Fractional Brownian motion with mean-density interaction.

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1 Introduction

Consider an ensemble of N random walkers undergoing FBM with an anomalous diffusion exponent α - such that $\langle x^2 \rangle \sim t^\alpha$ - in 1-dimension. This ensemble of walkers will interact via mean-field type interaction. Specifically, the recursion for the position contains a density dependent term as follows:

$$x_{n+1}^{(i)} = x_n^{(i)} + \xi_n^{(i)} - \frac{A}{N}\frac{d}{dx}P_c(x,n) \tag{1}$$

Where $i = 1...N$ with N being the number of walkers in the ensemble, and $\xi_n^{(i)}$ as the Fractional Gaussian Noise (FGN) for the i'th walker at time step n. Here, $P_c(x,n)$ is the cumulative distribution of the walker density over the trajectories since $t = 0$:

$$P_c(x,n) = \sum_{i=1}^{N}\sum_{m=1}^{n}\delta(x - x_n^{(i)}) \tag{2}$$

Furthermore, $-\frac{A}{N}\frac{d}{dx}P_c(x,n)$ models the force term. Here, the factor of $\frac{1}{N}$ removes the dependence on the ensemble size. Additionally, for $A > 0$, the force term pushes particles towards smaller densities.

This choice of mean-field interaction can be motivated by applications where walkers leave behind a chemical trail along their trajectory, or a similar idea.

Normalization of this cumulative distribution is chosen to be the following:

$$\int_{-\infty}^{\infty} P_c(x,n)dx = \sum_{i=1}^{N}\sum_{m=1}^{n}\int_{-\infty}^{\infty}\delta(x - x_n^{(i)})dx = Nn \tag{3}$$

2 Scaling Theory

Consider an experiment in which all walkers start at $x = 0$ at $n = 1$. Assume that the cumulative distribution $P_c(x,n)$ approaches a universal functional form characterized by a single length scale b_n that increases with time n. This characterization can be written as:

$$P_c(x,n) = \frac{Nn}{b_n} Y(\frac{x}{b_n}) \tag{4}$$

We then apply equation 3 to normalize this definition.

$$\int_{-\infty}^{\infty} P_c(x,n)dx = \int_{-\infty}^{\infty} \frac{Nn}{b_n} Y(\frac{x}{b_n})dx = Nn \tag{5}$$

$$\int_{-\infty}^{\infty} Y(y)dy = 1 \tag{6}$$

Likewise, the force term can be re-written as follows:

$$-\frac{A}{N}\frac{d}{dx}P_c(x,n) = -\frac{A}{N}\frac{d}{dx}\frac{Nn}{b_n}Y(\frac{x}{b_n}) = -A\frac{n}{b_n^2}Y'(x/b_n) \tag{7}$$

Next, let's assume that the length scale goes like $b_n \sim n^\sigma$ with unknown exponent σ. The force term f_n can be written as:

$$f_n \sim n^{1-2\sigma} \tag{8}$$

If the force term dominates, motion will be ballistic:

$$x \sim \int_0^n f_{n'}dn' \sim n^{2-2\sigma} \tag{9}$$

We also know that to be self-consistent, typical x must behave like b_n:

$$n^\sigma = n^{2-2\sigma} \tag{10}$$

$$\sigma = 2 - 2\sigma \tag{11}$$

$$\sigma = 2/3 \tag{12}$$

The mean-square displacement thus behaves as:

$$\langle x_n^2 \rangle \sim n^{2\sigma} = n^{4/3} \tag{13}$$

If the length scale b_n increases faster than $n^{2/3}$, forces decay faster than $n^{-1/3}$. Integrating the force term leads to displacement that grows more slowly than $n^{2/3}$. Forces are then sub-leading to whatever process creates the increase in length scales faster than $n^{2/3}$

If b_n increases more slowly than $n^{2/3}$, forces would decay more slowly than $n^{-1/3}$ leading to displacement that grows faster than $n^{2/3}$. This is a contradiction.

3 Expected behavior for FBM with mean-density interaction

Now, we combine our scaling theory with FBM. For FBM $\alpha < 4/3$, forces will dominate and give $\langle x^2 \rangle \sim t^{4/3}$. For FBM $\alpha > 4/3$, FGN will dominate and give $\langle x^2 \rangle \sim t^{\alpha}$.

3.1 2-dimensions

With this, we can also extend our theory to higher spatial dimensions. In 2-dimensions, the cumulative distribution becomes:

$$P_c(\vec{x}, n) = \frac{Nn}{b_n^2} Y(\vec{x}/b_n) \tag{14}$$

Normalization becomes:

$$\int_{-\infty}^{\infty} P_c(\vec{x}, n) dx_1 dx_2 = \int_{-\infty}^{\infty} \frac{Nn}{b_n^2} Y(\vec{x}/b_n) dx_1 dx_2 = Nn \tag{15}$$

$$\int_{-\infty}^{\infty} Y(\vec{y}) dy_1 dy_2 = 1 \mid \vec{y} = \vec{x}/b_n \tag{16}$$

Force becomes:

$$-\frac{A}{N}\frac{d}{d\vec{x}} P_c(\vec{x}, n) = -\frac{An}{b_n^2}\frac{d}{d\vec{x}}\frac{Nn}{b_n} Y(\frac{\vec{x}}{b_n}) = \frac{-An}{b_n^3}\frac{d}{d\vec{y}} Y(\vec{y}) \tag{17}$$

If we still assume $b_n \sim n^{\sigma}$, force $f_n \sim n^{1-3\sigma}$:

$$x \sim \int_0^n f_{n'} dn' \sim n^{2-3\sigma} \tag{18}$$

$$n^{\sigma} = n^{2-3\sigma} \tag{19}$$

$$\sigma = 2 - 3\sigma \tag{20}$$

$$\sigma = 1/2 \tag{21}$$

3.2 General d-dimension

We repeat this for a general dimension d.

$$P_c(\vec{x}, n) = \frac{Nn}{b_n^d} Y(\vec{x}/b_n) \tag{22}$$

Force becomes:

$$-\frac{A}{N}\frac{d}{d\vec{x}} P_c(\vec{x}, n) = -\frac{An}{b_n^d}\frac{d}{d\vec{x}}\frac{Nn}{b_n} Y(\frac{\vec{x}}{b_n}) = \frac{-An}{b_n^{d+1}}\frac{d}{d\vec{y}} Y(\vec{y}) \tag{23}$$

If we still assume $b_n \sim n^\sigma$, force $f_n \sim n^{1-(d+1)\sigma}$:

$$x \sim \int_0^n f_{n'} dn' \sim n^{2-(d+1)\sigma} \tag{24}$$

$$n^\sigma = n^{2-(d+1)\sigma} \tag{25}$$

$$\sigma = 2 - (d+1)\sigma \tag{26}$$

$$\sigma = \frac{2}{d+2} \tag{27}$$

This - in general - produces a mean-squared displacement with the form:

$$n^{2\sigma} = n^{4/(d+2)} \tag{28}$$

$$\begin{cases} \langle x^2 \rangle \sim t^{4/(d+2)} & \alpha < \frac{4}{d+2} \\ \langle x^2 \rangle \sim t^{\alpha} & \alpha > \frac{4}{d+2} \end{cases}$$

4 Generalization to non-linear forces

For force f_n proportional to a power of the gradient of P_c, we get:

$$f_n = -A|\frac{1}{N}\frac{d}{dx}P_c(x,n)|^\lambda sign(\frac{d}{dx}P_c(x,n)) \tag{29}$$

4.1 Scaling theory for $d = 1$

$$|f_n| = A|\frac{1}{N}\frac{d}{dx}P_c(x,n)|^\lambda = A|\frac{n}{b_n}\frac{d}{dx}Y(x/b_n)|^\lambda = A(\frac{n}{b_n})^\lambda|\frac{d}{dy}Y(y)|^\lambda \tag{30}$$

Assume $b_n \sim n^\sigma$. We than have that:

$$\text{force } f_n \sim n^{\lambda - 2\sigma\lambda} \tag{31}$$

$$\text{displacement } x \sim \int f_n dn \sim n^{1+\lambda-2\sigma\lambda} \tag{32}$$

$$\text{self-consistency } n^\sigma = n^{1+\lambda-2\sigma\lambda} \tag{33}$$

$$\sigma = 1 + \lambda - 2\sigma\lambda \tag{34}$$

$$\sigma = \frac{1+\lambda}{1+2\lambda} \tag{35}$$

$$\begin{cases} \sigma = 2/3 \text{ (as before)} & \lambda = 1 \\ \sigma = \frac{1+\lambda}{1+2\lambda} = \frac{2+\Delta\lambda}{3+2\Delta\lambda} \mid \Delta\lambda = \lambda - 1 & \lambda > 1 \\ \sigma = \frac{2}{3}\frac{6+3\Delta\lambda}{6+4\Delta\lambda} > \frac{2}{3} & \lambda < 1 \end{cases}$$

4.2 Interpretation

If $\lambda > 1$, forces drop off faster with decreasing density, will be less dominant for long times. In this case, b_n **increases more slowly** with n than in the linear case.
If $\lambda < 1$, forces decay more slowly with decreasing density, will be less dominant for long times. In this case, b_n **increases faster** with n than in the linear case.

4.3 General dimension and nonlinear force

$$P_c(\vec{x}, n) = \frac{Nn}{b_n^d} Y(\vec{x}/b_n) \tag{36}$$

$$\text{force } |f_n| = A|\frac{1}{N}\frac{d}{d\vec{x}} P_c(\vec{x}, n)|^2 = An^\lambda b_n^{-(d+1)\lambda} |\frac{d}{d\vec{y}} Y(\vec{y})|^\lambda \tag{37}$$

Under assumption $b_n \sim n^\sigma$

$$\text{force } f_n \sim n^{\lambda-(d+1)\lambda\sigma} \tag{38}$$

$$\text{displacement } x \sim \int f_n dn \sim n^{1+\lambda-(d+1)\lambda\sigma} \tag{39}$$

$$\text{self consistency } n^\sigma = n^{1+\lambda-(d+1)\lambda\sigma} \tag{40}$$

$$\sigma(1 + (d + 1)\lambda) = 1 + \lambda \tag{41}$$

$$\sigma = \frac{1 + \lambda}{1 + (d + 1)\lambda} \tag{42}$$

This agrees with the $d = 1$ for $\lambda = 1$ case result of $\sigma = 2/3$.