Chapter 10

Surfaces in Euclidean Space

The 2-dimensional analog of a curve is a surface. However, surfaces in general are much more complicated than curves. In this introductory chapter, we give basic definitions that will be used throughout the rest of the book.

The intuitive idea of a surface is a 2-dimensional set of points. Globally, the surface may be rather complicated and not look at all like a plane, but any sufficiently small piece of the surface should look like a 'warped' portion of a plane. The most straightforward 2-dimensional generalization of a curve in $\mathbb{R}^n$ is a *patch* or local surface, which we define in Section 10.1. A given partial derivative of the patch function constitutes a column of the associated Jacobian matrix, which can then be used to characterize a *regular patch.*

Patches in $\mathbb{R}^3$ are discussed in Section 10.2, and the definition of an associated unit normal vector leads to a first encounter with the Gauss map of a surface, which is an analog of the mapping used in Chapters 1 and 6 to define the turning angle and turning number of a curve.

A second generalization of a curve in $\mathbb{R}^n$ is a *regular surface* in $\mathbb{R}^n$; this is a notion defined in Section 10.3 that adopts a more global perspective. A selection of surfaces in $\mathbb{R}^3$ is presented in Sections 10.4 and 10.6. As well as discussing familiar cases, we shall plot a few interesting but less well-known examples. In later chapters, we shall be showing how to compute associated geometric quantities, such as curvature, for all the surfaces that we have introduced.

In Section 10.5, we define a tangent vector to a regular surface, and a surface mapping; these are the surface analogs of a tangent vector to $\mathbb{R}^n$ and a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. As well as providing examples, Section 10.6 is devoted to the *nonparametric representation* of level surfaces as the zero sets of differentiable functions on $\mathbb{R}^3$. The Gauss map of such a surface is determined by the *gradient* of the defining function.

10.1 Patches in $\mathbb{R}^n$

Since a curve in $\mathbb{R}^n$ is a vector-valued function of one variable, it is reasonable to consider vector-valued functions of *two* variables. Such an object is called a patch. First, we give the precise definition.

Definition 10.1. *A* ***patch*** *or* ***local surface*** *is a differentiable mapping*

$$\mathbf{x}\colon \mathcal{U} \longrightarrow \mathbb{R}^n,$$

where $\mathcal{U}$ is an open subset of $\mathbb{R}^2$. More generally, if A is any subset of $\mathbb{R}^2$, we say that a map $\mathbf{x}\colon A \to \mathbb{R}^n$ is a patch provided that $\mathbf{x}$ can be extended to a differentiable mapping from $\mathcal{U}$ into $\mathbb{R}^n$, where $\mathcal{U}$ is an open set containing A. We call $\mathbf{x}(\mathcal{U})$ (or more generally $\mathbf{x}(A)$) the ***trace*** *of $\mathbf{x}$.*

Although the domain of definition of a patch can in theory be any set, more often than not it is an open or closed rectangle.

The standard parametrization of the ***sphere*** $S^2(a)$ of radius a is the mapping

$$\begin{gathered}\mathsf{sphere}[a]\colon [0,2\pi]\times[-\frac{\pi}{2},\frac{\pi}{2}] \longrightarrow \mathbb{R}^3\\ (u,v) \mapsto (a\cos v\cos u,\ a\cos v\sin u,\ a\sin v).\end{gathered} \tag{10.1}$$

This is chosen so that u measures longitude and v measures latitude. Although $\mathsf{sphere}[a]$ is defined on the closed rectangle $\mathcal{R} = [0,2\pi]\times[-\pi/2,\pi/2]$, it has a differentiable extension to an open set containing $\mathcal{R}$. Any such extension will multiply cover substantial parts of $S^2(a)$, while the image of $\mathcal{R}$ multiply covers only a semicircle from the north pole $(0,0,a)$ to the south pole $(0,0,-a)$. By restricting the parametrization to an open *subset* of $\mathcal{R}$, it is possible to visualize the inside of the sphere.

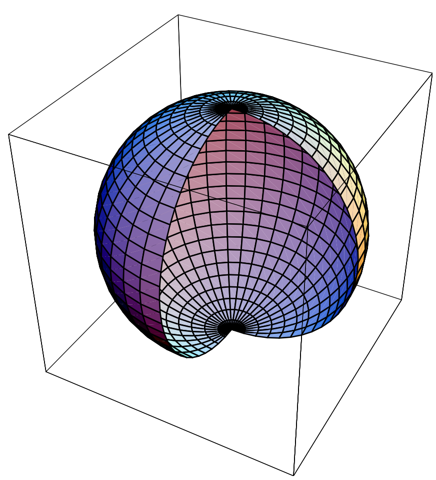

Figure 10.1: The image of $\mathsf{sphere}[1]$ for $0 \leqslant u \leqslant \dfrac{3\pi}{2}$

Since a patch can be written as an n-tuple of functions

$$\mathbf{x}(u,v) = \big(x_1(u,v), \ldots, x_n(u,v)\big), \tag{10.2}$$

we can define the partial derivative $\mathbf{x}_u$ of $\mathbf{x}$ with respect to u by

$$\mathbf{x}_u(u,v) = \left(\frac{\partial x_1}{\partial u}(u,v), \ldots, \frac{\partial x_n}{\partial u}(u,v)\right). \tag{10.3}$$

The other partial derivatives of $\mathbf{x}$, namely $\mathbf{x}_v, \mathbf{x}_{uu}, \mathbf{x}_{uv}, \ldots$, are defined similarly. Frequently, we abbreviate (10.2) and (10.3) to

$$\mathbf{x} = (x_1, \ldots, x_n) \qquad \text{and} \qquad \mathbf{x}_u = \left(\frac{\partial x_1}{\partial u}, \ldots, \frac{\partial x_n}{\partial u}\right).$$

The partial derivatives $\mathbf{x}_u$ and $\mathbf{x}_v$ can be expressed in terms of the tangent map of the patch $\mathbf{x}$.

Lemma 10.2. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ be a patch, and let $\mathbf{q} \in \mathcal{U}$. Then*

$$\mathbf{x}_*\big(\mathbf{e}_1(\mathbf{q})\big) = \mathbf{x}_u(\mathbf{q}) \qquad \textit{and} \qquad \mathbf{x}_*\big(\mathbf{e}_2(\mathbf{q})\big) = \mathbf{x}_v(\mathbf{q}),$$

where $\mathbf{x}_*$ denotes the tangent map of $\mathbf{x}$, and $\{\mathbf{e}_1, \mathbf{e}_2\}$ denotes the natural frame field of $\mathbb{R}^2$.

Proof. By Lemma 9.10, we have

$$\begin{aligned}\mathbf{x}_*\big(\mathbf{e}_1(\mathbf{q})\big) = \big(\mathbf{e}_1(\mathbf{q})[x_1], \ldots, \mathbf{e}_1(\mathbf{q})[x_n]\big)_{\mathbf{x}(\mathbf{q})} &= \left(\frac{\partial x_1}{\partial u}(\mathbf{q}), \ldots, \frac{\partial x_n}{\partial u}(\mathbf{q})\right)_{\mathbf{x}(\mathbf{q})} \\ &= \mathbf{x}_u(\mathbf{q}),\end{aligned}$$

and similarly for $\mathbf{x}_*\big(\mathbf{e}_2(\mathbf{q})\big)$. ∎

On page 271, we defined the Jacobian matrix of a differentiable map. We now specialize to the case of a patch.

Definition 10.3. *The **Jacobian matrix** of a patch $\mathbf{x}\colon\mathcal{U} \to \mathbb{R}^n$ is the matrix-valued function $\mathscr{J}(\mathbf{x})$ given by*

$$\mathscr{J}(\mathbf{x})(u,v) = \begin{pmatrix} \dfrac{\partial x_1}{\partial u}(u,v) & \dfrac{\partial x_1}{\partial v}(u,v) \\ \vdots & \vdots \\ \vdots & \vdots \\ \dfrac{\partial x_n}{\partial u}(u,v) & \dfrac{\partial x_n}{\partial v}(u,v) \end{pmatrix} = \begin{pmatrix} \mathbf{x}_u(u,v) \\ \mathbf{x}_v(u,v) \end{pmatrix}^T. \tag{10.4}$$

Observe that differentiation with respect to each of the two coordinates u, v corresponds to a *column* of (10.4), or a row of its transpose indicated with the superscript ^{T}. This is slightly inconvenient typographically, but consistent with the definition on page 271.

An equivalent definition of the ***rank*** of a matrix A is that it is the largest integer m such that a has an $m \times m$ submatrix whose determinant is nonzero. The following lemma is a consequence of well-known facts from linear algebra.

Lemma 10.4. *Let $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$. The following conditions are equivalent:*

(i) $\mathbf{p}$ *and* $\mathbf{q}$ *are linearly dependent;*

(ii) $\det\begin{pmatrix} \mathbf{p}\cdot\mathbf{p} & \mathbf{p}\cdot\mathbf{q} \\ \mathbf{q}\cdot\mathbf{p} & \mathbf{q}\cdot\mathbf{q} \end{pmatrix} = 0;$

(iii) *the $n \times 2$ matrix $\begin{pmatrix}\mathbf{p}, & \mathbf{q}\end{pmatrix}$ has rank less than 2.*

Proof. If $\mathbf{p}$ and $\mathbf{q}$ are linearly dependent, then either $\mathbf{p}$ is a multiple of $\mathbf{q}$, or vice versa. For example, if $\mathbf{p} = \lambda\mathbf{q}$, then

$$\det\begin{pmatrix} \mathbf{p}\cdot\mathbf{p} & \mathbf{p}\cdot\mathbf{q} \\ \mathbf{q}\cdot\mathbf{p} & \mathbf{q}\cdot\mathbf{q} \end{pmatrix} = \det\begin{pmatrix} \lambda\mathbf{q}\cdot\lambda\mathbf{q} & \lambda\mathbf{q}\cdot\mathbf{q} \\ \lambda\mathbf{q}\cdot\mathbf{q} & \mathbf{q}\cdot\mathbf{q} \end{pmatrix} = \lambda^2\det\begin{pmatrix} \mathbf{q}\cdot\mathbf{q} & \mathbf{q}\cdot\mathbf{q} \\ \mathbf{q}\cdot\mathbf{q} & \mathbf{q}\cdot\mathbf{q} \end{pmatrix} = 0.$$

Thus (i) implies (ii).

Next, suppose that (ii) holds. Write $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$. Then

$$\begin{aligned} 0 = \|\mathbf{p}\|^2\,\|\mathbf{q}\|^2 - (\mathbf{p}\cdot\mathbf{q})^2 &= \left(\sum_{i=1}^n p_i^2\right)\left(\sum_{j=1}^n q_j^2\right) - \left(\sum_{i=1}^n p_i\,q_i\right)^2 \\ &= \sum_{1\leqslant i<j\leqslant n} (p_iq_j - p_jq_i)^2\,. \end{aligned}$$

It follows that $p_iq_j = p_jq_i$ for all i and j. Hence (ii) implies (iii).

Finally, suppose (iii) holds. Then $p_iq_j = p_jq_i$ for all i and j. Without loss of generality, we can suppose that $q_i \neq 0$ for some i. Then $p_j = (p_i/q_i)q_j$ for all j, and

$$\mathbf{p} = (p_1, \ldots, p_n) = \frac{p_i}{q_i}(q_1, \ldots, q_n) = \frac{p_i}{q_i}\mathbf{q},$$

and $\mathbf{p}$ and $\mathbf{q}$ are linearly dependent. ∎

Corollary 10.5. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ be a patch. Then the following conditions are equivalent:*

(i) $\mathbf{x}_u(u_0, v_0)$ *and* $\mathbf{x}_v(u_0, v_0)$ *are linearly dependent;*

(ii) $\det\begin{pmatrix} \mathbf{x}_u \cdot \mathbf{x}_u & \mathbf{x}_u \cdot \mathbf{x}_v \\ \mathbf{x}_v \cdot \mathbf{x}_u & \mathbf{x}_v \cdot \mathbf{x}_v \end{pmatrix}$ *vanishes at* (u_0, v_0);

(iii) *the Jacobian matrix* $\mathscr{J}(\mathbf{x})$ *has rank less than* 2 *at* (u_0, v_0).

We shall need a stronger notion of patch in order that $\mathbf{x}(\mathcal{U})$ resemble the open set $\mathcal{U}$ more closely.

Definition 10.6. *A* ***regular patch*** *is a patch* $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ *for which* $\mathscr{J}(\mathbf{x})(u, v)$ *has rank* 2 *for all* $(u, v) \in \mathcal{U}$*. An* ***injective patch*** *is a patch such that* $\mathbf{x}(u_1, v_1) = \mathbf{x}(u_2, v_2)$ *implies that* $u_1 = u_2$ *and* $v_1 = v_2$.

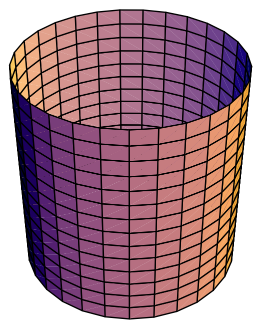

Figure 10.2: $(u, v) \mapsto (\cos u,\ \sin u,\ v)$

There are regular patches which are not injective. Consider, for example, the circular cylinder of Figure 10.2 defined by $\mathbf{x}(u, v) = (\cos u, \sin u, v)$, where $u \in \mathbb{R}$ and $-2 < v < 2$. There are also injective patches which are not regular. This is illustrated by the function $\mathbf{y}(u, v) = (u^3,\ v^3,\ uv)$ of Figure 10.3, with $-1 < u, v < 1$.

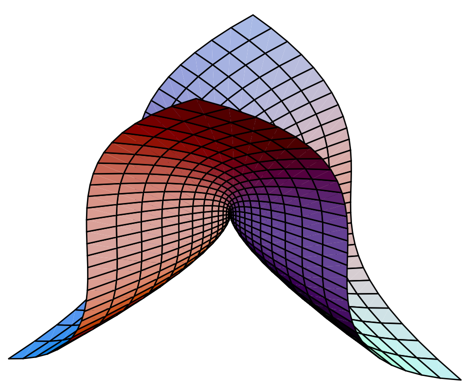

Figure 10.3: $(u, v) \mapsto (u^3,\ v^3,\ uv)$

The patch sphere$[a]$ fails to be regular when $v = \pm\pi/2$, that is, at the north and south poles. This is typical: often it is necessary to deal with patches that fail to be regular at just a few points. To handle such cases, we record the following obvious variant of Definition 10.6.

Definition 10.7. *A patch* $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ *is* ***regular*** *at a point* $(u_0, v_0) \in \mathcal{U}$ *(or sometimes we say at* $\mathbf{x}(u_0, v_0)$*) provided* $\mathscr{J}(\mathbf{x})(u_0, v_0)$ *has rank* 2.

As a consequence of Lemma 10.2 and Corollary 10.5 we have

Lemma 10.8. *A patch* $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ *is regular at* $\mathbf{q} \in \mathcal{U}$ *if and only if its tangent map* $\mathbf{x}_*\colon \mathbb{R}^2_{\mathbf{q}} \to \mathbb{R}^n_{\mathbf{x}(\mathbf{q})}$ *is injective.*

The following lemma is a consequence of the inverse function theorem (as given, for example, in [Buck, page 276]), and refers to Definition 9.7 on page 268.

Lemma 10.9. *Let* $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ *be a regular patch and let* $\mathbf{q} \in \mathcal{U}$*. There exists a neighborhood* $\mathcal{U}_{\mathbf{q}}$ *of* $\mathbf{q}$ *such that* $\mathbf{x}\colon \mathcal{U}_{\mathbf{q}} \to \mathbf{x}(\mathcal{U}_{\mathbf{q}})$ *is the restriction of a diffeomorphism between open sets of* $\mathbb{R}^n$*.*

Proof. Write $\mathbf{x} = (x_1, \ldots, x_n)$. Since $\mathbf{x}$ is regular, its Jacobian matrix has a 2×2 submatrix with nonzero determinant. By renaming $x_1, \ldots, x_n$ if necessary, we can suppose that

$$\det\begin{pmatrix} \dfrac{\partial x_1}{\partial u}(u,v) & \dfrac{\partial x_1}{\partial v}(u,v) \\[2ex] \dfrac{\partial x_2}{\partial u}(u,v) & \dfrac{\partial x_2}{\partial v}(u,v) \end{pmatrix} \tag{10.5}$$

is nonzero for $(u, v) \in \mathcal{U}$. We extend $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ to a map

$$\widetilde{\mathbf{x}}\colon \mathcal{U} \times \mathbb{R}^{n-2} \longrightarrow \mathbb{R}^n \tag{10.6}$$

by defining

$$\mathbf{x}(u, v, t_3, \ldots, t_n) \mapsto \big(x_1(u,v),\ x_2(u,v),\ x_3(u,v)+t_3, \ldots, x_n(u,v)+t_n\big).$$

It is clear that $\widetilde{\mathbf{x}}$ is differentiable; moreover, the determinant

$$\det\left(\mathscr{J}(\widetilde{\mathbf{x}})^T\right) = \det\begin{pmatrix} \dfrac{\partial x_1}{\partial u}(u,v) & \dfrac{\partial x_2}{\partial u}(u,v) & \dfrac{\partial x_3}{\partial u}(u,v) & \cdots & \dfrac{\partial x_n}{\partial u}(u,v) \\[2ex] \dfrac{\partial x_1}{\partial v}(u,v) & \dfrac{\partial x_2}{\partial v}(u,v) & \dfrac{\partial x_3}{\partial v}(u,v) & \cdots & \dfrac{\partial x_n}{\partial v}(u,v) \\[2ex] 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

equals (10.5). Because $\det\big(\mathscr{J}(\widetilde{\mathbf{x}})\big)(\mathbf{q}) \neq 0$, the inverse function theorem says that there is a neighborhood $\widetilde{\mathcal{U}}_{\mathbf{q}}$ of $(\mathbf{q},\mathbf{0})$ upon which $\widetilde{\mathbf{x}}$ has a differentiable inverse. It follows that $\widetilde{\mathbf{x}}\colon \widetilde{\mathcal{U}}_{\mathbf{q}} \to \widetilde{\mathbf{x}}(\widetilde{\mathcal{U}}_{\mathbf{q}})$ is a diffeomorphism, and we may take $\mathcal{U}_{\mathbf{q}} = \widetilde{\mathcal{U}}_{\mathbf{q}} \cap \mathcal{U}$ to complete the proof. ▮

The conclusion of Lemma 10.9 can be expressed by saying that $\mathbf{x}\colon \mathcal{U}_{\mathbf{q}} \to \mathbf{x}(\mathcal{U}_{\mathbf{q}})$ is itself a diffeomorphism, anticipating Definition 10.42 on page 309. We may also state

Corollary 10.10. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ be an injective regular patch. Then $\mathbf{x}$ maps $\mathcal{U}$ diffeomorphically onto $\mathbf{x}(\mathcal{U})$.*

Associated with any patch are some naturally-defined curves.

Definition 10.11. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ be a patch, and fix $(u_0, v_0) \in \mathcal{U}$. The curves*

$$u \mapsto \mathbf{x}(u, v_0) \qquad \textit{and} \qquad v \mapsto \mathbf{x}(u_0, v)$$

*are called u- and v- **parameter curves** or **coordinate curves**.*

These are the curves that are usually displayed by computer graphics.

There is also a convenient way to represent a general curve whose trace is contained in the trace of a patch.

Lemma 10.12. *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$ be a curve whose trace lies on the image $\mathbf{x}(\mathcal{U})$ of a regular patch $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ such that $\mathbf{x}\colon \mathcal{U} \to \mathbf{x}(\mathcal{U})$ is a homeomorphism. Then there exist unique differentiable functions $u, v\colon (a,b) \to \mathbb{R}$ such that*

$$\boldsymbol{\alpha}(t) = \mathbf{x}\big(u(t), v(t)\big) \tag{10.7}$$

for $a < t < b$.

Proof. For $a < t < b$ we can write

$$(\mathbf{x}^{-1} \circ \boldsymbol{\alpha})(t) = \big(u(t), v(t)\big);$$

this equation is equivalent to (10.7). It is clear that u and v are unique; by Lemma 10.9 they are also differentiable. ▮

Now we can define the notion of tangent vector to a patch.

Definition 10.13. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ be an injective patch, and let $\mathbf{p} \in \mathbf{x}(\mathcal{U})$. A **tangent vector** to $\mathbf{x}$ at $\mathbf{p}$ is a tangent vector $\mathbf{v_p} \in \mathbb{R}^n_{\mathbf{p}}$ for which there exists a curve $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$ that can be written as*

$$\boldsymbol{\alpha}(t) = \mathbf{x}\big(u(t), v(t)\big) \qquad (a < t < b), \tag{10.8}$$

such that $\boldsymbol{\alpha}(0) = \mathbf{p}$ and $\boldsymbol{\alpha}'(0) = \mathbf{v_p}$. We denote the set of tangent vectors to $\mathbf{x}$ at $\mathbf{p}$ by $\mathbf{x}(\mathcal{U})_{\mathbf{p}}$.

Lemma 10.14. *The set $\mathbf{x}(\mathcal{U})_{\mathbf{p}}$ of all tangent vectors to a patch $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ at a regular point $\mathbf{p} = \mathbf{x}(u_0, v_0) \in \mathbf{x}(\mathcal{U})$ forms a vector space that is spanned by $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$.*

Proof. By definition of tangent vector $\mathbf{v_p}$ to $\mathbf{x}$, there exists a curve $\boldsymbol{\alpha}$ of the form (10.8) such that $\boldsymbol{\alpha}(0) = \mathbf{p}$ and $\boldsymbol{\alpha}'(0) = \mathbf{v_p}$. The chain rule for curves (Lemma 1.9, page 7) implies that

$$\boldsymbol{\alpha}'(t) = u'(t)\mathbf{x}_u(u(t), v(t)) + v'(t)\mathbf{x}_v(u(t), v(t)).$$

In particular,

$$\mathbf{v_p} = \boldsymbol{\alpha}'(0) = u'(0)\mathbf{x}_u(u_0, v_0) + v'(0)\mathbf{x}_v(u_0, v_0).$$

Conversely, if $\mathbf{v_p} = c_1\mathbf{x}_u(u_0, v_0) + c_2\mathbf{x}_v(u_0, v_0)$, then $\mathbf{v_p}$ is the velocity vector at $\mathbf{x}(u_0, v_0) = \mathbf{p}$ of the curve $t \mapsto \mathbf{x}(u_0 + tc_1,\ v_0 + tc_2)$. ∎

The proof of Lemma 10.14 yields

Corollary 10.15. *Let $\boldsymbol{\alpha}\colon (a, b) \to \mathbb{R}^n$ be a curve whose trace lies on the image $\mathbf{x}(\mathcal{U})$ of a regular patch $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ such that $\mathbf{x}\colon \mathcal{U} \to \mathbf{x}(\mathcal{U})$ is a homeomorphism. Then there exist unique differentiable functions $u, v\colon (a, b) \to \mathbb{R}$ such that*

$$\boldsymbol{\alpha}' = u'\mathbf{x}_u + v'\mathbf{x}_v. \tag{10.9}$$

Definition 10.16. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ be an injective patch, and let $\mathbf{z_p} \in \mathbb{R}^n_{\mathbf{p}}$ with $\mathbf{p} \in \mathbf{x}(\mathcal{U})$. We say that $\mathbf{z_p}$ is **normal** or **perpendicular** to $\mathbf{x}$ at $\mathbf{p}$, provided $\mathbf{z_p} \cdot \mathbf{v_p} = 0$ for all vectors $\mathbf{v_p}$ tangent to $\mathbf{x}$ at $\mathbf{p}$.*

Let $\mathbf{x}(\mathcal{U})^{\perp}_{\mathbf{p}}$ denote the ***orthogonal complement*** of $\mathbf{x}(\mathcal{U})_{\mathbf{p}}$, namely the space of vectors in $\mathbb{R}^n_{\mathbf{p}}$ perpendicular to $\mathbf{x}(\mathcal{U})_{\mathbf{p}}$. It is easy to verify that there is a direct sum

$$\mathbb{R}^n_{\mathbf{p}} = \mathbf{x}(\mathcal{U})_{\mathbf{p}} \oplus \mathbf{x}(\mathcal{U})^{\perp}_{\mathbf{p}} \tag{10.10}$$

of vector spaces.

We shall also need vector fields on patches:

Definition 10.17. *A **vector field** $\mathbf{V}$ on a patch $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ is a function that assigns to each $\mathbf{q} \in \mathcal{U}$ a tangent vector $\mathbf{V}(\mathbf{q}) \in \mathbb{R}^n_{\mathbf{p}}$, where $\mathbf{p} = \mathbf{x}(\mathbf{q})$. We say that $\mathbf{V}$ is **tangent** to $\mathbf{x}$ if $\mathbf{V}(\mathbf{q}) \in \mathbf{x}(\mathcal{U})_{\mathbf{p}}$ for all $\mathbf{q} \in \mathcal{U}$. Similarly, a vector field $\mathbf{W}$ on $\mathbf{x}$ is **normal** or **perpendicular** to $\mathbf{x}$ if $\mathbf{W}(\mathbf{q}) \cdot \mathbf{v_p} = 0$ for all $\mathbf{v_p} \in \mathbf{x}(\mathcal{U})_{\mathbf{p}}$ and $\mathbf{q} \in \mathcal{U}$.*

10.2 Patches in $\mathbb{R}^3$ and the Local Gauss Map

We now restrict our attention to patches in $\mathbb{R}^3$ because they are the ones that are easiest to visualize. Computations are also simpler because there is a vector cross product on $\mathbb{R}^3$. Here is a useful criterion for the regularity of a patch:

Lemma 10.18. *A patch $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ is regular at $(u_0, v_0) \in \mathcal{U}$ if and only if $\mathbf{x}_u \times \mathbf{x}_v$ is nonzero at (u_0, v_0).*

Proof. This follows immediately from the equation

$$\mathbf{x}_u \times \mathbf{x}_v = \det \begin{pmatrix} \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_1}{\partial v} & \mathbf{i} \\ \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_2}{\partial v} & \mathbf{j} \\ \dfrac{\partial x_3}{\partial u} & \dfrac{\partial x_3}{\partial v} & \mathbf{k} \end{pmatrix},$$

confirming that $(\mathbf{x}_u \times \mathbf{x}_v)(u_0, v_0) = \mathbf{0}$ if and only if the rank of $\mathscr{J}(\mathbf{x})(u_0, v_0)$ is less than 2. ▮

The vector cross product gives rise to a convenient way of finding a vector satisfying Definition 10.16 when $n = 3$.

Lemma 10.19. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ be an injective regular patch. Then the vector field $(u, v) \mapsto \mathbf{x}_u \times \mathbf{x}_v$ is everywhere perpendicular to $\mathbf{x}(\mathcal{U})$.*

Proof. Certainly, $\mathbf{x}_u \times \mathbf{x}_v$ is perpendicular to both $\mathbf{x}_u$ and $\mathbf{x}_v$. Since any tangent vector $\mathbf{v_p}$ to $\mathbf{x}$ is a linear combination of $\mathbf{x}_u$ and $\mathbf{x}_v$ at $\mathbf{p}$, it follows from Lemma 10.14 that $(u, v) \mapsto \mathbf{x}_u \times \mathbf{x}_v$ is perpendicular to $\mathbf{x}(\mathcal{U})$. ▮

Now we can define a perpendicular vector field of unit length.

Definition 10.20. *For an injective patch $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ the **unit normal vector field** or **surface normal** $\mathbf{U}$ is defined by*

$$\mathbf{U}(u, v) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}(u, v) \tag{10.11}$$

at those points $(u, v) \in \mathcal{U}$ at which $\mathbf{x}_u \times \mathbf{x}_v$ does not vanish.

The notion of regularity of a patch has the following geometric interpretation.

Corollary 10.21. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ be an injective patch. Then $\mathbf{x}$ is regular if and only if the unit normal vector field $\mathbf{U}$ is everywhere well defined.*

The points for which $\mathbf{x}_u \times \mathbf{x}_v$ vanishes will be called ***singular*** in the next section.

One of the key elements necessary for the study of surfaces is the map that assigns to each point $\mathbf{p}$ on a surface $\mathcal{M} \subset \mathbb{R}^3$ the point on the unit sphere $S^2(1) \subset \mathbb{R}^3$ that is parallel to the unit normal $\mathbf{U}(\mathbf{p})$. It is called the Gauss[1] map. It is an analog of the mapping illustrated in Figure 6.2 on page 157, that associates to a point on a curve its unit tangent vector. It is not always possible to define this map at all points of a surface, for reasons that we take up in Chapter 11, but it *is* defined for any patch.

Definition 10.22. *Let* $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ *be an injective patch. Then the unit normal* $\mathbf{U}$ *to* $\mathbf{x}$ *viewed as a mapping from* $\mathcal{U}$ *to the unit sphere* $S^2(1) \subset \mathbb{R}^3$ *is called the* ***local Gauss map*** *of* $\mathbf{x}$.

Note that $\mathbf{U}$ is undefined at singular points of $\mathbf{x}$.

We shall illustrate this concept for a well-known surface, that will be discussed again later in this chapter. The ***hyperbolic paraboloid*** is the trace of the patch

$$\mathbf{x}(u,v) = (u,\ v,\ uv). \tag{10.12}$$

Figure 10.4 shows the image of the rectangle $[-1,1] \times [-1,1]$, superimposed with the corresponding portion of the unit sphere $S^2(1)$ determined by the unit normal $\mathbf{U}(u,v)$. The patch (10.12) is closely related to the more general one (10.16) given below, and the hyperbolic paraboloid is an example of a ruled surfaces (see page 434 in Chapter 16).

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Carl Friedrich Gauss (1777–1855). Gauss's *Disquisitiones Generales Circa Superficies Curvas* (published in 1828) revolutionized differential geometry as his *Disquisitiones Arithmeticae* had revolutionized number theory. In particular, his approach depended only on the intrinsic properties of surfaces, not on their embedding in Euclidean 3-space. Gauss' genius extended into nearly every branch of mathematics. While a student at Göttingen he made his first major original discovery – the constructibility of the 17-sided regular polygon, closely followed by the first of his four proofs of the fundamental theorem of algebra (his doctoral dissertation) and his calculation of the orbit of the newly discovered asteroid Ceres. Gauss's interest in geodesy led to the invention of the heliotrope and the development of least squares approximation, a technique he also used in his investigation of the distribution of prime numbers. Gauss' work on terrestrial magnetism led to his construction, in collaboration with Wilhelm Weber, of the first operating electric telegraph. Other applied discoveries of significance were those in optics, potential theory, and astronomy. Gauss spent his entire career in Brunswick, under the patronage of the local royalty; for many years he directed the observatory there. Perhaps due to his deeply ingrained conservatism or to his distaste for controversy, Gauss did not publish his development of the theory of non-Euclidean geometry nor his work anticipating Hamilton's investigation of quaternions. Although he worked for the most part in mathematical isolation, his results laid the basis for new departures in number theory and statistics, as well as in differential geometry.

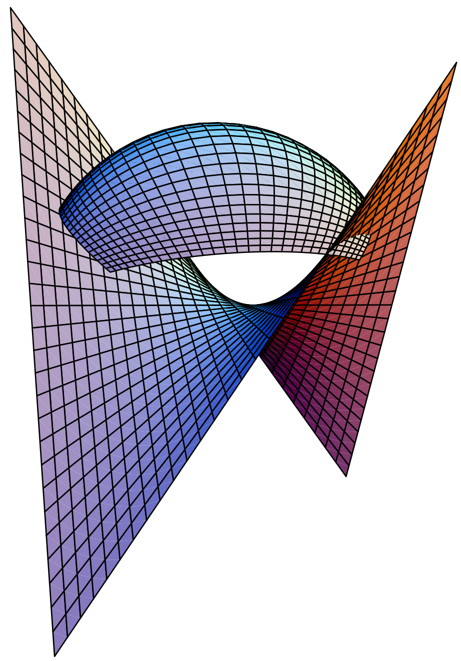


Figure 10.4: Gauss image of the hyperbolic paraboloid

10.3 The Definition of a Regular Surface

There are some subsets of $\mathbb{R}^n$ that we would like to call surfaces, but they do not fit into the framework of Section 10.1. Such a subset cannot be expressed as the image of a single regular one-to-one patch defined on an open set; spheres and tori are examples. In this section we define the notion of regular surface; roughly speaking, what we shall do is to combine several patches. The resulting definition is more complicated, but it is what is needed in many situations.

Definition 10.23. *A subset $\mathcal{M} \subset \mathbb{R}^n$ is a **regular surface**, provided that for each point $\mathbf{p} \in \mathcal{M}$ there exist a neighborhood $\mathcal{V}$ of $\mathbf{p}$ in $\mathbb{R}^n$ and a map $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ of an open set $\mathcal{U} \subset \mathbb{R}^2$ onto $\mathcal{V} \cap \mathcal{M}$ such that:*

(i) *$\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ is a regular patch;*

(ii) *$\mathbf{x}\colon \mathcal{U} \to \mathcal{V} \cap \mathcal{M}$ is a homeomorphism; thus $\mathbf{x}$ has a continuous inverse $\mathbf{x}^{-1}\colon \mathcal{V} \cap \mathcal{M} \to \mathcal{U}$ such that $\mathbf{x}^{-1}$ is the restriction to $\mathcal{V} \cap \mathcal{M}$ of a continuous mapping $F\colon \mathcal{W} \to \mathbb{R}^2$, where $\mathcal{W}$ is an open subset of $\mathbb{R}^n$ that contains $\mathcal{V} \cap \mathcal{M}$.*

*Each map $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ is called a **local chart** or **system of local coordinates** in a neighborhood of $\mathbf{p} \in \mathcal{M}$.*

We shall need the following fact.

Lemma 10.24. *Let $\mathcal{W}$ be an open subset of $\mathbb{R}^n$, and suppose that $G\colon \mathcal{W} \to \mathbb{R}^m$ is a map such that $G\colon \mathcal{W} \to G(\mathcal{W})$ is a diffeomorphism. If $\mathcal{M} \subseteq \mathcal{W}$ is a regular surface, then $G(\mathcal{M})$ is also a regular surface.*

Proof. If $\mathbf{x}$ is a patch on $\mathcal{M}$ satisfying (i) and (ii) in the definition of regular surface, then $G \circ \mathbf{x}\colon \mathcal{W} \to \mathbb{R}^m$ also satisfies (i) and (ii). ▮

There is an easy way to find new regular surfaces inside a given regular surface $\mathcal{M}$. A subset $\mathcal{V}$ of a regular surface $\mathcal{M} \subset \mathbb{R}^n$ is said to be ***open*** provided it is the intersection of an open subset of $\mathbb{R}^n$ with $\mathcal{M}$.

Lemma 10.25. *An open subset $\mathcal{W}$ of a regular surface $\mathcal{M}$ is also a regular surface.*

Proof. Let $\mathbf{x}$ be a patch on $\mathcal{M}$ satisfying (i) and (ii) in the definition of regular surface. Then $\mathbf{x}$ is continuous, and the inverse image of an open set is open, so $\mathcal{U} = \mathbf{x}^{-1}(\mathcal{W})$ is open. If $\mathcal{U}$ is nonempty, the restriction $\mathbf{x}|\mathcal{U}$ becomes a patch on $\mathcal{W}$. It is clear that $\mathbf{x}|\mathcal{U}$ satisfies (i) and (ii) in the definition of regular surface. Thus $\mathcal{W}$ becomes a regular surface. ▮

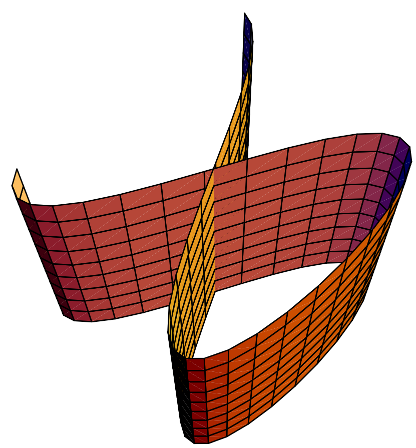


Figure 10.5: The patch $(u, v) \mapsto (\sin u, \sin 2u, v)$ with $-\dfrac{\pi}{3} \leqslant u \leqslant \dfrac{5\pi}{4}$

Let us clarify the relation between patches and regular surfaces. Certainly an arbitrary patch will fail to be a regular surface if its trace is 0- or 1-dimensional. Figure 10.5 shows that even a regular patch can fail to be a regular surface if it has self-intersections (which are precluded by condition (ii) in Definition 10.23). Nevertheless, let us prove that when we restrict the domain of definition of a regular patch we obtain a regular surface.

Lemma 10.26. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ be a regular patch. For any $\mathbf{q} \in \mathcal{U}$ there exists a neighborhood $\mathcal{U}_{\mathbf{q}}$ of $\mathbf{q}$ such that $\mathbf{x}(\mathcal{U}_{\mathbf{q}})$ is a regular surface.*

Proof. Lemma 10.9 states that $\mathbf{q}$ has a neighborhood $\mathcal{U}_{\mathbf{q}}$ such that $\mathbf{x}$ is a diffeomorphism between $\mathcal{U}_{\mathbf{q}}$ and $\mathbf{x}(\mathcal{U}_{\mathbf{q}})$. The neighborhood $\mathcal{U}_{\mathbf{q}}$ is an open set in the plane, and hence a regular surface by Lemma 10.25. Then Lemma 10.24 implies that $\mathbf{x}(\mathcal{U}_{\mathbf{q}})$ is a regular surface. ▮

Corollary 10.27. *Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ be an regular injective patch. Then $\mathbf{x}(\mathcal{U}_{\mathbf{q}})$ is a regular surface.*

Definition 10.28. *Let $\mathcal{M}$ be a regular surface in $\mathbb{R}^n$. If there exists a single regular injective patch $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ such that $\mathbf{x}(\mathcal{U}) = \mathcal{M}$, we say that $\mathcal{M}$ is* ***parametrized by*** $\mathbf{x}$.

This definition applies to the case of a Monge patch defined in Section 10.4 below. On the other hand, a sphere is an example of a regular surface which needs at least two patches to cover it. Figure 10.6 shows that the patches

$$\begin{aligned}(u,v) &\mapsto \big(\cos u \cos v,\ \sin u \cos v,\ \sin v\big)\\ (u,v) &\mapsto \big(\cos\big(\tfrac{3}{5}\pi - u\big)\cos v,\ \sin v,\ \sin\big(\tfrac{3}{5}\pi - u\big)\cos v\big),\end{aligned}$$

both defined for $0 < u < 7\pi/4$ and $-\pi/2 < v < \pi/2$, cover the sphere. For plotting purposes, the second patch is best regarded as the composition of the first with a rotation.

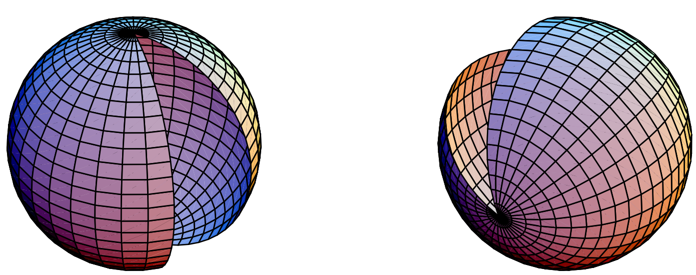


Figure 10.6: Two local charts covering the sphere

Next, we establish a fact about differentiable maps into a regular surface.

Theorem 10.29. *Let $\mathcal{M} \subset \mathbb{R}^n$ be a regular surface, and let $\mathcal{V} \subset \mathbb{R}^m$ be an open subset. Suppose that $F\colon \mathcal{V} \to \mathbb{R}^n$ is a differentiable mapping such that $F(\mathcal{V}) \subseteq \mathcal{M}$, and that $\mathbf{x}\colon\mathcal{U} \to \mathcal{M}$ is a regular patch on $\mathcal{M}$ such that $F(\mathcal{V}) \subseteq \mathbf{x}(\mathcal{U})$. Then $\mathbf{x}^{-1} \circ F\colon \mathcal{V} \to \mathcal{U}$ is differentiable.*

Proof. It is tempting to say that since $\mathbf{x}^{-1}$ and F are differentiable, so is $\mathbf{x}^{-1} \circ F$. Unfortunately, this is not valid, because we do not know at this point what it means for a function defined on a regular surface to be differentiable. Therefore, we are forced to prove the theorem in a more roundabout fashion, modifying the proof of Lemma 10.9.

Write $\mathbf{x} = (x_1, \ldots, x_n)$. Since $\mathbf{x}$ is regular, its Jacobian matrix has a 2×2 submatrix with nonzero determinant. By renaming $x_1, \ldots, x_n$ if necessary, we can suppose that (10.5) holds, and we define (10.6) in exactly the same way. Once again, $\widetilde{\mathbf{x}}$ is differentiable, and

$$\det\bigl(\mathscr{J}(\widetilde{\mathbf{x}})(u,v)\bigr) \neq 0.$$

Let $\mathbf{p} \in \mathcal{V}$; then $F(\mathbf{p}) \in \mathbf{x}(\mathcal{U}) \subseteq \mathcal{M}$. Because $\det\bigl(\mathscr{J}(\widetilde{\mathbf{x}})\bigr) \neq 0$ on $\mathcal{U}$, the inverse function theorem says that $\widetilde{\mathbf{x}}$ has an inverse $\widetilde{\mathbf{x}}^{-1}$ on a neighborhood $\widetilde{\mathcal{U}} \subseteq \mathbb{R}^n$ of $F(\mathbf{p})$, and that $\widetilde{\mathbf{x}}^{-1}$ is differentiable on $\widetilde{\mathcal{U}}$. Since F is continuous, there exists a neighborhood $\widetilde{\mathcal{V}}$ of $\mathbf{p}$ such that $F(\widetilde{\mathcal{V}}) \subset \widetilde{\mathcal{U}}$. Moreover,

$$\widetilde{\mathbf{x}}^{-1} \circ F \,|\, \widetilde{\mathcal{V}} = \mathbf{x}^{-1} \circ F \,|\, \widetilde{\mathcal{V}}.$$

Since $\widetilde{\mathbf{x}}^{-1} \circ F \,|\, \widetilde{\mathcal{V}}$ is differentiable at $\mathbf{p}$, so is $\mathbf{x}^{-1} \circ F \,|\, \widetilde{\mathcal{V}}$. Since $\mathbf{p} \in \mathcal{V}$ is arbitrary, it follows that $\mathbf{x}^{-1} \circ F$ is differentiable on all of $\mathcal{V}$. ▮

An important special case of Theorem 10.29 is:

Corollary 10.30. *Let $\mathcal{M}$ be a regular surface, and suppose we are given regular patches $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ and $\mathbf{y}\colon \mathcal{V} \to \mathcal{M}$ such that $\mathbf{x}(\mathcal{U}) \cap \mathbf{y}(\mathcal{V}) = \mathcal{W}$ is nonempty. Then the* ***change of coordinates***

$$\mathbf{x}^{-1} \circ \mathbf{y} : \ \mathbf{y}^{-1}(\mathcal{W}) \longrightarrow \mathbf{x}^{-1}(\mathcal{W}) \tag{10.13}$$

is a diffeomorphism between open subsets of $\mathbb{R}^2$.

Proof. That $\mathbf{x}^{-1} \circ \mathbf{y}$ and $\mathbf{y}^{-1} \circ \mathbf{x}$ are differentiable is a consequence of Theorem 10.29. Since both $\mathbf{x}$ and $\mathbf{y}$ are homeomorphisms, so are $\mathbf{x}^{-1} \circ \mathbf{y}$ and $\mathbf{y}^{-1} \circ \mathbf{x}$. Moreover, these maps are inverses of each other. Hence $\mathbf{x}^{-1} \circ \mathbf{y}$ and $\mathbf{y}^{-1} \circ \mathbf{x}$ are diffeomorphisms. ▮

We can write (10.13) more explicitly as follows: there exist differentiable functions $\bar{u}, \bar{v}$ such that

$$\mathbf{y}(u,v) = \mathbf{x}\bigl(\bar{u}(u,v),\ \bar{v}(u,v)\bigr).$$

Note that $(\bar{u}, \bar{v}) = (\mathbf{x}^{-1} \circ \mathbf{y})(u,v)$.

Lemma 10.31. *Let $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ and $\mathbf{y}\colon \mathcal{V} \to \mathcal{M}$ be patches on a regular surface $\mathcal{M}$ with $\mathbf{x}(\mathcal{U}) \cap \mathbf{y}(\mathcal{V})$ nonempty. Let $\mathbf{x}^{-1} \circ \mathbf{y} = (\bar{u}, \bar{v})\colon \mathcal{U} \cap \mathcal{V} \to \mathcal{U} \cap \mathcal{V}$ be the associated change of coordinates, so that*

$$\mathbf{y}(u,v) = \mathbf{x}\bigl(\bar{u}(u,v),\ \bar{v}(u,v)\bigr). \tag{10.14}$$

Then

$$\mathbf{y}_u = \frac{\partial \bar{u}}{\partial u}\mathbf{x}_{\bar{u}} + \frac{\partial \bar{v}}{\partial u}\mathbf{x}_{\bar{v}} \quad \textit{and} \quad \mathbf{y}_v = \frac{\partial \bar{u}}{\partial v}\mathbf{x}_{\bar{u}} + \frac{\partial \bar{v}}{\partial v}\mathbf{x}_{\bar{v}}. \tag{10.15}$$

Proof. (10.15) is an immediate consequence of (10.14) and the chain rule. ▮

Usually there are only a few points on a general surface which cannot be in the image of a regular patch. We refer to these as ***nonregular points*** or ***singular points*** of the surface. All other points of the surface are called ***regular points***. Note that by Corollary 10.30, the definition of regular point and singular point does not depend on the choice of patch. Furthermore, if we remove all of the nonregular points from a general surface, we obtain a surface.

We need to extend the calculus that we developed in Chapter 9 for $\mathbb{R}^n$ to regular surfaces. Our first task is to define what it means for a real-valued function on a regular surface to be differentiable.

Definition 10.32. *Let $f\colon \mathcal{W} \to \mathbb{R}$ be a function defined on an open subset $\mathcal{W}$ of a regular surface $\mathcal{M}$. We say that f is* ***differentiable*** *at $\mathbf{p} \in \mathcal{W}$ provided that for some patch $\mathbf{x}\colon \mathcal{U} \subset \mathbb{R}^2 \to \mathcal{M}$ with $\mathbf{p} \in \mathbf{x}(\mathcal{U}) \subset \mathcal{W}$, the composition*

$$f \circ \mathbf{x}\colon \mathcal{U} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$

is differentiable at $\mathbf{x}^{-1}(\mathbf{p})$. If f is differentiable at all points of $\mathcal{W}$, we say that f is ***differentiable*** *on $\mathcal{W}$.*

It is important to realize that the definition of differentiability of a real-valued function on a regular surface does not depend on the choice of patch. If $\mathbf{x}$ and $\mathbf{y}$ are patches on a regular surface $\mathcal{M}$, then $\mathbf{y}^{-1} \circ \mathbf{x}$ is differentiable by Corollary 10.30, and the composition of differentiable functions is differentiable.

Lemma 10.33. *Let $\mathcal{M}$ be a regular surface in $\mathbb{R}^n$. Then the restriction $f|\mathcal{M}$ of any differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ to $\mathcal{M}$ is differentiable.*

Proof. If $\mathbf{x}$ is any patch on $\mathcal{M}$, then $(f|\mathcal{M}) \circ \mathbf{x} = f \circ \mathbf{x}$ is differentiable, since it is the composition of differentiable functions. Thus by definition $f|\mathcal{M}$ is differentiable. ▮

As a simple consequence of Lemma 10.33, we see that the restrictions to any regular surface $\mathcal{M} \subset \mathbb{R}^n$ of the natural coordinate functions $u_1, \ldots, u_n$ of $\mathbb{R}^n$ are differentiable.

Here are two important examples of differentiable functions.

Definition 10.34. *Let $\mathcal{M}$ be a regular surface and let $\mathbf{v} \in \mathbb{R}^n$. The* ***height function*** *and the* ***square of the distance function*** *of $\mathcal{M}$ relative to $\mathbf{v}$ are the functions $h, f\colon \mathcal{M} \to \mathbb{R}$ defined by*

$$h(\mathbf{p}) = \mathbf{p} \cdot \mathbf{v} \qquad \textit{and} \qquad f(\mathbf{p}) = \|\mathbf{p} - \mathbf{v}\|^2.$$

Both the height and the square of the distance function are algebraic combinations of the natural coordinate functions of $\mathbb{R}^n$. Their differentiability therefore follows from Lemma 10.33.

We shall also need the notion of a curve on a regular surface.

Definition 10.35. *A* ***curve on a regular surface*** $\mathcal{M} \subset \mathbb{R}^n$ *is simply a curve* $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$ *such that* $\boldsymbol{\alpha}(t) \in \mathcal{M}$ *for* $a < t < b$. *The curve* $\boldsymbol{\alpha}$ *is said to be* ***differentiable*** *provided that* $\mathbf{x}^{-1} \circ \boldsymbol{\alpha}\colon (a,b) \to \mathcal{U}$ *is differentiable for any patch* $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$, *where* $\mathcal{U} \subseteq \mathbb{R}^2$.

10.4 Examples of Surfaces

An important class of surfaces in $\mathbb{R}^3$ consists of those that are graphs of a real-valued function of two variables.

Definition 10.36. *A* ***Monge***[2] ***patch*** *is a patch* $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ *of the form*

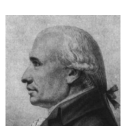
$$\mathbf{x}(u,v) = \big(u,\ v,\ h(u,v)\big),$$

where $\mathcal{U}$ *is an open set in* $\mathbb{R}^2$ *and* $h\colon \mathcal{U} \to \mathbb{R}$ *is a differentiable function.*

This is a regular patch, because the Jacobian matrix

$$\mathscr{J}(\mathbf{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{\partial h}{\partial u} & \dfrac{\partial h}{\partial v} \end{pmatrix}$$

obviously has rank 2. Its trace is the graph of h, whose nonparametric form is simply $z = h(x,y)$.

2  Gaspard Monge (1746–1818). French mathematician, erstwhile secretary of the navy, founding director of the École Polytechnique, which played a leading role in the development and organization of scientific research and education in France. Monge's work on fortifications led him to descriptive geometry, and from there he went on to a broad exposition of the differential geometry of space curves. His major contribution to the development of differential geometry was the integration of geometrical facts and intuition with the use of partial differential equations. A fervent supporter first of the Revolution and then of Napoleon, Monge was expelled from the Institut de France following the Battle of Waterloo. Many consider Monge to be the father of French differential geometry.

Paraboloids and Monkey Saddles

The concept of Monge patch is well illustrated by taking $h(u,v)$ proportional to au^2+bv^2. The trace of the parametrization

$$\text{paraboloid}[a,b](u,v)=\big(u,\ v,\ a\,u^2+b\,v^2\big), \tag{10.16}$$

or equivalently the graph of $z=ax^2+by^2$, is a ***paraboloid*** provided both a,b are nonzero. The paraboloid is ***circular*** if $a=b\neq 0$.

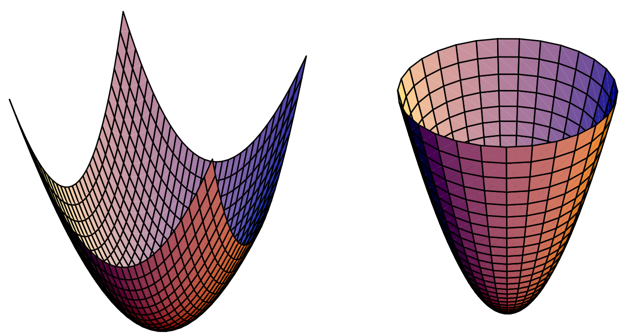


Figure 10.7: Subsets of the circular paraboloid paraboloid$[a,a]$

A clearer representation of a circular paraboloid is obtained by using polar coordinates (r,θ) instead of the rectangular coordinates (u,v). The relation between the two systems is of course

$$u=r\cos\theta,\qquad v=r\sin\theta,$$

and the polar parametric representation is merely

$$(r,\theta)\mapsto(r\cos\theta,\ r\sin\theta,\ ar^2).$$

Plotting this with a bound on the radius r provides the more symmetrical effect visible on the right of Figure 10.7.

Because of its shape, the hyperbolic paraboloid defined by taking a,b to have different signs in (10.16), is a type of a ***saddle surface***. A person can sit comfortably on a hyperbolic paraboloid because there are indentations for two legs. The ***monkey saddle***, which also takes account of a tail, is represented by the patch

$$\text{monkeysaddle}(u,v)=(u,\ v,\ u^3-3uv^2). \tag{10.17}$$

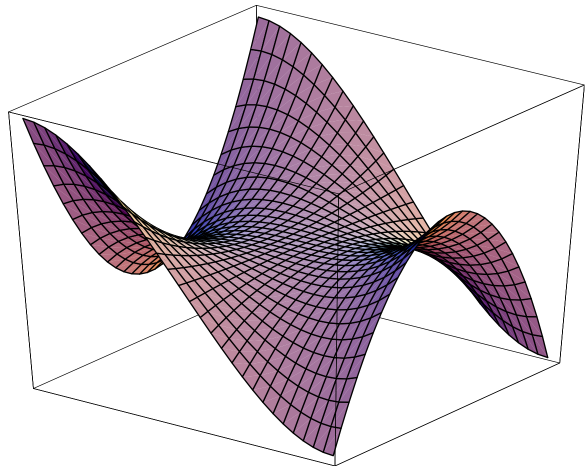


Figure 10.8: The monkey saddle

The plane, hyperbolic paraboloid ((10.16) with $a = 1$, $b = -1$) and monkey saddle are the first three members of a series of surfaces, namely the graphs of the real part of the function $z \mapsto z^n$, where z is a complex number and n a positive integer. Explicitly, we set

$$\mathsf{monkey}[n](u, v) = \big(u,\ v,\ \mathfrak{Re}\left[(u + iv)^n\right]\big),$$

so that $\mathsf{monkey}[2] = \mathsf{paraboloid}[1, -1]$. The imaginary part of z^n gives rise to a similar surface, namely the image of $\mathsf{monkey}[n]$ under a rotation by $\pi/(2n)$ radians.

The surface defined when $n = 5$ is illustrated on the left in Figure 10.9. On the right is the corresponding contour plot, in which the lighter the shading, the greater the value of the function $\mathsf{monkey}[5]$.

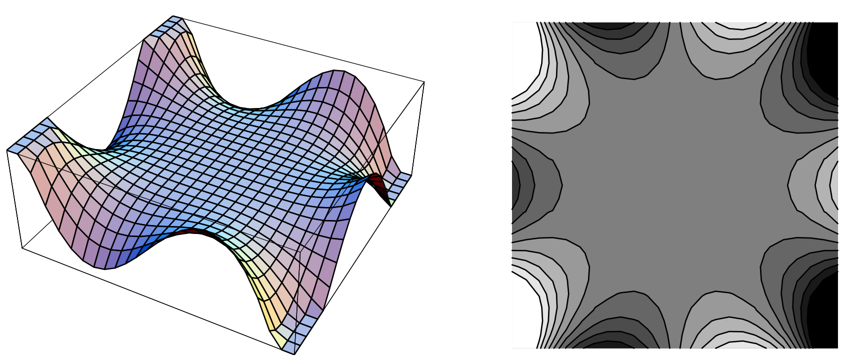


Figure 10.9: The $\mathsf{monkey}[5]$ surface

Elliptical Tori

The patch

$$\text{torus}[a,b,c]\colon\ [0,2\pi)\times[0,2\pi)\longrightarrow\mathbb{R}^3 \tag{10.18}$$
$$(u,v)\mapsto\big((a+b\cos v)\cos u,\ (a+b\cos v)\sin u,\ c\sin v\big)$$

is a parametrization of an ***elliptical torus*** in $\mathbb{R}^3$. Examples were encountered in Section 7.6.

When $b=c<a$, we get a standard round torus, where a is the ***wheel radius*** and b the ***tube radius*** of the torus. A torus with $a>b$ and $a>c$ is called a ***ring torus***, and is a regular surface in the sense of Section 10.3. The domain in (10.18) can be extended to any subset of $\mathbb{R}^2$, and including the rectangles

$$(0,\tfrac{3\pi}{2})\times(0,\tfrac{3\pi}{2}) \qquad (\pi,\tfrac{5\pi}{2})\times(0,\tfrac{3\pi}{2})$$
$$(0,\tfrac{3\pi}{2})\times(\pi,\tfrac{5\pi}{2}) \qquad (\pi,\tfrac{5\pi}{2})\times(\pi,\tfrac{5\pi}{2}).$$

Their images are displayed in the same order in Figure 10.10 for an elliptical torus with $a=8$, $b=3$ and $c=7$.

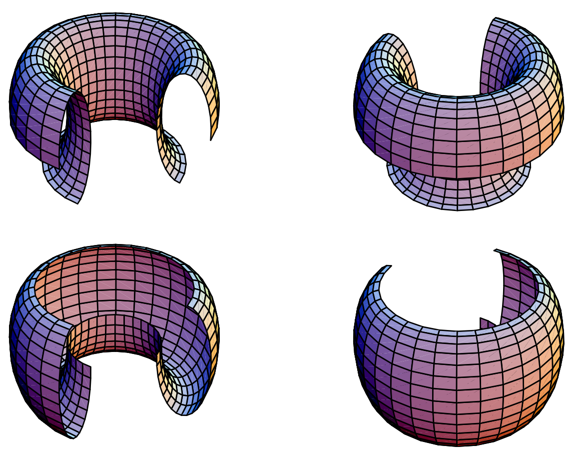


Figure 10.10: Four local charts covering torus$[8,3,7]$

If the wheel radius is less than the tube radius, the resulting torus becomes a self-intersecting surface and appears ‘inside-out’. It is usually called a ***horn torus***, though the intermediate case when the wheel radius equals the tube radius is called a ***spindle torus***. Figure 10.11 shows both a horn and spindle torus. The number of self-intersections is 0 for a ring torus, 1 for a spindle torus and 2 for a horn torus. These facts are checked analytically in Notebook 10, using the determinant of the matrix in Exercise 1. See also [Fischer, page 28].

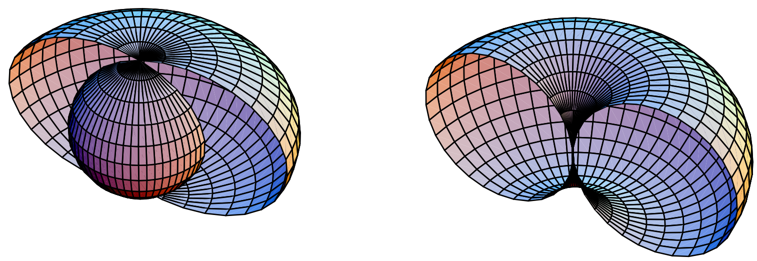


Figure 10.11: torus$[3, 8, 8]$ and torus$[4, 4, 4]$

Patches with Singularities

From the discussion in the previous section, we know that a surface fails to be regular at points of self-intersection, though the *patch* may or may not be singular on a 1-dimensional array of such points. Figure 10.11 shows isolated points of self-intersection where the patch is indeed singular, and we now give two more examples of this phenomenon.

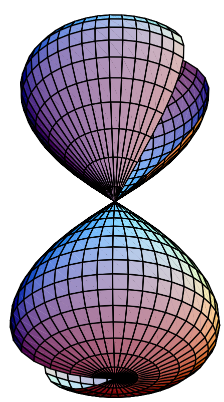


Figure 10.12: The eight surface

The ***eight surface*** consists of a figure eight revolved about the z-axis, and is defined as the trace of the patch

$$\text{eightsurface}(u, v) = \big(\cos u \cos v \sin v,\ \sin u \cos v \sin v,\ \sin v\big), \tag{10.19}$$

The eight surface becomes a regular surface only when the central 'vertex' is excluded. In spite of the existence of a singular point, there is no difficulty in plotting the surface.

An example of a surface with a ***pinch point*** is the ***Whitney umbrella***, defined by

$$\text{whitneyumbrella}(u,v) = (uv,\ u,\ v^2).$$

The Jacobian matrix (10.4) of this patch has rank 2 unless $(u,v) = (0,0)$, though whitneyumbrella maps $(0, \pm v)$ to the same point for all $v \neq 0$.

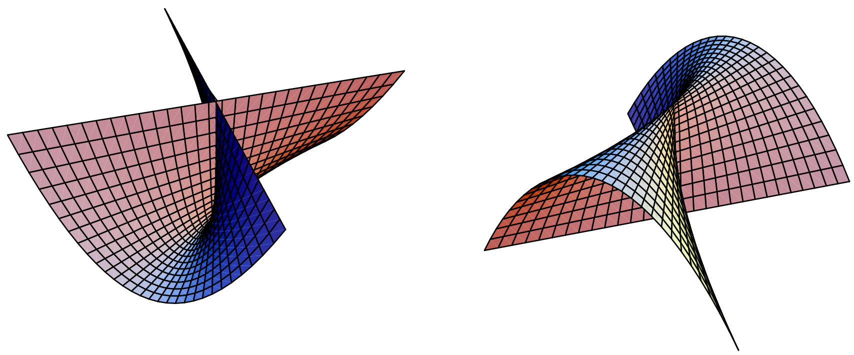


Figure 10.13: Two views of the Whitney umbrella

For a discussion of this surface see [Francis, pages 8–9].

10.5 Tangent Vectors and Surface Mappings

We are now ready to discuss the important notion of tangent vector to a regular surface. This is the next step in the extension to regular surfaces of the calculus of $\mathbb{R}^n$ that we developed in Chapter 9.

*Definition 10.37. Let $\mathcal{M}$ be a regular surface in $\mathbb{R}^n$ and let $\mathbf{p} \in \mathcal{M}$. We say that $\mathbf{v_p} \in \mathbb{R}^n_{\mathbf{p}}$ is **tangent to** $\mathcal{M}$ at $\mathbf{p}$ provided there exists a curve $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$ such that $\boldsymbol{\alpha}(0) = \mathbf{p}$, $\boldsymbol{\alpha}'(0) = \mathbf{v_p}$ and $\boldsymbol{\alpha}(t) \in \mathcal{M}$ for $a < t < b$. The **tangent space** to $\mathcal{M}$ at $\mathbf{p}$ is the set*

$$\mathcal{M}_{\mathbf{p}} = \left\{ \mathbf{v_p} \in \mathbb{R}^n_{\mathbf{p}} \mid \mathbf{v_p} \textit{ is tangent to } \mathcal{M} \textit{ at } \mathbf{p} \right\}.$$

On page 289, we defined $\mathbf{x}_u(u,v)$ and $\mathbf{x}_v(u,v)$ to be vectors in $\mathbb{R}^n$. Sometimes it is useful to modify this definition so that $\mathbf{x}_u(u,v)$ and $\mathbf{x}_v(u,v)$ are tangent vectors at $\mathbf{x}(u,v)$. Thus if $\mathbf{x}(u,v) = \big(x_1(u,v), \ldots, x_n(u,v)\big)$ we can redefine

$$\begin{aligned}
\mathbf{x}_u(u,v) &= \left(\frac{\partial x_1}{\partial u}(u,v), \ldots, \frac{\partial x_n}{\partial u}(u,v) \right)_{\mathbf{x}(u,v)}, \\
\mathbf{x}_v(u,v) &= \left(\frac{\partial x_1}{\partial v}(u,v), \ldots, \frac{\partial x_n}{\partial v}(u,v) \right)_{\mathbf{x}(u,v)}.
\end{aligned}$$

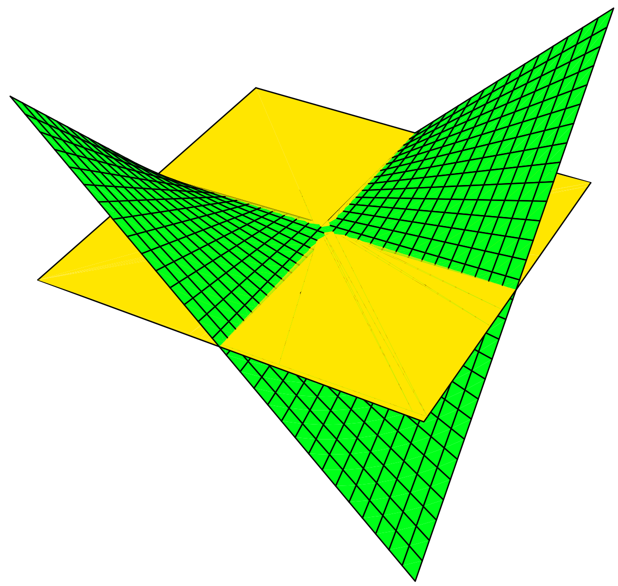


Figure 10.14: Tangent plane at the saddle point of a hyperbolic paraboloid

Similar remarks hold for $\mathbf{x}_{uu}$ and other higher partial derivatives of $\mathbf{x}$. However, such fine distinctions are unnecessary in practice, since when we calculate the partial derivatives of a patch explicitly (either by hand or computer) we consider them to be vectors in $\mathbb{R}^n$. In summary, we regard $\mathbf{x}_u$ and the other partial derivatives of $\mathbf{x}$ to be vectors in $\mathbb{R}^n$ unless for some theoretical reason we require them to be tangent vectors at the point of application.

Lemma 10.38. *Let $\mathbf{p}$ be a point on a regular surface $\mathcal{M} \subset \mathbb{R}^n$. Then the tangent space $\mathcal{M}_{\mathbf{p}}$ to $\mathcal{M}$ at $\mathbf{p}$ is a 2-dimensional vector subspace of $\mathbb{R}^n_{\mathbf{p}}$. If $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ is any regular patch on $\mathcal{M}$ with $\mathbf{p} = \mathbf{x}(\mathbf{q})$, then*

$$\mathbf{x}_*(\mathbb{R}^2_{\mathbf{q}}) = \mathcal{M}_{\mathbf{p}}.$$

Proof. It follows from Lemma 10.2 that $\mathbf{x}_*(\mathbb{R}^2_{\mathbf{q}})$ is spanned by $\mathbf{x}_u(\mathbf{q})$ and $\mathbf{x}_v(\mathbf{q})$. The regularity of $\mathbf{x}$ at $\mathbf{q}$ implies that $\mathbf{x}_*\colon \mathbb{R}^2_{\mathbf{q}} \to \mathbb{R}^n_{\mathbf{p}}$ is injective; consequently, $\mathbf{x}_u(\mathbf{q})$ and $\mathbf{x}_v(\mathbf{q})$ are linearly independent and $\dim\bigl(\mathbf{x}_*(\mathbb{R}^2_{\mathbf{q}})\bigr) = 2$ by Lemma 10.8. On the other hand, Lemma 10.14 implies that any tangent vector to $\mathcal{M}$ at $\mathbf{q}$ is a linear combination of $\mathbf{x}_u(\mathbf{q})$ and $\mathbf{x}_v(\mathbf{q})$. ▮

Frequently, we shall need the notion of tangent vector perpendicular to a surface, in complete analogy to Definition 10.16.

Definition 10.39. *Let $\mathcal{M}$ be a regular surface in $\mathbb{R}^n$ and let $\mathbf{z_p} \in \mathbb{R}^n_{\mathbf{p}}$ with $\mathbf{p} \in \mathcal{M}$. We say that $\mathbf{z_p}$ is **normal** or **perpendicular** to $\mathcal{M}$ at $\mathbf{p}$, provided $\mathbf{z_p} \cdot \mathbf{v_p} = 0$ for all tangent vectors $\mathbf{v_p} \in \mathcal{M}_{\mathbf{p}}$.*

We denote the set of normal vectors to $\mathcal{M}$ at $\mathbf{p}$ by $\mathcal{M}^{\perp}_{\mathbf{p}}$, so that

$$\mathbb{R}^n_{\mathbf{p}} = \mathcal{M}_{\mathbf{p}} \oplus \mathcal{M}^{\perp}_{\mathbf{p}},$$

just as in (10.10).

The notions of tangent and normal vector fields also make sense.

Definition 10.40. *A **vector field** $\mathbf{V}$ on a regular surface $\mathcal{M}$ is a function which assigns to each $\mathbf{p} \in \mathcal{M}$ a tangent vector $\mathbf{V}(\mathbf{p}) \in \mathbb{R}^n_{\mathbf{p}}$. We say that $\mathbf{V}$ is **tangent** to $\mathcal{M}$ if $\mathbf{V}(\mathbf{p}) \in \mathcal{M}_{\mathbf{p}}$ for all $\mathbf{p} \in \mathcal{M}$ and that $\mathbf{V}$ is **normal** or **perpendicular** to $\mathcal{M}$ if $\mathbf{V}(\mathbf{p}) \in \mathcal{M}^{\perp}_{\mathbf{p}}$ for all $\mathbf{p} \in \mathcal{M}$.*

The definition of differentiability of a mapping between regular surfaces is similar to that of a real-valued function on a regular surface.

Definition 10.41. *A function $F \colon \mathcal{M} \to \mathcal{N}$ from one regular surface to another is **differentiable**, provided that, for any two regular injective patches $\mathbf{x}$ of $\mathcal{M}$ and $\mathbf{y}$ of $\mathcal{N}$, the composition $\mathbf{y}^{-1} \circ F \circ \mathbf{x}$ is differentiable. When this is the case, we call F a **surface mapping**.*

The simplest example of a surface mapping is the identity map $\mathbf{1}_{\mathcal{M}} \colon \mathcal{M} \to \mathcal{M}$, defined by $\mathbf{1}_{\mathcal{M}}(\mathbf{p}) = \mathbf{p}$ for $\mathbf{p} \in \mathcal{M}$.

Definition 10.42. *A **diffeomorphism** between regular surfaces $\mathcal{M}$ and $\mathcal{N}$ is a differentiable map $F \colon \mathcal{M} \to \mathcal{N}$ which has a differentiable inverse, that is, a surface mapping $G \colon \mathcal{N} \to \mathcal{M}$ such that*

$$G \circ F = \mathbf{1}_{\mathcal{M}} \qquad \textit{and} \qquad F \circ G = \mathbf{1}_{\mathcal{N}},$$

where $\mathbf{1}_{\mathcal{M}}$ and $\mathbf{1}_{\mathcal{N}}$ denote the identity maps of $\mathcal{M}$ and $\mathcal{N}$.

Definition 10.43. *Let $\mathcal{M}$, $\mathcal{N}$ be regular surfaces in $\mathbb{R}^n$, and $\mathcal{W}$ an open subset of $\mathcal{M}$. We shall say that a mapping*

$$F \colon \mathcal{W} \longrightarrow \mathcal{N}$$

*is a **local diffeomorphism** at $\mathbf{p} \in \mathcal{W}$, if there exists a neighborhood $\mathcal{W}' \subset \mathcal{W}$ of $\mathbf{p}$ such that the restriction of F to $\mathcal{W}'$ is a diffeomorphism of $\mathcal{W}'$ onto an open subset $F(\mathcal{W}') \subseteq \mathcal{N}$.*

Just as a regular surface has a tangent *space*, a surface mapping has a tangent *map*.

Definition 10.44. *Let $\mathcal{M}$, $\mathcal{N}$ be regular surfaces in $\mathbb{R}^n$, and let $\mathbf{p} \in \mathcal{M}$. Let $F\colon \mathcal{M} \to \mathcal{N}$ be a surface mapping. Then the* ***tangent map*** *of F at $\mathbf{p}$ is the map*

$$F_*\colon \mathcal{M}_{\mathbf{p}} \longrightarrow \mathcal{N}_{F(\mathbf{p})}$$

given as follows. For $\mathbf{v_p} \in \mathcal{M}_{\mathbf{p}}$, choose a curve $\boldsymbol{\alpha}\colon (a,b) \to \mathcal{M}$ such that $\boldsymbol{\alpha}'(0) = \mathbf{v_p}$. Then we define $F_*(\mathbf{v_p})$ to be the initial velocity of the image curve $F \circ \boldsymbol{\alpha}\colon \to \mathcal{N}$; that is,

$$F_*(\mathbf{v_p}) = (F \circ \boldsymbol{\alpha})'(0).$$

Lemma 10.45. *Let $F\colon \mathcal{M} \to \mathcal{N}$ be a surface mapping, and let $\mathbf{v_p} \in \mathcal{M}_{\mathbf{p}}$. The definition of $F_*(\mathbf{v_p})$ is independent of the choice of curve $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}'(0) = \mathbf{v_p}$. Furthermore, $F_*\colon \mathcal{M}_{\mathbf{p}} \to \mathcal{N}_{F(\mathbf{p})}$ is a linear map.*

Proof. Let $\boldsymbol{\alpha}$ and $\widetilde{\boldsymbol{\alpha}}$ be curves in $\mathcal{M}$ with $\boldsymbol{\alpha}(0) = \widetilde{\boldsymbol{\alpha}}(0) = \mathbf{p}$ and $\boldsymbol{\alpha}'(0) = \widetilde{\boldsymbol{\alpha}}'(0) = \mathbf{v_p}$. Let $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ be a regular patch on $\mathcal{M}$ such that $\mathbf{x}(u_0, v_0) = \mathbf{p}$. Write

$$\boldsymbol{\alpha}(t) = \mathbf{x}\big(u(t), v(t)\big) \qquad \text{and} \qquad \widetilde{\boldsymbol{\alpha}}(t) = \mathbf{x}\big(\widetilde{u}(t), \widetilde{v}(t)\big).$$

Just as in the proof of Lemma 10.14, the chain rule for curves implies that

$$(F \circ \boldsymbol{\alpha})'(t) = u'(t)\big(F \circ \mathbf{x})_u(u(t), v(t)\big) + v'(t)\big(F \circ \mathbf{x})_v(u(t), v(t)\big)$$

and

$$(F \circ \widetilde{\boldsymbol{\alpha}})'(t) = \widetilde{u}'(t)(F \circ \mathbf{x})_u\big(\widetilde{u}(t), \widetilde{v}(t)\big) + \widetilde{v}'(t)(F \circ \mathbf{x})_v\big(\widetilde{u}(t), \widetilde{v}(t)\big).$$

Since $u(0) = \widetilde{u}(0) = u_0$, $v(0) = \widetilde{v}(0) = v_0$, $u'(0) = \widetilde{u}'(0)$ and $v'(0) = \widetilde{v}'(0)$, it follows that

$$F_*(\mathbf{v_p}) = (F \circ \boldsymbol{\alpha})'(0) = (F \circ \widetilde{\boldsymbol{\alpha}})'(0).$$

This proves that $F_*(\mathbf{v_p})$ does not depend on the choice of $\boldsymbol{\alpha}$. To prove the linearity of F_*, we note that

$$F_*(\mathbf{v_p}) = u'(0)(F \circ \mathbf{x})_u\big(u(0), v(0)\big) + v'(0)(F \circ \mathbf{x})_v\big(u(0), v(0)\big).$$

Since $u'(0), v'(0)$ depend linearly on $\mathbf{v_p}$, it follows that F_* must be linear. ∎

We have the following consequence of the inverse function theorem for $\mathbb{R}^2$.

Theorem 10.46. *If $\mathcal{M}$, $\mathcal{N}$ are regular surfaces and $F\colon \mathcal{W} \to \mathcal{N}$ is a differentiable mapping of an open subset $\mathcal{W} \subseteq \mathcal{M}$ such that the tangent map F_* of F at $\mathbf{p} \in \mathcal{W}$ is an isomorphism, then F is a local diffeomorphism at $\mathbf{p}$.*

Proof. Let $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ and $\mathbf{y}\colon \mathcal{V} \to \mathcal{N}$ be injective regular patches on $\mathcal{M}$ and $\mathcal{N}$ such that $(F \circ \mathbf{x})(\mathcal{U}) \cap \mathbf{y}(\mathcal{V})$ is nonempty. By restricting the domains of definition of $\mathbf{x}$ and $\mathbf{y}$ if necessary, we can assume that $(F \circ \mathbf{x})(\mathcal{U}) = \mathbf{y}(\mathcal{V})$. Then the tangent map of $\mathbf{y}^{-1} \circ F \circ \mathbf{x}$ is an isomorphism, and the inverse function theorem for $\mathbb{R}^2$ implies that the map $\mathbf{y}^{-1} \circ F \circ \mathbf{x}$ possesses a local inverse. Hence F also has a local inverse. ∎

Examples of Surface Mappings

1. Let $\mathbf{y}: \mathcal{U} \to \mathcal{M}$ be a regular patch on a regular surface $\mathcal{M}$. Theorem 10.29 implies that $\mathbf{y}$ is a diffeomorphism between the regular surfaces $\mathcal{U}$ and $\mathbf{y}(\mathcal{U})$.

2. Let $\mathbf{y}: \mathcal{U} \to \mathcal{M}$ be a regular patch on a regular surface $\mathcal{M}$. Then the local Gauss map $\mathbf{U} \circ \mathbf{y}^{-1}$ is a surface mapping from $\mathbf{y}(\mathcal{U})$ to the unit sphere $S^2(1)$ (see Definition 10.22 and Exercise 5).

3. Let $S^2(a) = \{ \mathbf{p} \mid \|\mathbf{p}\| = a\}$ be the sphere of radius a in $\mathbb{R}^3$. Then the ***antipodal map*** is the function $S^2(a) \to S^2(a)$ defined by $\mathbf{p} \mapsto -\mathbf{p}$. It is a diffeomorphism.

4. Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism, and let $\mathcal{M} \subset \mathbb{R}^n$ be a regular surface. Then the restriction $F|\mathcal{M}\colon \mathcal{M} \to F(\mathcal{M})$ is a diffeomorphism of regular surfaces. In particular, any Euclidean motion of $\mathbb{R}^n$ gives rise to a surface mapping.

10.6 Level Surfaces in $\mathbb{R}^3$

So far we have been considering regular surfaces defined by patches. Such a description is called a ***parametric representation***. Another way to describe a regular surface $\mathcal{M}$ is by means of a ***nonparametric representation***. For a regular surface in $\mathbb{R}^3$, this means that $\mathcal{M}$ is the set of points mapped by a differentiable function $g\colon \mathbb{R}^3 \to \mathbb{R}$ into the same real number.

Definition 10.47. *Let $g\colon \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function and c a real number. Then the set*

$$\mathcal{M}(c) = \left\{ \mathbf{p} \in \mathbb{R}^3 \mid g(\mathbf{p}) = c \right\}$$

is called the ***level surface*** *of g corresponding to c.*

Theorem 10.48. *Let $g\colon \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function and c a real number. Then the level surface $\mathcal{M}(c)$ of g is a regular surface if it is nonempty and the gradient* $\operatorname{grad} g$ *is nonzero at all points of $\mathcal{M}(c)$. When these conditions are satisfied,* $\operatorname{grad} g$ *is everywhere perpendicular to $\mathcal{M}(c)$.*

Proof. For each $\mathbf{p} \in \mathcal{M}(c)$, we must find a regular patch on a neighborhood of $\mathbf{p}$. The hypothesis that $(\operatorname{grad} g)(\mathbf{p}) \neq 0$ is equivalent to saying that at least one of the partial derivatives $\partial g/\partial x$, $\partial g/\partial y$, $\partial g/\partial z$ does not vanish at $\mathbf{p}$. Let us suppose that $\big(\partial g/\partial z\big)(\mathbf{p}\big) \neq 0$. The implicit function theorem states that the equation $g(x,y,z) = c$ can be solved for z. More precisely, there exists a function h such that

$$g\big(x,\, y,\, h(x,y)\big) = c.$$

Then the required patch is defined by $\mathbf{x}(u,v) = \big(u,\ v,\ h(u,v)\big)$.

Furthermore, let $\mathbf{v_p} = (v_1, v_2, v_3)_{\mathbf{p}} \in \mathcal{M}(c)_{\mathbf{p}}$. Then there exists a curve $\boldsymbol{\alpha}$ on $\mathcal{M}(c)$ with $\boldsymbol{\alpha}(0) = \mathbf{p}$ and $\boldsymbol{\alpha}'(0) = \mathbf{v_p}$. Write $\boldsymbol{\alpha}(t) = (a_1(t), a_2(t), a_3(t))$. Since $\boldsymbol{\alpha}$ lies on $\mathcal{M}(c)$, we have $g(a_1(t), a_2(t), a_3(t)) = c$ for all t. The chain rule implies that

$$\sum_{i=1}^{3}\left(\frac{\partial g}{\partial u_i}\circ\boldsymbol{\alpha}\right)\frac{da_i}{dt} = 0.$$

In particular,

$$0 = \sum_{i=1}^{3}\frac{\partial g}{\partial u_i}\big(\boldsymbol{\alpha}(0)\big)\frac{da_i}{dt}(0) = \sum_{i=1}^{3}\frac{\partial g}{\partial u_i}(\mathbf{p})v_i = (\mathsf{grad}\, g)(\mathbf{p})\cdot\mathbf{v_p}.$$

Hence $(\mathsf{grad}\, g)(\mathbf{p})$ is perpendicular to $\mathcal{M}(c)_{\mathbf{p}}$ for each $\mathbf{p} \in \mathcal{M}(c)$. ▌

To conclude this final section, we return to some examples.

Ellipsoids

The nonparametric equation that defines ***the ellipsoid*** is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{10.20}$$

Here, a, b and c are the lengths of the semi-axes of the ellipsoid. Ellipsoids with only two of a, b and c distinct are considerably simpler than general ellipsoids. Such an ellipsoid is called an ***ellipsoid of revolution***, and can be obtained by rotating an ellipse about one of its axes.

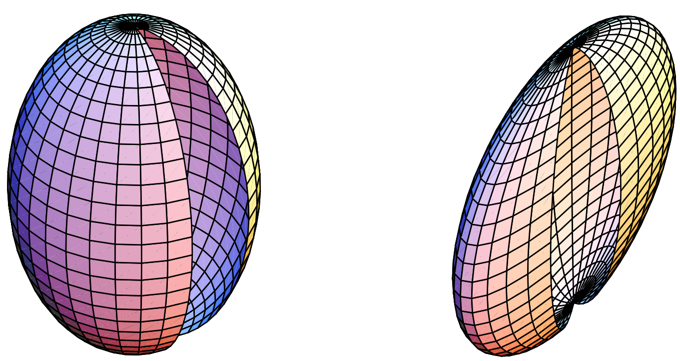


Figure 10.15: Regions of $\mathsf{ellipsoid}[1,1,2]$ and $\mathsf{ellipsoid}[1,2,3]$

We shall study surfaces of revolution in detail in Chapter 15. Figure 10.15 illustrates the difference between an ellipsoid of revolution and a general ellipsoid. A parametric form of (10.20) is

$$\text{ellipsoid}[a,b,c](u,v) = \big(a\cos v\cos u,\ b\cos v\sin u,\ c\sin v\big).$$

The patch is regular except at the north and south poles, which are $(0,0,\pm c)$.

We next define a different parametrization of an ellipsoid, the ***stereographic ellipsoid***. It is a generalization of the stereographic projection of the sphere that can be found in books on complex variables. It is often useful to change parametrization in order to highlight a particular property of a surface, and we shall see several instances of this procedure in the sequel. The ellipsoid, for reasons of physics and cartography, is one of the surfaces most amenable to the procedure (see Exercise 6 and Section 19.6).

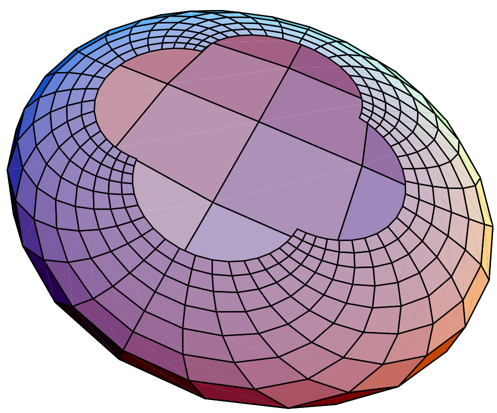


Figure 10.16: stereographicellipsoid$[5,3,1]$

For viewing convenience, the stereographic ellipse has been rotated so that the 'north pole' appears at the front of the picture. Its definition is:

$$\text{stereographicellipsoid}[a,b,c](u,v) = \left(\frac{2a\,u}{1+u^2+v^2},\ \frac{2b\,v}{1+u^2+v^2},\ \frac{c(u^2+v^2-1)}{1+u^2+v^2}\right).$$

Hyperboloids

Consider the following patches:

$$(10.21)\quad \begin{aligned} \text{hyperboloid1}[a,b,c](u,v) &= \big(a\cosh v\cos u,\ b\cosh v\sin u,\ c\sinh v\big),\\ \text{hyperboloid2}[a,b,c](u,v) &= \big(a\sinh v\cos u,\ b\sinh v\sin u,\ c\cosh v\big). \end{aligned}$$

They describe surfaces whose shape is altered by varying the three parameters a, b, c. Assuming all three are positive, hyperboloid1 describes a hyperboloid of one sheet, whereas hyperboloid2 describes the upper half of a hyperboloid of two sheets (the lower half is obtained by taking $c < 0$). As $|v|$ becomes large, the surfaces are asymptotic to

$$\text{ellipticalcone}[a,b,c](u,v) = \big(av\cos u,\ bv\cos u,\ cv\big),$$

and rotational symmetry is present if $a = b$. In this case, the surfaces are formed by revolving a branch of a hyperbola (or a line) about the z-axis, as is the case in Figure 10.17.

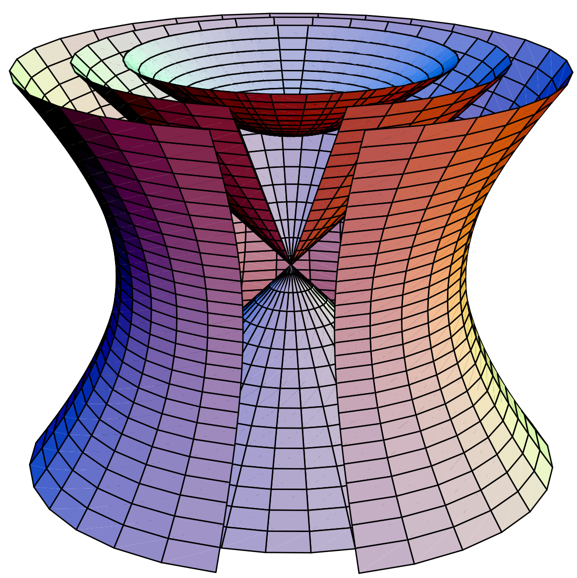


Figure 10.17: Hyperboloids in and outside of a cone

The nonparametric equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1,$$

describe a hyperboloid of one sheet and two sheets respectively. The equations can easily be distinguished, as the second has no solutions for $z = 0$, indicating that no points of the surface lie in the xy-plane.

Higher Order Surfaces

Consider the function $g_n\colon \mathbb{R}^3 \to \mathbb{R}$ defined by

$$g_n(x, y, z) = x^n + y^n + z^n,$$

where $n \geqslant 2$ is an even integer. Using Definition 9.24, the gradient of g_n is given by

$$(\mathsf{grad}\, g_n)(x, y, z) = \big(nx^{n-1},\ ny^{n-1},\ nz^{n-1}\big); \tag{10.22}$$

it vanishes if and only if $x = y = z = 0$, so the set

$$\{\, \mathbf{p} \in \mathbb{R}^3 \mid g_n(\mathbf{p}) = c \,\}$$

is a regular surface for any $c > 0$. It is a sphere for $n = 2$, but as n becomes larger and larger, (10.22) becomes more and more cube-like; see Figure 10.18.

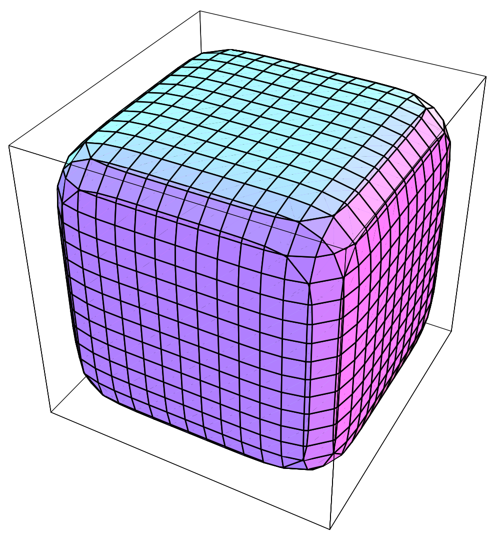


Figure 10.18: The regular surface $x^6 + y^6 + z^6 = 1$

There are functions g for which $\{\mathbf{p} \in \mathbb{R}^3 \mid g(\mathbf{p}) = c\}$ has several components. An obvious example is the regular surface defined by

$$(x^2 + y^2 + z^2 - 1)\big((x-3)^2 + y^2 + z^2 - 1\big) = 0,$$

which consists of two disjoint spheres.

Because the gradient of a function $g\colon \mathbb{R}^3 \to \mathbb{R}$ is always perpendicular to each of its level surfaces, there is an easy way to obtain the Gauss map for such surfaces.

Lemma 10.49. *Let $g\colon \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function and c a number such that $\operatorname{grad} g$ is nonzero on all points of the level surface $\mathcal{M}(c) = \{\, \mathbf{p} \in \mathbb{R}^3 \mid g(\mathbf{p}) = c \,\}$. Then the vector field $\mathbf{U}$ defined by*

$$\mathbf{U}(\mathbf{p}) = \frac{\operatorname{grad} g(\mathbf{p})}{\|\operatorname{grad} g(\mathbf{p})\|}$$

is a globally defined unit normal on $\mathcal{M}(c)$. Hence we can define the Gauss map of $\mathcal{M}(c)$ to be $\mathbf{U}$ considered as a mapping $\mathbf{U}\colon \mathcal{M}(c) \to S^2(1)$.

Steven Wilkinson's program ImplicitPlot3D is extremely effective in plotting surfaces such as those in Figures 10.18, 10.19 and 10.20. Details are given in Notebook 10. The reader is invited to test it out in Exercises 10–13.

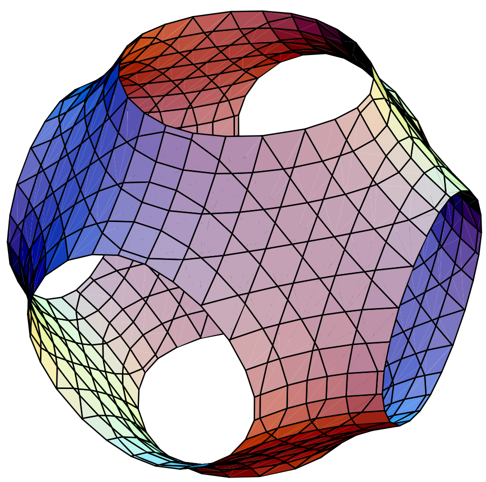


Figure 10.19: $(yz)^2 + (zx)^2 + (xy)^2 = 1$

10.7 Exercises

1. Show that the Jacobian matrix of a patch $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^n$ is related to $\mathbf{x}_u$ and $\mathbf{x}_v$ by the formula

$$\mathscr{J}(\mathbf{x})^T \mathscr{J}(\mathbf{x}) = \begin{pmatrix} \mathbf{x}_u \cdot \mathbf{x}_u & \mathbf{x}_u \cdot \mathbf{x}_v \\ \mathbf{x}_v \cdot \mathbf{x}_u & \mathbf{x}_v \cdot \mathbf{x}_v \end{pmatrix},$$

where A^T denotes the transpose of a matrix A (see Corollary 10.5(ii)). Show that when $n = 3$ then $\det\big(\mathscr{J}(\mathbf{x})^T \mathscr{J}(\mathbf{x})\big) = \|\mathbf{x}_u \times \mathbf{x}_v\|^2$.

2. Determine where the patch

$$(u,v) \mapsto \big(\cos u \cos v \sin v,\ \sin u \cos v \sin v,\ \sin v\big)$$

is regular.

3. A torus can be defined as a level surface $\mathcal{M}(b^2)$ of the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x,y,z) = z^2 + \left(\sqrt{x^2+y^2} - a\right)^2,$$

where $a > b > 0$. Show that such a torus is a regular surface and compute its unit normal.

4. Show that the Gauss map of a regular surface $\mathcal{M} \subset \mathbb{R}^3$ is a surface mapping from $\mathcal{M}$ to the unit sphere $S^2(1) \subset \mathbb{R}^3$ in the sense of Definition 10.41.

5. Find unit normals to the monkey saddle, the sphere, the torus and the eight surface.

6. Define the ***Mercator***[3] ***parametrization*** by

$$\text{mercatorellipsoid}[a,b,c](u,v) = \big(a \operatorname{sech} v \cos u,\ b \operatorname{sech} v \sin u,\ c \tanh v\big).$$

Verify that its trace is an ellipsoid.

7. Determine appropriate domains for the patch (10.19) so as to obtain two local charts (satisfying Definition 10.23) that cover the eight surface illustrated in Figure 10.5.

M **8.** Given a patch $\mathbf{x}(u,v)$, let $\mathbf{x}^{[n]}(u,v)$ denote the surface whose x-, y- and z-entries are the n^{th} powers of those of $\mathbf{x}(u,v)$. Plot the 'cubed surface' $\text{torus}^{[3]}[8,3,8]$.

M **9.** Verify that the following quartic factorizes into two quadratic polynomials, and hence plot the surface described by setting it equal to zero:

$$\frac{x^4}{9} + \frac{y^4}{9} + z^4 - \frac{82x^2y^2}{81} - \frac{10x^2z^2}{9} - \frac{10y^2z^2}{9} - \frac{10x^2}{9} + \frac{10y^2}{9} + 2z^2 + 1.$$

---

3  Gerardus Mercator (Latinized name of Gerhard Kremer) (1512–1594). Flemish cartographer. In 1569 he first used the map projection which bears his name.

M **10.** Use ImplicitPlot3D to plot ***Kummer's surface***[4], defined implicitly by

$$x^4 + y^4 + z^4 - (y^2z^2 + z^2x^2 + x^2y^2) - (x^2 + y^2 + z^2) + 1 = 0.$$

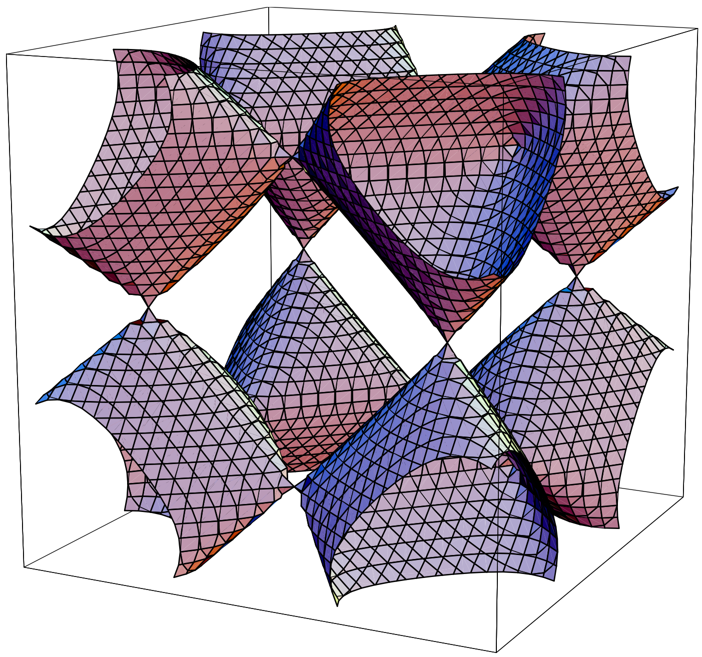
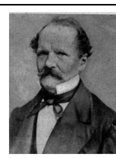


Figure 10.20: Part of Kummer's surface

M **11.** ***Goursat's surface***[5] is defined by

$$x^4 + y^4 + z^4 + a(x^2 + y^2 + z^2)^2 + b(x^2 + y^2 + z^2) + c = 0.$$

Plot it for $a = -1/5$, $b = 0$ and $c = -1$ in the range $-1 \leqslant x, y, z \leqslant 1$.

4  Ernst Eduard Kummer(1810–1893). Kummer is remembered for his work in hypergeometric functions, number theory and algebraic geometry.

5  Édouard Jean Babtiste Goursat (1858–1936). Goursat was a leading analyst of his day. His *Cours d'analyse mathématique* has long been a classic text in France and elsewhere.

M **12.** Plot the ***two-cusp surface***

$$(z-1)^2(x^2-z^2)-(x^2-z)^2-y^4-y^2(2x^2+z^2+2z-1)=0$$

and verify that it does indeed possess two singular points.

M **13.** Plot the ***sine surface*** defined by

$$\mathsf{sinsurface}(u,v)=\big(\sin u,\,\sin v,\,\sin(u+v)\big).$$

Find a nonparametric form of the surface. Describe the singular points of the sine surface.

14. Let $\mathcal{V}$ be an open subset of the monkey saddle containing its planar point $(0,0,0)$. Describe the image of $\mathcal{V}$ under the Gauss map.

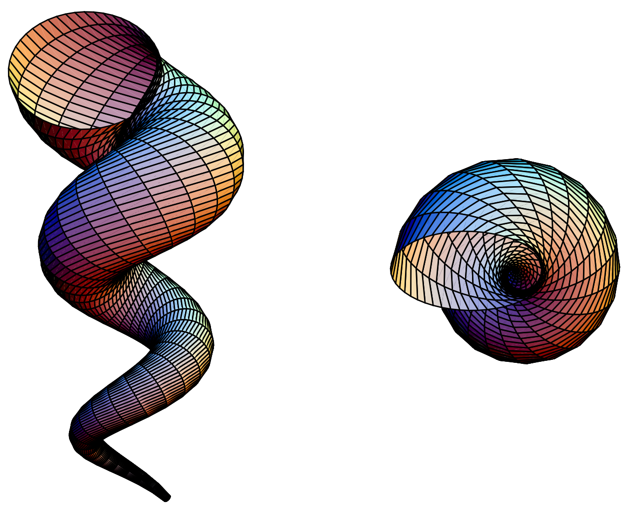


Figure 10.21: The $\mathsf{seashell}[\mathsf{helix}[1,0.6]][0.1]$

M **15.** In Section 7.6 we showed how to construct tubes about curves in $\mathbb{R}^3$. The construction can be easily modified to allow the radius of the tube to change from point to point. For example, given a curve $\boldsymbol{\gamma}\colon (a,b)\to\mathbb{R}^3$, let us define a ***sea shell***

$$\mathsf{seashell}[\boldsymbol{\gamma},r](t,\theta)=\boldsymbol{\gamma}(t)+rt\big(-\cos\theta\,\mathbf{N}(t)+\sin\theta\,\mathbf{B}(t)\big),$$

where $\mathbf{N}$ and $\mathbf{B}$ are the normal and binormal to $\boldsymbol{\gamma}$. Explain why the sea shell of a helix resembles Figure 10.21.