Chapter 3

Alternative Ways of Plotting Curves

So far, we have mainly been plotting curves in *parametric* form. In Section 3.1, we discuss implicitly defined curves, and contrast them with those parametrized in the previous chapters. A curve is defined implicitly as the 'zero set' or set of zeros of a differentiable function of two variables. In particular cases, this is the more natural way of defining the curve. Even so, if one is given the nonparametric form of a curve, it is sometimes easier to find a suitable parametrization before attempting to plot it. In any case, it can be important to be able to switch between the two representations of the curve.

As an initial example, after lines and circles, we recall that the cissoid was first defined as the set of zeros of a cubic equation. Similar techniques can be applied to another cubic curve, the folium of Descartes. The latter is studied in Section 3.2, which provides an example of what can happen to a curve as one deforms its initial equation. Cassinian ovals, which we consider in Section 3.3, form a further class of curves that are easily plotted implicitly. This generalizes the lemniscate by requiring that the product of the distances from a point on the curve to two fixed foci be constant without regard to the distance between the foci; the resulting equation is quartic.

As the second topic of this chapter, we shall use polar coordinates to describe curves in Section 3.4. We use this approach to give a simple generalization of a cardioid and limaçon, and to study related families of closed curves. We also compute lengths and curvature in polar coordinates, using the formalism of complex numbers to prove our results.

The use of polar coordinates leads to an investigation of different types of spirals in Section 3.5. A number of these have especially simple polar equations. Further examples of implicit and polar plots can be found in this chapter's exercises and notebook.

3.1 Implicitly Defined Plane Curves

We now consider a way of representing a plane curve that does not involve an explicit parametrization.

Definition 3.1. *Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be any function. The* ***set of zeros*** *of F is*

$$F^{-1}(0) = \{\, \mathbf{p} \in \mathbb{R}^2 \mid F(\mathbf{p}) = 0 \,\}.$$

If no restrictions are placed on F, then not much can be said about its set of zeros, so we make an additional assumption.

Definition 3.2. *An* ***implicitly defined curve*** *in $\mathbb{R}^2$ is the set of zeros of a differentiable function $F\colon \mathbb{R}^2 \to \mathbb{R}$. Frequently, we refer to the set of zeros as 'the curve $F(x, y) = 0$'.*

We point out that the analogous theory for nonparametrically defined surfaces will be presented in Section 10.6.

Even when F is assumed to be differentiable, the set of zeros of F may have cusps, and hence appear to be nondifferentiable. Moreover, a theorem of Whitney[1] states that any closed subset of $\mathbb{R}^2$ is the set of zeros of some differentiable function (see [BrLa, page 56]). However, there is an important case when it is possible to find a parametrized curve whose trace is the set of zeros of F.

Theorem 3.3. *Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function, and let $\mathbf{q} = (q_1, q_2)$ be a point such that $F(\mathbf{q}) = 0$. Assume that at least one of the partial derivatives F_x, F_y is nonzero at $\mathbf{q}$. Then there is a neighborhood $\mathcal{U}$ of $\mathbf{q}$ in $\mathbb{R}^2$ and a parametrized curve $\boldsymbol{\alpha}\colon (a, b) \to \mathbb{R}^2$ such that the trace of $\boldsymbol{\alpha}$ is precisely*

$$\{\, \mathbf{p} \in \mathcal{U} \mid F(\mathbf{p}) = 0 \,\}.$$

Proof. Suppose, for example, that $F_v(\mathbf{q}) \neq 0$. The Implicit Function Theorem states that there is a differentiable real-valued function g defined on a small neighborhood of q_1 in $\mathbb{R}$ such that $g(q_1) = q_2$ and $t \mapsto F(t, g(t))$ vanishes identically. Then we define $\boldsymbol{\alpha}(t) = (t,\ g(t))$. ▮

Definition 3.4. *Let $\mathscr{C}$ be a subset of $\mathbb{R}^2$. If $F\colon \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function with $F^{-1}(0) = \mathscr{C}$, we say that the equation $F(x, y) = 0$ is a* ***nonparametric form*** *or* ***implicit form*** *of $\mathscr{C}$. If $\boldsymbol{\alpha}\colon (a, b) \to \mathbb{R}^2$ is a curve whose trace is $\mathscr{C}$, we say that $t \mapsto \boldsymbol{\alpha}(t)$ is a* ***parametrization*** *or* ***parametric form*** *of $\mathscr{C}$.*

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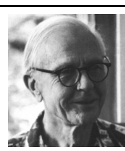
[1]  Hassler Whitney (1907–1989). Influential American differential-topologist.

To start with some examples using the operations of vector calculus, let $\mathbf{p}, \mathbf{v} \in \mathbb{R}^2$ with $\mathbf{v} \neq 0$, and define functions

$$F[\mathbf{p}, \mathbf{v}],\ G[\mathbf{p}, \mathbf{v}] \colon \mathbb{R}^2 \longrightarrow \mathbb{R}$$

by

$$\begin{aligned} F[\mathbf{p}, \mathbf{v}](\mathbf{q}) &= (\mathbf{q} - \mathbf{p}) \cdot \mathbf{v} \\ G[\mathbf{p}, \mathbf{v}](\mathbf{q}) &= \|\mathbf{q} - \mathbf{v}\|^2 - \|\mathbf{p} - \mathbf{v}\|^2. \end{aligned} \tag{3.1}$$

Then $F[\mathbf{p}, \mathbf{v}]^{-1}(0)$ is the straight line through $\mathbf{p}$ perpendicular to $\mathbf{v}$ and the curve $G[\mathbf{p}, \mathbf{v}]^{-1}(0)$ is the circle with center $\mathbf{v}$ that contains $\mathbf{p}$.

Some curves are more naturally defined implicitly. This was the case of the cissoid, for which it is easy to pass between the two representations. For given the parametric form (2.12), we have no difficulty in spotting that $t = y/x$ and reversing the derivation on page 47. Substituting this expression for t into the equation $x = 2at/(1 + t^2)$ yields the nonparametric form

$$x = \frac{2a\,y^2}{x^2 + y^2}$$

of the cissoid, which is equivalent to the cubic equation (2.11).

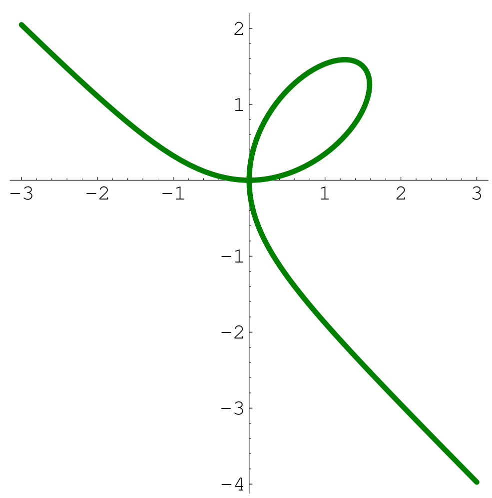


Figure 3.1: The folium of Descartes, $x^3 + y^3 = 3xy$

The trace of a curve $\boldsymbol{\alpha} \colon (a, b) \to \mathbb{R}^2$ is always connected. This is a special case of the theorem from topology that states that the image under a continuous map of a connected set remains connected [Kelley]. It can easily happen however that the set of zeros of a differentiable function $F \colon \mathbb{R}^2 \to \mathbb{R}$ is *disconnected.*

This means that the notions of parametric form and nonparametric form are essentially different.

We give an example of a disconnected zero set immediately below, by perturbing a slightly different cubic equation, that used to plot Figure 3.1.

3.2 The Folium of Descartes

The ***folium of Descartes***[2] is defined nonparametrically by the equation

$$x^3 + y^3 = 3xy. \tag{3.2}$$

In order to formalize this definition, we consider the family F_ε of functions defined by

$$F_\varepsilon(x,y) = x^3 + y^3 - 3xy - \varepsilon,$$

and their associated curves

$$\mathscr{F}_\varepsilon = \{(x,y) \in \mathbb{R}^2 \mid F_\varepsilon(x,y) = 0\}.$$

Thus, $\mathscr{F}_\varepsilon = F_\varepsilon^{-1}(0)$ is the set of zeros of the function F_ε, and $\mathscr{F}_0$ is the folium of Descartes. Whilst the latter is connected, $\mathscr{F}_\varepsilon$ is disconnected for general values of ε; Figure 3.2 illustrates the case $\varepsilon = -\frac{1}{10}$.

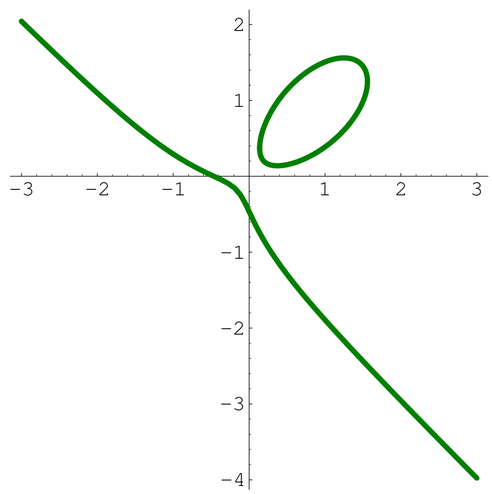


Figure 3.2: Perturbed folium, $\mathscr{F}_{-0.1}$

[2] It was Huygens who first drew the curve correctly. See [Still1, pages 67–68]

We can find a parametrization for $\mathscr{F}_0$, by observing that F_0 is close to a homogeneous polynomial. Dividing the equation $F_0 = 0$ by x^3, we obtain

$$1 + t^3 = \frac{3}{x}t,$$

where $t = y/x$ (as for the cissoid, in the previous section). Solving for x, and then $y = tx$, yields the parametrization

$$\mathsf{folium}(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right). \tag{3.3}$$

Such a ***rational*** parametrization cannot be found for the curve $\mathscr{F}_\varepsilon$, unless ε equals 0 or 1. This follows from the fact that $\mathscr{F}_\varepsilon$ is a nonsingular cubic curve if $\varepsilon \neq 0, 1$, and from standard results on the projective geometry of cubic curves. We explain briefly how this theory applies to our example, but refer the reader to [BrKn, Kir] for more details.

The equation $F_\varepsilon = 0$ can be fully 'homogenized' by inserting powers of a third variable z so that each term has total degree equal to 3; the result is

$$x^3 + y^3 - 3xyz - \varepsilon z^3 = 0. \tag{3.4}$$

This equation actually defines a curve not in $\mathbb{R}^2$, but in the *real projective plane* $\mathbb{RP}^2$, a surface that we define in Chapter 11. Linear transformations in the variables x, y, z determine *projective transformations* of $\mathbb{RP}^3$, and can be used to find an equivalent form of the equation of the cubic. For example, applying the linear change of coordinates

$$\begin{cases} x = \frac{1}{2}X + Y, \\ y = \frac{1}{2}X - Y, \\ z = Z - X, \end{cases} \tag{3.5}$$

converts (3.4) into

$$(1+\varepsilon)^2 X^3 - (3\varepsilon + \tfrac{3}{4})X^2 Z + 3\varepsilon XZ^2 - \varepsilon Z^3 + 3Y^2 Z = 0 \tag{3.6}$$

(Exercise 8). Finally, we set $Z = 1$ in order to return to an equation in two variables, namely

$$Y^2 = -\tfrac{1}{3}(1+\varepsilon)^2 X^3 + (\varepsilon + \tfrac{1}{4})X^2 - \varepsilon X + \tfrac{1}{3}\varepsilon. \tag{3.7}$$

This is a type of 'canonical equation' of the cubic curve $\mathscr{F}_\varepsilon$.

If we write (3.7) as $Y^2 = p(X)$, then p is a cubic polynomial provided $\varepsilon \neq -1$; if $\varepsilon = -1$, (3.6) is divisible by Z, and p arises from the remaining quadratic factor. The cubic curve $\mathscr{F}_\varepsilon$ is said to be *nonsingular* if p has three distinct roots (two of which may be complex), and it can be checked in Notebook 3 that this is the case provided $\varepsilon \neq 0, 1$. If $\varepsilon = 0$, then the origin $(0, 0)$ counts as a singular point in the implicit theory because there is no unique tangent line there (see Figure 3.3). The cissoid is a cubic curve with a different type of singularity. To parametrize a nonsingular cubic curve, one needs the Weierstrass $\wp$ function, a special case of which is defined (albeit with complex numbers) in Section 22.7. The analysis there should make it clear that there is no elementary parametrization of the curve $\mathscr{F}_\varepsilon$ in general.

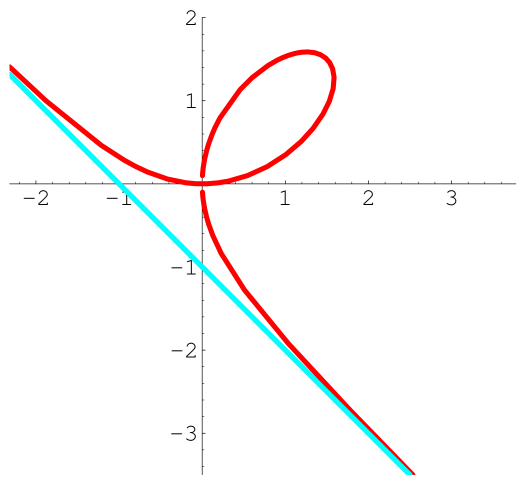
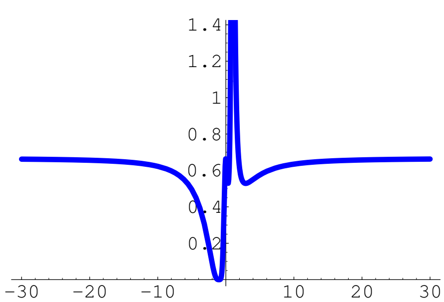


Figure 3.3: Folium of Descartes with asymptote

We conclude this section with some remarks of a more practical nature. The vector-valued function (3.3) was used to produce Figure 3.3, in which the asymptote is generated automatically by the plotting program. The latter is unable to distinguish a pair of points far apart in opposite quadrants from the many pairs of adjacent points selected on the curve. This phenomenon is one possible disadvantage of parametric plotting, though we explain in Notebook 3 how the problem can be overcome to advantage.



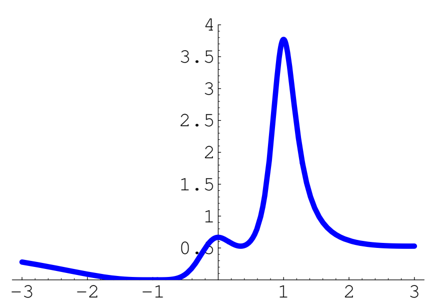


Figure 3.4: Folium curvature

The curvature of $\mathscr{F}_0$ can be obtained from the parametrization (3.3). To understand the resulting function $t \mapsto \boldsymbol{\kappa 2}[\mathsf{folium}](t)$, we plot it first over the range $-30 < t < 30$ and then over the range $-3 < t < 3$. These two plots constitute Figure 3.4. The determination of the maxima and minima of the curvature can be carried out by computer, and is explained in Notebook 3.

3.3 Cassinian Ovals

A ***Cassinian oval***[3] is a generalization of the lemniscate of Bernoulli, and therefore of the ellipse. It is the locus

$$\mathscr{C}_{a,b} = \big\{(x,y) \mid \mathsf{distance}\big((x,y),\ F_1\big)\mathsf{distance}\big((x,y),\ F_2\big) = b^2\big\},$$

where F_1, F_2 are two fixed points with $\mathsf{distance}(F_1, F_2) = 2a$. The constants a, b are unrelated, though setting $a = b = f$, the curve $\mathscr{C}_{f,f}$ coincides with the lemniscate $\mathscr{L}$ defined on page 43.

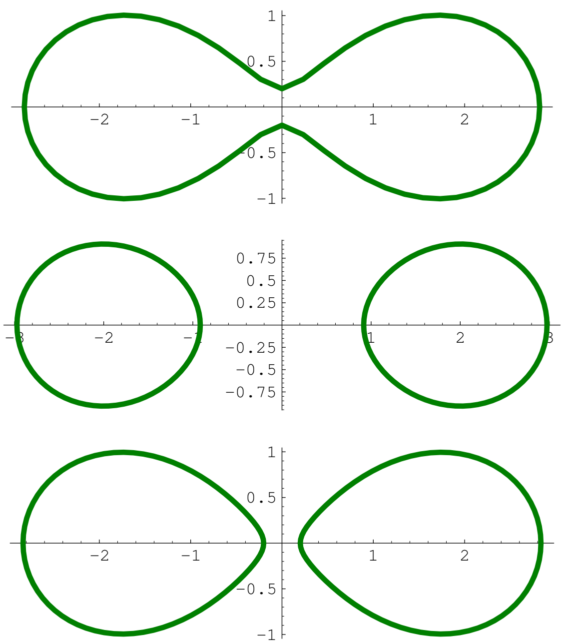


Figure 3.5: Ovals with $b = 2$ and $200a = 199, 200, 201$ respectively

3  Gian Domenico Cassini (1625–1712). Italian astronomer, who did his most important work in France. He proposed the fourth degree curves now called ovals of Cassini to describe planetary motion.

To study Cassinian ovals, we first define

$$\mathsf{cassiniimplicit}[a,b](x,y) = (x^2+y^2+a^2)^2 - b^4 - 4a^2x^2.$$

Generalizing the computations of Section 2.2, one can prove that $\mathscr{C}_{a,b}$ is the zero set of this function. This is the next result, whose proof we omit.

Lemma 3.5. *An oval of Cassini with* $(\pm a, 0)$ *as foci is the implicitly defined curve*

$$(x^2+y^2)^2 + 2a^2(y^2-x^2) = b^4 - a^4.$$

The zero set of

$$\mathsf{cassiniimplicit}[f,f](x,y) = (x^2+y^2)^2 + 2f^2(y^2-x^2)$$

is the lemniscate $\mathscr{L} = \mathscr{C}_{f,f}$, while cassiniimplicit$[0,b]$ is of course a circle.

The horizontal intercepts of cassiniimplicit$[a,b]$ are $(\pm\sqrt{a^2 \pm b^2},\ 0)$. When $a^2 > b^2$, there are four horizontal intercepts, but when $a^2 < b^2$ there are only two horizontal intercepts, since we must exclude imaginary solutions. Similarly, cassiniimplicit$[a,b]$ has no vertical intercepts when $a^2 > b^2$, whereas for $a^2 < b^2$ the vertical intercepts are $(0,\ \pm\sqrt{b^2-a^2})$. This is shown in Figure 3.5.

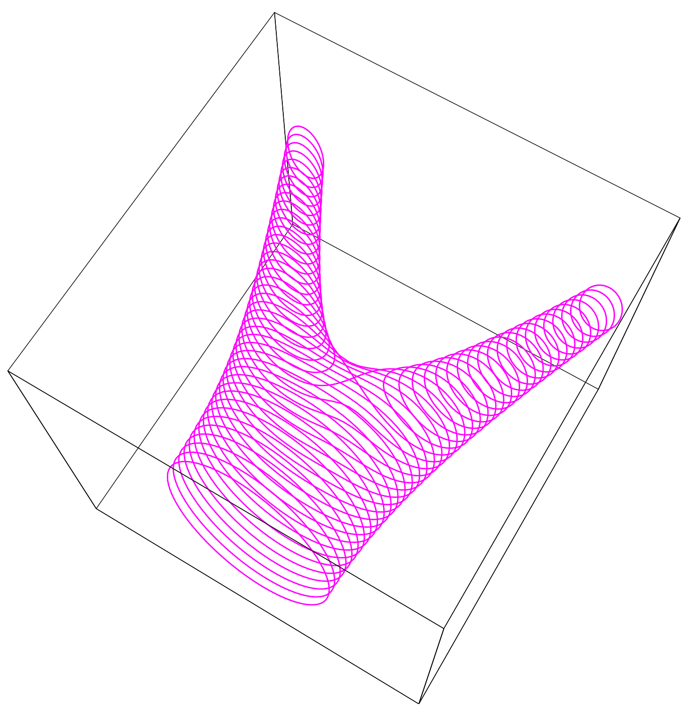


Figure 3.6: A Cassinian family

3.4 Plane Curves in Polar Coordinates

In this section we show how to study and compute the length and curvature of a plane curve using polar coordinates.

Definition 3.6. *A* ***polar parametrization*** *is a curve* $\boldsymbol{\gamma}\colon (a,b) \to \mathbb{R}^2$ *of the form*

$$\boldsymbol{\gamma}(\theta) = \mathbf{r}[\boldsymbol{\gamma}](\theta)\,(\cos\theta,\ \sin\theta), \tag{3.8}$$

where $\mathbf{r}[\boldsymbol{\gamma}](\theta) \geqslant 0$ *for* $a < \theta < b$. *We call* $\mathbf{r}[\boldsymbol{\gamma}]$ *the* ***radius function*** *of the curve* $\boldsymbol{\gamma}$, *and abbreviate it to* $\mathbf{r}$ *when there is no danger of confusion.*

The radius function completely determines the polar parametrization, so usually a curve is described in polar coordinates simply by giving the definition of $\mathbf{r}$.

A polar parametrization of a plane curve is often very simple. Its description is simplified further by writing just the definition of the radius function of the curve. We shall see shortly that there are formulas for the arc length and curvature of a polar parametrization in terms of the radius function alone.

Let us give a generalization of the cardioid and limaçon (see Exercise 4 of Chapter 2) by defining the radius function

$$\mathsf{limaconpolar}[n,a,b](\theta) = 2a\cos n\theta + b,$$

Observe that $\mathsf{limaconpolar}[1,a,2a](\theta) = 2a(\cos\theta + 1)$ is the radius function of a standard cardioid. More complicated examples are exhibited in Figure 3.7. An analogous polar parametrization is

$$\mathsf{pacman}[n](\theta) = 1 + \cos^n\theta, \tag{3.9}$$

and its name is justified by the example shown in Figure 3.13 and Exercise 10.

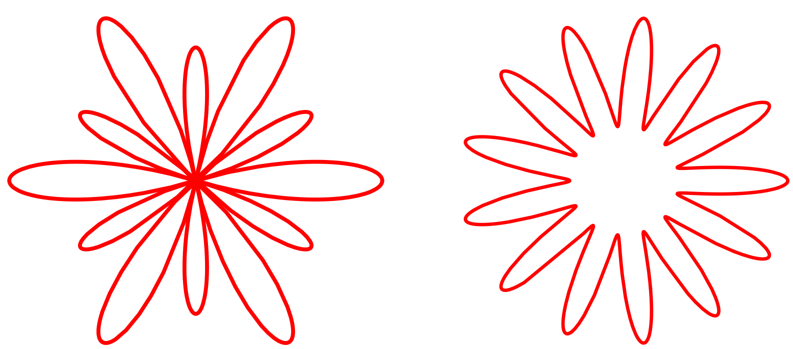


Figure 3.7: The polar limaçons $\mathsf{limaconpolar}[6,3,1]$ and $\mathsf{limaconpolar}[13,\frac{1}{2},2]$

Next, we derive the polar coordinate formulas for arc length and curvature.

Lemma 3.7. *The length and curvature of the polar parametrization* (3.8) *are given in terms of the radius function* $\mathbf{r} = \mathbf{r}[\gamma]$ *by the formulas*

$$\mathsf{length}[\gamma] = \int_a^b \sqrt{\mathbf{r}'(\theta)^2 + \mathbf{r}(\theta)^2}\, d\theta, \tag{3.10}$$

$$\boldsymbol{\kappa 2}[\gamma] = \frac{-\mathbf{r}''\mathbf{r} + 2\mathbf{r}'^2 + \mathbf{r}^2}{(\mathbf{r}'^2 + \mathbf{r}^2)^{3/2}}. \tag{3.11}$$

Proof. The calculations can be carried out most easily using complex numbers. Equation (3.8) can be written more succinctly as

$$\gamma(\theta) = \mathbf{r}(\theta)e^{i\theta}.$$

We get

$$\begin{aligned} \gamma'(\theta) &= \big(\mathbf{r}'(\theta) + i\mathbf{r}(\theta)\big)e^{i\theta} \\ \gamma''(\theta) &= \big(\mathbf{r}''(\theta) + 2i\mathbf{r}'(\theta) - \mathbf{r}(\theta)\big)e^{i\theta}. \end{aligned}$$

Therefore

$$\|\gamma'(\theta)\|^2 = \mathbf{r}'(\theta)^2 + \mathbf{r}(\theta)^2, \tag{3.12}$$

and (3.10) follows immediately from the definition of length. Furthermore, using Lemma 1.2 we find that

$$\begin{aligned} \gamma''(\theta) \cdot J\gamma'(\theta) &= \mathfrak{Re}\left\{\big(\mathbf{r}''(\theta) + 2i\mathbf{r}'(\theta) - \mathbf{r}(\theta)\big)e^{i\theta}\big(\,\overline{i(\mathbf{r}'(\theta) + i\mathbf{r}(\theta))e^{i\theta}}\,\big)\right\} \\ &= \mathfrak{Re}\left\{\big(\mathbf{r}''(\theta) + 2i\mathbf{r}'(\theta) - \mathbf{r}(\theta)\big)\big(-i\mathbf{r}'(\theta) - \mathbf{r}(\theta)\big)\right\} \\ &= -\mathbf{r}''(\theta)\mathbf{r}(\theta) + 2\mathbf{r}'(\theta)^2 + \mathbf{r}(\theta)^2. \end{aligned}$$

Then (3.11) now follows from (3.12) and (1.12). ∎

3.5 A Selection of Spirals

A good example of a polar parametrization is the logarithmic spiral, that was studied in some detail towards the end of Chapter 1. There, we saw that its radius function is given by

$$\mathsf{logspiralpolar}[a, b](\theta) = a\,e^{b\theta}.$$

The curve is characterized by Lemma 1.29 on page 24, and for this reason is sometimes called the ***equiangular spiral*** [Coxeter, 8.7].

A somewhat more naïve equation for a spiral in polar coordinates is

$$\mathbf{r}^n = a^n\theta, \tag{3.13}$$

where n is a nonzero integer. Several of the resulting curves have been given particular names, and we now plot a few of them.

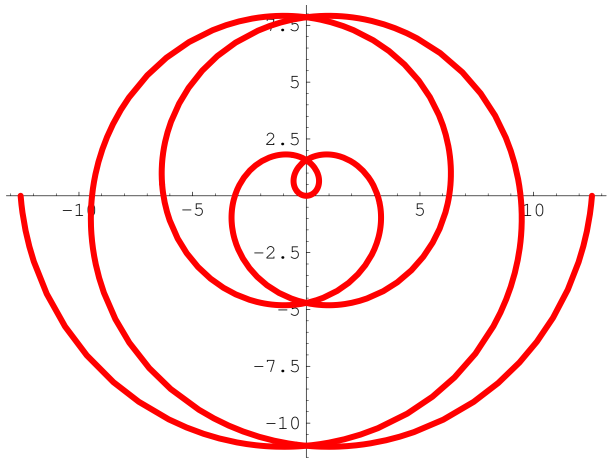


Figure 3.8: The spiral of Archimedes[4], $r = \theta$

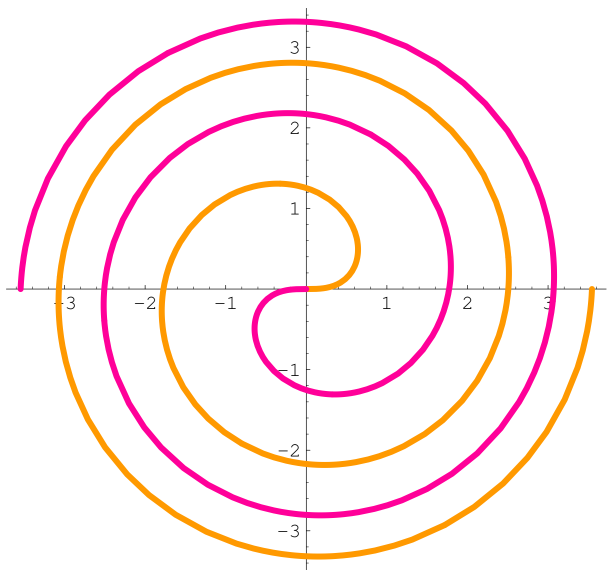


Figure 3.9: Fermat's spiral $r^2 = \theta$

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4  Archimedes of Syracuse (287–212 BC). Archimedes is credited with the creation of many mechanical devices such as compound pulleys, water clocks, catapults and burning mirrors. Legend has it that he was killed by a Roman soldier as he traced mathematical figures in the sand during the siege of Syracuse, at that time a Greek colony on what is now Sicily. His mathematical work included finding the area of a circle and the area under a parabola using the method of exhaustion.

The spiral (3.13) has two branches, the second obtained by allowing the radius function to be negative. The two branches join up provided the exponent n is positive. For Archimedes' spiral, the two branhes are reflections of each other in the vertical y-axis. By contrast, for Fermat's spiral[5] in Figure 3.9, the two branches are rotations of each other by 180^{o}.

Similar behavior is seen in Figure 3.10. However, when n is negative, θ cannot be allowed to pass through zero, and this explains why the two branches fail to join up.

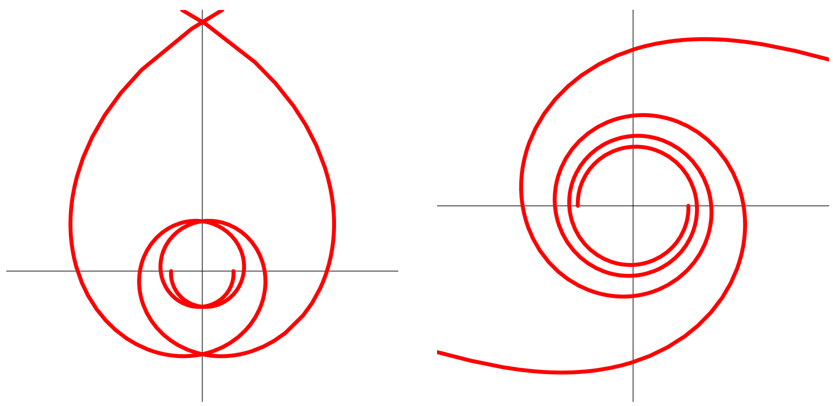

Figure 3.10: The hyperbolic spiral $r = 3/\theta$ and lituus $r^2 = 1/\theta$

Finally, we use the formula (3.11) to compute the curvature of a spiral given by (3.13), as a function of the angle θ. The result,

$$\boldsymbol{\kappa 2}(\theta) = \frac{n\theta^{1-a/n}(1+n+n^2\theta^2)}{a(1+n^2\theta^2)^{3/2}},$$

is finite for all θ. In the special case $n=-1$ and $a=3$, we get

$$\boldsymbol{\kappa 2}(\theta) = -\frac{\theta^6}{3(1+\theta^2)^{3/2}},$$

corresponding to Figure 3.10 (left).

[5]  Pierre de Fermat (1601–1665). Fermat, like his contemporary Descartes, was trained as a lawyer. Fermat generalized the work of Archimedes on spirals. Like Archimedes he had the germs of the ideas of both differential and integral calculus – using the new techniques of analytic geometry – but failed to see the connections. Fermat is most remembered for stating the theorem that $x^n + y^n = z^n$ has no solution in integers for $n > 2$.

3.6 Exercises

1. Find the nonparametric form of the following curves:

(a) $\boldsymbol{\alpha}(t) = (t^{15}, t^6)$.

(b) The hyperbola-like curve $\boldsymbol{\gamma}(t) = (t^3, t^{-4})$.

(c) The ***strophoid*** defined by

$$\mathsf{strophoid}[a](t) = a\left(\frac{t^2-1}{t^2+1}, \frac{t(t^2-1)}{t^2+1}\right).$$

2. The ***devil's curve*** is defined nonparametrically as the zeros of the function

$$\mathsf{devilimplicit}[a,b](x,y) = y^2(y^2-b^2) - x^2(x^2-a^2).$$

Find its intercepts, and compare the result with Figure 3.11.

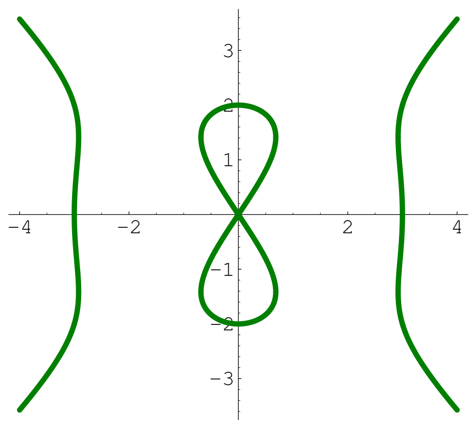

Figure 3.11: The devil's curve

M **3.** ***Kepler's folium***[6] is the curve defined nonparametrically as the set of zeros of the function

$$\mathsf{keplerimplicit}[a,b](x,y) = \big((x-b)^2+y^2\big)\big(x(x-b)+y^2\big) - 4a(x-b)y^2.$$

Plot Kepler's folium and some of its perturbations.

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6  Johannes Kepler (1571–1630). German astronomer, who lived in Prague. By empirical observations Kepler showed that a planet moves around the sun in an elliptical orbit having the sun at one of its two foci, and that a line joining the planet to the sun sweeps out equal areas in equal times as the planet moves along its orbit.

M **4.** Plot the following curves with interesting singularities ([Walk, page 56]):

(a) $x^3 - x^2 + y^2 = 0$,

(b) $x^4 + x^2y^2 - 2x^2y - xy^2 + y^2 = 0$,

(c) $2x^4 - 3x^2y + y^2 - 2y^3 - y^4 = 0$,

(d) $x^3 - y^2 = 0$,

(e) $(x^2 + y^2)^2 + 3x^2y - y^3 = 0$,

(f) $(x^2 + y^2)^3 - 4x^2y^2 = 0$.

M **5.** The parametric equations for the ***trisectrix of Maclaurin***[7] are

$$\text{trisectrix}[a](t) = \big(a(4\cos^2 t - 3),\ a(1 - 4\cos^2 t)\tan t\big).$$

Plot it, using its simpler polar equation

$$\text{polartrisectrix}[a](\theta) = a\sec\frac{\theta}{3}.$$

6. A ***nephroid*** is the kidney-shaped curve

$$\text{epicycloid}[2b, b] = b\big(3\cos t - \cos 3t,\ 2\sin t - \sin 2t\big),$$

that results from setting $a = 2b$ in Exercise 10 on page 144. Show that its equation in polar coordinates is

$$r = 2b\left(\left(\sin\frac{\theta}{2}\right)^{\frac{2}{3}} + \left(\cos\frac{\theta}{2}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}}.$$

7. The general semicubical parabola is defined implicitly by

$$y^2 = ax^3 + bx^2 + cx + d$$

Find appropriate values of a, b, c, d so as to duplicate Figure 3.12.

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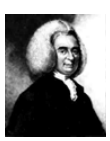
7  Colin Maclaurin (1698–1746). Scottish mathematician. His *Geometrica organica, sive discriptio linearum curvarum universalis* dealt with general properties of conics and higher plane curves. In addition to Maclaurin's own results, this book contained the proofs of many important theorems that Newton had given without proofs. Maclaurin is also known for his work on power series and the defense of his book *Theory of Fluxions* against the religious attacks of Bishop George Berkeley.

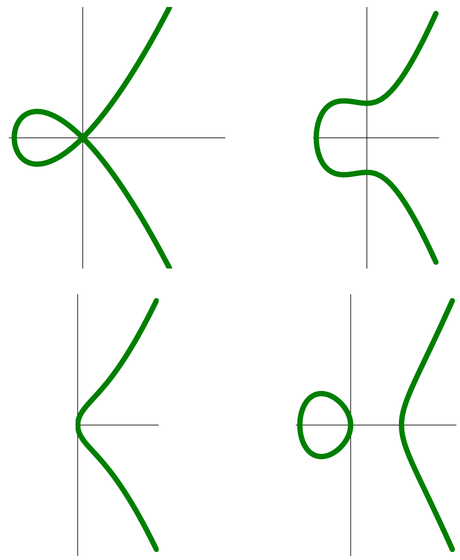


Figure 3.12: Semicubical curves $y^2 = a\,x^3 + b\,x^2 + c\,x + d$

M **8.** Verify that the transformation (3.5) does indeed convert the equation (3.4) on page 77 into (3.6) and (3.7).

9. Show that, up to sign, the signed curvature of the curve $g(x,y) = 0$ is

$$\boldsymbol{\kappa 2}(x,y) = \frac{g_{xx}g_y^2 - 2g_{xy}g_xg_y + g_{yy}g_x^2}{\left(g_x^2 + g_y^2\right)^{3/2}}.$$

M **10.** Draw a selection of the curves (3.9) for n between 1 and 1000, paying attention to the difference in behavior when n is even or odd.

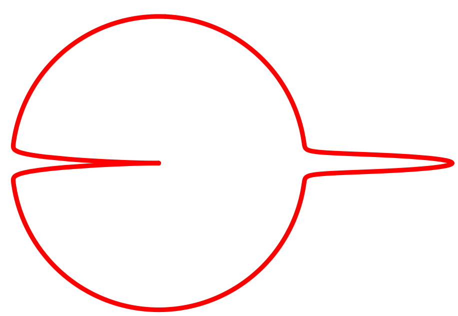


Figure 3.13: pacman$[999]$

Show that pacman$[n]$ has finite curvature at $\theta = 0$ but infinite curvature at $\theta = \pi$.