Chapter 1

Curves in the Plane

Geometry before calculus involves only the simplest of curves: straight lines, broken lines, arcs of circles, ellipses, hyperbolas and parabolas. These are the curves that form the basis of Greek geometry. Although other curves (such as the cissoid of Diocles and the spiral of Archimedes) were known in antiquity, the general theory of plane curves began to be developed only after the invention of Cartesian coordinates by Descartes[1] in the early 1600s.

Interesting plane curves arise in everyday life. Examples include the trajectory of a projectile (a parabola), the form of a suspension bridge (a catenary), the path of a point on the wheel of a car (a cycloid), and the orbit of a planet (an ellipse). In this chapter we get underway with the study of the local properties of curves that has developed over the last few centuries.

We begin in Section 1.1 by recalling some standard operations on Euclidean space $\mathbb{R}^n$, and in Section 1.2 define the notions of curves in $\mathbb{R}^n$ and vector fields along them. Arc length for curves in $\mathbb{R}^n$ is defined and discussed in Section 1.3. In Section 1.4, we define the important notion of signed curvature of a curve in the plane $\mathbb{R}^2$. Section 1.5 is devoted to the problem of defining an angle function between plane curves, and this allows us to show that the signed curvature is the derivative of the turning angle. The examples discussed in Section 1.6 include the logarithmic spiral, while the arc length of the semicubical parabola is computed in Section 1.7.

Our basic approach in Chapters 1 and 2 is to study curves by means of their parametric representations, whereas implicitly defined curves in $\mathbb{R}^2$ are discussed in Section 3.1. In this chapter, we must therefore show that any

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1  René du Perron Descartes (1596–1650). French mathematician and philosopher. Descartes developed algebraic techniques to solve geometric problems, thus establishing the foundations of analytic geometry. Although most widely known as a philosopher, he also made important contributions to physiology, optics and mechanics.

geometric invariant depending only on the point set traced out by the curve is independent of the parametrization, at least up to sign. The most important geometric quantities associated with a curve are of two types: (1) those that are totally independent of parametrization, and (2) those that do not change under a positive reparametrization, but change sign under a negative reparametrization. For example, we show in Section 1.3 that the length of a curve is independent of the parametrization chosen. The curvature of a plane curve is a more subtle invariant, because it does change sign if the curve is traversed in the opposite direction, but is otherwise independent of the parametrization. This fact we prove in Section 1.4.

In Section 1.6 we give the some examples of plane curves that generalize circles, discuss their properties and graph associated curvature functions.

1.1 Euclidean Spaces

Since we shall be studying curves and surfaces in a Euclidean space, we summarize some of the algebraic properties of Euclidean space in this section.

Definition 1.1. ***Euclidean** n**-space*** $\mathbb{R}^n$ *consists of the set of all real n-tuples*

$$\mathbb{R}^n = \big\{ (p_1, \dots, p_n) \mid p_j \text{ is a real number for } j = 1, \dots, n \big\}.$$

*We write $\mathbb{R}$ for $\mathbb{R}^1$; it is simply the set of all real numbers. $\mathbb{R}^2$ is frequently called the **plane**.*

Elements of $\mathbb{R}^n$ represent both points in n-dimensional space, and the *position vectors* of points. As a consequence, $\mathbb{R}^n$ is a vector space, so that the operations of *addition* and *scalar multiplication* are defined. Thus if

$$\mathbf{p} = (p_1, \dots, p_n) \quad \text{and} \quad \mathbf{q} = (q_1, \dots, q_n),$$

then $\mathbf{p} + \mathbf{q}$ is the element of $\mathbb{R}^n$ given by

$$\mathbf{p} + \mathbf{q} = (p_1 + q_1, \dots, p_n + q_n).$$

Similarly, for $\lambda \in \mathbb{R}$ the vector $\lambda\mathbf{p}$ is defined by

$$\lambda\mathbf{p} = (\lambda p_1, \dots, \lambda p_n).$$

Furthermore, we shall denote by $\cdot$ the ***dot product*** (or scalar product) of $\mathbb{R}^n$. It is an operation that assigns to each pair of vectors $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{q} = (q_1, \dots, q_n)$ the real number

$$\mathbf{p} \cdot \mathbf{q} = \sum_{j=1}^{n} p_j q_j.$$

The ***norm*** and ***distance*** functions of $\mathbb{R}^n$ can then be defined by

$$\|\mathbf{p}\| = \sqrt{\mathbf{p} \cdot \mathbf{p}}, \qquad \text{and} \qquad \mathsf{distance}(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|$$

for $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$.

These functions have the properties

$$\begin{aligned} \mathbf{p} \cdot \mathbf{q} &= \mathbf{q} \cdot \mathbf{p}, & (\mathbf{p} + \mathbf{r}) \cdot \mathbf{q} &= \mathbf{p} \cdot \mathbf{q} + \mathbf{r} \cdot \mathbf{q}, \\ (\lambda \mathbf{p}) \cdot \mathbf{q} &= \lambda(\mathbf{p} \cdot \mathbf{q}) = \mathbf{p} \cdot (\lambda \mathbf{q}), & \|\lambda \mathbf{p}\| &= |\lambda| \, \|\mathbf{p}\|, \end{aligned}$$

for $\lambda \in \mathbb{R}$ and $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbb{R}^n$. Furthermore, the ***Cauchy–Schwarz*** and ***triangle*** inequalities state that for all $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ we have

$$|\mathbf{p} \cdot \mathbf{q}| \leqslant \|\mathbf{p}\| \, \|\mathbf{q}\| \qquad \text{and} \qquad \|\mathbf{p} + \mathbf{q}\| \leqslant \|\mathbf{p}\| + \|\mathbf{q}\|.$$

(See, for example, [MS, pages 22–24].)

We shall also need the notion of the ***angle*** between nonzero vectors $\mathbf{p}, \mathbf{q}$ of $\mathbb{R}^n$, which is a number θ defined by:

$$\cos\theta = \frac{\mathbf{p} \cdot \mathbf{q}}{\|\mathbf{p}\| \, \|\mathbf{q}\|}.$$

The Cauchy–Schwarz inequality implies that

$$-1 \leqslant \frac{\mathbf{p} \cdot \mathbf{q}}{\|\mathbf{p}\| \, \|\mathbf{q}\|} \leqslant 1$$

for nonzero $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$, so that the definition of the angle makes sense. At times, it is convenient to specify the range of θ using Lemma 1.3 below.

A ***linear map*** of $\mathbb{R}^n$ into $\mathbb{R}^m$ is a function $A\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$A(\lambda \mathbf{p} + \mu \mathbf{q}) = \lambda A \mathbf{p} + \mu A \mathbf{q}$$

for $\lambda, \mu \in \mathbb{R}$ and $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$. We may regard such a map A as a $m \times n$ matrix, though to make the above definition consistent, one needs to represent elements of $\mathbb{R}^n$ by columns rather than rows. This is because the usual convention dictates that matrices, like functions, should act on the left.

So far we have been dealing with $\mathbb{R}^n$ for general n. When n is 2 or 3, the vector space $\mathbb{R}^n$ has special structures, namely a complex structure and a vector cross product, which are useful for describing curves and surfaces respectively. In Chapters 1 – 6 we shall be studying curves in the plane, so we concentrate attention on $\mathbb{R}^2$ now. Algebraic properties of $\mathbb{R}^3$ will be considered in Section 7.1, and those of $\mathbb{R}^4$ in Chapter 23.

For the differential geometry of curves in the plane an essential tool is the ***complex structure*** of $\mathbb{R}^2$; it is the linear map $J\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$J(p_1, \, p_2) = (-p_2, \, p_1).$$

Geometrically, J is merely a rotation by $\pi/2$ in a counterclockwise direction. It is easy to show that the complex structure J has the properties

$$J^2 = -\mathbf{1},$$
$$(J\mathbf{p}) \cdot (J\mathbf{q}) = \mathbf{p} \cdot \mathbf{q},$$
$$(J\mathbf{p}) \cdot \mathbf{p} = 0,$$

for $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2$ (where $\mathbf{1} \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the identity map).

A point in the plane $\mathbb{R}^2$ can be considered a complex number via the canonical isomorphism

$$\mathbf{p} = (p_1, p_2) \;\leftrightarrow\; p_1 + i p_2 = \mathfrak{Re}\,\mathbf{p} + i\,\mathfrak{Im}\,\mathbf{p}, \tag{1.1}$$

where $\mathfrak{Re}\,\mathbf{p}$ and $\mathfrak{Im}\,\mathbf{p}$ denote the real and imaginary parts of $\mathbf{p}$. In a moment, we shall need descriptions of the dot product $\cdot$ and the complex structure J in terms of complex numbers. Recall that the ***complex conjugate*** and ***absolute value*** of a complex number $\mathbf{p}$ are defined by

$$\overline{\mathbf{p}} = \mathfrak{Re}\,\mathbf{p} - i\,\mathfrak{Im}\,\mathbf{p} \qquad \text{and} \qquad |\mathbf{p}| = \sqrt{\mathbf{p}\overline{\mathbf{p}}}.$$

The proof of the following lemma is elementary.

Lemma 1.2. Identify the plane $\mathbb{R}^2$ with the set of complex numbers $\mathbb{C}$, and let $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2 = \mathbb{C}$. Then

$$J\mathbf{p} = i\,\mathbf{p}, \qquad |\mathbf{p}| = \|\mathbf{p}\| \qquad \textit{and} \qquad \mathbf{p}\overline{\mathbf{q}} = \mathbf{p} \cdot \mathbf{q} + i\,\mathbf{p} \cdot (J\mathbf{q}). \tag{1.2}$$

The angle between vectors in $\mathbb{R}^n$ does not distinguish between the order of the vectors, but there is a refined notion of angle between vectors in $\mathbb{R}^2$ that makes this distinction.

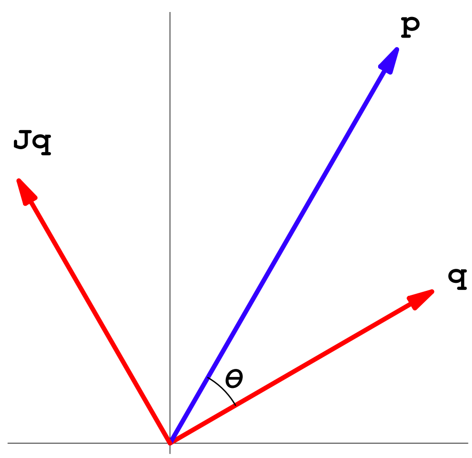

Figure 1.1: The oriented angle

Lemma 1.3. *Let $\mathbf{p}$ and $\mathbf{q}$ be nonzero vectors in $\mathbb{R}^2$. There exists a unique number θ with the properties*

$$\cos\theta = \frac{\mathbf{p}\cdot\mathbf{q}}{\|\mathbf{p}\|\,\|\mathbf{q}\|}, \qquad \sin\theta = \frac{\mathbf{p}\cdot J\mathbf{q}}{\|\mathbf{p}\|\,\|\mathbf{q}\|}, \qquad 0 \leqslant \theta < 2\pi. \tag{1.3}$$

We call θ the ***oriented angle*** *from $\mathbf{q}$ to $\mathbf{p}$.*

Proof. The oriented angle θ defines $\mathbf{p}$ in relation to the frame $(\mathbf{q},\, J\mathbf{q})$, as indicated in Figure 1.1. A formal proof proceeds as follows.

Since $\mathbf{p}\overline{\mathbf{q}}/(|\mathbf{p}|\,|\mathbf{q}|)$ is a complex number of absolute value 1, it lies on the unit circle in $\mathbb{C}$; thus there exists a unique θ with $0 \leqslant \theta < 2\pi$ such that

$$\frac{\mathbf{p}\overline{\mathbf{q}}}{|\mathbf{p}|\,|\mathbf{q}|} = e^{i\theta}. \tag{1.4}$$

Then using the expressions for $\mathbf{p}\overline{\mathbf{q}}$ in (1.2) and (1.4), we find that

$$\mathbf{p}\cdot\mathbf{q} + i\mathbf{p}\cdot(J\mathbf{q}) = e^{i\theta}|\mathbf{p}|\,|\mathbf{q}| = |\mathbf{p}|\,|\mathbf{q}|\cos\theta + i\,|\mathbf{p}|\,|\mathbf{q}|\sin\theta. \tag{1.5}$$

When we take the real and imaginary parts in (1.5), we get (1.3). ▮

1.2 Curves in Space

In the previous section we reviewed some of the algebraic properties of $\mathbb{R}^n$. We need to go one step further and study differentiation. In this chapter, and generally throughout the rest of the book, we shall use the word 'differentiable' to mean 'possessing derivatives or partial derivatives of all orders'. We begin by studying $\mathbb{R}^n$-valued functions of one variable.

Definition 1.4. *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$ be a function, where (a,b) is an open interval in $\mathbb{R}$. We write*

$$\boldsymbol{\alpha}(t) = \big(a_1(t),\dots,a_n(t)\big),$$

where each a_j is an ordinary real-valued function of a real variable. We say that $\boldsymbol{\alpha}$ is ***differentiable*** *provided a_j is differentiable for $j = 1,\dots,n$. Similarly, $\boldsymbol{\alpha}$ is* ***piecewise-differentiable*** *provided a_j is piecewise-differentiable for $j = 1,\dots,n$.*

Definition 1.5. *A* ***parametrized curve*** *in $\mathbb{R}^n$ is a piecewise-differentiable function*

$$\boldsymbol{\alpha}\colon\ (a,b) \longrightarrow \mathbb{R}^n,$$

where (a,b) is an open interval in $\mathbb{R}$. We allow the interval to be finite, infinite or half-infinite. If I is any other subset of $\mathbb{R}$, we say that

$$\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^n$$

is a ***curve*** *provided there is an open interval (a,b) containing I such that $\boldsymbol{\alpha}$ can be extended as a piecewise-differentiable function from (a,b) into $\mathbb{R}^n$.*

It is important to distinguish a curve $\boldsymbol{\alpha}$, which is a function, from the set of points traced out by $\boldsymbol{\alpha}$, which we call the ***trace*** of $\boldsymbol{\alpha}$. The trace of $\boldsymbol{\alpha}$ is just the image $\boldsymbol{\alpha}\big((a,b)\big)$, or more generally $\boldsymbol{\alpha}(I)$. We say that a subset $\mathscr{C}$ of $\mathbb{R}^n$ is ***parametrized by*** $\boldsymbol{\alpha}$ provided there is a subset $I \subseteq \mathbb{R}$ such that $\boldsymbol{\alpha}\colon I \to \mathbb{R}^n$ is a curve for which $\boldsymbol{\alpha}(I) = \mathscr{C}$.

Definition 1.6. *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$ be a curve with $\boldsymbol{\alpha}(t) = \big(a_1(t), \dots, a_n(t)\big)$. Then the **velocity** of $\boldsymbol{\alpha}$ is the function $\boldsymbol{\alpha}'\colon (a,b) \to \mathbb{R}^n$ given by*

$$\boldsymbol{\alpha}'(t) = \big(a_1'(t), \dots, a_n'(t)\big).$$

*The function v defined by $v(t) = \big\|\boldsymbol{\alpha}'(t)\big\|$ is called the **speed** of $\boldsymbol{\alpha}$. The **acceleration** of $\boldsymbol{\alpha}$ is given by $\boldsymbol{\alpha}''(t)$.*

Notice that $\boldsymbol{\alpha}'(t)$ is defined for those t for which $a_1'(t), \dots, a_n'(t)$ are defined. When $n = 2$, the acceleration can be compared with the vector $J\boldsymbol{\alpha}'(t)$ as shown in Figure 1.2; this will be the basis of the definition of curvature on page 14.

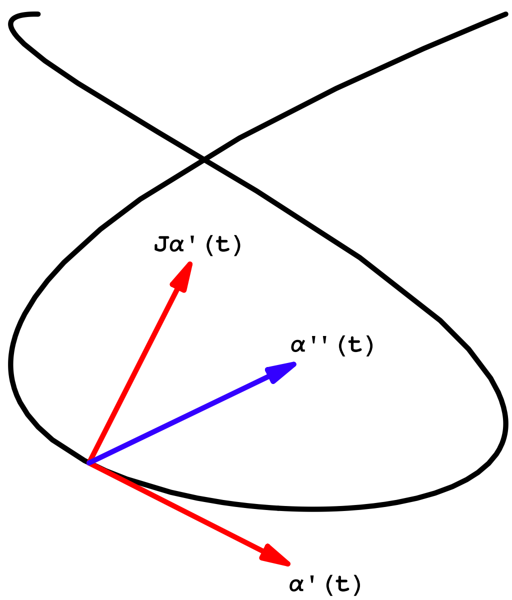

Figure 1.2: Velocity and acceleration of a plane curve

Some curves are 'better' than others:

Definition 1.7. *A curve $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$ is said to be **regular** if it is differentiable and its velocity is everywhere defined and nonzero. If $\big\|\boldsymbol{\alpha}'(t)\big\| = 1$ for $a < t < b$, then $\boldsymbol{\alpha}$ is said to have **unit speed**.*

The simplest example of a parametrized curve in $\mathbb{R}^n$ is a ***straight line***. If it contains distinct points $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$, it is most naturally parametrized by

$$\boldsymbol{\beta}(t) = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = (1-t)\mathbf{p} + t\,\mathbf{q}, \tag{1.6}$$

with $t \in \mathbb{R}$. In the plane $\mathbb{R}^2$, the second simplest curve is a ***circle***. If it has radius r and center $(p_1, p_2) \in \mathbb{R}^2$, it can be parametrized by

$$\boldsymbol{\gamma}(t) = \big(p_1 + r\cos t,\ p_2 + r\sin t\big), \tag{1.7}$$

with $0 \leqslant t < 2\pi$.

Our definition of a curve in $\mathbb{R}^n$ is the *parametric form* of a curve. In contrast, the ***nonparametric form*** of a curve would typically consist of a system of $n-1$ equations

$$F_1(p_1, \ldots, p_n) = \ \cdots\ = F_{n-1}(p_1, \ldots, p_n) = 0,$$

where each F_i is a differentiable function $\mathbb{R}^n \to \mathbb{R}$. We study the nonparametric form of curves in $\mathbb{R}^2$ in Section 3.1. It is usually easier to work with the parametric form of a curve, but there is one disadvantage: distinct parametrizations may trace out the same point set. Therefore, it is important to know when two curves are equivalent under a change of variables.

Definition 1.8. *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$ and $\boldsymbol{\beta}\colon (c,d) \to \mathbb{R}^n$ be differentiable curves. Then $\boldsymbol{\beta}$ is said to be a* ***positive reparametrization*** *of $\boldsymbol{\alpha}$ provided there exists a differentiable function $h\colon (c,d) \to (a,b)$ such that $h'(u) > 0$ for all $c < u < d$ and $\boldsymbol{\beta} = \boldsymbol{\alpha} \circ h$.*

Similarly, $\boldsymbol{\beta}$ is called a ***negative reparametrization*** *of $\boldsymbol{\alpha}$ provided there exists a differentiable function $h\colon (c,d) \to (a,b)$ such that $h'(u) < 0$ for all $c < u < d$ and $\boldsymbol{\beta} = \boldsymbol{\alpha} \circ h$.*

We say that β is a ***reparametrization*** *of $\boldsymbol{\alpha}$ if it is either a positive or negative reparametrization of $\boldsymbol{\alpha}$.*

Next, we determine the relation between the velocity of a curve $\boldsymbol{\alpha}$ and the velocity of a reparametrization of $\boldsymbol{\alpha}$.

Lemma 1.9. (The chain rule for curves.) *Suppose that $\boldsymbol{\beta}$ is a reparametrization of $\boldsymbol{\alpha}$. Write $\boldsymbol{\beta} = \boldsymbol{\alpha} \circ h$, where $h\colon (c,d) \to (a,b)$ is differentiable. Then*

$$\boldsymbol{\beta}'(u) = h'(u)\boldsymbol{\alpha}'\big(h(u)\big) \tag{1.8}$$

for $c < u < d$.

Proof. Write $\boldsymbol{\alpha}(t) = \big(a_1(t), \ldots, a_n(t)\big)$ and $\boldsymbol{\beta}(u) = \big(b_1(u), \ldots, b_n(u)\big)$. Then we have $b_j(u) = a_j\big(h(u)\big)$ for $j = 1, \ldots, n$. The ordinary chain rule implies that $b_j'(u) = a_j'\big(h(u)\big)h'(u)$ for $j = 1, \ldots, n$, and so we get (1.8). ∎

For example, the straight line (1.6) and circle (1.7) have slightly more complicated unit-speed parametrizations:

$$\begin{aligned} s &\mapsto \frac{\|\mathbf{p}-\mathbf{q}\|-s}{\|\mathbf{p}-\mathbf{q}\|}\,\mathbf{p}+\frac{s}{\|\mathbf{p}-\mathbf{q}\|}\,\mathbf{q} \\ s &\mapsto \Big(p_1+r\cos\frac{s}{r},\ p_2+r\sin\frac{s}{r}\Big). \end{aligned}$$

The quantities $\boldsymbol{\alpha}'$, $J\boldsymbol{\alpha}'$, $\boldsymbol{\alpha}''$ are examples of ***vector fields*** along a curve, because they determine a vector at *every* point of the curve, not just at the one point illustrated. We now give the general definition.

Definition 1.10. *Let* $\boldsymbol{\alpha}\colon (a,b)\to\mathbb{R}^n$ *be a curve. A* ***vector field along*** $\boldsymbol{\alpha}$ *is a function* $\mathbf{Y}$ *that assigns to each* t *with* $a<t<b$ *a vector* $\mathbf{Y}(t)$ *at the point* $\boldsymbol{\alpha}(t)$.

At this stage, we shall not distinguish between a vector at $\boldsymbol{\alpha}(t)$ and the vector parallel to it at the origin. This means a vector field $\mathbf{Y}$ along a curve $\boldsymbol{\alpha}$ is really an n-tuple of functions:

$$\mathbf{Y}(t)=\big(y_1(t),\ \dots,\ y_n(t)\big).$$

(Later, for example in Section 9.6, we *will* need to distinguish between a vector at $\boldsymbol{\alpha}(t)$ and the vector parallel to it at the origin.) Differentiability of $\mathbf{Y}$ means that each of the functions $y_1,\dots,y_n$ is differentiable, and in this case $\mathbf{Y}$ is effectively another curve and its derivative is defined in the obvious way:

$$\mathbf{Y}'(t)=\big(y_1'(t),\ \dots,\ y_n'(t)\big).$$

Addition, scalar multiplication and dot product of vector fields along a curve $\boldsymbol{\alpha}\colon (a,b)\to\mathbb{R}^n$ are defined in the obvious way. Furthermore, we can multiply a vector field $\mathbf{Y}$ along $\boldsymbol{\alpha}$ by a function $f\colon (a,b)\to\mathbb{R}$: we define the vector field $f\mathbf{Y}$ by $(f\mathbf{Y})(t)=f(t)\mathbf{Y}(t)$. Finally, if $n=2$ and $\mathbf{X}$ is a vector field along a curve with $\mathbf{X}(t)=(x(t),y(t))$, then another vector field $J\mathbf{X}$ can be defined by $J\mathbf{X}(t)=(-y(t),x(t))$. Taking the derivative is related to these operations in the obvious way:

Lemma 1.11. *Suppose that* $\mathbf{X}$ *and* $\mathbf{Y}$ *are differentiable vector fields along a curve* $\boldsymbol{\alpha}\colon (a,b)\to\mathbb{R}^n$, *and let* $f\colon (a,b)\to\mathbb{R}$ *be differentiable. Then*

(i) $(f\mathbf{Y})'=f'\mathbf{Y}+f\mathbf{Y}'$;

(ii) $(\mathbf{X}+\mathbf{Y})'=\mathbf{X}'+\mathbf{Y}'$;

(iii) $(\mathbf{X}\cdot\mathbf{Y})'=\mathbf{X}'\cdot\mathbf{Y}+\mathbf{X}\cdot\mathbf{Y}'$;

(iv) *for* $n=2$, $(J\mathbf{Y})'=J\mathbf{Y}'$.

Proof. For example, we prove (i):

$$\begin{aligned}(f\mathbf{Y})' &= \big((fy_1)', \ldots, (fy_n)'\big) = (f'y_1 + fy_1', \ldots, f'y_n + fy_n')\\ &= (y_1, \ldots, y_n)f' + f(y_1', \ldots, y_n') = f'\mathbf{Y} + f\mathbf{Y}'. \blacksquare\end{aligned}$$

1.3 The Length of a Curve

One of the simplest and most important geometric quantities associated with a curve is its length. Everyone has a natural idea of what is meant by length, but this idea must be converted into an exact definition. For our purposes the simplest definition is as follows:

*Definition 1.12. Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$ be a curve. Assume that $\boldsymbol{\alpha}$ is defined on a slightly larger interval containing (a,b), so that $\boldsymbol{\alpha}$ is defined and differentiable at a and b. Then the **length** of $\boldsymbol{\alpha}$ over the interval $[a,b]$ is given by*

$$\mathsf{length}[\boldsymbol{\alpha}] = \int_a^b \big\|\boldsymbol{\alpha}'(t)\big\|\,dt. \tag{1.9}$$

One can emphasize the role of the endpoints with the notation $\mathsf{length}[a,b][\boldsymbol{\alpha}]$ (compatible with Notebook 1); this is essential when we wish to compute the length of a curve restricted to an interval smaller than that on which it is defined. It is clear intuitively that length should not depend on the parametrization of the curve. We now prove this.

Theorem 1.13. Let $\boldsymbol{\beta}$ be a reparametrization of $\boldsymbol{\alpha}$. Then

$$\mathsf{length}[\boldsymbol{\alpha}] = \mathsf{length}[\boldsymbol{\beta}].$$

Proof. We do the positive reparametrization case first. Let $\boldsymbol{\beta} = \boldsymbol{\alpha} \circ h$, where $h\colon (c,d) \to (a,b)$ and $h'(u) > 0$ for $c < u < d$. Then by Lemma 1.9 we have

$$\big\|\boldsymbol{\beta}'(u)\big\| = \big\|\boldsymbol{\alpha}'\big(h(u)\big)h'(u)\big\| = \big\|\boldsymbol{\alpha}'\big(h(u)\big)\big\|h'(u).$$

Using the change of variables formula for integrals, we compute

$$\begin{aligned}\mathsf{length}[\boldsymbol{\alpha}] = \int_a^b \big\|\boldsymbol{\alpha}'(t)\big\|\,dt &= \int_c^d \big\|\boldsymbol{\alpha}'\big(h(u)\big)\big\|h'(u)\,du\\ &= \int_c^d \big\|\boldsymbol{\beta}'(u)\big\|\,du = \mathsf{length}[\boldsymbol{\beta}].\end{aligned}$$

In the negative reparametrization case, we have

$$\lim_{u \downarrow c} h(u) = b, \qquad \lim_{u \uparrow d} h(u) = a,$$

and

$$\|\boldsymbol{\beta}'(u)\| = \|\boldsymbol{\alpha}'\big(h(u)\big)h'(u)\| = -\|\boldsymbol{\alpha}'\big(h(u)\big)\|h'(u).$$

Again, using the change of variables formula for integrals,

$$\begin{aligned}\mathsf{length}[\boldsymbol{\alpha}] = \int_a^b \|\boldsymbol{\alpha}'(t)\|\,dt &= \int_d^c \|\boldsymbol{\alpha}'\big(h(u)\big)\|h'(u)\,du\\ &= \int_c^d \|\boldsymbol{\beta}'(u)\|\,du = \mathsf{length}[\boldsymbol{\beta}]. \blacksquare\end{aligned}$$

The trace of a curve $\boldsymbol{\alpha}$ can be approximated by a series of line segments connecting a sequence of points on the trace of $\boldsymbol{\alpha}$. Intuitively, the length of this approximation to $\boldsymbol{\alpha}$ should tend to the length of $\boldsymbol{\alpha}$ as the individual line segments become smaller and smaller. This is in fact true. To explain exactly what happens, let $\boldsymbol{\alpha}\colon (c,d) \to \mathbb{R}^n$ be a curve and $[a,b] \subset (c,d)$ a closed finite interval. For every partition

$$P = \{\, a = t_0 < t_1 < \cdots < t_N = b \,\} \tag{1.10}$$

of $[a,b]$, let

$$|P| = \max_{1\leqslant j\leqslant N}(t_j - t_{j-1}) \qquad \text{and} \qquad \ell(\boldsymbol{\alpha},P) = \sum_{j=1}^N \|\boldsymbol{\alpha}(t_j) - \boldsymbol{\alpha}(t_{j-1})\|.$$

Geometrically, $\ell(\boldsymbol{\alpha},P)$ is the length of the polygonal path in $\mathbb{R}^n$ whose vertices are $\boldsymbol{\alpha}(t_j)$, $j = 1,\ldots,N$. Then $\mathsf{length}[\boldsymbol{\alpha}]$ is the limit of the lengths of inscribed polygonal paths in the sense of

Theorem 1.14. *Let $\boldsymbol{\alpha}\colon (c,d) \to \mathbb{R}^n$ be a curve, and let $\mathsf{length}[\boldsymbol{\alpha}]$ denote the length of the restriction of $\boldsymbol{\alpha}$ to a closed subinterval $[a,b]$. Given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|P| < \delta \qquad \textit{implies} \qquad \big|\mathsf{length}[\boldsymbol{\alpha}] - \ell(\boldsymbol{\alpha},P)\big| < \epsilon. \tag{1.11}$$

Proof. Given a partition (1.10), write $\boldsymbol{\alpha} = (a_1,\ldots,a_n)$. For each i, the Mean Value Theorem implies that for $1 \leqslant j \leqslant N$ there exists $t_j^{(i)}$ with $t_{j-1} < t_j^{(i)} < t_j$ such that

$$a_i(t_j) - a_i(t_{j-1}) = a_i'(t_j^{(i)})(t_j - t_{j-1}).$$

Then

$$\begin{aligned}\|\boldsymbol{\alpha}(t_j) - \boldsymbol{\alpha}(t_{j-1})\|^2 &= \sum_{i=1}^n \big(a_i(t_j) - a_i(t_{j-1})\big)^2\\ &= \sum_{i=1}^n a_i'(t_j^{(i)})^2(t_j - t_{j-1})^2\\ &= (t_j - t_{j-1})^2\sum_{i=1}^n a_i'(t_j^{(i)})^2.\end{aligned}$$

Hence

$$\|\boldsymbol{\alpha}(t_j)-\boldsymbol{\alpha}(t_{j-1})\| \;=\; (t_j-t_{j-1})\sqrt{\sum_{i=1}^n a_i'(t_j^{(i)})^2} \;=\; (t_j-t_{j-1})(A_j+B_j),$$

where

$$A_j=\sqrt{\sum_{i=1}^n a_i'(t_j)^2}=\|\boldsymbol{\alpha}'(t_j)\| \quad\text{and}\quad B_j=\sqrt{\sum_{i=1}^n a_i'(t_j^{(i)})^2}-\sqrt{\sum_{i=1}^n a_i'(t_j)^2}.$$

Two separate applications of the triangle inequality allow us to deduce that

$$\begin{aligned}
|B_j| \;&=\; \Big|\big\|\big(a_1'(t_j^{(1)}),\ldots,a_n'(t_j^{(n)})\big)\big\|-\|\boldsymbol{\alpha}'(t_j)\|\Big| \\
&\leqslant\; \big\|\big(a_1'(t_j^{(1)}),\ldots,a_1'(t_j^{(n)})\big)-\boldsymbol{\alpha}'(t_j)\big\| \;=\; \left(\sum_{i=1}^n\big(a_i'(t_j^{(i)})-a_i'(t_j)\big)^2\right)^{1/2} \\
&\leqslant\; \sum_{i=1}^n\big|a_i'(t_j^{(i)})-a_i'(t_j)\big|.
\end{aligned}$$

Let $\epsilon>0$. Since $\boldsymbol{\alpha}'$ is continuous on the closed interval $[a,b]$, there exists a positive number δ_1 such that

$$|u-v|<\delta_1 \qquad\text{implies}\qquad \big|a_i'(u)-a_i'(v)\big|<\frac{\epsilon}{2n(b-a)}$$

for $i=1,\ldots,n$. Then $|P|<\delta_1$ implies that $|B_j|<\epsilon/(2(b-a))$, for each j, so that

$$\left|\sum_{j=1}^N(t_j-t_{j-1})B_j\right| \;\leqslant\; \sum_{j=1}^N(t_j-t_{j-1})\frac{\epsilon}{2(b-a)} \;=\; \frac{\epsilon}{2}.$$

Hence

$$\begin{aligned}
\big|\,\mathsf{length}[\boldsymbol{\alpha}]-\ell(\boldsymbol{\alpha},P)\big| \;&=\; \left|\mathsf{length}[\boldsymbol{\alpha}]-\sum_{j=1}^N\|\boldsymbol{\alpha}(t_j)-\boldsymbol{\alpha}(t_{j-1})\|\right| \\
&=\; \left|\mathsf{length}[\boldsymbol{\alpha}]-\sum_{j=1}^N(A_j+B_j)(t_j-t_{j-1})\right| \\
&\leqslant\; \left|\mathsf{length}[\boldsymbol{\alpha}]-\sum_{j=1}^N A_j(t_j-t_{j-1})\right|+\left|\sum_{j=1}^N B_j(t_j-t_{j-1})\right| \\
&\leqslant\; \left|\int_a^b\|\boldsymbol{\alpha}'(t)\|\,dt-\sum_{j=1}^N\|\boldsymbol{\alpha}'(t_j)(t_j-t_{j-1})\|\right|+\frac{\epsilon}{2}.
\end{aligned}$$

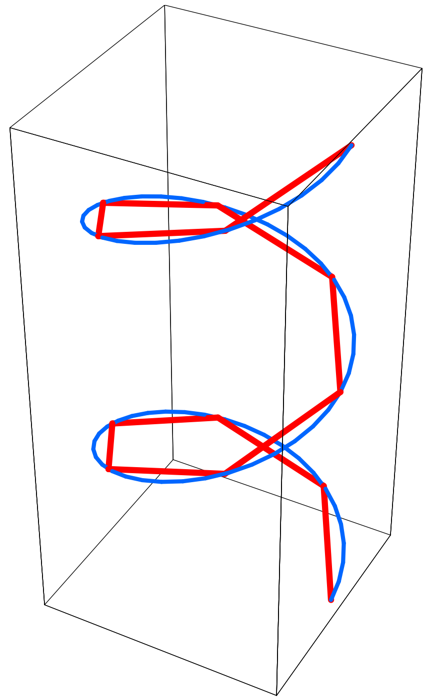

Figure 1.3: Polygonal approximation of the curve $t \mapsto (\cos t,\ \sin t,\ t/3)$

But

$$\sum_{j=1}^{N} \left\|\boldsymbol{\alpha}'(t_j)(t_j - t_{j-1})\right\|$$

is a Riemann sum approximation to the right-hand side of (1.9). Hence there exists $\delta_2 > 0$ such that $|P| < \delta_2$ implies that

$$\left| \sum_{j=1}^{N} \left\|\boldsymbol{\alpha}'(t_j)(t_j - t_{j-1})\right\| - \int_a^b \left\|\boldsymbol{\alpha}'(t)\right\| dt \right| < \frac{\epsilon}{2}.$$

Thus if we take $\delta = \min(\delta_1, \delta_2)$, we get (1.11). ▮

We need to know how the length of a curve varies when we change the limits of integration.

Definition 1.15. *Fix a number c with $a < c < b$. The* ***arc length function*** *s_α of a curve $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$, starting at c, is defined by*

$$s_\alpha(t) = \mathsf{length}[c,t][\boldsymbol{\alpha}] = \int_a^t \left\|\boldsymbol{\alpha}'(u)\right\| du,$$

for $a \leqslant t \leqslant b$. When there is no danger of confusion, we simplify s_α to s.

Note that if $c \leqslant d \leqslant f \leqslant b$, then $\mathsf{length}[d,f][\boldsymbol{\alpha}] = s_\alpha(f) - s_\alpha(d) \geqslant 0$.

Theorem 1.16. *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$ be a regular curve. Then there exists a unit-speed reparametrization $\boldsymbol{\beta}$ of $\boldsymbol{\alpha}$.*

Proof. By the fundamental theorem of calculus, any arc length function s of $\boldsymbol{\alpha}$ satisfies

$$\frac{ds}{dt}(t) = s'(t) = \big\|\boldsymbol{\alpha}'(t)\big\|.$$

Since $\boldsymbol{\alpha}$ is regular, $\boldsymbol{\alpha}'(t)$ is never zero; hence ds/dt is always positive. The Inverse Function Theorem implies that $t \mapsto s(t)$ has an inverse $s \mapsto t(s)$, and that

$$\left.\frac{dt}{ds}\right|_{s(t)} = \frac{1}{\left.\dfrac{ds}{dt}\right|_{t(s)}}.$$

Now define $\boldsymbol{\beta}$ by $\boldsymbol{\beta}(s) = \boldsymbol{\alpha}\big(t(s)\big)$. By Lemma 1.9, we have $\boldsymbol{\beta}'(s) = (dt/ds)\boldsymbol{\alpha}'\big(t(s)\big)$. Hence

$$\big\|\boldsymbol{\beta}'(s)\big\| = \left\|\frac{dt}{ds}\boldsymbol{\alpha}'\big(t(s)\big)\right\| = \frac{dt}{ds}\big\|\boldsymbol{\alpha}'\big(t(s)\big)\big\| = \left.\frac{dt}{ds}\right|_s \left.\frac{ds}{dt}\right|_{t(s)} = 1. \quad \blacksquare$$

The arc length function of any unit-speed curve $\boldsymbol{\beta}\colon (c,d) \to \mathbb{R}^n$ starting at c satisfies

$$s_\beta(t) = \int_c^t \big\|\boldsymbol{\beta}'(u)\big\|\, du = t - c.$$

Thus the function s_β actually measures length along $\boldsymbol{\beta}$. This is the reason why unit-speed curves are said to be ***parametrized by arc length***.

We conclude this section by showing that unit-speed curves are unique up to starting point and direction.

Lemma 1.17. *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$ be a unit-speed curve, and let $\boldsymbol{\beta}\colon (c,d) \to \mathbb{R}^n$ be a reparametrization of $\boldsymbol{\alpha}$ such that $\boldsymbol{\beta}$ also has unit speed. Then*

$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(\pm s + s_0)$$

for some constant real number s_0.

Proof. By hypothesis, there exists a differentiable function $h\colon (c,d) \to (a,b)$ such that $\boldsymbol{\beta} = \boldsymbol{\alpha} \circ h$ and $h'(u) \neq 0$ for $c < u < d$. Then Lemma 1.9 implies that

$$1 = \big\|\boldsymbol{\beta}'(u)\big\| = \big\|\boldsymbol{\alpha}'\big(h(u)\big)\big\|\, |h'(u)| = |h'(u)|.$$

Therefore, $h'(u) = \pm 1$ for all t. Since h' is continuous on a connected open set, its sign is constant. Thus there is a constant s_0 such that $h(u) = \pm s + s_0$ for $c < u < d$, s being arc length for $\boldsymbol{\alpha}$. $\blacksquare$

1.4 Curvature of Plane Curves

Intuitively, the curvature of a curve measures the failure of a curve to be a straight line. The faster the velocity $\boldsymbol{\alpha}'$ turns along a curve $\boldsymbol{\alpha}$, the larger the curvature[2].

In Chapter 7, we shall define the notion of curvature $\boldsymbol{\kappa}[\boldsymbol{\alpha}]$ of a curve $\boldsymbol{\alpha}$ in $\mathbb{R}^n$ for arbitrary n. In the special case that $n = 2$ there is a refined version $\boldsymbol{\kappa 2}$ of $\boldsymbol{\kappa}$, which we now define. We first give the definition that is the easiest to use in computations. In the next section, we show that the curvature can be interpreted as the derivative of a turning angle.

Definition 1.18. *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$ be a regular curve. The* ***curvature*** $\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]$ *of $\boldsymbol{\alpha}$ is given by the formula*

$$\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t) = \frac{\boldsymbol{\alpha}''(t) \cdot J\boldsymbol{\alpha}'(t)}{\|\boldsymbol{\alpha}'(t)\|^3}. \tag{1.12}$$

The positive function $1/|\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]|$ is called the ***radius of curvature*** *of $\boldsymbol{\alpha}$.*

The notation $\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]$ is consistent with that of Notebook 1, though there is no danger in omitting $\boldsymbol{\alpha}$ if there is only one curve under consideration.

We can still speak of the curvature of a plane curve $\boldsymbol{\alpha}$ that is only piecewise-differentiable. At isolated points where the velocity vector $\boldsymbol{\alpha}'(t)$ vanishes or does not exist, the curvature $\boldsymbol{\kappa 2}(t)$ remains undefined. Similarly, the radius of curvature $1/|\boldsymbol{\kappa 2}(t)|$ is undefined at those t for which either $\boldsymbol{\alpha}'(t)$ or $\boldsymbol{\alpha}''(t)$ vanishes or is undefined.

Notice that the curvature is proportional (with a positive constant) to the orthogonal projection of $\boldsymbol{\alpha}''(t)$ in the direction of $J\boldsymbol{\alpha}'(t)$ (see Figure 1.2). If the acceleration $\boldsymbol{\alpha}''(t)$ vanishes, so does $\boldsymbol{\kappa 2}(t)$. In particular, the curvature of the straight line parametrized by (1.6) vanishes identically.

It is important to realize that the curvature can assume both positive and negative values. Often $\boldsymbol{\kappa 2}$ is called the ***signed curvature*** in order to distinguish it from a curvature function $\boldsymbol{\kappa}$ which we shall define in Chapter 7. It is useful to have some pictures to understand the meaning of positive and negative curvature for plane curves. There are four cases, each of which we illustrate by a parabola in Figures 1.4 and 1.5. These show that, briefly, positive curvature means 'bend to the left' and negative curvature means 'bend to the right'.

---

[2]The notion of curvature of a plane curve first appears implicitly in the work of Apollonius of Perga (262–180 BC). Newton (1642–1727) was the first to study the curvature of plane curves explicitly, and found in particular formula (1.13).

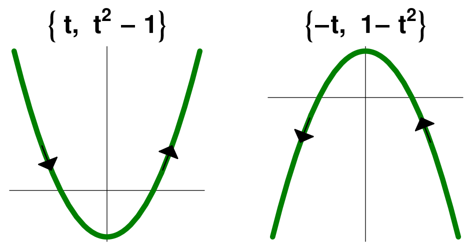

Figure 1.4: Parabolas with $\boldsymbol{\kappa}\mathbf{2} > 0$

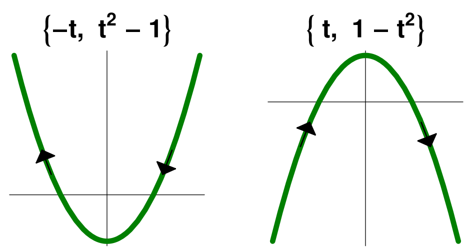

Figure 1.5: Parabolas with $\boldsymbol{\kappa}\mathbf{2} < 0$

Next, we derive the formula for curvature that can be found in most calculus books, and a version using complex numbers and the notation of Lemma 1.2.

Lemma 1.19. If $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$ is a regular curve with $\boldsymbol{\alpha}(t) = (x(t),\, y(t))$, then the curvature of $\boldsymbol{\alpha}$ is given in the equivalent ways

$$\boldsymbol{\kappa}\mathbf{2}[\boldsymbol{\alpha}](t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{\big(x'^2(t) + y'^2(t)\big)^{3/2}}, \tag{1.13}$$

$$\boldsymbol{\kappa}\mathbf{2}[\boldsymbol{\alpha}](t) = \mathfrak{Im}\, \frac{\boldsymbol{\alpha}''(t)\,\overline{\boldsymbol{\alpha}'(t)}}{\big\|\boldsymbol{\alpha}'(t)\big\|^3}. \tag{1.14}$$

Proof. We have $\boldsymbol{\alpha}''(t) = (x''(t), y''(t))$ and $J\boldsymbol{\alpha}'(t) = (-y'(t), x'(t))$, so that

$$\boldsymbol{\kappa}\mathbf{2}[\boldsymbol{\alpha}](t) = \frac{\big(x''(t), y''(t)\big)\cdot\big(-y'(t), x'(t)\big)}{\big(x'^2(t) + y'^2(t)\big)^{3/2}} = \frac{x'(t)y''(t) - x''(t)y'(t)}{\big(x'^2(t) + y'^2(t)\big)^{3/2}},$$

proving (1.13). Equation (1.14) follows from (1.12) and (1.2) (Exercise 5). ∎

Like the function $\mathsf{length}[\boldsymbol{\alpha}]$ the curvature $\boldsymbol{\kappa}\mathbf{2}[\boldsymbol{\alpha}]$ is a geometric quantity associated with a curve $\boldsymbol{\alpha}$. As such, it should be independent of the parametrization of the curve $\boldsymbol{\alpha}$. We now show that this is almost true. If λ is a nonzero real number, then $\lambda/|\lambda|$ is understandably denoted by $\operatorname{sign}\lambda$.

Theorem 1.20. *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$ be a regular curve, and let $\boldsymbol{\beta}\colon (c,d) \to \mathbb{R}^2$ be a reparametrization of $\boldsymbol{\alpha}$. Write $\boldsymbol{\beta} = \boldsymbol{\alpha} \circ h$, where $h\colon (c,d) \to (a,b)$ is differentiable. Then*

$$\boldsymbol{\kappa 2}[\beta](u) = \big(\operatorname{sign} h'(u)\big)\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]\big(h(u)\big), \tag{1.15}$$

wherever $h'(u) \neq 0$. Thus the curvature of a plane curve is independent of the parametrization up to sign.

Proof. We have $\boldsymbol{\beta}' = (\boldsymbol{\alpha}' \circ h)h'$, so that $J\boldsymbol{\beta}' = J(\boldsymbol{\alpha}' \circ h)\, h'$, and

$$\boldsymbol{\beta}'' = (\boldsymbol{\alpha}'' \circ h)h'^2 + (\boldsymbol{\alpha}' \circ h)h''. \tag{1.16}$$

Thus

$$\begin{aligned}\boldsymbol{\kappa 2}[\beta] &= \frac{\big((\boldsymbol{\alpha}'' \circ h)h'^2 + (\boldsymbol{\alpha}' \circ h)h''\big) \cdot J(\boldsymbol{\alpha}' \circ h)\, h'}{\big\|(\boldsymbol{\alpha}' \circ h)h'\big\|^3}\\ &= \left(\frac{h'^3}{|h'|^3}\right)\frac{(\boldsymbol{\alpha}'' \circ h) \cdot J(\boldsymbol{\alpha}' \circ h)}{\big\|(\boldsymbol{\alpha}' \circ h)\big\|^3}.\end{aligned}$$

Hence we get (1.15). ∎

The formula for $\boldsymbol{\kappa 2}$ simplifies dramatically for a unit-speed curve:

Lemma 1.21. *Let $\boldsymbol{\beta}$ be a unit-speed curve in the plane. Then*

$$\boldsymbol{\beta}'' = \boldsymbol{\kappa 2}[\boldsymbol{\beta}]\, J\boldsymbol{\beta}'. \tag{1.17}$$

Proof. Differentiating $\boldsymbol{\beta}' \cdot \boldsymbol{\beta}' = 1$, we obtain $\boldsymbol{\beta}'' \cdot \boldsymbol{\beta}' = 0$. Thus $\boldsymbol{\beta}''$ must be a multiple of $J\boldsymbol{\beta}'$. In fact, it follows easily from (1.12) that this multiple is $\boldsymbol{\kappa 2}[\boldsymbol{\beta}]$, and we obtain (1.17) . ∎

Finally, we give simple characterizations of straight lines and circles by means of curvature.

Theorem 1.22. *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$ be a regular curve.*

(i) *$\boldsymbol{\alpha}$ is part of a straight line if and only if $\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t) \equiv 0$.*

(ii) *$\boldsymbol{\alpha}$ is part of a circle of radius $r > 0$ if and only if $\big|\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]\big|(t) \equiv 1/r$.*

Proof. It is easy to show that the curvature of the straight line is zero, and that the curvature of a circle of radius r is $+1/r$ for a counterclockwise circle and $-1/r$ for a clockwise circle (see Exercise 3).

For the converse, we can assume without loss of generality that $\boldsymbol{\alpha}$ is a unit-speed curve. Suppose $\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t) \equiv 0$. Lemma 1.21 implies that $\boldsymbol{\alpha}''(t) = 0$ for $a < t < b$. Hence there exist constant vectors $\mathbf{p}$ and $\mathbf{q}$ such that

$$\boldsymbol{\alpha}(t) = \mathbf{p}\,t + \mathbf{q}$$

for $a < t < b$; thus $\boldsymbol{\alpha}$ is part of a straight line.

Next, assume that $\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t)$ is identically equal to a positive constant $1/r$. Without loss of generality, $\boldsymbol{\alpha}$ has unit speed. Define a curve $\boldsymbol{\gamma}\colon (b,c) \to \mathbb{R}^2$ by

$$\boldsymbol{\gamma}(t) = \boldsymbol{\alpha}(t) + rJ\boldsymbol{\alpha}'(t).$$

Lemma 1.21 implies that $\boldsymbol{\gamma}'(t) = 0$ for $a < t < b$. Hence there exists $\mathbf{q}$ such that $\boldsymbol{\gamma}(t) = \mathbf{q}$ for $a < t < b$. Then $\|\boldsymbol{\alpha}(t) - \mathbf{q}\| = \|rJ\boldsymbol{\alpha}'(t)\| = r$. Hence $\boldsymbol{\alpha}(t)$ lies on a circle of radius r centered at $\mathbf{q}$.

Similarly, $\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t) \equiv -1/r$ also implies that $\boldsymbol{\alpha}$ is part of a circle of radius r (see Exercise 4). ▮

1.5 Angle Functions

In Section 1.1 we defined the notion of oriented angle between vectors in $\mathbb{R}^2$; now we wish to define a similar notion for curves. Clearly, we can compute the oriented angle between corresponding velocity vectors. This oriented angle θ satisfies $0 \leqslant \theta < 2\pi$, but such a restriction is not desirable for curves. The problem is that the two curves can twist and turn so that eventually the angle between them lies outside the interval $[0, 2\pi)$. To arrive at the proper definition, we need the following useful lemma of O'Neill [ON1].

Lemma 1.23. *Let $f, g\colon (a,b) \to \mathbb{R}$ be differentiable functions with $f^2 + g^2 = 1$. Fix t_0 with $a < t_0 < b$ and let θ_0 be such that $f(t_0) = \cos\theta_0$ and $g(t_0) = \sin\theta_0$. Then there exists a unique continuous function $\theta\colon (a,b) \to \mathbb{R}$ such that*

$$\theta(t_0) = \theta_0, \quad f(t) = \cos\theta(t) \quad \textit{and} \quad g(t) = \sin\theta(t) \tag{1.18}$$

for $a < t < b$.

Proof. We use complex numbers. Let $h = f + ig$, so $h\overline{h} = 1$. Define

$$\theta(t) = \theta_0 \,-\, i\int_{t_0}^{t} h'(s)\overline{h(s)}\,ds. \tag{1.19}$$

Then

$$\frac{d}{dt}(h e^{-i\theta}) = e^{-i\theta}(h' - i\,h\theta') = e^{-i\theta}(h' - h\,h'\overline{h}) = 0.$$

Thus $h e^{-i\theta} = c$ for some constant c. Since $h(t_0) = e^{i\theta_0}$, it follows that

$$c = h(t_0)e^{-i\theta(t_0)} = 1.$$

Hence $h(t) = e^{i\theta(t)}$, and so we get (1.18).

Let $\hat{\theta}$ be another continuous function such that

$$\hat{\theta}(t_0) = \theta_0, \quad f(t) = \cos\hat{\theta}(t) \quad \text{and} \quad g(t) = \sin\hat{\theta}(t)$$

for $a < t < b$. Then $e^{i\theta(t)} = e^{i\hat{\theta}(t)}$ for $a < t < b$. Since both θ and $\hat{\theta}$ are continuous, there is an integer n such that

$$\theta(t) - \hat{\theta}(t) = 2\pi n$$

for $a < t < b$. But $\theta(t_0) = \hat{\theta}(t_0)$, so that $n = 0$. Hence θ and $\hat{\theta}$ coincide. ▮

We can now apply this lemma to deduce the existence and uniqueness of a differentiable angle function between curves in $\mathbb{R}^2$.

Corollary 1.24. Let $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *be regular curves in* $\mathbb{R}^2$ *defined on the same interval* (a, b)*, and let* $a < t_0 < b$*. Choose* θ_0 *such that*

$$\frac{\boldsymbol{\alpha}'(t_0)\cdot\boldsymbol{\beta}'(t_0)}{\|\boldsymbol{\alpha}'(t_0)\|\,\|\boldsymbol{\beta}'(t_0)\|} = \cos\theta_0 \quad \textit{and} \quad \frac{\boldsymbol{\alpha}'(t_0)\cdot J\boldsymbol{\beta}'(t_0)}{\|\boldsymbol{\alpha}'(t_0)\|\,\|\boldsymbol{\beta}'(t_0)\|} = \sin\theta_0.$$

Then there exists a unique differentiable function $\theta\colon (a, b) \to \mathbb{R}$ *such that*

$$\theta(t_0) = \theta_0, \quad \frac{\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\beta}'(t)}{\|\boldsymbol{\alpha}'(t)\|\,\|\boldsymbol{\beta}'(t)\|} = \cos\theta(t) \quad \textit{and} \quad \frac{\boldsymbol{\alpha}'(t)\cdot J\boldsymbol{\beta}'(t)}{\|\boldsymbol{\alpha}'(t)\|\,\|\boldsymbol{\beta}'(t)\|} = \sin\theta(t)$$

for $a < t < b$*. We call* θ *the* ***angle function*** *from* $\boldsymbol{\beta}$ *to* $\boldsymbol{\alpha}$ *determined by* θ_0*.*

Proof. We take

$$f(t) = \frac{\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\beta}'(t)}{\|\boldsymbol{\alpha}'(t)\|\,\|\boldsymbol{\beta}'(t)\|} \quad \text{and} \quad g(t) = \frac{\boldsymbol{\alpha}'(t)\cdot J\boldsymbol{\beta}'(t)}{\|\boldsymbol{\alpha}'(t)\|\,\|\boldsymbol{\beta}'(t)\|}$$

in Lemma 1.23. ▮

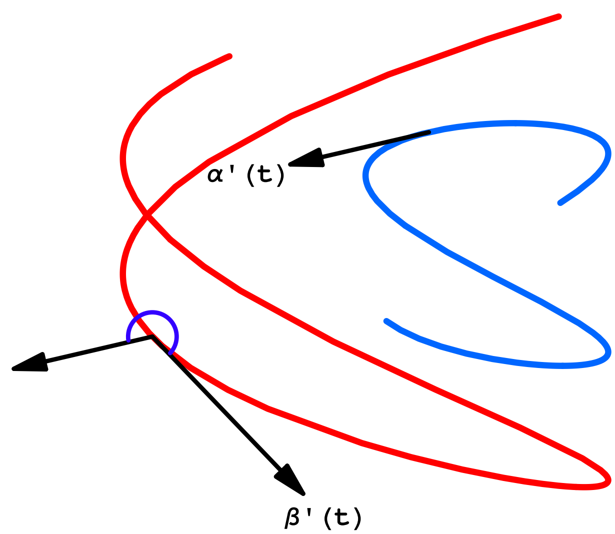

Figure 1.6: The angle function between curves

Intuitively, it is clear that the velocity of a highly curved plane curve changes rapidly. This idea can be made precise with the concept of ***turning angle***.

Lemma 1.25. *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$ be a regular curve and fix t_0 with $a < t_0 < b$. Let θ_0 be a number such that*

$$\frac{\boldsymbol{\alpha}'(t_0)}{\|\boldsymbol{\alpha}'(t_0)\|} = (\cos\theta_0,\ \sin\theta_0).$$

Then there exists a unique differentiable function $\boldsymbol{\theta} = \boldsymbol{\theta}[\boldsymbol{\alpha}]\colon (a,b) \to \mathbb{R}$ such that $\boldsymbol{\theta}(t_0) = \theta_0$ and

$$\frac{\boldsymbol{\alpha}'(t)}{\|\boldsymbol{\alpha}'(t)\|} = \big(\cos\boldsymbol{\theta}(t),\ \sin\boldsymbol{\theta}(t)\big). \tag{1.20}$$

*for $a < t < b$. We call $\boldsymbol{\theta}[\boldsymbol{\alpha}]$ the **turning angle** determined by θ_0.*

Proof. Let $\boldsymbol{\beta}(t) = (t, 0)$; then $\boldsymbol{\beta}$ parametrizes a horizontal straight line and $\boldsymbol{\beta}'(t) = (1,0)$ for all t. Write $\boldsymbol{\alpha}(t) = (a_1(t), a_2(t))$; then

$$\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\beta}'(t) = a_1'(t) \qquad \text{and} \qquad \boldsymbol{\alpha}'(t)\cdot J\boldsymbol{\beta}'(t) = a_2'(t).$$

Corollary 1.24 implies the existence of a unique function $\boldsymbol{\theta}\colon (a,b) \to \mathbb{R}$ such that $\boldsymbol{\theta}(t_0) = \theta_0$ and

$$\begin{cases} \cos\boldsymbol{\theta}(t) = \dfrac{\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\beta}'(t)}{\|\boldsymbol{\alpha}'(t)\|\,\|\boldsymbol{\beta}'(t)\|} = \dfrac{a_1'(t)}{\|\boldsymbol{\alpha}'(t)\|}, \\[2ex] \sin\boldsymbol{\theta}(t) = \dfrac{\boldsymbol{\alpha}'(t)\cdot J\boldsymbol{\beta}'(t)}{\|\boldsymbol{\alpha}'(t)\|\,\|\boldsymbol{\beta}'(t)\|} = \dfrac{a_2'(t)}{\|\boldsymbol{\alpha}'(t)\|}. \end{cases} \tag{1.21}$$

Then (1.21) is equivalent to (1.20). ▮

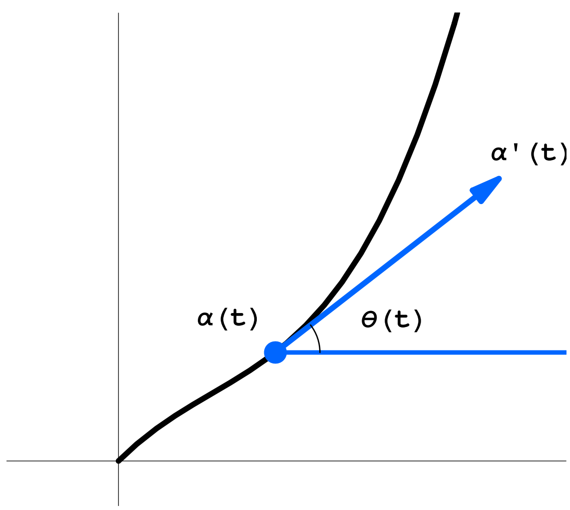

Figure 1.7: The turning angle of a plane curve

Geometrically, the turning angle $\boldsymbol{\theta}[\boldsymbol{\alpha}](t)$ is the angle between the horizontal and $\boldsymbol{\alpha}'(t)$. Next, we derive a relation (first observed by Kästner[3]) between the turning angle and the curvature of a plane curve.

Lemma 1.26. *The turning angle and curvature of a regular curve $\boldsymbol{\alpha}$ in the plane are related by*

$$\boldsymbol{\theta}[\boldsymbol{\alpha}]'(t) \ = \big\|\boldsymbol{\alpha}'(t)\big\|\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t). \tag{1.22}$$

Proof. The derivative of the left-hand side of (1.20) is

$$\frac{\boldsymbol{\alpha}''(t)}{\big\|\boldsymbol{\alpha}'(t)\big\|} + \boldsymbol{\alpha}'(t)\frac{d}{dt}\Big(\frac{1}{\big\|\boldsymbol{\alpha}'(t)\big\|}\Big). \tag{1.23}$$

Furthermore, by the chain rule (Lemma 1.9) it follows that the derivative of the right-hand side of (1.20) is

$$\boldsymbol{\theta}'(t)\big(-\sin\boldsymbol{\theta}(t),\ \cos\boldsymbol{\theta}(t)\big) = \boldsymbol{\theta}'(t)\frac{J\boldsymbol{\alpha}'(t)}{\big\|J\boldsymbol{\alpha}'(t)\big\|}. \tag{1.24}$$

Setting (1.24) equal to (1.23) and taking the dot product with $J\boldsymbol{\alpha}'(t)$, we obtain

$$\boldsymbol{\theta}'(t)\big\|\boldsymbol{\alpha}'(t)\big\| = \frac{\boldsymbol{\alpha}''(t)\cdot J\boldsymbol{\alpha}'(t)}{\big\|\boldsymbol{\alpha}'(t)\big\|} = \big\|\boldsymbol{\alpha}'(t)\big\|^2\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t).$$

Equation (1.22) now follows.. ▮

Corollary 1.27. *The turning angle and curvature of a unit-speed curve $\boldsymbol{\beta}$ in the plane are related by*

$$\boldsymbol{\kappa 2}[\boldsymbol{\beta}](s) = \frac{d\boldsymbol{\theta}[\boldsymbol{\beta}](s)}{ds}.$$

Curvature therefore measures rate of change of the turning angle with respect to arc length.

1.6 First Examples of Plane Curves

The equation of a circle of radius a with center $(0,0)$ is

$$x^2 + y^2 = a^2.$$

Its simplest parametrization is

$$\mathsf{circle}[a](t) = a(\cos t,\ \sin t), \qquad 0 \leqslant t < 2\pi, \tag{1.25}$$

---

3  Abraham Gotthelf Kästner (1719–1800). German mathematician, professor at Leipzig and Göttingen. Although Kästner's German contemporaries ranked his mathematical and expository skills very high, Gauss found his lectures too elementary to attend. Gauss declared that Kästner was first mathematician among poets and first poet among mathematicians.

as in (1.7) but with a now representing the radius (not an initial parameter value). A more interesting parametrization is obtained by setting $\tau = \tan(t/2)$, so as to give

$$\boldsymbol{\beta}(\tau) = a\left(\frac{1-\tau^2}{1+\tau^2}, \frac{2\tau}{1+\tau^2}\right), \qquad \tau \in \mathbb{R},$$

though the trace of $\boldsymbol{\beta}$ omits the point $(-a, 0)$.

We shall now study two classes of curves that include the circle as a special case. In both cases, we shall graph their curvature.

Ellipses

These are perhaps the next simplest curves after the circle. The name 'ellipse' (which means 'falling short') is due to Apollonius[4]. The nonparametric form of the ellipse centered at the origin with axes of lengths a and b is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and the standard parametrization is

$$\mathsf{ellipse}[a,b](t) = \big(a\cos t,\ b\sin t\big), \qquad 0 \leqslant t < 2\pi. \tag{1.26}$$

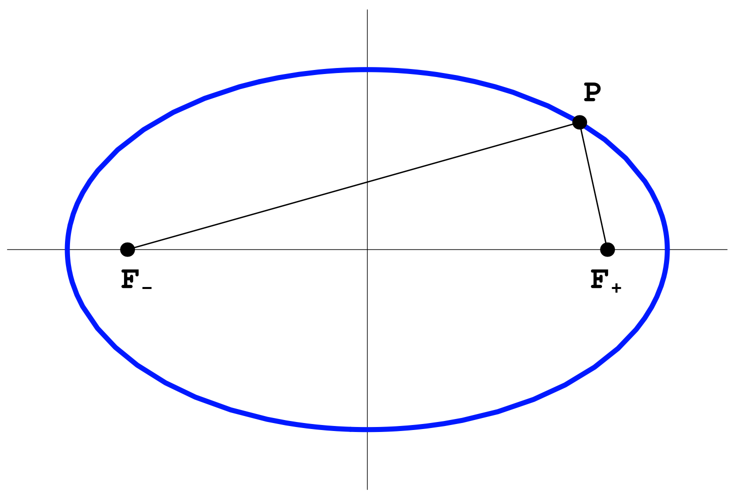

Figure 1.8: An ellipse and its foci

The curvature of this ellipse is

$$\boldsymbol{\kappa 2}[\mathsf{ellipse}[a,b]](t) = \frac{ab}{(b^2\cos^2 t + a^2\sin^2 t)^{3/2}}.$$

[4]Apollonius of Perga (262–180 BC). His eight volume treatise on conic sections is the standard ancient source of information about ellipses, hyperbolas and parabolas.

Its graph for $a = 3,\ b = 4$ (the ellipse in Figure 1.8) is illustrated below with an exaggerated vertical scale. The fact that there are two maxima and two minima is related to the four vertex theorem, which will be discussed in Section 6.5.

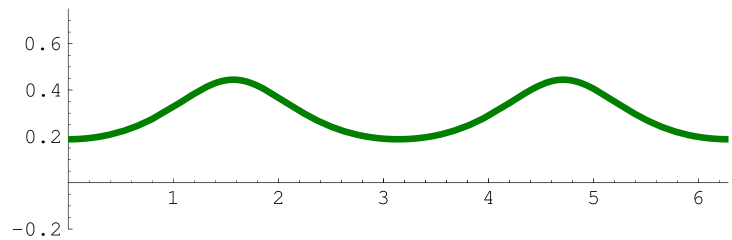

Figure 1.9: The curvature of an ellipse

An ellipse is traditionally defined as the locus of points such that the sum of the distances from two fixed points F_- and F_+ is constant. The points F_- and F_+ are called the ***foci*** of the ellipse. In the case of $\mathsf{ellipse}[a, b]$ with $a > b$ the foci are $\big(\pm\sqrt{a^2 - b^2},\, 0\big)$:

Theorem 1.28. *Let $\mathscr{E}$ be an ellipse with foci F_- and F_+. Then*

(i) *for any point P on $\mathscr{E}$, the sum*

$$\mathsf{distance}(F_-, P) + \mathsf{distance}(F_+, P)$$

is constant.

(ii) *the tangent line to $\mathscr{E}$ at P makes equal angles with the line segments connecting P to F_- and F_+.*

Proof. To establish these two properties of an ellipse, it is advantageous to use complex numbers. Clearly, the parametrization (1.26) is equivalent to

$$\mathbf{z}(t) = \tfrac{1}{2}(a + b)e^{it} + \tfrac{1}{2}(a - b)e^{-it} \qquad (0 \leqslant t < 2\pi). \tag{1.27}$$

The vector from $F_\mp$ to P is given by

$$\mathbf{z}_\pm(t) = \tfrac{1}{2}(a + b)e^{it} + \tfrac{1}{2}(a - b)e^{-it} \ \pm\ \sqrt{a^2 - b^2}.$$

It is remarkable that this can be written as a perfect square

$$\mathbf{z}_\pm(t) = \tfrac{1}{2}\left(\sqrt{a + b}\, e^{it/2} \pm \sqrt{a - b}\, e^{-it/2}\right)^2. \tag{1.28}$$

An easy calculation now shows that

$$|\mathbf{z}_\pm(t)| = a \pm \sqrt{a^2 - b^2}\,\cos t.$$

Consequently, $|\mathbf{z}_+(t)| + |\mathbf{z}_-(t)| = 2a$; this proves (i).

To prove (ii), we first use (1.27) to compute

$$\begin{aligned}\mathbf{z}'(t) &= \frac{i}{2}(a+b)\,e^{it} - \frac{i}{2}(a-b)\,e^{-it}\\ &= \frac{i}{2}\left(\sqrt{a+b}\,e^{it/2}+\sqrt{a+b}\,e^{-it/2}\right)\left(\sqrt{a+b}\,e^{it/2}-\sqrt{a+b}\,e^{-it/2}\right).\end{aligned}$$

It follows from (1.28) that

$$i\,\frac{\mathbf{z}_+(t)}{\mathbf{z}'(t)} = \frac{\sqrt{a+b}\,e^{it/2}+\sqrt{a+b}\,e^{-it/2}}{\sqrt{a+b}\,e^{it/2}-\sqrt{a+b}\,e^{-it/2}} = -i\,\frac{\mathbf{z}'(t)}{\mathbf{z}_-(t)},$$

and (ii) follows once one expresses the three complex numbers $\mathbf{z}_-(t), \mathbf{z}'(t), \mathbf{z}_-(t)$ in exponential form. ▮

Logarithmic spirals

A ***logarithmic spiral*** is parametrized by

(1.29) $$\mathsf{logspiral}[a,b](t) = a(e^{bt}\cos t,\ e^{bt}\sin t).$$

Changing the sign of b makes the sprial unwind the other way, with $b = 0$ corresponding to the circle. If $b > 0$, it is evident that the curvature decreases as t increases. We check this by graphing the function

$$\boldsymbol{\kappa}\mathbf{2}[\mathsf{logspiral}[a,b]](t) = \frac{1}{a\,e^{bt}\sqrt{1+b^2}},$$

obtained from Notebook 1, in the case corresponding to Figure 1.10.

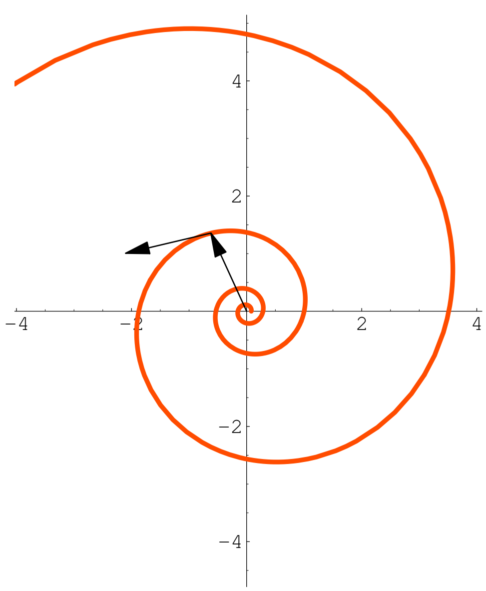

Figure 1.10: The curve $\mathsf{logspiral}[1, \frac{1}{5}]$

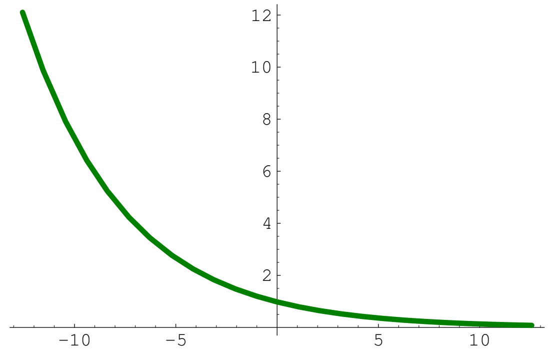

Figure 1.11: The function $\boldsymbol{\kappa}\mathbf{2}[\text{logspiral}[1, \frac{1}{5}]](t) = 0.98e^{-0.2t}$

Logarithmic spirals were first discussed by Descartes in 1638 in connection with a problem from dynamics. He was interested in determining all plane curves $\boldsymbol{\alpha}$ with the property that the angle between $\boldsymbol{\alpha}(t)$ and the tangent vector $\boldsymbol{\alpha}'(t)$ is constant. Such curves turn out to be precisely the logarithmic spirals, as we proceed to prove.

Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$ be a curve which does not pass through the origin. The ***tangent-radius angle*** of $\boldsymbol{\alpha}$ is defined to be the function $\boldsymbol{\phi} = \boldsymbol{\phi}[\boldsymbol{\alpha}]\colon (a,b) \to \mathbb{R}^2$ such that

$$\frac{\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}(t)}{\|\boldsymbol{\alpha}'(t)\|\,\|\boldsymbol{\alpha}(t)\|} = \cos\boldsymbol{\phi}(t) \quad \text{and} \quad \frac{\boldsymbol{\alpha}'(t)\cdot J\boldsymbol{\alpha}(t)}{\|\boldsymbol{\alpha}'(t)\|\,\|\boldsymbol{\alpha}(t)\|} = \sin\boldsymbol{\phi}(t)$$

for $a < t < b$. The existence of $\boldsymbol{\phi}[\boldsymbol{\alpha}]$ follows from Lemma 1.23; see Exercise 6.

Lemma 1.29. *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$ be a curve which does not pass through the origin. The following conditions are equivalent*:

(i) *The tangent-radius angle $\boldsymbol{\phi}[\boldsymbol{\alpha}]$ is constant*;

(ii) *$\boldsymbol{\alpha}$ is a reparametrization of a logarithmic spiral.*

Proof. We write $\boldsymbol{\alpha}(t) = r\,e^{i\theta}$ with $r = r(t)$ and $\theta = \theta(t)$ both functions of t. Then $\boldsymbol{\alpha}'(t) = e^{i\theta}(r' + i\,r\,\theta')$, so that

$$\left|\boldsymbol{\alpha}(t)\right| = r, \quad \left|\boldsymbol{\alpha}'(t)\right| = \sqrt{r'^2 + r^2\theta'^2} \quad \text{and} \quad \overline{\boldsymbol{\alpha}(t)}\,\boldsymbol{\alpha}'(t) = r(r' + i\,r\,\theta').$$

Assume that (i) holds and let γ be the constant value of $\boldsymbol{\phi}[\boldsymbol{\alpha}](t)$. Equation (1.4) implies that

$$e^{i\gamma} = \frac{\boldsymbol{\alpha}'(t)\,\overline{\boldsymbol{\alpha}(t)}}{\left|\boldsymbol{\alpha}'(t)\right|\,\left|\boldsymbol{\alpha}(t)\right|} = \frac{r' + i\,r\,\theta'}{\sqrt{r'^2 + r^2\theta'^2}}. \tag{1.30}$$

From (1.30) we get

$$\cos\gamma = \frac{r'}{\sqrt{r'^2 + r^2\theta'^2}} \qquad \text{and} \qquad \sin\gamma = \frac{r\,\theta'}{\sqrt{r'^2 + r^2\theta'^2}},$$

so that

$$\frac{r'}{r} = \theta' \cot\gamma.$$

The solution of this differential equation is found to be

$$r = a\exp(\theta\cot\gamma),$$

where a is a constant, and we obtain

$$\boldsymbol{\alpha}(t) = a\exp\big((i+\cot\gamma)\theta(t)\big), \tag{1.31}$$

which is equivalent to (1.29).

Conversely, one can check that the tangent-radius angle of any logarithmic spiral is a constant γ. Under a reparametrization, the tangent-radius angle has the constant value $\pm\gamma$. ▮

1.7 The Semicubical Parabola and Regularity

It is informative to calculate lengths and curvatures by hand in a simple case.

We choose the ***semicubical parabola*** defined parametrically by

$$\mathsf{sc}(t) = (t^2,\ t^3). \tag{1.32}$$

The nonparametric version of sc is obviously $x^3 = y^2$. The semicubical parabola has a special place in the history of curves, because it was the first algebraic curve (other than a straight line) to be found whose arc length function is algebraic. See [Lock, pages 3–11] for an extensive discussion.

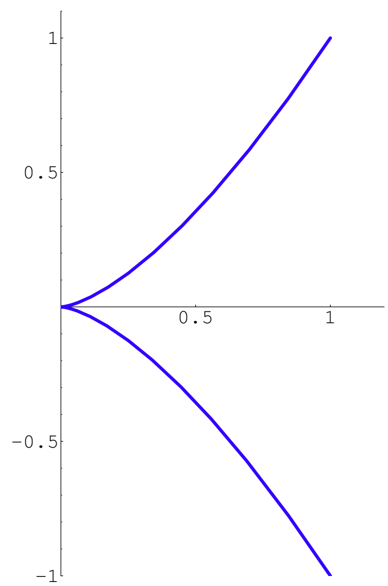

Figure 1.12: The semicubical parabola

We compute easily

$$\mathsf{sc}'(t) = (2t,\ 3t^2), \qquad J\,\mathsf{sc}'(t) = (-3t^2,\ 2t), \qquad \mathsf{sc}''(t) = (2, 6t),$$

so that

$$\boldsymbol{\kappa 2}[\mathsf{sc}](t) = \frac{(2,6t)\cdot(-3t^2,2t)}{\|(2t,3t^2)\|^3} = \frac{6}{|t|(4+9t^2)^{3/2}}.$$

Also, the length of sc over the interval $[0,t]$ is

$$s_{\mathrm{sc}}(t) = \int_0^t \|\mathsf{sc}'(u)\|\,du = \int_0^t u\sqrt{4+9\,u^2}\,du = \frac{1}{27}(4+9t^2)^{3/2} - \frac{8}{27}.$$

Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$ be a curve, and let $a < t_0 < b$. There are two ways that $\boldsymbol{\alpha}$ can fail to be regular at t_0. On the one hand, regularity can fail because of the particular parametrization. For example, a horizontal straight line, when parametrized as $t \mapsto (t^3, 0)$, is not regular at 0. This kind of nonregularity can be avoided by changing the parametrization. A more fundamental kind of nonregularity occurs when it is impossible to find a regular reparametrization of a curve near a point.

Definition 1.30. *We say that a curve $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$ is **regular** at t_0 provided that it is possible to extend the function*

$$t \mapsto \frac{\boldsymbol{\alpha}'(t)}{\|\boldsymbol{\alpha}'(t)\|}$$

*to be a differentiable function at t_0. Otherwise $\boldsymbol{\alpha}$ is said to be **singular** at t_0.*

It is easy to see that the definitions of regular and singular at t_0 are independent of the parametrization of $\boldsymbol{\alpha}$. The notion of ***regular curve*** $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$, as defined on page 6, implies regularity at each t_0 for $a < t_0 < b$, but is stronger, because it implies that $\boldsymbol{\alpha}'(t_0)$ must be nonzero.

The following result may easily be deduced from the previous definition:

Lemma 1.31. *A curve $\boldsymbol{\alpha}$ is regular at t_0 if and only if there exists a unit-speed parametrization of $\boldsymbol{\alpha}$ near t_0.*

For example, in seeking a unit-speed parametrization of sc, a computation shows that

$$\frac{\mathsf{sc}'(t)}{\|\mathsf{sc}'(t)\|} = \frac{(2t,3t^2)}{\sqrt{t^2(4+9t^2)}}.$$

However, this is discontinuous at the origin because

$$(-1,0) = \lim_{t\uparrow 0} \frac{(2t,3t^2)}{\sqrt{t^2(4+9t^2)}} \neq \lim_{t\downarrow 0} \frac{(2t,3t^2)}{\sqrt{t^2(4+9t^2)}} = (1,0),$$

reflecting the singularity of (1.32) at $t=0$.

1.8 Exercises

1. Let $a > 0$. Find the arc length function and the curvature for each of the following curves, illustrated in Figure 1.13.

(a) $t \mapsto a\big(\cos t + t\sin t,\ \sin t - t\cos t\big)$;

(b) $t \mapsto \left(a\cosh\dfrac{t}{a},\ t\right)$;

(c) $t \mapsto a\big(\cos^3 t,\ \sin^3 t\big)$;

(d) $t \mapsto a\big(2t,\ t^2\big)$.

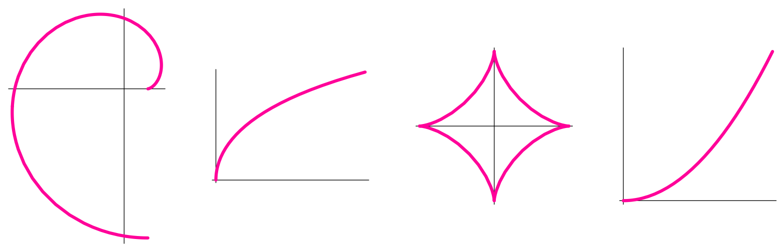

Figure 1.13: The curves of Exercise 1

2. Show that the velocity $\boldsymbol{\alpha}'$ of a differentiable curve $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$ is given by

$$\boldsymbol{\alpha}'(t) = \lim_{\delta\to 0}\frac{\boldsymbol{\alpha}(t+\delta) - \boldsymbol{\alpha}(t)}{\delta}, \qquad a < t < b.$$

3. Let $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^2$ be a circle of radius $r > 0$ centered at $\mathbf{q} \in \mathbb{R}^2$. Using the fact that $\big\|\boldsymbol{\beta}(t) - \mathbf{q}\big\| = r$, show that $\big|\boldsymbol{\kappa}\mathbf{2}[\boldsymbol{\beta}](t)\big| = 1/r$ for $a < t < b$.

4. Suppose that $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^2$ is a unit-speed curve whose curvature is given by $\big|\boldsymbol{\kappa}\mathbf{2}[\boldsymbol{\beta}](s)\big| = 1/r$ for all s, where $r > 0$ is a constant. Show that $\boldsymbol{\beta}$ is part of a circle of radius r centered at some point $\mathbf{q} \in \mathbb{R}^2$.

5. Verify the formula (1.14) for the curvature of a plane curve $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{C}$.

6. Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$ be a regular curve which does not pass through the origin. Fix t_0 and choose ϕ_0 such that

$$\frac{\boldsymbol{\alpha}'(t_0)\cdot\boldsymbol{\alpha}(t_0)}{\big\|\boldsymbol{\alpha}'(t_0)\big\|\,\big\|\boldsymbol{\alpha}(t_0)\big\|} = \cos\phi_0 \quad\text{and}\quad \frac{\boldsymbol{\alpha}'(t_0)\cdot J\boldsymbol{\alpha}(t_0)}{\big\|\boldsymbol{\alpha}'(t_0)\big\|\,\big\|\boldsymbol{\alpha}(t_0)\big\|} = \sin\phi_0.$$

Establish the existence and uniqueness of a differentiable function $\boldsymbol{\phi} = \boldsymbol{\phi}[\boldsymbol{\alpha}]\colon (a,b) \to \mathbb{R}$ such that $\boldsymbol{\phi}(t_0) = \phi_0$,

$$\frac{\boldsymbol{\alpha}'(t)\cdot\boldsymbol{\alpha}(t)}{\big\|\boldsymbol{\alpha}'(t)\big\|\,\big\|\boldsymbol{\alpha}(t)\big\|} = \cos\boldsymbol{\phi}(t) \quad\text{and}\quad \frac{\boldsymbol{\alpha}'(t)\cdot J\boldsymbol{\alpha}(t)}{\big\|\boldsymbol{\alpha}'(t)\big\|\,\big\|\boldsymbol{\alpha}(t)\big\|} = \sin\boldsymbol{\phi}(t)$$

for $a < t < b$. Geometrically, $\boldsymbol{\phi}(t)$ represents the angle between the radius vector $\boldsymbol{\alpha}(t)$ and the tangent vector $\boldsymbol{\alpha}'(t)$.

7. Find a unit-speed parametrization of the semicubical parabola $t \mapsto (t^2,\ t^3)$, valid for $t > 0$.

8. Let $\boldsymbol{\gamma}_a \colon (-\infty, \infty) \to \mathbb{R}^2$ be the curve defined by

$$\boldsymbol{\gamma}_a(t) = \begin{cases} \left(t,\ |t|^a \sin\dfrac{1}{t}\right) & \text{if } t \neq 0, \\ (0,0) & \text{if } t = 0. \end{cases}$$

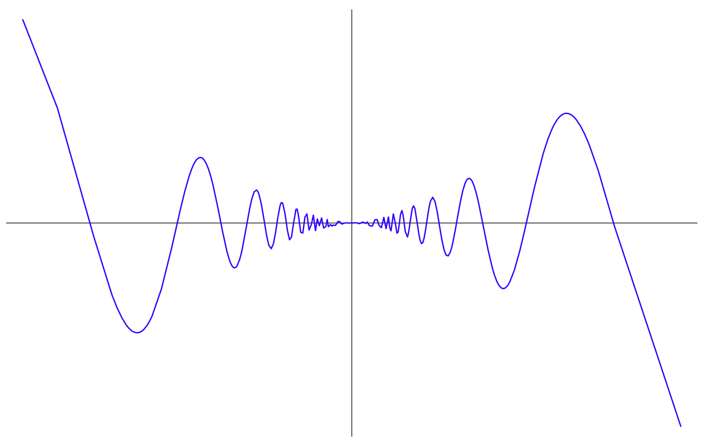

Figure 1.14: The curve $\boldsymbol{\gamma}_{3/2}$

(i) Show that $\boldsymbol{\gamma}_a$ is continuous if $a > 0$, but discontinuous if $a \leqslant 0$.

(ii) Show that $\boldsymbol{\gamma}_a$ is differentiable if $1 < a < 2$, but that the curve $(\boldsymbol{\gamma}_a)'$ is discontinuous.