Chapter 14

Ruled Surfaces

We describe in this chapter the important class of surfaces, consistng of those which contain infinitely many straight lines. The most obvious examples of ruled surfaces are cones and cylinders (see pages 314 and 375).

Ruled surfaces are arguably the easiest of all surfaces to parametrize. A chart can be defined by choosing a curve in $\mathbb{R}^3$ and a vector field along that curve, and the resulting parametrization is linear in one coordinate. This is the content of Definition 14.1, that provides a model for the definition of both the helicoid and the Möbius strip (described on pages 376 and 339).

Several quadric surfaces, including the hyperbolic paraboloid and the hyperboloid of one sheet are also ruled, though this fact does not follow so readily from their definition. These particular quadric surfaces are in fact 'doubly ruled' in the sense that they admit two one-parameter families of lines. In Section 14.1, we explain how to parametrize them in such a way as to visualize the straight line rulings. We also define the Plücker conoid and its generalizations.

It is a straightforward matter to compute the Gaussian and mean curvature of a ruled surface. This we do in Section 14.2, quickly turning attention to flat ruled surfaces, meaning ruled surfaces with zero Gaussian curvature. There are three classes of such surfaces, the least obvious but most interesting being the class of tangent developables.

Tangent developables are the surfaces swept out by the tangent lines to a space curve, and the two halves of each tangent line effectively divide the surface into two sheets which meet along the curve in a singular fashion described by Theorem 14.7. This behaviour is examined in detail for Viviani's curve, and the linearity inherent in the definition of a ruled surface enables us to highlight the useful role played by certain plane curves in the description of tangent developables.

In the final Section 14.4, we shall study a class of surfaces that will help us to better understand how the Gaussian curvature varies on a ruled surface.

14.1 Definitions and Examples

A ruled surface is a surface generated by a straight line moving along a curve.

Definition 14.1. *A* ***ruled surface*** $\mathcal{M}$ *in* $\mathbb{R}^3$ *is a surface which contains at least one 1-parameter family of straight lines. Thus a ruled surface has a parametrization* $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ *of the form*

$$\mathbf{x}(u,v) = \boldsymbol{\alpha}(u) + v\,\boldsymbol{\gamma}(u), \tag{14.1}$$

where $\boldsymbol{\alpha}$ *and* $\boldsymbol{\gamma}$ *are curves in* $\mathbb{R}^3$*. We call* $\mathbf{x}$ *a* ***ruled patch****. The curve* $\boldsymbol{\alpha}$ *is called the* ***directrix*** *or* ***base curve*** *of the ruled surface, and* $\boldsymbol{\gamma}$ *is called the* ***director curve****. The* ***rulings*** *are the straight lines* $v \mapsto \boldsymbol{\alpha}(u) + v\,\boldsymbol{\gamma}(u)$*.*

In using (14.1), we shall assume that $\boldsymbol{\alpha}'$ is never zero, and that $\boldsymbol{\gamma}$ is not identically zero.

We shall see later that any straight line in a surface is necessarily an asymptotic curve, pointing as it does in direction for which the normal curvature vanishes (see Definition 13.9 on page 390 and Corollary 18.6 on page 560). It follows that the rulings are asymptotic curves. Sometimes a ruled surface $\mathcal{M}$ has two distinct ruled patches on it, so that a ruling of one patch does not belong to the other patch. In this case, we say that $\mathbf{x}$ is ***doubly ruled***.

We now investigate a number of examples.

The Helicoid and Möbius Strip Revisited

We can rewrite the definition (12.27) as

$$\mathsf{helicoid}[a,b](u,v) = \boldsymbol{\alpha}(u) + v\,\boldsymbol{\gamma}(u),$$

where

$$\begin{cases} \boldsymbol{\alpha}(u) = (0,\ 0,\ bu), \\ \boldsymbol{\gamma}(u) = a(\cos u,\ \sin u,\ 0). \end{cases} \tag{14.2}$$

In this way, $\mathsf{helicoid}[a,b]$ is a ruled surface whose base curve has the z-axis as its trace, and director curve $\boldsymbol{\gamma}$ that describes a circle.

In a similar fashion, our definition (11.3), page 339, of the Möbius strip becomes

$$\mathsf{moebiusstrip}(u,v) = \boldsymbol{\alpha}(u) + v\,\boldsymbol{\gamma}(u),$$

where

$$\begin{cases} \boldsymbol{\alpha}(u) = (\cos u,\ \sin u,\ 0), \\ \boldsymbol{\gamma}(u) = \Big(\cos\dfrac{u}{2}\cos u,\ \cos\dfrac{u}{2}\sin u,\ \sin\dfrac{u}{2}\Big). \end{cases} \tag{14.3}$$

This time, it is the circle that is the *base* curve of the ruled surface. The Möbius strip has a director curve $\boldsymbol{\gamma}$ which lies on a unit sphere, shown in two equivalent ways in Figure 14.1.

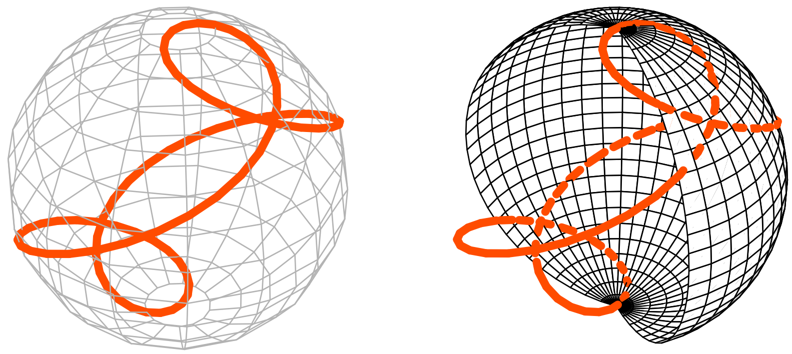

Figure 14.1: Director curve for the Möbius strip

The curve $\boldsymbol{\gamma}$ is not one that we encountered in the study of curves on the sphere in Section 8.4. It does however have the interesting property that whenever $\mathbf{p}$ belongs to its trace, so does the antipodal point $-\mathbf{p}$.

The Hyperboloid of One Sheet

We have seen that the *elliptical hyperboloid of one sheet* is defined nonparametrically by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1. \tag{14.4}$$

Planes perpendicular to the z-axis intersect the surface in ellipses, while planes parallel to the z-axis intersect it in hyperbolas. We gave the standard parametrization of the hyperboloid of one sheet on page 313, but this has the disadvantage of not showing the rulings.

Let us show that the hyperboloid of one sheet is a doubly-ruled surface by finding two ruled patches on it. This can be done by fixing $a,b,c>0$ and defining

$$\mathbf{x}^{\pm}(u,v)=\boldsymbol{\alpha}(u)\pm v\,\boldsymbol{\alpha}'(u)+v\,(0,0,c),$$

where

$$\boldsymbol{\alpha}(u)=\mathsf{ellipse}[a,b](u)=(a\cos u,\ b\sin u,\ 0). \tag{14.5}$$

is the standard parametrization of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

in the xy-plane. It is readily checked that

$$\mathbf{x}^{\pm}(u,v)=\Big(a(\cos u\mp v\sin u),\ b(\sin u\pm v\cos u),\ cv\Big), \tag{14.6}$$

and that both $\mathbf{x}^+$ and $\mathbf{x}^-$ are indeed parametrizations of the hyperboloid (14.4). Hence the elliptic hyperboloid is doubly ruled; in both cases, the base curve can be taken to be the ellipse (14.5). We also remark that $\mathbf{x}^+$ can be obtained from $\mathbf{x}^-$ by simultaneously changing the signs of the parameter c and the variable v.

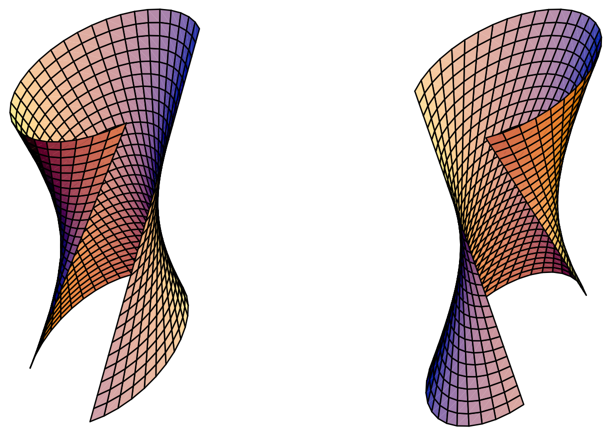

Figure 14.2: Rulings on a hyperboloid of one sheet

The Hyperbolic Paraboloid

The hyperbolic paraboloid is defined nonparametrically by

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2} \tag{14.7}$$

(though the constants a, b here are different from those in (10.16)). It is doubly ruled, since it can be parametrized in the two ways

$$\begin{aligned} \mathbf{x}^{\pm}(u,v) &= (au,\ 0,\ u^2) + v(a,\ \pm b,\ 2u) \\ &= (a(u+v),\ \pm bv,\ u^2 + 2uv). \end{aligned}$$

Although we have tacitly assumed that $b > 0$, both parametrizations can obviously be obtained from the same formula by changing the sign of b, a fact exploited in Notebook 14.

A special case corresponds to taking $a = b$ and carrying out a rotation by $\pi/2$ about the z-axis, so as to define new coordinates

$$\begin{cases} \bar{x} = \dfrac{1}{\sqrt{2}}(x - y) \\ \bar{y} = \dfrac{1}{\sqrt{2}}(x + y), \\ \bar{z} = z. \end{cases}$$

This transforms (14.7) into the equation $a^2\bar{z} = 2\bar{x}\bar{y}$.

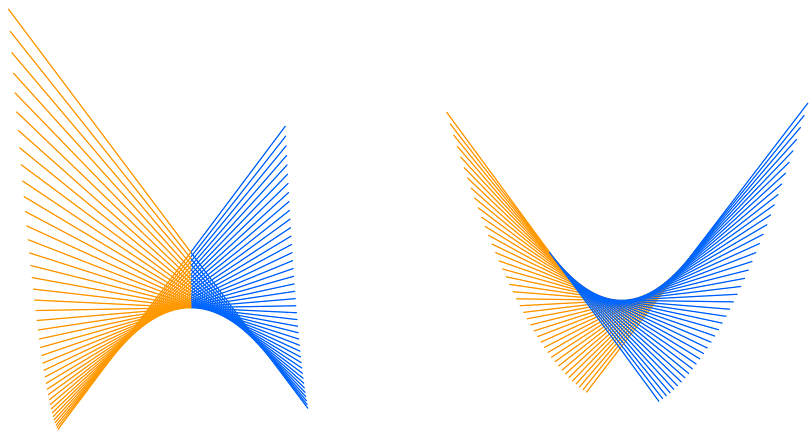

Figure 14.3: Rulings on a hyperbolic paraboloid

Plücker's Conoid

The surface defined nonparametrically by

$$z = \frac{2xy}{x^2 + y^2}$$

is called ***Plücker's conoid***[1] [BeGo, pages 352,363]. Its Monge parametrization is obviously

$$\mathsf{pluecker}(u, v) = \left(u,\ v,\ \frac{2uv}{u^2 + v^2}\right) \tag{14.8}$$

A computer plot using (14.8) does not reveal any rulings (Figure 14.4, left). To see that this conoid is in fact ruled, one needs to convert (u, v) to polar coordinates, as explained in Section 10.4. Let us write

$$\begin{aligned} \mathsf{pluecker}(r\cos\theta,\ r\sin\theta) &= (r\cos\theta,\ r\sin\theta,\ 2\cos\theta\sin\theta) \\ &= (0,\ 0,\ \sin 2\theta) + r(\cos\theta,\ \sin\theta, 0). \end{aligned}$$

Thus, the z-axis acts as base curve and the circle $\theta \mapsto (\cos\theta, \sin\theta)$ as director curve for the parametrization in terms of (r, θ). Using this parametrization, the rulings are clearly visible passing through the z-axis (Figure 14.4, right).

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[1]  Julius Plücker (1801–1868). German mathematician. Until 1846 Plücker's original research was in analytic geometry, but starting in 1846 as professor of physics in Bonn, he devoted his energies to experimental physics for nearly twenty years. At the end of his life he returned to mathematics, inventing line geometry.

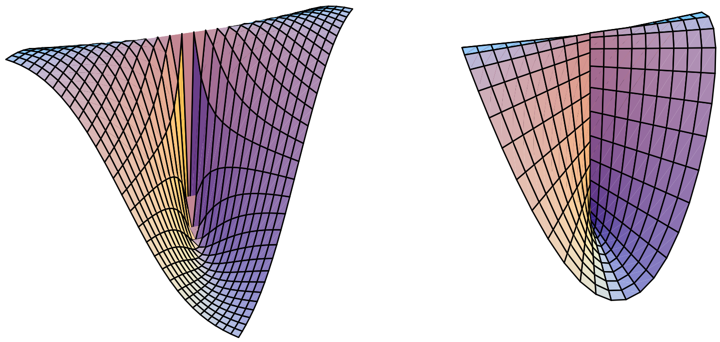

Figure 14.4: Parametrizations of the Plücker conoid

It is now an easy matter to define a generalization of Plücker's conoid that has n folds instead of 2:

$$\text{pluckerpolar}[n](r,\theta) = (r\cos\theta,\ r\sin\theta,\ \sin n\theta). \tag{14.9}$$

Each surface pluckerpolar$[n]$ is a ruled surface with the rulings passing through the z-axis. For a generalization of (14.9) that includes a variant of the monkey saddle as a special case, see Exercise 5.

Even more general than (14.9) is the ***right conoid***, which is a ruled surface with rulings parallel to a plane and passing through a line that is perpendicular to the plane. For example, if we take the plane to be the xy-plane and the line to be the z-axis, a right conoid will have the form

$$\text{rightconoid}[\vartheta, h](u,v) = \big(v\cos\vartheta(u),\ v\sin\vartheta(u),\ h(u)\big). \tag{14.10}$$

This is investigated in Notebook 14 and illustrated in Figure 14.12 on page 449.

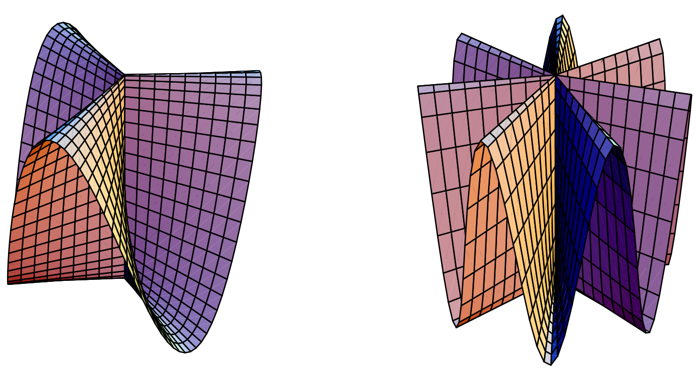

Figure 14.5: The conoids pluckerpolar$[n]$ with $n = 3$ and 7

14.2 Curvature of a Ruled Surface

Lemma 14.2. *The Gaussian curvature of a ruled surface $\mathcal{M} \subset \mathbb{R}^3$ is everywhere nonpositive.*

Proof. If $\mathbf{x}$ is a ruled patch on $\mathcal{M}$, then $\mathbf{x}_{vv} = 0$; consequently $g = 0$. Hence it follows from Theorem 13.25, page 400, that

$$K = \frac{-f^2}{EG - F^2} \leqslant 0. \blacksquare \tag{14.11}$$

As an example, we consider the parametrization (14.3) of the Möbius strip. The coefficients of its first and second fundamental forms, its Gaussian and mean curvatures are all computed in Notebook 14. For example, with $a = 2$ as in Figure 14.6, we obtain

$$K(u,v) = -\frac{16}{\left(16 + 3v^2 + 16v\cos\frac{u}{2} + 2v^2\cos u\right)^2}$$

$$H(u,v) = -2\frac{\left(8 + 2v^2 + 8v\cos\frac{u}{2} + v^2\cos u\right)\sin\frac{u}{2}}{\left(16 + 3v^2 + 16v\cos\frac{u}{2} + 2v^2\cos u\right)^{3/2}}.$$

It is clear that, for this particular parametrization, K is never zero. The right-hand plot of K in Figure 14.6 exhibits its minima, which correspond to definite regions of the strip which are especially distorted.

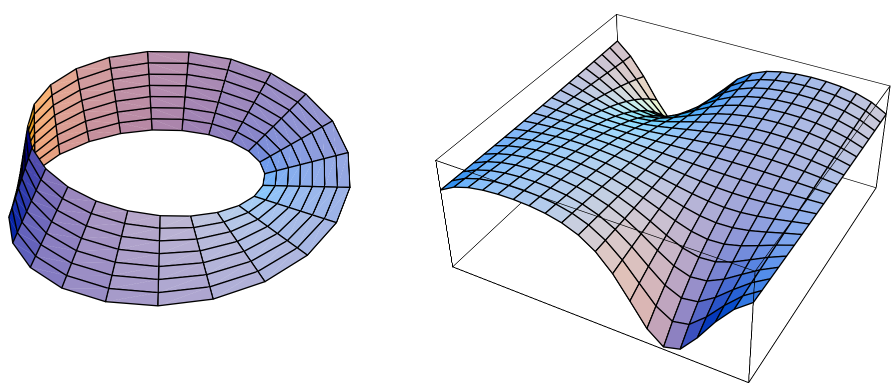

Figure 14.6: Curvature of a Möbius strip

Recall that a ***flat surface*** is a surface whose Gaussian curvature is everywhere zero. Such a surface is classically called a ***developable surface***. An immediate consequence of (14.11) is the following criterion for flatness.

Corollary 14.3. *$\mathcal{M}$ is flat if and only if $f = 0$.*

The most obvious examples of flat surfaces, other than the plane, are circular cylinders and cones (see the discussion after Figure 14.7). Paper models of both can easily be constructed by bending a sheet of paper and, as we shall see in Section 17.2, this operation leaves the Gaussian curvature unchanged and identically zero. Circular cylinders and cones are subsumed into parts (ii) and (iii) of the following list of flat ruled surfaces.

Definition 14.4. *Let $\mathcal{M} \subset \mathbb{R}^3$ be a surface. Then:*

(i) *$\mathcal{M}$ is said to be the* ***tangent developable*** *of a curve $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^3$ if $\mathcal{M}$ can be parametrized as*

$$\mathbf{x}(u,v) = \boldsymbol{\alpha}(u) + v\,\boldsymbol{\alpha}'(u); \tag{14.12}$$

(ii) *$\mathcal{M}$ is a* ***generalized cylinder*** *over a curve $\boldsymbol{\alpha}\colon (a,b) \mapsto \mathbb{R}^3$ if $\mathcal{M}$ can be parametrized as*

$$\mathbf{y}(u,v) = \boldsymbol{\alpha}(u) + v\,\mathbf{q},$$

where $\mathbf{q} \in \mathbb{R}^3$ is a fixed vector;

(iii) *$\mathcal{M}$ is a* ***generalized cone*** *over a curve $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^3$, provided $\mathcal{M}$ can be parametrized as*

$$\mathbf{z}(u,v) = \mathbf{p} + v\,\boldsymbol{\alpha}(u),$$

where $\mathbf{p} \in \mathbb{R}^3$ is fixed (it can be interpreted as the ***vertex*** *of the cone).*

Our next result gives criteria for the regularity of these three classes of surfaces. We base it on Lemma 10.18.

Lemma 14.5. (i) *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^3$ be a regular curve whose curvature $\boldsymbol{\kappa}[\boldsymbol{\alpha}]$ is everywhere nonzero. The tangent developable $\mathbf{x}$ of $\boldsymbol{\alpha}$ is regular everywhere except along $\boldsymbol{\alpha}$.*

(ii) *A generalized cylinder $\mathbf{y}(u,v) = \boldsymbol{\alpha}(u) + v\,\mathbf{q}$ is regular wherever $\boldsymbol{\alpha}' \times \mathbf{q}$ does not vanish.*

(iii) *A generalized cone $\mathbf{z}(u,v) = \mathbf{p} + v\,\boldsymbol{\alpha}(u)$ is regular wherever $v\,\boldsymbol{\alpha} \times \boldsymbol{\alpha}'$ is nonzero, and is never regular at its vertex.*

Proof. For a tangent developable $\mathbf{x}$, we have

$$(\mathbf{x}_u \times \mathbf{x}_v)(u,v) = (\boldsymbol{\alpha}' + v\,\boldsymbol{\alpha}'') \times \boldsymbol{\alpha}' = v\,\boldsymbol{\alpha}'' \times \boldsymbol{\alpha}'. \tag{14.13}$$

If $\boldsymbol{\kappa}[\boldsymbol{\alpha}] \neq 0$, then $\boldsymbol{\alpha}'' \times \boldsymbol{\alpha}'$ is everywhere nonzero by (7.26). Thus (14.13) implies that $\mathbf{x}$ is regular whenever $v \neq 0$. The other statements have similar proofs. ∎

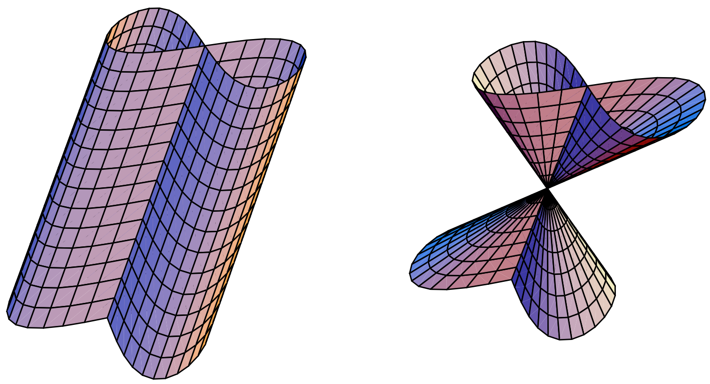

Figure 14.7: Cylinder and cone over a figure eight

Figure 14.7 illustrates cases (ii) and (iii) of Definition 14.4.

Proposition 14.6. *If $\mathcal{M}$ is a tangent developable, a generalized cylinder or a generalized cone, then $\mathcal{M}$ is flat.*

Proof. By Corollary 14.3, it suffices to show that $f = 0$ in each of the three cases. For a tangent developable $\mathbf{x}$, we have $\mathbf{x}_u = \boldsymbol{\alpha}' + v\boldsymbol{\alpha}''$, $\mathbf{x}_v = \boldsymbol{\alpha}'$ and $\mathbf{x}_{uv} = \boldsymbol{\alpha}''$; hence the triple product

$$f = \frac{[\boldsymbol{\alpha}''\ (\boldsymbol{\alpha}' + v\boldsymbol{\alpha}'')\ \boldsymbol{\alpha}']}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

is zero. It is obvious that $f = 0$ for a generalized cylinder $\mathbf{y}$, since $\mathbf{y}_{uv} = 0$. Finally, for a generalized cone, we compute

$$f = \frac{[\boldsymbol{\alpha}'\ v\boldsymbol{\alpha}'\ \boldsymbol{\alpha}]}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = 0. \ \blacksquare$$

The general developable surface is in some sense the union of tangent developables, generalized cylinders and generalized cones. This remark is explained in the paragraph directly after the proof of Theorem 14.15.

The ***normal surface*** and the ***binormal surface*** to a space curve $\boldsymbol{\alpha}$ can be defined by mimicking the construction of the tangent developable. It suffices to replace the tangent vector $\boldsymbol{\alpha}'(u)$ in (14.12) by the unit normal or binormal vector $\mathbf{N}(u)$ or $\mathbf{B}(u)$. However, unlike a tangent developable, the normal and binormal surfaces are not in general flat.

Consider the normal and binormal surfaces to Viviani's curve, defined in Section 7.5. Output from Notebook 14 gives the following expression for the binormal surface of viviani[1]:

$$\left(1+\cos u+\frac{v\left(3\sin\frac{u}{2}+\sin\frac{3u}{2}\right)}{\sqrt{26+6\cos u}},\ \frac{-2\sqrt{2}\,v\cos^3\frac{u}{2}}{\sqrt{13+3\cos u}}+\sin u,\right.$$
$$\left.\frac{2\sqrt{2}\,v}{\sqrt{13+3\cos u}}+2\sin\frac{u}{2}\right)$$

Figure 14.8 shows parts of the normal and binormal surfaces together, with the small gap representing Viviani's curve itself. The curvature of these surfaces is computed in Notebook 14, and shown to be nonzero.

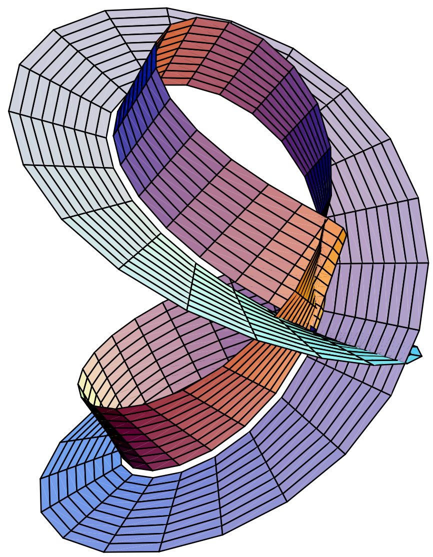

Figure 14.8: Normal and binormal surfaces to Viviani's curve

14.3 Tangent Developables

Consider the surface

$$\mathsf{tandev}[\boldsymbol{\alpha}]=\boldsymbol{\alpha}(u)+v\,\boldsymbol{\alpha}'(u);$$

this was first defined in (14.12), though we now use notation from Notebook 14. We know from Lemma 14.5 that $\mathsf{tandev}[\boldsymbol{\alpha}]$ is singular along the curve $\boldsymbol{\alpha}$. We prove next that it is made up of two sheets which meet along the trace of $\boldsymbol{\alpha}$ in a sharp edge, called the ***edge of regression***.

Theorem 14.7. *Let $\boldsymbol{\alpha}\colon (a,b)\to\mathbb{R}^3$ be a unit-speed curve with $a<0<b$, and let $\mathbf{x}$ be the tangent developable of $\boldsymbol{\alpha}$. Suppose that $\boldsymbol{\alpha}$ is differentiable at 0 and that the curvature and torsion of $\boldsymbol{\alpha}$ are nonzero at 0. Then the intersection of the trace of $\mathbf{x}$ with the plane perpendicular to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(0)$ is approximated by a semicubical parabola with a cusp at $\boldsymbol{\alpha}(0)$.*

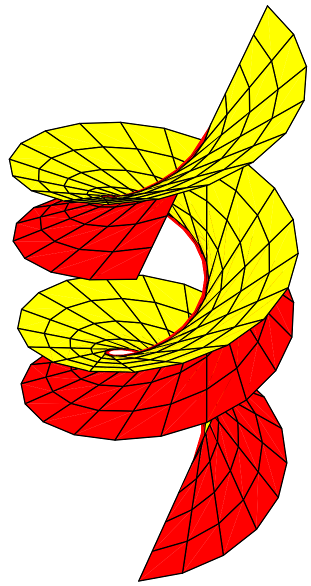

Figure 14.9: Tangent developable to a circular helix

Proof. Since $\boldsymbol{\alpha}$ has unit speed, we have $\boldsymbol{\alpha}'(s) = \mathbf{T}(s)$. The Frenet formulas (Theorem 7.10 on page 197) tell us that $\boldsymbol{\alpha}''(s) = \boldsymbol{\kappa}(s)\mathbf{N}(s)$ and moreover

$$\begin{aligned}
\boldsymbol{\alpha}'''(s) &= -\boldsymbol{\kappa}(s)^2\mathbf{T}(s) + \boldsymbol{\kappa}'(s)\mathbf{N}(s) + \boldsymbol{\kappa}(s)\boldsymbol{\tau}(s)\mathbf{B}(s),\\
\boldsymbol{\alpha}''''(s) &= -3\boldsymbol{\kappa}(s)\boldsymbol{\kappa}'(s)\mathbf{T}(s) + \big(-\boldsymbol{\kappa}(s)^3 + \boldsymbol{\kappa}''(s) - \boldsymbol{\kappa}(s)\boldsymbol{\tau}(s)^2\big)\mathbf{N}(s)\\
&\qquad + \big(2\boldsymbol{\kappa}'(s)\boldsymbol{\tau}(s) + \boldsymbol{\kappa}(s)\boldsymbol{\tau}'(s)\big)\mathbf{B}(s).
\end{aligned}$$

We next substitute these formulas into the power series expansion

$$\boldsymbol{\alpha}(s) = \boldsymbol{\alpha}_0 + s\,\boldsymbol{\alpha}_0' + \frac{s^2}{2}\boldsymbol{\alpha}_0'' + \frac{s^3}{6}\boldsymbol{\alpha}_0''' + \frac{s^4}{24}\boldsymbol{\alpha}_0'''' + O\big(s^5\big)\,,$$

where the subscript $_0$ denotes evaluation at $s = 0$. We get

$$\begin{aligned}
\boldsymbol{\alpha}(s) = \boldsymbol{\alpha}_0 + {}& \mathbf{T}_0\left(s - \frac{s^3}{6}\boldsymbol{\kappa}_0{}^2 - \frac{s^4}{8}\boldsymbol{\kappa}_0\boldsymbol{\kappa}_0'\right)\\
{}+{}& \mathbf{N}_0\left(\frac{s^2}{2}\boldsymbol{\kappa}_0 + \frac{s^3}{6}\boldsymbol{\kappa}_0' + \frac{s^4}{24}\big(-\boldsymbol{\kappa}_0{}^3 + \boldsymbol{\kappa}_0'' - \boldsymbol{\kappa}_0\boldsymbol{\tau}_0{}^2\big)\right)\\
{}+{}& \mathbf{B}_0\left(\frac{s^3}{6}\boldsymbol{\kappa}_0\boldsymbol{\tau}_0 + \frac{s^4}{24}\big(2\boldsymbol{\kappa}_0'\boldsymbol{\tau}_0 + \boldsymbol{\kappa}_0\boldsymbol{\tau}_0'\big)\right) + O\big(s^5\big)\,.
\end{aligned} \tag{14.14}$$

This is a significant formula that gives a local expression for a space curve relative to a fixed Frenet frame; see [dC1, §1-6]. Taking the derivative of (14.14) gives

$$\begin{aligned}\boldsymbol{\alpha}'(s) \;=\; \mathbf{T}_0\left(1-\frac{s^2}{2}\boldsymbol{\kappa}_0{}^2-\frac{s^3}{2}\boldsymbol{\kappa}_0\boldsymbol{\kappa}_0'\right)\\ +\,\mathbf{N}_0\left(s\boldsymbol{\kappa}_0+\frac{s^2}{2}\boldsymbol{\kappa}_0'+\frac{s^3}{6}\big(-\boldsymbol{\kappa}_0{}^3+\boldsymbol{\kappa}_0''-\boldsymbol{\kappa}_0\boldsymbol{\tau}_0{}^2\big)\right)\\ +\,\mathbf{B}_0\left(\frac{s^2}{2}\boldsymbol{\kappa}_0\boldsymbol{\tau}_0+\frac{s^3}{6}\big(2\boldsymbol{\kappa}_0'\boldsymbol{\tau}_0+\boldsymbol{\kappa}_0\boldsymbol{\tau}_0'\big)\right)+O\big(s^4\big)\,.\end{aligned}$$

The tangent developable of $\boldsymbol{\alpha}$ is therefore given by

$$\begin{aligned}\mathbf{x}(u,v) \;&=\; \boldsymbol{\alpha}(u)+v\,\boldsymbol{\alpha}'(u)\\ &=\; \boldsymbol{\alpha}_0+\mathbf{T}_0\bigg(u-\frac{u^3}{6}\boldsymbol{\kappa}_0{}^2+v-\frac{u^2v}{2}\boldsymbol{\kappa}_0{}^2-\frac{u^3v}{2}\boldsymbol{\kappa}_0\boldsymbol{\kappa}_0'\bigg)\\ &+\mathbf{N}_0\bigg(\frac{u^2}{2}\boldsymbol{\kappa}_0+\frac{u^3}{6}\boldsymbol{\kappa}_0'+u\,v\,\boldsymbol{\kappa}_0+\frac{u^2v}{2}\boldsymbol{\kappa}_0'+\frac{u^3v}{6}\big(-\boldsymbol{\kappa}_0{}^3+\boldsymbol{\kappa}_0''-\boldsymbol{\kappa}_0\boldsymbol{\tau}_0{}^2\big)\bigg)\\ &+\mathbf{B}_0\bigg(\Big(\frac{u^3}{6}+\frac{u^2v}{2}\Big)\boldsymbol{\kappa}_0\boldsymbol{\tau}_0+\frac{u^3v}{6}\big(2\boldsymbol{\kappa}_0'\boldsymbol{\tau}_0+\boldsymbol{\kappa}_0\boldsymbol{\tau}_0'\big)\bigg)+O\big(u^4\big)\,.\end{aligned}$$

We want to determine the intersection of $\mathbf{x}$ with the plane perpendicular to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}_0$. Therefore, we set the above coefficient of $\mathbf{T}_0$ equal to zero and solve for v to give

$$v \;=\; -\frac{u-\frac{1}{6}u^3\boldsymbol{\kappa}_0{}^2+O\big(u^4\big)}{1-\frac{1}{2}u^2\boldsymbol{\kappa}_0{}^2-\frac{1}{2}u^3\boldsymbol{\kappa}_0\boldsymbol{\kappa}_0'+O\big(u^4\big)} \;=\; -u-\boldsymbol{\kappa}_0{}^2\frac{u^3}{3}+O\big(u^4\big)\,.$$

Substituting back this value yields the following power series expansions:

$$\text{(14.15)}\qquad \begin{cases} x \;=\; \text{coefficient of } \mathbf{N}_0 \;=\; -\dfrac{u^2}{2}\boldsymbol{\kappa}_0-\cdots\\[2ex] y \;=\; \text{coefficient of } \mathbf{B}_0 \;=\; -\dfrac{u^3}{3}\boldsymbol{\kappa}_0\boldsymbol{\tau}_0+\cdots\end{cases}$$

By hypothesis, $\boldsymbol{\kappa}_0 \neq 0 \neq \boldsymbol{\tau}_0$, so that we can legitimately ignore higher order terms. Doing this, the plane curve described by (14.15) is approximated by the implicit equation

$$8\boldsymbol{\tau}_0{}^2x^3+9\boldsymbol{\kappa}_0\,y^2=0,$$

which is a semicubical parabola. ∎

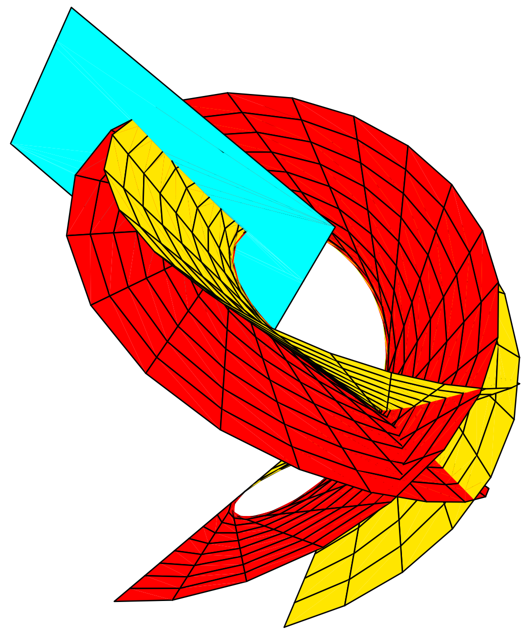

Figure 14.10: Tangent developable to Viviani's curve

Figures 14.10 illustrates a case where the hypotheses of Theorem 14.7 are not valid. It shows the plane generated by the normal $\mathbf{N}$ and binormal $\mathbf{B}$ unit vectors to Viviani's curve

$$\boldsymbol{\alpha}(t) = \Big(1-\cos t,\ -\sin t,\ 2\cos\frac{t}{2}\Big), \qquad -3\pi \leqslant t \leqslant \pi \tag{14.16}$$

at $\boldsymbol{\alpha}(0) = (0,0,2)$. We have reparametrized the curve (7.35) on page 207 so that the point at which the torsion vanishes has parameter value 0 rather than π (see Figure 7.7). We can find the intersection of this plane with the tangent developable explicitly, using the following general considerations.

We seek points $\mathbf{x}(u,v) = \boldsymbol{\alpha}(u) + v\boldsymbol{\alpha}'(u)$ on the tangent developable surface satisfying the orthogonality condition

$$(\mathbf{x}(u,v) - \boldsymbol{\alpha}(0)) \cdot \boldsymbol{\alpha}'(0) = 0.$$

The resulting equation is linear in v, and has solution

$$v = v(u) = \frac{(\boldsymbol{\alpha}(0) - \boldsymbol{\alpha}(u)) \cdot \boldsymbol{\alpha}'(0)}{\boldsymbol{\alpha}'(u) \cdot \boldsymbol{\alpha}'(0)}.$$

The cuspidal section is now represented by the plane curve (x,y) where

$$\begin{cases} x(u) = (\mathbf{x}(u,v(u)) - \boldsymbol{\alpha}(0)) \cdot \mathbf{N}, \\ y(u) = (\mathbf{x}(u,v(u)) - \boldsymbol{\alpha}(0)) \cdot \mathbf{B}. \end{cases}$$

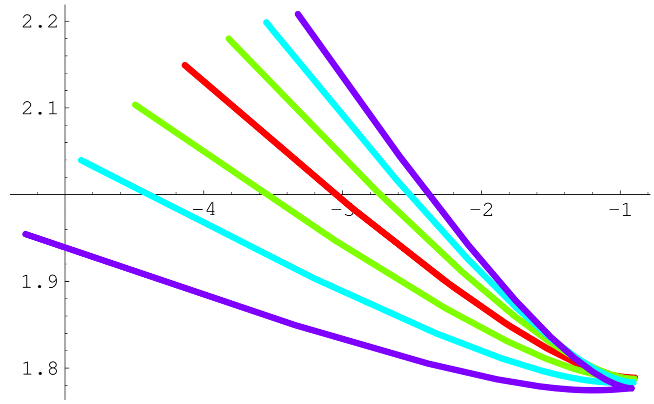

Figure 14.11: Viviani cusps for $t = 0.3$, 0.2, 0.1 and (in center) $t = 0$

Carrying out the calculations for (14.16) in Notebook 14 at $t = 0$ shows that (x, y) equals

$$\frac{1}{2\sqrt{5}\cos u}\left(-4 + 4\cos u - 3\cos\frac{u}{2} - \cos\frac{3u}{2},\ -2 + 2\cos u + 6\cos\frac{u}{2} + 2\cos\frac{3u}{2}\right).$$

It follows that $(x(-u),\, y(-u)) = (x(u),\, y(u))$, and so the two branches of the cusp coincide. The situation is illustrated by Figure 14.11 for neighboring values of t. For a fuller discussion of the two sheets that form the tangent developable, see [Spivak, vol 3, pages 207–213] and [Stru2, pages 66–73].

The following lemma is easy to prove.

Lemma 14.8. *Let $\boldsymbol{\beta}\colon (c, d) \to \mathbb{R}^3$ be a unit-speed curve. The metric of the tangent developable* tandev[$\boldsymbol{\beta}$] *depends only on the curvature of $\boldsymbol{\beta}$. Explicitly:*

$$E = 1 + v^2\boldsymbol{\kappa}[\boldsymbol{\beta}]^2, \qquad F = 1, \qquad G = 1.$$

The helix on page 200 has the same constant curvature as a circle of radius $(a^2 + b^2)/a$. Lemma 14.8 can then be used to provide a local isometry between a portion of the plane (the tangent developable to the circle) and the tangent developable to the helix.

14.4 Noncylindrical Ruled Surfaces

We shall now impose the following hypothesis on a class of ruled surfaces to be studied further.

Definition 14.9. *A ruled surface parametrized by $\mathbf{x}(u, v) = \boldsymbol{\beta}(u) + v\boldsymbol{\gamma}(u)$ is said to be* ***noncylindrical*** *provided $\boldsymbol{\gamma} \times \boldsymbol{\gamma}'$ never vanishes.*

The rulings are always changing directions on a noncylindrical ruled surface. We show how to find a useful reference curve on a noncylindrical ruled surface. This curve, called a striction curve, is a generalization of the edge of regression of a tangent developable.

Lemma 14.10. *Let* $\widetilde{\mathbf{x}}$ *be a parametrization of a noncylindrical ruled surface of the form* $\widetilde{\mathbf{x}}(u,v) = \boldsymbol{\beta}(u) + v\boldsymbol{\gamma}(u)$. *Then* $\widetilde{\mathbf{x}}$ *has a reparametrization of the form*

$$\mathbf{x}(u,v) = \boldsymbol{\sigma}(u) + v\,\boldsymbol{\delta}(u), \tag{14.17}$$

where $\|\boldsymbol{\delta}\| = 1$ *and* $\boldsymbol{\sigma}' \cdot \boldsymbol{\delta}' = 0$. *The curve* $\boldsymbol{\sigma}$ *is called the* ***striction curve*** *of* $\widetilde{\mathbf{x}}$.

Proof. Since $\boldsymbol{\gamma}\times\boldsymbol{\gamma}'$ is never zero, $\boldsymbol{\gamma}$ is never zero. We define a reparametrization $\widetilde{\widetilde{\mathbf{x}}}$ of $\widetilde{\mathbf{x}}$ by

$$\widetilde{\widetilde{\mathbf{x}}}(u,v) = \widetilde{\mathbf{x}}\Big(u,\ \frac{v}{\|\boldsymbol{\gamma}(u)\|}\Big) = \boldsymbol{\beta}(u) + \frac{v\,\boldsymbol{\gamma}(u)}{\|\boldsymbol{\gamma}(u)\|}.$$

Clearly, $\widetilde{\widetilde{\mathbf{x}}}$ has the same trace as $\widetilde{\mathbf{x}}$. If we put $\boldsymbol{\delta}(u) = \boldsymbol{\gamma}(u)/\|\boldsymbol{\gamma}(u)\|$, then

$$\widetilde{\widetilde{\mathbf{x}}}(u,v) = \boldsymbol{\beta}(u) + v\,\boldsymbol{\delta}(u).$$

Furthermore, $\|\boldsymbol{\delta}(u)\| = 1$ and so $\boldsymbol{\delta}(u)\cdot\boldsymbol{\delta}'(u) = 0$.

Next, we need to find a curve $\boldsymbol{\sigma}$ such that $\boldsymbol{\sigma}'(u)\cdot\boldsymbol{\delta}'(u) = 0$. To this end, we write

$$\boldsymbol{\sigma}(u) = \boldsymbol{\beta}(u) + t(u)\boldsymbol{\delta}(u) \tag{14.18}$$

for some function $t = t(u)$ to be determined. We differentiate (14.18), obtaining

$$\boldsymbol{\sigma}'(u) = \boldsymbol{\beta}'(u) + t'(u)\boldsymbol{\delta}(u) + t(u)\boldsymbol{\delta}'(u).$$

Since $\boldsymbol{\delta}(u)\cdot\boldsymbol{\delta}'(u) = 0$, it follows that

$$\boldsymbol{\sigma}'(u)\cdot\boldsymbol{\delta}'(u) = \boldsymbol{\beta}'(u)\cdot\boldsymbol{\delta}'(u) + t(u)\boldsymbol{\delta}'(u)\cdot\boldsymbol{\delta}'(u).$$

Since $\boldsymbol{\gamma}\times\boldsymbol{\gamma}'$ never vanishes, $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ are always linearly independent, and consequently $\boldsymbol{\delta}'$ never vanishes. Thus if we define t by

$$t(u) = -\frac{\boldsymbol{\beta}'(u)\cdot\boldsymbol{\delta}'(u)}{\|\boldsymbol{\delta}'(u)\|^2}, \tag{14.19}$$

we get $\boldsymbol{\sigma}'(u)\cdot\boldsymbol{\delta}'(u) = 0$. Now define

$$\mathbf{x}(u,v) = \widetilde{\widetilde{\mathbf{x}}}(u,\ t(u)+v).$$

Then $\mathbf{x}(u,v) = \boldsymbol{\beta}(u) + (t(u)+v)\boldsymbol{\delta}(u) = \boldsymbol{\sigma}(u) + v\,\boldsymbol{\delta}(u)$, so that $\mathbf{x}$, $\widetilde{\mathbf{x}}$ and $\widetilde{\widetilde{\mathbf{x}}}$ all have the same trace, and $\mathbf{x}$ satisfies (14.17). ∎

Lemma 14.11. The striction curve of a noncylindrical ruled surface $\mathbf{x}$ *does not depend on the choice of base curve.*

Proof. Let $\boldsymbol{\beta}$ and $\widetilde{\boldsymbol{\beta}}$ be two base curves for $\mathbf{x}$. In the notation of the previous proof, we may write

$$\boldsymbol{\beta}(u) + v\,\boldsymbol{\delta}(u) = \widetilde{\boldsymbol{\beta}}(u) + w(v)\boldsymbol{\delta}(u) \tag{14.20}$$

for some function $w = w(v)$. Let $\boldsymbol{\sigma}$ and $\widetilde{\boldsymbol{\sigma}}$ be the corresponding striction curves. Then

$$\boldsymbol{\sigma}(u) = \boldsymbol{\beta}(u) - \frac{\boldsymbol{\beta}'(u)\cdot\boldsymbol{\delta}'(u)}{\|\boldsymbol{\delta}'(u)\|^2}\boldsymbol{\delta}(u)$$

and

$$\widetilde{\boldsymbol{\sigma}}(u) = \widetilde{\boldsymbol{\beta}}(u) - \frac{\widetilde{\boldsymbol{\beta}}'(u)\cdot\boldsymbol{\delta}'(u)}{\|\boldsymbol{\delta}'(u)\|^2}\boldsymbol{\delta}(u),$$

so that

$$\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}} = \boldsymbol{\beta} - \widetilde{\boldsymbol{\beta}} - \frac{(\boldsymbol{\beta}' - \widetilde{\boldsymbol{\beta}}')\cdot\boldsymbol{\delta}'}{\|\boldsymbol{\delta}'\|^2}\boldsymbol{\delta}. \tag{14.21}$$

On the other hand, it follows from (14.20) that

$$\boldsymbol{\beta} - \widetilde{\boldsymbol{\beta}} = (w(v) - v)\boldsymbol{\delta}. \tag{14.22}$$

The result follows by substituting (14.22) and its derivative into (14.21).

There is a nice geometric interpretation of the striction curve $\boldsymbol{\sigma}$ of a ruled surface $\mathbf{x}$, which we mention without proof. Let $\varepsilon > 0$ be small. Since nearby rulings are not parallel to each other, there is a unique point $P(\varepsilon)$ on the straight line $v \mapsto \mathbf{x}(u,v)$ that is closest to the line $v \mapsto \mathbf{x}(u+\varepsilon,\, v)$. Then $P(\varepsilon) \to \boldsymbol{\sigma}(u)$ as $\varepsilon \to 0$. This follows because (14.19) is the equation that results from minimizing $\|\boldsymbol{\sigma}'\|$ to first order.

Definition 14.12. Let $\mathbf{x}$ *be a noncylindrical ruled surface given by* (14.17)*. Then the* ***distribution parameter*** *of* $\mathbf{x}$ *is the function* $p = p(u)$ *defined by*

$$p = \frac{[\boldsymbol{\sigma}'\ \boldsymbol{\delta}\ \boldsymbol{\delta}']}{\boldsymbol{\delta}'\cdot\boldsymbol{\delta}'}. \tag{14.23}$$

Whilst the definition of $\boldsymbol{\sigma}$ requires $\boldsymbol{\sigma}'$ to be perpendicular to $\boldsymbol{\delta}'$, the function p measures the component of $\boldsymbol{\sigma}'$ perpendicular to $\boldsymbol{\delta}\times\boldsymbol{\delta}'$.

Lemma 14.13. Let $\mathcal{M}$ *be a noncylindrical ruled surface, parametrized by a patch* $\mathbf{x}$ *of the form* (14.17)*. Then* $\mathbf{x}$ *is regular whenever* $v \neq 0$*, or when* $v = 0$ *and* $p(u) \neq 0$*. Furthermore, the Gaussian curvature of* $\mathbf{x}$ *is given in terms of its distribution parameter by*

$$K = \frac{-p(u)^2}{\big(p(u)^2 + v^2\big)^2}. \tag{14.24}$$

Also,

$$E = \|\boldsymbol{\sigma}'\|^2 + v^2\|\boldsymbol{\delta}'\|^2, \qquad F = \boldsymbol{\sigma}' \cdot \boldsymbol{\delta}, \qquad G = 1,$$

$$EG - F^2 = (p^2 + v^2)\|\boldsymbol{\delta}'\|^2,$$

and

$$g = 0, \qquad f = \frac{p\|\boldsymbol{\delta}'\|}{\sqrt{p^2 + v^2}}.$$

Proof. First, we observe that both $\boldsymbol{\sigma}' \times \boldsymbol{\delta}$ and $\boldsymbol{\delta}'$ are perpendicular to both $\boldsymbol{\delta}$ and $\boldsymbol{\sigma}'$. Therefore, $\boldsymbol{\sigma}' \times \boldsymbol{\delta}$ must be a multiple of $\boldsymbol{\delta}'$, and

$$\boldsymbol{\sigma}' \times \boldsymbol{\delta} = p\,\boldsymbol{\delta}' \qquad \text{where} \qquad p = \frac{[\boldsymbol{\sigma}'\ \boldsymbol{\delta}\ \boldsymbol{\delta}']}{\boldsymbol{\delta}' \cdot \boldsymbol{\delta}'}.$$

Since $\mathbf{x}_u = \boldsymbol{\sigma}' + v\,\boldsymbol{\delta}'$ and $\mathbf{x}_v = \boldsymbol{\delta}$, we have

$$\mathbf{x}_u \times \mathbf{x}_v = p\,\boldsymbol{\delta}' + v\,\boldsymbol{\delta}' \times \boldsymbol{\delta},$$

so that

$$\|\mathbf{x}_u \times \mathbf{x}_v\|^2 = \|p\,\boldsymbol{\delta}'\|^2 + \|v\,\boldsymbol{\delta}' \times \boldsymbol{\delta}\|^2 = (p^2 + v^2)\|\boldsymbol{\delta}'\|^2.$$

It is now clear that the regularity of $\mathbf{x}$ is as stated.

Next, $\mathbf{x}_{uv} = \boldsymbol{\delta}'$ and $\mathbf{x}_{vv} = 0$, so that $g = 0$ and

$$f = \frac{[\mathbf{x}_{uv}\ \mathbf{x}_u\ \mathbf{x}_v]}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \frac{\boldsymbol{\delta}' \cdot (p\,\boldsymbol{\delta}' + v\,\boldsymbol{\delta}' \times \boldsymbol{\delta})}{\sqrt{p^2 + v^2}\|\boldsymbol{\delta}'\|} = \frac{p\|\boldsymbol{\delta}'\|}{\sqrt{p^2 + v^2}}.$$

Therefore,

$$K = \frac{-f^2}{\|\mathbf{x}_u \times \mathbf{x}_v\|^2} = \frac{-\left(\dfrac{p\|\boldsymbol{\delta}'\|}{\sqrt{p^2 + v^2}}\right)^2}{(p^2 + v^2)\|\boldsymbol{\delta}'\|^2},$$

which simplifies into (14.24). ▮

Equation (14.24) tells us that the Gaussian curvature of a noncylindrical ruled surface is generally negative. However, more can be said.

Corollary 14.14. *Let $\mathcal{M}$ be a noncylindrical ruled surface given by* (14.17) *with distribution parameter p, and Gaussian curvature $K(u, v)$.*

(i) *Along a ruling (so u is fixed), $K(u, v) \to 0$ as $v \to \infty$.*

(ii) *$K(u, v) = 0$ if and only if $p(u) = 0$.*

(iii) *If p never vanishes, then $K(u, v)$ is continuous and $|K(u, v)|$ assumes its maximum value $1/p^2$ at $v = 0$.*

Proof. All of these statements follow from (14.24). ▮

Next, we prove a partial converse of Proposition 14.6.

Theorem 14.15. *Let* $\mathbf{x}(u,v) = \boldsymbol{\beta}(u) + v\,\boldsymbol{\delta}(u)$ *with* $\|\boldsymbol{\delta}(u)\| = 1$ *parametrize a flat ruled surface* $\mathcal{M}$*.*

(i) *If* $\boldsymbol{\beta}'(u) \equiv 0$*, then* $\mathcal{M}$ *is a cone.*

(ii) *If* $\boldsymbol{\delta}'(u) \equiv 0$*, then* $\mathcal{M}$ *is a cylinder.*

(iii) *If both* $\boldsymbol{\beta}'$ *and* $\boldsymbol{\delta}'$ *never vanish, then* $\mathcal{M}$ *is the tangent developable of its striction curve.*

Proof. Parts (i) and (ii) are immediate from the definitions, so it suffices to prove (iii). We can assume that $\boldsymbol{\beta}$ is a unit-speed striction curve, so that

$$\boldsymbol{\beta}' \cdot \boldsymbol{\delta}' \equiv 0. \tag{14.25}$$

Since $K \equiv 0$, it follows from (14.24) and (14.23) that

$$[\boldsymbol{\beta}'\ \boldsymbol{\delta}\ \boldsymbol{\delta}'] \equiv 0. \tag{14.26}$$

Then (14.25) and (14.26) imply that $\boldsymbol{\beta}'$ and $\boldsymbol{\delta}$ are collinear. ▮

Of course, cases (i), (ii) and (iii) of Theorem 14.15 do not exhaust all of the possibilities. If there is a clustering of the zeros $\boldsymbol{\beta}$ or $\boldsymbol{\delta}$, the surface can be complicated. In any case, away from the cluster points a developable surface is the union of pieces of cylinders, cones and tangent developables. Indeed, the following result is proved in [Krey1, page 185]. Every flat ruled patch $(u,v) \mapsto \mathbf{x}(u,v)$ can be subdivided into sufficiently small u-intervals so that the portion of the surface corresponding to each interval is a portion of one of the following: a plane, a cylinder, a cone, a tangent developable.

Examples of Striction Curves

The parametrization (12.27) of the circular helicoid can be rewritten as

$$\mathbf{x}(u,v) = (0,0,bu) + av(\cos u, \sin u, 0),$$

which shows that it is a ruled surface. The striction curve $\boldsymbol{\sigma}$ and director curve $\boldsymbol{\delta}$ are given by

$$\boldsymbol{\sigma}(u) = (0,\ 0,\ bu) \qquad \text{and} \qquad \boldsymbol{\delta}(u) = (\cos u,\ \sin u,\ 0).$$

Then

$$\mathbf{x}\Big(u,\ \frac{v}{a}\Big) = \boldsymbol{\sigma}(u) + v\,\boldsymbol{\delta}(u) = (v\cos u,\ v\sin u,\ bu),$$

and the distribution parameter assumes the constant value b.

The hyperbolic paraboloid (10.12), page 296, when parametrized as

$$\mathbf{x}(u,v) = (u,0,0) + v(0,1,u),$$

has $\boldsymbol{\sigma}(u) = (u,0,0)$ as its striction curve and

$$\boldsymbol{\delta}(u) = \frac{(0,1,u)}{\sqrt{1+u^2}}$$

as its director curve. Thus

$$\mathbf{x}(u,\ v\sqrt{1+u^2}) = \boldsymbol{\sigma}(u) + v\,\boldsymbol{\delta}(u),$$

and the distribution parameter is given by $p(u) = 1 + u^2$.

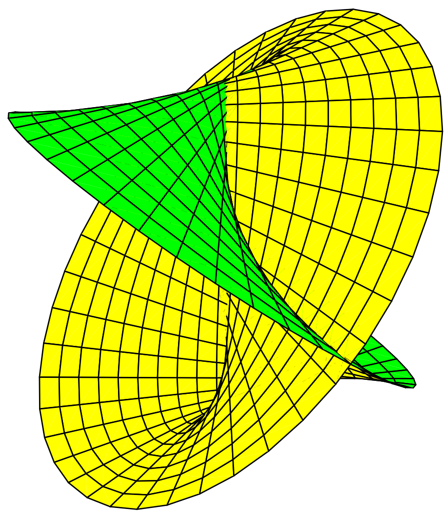

Figure 14.12: Right conoid

14.5 Exercises

1. Compute the Gaussian and mean curvatures of the generalized hyperbolic paraboloid defined and plotted on page 434.

M **2.** Compute the Gaussian and mean curvatures of the right conoid defined by equation (14.10) and illustrated in Figure 14.12.

3. Explain why Lemma 14.2 is also a consequence of the fact (mentioned on page 432) that a ruled surface has asymptotic curves, namely, the rulings.

M **4.** Further to Exercise 2 of the previous chapter, describe the sets of elliptic, hyperbolic, parabolic and planar points for the following surfaces:

(a) a cylinder over an ellipse,
(b) a cylinder over a parabola,
(c) a cylinder over a hyperbola,
(d) a cylinder over $y = x^3$,
(e) a cone over a circle.

M **5.** Compute the Gaussian and mean curvature of the patch

$$\mathsf{plueckerpolar}[m,n,a](r,\theta) = a\big(r\cos\theta,\ r\sin\theta,\ r^m \sin n\theta\big)$$

that generalizes plueckerpolar on page 436. Relate the case $m = n = 3$ to the monkey saddle on page 304.

6. Complete the proof of Lemma 14.5 and prove Lemma 14.8.

M **7.** ***Wallis' conical edge***[2] is defined by

$$\mathsf{wallis}[a,b,c](u,v) = \left(v\cos u,\ v\sin u,\ c\sqrt{a^2 - b^2\cos^2 u}\right).$$

Show that wallis$[a,b,c]$ is a right conoid, as in (14.10). Figure 14.13 illustrates the case $a = 1 - c$ and $b = \sqrt{3}$. Compute and plot its Gaussian and mean curvatures.

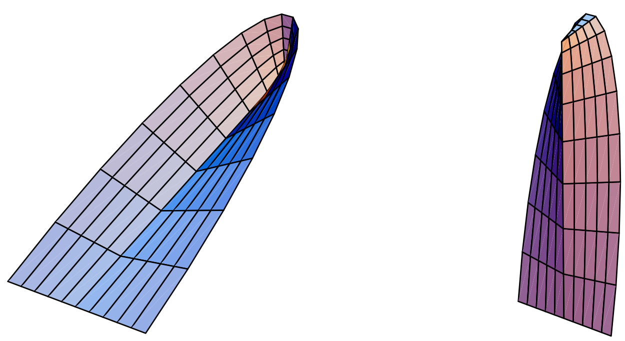

Figure 14.13: Back and front of Wallis's conical edge

[2]  John Wallis (1616–1703). English mathematician. Although ordained as a minister, Wallis was appointed Savilian professor of geometry at Oxford in 1649. He was one of the first to use Cartesian methods to study conic sections instead of employing the traditional synthetic approach. The sign ∞ for infinity (probably adapted from the late Roman symbol for 1000) was first introduced by Wallis.

8. Show that the parametrization exptwist$[a, c]$ of the expondentially twisted helicoid described on page 563 of the next chapter is a ruled patch. Find the rulings.

M **9.** Plot the tangent developable to the twisted cubic defined on page 202.

M **10.** Plot the tangent developable to the Viviani curve defined on page 207 for the complete range $-2\pi \leqslant t \leqslant 2\pi$ (see Figure 14.10).

11. Plot the tangent developable to the bicylinder defined on page 214.

M **12.** Carry out the calculations of Theorem 14.7 by computer.

M **13.** Show that the normal surface to a circular helix is a helicoid. Draw the binormal surface to a circular helix and compute its Gaussian curvature.

M **14.** For any space curve $\boldsymbol{\alpha}$, there are other surfaces which lie between the normal surface and the binormal surface. Consider

$$\text{perpsurf}[\phi, \boldsymbol{\alpha}](u, v) = \boldsymbol{\alpha}(u) + v \cos\phi \mathbf{N}(u) + \sin\phi \mathbf{B}(u),$$

where $\mathbf{N}, \mathbf{B}$ are the normal and binormal vector fields to $\boldsymbol{\alpha}$. Clearly, perpsurf$[0, \boldsymbol{\alpha}]$ is the normal surface of $\boldsymbol{\alpha}$, whilst perpsurf$[\frac{\pi}{2}, \boldsymbol{\alpha}]$ is its binormal surface. Use perpsurf to construct several of these intermediate surfaces for a helix.