Complementary error function for binned Gaussian integral notes

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The complementary error function - denoted as erfc(x) - is defined as:

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty dt e^{-t^2}$$

We want to utilize this function to compute the area under a Gaussian PDF between $a \leq x \leq b$. This can be written as:

$$P(a \leq x \leq b) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b dx e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Then, to re-write in terms of the erfc(x) function, we expand the integral and apply a variable transformation $t = \frac{(x-\mu)}{\sqrt{2\sigma^2}}$:

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b dx e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^\infty dx e^{\frac{-(x-\mu)^2}{2\sigma^2}} - \frac{1}{\sqrt{2\pi\sigma^2}} \int_b^\infty dx e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{(a-\mu)/\sqrt{2\sigma^2}}^\infty (dt\sqrt{2\sigma^2}) e^{-t^2} - \frac{1}{\sqrt{2\pi\sigma^2}} \int_{(b-\mu)/\sqrt{2\sigma^2}}^\infty (dt\sqrt{2\sigma^2}) e^{-t^2}$$

$$= \frac{1}{\sqrt{\pi}} \int_{(a-\mu)/\sqrt{2\sigma^2}}^\infty dt e^{-t^2} - \frac{1}{\sqrt{\pi}} \int_{(b-\mu)/\sqrt{2\sigma^2}}^\infty dt e^{-t^2}$$

$$= \frac{1}{2} erfc\left(\frac{a-\mu}{\sqrt{2\sigma^2}}\right) - \frac{1}{2} erfc\left(\frac{b-\mu}{\sqrt{2\sigma^2}}\right)$$

$$= \frac{1}{2}\left[erfc\left(\frac{a-\mu}{\sqrt{2\sigma^2}}\right) - erfc\left(\frac{b-\mu}{\sqrt{2\sigma^2}}\right)\right]$$

If we have discrete bins, we can also defined our bounds as:

$$ibin = nint(\mu/\Delta x)$$

$$a = (ibin - 0.5 + j)\Delta x$$

$$b = (ibin + 0.5 + j)\Delta x$$

Where μ is the mean of the Gaussian (here, the walker's current position), Δx is the unit length per bin, and j is a dummy offset from that bin. Additionally, $nint(x)$ is a function which rounds a floating-point number to the closest integer - therefore we must integrate to the appropriate 0.5 edges of the bins. If we wanted to calculate the area under the Gaussian PDF a distance of σ from the mean, $j = nint(\sigma/\Delta x)$