Chapter 11

Nonorientable Surfaces

An important discovery of the nineteenth century was that nonorientable surfaces exist. The goal of the present chapter is to gain an understanding of such surfaces. We begin in Section 11.1 by saying precisely what it means for a regular surface to be *orientable*, and defining the associated Gauss map. Examples of nonorientable surfaces are first described in Section 11.2 by means of various identifications of the edges of a square. We obtain topological descriptions of some of the simpler nonorientable surfaces, without reference to an underlying space like $\mathbb{R}^n$ with $n \geqslant 3$. The analytical theory of surfaces described in such an abstract way is developed in Chapter 26.

The rest of the chapter is devoted to realizing nonorientable surfaces inside 3-dimensional space. Section 11.3 is devoted to the Möbius strip, a building block for all nonorientable surfaces. Having parametrized the Möbius strip explicitly, we are able to depict its Gauss image in Figures 11.8 and 11.9. The Möbius strip has become an icon that has formed the basis of many artistic designs, including one on the Mall in Washington DC.

Sections 11.4 and 11.5 describe two basic compact nonorientable surfaces, namely the Klein bottle and real projective plane. These necessarily have points of self-intersection in $\mathbb{R}^3$, and two different models of the Klein bottle are discussed. Different realizations of the projective plane are described using the notion of a map with the antipodal property (Definition 11.5).

11.1 Orientability of Surfaces

If V is a 2-dimensional vector space, we call a linear map $J\colon V \to V$ such that $J^2 = -\mathbf{1}$ a *complex structure* on V. We used this notion from the beginning of our study of curves, and observed on page 265 that each tangent space to $\mathbb{R}^2$ has a complex structure. More generally, because all 2-dimensional vector spaces are isomorphic, each tangent space $\mathcal{M}_{\mathbf{p}}$ to a regular surface $\mathcal{M}$ admits a complex structure $J_{\mathbf{p}}\colon \mathcal{M}_{\mathbf{p}} \to \mathcal{M}_{\mathbf{p}}$. However, it may or may not be possible to make a *continuous* choice of $J_{\mathbf{p}}$. Continuity here means that the vector field $\mathbf{p} \mapsto J_{\mathbf{p}}\mathbf{X}$ is continuous for each continuous vector field $\mathbf{X}$ tangent to $\mathcal{M}$. This leads to

Definition 11.1. *A regular surface $\mathcal{M} \subset \mathbb{R}^3$ is called* ***orientable*** *provided each tangent space $\mathcal{M}_{\mathbf{p}}$ has a complex structure $J_{\mathbf{p}}\colon \mathcal{M}_{\mathbf{p}} \to \mathcal{M}_{\mathbf{p}}$ such that $\mathbf{p} \mapsto J_{\mathbf{p}}$ is a continuous function. An* ***oriented*** *regular surface $\mathcal{M} \subset \mathbb{R}^3$ is an orientable regular surface together with a choice of the complex structure $\mathbf{p} \mapsto J_{\mathbf{p}}$.*

We shall see in Chapter 26 that this definition works equally well for surfaces *not* contained in $\mathbb{R}^3$. For regular surfaces contained in $\mathbb{R}^3$, there is a more intuitive way to describe orientability using the vector cross product.

Theorem 11.2. *A regular surface $\mathcal{M} \subset \mathbb{R}^3$ is orientable if and only if there is a continuous map $\mathbf{p} \mapsto \mathbf{U}(\mathbf{p})$ that assigns to each $\mathbf{p} \in \mathcal{M}$ a unit normal vector $\mathbf{U}(\mathbf{p}) \in \mathcal{M}_{\mathbf{p}}^{\perp}$.*

Proof. Suppose we are given $\mathbf{p} \mapsto \mathbf{U}(\mathbf{p})$. Then for each $\mathbf{p} \in \mathcal{M}$ we define $J_{\mathbf{p}}\colon \mathcal{M}_{\mathbf{p}} \to \mathcal{M}_{\mathbf{p}}$ by

$$J_{\mathbf{p}}\mathbf{v}_{\mathbf{p}} = \mathbf{U}(\mathbf{p}) \times \mathbf{v}_{\mathbf{p}}. \tag{11.1}$$

It is easy to check that $J_{\mathbf{p}}$ maps $\mathcal{M}_{\mathbf{p}}$ into $\mathcal{M}_{\mathbf{p}}$ and not merely into $\mathbb{R}^3_{\mathbf{p}}$, and that $\mathbf{p} \mapsto J_{\mathbf{p}}$ is continuous. From (7.3), page 193, it follows that

$$\begin{aligned} J_{\mathbf{p}}{}^2\mathbf{v}_{\mathbf{p}} &= \mathbf{U}(\mathbf{p}) \times \big(\mathbf{U}(\mathbf{p}) \times \mathbf{v}_{\mathbf{p}}\big) \\ &= \big(\mathbf{U}(\mathbf{p}) \cdot \mathbf{v}_{\mathbf{p}}\big)\mathbf{U}(\mathbf{p}) - \big(\mathbf{U}(\mathbf{p}) \cdot \mathbf{U}(\mathbf{p})\big)\mathbf{v}_{\mathbf{p}} = -\mathbf{v}_{\mathbf{p}}. \end{aligned}$$

Conversely, if we are given a regular surface $\mathcal{M} \subset \mathbb{R}^3$ with a globally-defined continuous complex structure $\mathbf{p} \mapsto J_{\mathbf{p}}$, we define $\mathbf{U}(\mathbf{p}) \in \mathbb{R}^3_{\mathbf{p}}$ by

$$\mathbf{U}(\mathbf{p}) = \frac{\mathbf{v}_{\mathbf{p}} \times J_{\mathbf{p}}\mathbf{v}_{\mathbf{p}}}{\|\mathbf{v}_{\mathbf{p}} \times J_{\mathbf{p}}\mathbf{v}_{\mathbf{p}}\|} \tag{11.2}$$

for any nonzero $\mathbf{v}_{\mathbf{p}} \in \mathcal{M}_{\mathbf{p}}$. Then $\mathbf{U}(\mathbf{p})$ is perpendicular to both $\mathbf{v}_{\mathbf{p}}$ and $J_{\mathbf{p}}\mathbf{v}_{\mathbf{p}}$. Since $\mathbf{v}_{\mathbf{p}}$ and $J_{\mathbf{p}}\mathbf{v}_{\mathbf{p}}$ form a basis for $\mathcal{M}_{\mathbf{p}}$, it follows that $\mathbf{U}(\mathbf{p})$ is perpendicular to $\mathcal{M}_{\mathbf{p}}$. To check that the $\mathbf{U}(\mathbf{p})$ defined by (11.2) is independent of the choice of $\mathbf{v}_{\mathbf{p}}$, let $\mathbf{w}_{\mathbf{p}}$ be another nonzero tangent vector in $\mathcal{M}_{\mathbf{p}}$. Then $\mathbf{w}_{\mathbf{p}} = a\mathbf{v}_{\mathbf{p}} + bJ_{\mathbf{p}}\mathbf{v}_{\mathbf{p}}$ is a linear combination of $\mathbf{v}_{\mathbf{p}}$, and

$$\mathbf{w}_{\mathbf{p}} \times J_{\mathbf{p}}\mathbf{w}_{\mathbf{p}} = (a\,\mathbf{v}_{\mathbf{p}} + b\,J_{\mathbf{p}}\mathbf{v}_{\mathbf{p}}) \times (-b\,\mathbf{v}_{\mathbf{p}} + a\,J_{\mathbf{p}}\mathbf{v}_{\mathbf{p}}) = (a^2+b^2)(\mathbf{v}_{\mathbf{p}} \times J_{\mathbf{p}}\mathbf{v}_{\mathbf{p}}).$$

Hence

$$\frac{\mathbf{w}_{\mathbf{p}} \times J_{\mathbf{p}}\mathbf{w}_{\mathbf{p}}}{\|\mathbf{w}_{\mathbf{p}} \times J_{\mathbf{p}}\mathbf{w}_{\mathbf{p}}\|} = \frac{(a^2+b^2)(\mathbf{v}_{\mathbf{p}} \times J_{\mathbf{p}}\mathbf{v}_{\mathbf{p}})}{\|(a^2+b^2)(\mathbf{v}_{\mathbf{p}} \times J_{\mathbf{p}}\mathbf{v}_{\mathbf{p}})\|} = \frac{\mathbf{v}_{\mathbf{p}} \times J_{\mathbf{p}}\mathbf{v}_{\mathbf{p}}}{\|\mathbf{v}_{\mathbf{p}} \times J_{\mathbf{p}}\mathbf{v}_{\mathbf{p}}\|},$$

and (11.2) is unambiguous. Since $\mathbf{p} \mapsto J_{\mathbf{p}}$ is continuous, so is $\mathbf{p} \mapsto \mathbf{U}(\mathbf{p})$. ∎

Theorem 11.2 permits us to define the Gauss map of an arbitrary orientable regular surface in $\mathbb{R}^3$, rather than just a patch. Definition 10.22 on page 296 extends in an obvious fashion to

Definition 11.3. *Let $\mathcal{M}$ be an oriented regular surface in $\mathbb{R}^3$, and let $\mathbf{U}$ be a globally defined unit normal vector field on $\mathcal{M}$ that defines the orientation of $\mathcal{M}$. Let $S^2(1)$ denote the unit sphere in $\mathbb{R}^3$. Then $\mathbf{U}$, viewed as a map*

$$\mathbf{U}\colon \mathcal{M} \longrightarrow S^2(1),$$

*is called the **Gauss map** of $\mathcal{M}$.*

It is easy to find examples of orientable regular surfaces in $\mathbb{R}^3$.

Lemma 11.4. *Let $\mathcal{U} \subseteq \mathbb{R}^2$. The graph $\mathcal{M}_h$ of a function $h\colon \mathcal{U} \to \mathbb{R}$ is an orientable regular surface.*

Proof. As in Section 10.4, we have the Monge patch $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}_h$ by

$$\mathbf{x}(u,v) = \big(u,\ v,\ h(u,v)\big).$$

Then $\mathbf{x}$ covers all of $\mathcal{M}_h$; that is, $\mathbf{x}(\mathcal{U}) = \mathcal{M}_h$. Furthermore, $\mathbf{x}$ is regular and injective. The surface normal $\mathbf{U}$ to $\mathcal{M}_h$ is given by

$$\mathbf{U} \circ \mathbf{x} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \frac{(-h_u,\ -h_v,\ 1)}{\sqrt{1 + h_u^2 + h_v^2}}.$$

The unit vector $\mathbf{U}$ is everywhere nonzero, and it follows from Theorem 11.2 that $\mathcal{M}_h$ is orientable. ∎

More generally, the method of proof of Lemma 11.4 yields:

Lemma 11.5. *Any surface $\mathcal{M} \subset \mathbb{R}^3$ which is the trace of a single injective regular patch $\mathbf{x}$ is orientable.*

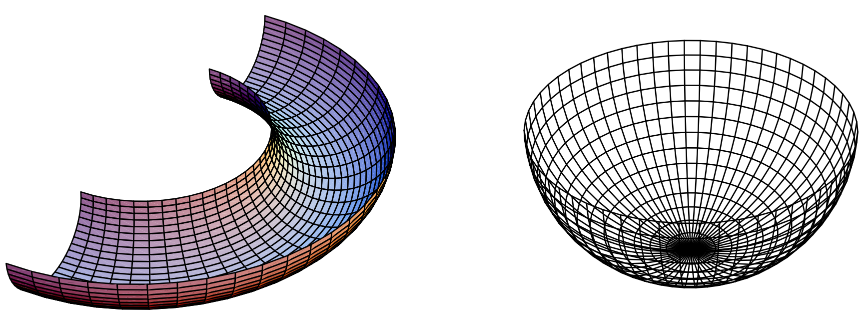

Figure 11.1: Gauss image of a quarter torus

As an example, Figure 11.1 shows that the Gauss map of a *quarter* of a torus covers *half* the unit sphere. Here is another generalization of Lemma 11.4.

Lemma 11.6. *Let $g\colon \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function and c a number such that* $\operatorname{grad} g$ *is nonzero on all points of* $\mathcal{M}(c) = \{\, \mathbf{p} \in \mathbb{R}^3 \mid g(\mathbf{p}) = c \,\}$. *Then* $\mathcal{M}(c)$*, as well as every component of* $\mathcal{M}(c)$*, is orientable.*

Proof. Theorem 10.48, page 311, implies that $\mathcal{M}(c)$ is a regular surface and that $\operatorname{grad} g$ is everywhere perpendicular to $\mathcal{M}(c)$. We are assuming that $\operatorname{grad} g$ never vanishes on $\mathcal{M}(c)$. Putting these facts together we see that

$$\frac{\operatorname{grad} g}{\|\operatorname{grad} g\|}$$

is a unit vector field that is well defined at all points of $\mathcal{M}(c)$ and everywhere perpendicular to $\mathcal{M}(c)$. Then Theorem 11.2 says that $\mathcal{M}(c)$ and each of its components are orientable. ∎

Recall that a subset X of $\mathbb{R}^n$ is said to be ***closed*** if it contains all its limit points. Equivalently, X is closed if and only if its complement $\mathbb{R}^n \setminus X$ is open (see page 268). A subset X is said to be ***connected*** if any decomposition $X = X_1 \cup X_2$ of X into closed subsets X_1, X_2 with $X_1 \cap X_2$ empty must be trivial; that is, either $X = X_1$ or $X = X_2$.

Lemma 11.7. *Let $\mathcal{M}$ be a connected orientable regular surface in $\mathbb{R}^3$. Then $\mathcal{M}$ has exactly two globally-defined unit normal vector fields.*

Proof. Since $\mathcal{M}$ is orientable, it has at least one globally-defined unit normal vector field $\mathbf{p} \mapsto \mathbf{U}(\mathbf{p})$. Let $\mathbf{p} \mapsto \mathbf{v}(\mathbf{p})$ be any other globally-defined unit normal vector field on $\mathcal{M}$. The sets $\mathcal{W}_{\pm} = \{\, \mathbf{p} \in \mathcal{M} \mid \mathbf{U}(\mathbf{p}) = \pm\mathbf{v}(\mathbf{p}) \,\}$ are closed, because $\mathbf{U}$ and $\mathbf{v}$ are continuous. Also $\mathcal{M} = \mathcal{W}_+ \cup \mathcal{W}_-$. The connectedness of $\mathcal{M}$ implies that $\mathcal{M}$ coincides with either $\mathcal{W}_+$ or $\mathcal{W}_-$. Consequently, the globally-defined unit vector fields on $\mathcal{M}$ are $\mathbf{p} \mapsto \mathbf{U}(\mathbf{p})$ and $\mathbf{p} \mapsto -\mathbf{U}(\mathbf{p})$. ∎

Definition 11.8. *Let $\mathcal{M}$ be an orientable regular surface in $\mathbb{R}^3$. An* ***orientation*** *of $\mathcal{M}$ is the choice of a globally-defined unit normal vector field on $\mathcal{M}$.*

In general, it may not be possible to cover a regular surface with a single patch. If we have a family of patches, we must know how their orientations are related in order to define the orientation of a regular surface.

Lemma 11.9. *Let $\mathcal{M} \subset \mathbb{R}^3$ be a regular surface. If $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ and $\mathbf{y}\colon \mathcal{V} \to \mathcal{M}$ are patches such that $\mathbf{x}(\mathcal{U}) \cap \mathbf{y}(\mathcal{V})$ is nonempty, then*

$$\mathbf{y}_u \times \mathbf{y}_v = \det\big(\mathscr{J}(\mathbf{x}^{-1} \circ \mathbf{y})\big)\, \mathbf{x}_{\bar{u}} \times \mathbf{x}_{\bar{v}},$$

where $\mathscr{J}(\mathbf{x}^{-1} \circ \mathbf{y})$ denotes the Jacobian matrix of $\mathbf{x}^{-1} \circ \mathbf{y}$.

Proof. It follows from Lemma 10.31, page 300, that

$$\begin{aligned}
\mathbf{y}_u \times \mathbf{y}_v &= \left(\frac{\partial \bar{u}}{\partial u}\mathbf{x}_{\bar{u}} + \frac{\partial \bar{v}}{\partial u}\mathbf{x}_{\bar{v}}\right) \times \left(\frac{\partial \bar{u}}{\partial v}\mathbf{x}_{\bar{u}} + \frac{\partial \bar{v}}{\partial v}\mathbf{x}_{\bar{v}}\right)\\
&= \det\begin{pmatrix} \dfrac{\partial \bar{u}}{\partial u} & \dfrac{\partial \bar{u}}{\partial v} \\ \dfrac{\partial \bar{v}}{\partial u} & \dfrac{\partial \bar{v}}{\partial v} \end{pmatrix} \mathbf{x}_{\bar{u}} \times \mathbf{x}_{\bar{v}}\\
&= \det\big(\mathscr{J}(\mathbf{x}^{-1} \circ \mathbf{y})\big)\, \mathbf{x}_{\bar{u}} \times \mathbf{x}_{\bar{v}}. \quad \blacksquare
\end{aligned}$$

Definition 11.10. *We say that patches $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ and $\mathbf{y}\colon \mathcal{V} \to \mathcal{M}$ on a regular surface $\mathcal{M}$ with $\mathbf{x}(\mathcal{U}) \cap \mathbf{y}(\mathcal{V})$ nonempty are* ***coherently oriented****, provided the determinant of the Jacobian matrix $\mathscr{J}(\mathbf{x}^{-1} \circ \mathbf{y})$ is positive on $\mathbf{x}(\mathcal{U}) \cap \mathbf{y}(\mathcal{V})$.*

Theorem 11.11. *A regular surface $\mathcal{M} \subset \mathbb{R}^3$ is orientable if and only if it is possible to cover $\mathcal{M}$ with a family $\mathfrak{B}$ of regular injective patches such that any two patches $(\mathbf{x},\mathcal{U})$, $(\mathbf{y},\mathcal{V})$ with $\mathbf{x}(\mathcal{U}) \cap \mathbf{y}(\mathcal{V}) \neq \emptyset$ are coherently oriented.*

Proof. Suppose that $\mathcal{M}$ is orientable. Theorem 11.2 implies that $\mathcal{M}$ has a globally-defined surface normal $\mathbf{p} \mapsto \mathbf{U}(\mathbf{p})$. Let $\mathfrak{A}$ be a family of regular injective patches whose union covers $\mathcal{M}$. Without loss of generality, we may suppose that the domain of definition of each patch in $\mathfrak{A}$ is connected. We must construct from $\mathfrak{A}$ a family $\mathfrak{B}$ of coherently-oriented patches whose union covers $\mathcal{M}$.

To this end, we first note that if $\mathbf{x}$ is any regular injective patch on $\mathcal{M}$, then $\widetilde{\mathbf{x}}$ defined by

$$\widetilde{\mathbf{x}}(u,v) = \mathbf{x}(v,u)$$

is also a regular injective patch on $\mathcal{M}$. Moreover, $\widetilde{\mathbf{x}}$ and $\mathbf{x}$ are oppositely oriented, because $\widetilde{\mathbf{x}}_u \times \widetilde{\mathbf{x}}_v = -\mathbf{x}_u \times \mathbf{x}_v$. Now it is clear how to choose the patches that are to be members of the family $\mathfrak{B}$. If $\mathbf{x}$ is a patch in $\mathfrak{A}$ and

$$\frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \mathbf{U}$$

we put $\mathbf{x}$ into $\mathfrak{B}$, but if

$$\frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = -\mathbf{U}$$

we put $\widetilde{\mathbf{x}}$ into $\mathfrak{B}$. Then $\mathfrak{B}$ is a family of coherently-oriented patches that covers $\mathcal{M}$.

Conversely, suppose that $\mathcal{M}$ is covered by a family $\mathfrak{B}$ of coherently-oriented patches. If $\mathbf{x}$ is a patch in $\mathfrak{B}$ defined on $\mathcal{U} \subset \mathbb{R}^2$, we define $\mathbf{U}_{\mathbf{x}}$ on $\mathbf{x}(\mathcal{U})$ by

$$\mathbf{U}_{\mathbf{x}}\big(\mathbf{x}(u,v)\big) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}(u,v).$$

Let $\mathbf{y}\colon \mathcal{V} \to \mathcal{M}$ be another patch in $\mathfrak{B}$ with $\mathbf{x}(\mathcal{U}) \cap \mathbf{y}(\mathcal{V}) \neq \emptyset$. Lemma 11.9 implies that on $\mathbf{x}(\mathcal{U}) \cap \mathbf{y}(\mathcal{V})$ we have

$$\begin{aligned} \mathbf{U}_{\mathbf{y}}\big(\mathbf{y}(u,v)\big) &= \frac{\mathbf{y}_u \times \mathbf{y}_v}{\|\mathbf{y}_u \times \mathbf{y}_v\|}(u,v) = \left(\frac{\det \mathscr{J}(\mathbf{x}^{-1}\circ \mathbf{y})}{|\det \mathscr{J}(\mathbf{x}^{-1}\circ \mathbf{y})|}\frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}\right)(u,v) \\ &= \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}(u,v) = \mathbf{U}_{\mathbf{x}}\big(\mathbf{x}(u,v)\big). \end{aligned}$$

Thus we get a well-defined surface normal $\mathbf{U}$ on $\mathcal{M}$ by putting $\mathbf{U} = \mathbf{U}_{\mathbf{x}}$ on $\mathbf{x}(\mathcal{U})$ for any patch $\mathbf{x}$ in $\mathfrak{B}$. By Theorem 11.2, $\mathcal{M}$ is orientable. ▮

The proof of Theorem 11.11 establishes

Corollary 11.12. *A family of coherently-oriented regular injective patches on a regular surface in $\mathbb{R}^3$ defines globally a unit normal vector field on a surface $\mathcal{M}$ in $\mathbb{R}^3$, that is, an orientation of $\mathcal{M}$.*

The image under the Gauss map of the equatorial region

$$\big\{\mathsf{hyperboloid1}[1,1,1](u,v) \ \ | \ \ 0 \leqslant u \leqslant 2\pi,\ -1 \leqslant v \leqslant 1\big\}$$

of a hyperboloid of one sheet is illustrated in Figure 11.2. The image of the whole hyperboloid is also a bounded equatorial region, because the normals to the surface approach those of the asymptotic cone (shown in Figure 10.17). The image of the whole hyperboloid of two sheets under the Gauss map consists of two antipodal disks (see Exercise 2).

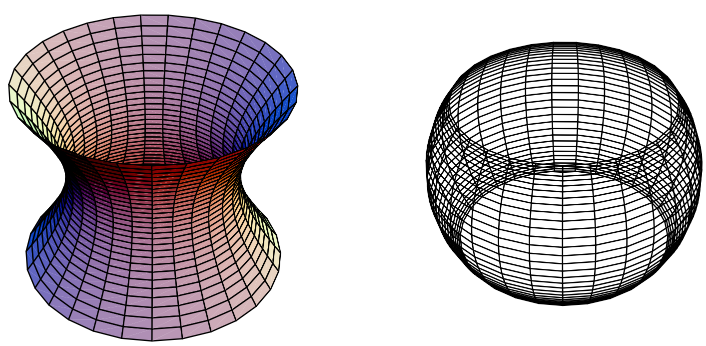

Figure 11.2: Gauss image of a hyperboloid

11.2 Surfaces by Identification

There are useful topological descriptions of some elementary surfaces that are obtained from identifying edges of a square. For example, if we identify the top and bottom edges, we obtain a cylinder. It is conventional to describe this identification by means of an arrow along the top edge and an arrow pointing in the same direction along the bottom edge, as in Figure 11.3 (left).

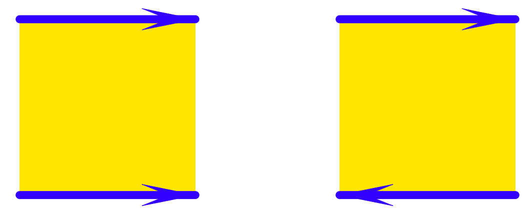

Figure 11.3: Models of the cylinder and Möbius strip

Now consider a square with the top and bottom edges identified, but in reverse order. This identification is indicated by means of an arrow along the top edge pointing in one direction and an arrow along the bottom edge pointing in the opposite direction. The resulting surface is called a ***Möbius***[1] ***strip*** or a ***Möbius band***. It is easy to put a cylinder into Euclidean space $\mathbb{R}^3$, but to find the actual parametrization of a Möbius strip in $\mathbb{R}^3$ is more difficult. We shall return to this problem shortly.

Now let us see what happens when we identify the vertical as well as the horizontal edges of a square. There are three possibilities. If the vertical arrows point in the same direction, and if the direction of the horizontal arrows is also the same, we obtain a torus. We have already seen on page 305 how to parametrize a torus in $\mathbb{R}^3$ using the function torus$[a, b, c]$. For different values of a, b, c, we get tori which are different in shape, but provided $a < b$ and $a < c$ they are topologically all the same.

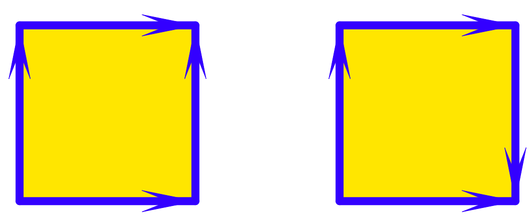

Figure 11.4: Models of the torus and Klein bottle

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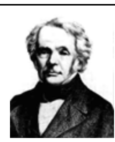
1  August Ferdinand Möbius (1790–1868). Professor at the University of Leipzig. Möbius' name is attached to several important mathematical objects such as the Möbius function, the Möbius inversion formula and Möbius transformations. He discovered the Möbius strip at age 71.

The ***Klein*[2] *bottle*** is the surface that results when the edges of the square are identified with the vertical arrows pointing in the same direction, but the horizontal arrows pointing in opposite directions. Clearly, the same surface results if we interchange horizontal with vertical, to obtain the right-hand side of Figure 11.4.

Finally, the surface that results when we identify the edges of the square with the two vertical arrows pointing in different directions *and* the two horizontal arrows pointing in different directions is called the ***real projective plane***, and sometimes denoted $\mathbb{RP}^2$. This surface can also be thought of as a sphere with antipodal points identified. It is one of a series of important topological objects, namely the real projective spaces $\mathbb{RP}^n$, that are defined in Exercise 12 on page 798. The superscript indicates dimension, a concept that will be defined rigorously in Chapter 24. Whilst $\mathbb{RP}^1$ is equivalent to a circle, it turns out that $\mathbb{RP}^3$ can be identified with the set of rotations $\mathbb{R}^3$ discussed in Chapter 23.

We have described the Möbius strip, the Klein bottle and the real projective plane topologically. All these surfaces turn out to be nonorientable. It is quite another matter, however, to find effective parametrizations of these surfaces in $\mathbb{R}^3$. That is the subject of the rest of this chapter.

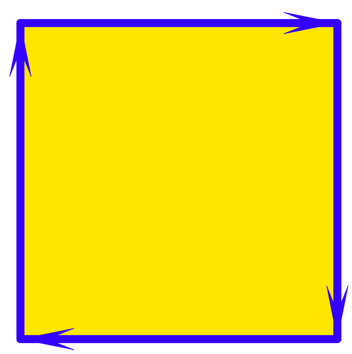


Figure 11.5: Model of the projective plane

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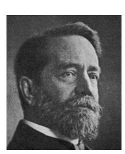
[2]  Christian Felix Klein (1849–1925). Klein made fruitful contributions to many branches of mathematics, including applied mathematics and mathematical physics. His Erlanger Programm (1872) instituted research directions in geometry for a half century; his subsequent work on Riemann surfaces established their essential role in function theory. In his writings, Klein concerned himself with what he saw as a developing gap between the increasing abstraction of mathematics and applied fields whose practitioners did not appreciate the fundamental rôle of mathematics, as well as with mathematics instruction at the secondary level. Klein discussed nonorientable surfaces in 1874 in [Klein].

11.3 The Möbius Strip

Another useful description of the Möbius strip is as the surface resulting from revolving a line segment around an axis, putting one twist in the line segment as it goes around the axis.

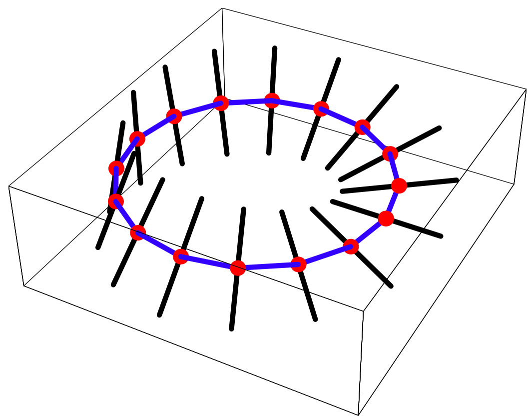


Figure 11.6: Lines twisting to form a Möbius strip

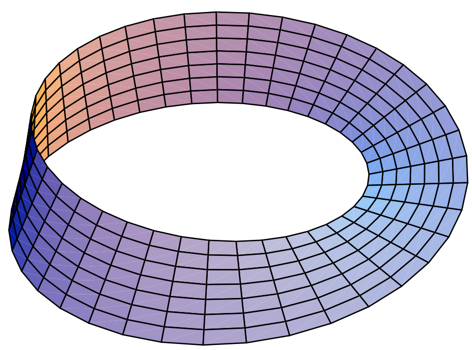


Figure 11.7: Möbius strip

We can use this model to get a parametrization of the strip:

$$\text{moebiusstrip}[a](u,v) = \Big(a\cos u + v\cos\frac{u}{2}\cos u,\quad a\sin u + v\cos\frac{u}{2}\sin u,\ v\sin\frac{u}{2}\Big). \tag{11.3}$$

We see that the points

$$\text{moebiusstrip}[a](u,0) = \big(a\cos u,\ a\sin u,\ 0\big), \quad 0 \leqslant u < 2\pi,$$

constitute the central circle which therefore has radius a. It is convenient to retain this parameter a in order to plot Möbius strips of different shape, and our notation reflects that of Notebook 11.

The points moebiusstrip$[a](u_0, v)$, as v varies for each fixed u_0, form a line segment meeting the central circle. As u_0 increases from 0 to 2π, the angle between this line and the xy-plane changes from 0 to π.

The nonorientability of the Möbius strip means that the Gauss map is not well defined on the whole surface: any attempt to define a unit normal vector on the *entire* Möbius strip is doomed to failure. However, the Gauss map is defined on any orientable portion, for example, on a Möbius strip minus a line orthogonal to the central circle.

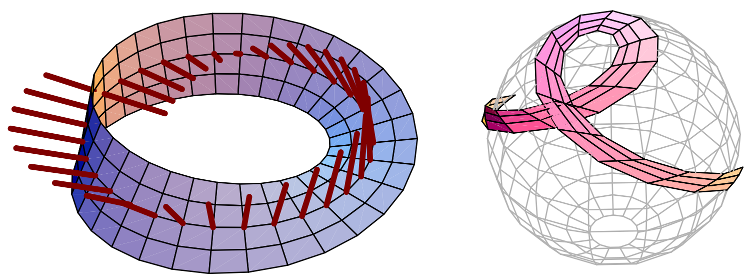


Figure 11.8: Gauss image of a Möbius strip traversed once

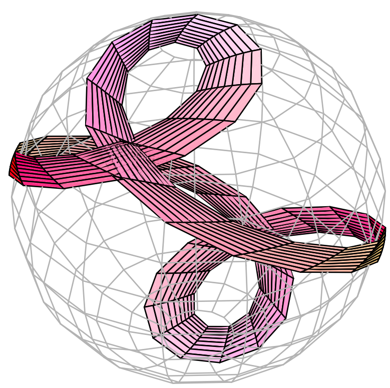


Figure 11.9: Gauss image of a Möbius strip traversed twice

It is possible to compute a normal vector field to the patch moebiusstrip$[1]$; the result is

$$\begin{aligned}(11.4)\qquad &\Big(\big(-v\cos\frac{u}{2}+2\cos u+v\cos\frac{3u}{2}\big)\sin\frac{u}{2},\\ &\qquad\cos\frac{u}{2}-\cos\frac{3u}{2}+v(\cos u+\sin^2 u),\ -2\cos\frac{u}{2}\big(1+v\cos\frac{u}{2}\big)\Big).\end{aligned}$$

The Gauss map is determined by normalizing this vector. Although (11.4) is complicated, it is used in Notebook 11 to plot the accompanying figures. If one tries to extend the definition of the unit normal so that it is defined on all of the Möbius strip by going around the center circle, then the unit normal comes back on the other side of the surface (Figure 11.8). If we repeat this operation to return to where we started on the sphere, we have effectively associated *two* unit normal vectors to each point of the strip (Figure 11.9).

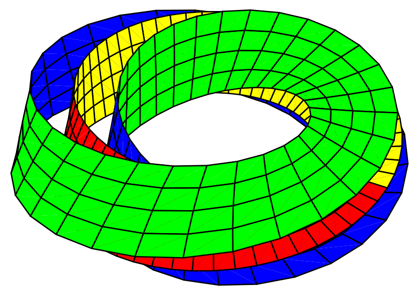


Figure 11.10: Parallel surface to the Möbius strip

Figure 11.10 displays a Möbius strip together with the surface formed by moving a small fixed distance along both of the unit normals emanating from each point of the strip. This construction is considered further in Notebook 11 and Section 19.8; here we merely observe that the resulting surface parallel to the Möbius strip is orientable.

11.4 The Klein Bottle

The Klein bottle is also a nonorientable surface, but in contrast to the Möbius strip it is ***compact***, equivalently closed without boundary, like the torus. An elementary account of the theory of compact surfaces can be found in [FiGa].

One way to define the Klein bottle is as follows. It is the surface that results from rotating a figure eight about an axis, but putting a twist in it. The rotation and twisting are the same as a Möbius strip but, instead of a line segment, one uses a figure eight.

Here is a parametrization of the Klein bottle that uses this construction.

$$\begin{aligned}\mathsf{kleinbottle}[a](u,v) \;=\; \Big(&\big(a+\cos\tfrac{u}{2}\sin v-\sin\tfrac{u}{2}\sin 2v\big)\cos u,\\ &\big(a+\cos\tfrac{u}{2}\sin v-\sin\tfrac{u}{2}\sin 2v\big)\sin u,\quad \sin\tfrac{u}{2}\sin v+\cos\tfrac{u}{2}\sin 2v\Big).\end{aligned}$$

The central circle of the Klein bottle is traversed twice by the curve

$$u \mapsto \mathsf{kleinbottle}[a](u,0), \quad 0 \leqslant u < 4\pi.$$

Each of the curves $v \mapsto \mathsf{kleinbottle}[a](u_0, v)$ is a figure eight. As u varies from 0 to 2π, the figure eights twist from 0 to π; this is the same twisting that we encountered in the parametrization (11.3) of the Möbius strip.

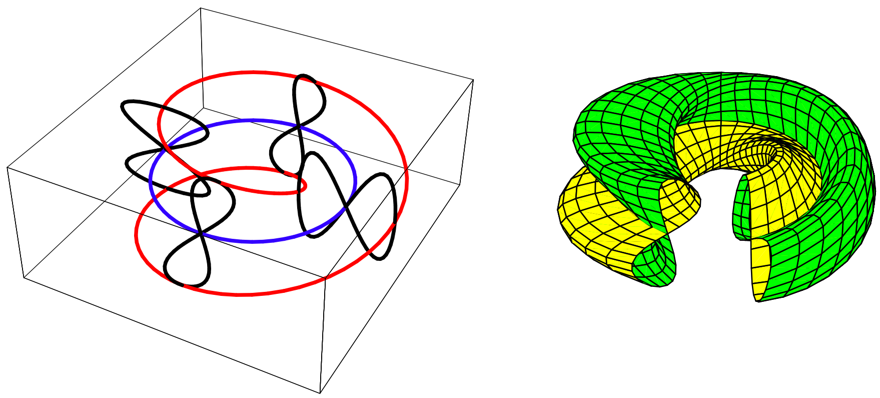


Figure 11.11: Figure eights twisting to form a Klein bottle

The Klein bottle is not a regular surface in $\mathbb{R}^3$ because it has self-intersections. However, the Klein bottle can be shown to be an abstract surface, a concept to be defined in Chapter 26. There are two kinds of Klein bottles in $\mathbb{R}^3$. The first one, $\mathcal{K}_1$ illustrated in Figure 11.11, has the feature that a neighborhood $\mathcal{V}_1$ of the self-intersection curve is nonorientable. In fact, $\mathcal{V}_1$ is formed from an 'X' that rotates and twists about an axis in the same way that the figure eights move when they form the surface.

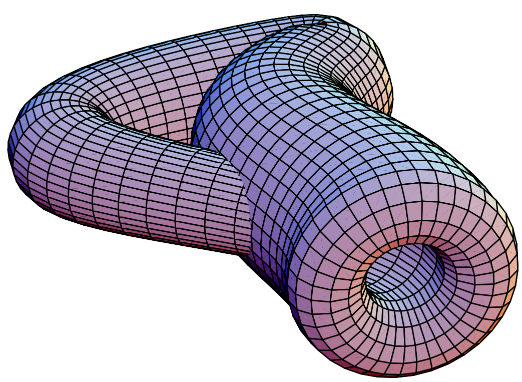


Figure 11.12: Klein bottle with an orientable neighborhood of the self-intersection curve

The original description of Klein (see [HC-V, pages 308-311]) of his surface was much different, and is described as follows. Consider a tube $\mathscr{T}$ of variable radius about a line. Topologically, a torus is formed from $\mathscr{T}$ by bending the tube until the ends meet and then gluing the boundary circles together. Another way to glue the ends is as follows. Let one end of $\mathscr{T}$ be a little smaller than the other. We bend the smaller end, then push it through the surface of the tube, and move it so that it is a concentric circle with the larger end, lying in the same plane. We complete the surface by adjoining a torus on the other side of the plane. The result is shown in Figures 11.12 and 11.13.

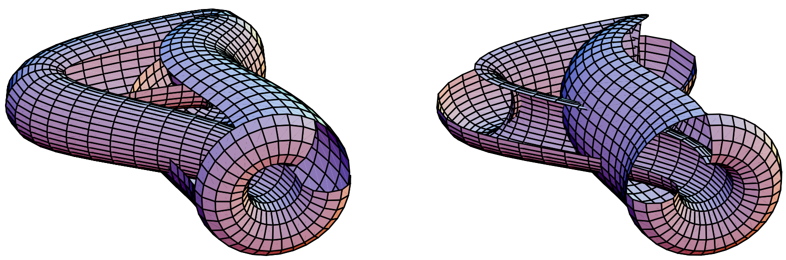


Figure 11.13: Open views displaying the self-intersection

To see that this new Klein bottle $\mathcal{K}_2$ is (as a surface in $\mathbb{R}^3$) distinct from the Klein bottle $\mathcal{K}_1$ formed by twisting figure eights, consider a neighborhood $\mathcal{V}_2$ of the self-intersection curve of $\mathcal{K}_2$ This neighborhood is again formed by rotating an 'X', but in an orientable manner. Thus the difference between $\mathcal{K}_1$ and $\mathcal{K}_2$ is that $\mathcal{V}_1$ is nonorientable, but $\mathcal{V}_2$ is orientable. Nevertheless, $\mathcal{K}_1$ and $\mathcal{K}_2$ are topologically the same surface, because each can be formed by identifying sides of squares, as described in Section 11.2. A parametrization of $\mathcal{K}_2$ is given in Notebook 11.

11.5 Realizations of the Real Projective Plane

Let

$$S^2 = S^2(1) = \{\, \mathbf{p} \mid \|\mathbf{p}\| = 1\}$$

be the sphere of unit radius in $\mathbb{R}^3$. Recall that the ***antipodal map*** of the sphere is the diffeomorphism

$$\begin{aligned} S^2 &\longrightarrow S^2 \\ \mathbf{p} &\mapsto -\mathbf{p}. \end{aligned}$$

The real projective plane $\mathbb{RP}^2$ can be defined as the set that results when antipodal points of S^2 are identified; thus

$$\mathbb{RP}^2 = \big\{\, \{\mathbf{p}, -\mathbf{p}\} \mid \|\mathbf{p}\| = 1 \,\big\}.$$

To realize the real projective plane as a surface in $\mathbb{R}^3$ let us look for a map of $\mathbb{R}^3$ into itself with a special property.

Definition 11.13. *A map* $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ *such that*

$$F(-\mathbf{p}) = F(\mathbf{p}) \tag{11.5}$$

is said to have the ***antipodal property***.

That we can use a map with the antipodal property to realize the real projective plane is a consequence of the following easily-proven lemma:

Lemma 11.14. *A map* $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ *with the antipodal property gives rise to a map* $\widetilde{F}\colon \mathbb{RP}^2 \to F\big(S^2\big) \subset \mathbb{R}^3$ *defined by*

$$\widetilde{F}(\{\mathbf{p}, -\mathbf{p}\}) = F(\mathbf{p}).$$

Thus we can realize $\mathbb{RP}^2$ as the image of S^2 under a map F which has the antipodal property. Moreover, any patch $\mathbf{x}\colon \mathcal{U} \to S^2$ will give rise to a patch $\widetilde{\mathbf{x}}\colon \mathcal{U} \to F\big(S^2\big)$ defined by

$$\widetilde{\mathbf{x}}(u,v) = F\big(\mathbf{x}(u,v)\big).$$

Ideally, one should choose F so that its Jacobian matrix is never zero. It turns out that this can be accomplished with quartic polynomials (see page 347), but the zeros can be kept to a minimum with well-chosen quadratic polynomials. We present three examples of maps with the antipodal property and associated realizations of $\mathbb{RP}^2$. The first is illustrated in Figure 11.14.

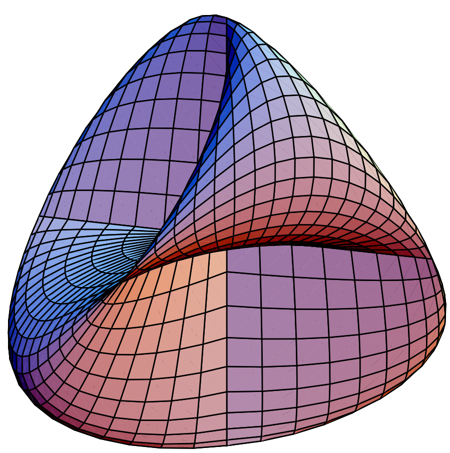


Figure 11.14: Steiner's Roman surface

Steiner's Roman Surface

When Jakob Steiner visited Rome in 1844 he developed the concept of a surface that now carries his name (see [Apéry, page 37]). It is a realization of the real projective plane. To describe it, we first define the map

$$\mathsf{romanmap}(x, y, z) = (yz,\ zx,\ xy) \tag{11.6}$$

from $\mathbb{R}^3$ to itself. It is obvious that romanmap has the antipodal property; it therefore induces a map $\mathbb{RP}^2 \to \mathsf{romanmap}\big(S^2\big)$. We call this image ***Steiner's Roman surface***[3] of radius 1. Moreover, we can plot a portion of $\mathsf{romanmap}(S^2)$ by composing romanmap with any patch on S^2, for example, the standard parametrization

$$\mathsf{sphere}[1]\colon\ (u, v) \mapsto \big(\cos v \cos u,\ \cos v \sin u,\ \sin v\big).$$

Then the composition $\mathsf{romanmap} \circ \mathsf{sphere}[1]$ parametrizes all of Steiner's Roman surface.

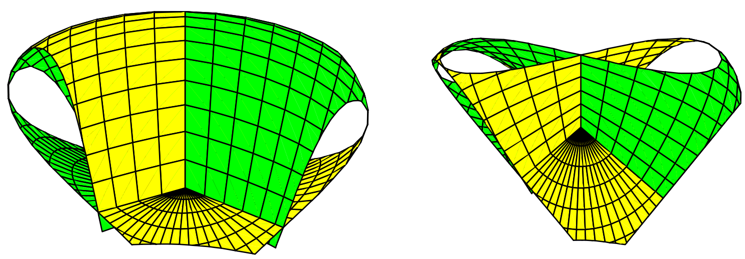


Figure 11.15: Two cut views of the Roman surface

Therefore, we define

$$\mathsf{roman}(u, v) = \tfrac{1}{2}\big(\sin u \sin 2v,\ \cos u \sin 2v,\ \sin 2u \cos^2 v\big)$$

Figure 11.15 cuts open Steiner's surface and displays the inside and outside in different colors to help visualize the intersections. Notice that the origin $(0, 0, 0)$ of $\mathbb{R}^3$ can be represented in one of the equivalent ways

$$\mathsf{romanmap}(\pm 1, 0, 0) = \mathsf{romanmap}(0, \pm 1, 0) = \mathsf{romanmap}(0, 0, \pm 1).$$

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3  Jakob Steiner (1796–1863). Swiss mathematician who was professor at the University of Berlin. Steiner did not learn to read and write until he was 14 and only went to school at the age of 18, against the wishes of his parents. Synthetic geometry was revolutionized by Steiner. He hated analysis as thoroughly as Lagrange hated geometry, according to [Cajori]. He believed that calculation replaces thinking while geometry stimulates thinking.

This confirms that the origin is a ***triple point*** of Steiner's surface, meaning that three separate branches intersect there. In addition, there are six singularities of the umbrella type illustrated in Figure 10.14. These occur at either end of each of the three 'axes' visible in Figure 11.15, and one is clearly visible on the left front in Figure 11.14.

The Cross Cap

A mapping with the antipodal property formed from homogeneous quadratic polynomials, and similar to (11.6), is given by

$$\mathsf{crosscapmap}(x,y,z)=\big(yz,\ 2xy,\ x^2-y^2\big).$$

We can easily get a parametrization of the cross cap, just as we did for Steiner's Roman surface. An explicit parametrization is

$$\mathsf{crosscap}(u,v)=\big(\tfrac{1}{2}\sin u\sin 2v,\ \sin 2u\cos^2 v,\ \cos 2u\cos^2 v\big). \tag{11.7}$$

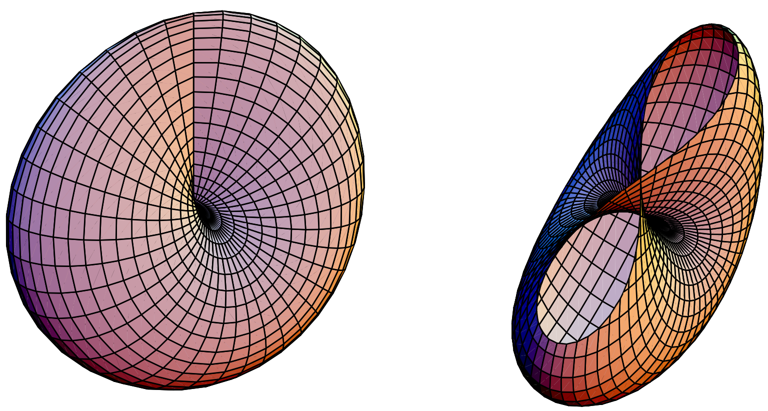

Figure 11.16: A cross cap and a cut view

The cross cap is perhaps the easiest realization of the sphere with antipodal points identified. Given a pair of points $\pm\mathbf{p}$ in S^2, either (i) both points belong to the equator (meaning $z=0$), or (ii) they correspond to a unique point in the 'southern hemisphere' (for which $z<0$). To obtain $\mathbb{RP}^2$ from S^2, it therefore suffices to remove the open northern hemisphere (for which $z>0$), and then deform the equator upwards towards where the north pole was, and 'sew' it to itself so that opposite points on (what was) the equator are placed next to each other. This requires a segment in which the surface intersects itself, but is the idea behind Figure 11.16.

Boy's Surface

A more complicated mapping

$$F = \big(F_1(x,y,z),\ F_2(x,y,z),\ F_3(x,y,z)\big),$$

with the antipodal property is one defining the remarkable surface discovered in 1901 by Boy[4], whose components are homogeneous quartic polynomials. This description of the surface was found by Apéry [Apéry], and is obtained by taking

$$\begin{aligned}
4F_1(x,y,z) &= (x+y+z)\big((x+y+z)^3+4(y-x)(z-y)(x-z)\big),\\
F_2(x,y,z) &= (2x^2-y^2-z^2)(x^2+y^2+x^2)+2yz(y^2-z^2)\\
&\qquad\qquad\qquad\qquad +zx(x^2-z^2)+xy(y^2-x^2),\\
\tfrac{1}{\sqrt{3}}F_3(x,y,z) &= (y^2-z^2)(x^2+y^2+z^2)+zx(z^2-x^2)+xy(y^2-x^2).
\end{aligned}$$

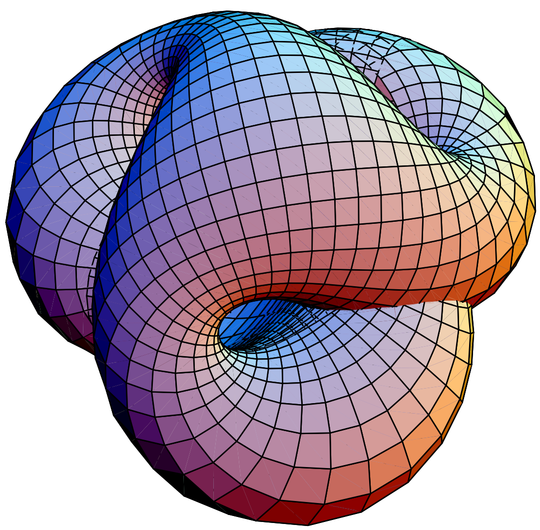

Figure 11.17: A view of Boy's surface

Figure 11.17 displays another realization of the surface, described in Notebook 11, using the concept of inversion from Section 20.4. Boy's surface can be covered by patches without singularities, in contrast to the two previous examples. Figure 11.18 helps to understand that curves of self-intersection meet in a triple point, at which the surface has a 3-fold symmetry, but there are no pinch points.

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[4]Werner Boy, a student of David Hilbert (see page 602).

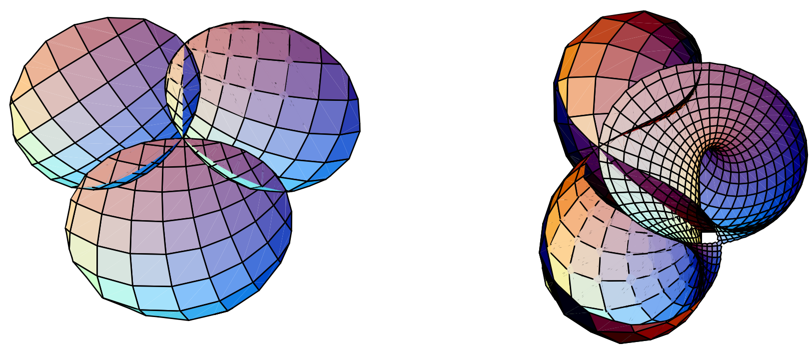

Figure 11.18: Two cut views of Boy's surface, revealing a triple self-intersection point

11.6 Twisted Surfaces

In this section we define a class of 'twisted' surfaces that generalize the Klein bottle and Möbius strip.

Definition 11.15. *Let $\boldsymbol{\alpha}$ be a plane curve with the property that*

$$\boldsymbol{\alpha}(-t) = -\boldsymbol{\alpha}(t) \tag{11.8}$$

*and write $\boldsymbol{\alpha}(t) = (\varphi(t), \psi(t))$. The **twisted surface** with **profile curve** $\boldsymbol{\alpha}$ and parameters a and b is defined by*

$$\begin{aligned}\mathsf{twist}[\boldsymbol{\alpha}, a, b](u,v) \;=\; & \Big(a + \cos(bu)\varphi(v) - \sin(bu)\psi(v)\Big)(\cos u,\ \sin u,\ 0)\\ & + \Big(\sin(bu)\varphi(v) + \cos(bu)\psi(v)\Big)(0,0,1).\end{aligned}$$

To understand the significance of this definition, take $a = 0$ and $b = 1/2$. The coordinate curve of the surface defined for each fixed value of u consists of a copy of $\boldsymbol{\alpha}$ mapping to the plane Π_u generated by the vectors $(\cos u, \sin u, 0)$ and $(0,0,1)$. This is the image of the xz-plane Π_0 under a rotation by u, and the curve $\boldsymbol{\alpha}$ in Π_u is rotated by the same angle u. Note that the curve $\boldsymbol{\alpha}$ starts from the xz-plane (corresponding to $u = 0$), and by the time it returns to the same plane ($u = 2\pi$) it has been rotated by only 180°, though the two traces coincide by (11.8). The resulting surface may or may not be orientable, depending on the choice of $\boldsymbol{\alpha}$.

We shall now show that both the Möbius strip and Klein bottle can be constructed in this way.

The Möbius Strip

Define $\boldsymbol{\alpha}$ by $\boldsymbol{\alpha}(t) = (at,\ 0)$. Then

$$\begin{aligned} \mathsf{twist}\big[\boldsymbol{\alpha}, a, \tfrac{1}{2}\big](u,v) &= a\Big(\cos u + v\cos\frac{u}{2}\cos u,\ \sin u + v\cos\frac{u}{2}\sin u,\ v\sin\frac{u}{2}\Big) \\ &= \mathsf{moebiusstrip}[a](u,v). \end{aligned}$$

The Klein Bottle

Define $\boldsymbol{\gamma}$ by $\boldsymbol{\gamma}(t) = (\sin t,\ \sin 2t)$. Then

$$\begin{aligned} \mathsf{twist}\big[\boldsymbol{\gamma}, a, \tfrac{1}{2}\big](u,v) &= \Big((a + \cos\frac{u}{2}\sin v - \sin\frac{u}{2}\sin 2v)\cos u, \\ &\quad (a + \cos\frac{u}{2}\sin v - \sin\frac{u}{2}\sin 2v)\sin u,\quad \sin\frac{u}{2}\sin v + \cos\frac{u}{2}\sin 2v\Big) \\ &= \mathsf{kleinbottle}[a](u,v). \end{aligned}$$

Twisted Surfaces of Lissajous Curves

Instead of twisting a figure eight around a circle we can twist a Lissajous curve. The latter is parametrized by

$$\mathsf{lissajous}[n,d,a,b](t) = \big(a\sin(nt+d),\ b\sin t\big).$$

For example, $\mathsf{lissajous}[2,0,1,1]$ resembles a figure eight, and the associated twisted surface is a Klein bottle. Figure 11.19 (left) displays the trace of the curve $\boldsymbol{\beta} = \mathsf{lissajous}[4,0,1,1]$, whilst the right-hand side consists of the points

$$\mathsf{twist}[\boldsymbol{\beta}, 2, \tfrac{1}{2}](u,v) \qquad 0 \leqslant u, v < 2\pi,$$

describing another self-intersecting surface.

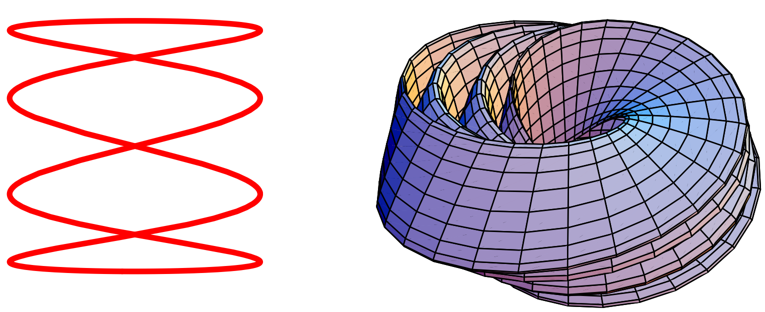

Figure 11.19: A Lissajous curve and twisted surface

11.7 Exercises

1. Prove Corollary 11.12.

M **2.** Sketch the image under the Gauss map of the region

$$\Big\{ \mathsf{hyperboloid2}[1,1,1](u,v) \mid 0 \leqslant u \leqslant 2\pi,\ 1 \leqslant v \Big\}$$

of the 2-sheeted hyperboloid. Show that the image of the whole hyperboloid of two sheets under the Gauss map consists of two antipodal disks, and is not therefore the entire sphere.

3. The image of the Gauss map of an ellipsoid is obviously the whole unit sphere. Nonetheless, plotting the result can be of interest since one can visualize the image of coordinate curves. Explain how the curves visible in Figure 11.20, left, describe the orientation of the unit normal to the patch $\mathsf{ellipsoid}[3,2,1]$. The associated approximation, right, is therefore an 'elliptical' polyhedral representation of the sphere.

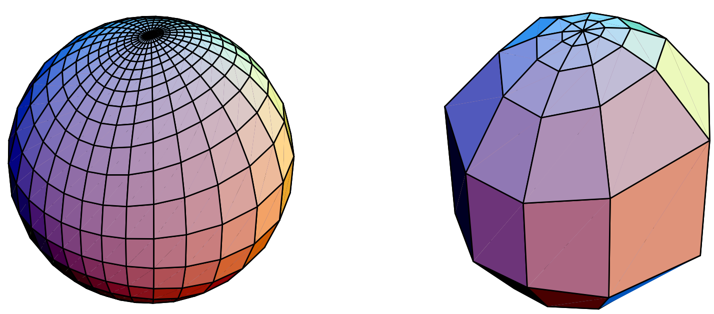

Figure 11.20: Gauss images of $\mathsf{ellipsoid}[3,2,1]$

M **4.** Let $\mathbf{x} = \mathsf{moebiusstrip}[1]$, and consider the two patches

$$\begin{aligned} &\mathbf{x}(u,v), \qquad 0 < u < \tfrac{3}{2}\pi,\ -1 < v < 1; \\ &\mathbf{x}(u,v), \qquad -\pi < u < \tfrac{1}{2}\pi,\ -1 < v < 1. \end{aligned}$$

Compute the unit normal (see (11.4)), and prove that the two normals have equal and opposite signs on their two respective regions of intersection. Deduce that the Möbius strip is nonorientable.

5. Verify that Equation (11.7) is indeed the composition of sphere$[a]$ with crosscapmap.

M **6.** A cross cap can be defined implicitly by the equation

$$(ax^2 + by^2)(x^2 + y^2 + z^2) - 2z(x^2 + y^2) = 0.$$

Plot this surface using ImplicitPlot3D.

7. Let us put a family of figure eight curves into $\mathbb{R}^3$ in such a way that the first and last figures eight reduce to points. This defines a surface, which we call a ***pseudo cross cap***, shown in Figure 11.21. It can be parametrized as follows:

$$\text{pseudocrosscap}(u, v) = \big((1 - u^2)\sin v,\ (1 - u^2)\sin 2v,\ u\big).$$

Show that pseudocrosscap is not regular at the points $(0, 0, \pm 1)$. Why should the pseudo cross cap be considered orientable, even though it is not a regular surface?

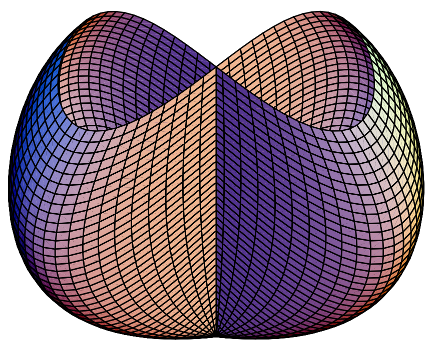

Figure 11.21: A pseudo cross cap

8. Show that the image under crosscapmap of a torus centered at the origin is topologically a Klein bottle. To do this, explain how points of the image can be made to correspond to those in Figure 11.4 with the right-hand identifications.