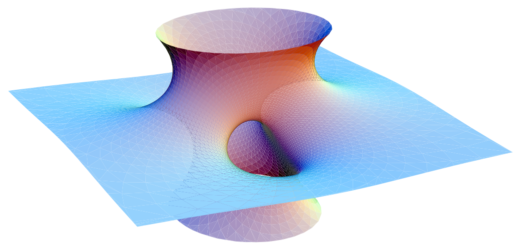
Alfred Gray



Modern Differential Geometry of
Curves and Surfaces
with Mathematica®

Third Edition by
Elsa Abbena and Simon Salamon

Preface to the Second Edition[1]

Modern Differential Geometry of Curves and Surfaces is a traditional text, but it uses the symbolic manipulation program *Mathematica*. This important computer program, available on PCs, Macs, NeXTs, Suns, Silicon Graphics Workstations and many other computers, can be used very effectively for plotting and computing. The book presents standard material about curves and surfaces, together with accurate interesting pictures, *Mathematica* instructions for making the pictures and *Mathematica* programs for computing functions such as curvature and torsion.

Although Curves and Surfaces makes use of *Mathematica*, the book should also be useful for those with no access to *Mathematica*. All calculations mentioned in the book can in theory be done by hand, but some of the longer calculations might be just as tedious as they were for differential geometers in the 19$^{\text{th}}$ century. Furthermore, the pictures (most of which were done with *Mathematica*) elucidate concepts, whether or not *Mathematica* is used by the reader.

The main prerequisite for the book is a course in calculus, both single variable and multi-variable. In addition, some knowledge of linear algebra and a few basic concepts of point set topology are needed. These can easily be obtained from standard sources. *No computer knowledge is presumed.* In fact, the book provides a good introduction to *Mathematica*; the book is compatible with both versions 2.2 and 3.0. For those who want to use Curves and Surfaces to learn *Mathematica*, it is advisable to have access to Wolfram's book Mathematica for reference. (In version 3.0 of *Mathematica*, Wolfram's book is available through the help menus.)

Curves and Surfaces is designed for a traditional course in differential geometry. At an American university such a course would probably be taught at the junior-senior level. When I taught a one-year course based on Curves and Surfaces at the University of Maryland, some of my students had computer experience, others had not. All of them had acquired sufficient knowledge of *Mathematica* after one week. I chose not to have computers in my classroom because I needed the classroom time to explain concepts. I assigned all of the problems at the end of each chapter. The students used workstations, PCs

[1] This is a faithful reproduction apart from the updating of chapter references. It already incorporated the Preface to the First Edition dating from 1993.

and Macs to do those problems that required *Mathematica*. They either gave me a printed version of each assignment, or they sent the assignment to me by electronic mail.

Symbolic manipulation programs such as *Mathematica* are very useful tools for differential geometry. Computations that are very complicated to do by hand can frequently be performed with ease in *Mathematica*. However, they are no substitute for the theoretical aspects of differential geometry. So Curves and Surfaces presents theory and uses *Mathematica* programs in a complementary way.

Some of the aims of the book are the following.

- To show how to use *Mathematica* to plot many interesting curves and surfaces, more than in the standard texts. Using the techniques described in Curves and Surfaces, students can understand concepts geometrically by plotting curves and surfaces on a monitor and then printing them. The effect of changes in parameters can be strikingly portrayed.

- The presentation of pictures of curves and surfaces that are informative, interesting and accurate. The book contains over 400 illustrations.

- The inclusion of as many topics of the classical differential geometry and surfaces as possible. In particular, the book contains many examples to illustrate important theorems.

- Alleviation of the drudgery of computing things such as the curvature and torsion of a curve in space. When the curvature and torsion become too complicated to compute, they can be graphed instead. There are more than 175 miniprograms for computing various geometric objects and plotting them.

- The introduction of techniques from numerical analysis into differential geometry. *Mathematica* programs for numerical computation and drawing of geodesics on an arbitrary surface are given. Curves can be found numerically when their torsion and curvature are specified.

- To place the material in perspective through informative historical notes. There are capsule biographies with portraits of over 75 mathematicians and scientists.

- To introduce interesting topics that, in spite of their simplicity, deserve to be better known. I mention triply orthogonal systems of surfaces (Chapter 19), Björling's formula for constructing a minimal surface containing a given plane curve as a geodesic (Chapter 22) and canal surfaces and cyclides of Dupin as Maxwell discussed them (Chapter 20).

- To develop a dialect of *Mathematica* for handling functions that facilitates the construction of new curves and surfaces from old. For example, there is a simple program to generate a surface of revolution from a plane curve.

- To provide explicit definitions of curves and surfaces. Over 300 *Mathematica* definitions of curves and surfaces can be used for further study.

The approach of Curves and Surfaces is admittedly more computational than is usual for a book on the subject. For example, Brioschi's formula for the Gaussian curvature in terms of the first fundamental form can be too complicated for use in hand calculations, but *Mathematica* handles it easily, either through computations or through graphing the curvature. Another part of *Mathematica* that can be used effectively in differential geometry is its special function library. For example, nonstandard spaces of constant curvature can be defined in terms of elliptic functions and then plotted.

Frequently, I have been asked if new mathematical results can be obtained by means of computers. Although the answer is generally no, it is certainly the case that computers can be an effective supplement to pure thought, because they allow experimentation and the graphs provide insights into complex relationships. I hope that many research mathematicians will find Curves and Surfaces useful for that purpose. Two results that I found with the aid of *Mathematica* are the interpretation of torsion in terms of tube twisting in Chapter 7 and the construction of a conjugate minimal surface without integration in Chapter 22. I have not seen these results in the literature, but they may not be new.

The programs in the book, as well as some descriptive *Mathematica* notebooks, will eventually be available on the web.

Sample Course Outlines

There is ample time to cover the whole book in three semesters at the undergraduate level or two semesters at the graduate level. Here are suggestions for courses of shorter length.

- One semester undergraduate differential geometry course: Chapters 1, 2, 7, 9 – 13, parts of 14 – 16, 27.

- Two semester undergraduate differential geometry course: Chapters 1 – 3, 9 – 19, 27.

- One semester graduate differential geometry course: Chapters 1, 2, 7 – 13, 15 – 19, parts of 22 – 27.

- One semester course on curves and their history: Chapters 1 – 8.

- One semester course on *Mathematica* and graphics Chapters $1-6$, $7-11$, parts of $14-19$, 23, and their notebooks.

I have tried to include more details than are usually found in mathematics books. This, plus the fact that the *Mathematica* programs can be used to elucidate theoretical concepts, makes the book easy to use for independent study.

Curves and Surfaces is an ongoing project. In the course of writing this book, I have become aware of the vast amount of material that was well-known over a hundred years ago, but now is not as popular as it should be. So I plan to develop a web site, and to write a problem book to accompany the present text. Spanish, German, Japanese and Italian versions of Curves and Surfaces are already available.

Graphics

Although *Mathematica* graphics are very good, and can be used to create *QuickTime* movies, the reader may also want to consider the following additional display methods:

- *Acrospin* is an inexpensive easy-to-use program that works on even the humblest PC.

- *Geomview* is a program for interactive display of curves and surfaces. It works on most unix-type systems, and can be freely downloaded from http://www.geomview.org

- *Dynamic Visualizer* is an add-on program to *Mathematica* that allows interactive display. Details are available from http://www.wolfram.com

- *AVS* programs (see the commercial site http://www.avs.com) have been developed by David McNabb at the University of Maryland (http://www.umd.edu) for spectacular stereo three-dimensional images of the surfaces described in this book.

A Perspective

Mathematical trends come and go. R. Osserman in his article ('The Geometry Renaissance in America: 1938–1988' in A Century of Mathematics in America, volume 2, American Mathematical Society, Providence, 1988) makes the point that in the 1950s when he was a student at Harvard, algebra dominated mathematics, the attention given to analysis was small, and the interest in differential geometry was converging to zero.

It was not always that way. In the last half of the 19^{th} century surface theory was a very important area of mathematics, both in research and teaching. Brill, then Schilling, made an extensive number of plaster models available to the mathematical public. Darboux's Leçons sur la Théorie Générale des Surfaces and Bianchi's Lezioni di Geometria Differenziale were studied intensely. I attribute the decline of differential geometry, especially in the United States, to the rise of tensor analysis. Instead of drawing pictures it became fashionable to raise and lower indices.

I strongly feel that pictures need to be much more stressed in differential geometry than is presently the case. It is unfortunate that the great differential geometers of the past did not share their extraordinary intuitions with others by means of pictures. I hope that the present book contributes in some way to returning the differential geometry of curves and surfaces to its proper place in the mathematics curriculum.

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Alfred Gray
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Preface to the Third Edition

Most of the material of this book can be found, in one form or another, in the Second Edition. The exceptions to this can be divided into three categories.

Firstly, a number of modifications and new items had been prepared by Alfred Gray following publication of the Second Edition, and we have been able to incorporate some of these in the Third Edition. The most obvious is Chapter 21. In addition, we have liberally expanded a number of sections by means of additional text or graphics, where we felt that this was warranted.

The second is Chapter 23, added by the editors to present the popular theory of quaternions. This brings together many of the techniques in the rest of the book, combining as it does the theory of space curves and surfaces.

The third concerns the *Mathematica* code presented in the notebooks. Whilst this is closely based on that written by the author and displayed in previous editions, many programs have been enhanced and sibling ones added. This is to take account of the progressive presentation that *Mathematica* notebooks offer, and a desire to publish instructions to generate every figure in the book.

The new edition does differ notably from the previous one in the *manner* in which the material is organized. All *Mathematica* code has been separated from the body of the text and organized into notebooks, so as to give readers interactive access to the material. There is one notebook to accompany each chapter, and it contains relevant programs in parallel with the text, section by section. An abridged version is printed at the end of the chapter, for close reference and to present a fair idea of the programs that ride in tandem with the mathematics. The distillation of computer code into notebooks also makes it easier to conceive of rewriting the programs in a different language, and a project is underway to do this for *Maple*.

The full notebooks can be downloaded from the publisher's site

http://www.crcpress.com

Their organization and layout is discussed in more detail in Notebook 0 below. They contain no output, as this can be generated at will. All the figures in the book were compiled automatically by merely evaluating the notebooks chapter by chapter, and this served to 'validate' the notebooks using Version 5.1 of *Mathematica*. It is the editors' intention to build up an on-line database of solutions to the exercises at the end of each chapter. Those marked M are designed to be solved with the help of a suitable *Mathematica* program.

The division of the material into chapters and the arrangement of later chapters has also been affected by the presence of the notebooks. We have chosen to shift the exposition of differentiable manifolds and abstract surfaces towards the end of the book. In addition, there are a few topics in the Second Edition that have been relegated to electronic form in an attempt to streamline the volume. This applies to the fundamental theorems of surfaces, and some more advanced material on minimal surfaces. (It is also a recognition of other valuable sources, such as [Oprea2] to mention one.) The Chapter Scheme overleaf provides an idea of the relationship between the various topics; touching blocks represent a group of chapters that are probably best read in numerical order, and $21, 22, 27$ are independent peaks to climb.

In producing the Third Edition, the editors were fortunate in having ready access to electronic versions of the author's files. For this, they are grateful to Mike Mezzino, as well as the editors of the Spanish and Italian versions, Marisa Ferńandez and Renzo Caddeo. Above all, they are grateful to Mary Gary and Bob Stern for entrusting them with the editing task.

As regards the detailed text, the editors acknowledge the work of Daniel Drucker, who diligently scanned the Second Edition for errors, and provided helpful comments on much of the Third Edition. Some of the material was used for the course 'Geometrical Methods for Graphics' at the University of Turin in 2004 and 2005, and we thank students of that course for improving some of the figures and associated computer programs. We are also grateful to Simon Chiossi, Sergio Console, Antonio Di Scala, Anna Fino, Gian Mario Gianella and Sergio Garbiero for proof-reading parts of the book, and to John Sullivan and others for providing photo images.

The years since publication of the First Edition have seen a profileration of useful websites containing information on curves and surfaces that complements the material of this book. In particular, the editors acknowledge useful visits to the Geometry Center's site http://www.geom.uiuc.edu, and Richard Palais' pages http://vmm.math.uci.edu/3D-XplorMath. Finally, we are grateful for support from Wolfram Research.

The editors initially worked with Alfred Gray at the University of Maryland in 1979, and on various occasions subsequently. Although his more abstract research had an enormous influence on many branches of differential geometry (a hint of which can be found in the volume [FeWo]), we later witnessed the pleasure he experienced in preparing material for both editions of this book. We hope that our more modest effort for the Third Edition will help extend this pleasure to others.

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Chapter Scheme