Chapter 5

Determining a Plane Curve from its Curvature

In this chapter we confront the following question: *to what extent does curvature determine a plane curve?* This question has two parts: *when do two curves have the same curvature*, and *when can a curve be determined from its curvature?*

To answer the first part, we begin by asking: *which transformations of the plane preserve curvature*? We have seen (in Theorem 1.20 on page 16) that the curvature of a plane curve is independent of the parametrization, at least up to sign. Therefore, without loss of generality, we can assume in this chapter that all curves have unit speed.

It is intuitively clear that the image of a plane curve $\boldsymbol{\alpha}$ under a rotation or translation has the same curvature as the original curve $\boldsymbol{\alpha}$. Rotations and translations are examples of Euclidean motions of $\mathbb{R}^2$, meaning those maps of $\mathbb{R}^2$ onto itself which do not distort distances. We discuss Euclidean motions in general in Section 5.1, and introduce the important concept of a *group*. We specialize Euclidean motions to the plane in Section 5.2, while the case of orthogonal transformations and rotations of $\mathbb{R}^3$ will be taken up in Chapter 23.

The invariance of curvature under Euclidean motions is established in Section 5.3, in which we also prove the *Fundamental Theorem of Plane Curves*. This important theorem states that two unit-speed curves in $\mathbb{R}^2$ that have the same curvature differ only by a Euclidean motion.

The second part of our question now becomes: *can a unit-speed curve in* $\mathbb{R}^2$ *be determined up to a Euclidean motion from its curvature*? In Section 5.3, we give an explicit system of differential equations for determining a plane curve from its curvature. In simple cases this system can be solved explicitly, but in the general case only a numerical solution is possible. Such solutions are illustrated in Section 5.4. So the answer to the second part of the question is effectively ‘yes’.

5.1 Euclidean Motions

In order to discuss the invariance of the curvature of plane curves under rotations and translations, we need to make precise what we mean by 'invariance'. This requires discussing various kinds of maps of $\mathbb{R}^2$ to itself. We begin by defining transformations of $\mathbb{R}^n$, before restricting to $n = 2$.

First, we define several important types of maps that are linear, a concept introduced in Chapter 1. The linearity property

$$A(a\,\mathbf{p} + b\,\mathbf{q}) = a\,A\mathbf{p} + b\,A\mathbf{q},$$

for $a, b \in \mathbb{R}$, implies that each of these maps takes the origin into the origin.

Definition 5.1. *Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be a nonsingular linear map.*

(i) *We say that A is* ***orientation-preserving*** *if* $\det A$ *is positive, or* ***orientation-reversing*** *if* $\det A$ *is negative.*

(ii) *A is called an* ***orthogonal transformation*** *if*

$$A\mathbf{p} \cdot A\mathbf{q} = \mathbf{p} \cdot \mathbf{q}$$

for all $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$.

(iii) *A* ***rotation*** *of $\mathbb{R}^n$ is an orientation-preserving orthogonal transformation.*

(iv) *A* ***reflection*** *of $\mathbb{R}^n$ is an orthogonal transformation of the form $\mathsf{refl}_{\mathbf{q}}$, where*

$$\mathsf{refl}_{\mathbf{q}}(\mathbf{p}) = \mathbf{p} - \frac{2(\mathbf{p} \cdot \mathbf{q})}{\|\mathbf{q}\|^2}\mathbf{q} \tag{5.1}$$

for all $\mathbf{p} \in \mathbb{R}^n$ and some fixed $\mathbf{q} \neq \mathbf{0}$.

Lemma 5.2. *Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal transformation. Then*

$$\det A = \pm 1.$$

Proof. This well-known result follows from the fact that, if we represent A by a matrix relative to an orthonormal basis, and $\mathbf{p}, \mathbf{q}$ by column vectors, then the orthogonality property is equivalent to asserting that

$$(A\mathbf{p})^T(A\mathbf{q}) = \mathbf{p}^T\mathbf{q}, \tag{5.2}$$

where A^T denotes the transpose of A. It follows that A is orthogonal if and only if A^TA is the identity matrix, and

$$1 = \det(A^TA) = \det(A^T)\det A = (\det A)^2. \quad \blacksquare$$

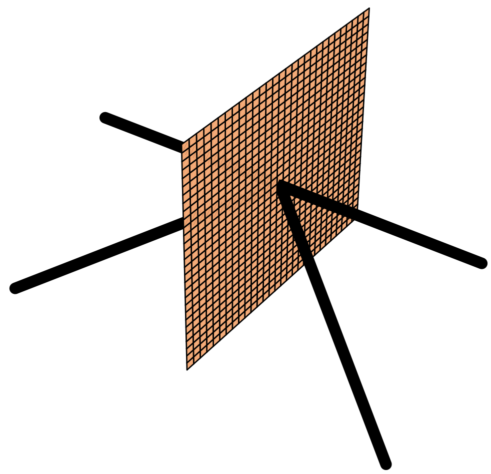

Figure 5.1: Reflection in 3 dimensions

We can characterize reflections geometrically.

Lemma 5.3. *Let* $\mathsf{refl}_{\mathbf{q}} \colon \mathbb{R}^n \to \mathbb{R}^n$ *be a reflection,* $\mathbf{q} \neq \mathbf{0}$*. Then* $\det(\mathsf{refl}_{\mathbf{q}}) = -1$*. Furthermore,* $\mathsf{refl}_{\mathbf{q}}$ *restricts to the identity map on the orthogonal complement* Π *of* $\mathbf{q}$ *(the hyperplane of* $\mathbb{R}^n$ *of dimension* $n-1$ *of the vectors orthogonal to* $\mathbf{q}$*) and reverses directions parallel to* $\mathbf{q}$*. Conversely, any hyperplane gives rise to a reflection* $\mathsf{refl}_{\mathbf{q}}$ *which restricts to the identity map on* Π *and reverses directions on the orthogonal complement of* Π*.*

Proof. Let Π be the orthogonal complement of $\mathbf{q}$, that is

$$\Pi = \{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{p} \cdot \mathbf{q} = \mathbf{0}\}.$$

Clearly, $\dim(\Pi) = n-1$, and it is easy to check that $\mathsf{refl}_{\mathbf{q}}(\mathbf{p}) = \mathbf{p}$ for all $\mathbf{p} \in \Pi$. Since $\mathsf{refl}_{\mathbf{q}}(\mathbf{q}) = -\mathbf{q}$, it follows that $\mathsf{refl}_{\mathbf{q}}$ reverses directions on the orthogonal complement of Π.

To show that $\det(\mathsf{refl}_{\mathbf{q}}) = -1$, choose a basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_{n-1}\}$ of Π. It follows that $\{\mathbf{q}, \mathbf{f}_1, \ldots, \mathbf{f}_{n-1}\}$ is a basis of $\mathbb{R}^n$. Since the matrix of $\mathsf{refl}_{\mathbf{q}}$ with respect to this basis is

$$\begin{pmatrix} -1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix},$$

we have $\det(\mathsf{refl}_{\mathbf{q}}) = -1$.

Conversely, given a hyperplane Π, let $\mathbf{q}$ be a nonzero vector perpendicular to Π. Then the reflection $\mathsf{refl}_{\mathbf{q}}$ defined by (5.1) has the required properties. ∎

Definition 5.4. *Let* $\mathbf{q} \in \mathbb{R}^n$*.*

(i) *An* ***affine transformation*** *of* $\mathbb{R}^n$ *is a map* $F\colon \mathbb{R}^n \to \mathbb{R}^n$ *of the form*

$$F(\mathbf{p}) = A\mathbf{p} + \mathbf{q}$$

for all $\mathbf{p} \in \mathbb{R}^n$*, where* A *is a linear transformation of* $\mathbb{R}^n$*. We shall call* A *the* ***linear part*** *of* F*. An affine transformation* F *is* ***orientation-preserving*** *if* $\det A$ *is positive, or* ***orientation-reversing*** *if* $\det A$ *is negative.*

(ii) *A* ***translation*** *is an affine map as above with* A *the identity, that is a mapping* $\mathsf{tran}_{\mathbf{q}}\colon \mathbb{R}^n \to \mathbb{R}^n$ *of the form*

$$\mathsf{tran}_{\mathbf{q}}(\mathbf{p}) = \mathbf{p} + \mathbf{q}$$

for all $\mathbf{p} \in \mathbb{R}^n$*.*

(iii) *A* ***Euclidean motion*** *of* $\mathbb{R}^n$ *is an affine transformation whose linear part is an* orthogonal *transformation.*

(iv) *An* ***isometry*** *of* $\mathbb{R}^n$ *is a map* $F\colon \mathbb{R}^n \to \mathbb{R}^n$ *that preserves distance, so that*

$$\|F(\mathbf{p}_1) - F(\mathbf{p}_2)\| = \|\mathbf{p}_1 - \mathbf{p}_2\|$$

for all $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^n$*. It is easy to see that an isometry must be an injective map; we shall see below that it is necessarily bijective.*

Given an affine transformation, we can study its effect on any geometrical object in $\mathbb{R}^n$. Figure 5.2 illustrates a logarithmic spiral in $\mathbb{R}^2$ and its image under an affine transformation F, a process formalized in Definition 5.11 below. Notice that F is orientation-reversing, as the inward spiralling is changed from clockwise to counterclockwise.

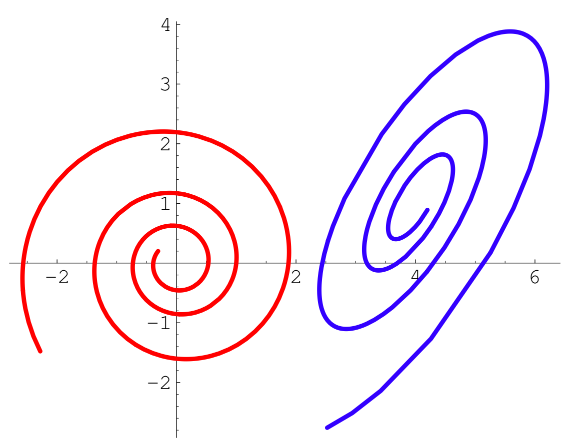

Figure 5.2: The effect of an affine transformation

To put these notions in their proper context, let us recall the following abstract algebra.

Definition 5.5. *A **group** consists of a nonempty set $\mathscr{G}$ and a multiplication*

$$\circ\colon \mathscr{G}\times\mathscr{G}\longrightarrow\mathscr{G}$$

with the following properties.

(i) *There exists an **identity element** $\mathbf{1}\in\mathscr{G}$, that is, an element $\mathbf{1}$ such that*

$$a\circ\mathbf{1}=\mathbf{1}\circ a=a$$

for all $a\in\mathscr{G}$.

(ii) *Multiplication is **associative**; that is,*

$$(a\circ b)\circ c=a\circ(b\circ c).$$

for all $a,b,c\in\mathscr{G}$.

(iii) *Every element $a\in\mathscr{G}$ has an **inverse**; that is, for each $a\in\mathscr{G}$ there exists an element $a^{-1}\in\mathscr{G}$ (that can be proved to be unique) such that*

$$a\circ a^{-1}=a^{-1}\circ a=\mathbf{1}.$$

*Moreover, a **subgroup** of a group $\mathscr{G}$ is a nonempty subset $\mathscr{S}$ of $\mathscr{G}$ closed under multiplication and inverses, that is*

$$a\circ b\in\mathscr{S}\qquad\text{and}\qquad a^{-1}\in\mathscr{S},$$

for all $a,b\in\mathscr{S}$.

It is easy to check that a subgroup of a group is itself a group. See any book on abstract algebra for more details on groups, for example [Scott] or [NiSh].

Many sets of maps form groups. Recall that if

$$F\colon\mathbb{R}^m\longrightarrow\mathbb{R}^n\qquad\text{and}\qquad G\colon\mathbb{R}^n\longrightarrow\mathbb{R}^p$$

are maps, their ***composition*** $G\circ F\colon\mathbb{R}^m\to\mathbb{R}^p$ is defined by

$$(G\circ F)(\mathbf{p})=G\big(F(\mathbf{p})\big)$$

for $\mathbf{p}\in\mathbb{R}^m$. Denote by $\mathsf{GL}(n)$ the set of all nonsingular linear maps of $\mathbb{R}^n$ into itself. It is easy to verify that composition makes $\mathsf{GL}(n)$ into a group. Similarly, we may speak of the following groups:

- the group $\mathsf{Affine}(\mathbb{R}^n)$ of affine maps of $\mathbb{R}^n$,

- the group $\mathsf{Orthogonal}(\mathbb{R}^n)$ of orthogonal transformations of $\mathbb{R}^n$,
- the group $\mathsf{Translation}(\mathbb{R}^n)$ of translations of $\mathbb{R}^n$,
- the group $\mathsf{Isometry}(\mathbb{R}^n)$ of isometries of $\mathbb{R}^n$,
- the group $\mathsf{Euclidean}(\mathbb{R}^n)$ of Euclidean motions of $\mathbb{R}^n$.

The verification that each is a group is easy. Moreover:

Theorem 5.6. *A map $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is an isometry of $\mathbb{R}^n$ if and only if it is a composition of a translation and an orthogonal transformation of $\mathbb{R}^n$. As a consequence, the group* $\mathsf{Euclidean}(\mathbb{R}^n)$ *coincides with the group* $\mathsf{Isometry}(\mathbb{R}^n)$.

Proof. It is easy to check that any orthogonal transformation or translation of $\mathbb{R}^n$ preserves the distance function. Hence the composition of an orthogonal transformation and a translation also preserves the distance function. This shows that $\mathsf{Euclidean}(\mathbb{R}^n) \subseteq \mathsf{Isometry}(\mathbb{R}^n)$.

Conversely, let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be an isometry, and define $G\colon \mathbb{R}^n \to \mathbb{R}^n$ by $G(\mathbf{p}) = F(\mathbf{p}) - F(\mathbf{0})$. Then G is an isometry and $G(\mathbf{0}) = \mathbf{0}$. To show that G preserves the scalar product of $\mathbb{R}^n$, we calculate

$$\begin{aligned} -2G(\mathbf{p})\cdot G(\mathbf{q}) &= \|G(\mathbf{p}) - G(\mathbf{q})\|^2 - \|G(\mathbf{p}) - G(\mathbf{0})\|^2 - \|G(\mathbf{q}) - G(\mathbf{0})\|^2 \\ &= \|\mathbf{p} - \mathbf{q}\|^2 - \|\mathbf{p} - \mathbf{0}\|^2 - \|\mathbf{q} - \mathbf{0}\|^2 \\ &= -2\,\mathbf{p}\cdot\mathbf{q}. \end{aligned}$$

To show that G is linear, let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an orthonormal basis of $\mathbb{R}^n$. By definition, each $\mathbf{e}_i$ has unit length and is perpendicular to every other $\mathbf{e}_j$. Then $\{G(\mathbf{e}_1), \dots, G(\mathbf{e}_n)\}$ is also an orthonormal basis of $\mathbb{R}^n$. Thus, for any $\mathbf{p} \in \mathbb{R}^n$ we have

$$G(\mathbf{p}) = \sum_{j=1}^{n} \big(G(\mathbf{p})\cdot G(\mathbf{e}_j)\big)G(\mathbf{e}_j) = \sum_{j=1}^{n} (\mathbf{p}\cdot\mathbf{e}_j)G(\mathbf{e}_j)$$

(see Exercise 1). Because each mapping $\mathbf{p} \mapsto \mathbf{p}\cdot\mathbf{e}_j$ is linear, so is G. Hence G is an orthogonal transformation of $\mathbb{R}^n$, so that F is the composition of the orthogonal transformation G and the translation $\mathsf{tran}_{F(\mathbf{0})}$. We have proved that $\mathsf{Isometry}(\mathbb{R}^n) \subseteq \mathsf{Euclidean}(\mathbb{R}^n)$. Since we have already proved the reverse inclusion, it follows that $\mathsf{Isometry}(\mathbb{R}^n) = \mathsf{Euclidean}(\mathbb{R}^n)$. ∎

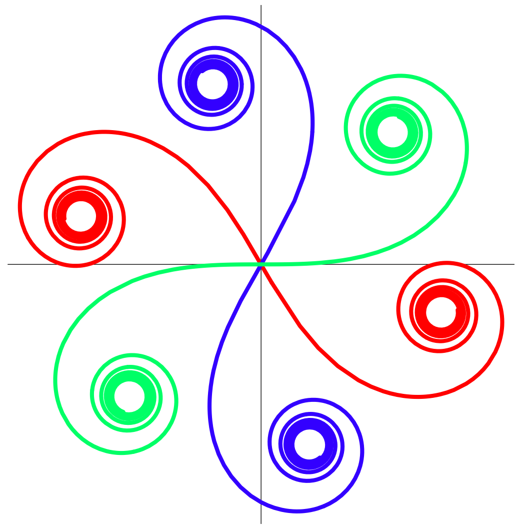

Figure 5.3: Bouquet of clothoids

5.2 Isometries of the Plane

We now specialize to $\mathbb{R}^2$. First, we give a characterization of transformations of $\mathbb{R}^2$ in terms of the linear map J, defined on page 3, represented by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Lemma 5.7. *Let A be an orthogonal transformation of $\mathbb{R}^2$. Then*

$$AJ = \varepsilon JA, \tag{5.3}$$

$$\det A = \varepsilon, \tag{5.4}$$

where

$$\varepsilon = \begin{cases} +1 & \textit{if } A \textit{ is orientation-preserving,} \\ -1 & \textit{if } A \textit{ is orientation-reversing.} \end{cases}$$

Furthermore, $\det A = -1$ *if and only if A is a reflection.*

Proof. Let $\mathbf{p} \in \mathbb{R}^2$ be a nonzero vector. Then $JA\mathbf{p} \cdot A\mathbf{p} = \mathbf{0}$; since A is orthogonal,

$$AJ\mathbf{p} \cdot A\mathbf{p} = J\mathbf{p} \cdot \mathbf{p} = \mathbf{0}.$$

Thus both $AJ\mathbf{p}$ and $JA\mathbf{p}$ are perpendicular to $A\mathbf{p}$, so that for some $\lambda \in \mathbb{R}$ we have $AJ\mathbf{p} = \lambda JA\mathbf{p}$. Since both A and J are orthogonal transformations, it is easy to see that $|\lambda| = 1$. One can now obtain both (5.3) and (5.4) by computing the matrix of A with respect to the basis $\{\mathbf{p}, J\mathbf{p}\}$ of $\mathbb{R}^2$.

Lemma 5.3 implies that the determinant of any reflection of $\mathbb{R}^2$ is -1. Conversely, an orthogonal transformation $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $\det A = -1$ must have 1 and -1 as its eigenvalues. Let $\mathbf{q} \in \mathbb{R}^2$ be such that $A(\mathbf{q}) = -\mathbf{q}$ and $\|\mathbf{q}\| = 1$. Then $A = \mathsf{refl}_{\mathbf{q}}$. ▮

The following theorem is due to Chasles[1].

Theorem 5.8. *Every isometry of $\mathbb{R}^2$ is the composition of translations, reflections and rotations.*

Proof. Theorem 5.6 implies that an isometry F of $\mathbb{R}^2$ is a Euclidean motion. There is an orthogonal transformation $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathbf{q} \in \mathbb{R}^2$ such that $F(\mathbf{p}) = A(\mathbf{p}) + \mathbf{q}$. Otherwise said,

$$F = \mathsf{tran}_{\mathbf{q}} \circ A.$$

From Lemma 5.7 we know that A is a rotation if $\det A = 1$ or a reflection if $\det A = -1$. ▮

A standard procedure for simplifying the equation

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

of a conic into 'normal form' corresponds to first rotating the vector (x, y) so as to eliminate the term in xy (in the new coordinates) and then translating it to eliminate the terms in x and y. But it is sometimes convenient to eliminate the linear terms first, so as determine the center, and then rotate. These two contrasting methods are illustrated in the Figure 5.4, starting with the ellipse in (say) the first quadrant.

By expressing an arbitrary vector $\mathbf{p} \in \mathbb{R}^2$ as a linear combination of two nonzero vectors $\mathbf{q}$, $J\mathbf{q}$, and using (5.3), we may deduce

Lemma 5.9. *Let A be a rotation of $\mathbb{R}^2$. Then the numbers*

$$\frac{A\mathbf{p}\cdot\mathbf{p}}{\|\mathbf{p}\|^2} \qquad \textit{and} \qquad \frac{AJ\mathbf{p}\cdot\mathbf{p}}{\|\mathbf{p}\|^2}$$

are each independent of the nonzero vector $\mathbf{p}$.

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1  Michel Chasles (1793–1880). French geometer and mathematical historian. He worked on algebraic and projective geometry.

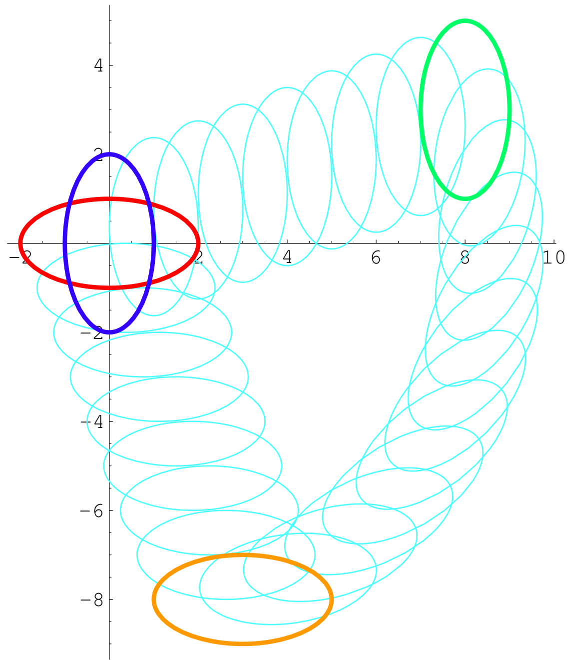

Figure 5.4: Translations and rotations of a conic

Definition 5.10. *Let A be a rotation of $\mathbb{R}^2$. The* ***angle of rotation*** *of A is any number θ such that*

$$\cos\theta = \frac{A\mathbf{p}\cdot\mathbf{p}}{\|\mathbf{p}\|^2} \qquad \textit{and} \qquad \sin\theta = \frac{AJ\mathbf{p}\cdot\mathbf{p}}{\|\mathbf{p}\|^2}$$

for all nonzero $\mathbf{p}\in\mathbb{R}^2$. Hence using the identification (1.1) *of $\mathbb{C}$ with $\mathbb{R}^2$, we can write*

$$A\mathbf{p} = (\cos\theta)\mathbf{p} + (\sin\theta)J\mathbf{p} = e^{i\theta}\mathbf{p}$$

for all $\mathbf{p}\in\mathbb{R}^2$.

It is clear intuitively that any mapping of $\mathbb{R}^2$ into itself maps a curve into another curve. Here is the exact definition:

Definition 5.11. *Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a map, and let $\boldsymbol{\alpha}\colon (a,b)\to\mathbb{R}^2$ be a curve. The* ***image of $\boldsymbol{\alpha}$ under*** *F is the curve $F\circ\boldsymbol{\alpha}$.*

If for example F is an affine transformation with linear part represented by a matrix A, then the image of $\boldsymbol{\alpha}(t)$ is found by *pre*multiplying the column vector $\boldsymbol{\alpha}(t)$ by A and then adding the translation vector. This fact is the basis of the practical implementation of rotations in Notebooks 5 and 23. It was used to produce Figure 5.3, which shows a clothoid and its images under two rotations.

5.3 Intrinsic Equations for Plane Curves

Since a Euclidean motion F of the plane preserves distance, it cannot stretch or otherwise distort a plane curve. We now show that F preserves the curvature up to sign.

Theorem 5.12. *The absolute value of the curvature and the derivative of arc length of a curve are invariant under Euclidean motions of* $\mathbb{R}^2$. *The signed curvature* $\boldsymbol{\kappa}\mathbf{2}$ *is preserved by an orientation-preserving Euclidean motion of* $\mathbb{R}^2$ *and changes sign under an orientation-reversing Euclidean motion.*

Proof. Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$ be a curve, and let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a Euclidean motion. Let A denote the linear part of F, so that for all $\mathbf{p} \in \mathbb{R}^2$ we have $F(\mathbf{p}) = A\mathbf{p} + F(0)$. Define a curve $\boldsymbol{\gamma}\colon (a,b) \to \mathbb{R}^2$ by $\boldsymbol{\gamma} = F \circ \boldsymbol{\alpha}$; then for $a < t < b$ we have

$$\boldsymbol{\gamma}(t) = A\boldsymbol{\alpha}(t) + F(0).$$

Hence $\boldsymbol{\gamma}'(t) = A\boldsymbol{\alpha}'(t)$ and $\boldsymbol{\gamma}''(t) = A\boldsymbol{\alpha}''(t)$. Let s_α and s_γ denote the arc length functions with respect to $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$. Since A is an orthogonal transformation, we have

$$s_\gamma'(t) = \big\|\boldsymbol{\gamma}'(t)\big\| = \big\|A\boldsymbol{\alpha}'(t)\big\| = s_\alpha'(t).$$

We compute the curvature of $\boldsymbol{\gamma}$ using (5.3):

$$\begin{aligned} \boldsymbol{\kappa}\mathbf{2}[\boldsymbol{\gamma}](t) \;&=\; \frac{\boldsymbol{\gamma}''(t)\cdot J\boldsymbol{\gamma}'(t)}{\big\|\boldsymbol{\gamma}'(t)\big\|^3} \;=\; \frac{A\boldsymbol{\alpha}''(t)\cdot \varepsilon A\,J\boldsymbol{\alpha}'(t)}{\big\|A\boldsymbol{\alpha}'(t)\big\|^3} \\ &\;=\; = \varepsilon\frac{\boldsymbol{\alpha}''(t)\cdot J\boldsymbol{\alpha}'(t)}{\big\|\boldsymbol{\alpha}'(t)\big\|^3} = \varepsilon\,\boldsymbol{\kappa}\mathbf{2}[\boldsymbol{\alpha}](t). \quad \blacksquare \end{aligned}$$

Next, we prove the converse of Theorem 5.12.

Theorem 5.13. (Fundamental Theorem of Plane Curves, Uniqueness) *Let* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\gamma}$ *be unit-speed regular curves in* $\mathbb{R}^2$ *defined on the same interval* (a,b), *and having the same signed curvature. Then there is an orientation-preserving Euclidean motion* F *of* $\mathbb{R}^2$ *mapping* $\boldsymbol{\alpha}$ *into* $\boldsymbol{\gamma}$.

Proof. Fix $s_0 \in (a,b)$. Clearly, there exists a translation of $\mathbb{R}^2$ taking $\boldsymbol{\alpha}(s_0)$ into $\boldsymbol{\gamma}(s_0)$. Moreover, we can find a rotation of $\mathbb{R}^2$ that maps $\boldsymbol{\alpha}'(s_0)$ into $\boldsymbol{\gamma}'(s_0)$. Thus there exists an orientation-preserving Euclidean motion F of $\mathbb{R}^2$ such that

$$F\big(\boldsymbol{\alpha}(s_0)\big) = \boldsymbol{\gamma}(s_0) \qquad \text{and} \qquad F\big(\boldsymbol{\alpha}'(s_0)\big) = \boldsymbol{\gamma}'(s_0).$$

To show that $F \circ \boldsymbol{\alpha}$ coincides with $\boldsymbol{\gamma}$, we define a real-valued function f by

$$f(s) = \big\|(F \circ \boldsymbol{\alpha})'(s) - \boldsymbol{\gamma}'(s)\big\|^2$$

for $a < s < b$. The derivative of f is easily computed to be

$$\begin{aligned} f'(s) &= 2\big((F \circ \boldsymbol{\alpha})''(s) - \boldsymbol{\gamma}''(s)\big) \cdot \big((F \circ \boldsymbol{\alpha})'(s) - \boldsymbol{\gamma}'(s)\big) \\ &= 2\big((F \circ \boldsymbol{\alpha})''(s) \cdot (F \circ \boldsymbol{\alpha})'(s) + \boldsymbol{\gamma}''(s) \cdot \boldsymbol{\gamma}'(s) \\ &\qquad - (F \circ \boldsymbol{\alpha})''(s) \cdot \boldsymbol{\gamma}'(s) - (F \circ \boldsymbol{\alpha})'(s) \cdot \boldsymbol{\gamma}''(s)\big). \end{aligned} \tag{5.5}$$

Since both $F \circ \boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ have unit speed, it follows that

$$(F \circ \boldsymbol{\alpha})''(s) \cdot (F \circ \boldsymbol{\alpha})'(s) = \boldsymbol{\gamma}''(s) \cdot \boldsymbol{\gamma}'(s) = 0.$$

Hence (5.5) reduces to

$$f'(s) = -2\big((F \circ \boldsymbol{\alpha})''(s) \cdot \boldsymbol{\gamma}'(s) + (F \circ \boldsymbol{\alpha})'(s) \cdot \boldsymbol{\gamma}''(s)\big). \tag{5.6}$$

Let A denote the linear part of the motion F. Since $(F \circ \boldsymbol{\alpha})'(s) = A\boldsymbol{\alpha}'(s)$ and $(F \circ \boldsymbol{\alpha})''(s) = A\boldsymbol{\alpha}''(s)$, we can rewrite (5.6) as

$$f'(s) = -2\big(A\boldsymbol{\alpha}''(s) \cdot \boldsymbol{\gamma}'(s) + A\boldsymbol{\alpha}'(s) \cdot \boldsymbol{\gamma}''(s)\big). \tag{5.7}$$

Now we use the assumption that $\boldsymbol{\kappa 2}[\boldsymbol{\alpha}] = \boldsymbol{\kappa 2}[\boldsymbol{\gamma}]$ and (5.7) to get

$$\begin{aligned} f'(s) &= -2\big(\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](s) A\, J\boldsymbol{\alpha}'(s) \cdot \boldsymbol{\gamma}'(s) + A\boldsymbol{\alpha}'(s) \cdot \boldsymbol{\kappa 2}[\boldsymbol{\gamma}](s) J\boldsymbol{\gamma}'(s)\big) \\ &= -2\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](s)\big(JA\boldsymbol{\alpha}'(s) \cdot \boldsymbol{\gamma}'(s) + A\boldsymbol{\alpha}'(s) \cdot J\boldsymbol{\gamma}'(s)\big) = 0. \end{aligned}$$

Since $f(s_0) = 0$, we conclude that $f(s) = 0$ for all s. Hence $(F \circ \boldsymbol{\alpha})'(s) = \boldsymbol{\gamma}'(s)$ for all s, and so there exists $\mathbf{q} \in \mathbb{R}^2$ such that $(F \circ \boldsymbol{\alpha})(s) = \boldsymbol{\gamma}(s) - \mathbf{q}$ for all s. In fact, $\mathbf{q} = 0$ because $F\big(\boldsymbol{\alpha}(s_0)\big) = \boldsymbol{\gamma}(s_0)$. Thus the Euclidean motion F maps $\boldsymbol{\alpha}$ into $\boldsymbol{\gamma}$. ∎

Next, we turn to the problem of explicitly determining a plane curve from its curvature. Theory from Chapter 1 enables us to prove

Theorem 5.14. (Fundamental Theorem of Plane Curves, Existence) *A unit-speed curve $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^2$ whose curvature is a given piecewise-continuous function $k\colon (a,b) \to \mathbb{R}$ is parametrized by*

$$\begin{cases} \boldsymbol{\beta}(s) = \left(\displaystyle\int \cos\theta(s)\,ds + c,\ \int \sin\theta(s)\,ds + d\right), \\ \theta(s) = \displaystyle\int k(s)\,ds + \theta_0, \end{cases} \tag{5.8}$$

where c, d, θ_0 are constants of integration.

Proof. We define $\boldsymbol{\beta}$ and θ by (5.8); it follows that

$$\begin{cases} \boldsymbol{\beta}'(s) = \big(\cos\theta(s),\ \sin\theta(s)\big), \\ \theta'(s) = k(s). \end{cases} \tag{5.9}$$

Thus $\boldsymbol{\beta}$ has unit speed, so Corollary 1.27 on page 20 tells us that the curvature of $\boldsymbol{\beta}$ is k. ∎

One classical way to describe a plane curve $\boldsymbol{\alpha}$ is by means of a ***natural equation***, which expresses the curvature $\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]$ (at least implicitly) in terms of the arc length s of $\boldsymbol{\alpha}$. Although such an equation may not be the most useful equation for calculational purposes, it shows clearly how curvature changes with arc length and is obviously invariant under translations and rotations. To find the natural equation of a curve $t \mapsto \boldsymbol{\alpha}(t)$, one first computes the curvature and arc length as functions of t and then one tries to eliminate t. Conversely, the ordinary equation can be found (at least in principle) by solving the natural equation and using Theorem 5.14.

The following table lists some examples, while Figure 5.5 plots variants of its last two entries.

Curve	Natural equation
straight line	$\boldsymbol{\kappa 2} = 0$
circle of radius a	$\boldsymbol{\kappa 2} = \dfrac{1}{a}$
catenary$[a]$	$\boldsymbol{\kappa 2} = -\dfrac{a}{a^2 + s^2}$
clothoid$[n, a]$	$\boldsymbol{\kappa 2} = -\dfrac{s^n}{a^{n+1}}$
logspiral$[a, b]$	$\boldsymbol{\kappa 2} = \dfrac{1}{bs}$
involute[circle$[a]$]	$\boldsymbol{\kappa 2} = \dfrac{1}{\sqrt{2as}}$

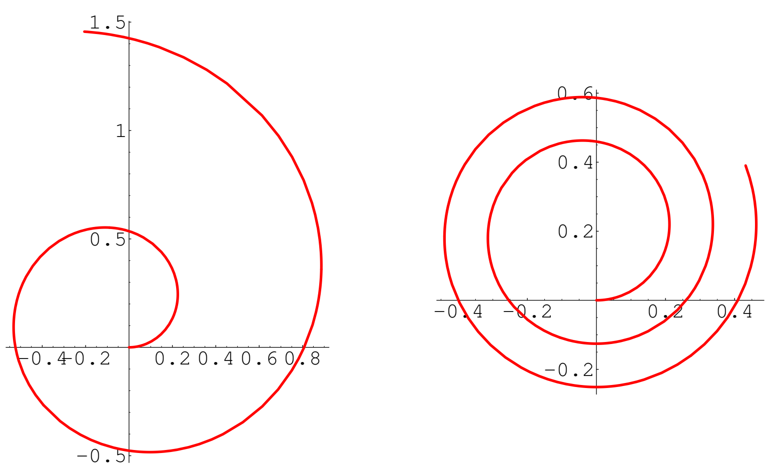

Figure 5.5: Curves with $\boldsymbol{\kappa 2}(s) = 5/(s+1)$ and $5/\sqrt{s+1}$

Natural equations arose as a reaction against the use of Cartesian and polar coordinates, which, despite their utility, were considered arbitrary. In 1736, Euler[2] proposed the use of the arc length s and the radius of curvature $1/|\boldsymbol{\kappa 2}|$ (see [Euler1]). Natural equations were also studied by Lacroix[3] and Hill[4], and excellent discussions of them are given in [Melz, volume 2, pages 33–41] and [Ces]. The book [Ces] of Cesàro[5] uses natural equations as its starting point; it contains many plots (quite remarkable for a book published at the end of the 19$^{\text{th}}$ century) of plane curves given by means of natural equations.

For example, let us find the natural equation of the catenary defined by

$$\mathsf{catenary}[a](t) = \left(a\cosh\frac{t}{a},\, t\right). \tag{5.10}$$

A computation tells us that the curvature and arc length are given by

$$\boldsymbol{\kappa 2}(t) = -\frac{\operatorname{sech}(t/a)^2}{a} \qquad \text{and} \qquad s(t) = a\sinh\frac{t}{a}. \tag{5.11}$$

Then (5.11) implies that

$$(s^2 + a^2)\boldsymbol{\kappa 2} = -a. \tag{5.12}$$

Although (5.10) and (5.12) both describe a catenary, (5.12) is considered more natural because it does not depend on the choice of parametrization of the catenary.

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2 
Leonhard Euler (1707–1783). Swiss mathematician. Euler was a geometer in the broad sense in which the term was used during his time. Not only did he contribute greatly to the evolution and systematization of analysis – in particular to the founding of the calculus of variation and the theories of differential equations, functions of complex variables, and special functions – but he also laid the foundations of number theory as a rigorous discipline. Moreover, he concerned himself with applications of mathematics to fields as diverse as lotteries, hydraulic systems, shipbuilding and navigation, actuarial science, demography, fluid mechanics, astronomy, and ballistics.

3 Sylvestre Francois Lacroix (1765–1843). French writer of mathematical texts who was a student of Monge.

4 Thomas Hill (1818–1891). American mathematician who became president of Harvard University.

5 
Ernesto Cesàro (1859–1906). Italian mathematician, born in Naples, professor in Palermo and Rome. Although his most important contribution was his monograph *Lezioni di Geometria Intrinseca*, he is also remembered for his work on divergent series.

5.4 Examples of Curves with Assigned Curvature

A curve $\boldsymbol{\beta}$ parametrized by arc length s whose curvature is a given function $\boldsymbol{\kappa 2} = k(s)$ can be found by solving the system (5.9). Notebook 5 finds numerical solutions of the resulting ordinary differential equations. In this way, we may readily draw plane curves with assigned curvature.

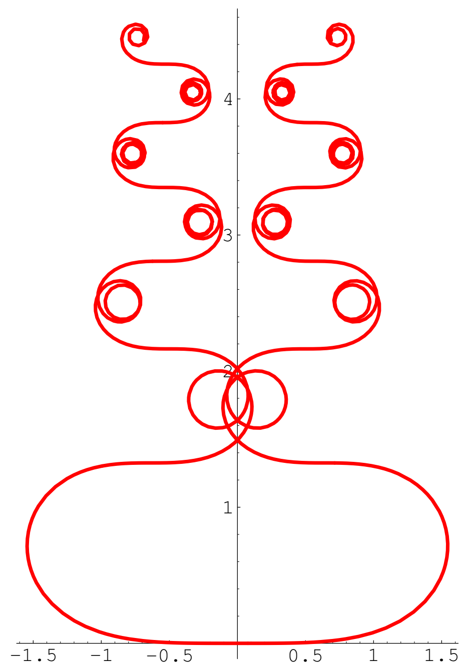

Figure 5.6: A curve with $\boldsymbol{\kappa 2}(s) = s \sin s$

Quite simple curvature functions can be used to produce very interesting curves. We have already seen in Section 2.7 that the curve clothoid$[1,1]$ has curvature given by $\boldsymbol{\kappa 2}(s) = -s$, where s denotes arc length. The clothoids in Figure 5.3 were in fact generated by using this program.

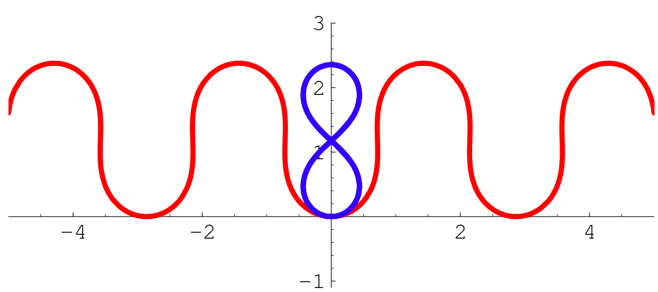

Figure 5.7: Curves with $\boldsymbol{\kappa 2}(s) = 1.6 \sin s$ and $\boldsymbol{\kappa 2}(s) = 2.4 \sin s$

Figure 5.6 displays the case $\boldsymbol{\kappa 2}(s) = s \sin s$. But setting $\boldsymbol{\kappa 2}$ to be a mere multiple of $\sin s$ presents an intriguing situation. Figure 5.7 plots simultaneously two curves whose curvature has the form $\boldsymbol{\kappa 2}(s) = c \sin s$ with c constant; the one in the middle with $c = 2.4$ appears to be a closed curve with this property. Notebook 5 includes an animation of all the intermediate values of c.

Finally, we consider several functions that oscillate more wildly, namely those of the form $s \mapsto s f(s)$, where f is one of the Bessel[6] functions J_i. As well as solving a natural differential equation, these functions occur naturally as coefficients of the Laurent series of the function $\exp\big(t(z - z^{-1})\big)$. This and other useful expressions for them can be found in books on complex analysis, such as [Ahlf]. Below we display both the function f and the resulting curve, omitting the axis scales for visual clarity.

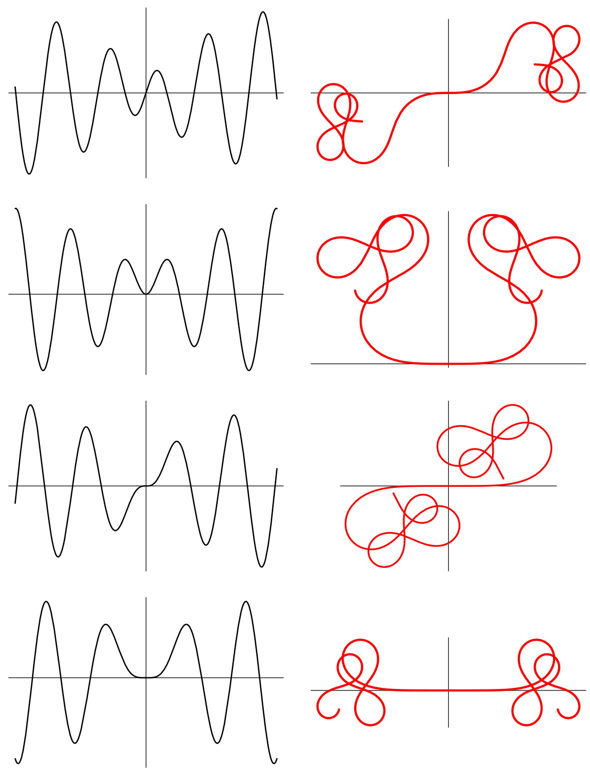

Figure 5.8: Curves with $\boldsymbol{\kappa 2}(s) = s\, J_n(s)$ with $n = 1, 2, 3, 4$

6  Friedrich Wilhelm Bessel (1784–1846). German astronomer and friend of Gauss. In 1800, Bessel was appointed director of the observatory at Königsberg, where he remained for the rest of his life. Bessel functions were introduced in 1817 in connection with a problem of Kepler of determining the motion of three bodies moving under mutual gravitation.

5.5 Exercises

1. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Show that any vector $\mathbf{v} \in \mathbb{R}^n$ can be written as

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 + \cdots + (\mathbf{v} \cdot \mathbf{e}_n)\mathbf{e}_n.$$

2. A ***homothety*** of $\mathbb{R}^n$ is a map $F\colon \mathbb{R}^n \to \mathbb{R}^n$ for which there exists a constant $\lambda > 0$ such that

$$\|F(\mathbf{p}_1) - F(\mathbf{p}_2)\| = \lambda \|\mathbf{p}_1 - \mathbf{p}_2\|$$

for all $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^n$. Show that a homothety is a bijective affine transformation.

3. Show that a Euclidean motion $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ preserves evolutes, involutes, parallel curves and pedal curves. More precisely, if $\boldsymbol{\alpha}\colon (a, b) \to \mathbb{R}^2$ is a curve, show that the following formulas hold for $a < t < b$:

$$\begin{aligned}
\mathsf{evolute}[F \circ \boldsymbol{\alpha}](t) &= \big(F \circ \mathsf{evolute}[\boldsymbol{\alpha}]\big)(t), \\
\mathsf{involute}[F \circ \boldsymbol{\alpha}, c](t) &= \big(F \circ \mathsf{involute}[\boldsymbol{\alpha}, c]\big)(t), \ . \\
\mathsf{pedal}[F(\mathbf{p}), F \circ \boldsymbol{\alpha}](t) &= \big(F \circ \mathsf{pedal}[\mathbf{p}, \boldsymbol{\alpha}]\big)(t),
\end{aligned}$$

and

$$\mathsf{parcurve}[F \circ \boldsymbol{\alpha}][s](t) = \begin{cases} \big(F \circ \mathsf{parcurve}[\boldsymbol{\alpha}][s]\big)(t) \\ \qquad\qquad \text{if } F \text{ is orientation-preserving,} \\ \big(F \circ \mathsf{parcurve}[\boldsymbol{\alpha}][-s]\big)(t) \\ \qquad\qquad \text{if } F \text{ is orientation-reversing.} \end{cases}$$

4. An ***algebraic curve*** is an implicitly-defined curve of the form $P(x, y) = 0$, where P is a polynomial in x and y. The ***order*** of an algebraic curve is the order of the polynomial P. Show that an affine transformation preserves the order of an algebraic curve.

5. Find the natural equations of the first three involutes of a circle.

6. ***Nielsen's spiral***[7] is defined as

$$\mathsf{nielsenspiral}[a](t) = a\big(\mathrm{Ci}(t),\ \mathrm{Si}(t)\big),$$

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[7]Niels Nielsen (1865–1931). Danish mathematician.

where the sine and cosine integrals are defined by

$$\mathrm{Si}(t) = \int_0^t \frac{\sin u}{u} du, \qquad \mathrm{Ci}(t) = -\int_t^\infty \frac{\cos u}{u} du$$

for $t > 0$. It is illustrated in Figure 5.9.

(a) Explain why the spiral is asymptotic to the x-axis as $t \to -\infty$, and find its limit as $t \to \infty$.

(b) Show that the function $\mathrm{Ci}(t)$ can be defined in the equivalent way

$$\mathrm{Ci}(t) = \int_0^t \frac{(\cos u - 1)}{u} du + \log t + \gamma,$$

again for $t > 0$, where γ is Euler's constant $0.5772\ldots$

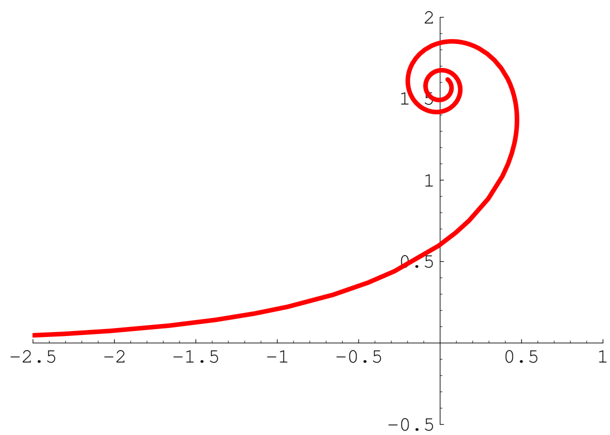

Figure 5.9: Nielson's spiral plotted with $-2.5 < t < 15$

7. Show that the natural equation of the spiral in the previous exercise is

$$\boldsymbol{\kappa 2}(s) = \frac{e^{s/a}}{a}.$$

M **8.** Plot curves whose curvature is $n\,s \sin s$ for $n = \pm 2, \pm 2.5, \pm 3$.

M **9.** Plot a curve whose curvature is $s \mapsto \Gamma(s + \frac{1}{2})$, where $-5 < s < 5$, and the ***Gamma function*** is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

M **10.** An ***epicycloid*** is the curve that is traced out by a point $\mathbf{p}$ on the circumference of a circle (of radius b) rolling *outside* another circle (of radius a). Similarly, a ***hypocycloid*** is the curve that is traced out by a point $\mathbf{p}$ on the circumference of a circle (of radius b) rolling *inside* another circle (of radius a).

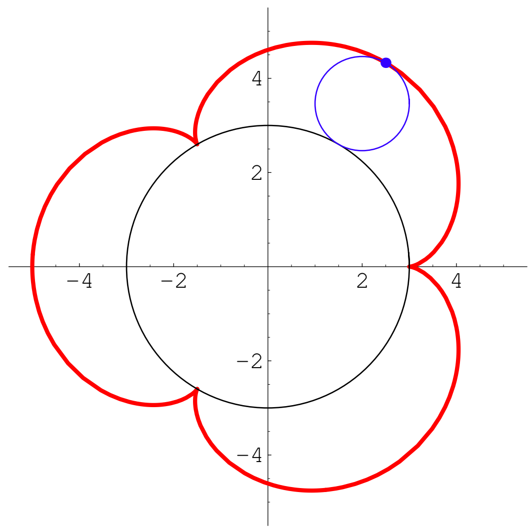

Figure 5.10: An epicycloid with 3 cusps

These curves are parametrized by

$$\text{epicycloid}[a,b](t) = \left((a+b)\cos t - b\cos\left(\frac{(a+b)t}{b}\right),\right.$$
$$\left.(a+b)\sin t - b\sin\left(\frac{(a+b)t}{b}\right)\right),$$

$$\text{hypocycloid}[a,b](t) = \left((a-b)\cos t + b\cos\left(\frac{(a-b)t}{b}\right),\right.$$
$$\left.(a-b)\sin t - b\sin\left(\frac{(a-b)t}{b}\right)\right),$$

and illustrated in Figures 5.10 and 5.11. Show that the natural equation of an epicycloid is

$$a^2 s^2 + \left(\frac{a+2b}{\boldsymbol{\kappa}\mathbf{2}}\right)^2 = 16b^2(a+b)^2, \tag{5.13}$$

and find that of the hypocycloid.

M **11.** Let $\boldsymbol{\alpha}_n\colon(-\infty,\infty)\to\mathbb{R}^2$ be a curve whose curvature is $n\sin^2 s$, where n is an integer and s denotes arc length. Show that $\boldsymbol{\alpha}_n$ is a closed curve if and only if n is *not* divisible by 4. Plot $\boldsymbol{\alpha}_n$ for $1\leqslant n\leqslant 12$.

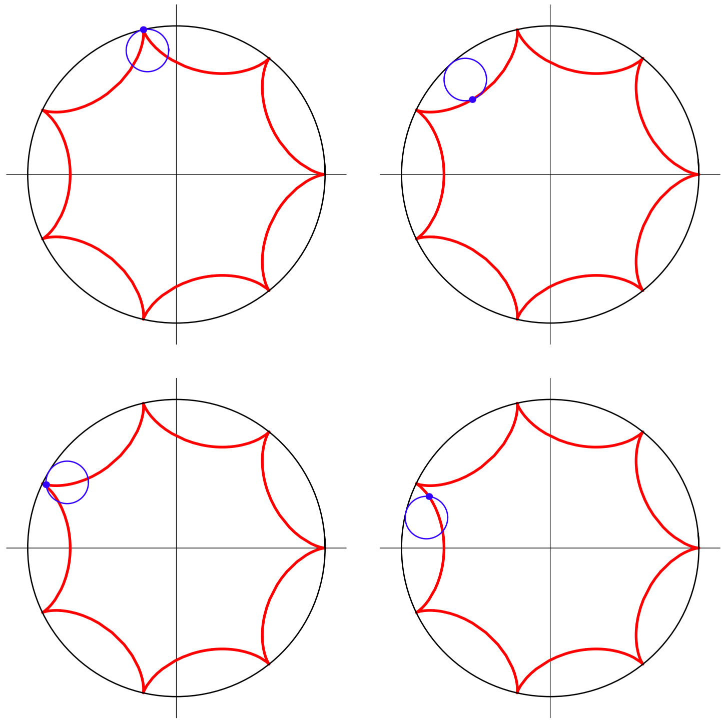

Figure 5.11: A hypocycloid with 7 cusps

12. Show that the natural equation of a tractrix is

$$\boldsymbol{\kappa 2} = -\frac{1}{a\sqrt{e^{2s/a}-1}}.$$

M **13.** The ***Airy***[8] ***functions***, denoted AiryAi and AiryBi, are linearly independent solutions of the differential equation $y''(t) - t\,y(t) = 0$. Plot curves whose signed curvature is respectively $s \mapsto \mathsf{AiryAi}(s)$ and $s \mapsto \mathsf{AiryBi}(s)$.

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[8]  Sir George Biddel Airy (1801–1892) Royal astronomer of England.