed in the intervening Section 7.3. A number of other space curves, including Viviani's curve, the intersection of a sphere and a cylinder, are introduced and studied in Section 7.5.

Construction of the Frenet frame of a space curve leads one naturally to consider tubes about such curves. They are introduced in Section 7.6, and constitute our first examples of surfaces in $\mathbb{R}^3$. The torus is a special case, and we quickly generalize its definition to the case of elliptical cross-sections, though the detailed study of such surfaces does not begin until Chapter 10.

Knots form one of the most interesting classes of space curves. In the final section of this chapter we study *torus knots*, that is, knots which lie on the surface of a torus. Visualization of torus knots is considerably enhanced by making them thicker, and in practice considering the associated tube. We observe that such a tube can twist on itself, and explain how this phenomenon is influenced by the curvature and torsion of the torus knot.

## 7.1 The Vector Cross Product

Let us recall the notion of vector cross product on $\mathbb{R}^3$.

First, let us agree to use the notation

$$\mathbf{i} = (1,0,0), \qquad \mathbf{j} = (0,1,0), \qquad \mathbf{k} = (0,0,1).$$

For $\mathbf{a} \in \mathbb{R}^3$ we can write either $\mathbf{a} = (a_1, a_2, a_3)$ or $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.

*Definition 7.1.* *Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be vectors in $\mathbb{R}^3$. Then the* ***vector cross product*** *of $\mathbf{a}$ and $\mathbf{b}$ is formally given by*

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

*More explicitly,*

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix} \mathbf{i} - \det \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix} \mathbf{j} + \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \mathbf{k}.$$

The vector cross product enjoys the following properties:

$$\begin{aligned} \mathbf{a} \times \mathbf{c} &= -\mathbf{c} \times \mathbf{a}, \\ (\mathbf{a} + \mathbf{b}) \times \mathbf{c} &= \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}, \end{aligned}$$

$$(\lambda\mathbf{a}) \times \mathbf{c} \;=\; \lambda(\mathbf{a} \times \mathbf{c}) = \mathbf{a} \times (\lambda\mathbf{c}),$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) \;=\; (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}), \tag{7.1}$$

$$\|\mathbf{a} \times \mathbf{b}\|^2 \;=\; \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2, \tag{7.2}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \;=\; (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}, \tag{7.3}$$

for $\lambda \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$. Frequently, (7.2), which is a special case of (7.1), is referred to as ***Lagrange's identity***[2].

For $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ we define the ***vector triple product*** $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$ by

$$[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = \det\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = \mathit{det}\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

where $\mathbf{a} = (a_1, a_2, a_3)$, and so forth. The vector triple product is related to the dot product and the cross product by the formulas

$$[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}), \tag{7.4}$$

reflecting properties of the determinant. In particular, $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, and its direction is uniquely determined by the 'right-hand rule'.

Finally, note the following consequence of (7.2):

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\| \sin\theta, \tag{7.5}$$

where $\theta$ is the angle $\theta$ with $0 \leqslant \theta \leqslant \pi$ between $\mathbf{a}$ and $\mathbf{b}$. As a trivial consequence $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any $\mathbf{a} \in \mathbb{R}^3$. Equation (7.5) can be interpreted geometrically; it says that $\|\mathbf{a} \times \mathbf{b}\|$ equals the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

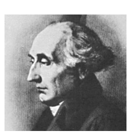
## Complex Vector Algebra

For the sequel, it will be helpful to point out the differences between real and complex vector algebra. This subsection can be omitted on a first reading.

*Definition 7.2.* ***Complex Euclidean*** $n$***-space*** $\mathbb{C}^n$ *consists of the set of all complex* $n$*-tuples*:

$$\mathbb{C}^n = \big\{\, (p_1, \ldots, p_n) \mid p_j \text{ is a complex number for } j = 1, \ldots, n \,\big\}.$$

---

[2] 
Joseph Louis Lagrange (1736–1813). Born in Turin, Italy, Lagrange succeeded Euler as director of mathematics of the Berlin Academy of Science in 1766. In 1787 he left Berlin to become a member of the Paris Academy of Science, where he remained for the rest of his life. He made important contributions to mechanics, the calculus of variations and differential equations, in addition to differential geometry. During the 1790s he worked on the metric system and advocated a decimal base. He died in the same month as Beethoven, March 1813

The elements of $\mathbb{C}^n$ are called ***complex vectors***, and one makes $\mathbb{C}^n$ into a complex vector space in the same way that we made $\mathbb{R}^n$ into a real vector space in Section 1.1. Thus if $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ are complex vectors, we define $\mathbf{p} + \mathbf{q}$ to be the element of $\mathbb{C}^n$ given by

$$\mathbf{p} + \mathbf{q} = (p_1 + q_1, \ldots, p_n + q_n).$$

Similarly, for $\lambda \in \mathbb{C}$ the vector $\lambda\mathbf{p}$ is defined by $\lambda\mathbf{p} = (\lambda p_1, \ldots, \lambda p_n)$. The ***complex dot product*** of $\mathbb{C}^n$ is given by the same formula as its real counterpart:

$$\mathbf{p} \cdot \mathbf{q} = \sum_{j=1}^{n} p_j q_j. \tag{7.6}$$

In addition to these operations, $\mathbb{C}^n$ also has a ***conjugation*** $\mathbf{p} \mapsto \overline{\mathbf{p}}$, defined by

$$\overline{\mathbf{p}} = (\overline{p}_1, \ldots, \overline{p}_n).$$

Conjugation has the following properties:

$$\overline{\mathbf{p} + \mathbf{q}} = \overline{\mathbf{p}} + \overline{\mathbf{q}}, \qquad \overline{\lambda\mathbf{p}} = \overline{\lambda}\,\overline{\mathbf{p}},$$
$$\overline{\mathbf{p} \cdot \mathbf{q}} = \overline{\mathbf{p}} \cdot \overline{\mathbf{q}}, \qquad \mathbf{p} \cdot \overline{\mathbf{p}} \geqslant 0.$$

The complex dot product should not be confused with the so-called Hermitian product of $\mathbb{C}^n$, in which one of the two vectors in (7.6) is replaced by its conjugate. The latter is present in the definition of the ***norm***

$$\|\mathbf{p}\| = \sqrt{\mathbf{p} \cdot \overline{\mathbf{p}}} = \sqrt{|p_1|^2 + \cdots + |p_n|^2}$$

of a vector in $\mathbb{C}^n$, to ensure that this number is nonnegative. Whilst $\mathbf{p} \cdot \mathbf{p} = 0$ does not imply that $\mathbf{p} = 0$, the latter does follow if $\|\mathbf{p}\| = 0$.

Definition 7.1 extends to complex 3-tuples. Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be vectors in $\mathbb{C}^3$. Written out in coordinates, the vector cross product of $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{a} \times \mathbf{b} = \big(a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1\big).$$

It is easy to check that

$$\overline{\mathbf{a} \times \mathbf{b}} = \overline{\mathbf{a}} \times \overline{\mathbf{b}},$$

and that all of the identities for the real vector cross product given on page 193, with one exception, hold also for the complex vector cross product. The exception is the Lagrange identity (7.2), whose modification we include in the following lemma:

*Lemma 7.3.* *For $\mathbf{a}, \mathbf{b} \in \mathbb{C}^3$ we have*

$$\|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - |\mathbf{a} \cdot \overline{\mathbf{b}}|^2, \tag{7.7}$$

$$\|\mathbf{a} \times \overline{\mathbf{a}}\| = \|\mathbf{a}\|^2 \quad \textit{if} \quad \mathbf{a} \cdot \mathbf{a} = 0, \tag{7.8}$$

$$\mathbf{a} \times \overline{\mathbf{a}} = 2i\big(\mathfrak{Im}(a_2 \overline{a}_3),\, \mathfrak{Im}(a_3 \overline{a}_1),\, \mathfrak{Im}(a_1 \overline{a}_2)\big), \tag{7.9}$$

*where $\mathbf{a} = (a_1, a_2, a_3)$.*

*Proof.* This is straightforward; for example, (7.8) follows directly because

$$\begin{aligned} \|\mathbf{a} \times \overline{\mathbf{a}}\|^2 = \mathbf{a} \times \overline{\mathbf{a}} \cdot \overline{\mathbf{a} \times \overline{\mathbf{a}}} &= -(\mathbf{a} \times \overline{\mathbf{a}}) \cdot (\mathbf{a} \times \overline{\mathbf{a}}) \\ &= -(\mathbf{a} \cdot \mathbf{a})(\overline{\mathbf{a}} \cdot \overline{\mathbf{a}}) + (\mathbf{a} \cdot \overline{\mathbf{a}})^2 = \|\mathbf{a}\|^4. \quad \blacksquare \end{aligned}$$

## 7.2 Curvature and Torsion of Unit-Speed Curves

We first define the curvature of a *unit-speed* curve in $\mathbb{R}^n$. The definitions are more straightforward in this case, though arbitrary-speed curves in $\mathbb{R}^3$ will be considered in Section 7.4.

*Definition 7.4.* *Let $\boldsymbol{\beta}\colon (c, d) \to \mathbb{R}^n$ be a unit-speed curve. Write*

$$\boldsymbol{\kappa}[\boldsymbol{\beta}](s) = \|\boldsymbol{\beta}''(s)\|.$$

*Then the function $\boldsymbol{\kappa}[\boldsymbol{\beta}]\colon (c, d) \to \mathbb{R}$ is called the* ***curvature*** *of $\boldsymbol{\beta}$.*

We abbreviate $\boldsymbol{\kappa}[\boldsymbol{\beta}]$ to $\boldsymbol{\kappa}$ when there is no danger of confusion.

Intuitively, curvature measures the failure of a curve to be a straight line. More precisely, a straight line in $\mathbb{R}^n$ is characterized by the fact that its curvature vanishes, as we show in

*Lemma 7.5.* *Let $\boldsymbol{\beta}\colon (c, d) \to \mathbb{R}^n$ be a unit-speed curve. The following conditions are equivalent:*

(i) $\boldsymbol{\kappa} \equiv 0$;

(ii) $\boldsymbol{\beta}'' \equiv 0$;

(iii) $\boldsymbol{\beta}$ *is a straight line segment.*

*Proof.* It is clear from the definition that (i) and (ii) are equivalent. To show that (ii) and (iii) are equivalent, suppose that $\boldsymbol{\beta}'' \equiv 0$. Integration of this $n$-tuple of differential equations yields

$$\boldsymbol{\beta}(s) = \mathbf{u}s + \mathbf{v}, \tag{7.10}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are constant vectors with $\|\mathbf{u}\| = 1$. Then (7.10) is a unit-speed parametrization of a straight line in $\mathbb{R}^n$. Conversely, any straight line in $\mathbb{R}^n$ has a parametrization of the form (7.10), which implies that $\boldsymbol{\beta}'' \equiv 0$. $\blacksquare$

In contrast to the signed curvature $\boldsymbol{\kappa 2}$ defined in Chapter 1, the curvature $\boldsymbol{\kappa}$ is manifestly nonnegative. For a curve in the plane, the two quantities are however related in the obvious way:

*Lemma 7.6.* *Let $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^2$ be a unit-speed curve. Then*

$$\boldsymbol{\kappa}[\boldsymbol{\beta}] = \big|\boldsymbol{\kappa 2}[\beta]\big|.$$

*Proof.* From Lemma 1.21 we have

$$\boldsymbol{\kappa}[\boldsymbol{\beta}] = \|\boldsymbol{\beta}''\| = \big\|\boldsymbol{\kappa 2}[\beta] J\boldsymbol{\beta}'\big\| = \big|\boldsymbol{\kappa 2}[\beta]\big|\,\|\boldsymbol{\beta}'\| = \big|\boldsymbol{\kappa 2}[\beta]\big|. \blacksquare$$

*Definition 7.7.* *Let $\boldsymbol{\beta}\colon (c,d) \to \mathbb{R}^n$ be a unit-speed curve. Then*

$$\mathbf{T} = \boldsymbol{\beta}'$$

*is called the* ***unit tangent vector field*** *of $\boldsymbol{\beta}$.*

We need a general fact about a vector field along a curve. (The definition of vector field along a curve was given in Section 1.2.)

*Lemma 7.8.* *If $\mathbf{F}$ is a vector field of unit length along a curve $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$, then $\mathbf{F}' \cdot \mathbf{F} = 0$.*

*Proof.* Differentiating the equation

$$\mathbf{F}(t) \cdot \mathbf{F}(t) = 1$$

gives $2\,\mathbf{F}(t) \cdot \mathbf{F}'(t) = 0$. $\blacksquare$

Taking $\mathbf{F} = \mathbf{T}$ in Lemma 7.8, we see that $\mathbf{T}' \cdot \mathbf{T} = 0$.

We now restrict our attention to unit-speed curves in $\mathbb{R}^3$ whose curvature is strictly positive. This implies that $\mathbf{T}' = \boldsymbol{\beta}''$ never vanishes. Now we can define the torsion, as well as the vector fields $\mathbf{N}$ and $\mathbf{B}$.

*Definition 7.9.* *Let $\boldsymbol{\beta}\colon (c,d) \to \mathbb{R}^3$ be a unit-speed curve, and suppose that $\boldsymbol{\kappa}(s) > 0$ for $c < s < d$. The vector field*

$$\mathbf{N} = \frac{1}{\boldsymbol{\kappa}}\mathbf{T}'$$

*is called the* ***principal normal vector field*** *and $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ is called the* ***binormal vector field****. The triple $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is called the* ***Frenet***[3] ***frame field*** *on $\boldsymbol{\beta}$.*

---

[3]Jean Frédéric Frenet (1816–1900). French mathematician. Professor at Toulouse and Lyon.

The Frenet frame at a given point of a space curve is illustrated in Figure 7.1.

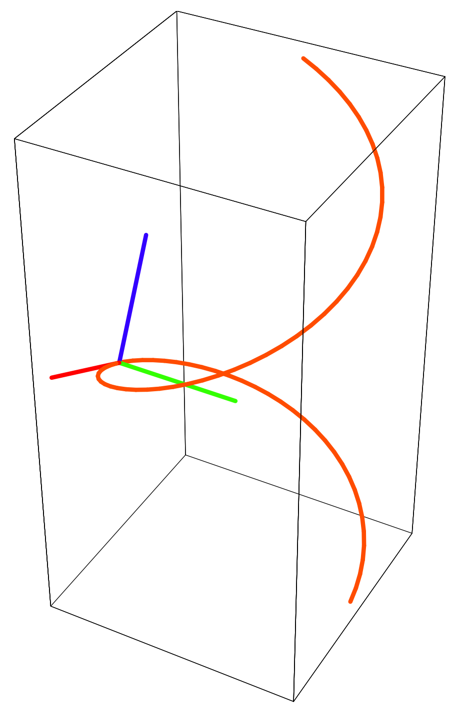


*Figure 7.1*: Frenet frame on the helix

We are now ready to establish the Frenet formulas; they form one of the basic tools for the differential geometry of space curves.

*Theorem 7.10.* *Let* $\boldsymbol{\beta}\colon (c,d) \to \mathbb{R}^3$ *be a unit-speed curve with* $\boldsymbol{\kappa}(s) > 0$ *for* $c < s < d$. *Then*:

(i) $\|\mathbf{T}\| = \|\mathbf{N}\| = \|\mathbf{B}\| = 1$ *and* $\mathbf{T}\cdot\mathbf{N} = \mathbf{N}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{T} = 0$.

(ii) *Any vector field* $\mathbf{F}$ *along* $\boldsymbol{\beta}$ *can be expanded as*

$$\mathbf{F} = (\mathbf{F}\cdot\mathbf{T})\mathbf{T} + (\mathbf{F}\cdot\mathbf{N})\mathbf{N} + (\mathbf{F}\cdot\mathbf{B})\mathbf{B}. \tag{7.11}$$

(iii) *The* ***Frenet formulas*** *hold*:

$$\begin{cases} \mathbf{T}' = & & \boldsymbol{\kappa}\mathbf{N}, & \\ \mathbf{N}' = & -\boldsymbol{\kappa}\mathbf{T} & & +\boldsymbol{\tau}\mathbf{B}, \\ \mathbf{B}' = & & -\boldsymbol{\tau}\mathbf{N}. & \end{cases} \tag{7.12}$$

*Here, the function* $\boldsymbol{\tau} = \boldsymbol{\tau}[\boldsymbol{\beta}]$ *is called the* ***torsion*** *of the curve* $\boldsymbol{\beta}$.

*Proof.* By definition $\|\mathbf{T}\| = 1$; furthermore,

$$\|\mathbf{N}\| = \left\| \frac{1}{\boldsymbol{\kappa}} \mathbf{T}' \right\| = \frac{\|\boldsymbol{\beta}''\|}{|\boldsymbol{\kappa}|} = 1,$$

and by Lemma 7.8 we have $\mathbf{T} \cdot \mathbf{N} = 0$. Therefore, the Lagrange identity (7.2) implies that

$$\|\mathbf{B}\|^2 = \|\mathbf{T} \times \mathbf{N}\|^2 = \|\mathbf{T}\|^2 \|\mathbf{N}\|^2 - (\mathbf{T} \cdot \mathbf{N})^2 = \|\mathbf{T}\|^2 \|\mathbf{N}\|^2 = 1.$$

Finally, (7.4) implies that $\mathbf{B} \cdot \mathbf{T} = \mathbf{B} \cdot \mathbf{N} = 0$.

Since $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are mutually orthogonal, they form a basis for the vector fields along $\boldsymbol{\beta}$. Hence there exist functions $\lambda, \mu, \nu$ such that

$$\mathbf{F} = \lambda \mathbf{T} + \mu \mathbf{N} + \nu \mathbf{B}. \tag{7.13}$$

When we take the dot product of both sides of (7.13) with $\mathbf{T}$ and use (i), we find that $\lambda = \mathbf{F} \cdot \mathbf{T}$. Similarly, $\mu = \mathbf{F} \cdot \mathbf{N}$ and $\nu = \mathbf{F} \cdot \mathbf{B}$, proving (ii).

For (iii), we first observe that the first equation of (7.12) holds by definition of $\mathbf{N}$. To prove the third equation of (7.12), we differentiate $\mathbf{B} \cdot \mathbf{T} = 0$, obtaining $\mathbf{B}' \cdot \mathbf{T} + \mathbf{B} \cdot \mathbf{T}' = 0$; then

$$\mathbf{B}' \cdot \mathbf{T} = -\mathbf{B} \cdot \mathbf{T}' = -\mathbf{B} \cdot \boldsymbol{\kappa} \mathbf{N} = 0.$$

Lemma 7.8 implies $\mathbf{B}' \cdot \mathbf{B} = 0$. Since $\mathbf{B}'$ is perpendicular to $\mathbf{T}$ and $\mathbf{B}$, it must be a multiple of $\mathbf{N}$ by part (ii). This means that we can define the torsion $\boldsymbol{\tau}$ by the equation

$$\mathbf{B}' = -\boldsymbol{\tau} \mathbf{N}.$$

We have established the first and third of the Frenet formulas. To prove the second, we use the orthonormal expansion of $\mathbf{N}'$ in terms of $\mathbf{T}, \mathbf{N}, \mathbf{B}$ given by (ii), namely,

$$\mathbf{N}' = (\mathbf{N}' \cdot \mathbf{T})\mathbf{T} + (\mathbf{N}' \cdot \mathbf{N})\mathbf{N} + (\mathbf{N}' \cdot \mathbf{B})\mathbf{B}. \tag{7.14}$$

The coefficients in (7.14) are easy to find. First, differentiating $\mathbf{N} \cdot \mathbf{T} = 0$ we get

$$\mathbf{N}' \cdot \mathbf{T} = -\mathbf{N} \cdot \mathbf{T}' = -\mathbf{N} \cdot \boldsymbol{\kappa} \mathbf{N} = -\boldsymbol{\kappa}.$$

That $\mathbf{N}' \cdot \mathbf{N} = 0$ follows from Lemma 7.8. Finally,

$$\mathbf{N}' \cdot \mathbf{B} = -\mathbf{N} \cdot \mathbf{B}' = -\mathbf{N} \cdot (-\boldsymbol{\tau} \mathbf{N}) = \boldsymbol{\tau}.$$

Hence (7.14) reduces to the second Frenet formula. ▮

The first book containing a systematic treatment of space curves is Clairaut's *Recherches sur les courbes à double courbure*[4]. After that the term 'courbe à double courbure' became a technical term for a space curve. The theory of space curves became much simpler after Frenet discovered the formulas named after him in 1847. Serret[5] found the formulas independently in 1851, and for this reason they are sometimes called the Frenet-Serret formulas (see [Frenet] and [Serret1]). In spite of their simplicity and usefulness, many years passed before they gained wide acceptance. The Frenet formulas were actually discovered for the first time in 1831 by Senff[6] and his teacher Bartels[7]. Needless to say, the isolation of Senff and Bartels in Dorpat, then a part of Russia, prevented their work from becoming widely known. (See [Reich] for details.)

It is impossible to define $\mathbf{N}$ for a straight line parametrized as a unit-speed curve $\boldsymbol{\beta}$, since $\mathbf{T}$ is a constant vector and $\boldsymbol{\kappa}$ vanishes. However, one is at liberty to take $\mathbf{N}$ and $\mathbf{B}$ to be arbitrary constant vector fields along $\boldsymbol{\beta}$, and to define the torsion of $\boldsymbol{\beta}$ to be zero. Then the formulas (7.12) remain valid (see Exercise 5 for details).

The following lemma shows that the torsion measures the failure of a curve to lie in a plane.

**Lemma 7.11.** *Let $\boldsymbol{\beta}\colon (c,d) \to \mathbb{R}^3$ be a unit-speed curve with $\boldsymbol{\kappa}(s) > 0$ for $c < s < d$. The following conditions are equivalent:*

(i) *$\boldsymbol{\beta}$ is a plane curve*;

(ii) $\boldsymbol{\tau} \equiv 0$.

*When* (i) *and* (ii) *hold,* $\mathbf{B}$ *is perpendicular to the plane containing* $\boldsymbol{\beta}$.

---

4 

Alexis Claude Clairaut (1713–1765). French mathematician and astronomer, who at the age of 18 was elected to the French Academy of Sciences for his work on curve theory. In 1736–1737 he took part in an expedition to Lapland led by Maupertuis, the purpose of which was to verify Newton's theoretical proof that the earth is an oblate spheroid. Clairaut's precise calculations led to a near perfect prediction of the arrival of Halley's comet in 1759.

5 

Joseph Alfred Serret (1819–1885). French mathematician. Serret with other Paris mathematicians greatly advanced differential calculus in the period 1840–1865. Serret also worked in number theory and mechanics. Another Serret (Paul Joseph (1827–1898)) wrote a book *Théorie nouvelle géométrique et mécanique des lignes à double courbure* emphasizing space curves.

[6] Karl Eduard Senff (1810–1849). German professor at the University of Dorpat (now Tartu) in Estonia.

[7] Johann Martin Bartels (1769–1836). Another German professor at the University of Dorpat. Bartels was the first teacher of Gauss (in Brunswick); he and Gauss kept in contact over the years.

*Proof.* The condition that a curve $\boldsymbol{\beta}$ lie in a plane $\Pi$ can be expressed analytically as

$$(\boldsymbol{\beta}(s) - \mathbf{p}) \cdot \mathbf{q} \equiv 0,$$

where $\mathbf{q}$ is a nonzero vector perpendicular to $\Pi$. Differentiation yields

$$\boldsymbol{\beta}'(s) \cdot \mathbf{q} \equiv 0 \equiv \boldsymbol{\beta}''(s) \cdot \mathbf{q}.$$

Thus both $\mathbf{T}$ and $\mathbf{N}$ are perpendicular to $\mathbf{q}$. Since $\mathbf{B}$ is also perpendicular to $\mathbf{T}$ and $\mathbf{N}$, it follows that $\mathbf{B}(s) = \pm\mathbf{q}/\|\mathbf{q}\|$ for all $s$. Therefore $\mathbf{B}' \equiv 0$, and from the definition of torsion, it follows that $\boldsymbol{\tau} \equiv 0$.

Conversely, suppose that $\boldsymbol{\tau} \equiv 0$; then (7.12) implies that $\mathbf{B}' \equiv -\boldsymbol{\tau}\mathbf{N} \equiv 0$. Thus $s \mapsto \mathbf{B}(s)$ is a constant curve; that is, there exists a unit vector $\mathbf{u} \in \mathbb{R}^3$ such that $\mathbf{B} \equiv \mathbf{u}$. Choose $t_0$ with $c < t_0 < d$ and consider the real-valued function $f\colon (c,d) \to \mathbb{R}^3$ given by

$$f(s) = \big(\boldsymbol{\beta}(s) - \boldsymbol{\beta}(t_0)\big) \cdot \mathbf{u}.$$

Then $f(t_0) = 0$ and $f'(s) \equiv \boldsymbol{\beta}'(s) \cdot \mathbf{u} \equiv \mathbf{T} \cdot \mathbf{B} \equiv 0$, so that $f$ is identically zero. Thus $\big(\boldsymbol{\beta}(s) - \boldsymbol{\beta}(t_0)\big) \cdot \mathbf{u} \equiv 0$, and it follows that $\boldsymbol{\beta}$ lies in the plane perpendicular to $\mathbf{u}$ that passes through $\boldsymbol{\beta}(t_0)$. ∎

## 7.3 The Helix and Twisted Cubic

The ***circular helix*** is a curve in $\mathbb{R}^3$ that resembles a spring. Its usual parametrization is

$$\mathsf{helix}[a,b](t) = \big(a\cos t,\ a\sin t,\ bt\big), \tag{7.15}$$

where $a > 0$ is the radius and $b$ is the ***slope*** or ***incline*** of the helix. Observe that the projection of $\mathbb{R}^3$ onto the $xy$-plane maps the helix onto the circle $(a\cos t,\ a\sin t)$ of radius $a$. Figure 7.2 displays the helix $t \mapsto (\cos t,\ \sin t,\ \frac{1}{5}t)$.

The helix is one of the few curves for which a unit-speed parametrization is easy to find. In fact, a unit-speed parametrization of the helix is given by

$$\boldsymbol{\beta}(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}},\ \frac{bs}{\sqrt{a^2+b^2}}\right).$$

We use Theorem 7.10 to compute $\boldsymbol{\kappa}, \boldsymbol{\tau}$ and $\mathbf{T}, \mathbf{N}, \mathbf{B}$ for $\boldsymbol{\beta}$. Firstly,

$$\mathbf{T}(s) = \boldsymbol{\beta}'(s) = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin\frac{s}{\sqrt{a^2+b^2}},\ a\cos\frac{s}{\sqrt{a^2+b^2}},\ b\right);$$

thus $\|\mathbf{T}(s)\| = 1$ and $\boldsymbol{\beta}$ is indeed a unit-speed curve. Moreover,

$$\mathbf{T}'(s) = \frac{1}{a^2+b^2}\left(-a\cos\frac{s}{\sqrt{a^2+b^2}},\ -a\sin\frac{s}{\sqrt{a^2+b^2}},\ 0\right).$$

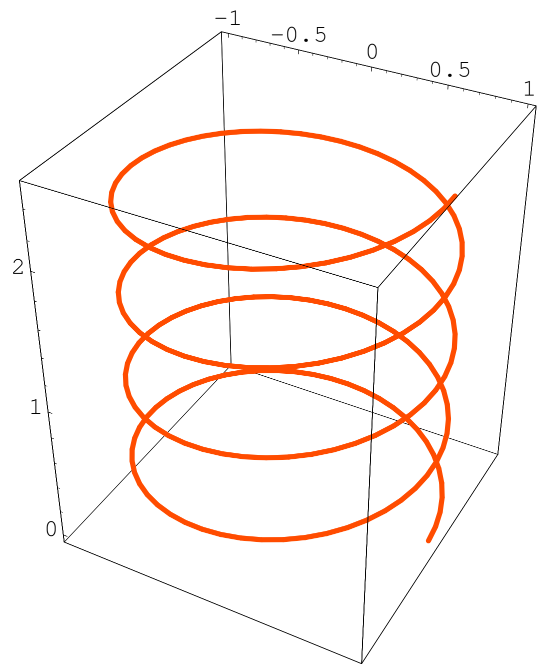



*Figure 7.2*: Part of the trace of helix$[1, 0.2]$

Since $a > 0$,

$$\boldsymbol{\kappa}(s) = \|\mathbf{T}'(s)\| = \frac{a}{a^2+b^2}, \tag{7.16}$$

and

$$\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} = \left(-\cos\frac{s}{\sqrt{a^2+b^2}},\ -\sin\frac{s}{\sqrt{a^2+b^2}},\ 0\right). \tag{7.17}$$

From the formula $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ we get

$$\mathbf{B}(s) = \frac{1}{\sqrt{a^2+b^2}}\left(b\sin\frac{s}{\sqrt{a^2+b^2}},\ -b\cos\frac{s}{\sqrt{a^2+b^2}},\ a\right).$$

Finally, to compute the torsion we note that

$$\mathbf{B}'(s) = \frac{1}{a^2+b^2}\left(b\cos\frac{s}{\sqrt{a^2+b^2}},\ b\sin\frac{s}{\sqrt{a^2+b^2}},\ 0\right).$$

When we compare this to (7.17), and use the Frenet formula $\mathbf{B}' = -\boldsymbol{\tau}\mathbf{N}$, we see that

$$\boldsymbol{\tau}(s) = \frac{b}{a^2+b^2}.$$

In conclusion, both the curvature and torsion of a helix are constant, and there is no need to graph them!

Although we know that every regular curve has a unit-speed parametrization, in practice it is very difficult to find it. For example, consider the ***twisted cubic*** defined by

$$\mathsf{twicubic}(t) = (t,\ t^2,\ t^3). \tag{7.18}$$

One can readily check that this curve does not lie in a plane. For any such plane would have to be parallel to all three vectors

$$\boldsymbol{\gamma}'(0) = (1,0,0), \quad \boldsymbol{\gamma}''(0) = (0,2,0), \quad \boldsymbol{\gamma}'''(0) = (0,0,6),$$

where $\boldsymbol{\gamma} = \mathsf{twicubic}$. But this is clearly impossible.

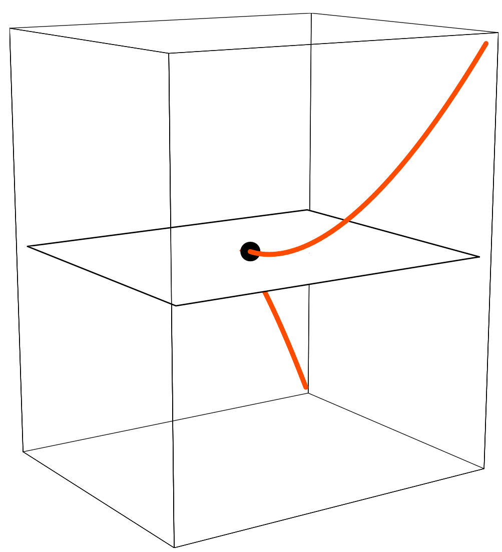


*Figure 7.3*: twicubic and the plane $z = 0$

Since $\boldsymbol{\gamma}'(0)$ is a unit vector, it coincides with $\mathbf{T}$ at $t = 0$. On the other hand, $\boldsymbol{\gamma}'(t)$ is not a unit vector unless $t = 0$, so $\boldsymbol{\gamma}''(0)$ is not parallel to $\mathbf{N}$. Nonetheless, $\boldsymbol{\gamma}''(0)$ is a linear combination of $\mathbf{T}$ and $\mathbf{N}$, and both pairs $\{\boldsymbol{\gamma}'(0),\ \boldsymbol{\gamma}''(0)\}$ and $\{\mathbf{T},\ \mathbf{N}\}$ (when applied to the origin) generate the $xy$-plane, shown in Figure 7.3.

The arc length function of the curve (7.18) is

$$s(t) = \int_0^t \|\boldsymbol{\gamma}'(u)\|\, du = \int_0^t \sqrt{1 + 4u^2 + 9u^4}\, du.$$

The inverse of $s(t)$, which is needed to find the unit-speed parametrization, is an elliptic function that too complicated to be of much use. We shall show how to compute the curvature and torsion of (7.18) more directly in the next section.

## 7.4 Arbitrary-Speed Curves in $\mathbb{R}^3$

For efficient computation of the curvature and torsion of an arbitrary-speed curve, we need formulas that bypass finding a unit-speed parametrization. Although we shall define the curvature and torsion of an arbitrary-speed curve in terms of the curvature and torsion of its unit-speed parametrization, ultimately we shall find formulas for these quantities that avoid finding a unit-speed parametrization explicitly.

The theoretical definition is

**Definition 7.12.** *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^3$ be a regular curve, and let $\widetilde{\boldsymbol{\alpha}}\colon (c,d) \to \mathbb{R}^3$ be a unit-speed reparametrization of $\boldsymbol{\alpha}$. Write $\boldsymbol{\alpha}(t) = \widetilde{\boldsymbol{\alpha}}\big(s(t)\big)$, where $s(t)$ is the arc length function. Denote by $\widetilde{\boldsymbol{\kappa}}$ and $\widetilde{\boldsymbol{\tau}}$ the curvature and torsion of $\widetilde{\boldsymbol{\alpha}}$, respectively. Also, let $\{\widetilde{\mathbf{T}}, \widetilde{\mathbf{N}}, \widetilde{\mathbf{B}}\}$ be the Frenet frame field of $\widetilde{\boldsymbol{\alpha}}$. Then we define*

$$\boldsymbol{\kappa}(t) = \widetilde{\boldsymbol{\kappa}}\big(s(t)\big), \qquad \boldsymbol{\tau}(t) = \widetilde{\boldsymbol{\tau}}\big(s(t)\big),$$
$$\mathbf{T}(t) = \widetilde{\mathbf{T}}\big(s(t)\big), \quad \mathbf{N}(t) = \widetilde{\mathbf{N}}\big(s(t)\big), \quad \mathbf{B}(t) = \widetilde{\mathbf{B}}\big(s(t)\big).$$

*Thus, the curvature, torsion and Frenet frame field of an arbitrary-speed curve $\boldsymbol{\alpha}$ are reparametrizations of those of a unit-speed parametrization of $\boldsymbol{\alpha}$.*

Next, we generalize the Frenet formulas (7.12) to arbitrary-speed curves.

**Theorem 7.13.** *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^3$ be a regular curve with speed $v = \|\boldsymbol{\alpha}'\| = s'$. Then the following generalizations of the Frenet formulas hold:*

$$\left\{\begin{array}{lcl} \mathbf{T}' &=& \qquad\quad v\boldsymbol{\kappa}\mathbf{N}, \\ \mathbf{N}' &=& -v\boldsymbol{\kappa}\mathbf{T} \qquad\qquad\quad +v\boldsymbol{\tau}\mathbf{B}, \\ \mathbf{B}' &=& \qquad\quad -v\boldsymbol{\tau}\mathbf{N}. \end{array}\right. \tag{7.19}$$

*Proof.* By the chain rule we have

$$\left\{\begin{array}{lcl} \mathbf{T}'(t) &=& s'(t)\widetilde{\mathbf{T}}'\big(s(t)\big) = v(t)\widetilde{\mathbf{T}}'\big(s(t)\big), \\ \mathbf{N}'(t) &=& s'(t)\widetilde{\mathbf{N}}'\big(s(t)\big) = v(t)\widetilde{\mathbf{N}}'\big(s(t)\big), \\ \mathbf{B}'(t) &=& s'(t)\widetilde{\mathbf{B}}'\big(s(t)\big) = v(t)\widetilde{\mathbf{B}}'\big(s(t)\big). \end{array}\right. \tag{7.20}$$

Thus from the Frenet formulas for $\widetilde{\boldsymbol{\alpha}}$, it follows that

$$\mathbf{T}'(t) = v(t)\widetilde{\boldsymbol{\kappa}}\big(s(t)\big)\widetilde{\mathbf{N}}\big(s(t)\big) = v(t)\boldsymbol{\kappa}(t)\mathbf{N}(t).$$

The other two formulas of (7.20) are proved similarly. ∎

*Lemma 7.14.* *The velocity $\boldsymbol{\alpha}'$ and acceleration $\boldsymbol{\alpha}''$ of a regular curve $\boldsymbol{\alpha}$ are given by*

$$\boldsymbol{\alpha}' = v\mathbf{T}, \tag{7.21}$$

$$\boldsymbol{\alpha}'' = \frac{dv}{dt}\mathbf{T} + v^2\boldsymbol{\kappa}\mathbf{N}, \tag{7.22}$$

*where $v$ denotes the speed of $\boldsymbol{\alpha}$.*

*Proof.* Write $\boldsymbol{\alpha}(t) = \widetilde{\boldsymbol{\alpha}}\big(s(t)\big)$, where $\widetilde{\boldsymbol{\alpha}}$ is a unit-speed parametrization of $\boldsymbol{\alpha}$. By the chain rule we have

$$\boldsymbol{\alpha}'(t) = \widetilde{\boldsymbol{\alpha}}'\big(s(t)\big)s'(t) = v(t)\widetilde{\mathbf{T}}\big(s(t)\big) = v(t)\mathbf{T}(t),$$

proving (7.21). Next, we take the derivative of (7.21) and use the first equation of (7.19) to get

$$\boldsymbol{\alpha}'' = v'\mathbf{T} + v\mathbf{T}' = v'\mathbf{T} + v^2\boldsymbol{\kappa}\mathbf{N},$$

and in stating the result we have chosen to write $v' = dv/dt$. ∎

Now we can derive useful formulas for the curvature and torsion for an arbitrary-speed curve. These formulas avoid finding a unit-speed reparametrization.

*Theorem 7.15.* *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^3$ be a regular curve with nonzero curvature. Then*

$$\mathbf{T} = \frac{\boldsymbol{\alpha}'}{\|\boldsymbol{\alpha}'\|}, \tag{7.23}$$

$$\mathbf{N} = \mathbf{B} \times \mathbf{T}, \tag{7.24}$$

$$\mathbf{B} = \frac{\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''}{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|}, \tag{7.25}$$

$$\boldsymbol{\kappa} = \frac{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|}{\|\boldsymbol{\alpha}'\|^3}, \tag{7.26}$$

$$\boldsymbol{\tau} = \frac{[\boldsymbol{\alpha}'\ \boldsymbol{\alpha}''\ \boldsymbol{\alpha}''']}{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|^2}. \tag{7.27}$$

*Proof.* Clearly, (7.23) is equivalent to (7.21), and (7.24) is an algebraic consequence of the definition of vector cross product. Furthermore, it follows from (7.21) and (7.22) that

$$\begin{aligned} \boldsymbol{\alpha}' \times \boldsymbol{\alpha}'' &= v\mathbf{T} \times \left(\frac{dv}{dt}\mathbf{T} + \boldsymbol{\kappa} v^2\mathbf{N}\right) \\ &= v\frac{dv}{dt}\mathbf{T} \times \mathbf{T} + \boldsymbol{\kappa} v^3\,\mathbf{T} \times \mathbf{N} = \boldsymbol{\kappa} v^3\,\mathbf{B}. \end{aligned} \tag{7.28}$$

Taking norms in (7.28), we get

$$\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\| = \|\boldsymbol{\kappa} v^3 \mathbf{B}\| = \boldsymbol{\kappa} v^3. \tag{7.29}$$

Then (7.29) implies (7.26). Furthermore, (7.28) and (7.29) imply that

$$\mathbf{B} = \frac{\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''}{v^3 \boldsymbol{\kappa}} = \frac{\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''}{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|},$$

and so (7.25) is proved.

To prove (7.27) we need a formula for $\boldsymbol{\alpha}'''$ analogous to (7.21) and (7.22). Actually, all we need is the component of $\boldsymbol{\alpha}'''$ in the $\mathbf{B}$ direction, because we want to take the dot product of $\boldsymbol{\alpha}'''$ with $\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''$. So we compute

$$\begin{aligned} \boldsymbol{\alpha}''' &= \left(\frac{dv}{dt}\mathbf{T} + \boldsymbol{\kappa} v^2 \mathbf{N}\right)' = \boldsymbol{\kappa} v^2 \mathbf{N}' + \cdots \\ &= \boldsymbol{\kappa\tau} v^3 \mathbf{B} + \cdots \end{aligned} \tag{7.30}$$

where the dots represent irrelevant terms. It follows from (7.28) and (7.30) that

$$(\boldsymbol{\alpha}' \times \boldsymbol{\alpha}'') \cdot \boldsymbol{\alpha}''' = \boldsymbol{\kappa}^2 \boldsymbol{\tau} v^6. \tag{7.31}$$

Now (7.27) follows from (7.31) and (7.29). ∎

To conclude this section, we compute the curvature and torsion of the curve $\boldsymbol{\gamma} =$ twicubic defined by (7.18). Instead of finding a unit-speed reparametrization of $\boldsymbol{\gamma}$, we use Theorem 7.15. The computations are easy enough to do by hand:

$$\begin{aligned} \boldsymbol{\gamma}'(t) &= (1,\ 2t,\ 3t^2), & \boldsymbol{\gamma}''(t) &= (0,\ 2,\ 6t), \\ \boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t) &= (6t^2,\ -6t,\ 2), & \boldsymbol{\gamma}'''(t) &= (0,\ 0,\ 6), \end{aligned}$$

so that

$$\begin{aligned} \|\boldsymbol{\gamma}'(t)\| &= (1 + 4t^2 + 9t^4)^{1/2}, \\ \|\boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t)\| &= (4 + 36t^2 + 36t^4)^{1/2}. \end{aligned}$$

Therefore, the curvature and torsion of the twisted cubic are given by

$$\begin{aligned} \boldsymbol{\kappa}(t) &= \frac{(4 + 36t^2 + 36t^4)^{1/2}}{(1 + 4t^2 + 9t^4)^{3/2}} \\ \boldsymbol{\tau}(t) &= \frac{3}{1 + 9t^2 + 9t^4}. \end{aligned}$$

Their respective maximum values are 2 and 3, and the graphs are plotted in Figure 7.4.

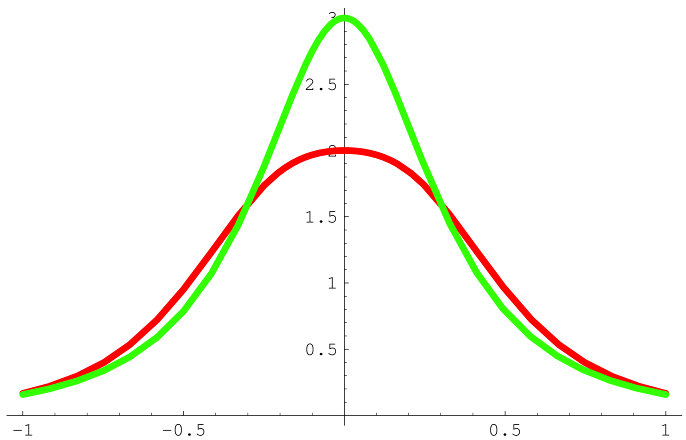



*Figure 7.4*: Curvature and torsion of a cubic

The behavior of the Frenet frame of the twisted cubic for large values of $|t|$ is discussed on page 794 in Chapter 23.

## 7.5 More Constructions of Space Curves

Just as the helix sits over the circle, there are helical analogs of many other plane curves. Suppose that we are given a plane curve $t \mapsto \boldsymbol{\alpha}(t) = (x(t), y(t))$. A general formula for the helix with slope $c$ over $\boldsymbol{\alpha}$ is

$$\mathsf{helical}[\boldsymbol{\alpha}, c](t) = \big(x(t),\ y(t),\ ct\big) \tag{7.32}$$

As an example, we may construct the helical curve over the logarithmic spiral. Using the paraemtrization on page 23, we obtain

$$\mathsf{helical}[\mathsf{logspiral}[a,b], c](t) = \big(ae^{bt}\cos t,\ ae^{bt}\sin t,\ ct\big).$$

A similar but more complicated example formed from the clothoid is shown in Figure 7.8, at the end of this section on page 209.

The formula (7.32) can of course be generalized by defining the third coordinate or 'height' $z$ to equal an arbitrary function of the parameter $t$. One can use this idea to graph the signed curvature of a plane curve. Suppose that $t \mapsto \boldsymbol{\alpha}(t) = (x(t),\ y(t))$ is a plane curve with signed curvature function $\boldsymbol{\kappa 2}(t)$, and define

$$\boldsymbol{\beta}(t) = \big(x(t),\ y(t),\ \boldsymbol{\kappa 2}(t)\big).$$

Figure 7.5 displays $\boldsymbol{\beta}$ when $\boldsymbol{\alpha}$ is one of the epitrochoids defined on page 157.

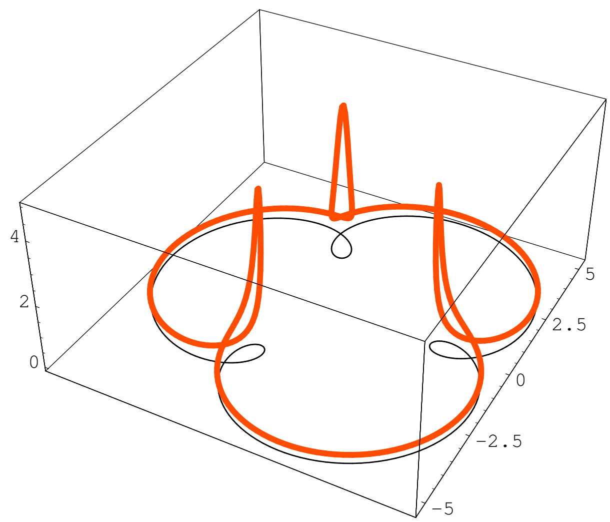



*Figure 7.5*: The curvature of epitrochoid$[3, 1, \frac{3}{2}]$

Let $a > 0$. Consider the sphere

$$x^2 + y^2 + z^2 = 4a^2 \tag{7.33}$$

of radius $2a$ and center the origin, and the cylinder

$$(x - a)^2 + y^2 = a^2 \tag{7.34}$$

of radius $a$ containing the $z$ axis and passing through the point $(2a, 0, 0)$. This point lies on ***Viviani's curve***, which is by definition the intersection of (7.33) and (7.34). Figure 7.6 shows one quarter of Viviani's curve; the remaining quarters are generated by reflection in the planes shown.

Using the identity $\cos t = 1 - 2\sin^2(t/2)$, one may verify that this intersection is parametrized by

$$\mathsf{viviani}[a](t) = \left(a(1 + \cos t),\ a\sin t,\ 2a\sin\frac{t}{2}\right), \quad -2\pi \leqslant t \leqslant 2\pi, \tag{7.35}$$

Viviani[8] studied the curve in 1692. See [Stru2, pages 10–11] and [Gomes, volume 2, pages 311–320]. Figure 7.6 displays part of the curve in a way that emphasizes that it lies on a sphere.

[8]Vincenzo Viviani (1622–1703). Student and biographer of Galileo. In 1660, together with Borelli, Viviani measured the velocity of sound by timing the difference between the flash and the sound of a cannon. In 1692, Viviani proposed the following problem which aroused much interest. How is it possible that a hemisphere has 4 equal windows of such a size that the remaining surface can be exactly squared? The answer involved the Viviani curve.

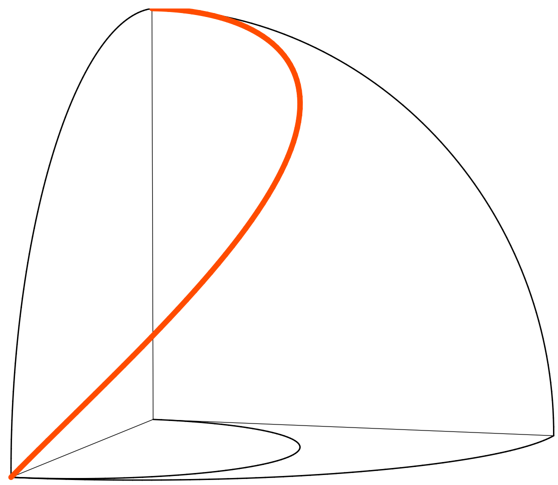


*Figure 7.6*: The trace of $t \mapsto \mathsf{viviani}[1](t)$ for $0 \leqslant t \leqslant \pi$

Computation of the curvature and torsion of Viviani's curve in Notebook 7 yields the results

$$\begin{aligned}\boldsymbol{\kappa}(t) &= \frac{\sqrt{13+3\cos t}}{a(3+\cos t)^{3/2}},\\ \boldsymbol{\tau}(t) &= \frac{6\cos(t/2)}{a(13+3\cos t)}.\end{aligned}$$

These functions are graphed simultaneously in Figure 7.7 for $a=1,\ 0 \leqslant t \leqslant 2\pi$. The torsion vanishes when $t=\pi$, and the most 'planar' parts of Viviani's curve occur at the sphere's poles. Notebook 7 includes an animation of the Frenet frame on Viviani's curve; see also Figure 24.8 on page 794.

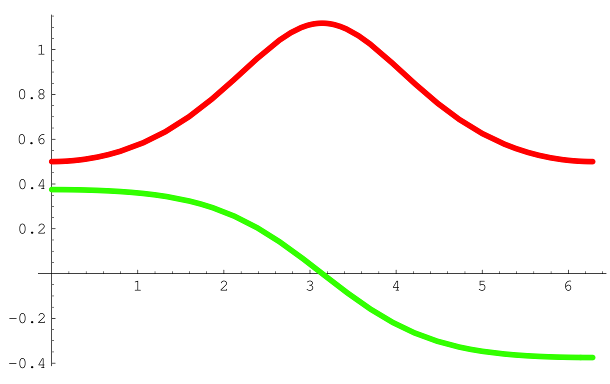



*Figure 7.7*: The curvature (above) and torsion (below) of $\mathsf{viviani}[1]$

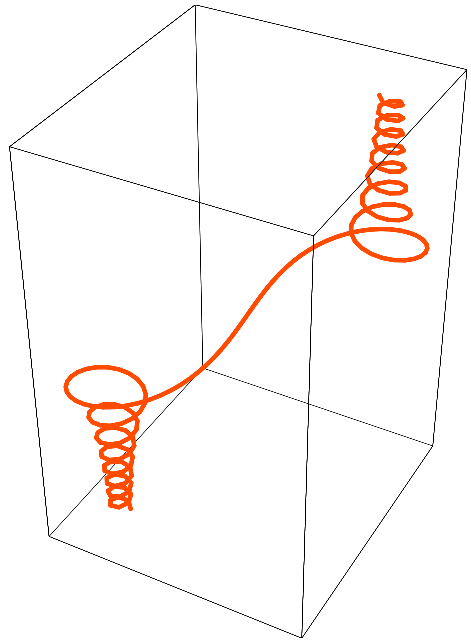


*Figure 7.8*: Helical curve over clothoid$[1,1]$

## 7.6 Tubes and Tori

By definition, the ***tube*** of radius $r$ around a set $\mathscr{C}$ is the set of points at a distance $r$ from $\mathscr{C}$. In particular, let $\boldsymbol{\gamma}\colon (a,b) \to \mathbb{R}^3$ be a regular curve whose curvature does not vanish. Since the normal $\mathbf{N}$ and binormal $\mathbf{B}$ are perpendicular to $\boldsymbol{\gamma}$, the circle

$$\theta \mapsto -\cos\theta\,\mathbf{N}(t) + \sin\theta\,\mathbf{B}(t)$$

is perpendicular to $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(t)$. As this circle moves along $\boldsymbol{\gamma}$ it traces out a surface about $\boldsymbol{\gamma}$, which will be the tube about $\boldsymbol{\gamma}$, provided $r$ is not too large.

We can parametrize this surface by

$$\text{tubecurve}[\boldsymbol{\gamma}, r](t,\theta) = \boldsymbol{\gamma}(t) + r\bigl(-\cos\theta\,\mathbf{N}(t) + \sin\theta\,\mathbf{B}(t)\bigr), \tag{7.36}$$

where $a \leqslant t \leqslant b$ and $0 \leqslant \theta \leqslant 2\pi$. Figure 7.9 displays the tube of radius $1/2$ around the trace of $helix[2, \frac{1}{2}]$.

A tube about a curve $\boldsymbol{\gamma}$ in $\mathbb{R}^3$ has the following interesting property: the volume depends only on the length of $\boldsymbol{\gamma}$ and radius of the tube. In particular, the volume of the tube does not depend on the curvature or torsion of $\boldsymbol{\gamma}$. Thus, for example, tubes of the same radius about a circle and a helix of the same length will have the same volume. For the proofs of these facts and the study of tubes in higher dimensions, see [Gray].

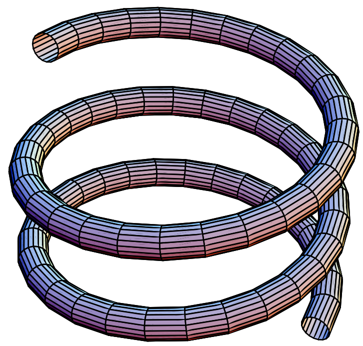


*Figure 7.9*: Tube around a helix

Figure 7.10 shows tubes around two ellipses. Actually, the 'horizontal' tube is a circular torus, formed by rotating a circle around another circle, but both tubes have circular cross sections. More generally, an elliptical torus of revolution is parametrized by the function

$$\text{torus}[a,b,c](u,v) = \big((a+b\cos v)\cos u,\ (a+b\cos v)\sin u,\ c\sin v\big), \tag{7.37}$$

where $u$ represents the angle of rotation about the $z$ axis. Setting $u=0$ gives the ellipse

$$(a+b\cos v,\ 0,\ c\sin v)$$

in the $xz$-plane centered at $(a,0,0)$, waiting to be rotated. Suppose for the time being that $a,b,c>0$. In accordance with Figure 7.11, the ratio $b/c$ determines how 'flat' ($b>c$) or how 'slim' ($b<c$) the torus is. On the other hand, the bigger $a$ is, the bigger the 'hole' in the middle.

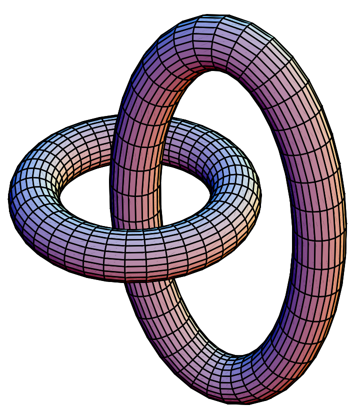


*Figure 7.10*: Tubes around linked ellipses

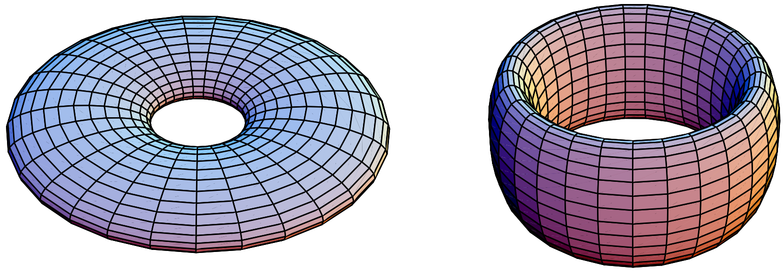


*Figure 7.11*: Tori formed by revolving ellipses

Elliptical tori will be investigated further in Section 10.4. Canal surfaces are generalizations of tubes that will be studied in Chapter 20.

## 7.7 Torus Knots

Curves that wind around a torus are frequently knotted. Let us define

$$\begin{aligned}\mathsf{torusknot}[a,b,c][p,q](t) &= \mathsf{torus}[a,b,c](pt,qt)\\ &= \Big(\big(a+b\cos(qt)\big)\cos(pt),\ \big(a+b\cos(qt)\big)\sin(pt),\ c\sin(qt)\Big)\end{aligned}$$

(refer to (7.37)). It follows that $\mathsf{torusknot}[a,b,c][p,q](t)$ lies on an elliptical torus. For this reason, we have called the curve a ***torus knot***; it may or may not be truly knotted, depending on $p$ and $q$. In fact, $\mathsf{torusknot}[a,b,c][1,q]$ is unknotted, whereas $\mathsf{torusknot}[8,3,5][2,3]$ is the ***trefoil knot***. Of the many books on the interesting subject of knot theory, we mention [BuZi], [Kauf1],[Kauf2] and [Rolf].

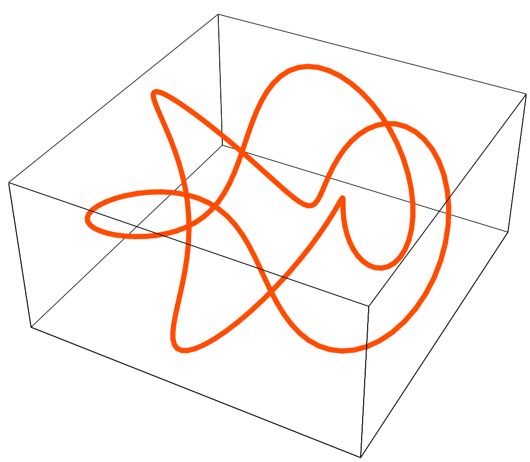


*Figure 7.12*: $\mathsf{torusknot}[8,3,5][2,5]$

For the remainder of this section, we shall study the curve

$$\boldsymbol{\alpha} = \mathsf{torusknot}[8, 3, 5][2, 5]$$

displayed in Figure 7.12. This representation is not as informative as we would like, because at an apparent crossing it is not clear which part of the curve is in front and which part is behind. To see what really is going on, let us draw a tube about this curve using the construction of the previous section. The crossings are now clear; furthermore, we can describe the winding a little more precisely by saying that $\boldsymbol{\alpha}$ spirals around the torus 2 times in the longitudinal sense and 5 times in the meridianal sense. (This is the terminology of [Costa2].) More generally, $\mathsf{torusknot}[a, b, c][p, q]$ will spiral around an elliptical torus $p$ times in the longitudinal sense and $q$ times in the meridianal sense.

Figure 7.13 raises a new issue. The tube itself seems to twist violently in several places. (This is even clearer when the graphics is rendered with fewer sample points in Notebook 7.) A careful look at the plot of the tube shows that this increased twisting occurs in five different places, in spite of the fact that other parts of the tube partially obscure two of these. Moreover, nearby to where the violent twisting occurs, the actual knot curve is straighter. This evidence leads us to suspect that there are five points on the original torus knot where simultaneously the curvature $\boldsymbol{\kappa}$ is small and the absolute value of the torsion $\boldsymbol{\tau}$ is large.

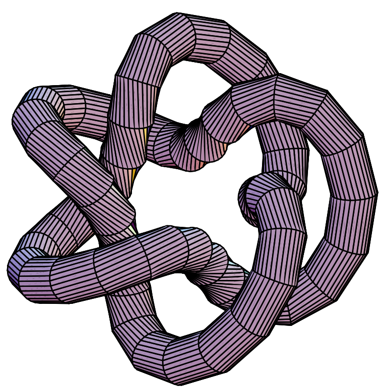


*Figure 7.13*: Tube around a torus knot

The graphs of the curvature and torsion of $\boldsymbol{\alpha}$ can be used to test this empirical evidence; they confirm that indeed there are five values of $t$ at which the curvature of $\boldsymbol{\alpha}(t)$ is small when the absolute value of the torsion is large. Furthermore, the graph seems to indicate these points are $\pi/5$, $3\pi/5$, $\pi$, $7\pi/5$ and $9\pi/5$. In fact, this is the case; the numerical values at $\pi/5$ are

$$\boldsymbol{\kappa}[\boldsymbol{\alpha}](\pi/5) = 0.076, \qquad \boldsymbol{\tau}[\boldsymbol{\alpha}](\pi/5) = -0.414.$$

At $t = 2\pi/5$ the curvature is a little larger and the absolute value of the torsion is quite small:

$$\boldsymbol{\kappa}[\boldsymbol{\alpha}](2\pi/5) = 0.107, \qquad \boldsymbol{\tau}[\boldsymbol{\alpha}](2\pi/5) = -0.002.$$

See Figure 7.14.

The Frenet formulas (7.19), Page 203, explain why small curvature and large absolute torsion produce so much twisting in the tube. The construction of the tube involves $\mathbf{N}$ and $\mathbf{B}$, but not $\mathbf{T}$. When the derivatives of $\mathbf{N}$ and $\mathbf{B}$ are large, the twisting in the tube will also be large. But a glance at the Frenet formulas shows that small curvature and large absolute torsion create exactly this effect.

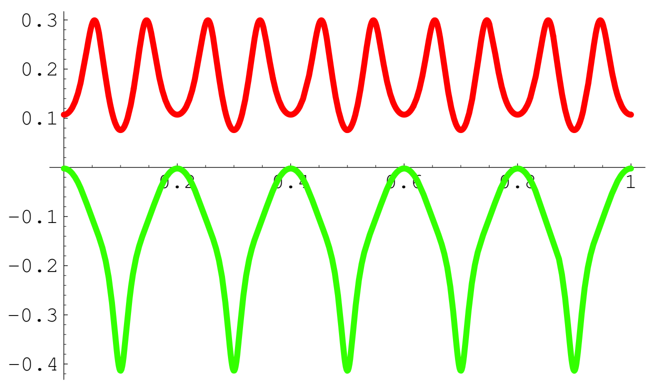



*Figure 7.14*: Curvature (positive) and torsion of torusknot$[8, 3, 5][2, 5]$

## 7.8 Exercises

**1.** Establish the following identities for $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$:

(a) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a}\,\mathbf{c}\,\mathbf{d}]\,\mathbf{b} - [\mathbf{b}\,\mathbf{c}\,\mathbf{d}]\,\mathbf{a}$.

(b) $(\mathbf{d} \times (\mathbf{a} \times \mathbf{b})) \cdot (\mathbf{a} \times \mathbf{c}) = [\mathbf{a}\,\mathbf{b}\,\mathbf{c}](\mathbf{a} \cdot \mathbf{d})$.

(c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = \mathbf{0}$.

(d) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}) = [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]^2$.

M **2.** Graph helical curves over a cycloid, a cardioid and a figure eight curve, and plot the curvature and torsion of each.

**3.** Let $\boldsymbol{\alpha}$ be an arbitrary-speed curve in $\mathbb{R}^n$. Define

$$\boldsymbol{\kappa n}[\boldsymbol{\alpha}] = \frac{\Big\| \|\boldsymbol{\alpha}'\|\boldsymbol{\alpha}'' - \|\boldsymbol{\alpha}'\|'\boldsymbol{\alpha}' \Big\|}{\|\boldsymbol{\alpha}'\|^3}.$$

Show that $\boldsymbol{\kappa n}[\boldsymbol{\alpha}]$ is the curvature of $\boldsymbol{\alpha}$ for $n = 3$.

M **4.** Plot the following space curves and graph their curvature and torsion:

(a) $s \mapsto \left(\dfrac{(1+s)^{3/2}}{3}, \dfrac{(1-s)^{3/2}}{3}, \dfrac{s}{\sqrt{2}}\right)$.

(b) $t \mapsto \big(f(t)\cos t,\ f(t)\sin t,\ a\cos 10t\big)$ where $f(t) = \sqrt{1 - a^2\cos^2 10t}$, and $a = 0.3$.

(c) $t \mapsto 0.5\big((1+\cos 10t)\cos t,\ (1+\cos 10t)\sin t,\ 1+\cos 10t\big)$.

(d) $t \mapsto (a\cosh t,\ a\sinh t,\ b\,t)$.

(e) $t \mapsto \big(te^t,\ e^{-t},\ \sqrt{2}\,t\big)$.

(f) $t \mapsto a\big(t - \sin t,\ 1 - \cos t,\ 4\cos(t/2)\big)$.

**5.** Let $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^3$ be a straight line parametrized as $\boldsymbol{\beta}(s) = s\,\mathbf{e}_1 + \mathbf{q}$, where $\mathbf{e}_1, \mathbf{q} \in \mathbb{R}^3$ with $\mathbf{e}_1$ a unit vector. Let $\mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3$ be such that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis of $\mathbb{R}^3$. Define constant vector fields by $\mathbf{T}(s) = \mathbf{e}_1$, $\mathbf{N}(s) = \mathbf{e}_2$ and $\mathbf{B}(s) = \mathbf{e}_3$ for $a < s < b$. Show that the Frenet formulas (7.12) hold provided we take $\boldsymbol{\kappa}(s) = \boldsymbol{\tau}(s) = 0$ for $a < s < b$.

**6.** Suppose that $a < b$ and choose $\epsilon$ to be 1 or $-1$. Show that the mapping

$$\mathsf{bicylinder}[a,b,\epsilon](t) = \left(a\cos t,\ a\sin t,\ \epsilon\sqrt{b^2 - a^2\sin^2 t}\,\right)$$

parametrizes one component of the intersection of a cylinder of radius $a$ and an appropriately-positioned cylinder of radius $b$.

M **7.** A 3-dimensional version of the astroid is defined by

$$\mathsf{ast3d}[n,a,b](t) = (a\cos^n t,\ b\sin^n t,\ \cos 2t).$$

Compute the curvature and torsion of $\mathsf{ast3d}[n,1,1]$ and plot $\mathsf{ast3d}[n,1,1]$ for $n > 3$.

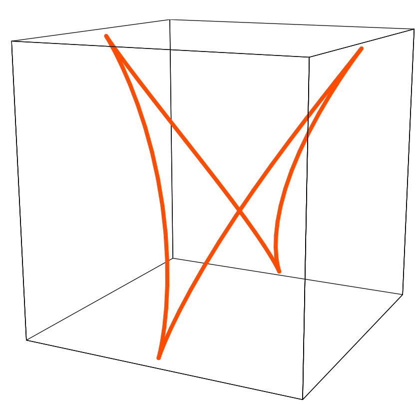


*Figure 7.15*: The 3-dimensional astroid $\mathsf{ast3d}[3,1,1]$

M **8.** An ***elliptical helix*** is given by

$$\text{(7.38)} \qquad \mathsf{helix}[a,b,c](t) = \big(a\cos t,\ b\sin t,\ ct\big).$$

Plot an elliptical helix, its curvature and torsion. Show that an elliptical helix has constant curvature if and only if $a^2 = b^2$. [Hint: Compute the derivative of the curvature of an elliptical helix and evaluate it at $\pi/4$.]

M **9.** The ***Darboux***[9] ***vectorfield*** along a unit-speed curve $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^3$ is defined by $\mathbf{D} = \boldsymbol{\tau}\mathbf{T} + \boldsymbol{\kappa}\mathbf{B}$. Show that

$$\begin{cases} \mathbf{T}' &= \mathbf{D}\times\mathbf{T}, \\ \mathbf{N}' &= \mathbf{D}\times\mathbf{N}, \\ \mathbf{B}' &= \mathbf{D}\times\mathbf{B}. \end{cases}$$

Plot the curves traced out by the Darboux vectors of an elliptical helix, Viviani's curve , and a bicylinder parametrized in Exercise 6.

M **10.** A curve constructed from the Bessel functions $J_a, J_b, J_c$ is defined by

$$\mathsf{besselcurve}[a,b,c](t) = \big(J_a(t), J_b(t), J_c(t)\big)$$

Plot $\mathsf{besselcurve}[0,1,2]$.

M **11.** A generalization of the twisted cubic (defined on page 202) is the curve $\mathsf{twistedn}[n,a]\colon \mathbb{R} \to \mathbb{R}^n$ given by

$$\mathsf{twistedn}[n,a](t) = \big(t,\ t^2,\ \ldots\ ,\ t^n\big).$$

Compute the curvature of $\mathsf{twistedn}[n,a]$ for $1 \leqslant n \leqslant 6$.

M **12.** Plot the torus knot with $p = 3$ and $q = 2$ and graph its curvature and torsion. Plot a tubular surface surrounding it.

**13.** Find a parametrization for the tube about Viviani's curve illustrated in Figure 7.16.

---

9 

Jean Gaston Darboux (1842–1917). French mathematician, who is best known for his contributions to differential geometry. His four-volume work *Leçons sur la théorie general de surfaces* remains the bible of surface theory. In 1875 he provided new insight into the Riemann integral, first defining upper and lower sums and then defining a function to be integrable if the difference between the upper and lower sums tends to zero as the mesh size gets smaller.

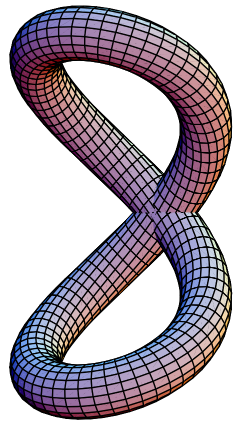


*Figure 7.16*: A tube of radius 1.5 around Viviani's curve

M **14.** The definition of a spherical spiral shown in Figure 7.17 is similar to that of a torus knot, namely

$$\text{sphericalspiral}[a][m,n](t) = a\Big(\cos(mt)\cos(nt),\ \sin(mt)\cos(nt),\ \sin(nt)\Big).$$

Find formulas for its curvature and torsion.

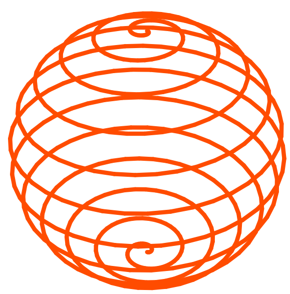


*Figure 7.17*: The spherical spiral with $m = 24$ and $n = 1$

M **15.** A ***figure eight knot*** can be parametrized by

$$\begin{aligned}\text{eightknot}(t) = \Big(&10(\cos t + \cos 3t) + \cos 2t + \cos 4t,\\ &6\sin t + 10\sin 3t,\ 4\sin 3t \sin\frac{5t}{2} + 4\sin 4t - 2\sin 6t\Big).\end{aligned}$$

Plot the tube about the figure eight knot.