Behavioral Expectations Equilibrium Toolkit: Conceptual Documentation

Jonathan J Adams*

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VERY EARLY DRAFT!

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github link

*University of Florida. Website: www.jonathanjadams.com Email: adamsjonathan@ufl.edu. This is an early draft, so please email me with comments and questions!

1 Introduction

This document describes the BEET toolkit for solving behavioral expectations.

2 Uhlig Form

I say a behavioral macroeconomic model is in "Uhlig" form if

$$0 = \mathbb{E}_t^k \left[F x_{t+1} + G x_t + H x_{t-1} + L z_{t+1} + M z_t \right] \tag{1}$$

where x_t is a $m \times 1$ vector of endogenous state variables chosen at time t, and z_t is a vector of exogenous state variables given by the *actual law of motion* (ALM)

$$[ALM]: \qquad z_{t+1} = N z_t + \epsilon_{t+1}$$

with ϵ_{t+1} an iid stochastic shock satsifying $\mathbb{E}_t^k[\epsilon_{t+1}] = 0$.

We would like to find the matrices P and Q^k to solve the model (1) resursively by

$$x_t = P x_{t-1} + Q^k z_t \tag{2}$$

Uhlig (2001) Theorem 1 solves this model in the case of rational expectations. The behavioral expectations case is similar, except it utilizes an incorrect *perceived law of motion* (PLM) for the exogenous state, namely:

$$[PLM]: \qquad z_{t+1} = N^k z_t + \epsilon_{t+1}$$

which is related to the ALM by the behavioral expectation operator $\mathbb{E}^k$ such that

$$N^k z_t = \mathbb{E}^k[z_{t+1} | z_t]$$

Depending on the behavioral expectations process, some extra work may be required to write the exogenous states in this way. See Appendix A.

Then, following Uhlig's proof, the solution to the model is given by:

Theorem 1 *A behavioral solution* (2) *is given by a matrix P satisfying the quadratic*

$$0 = FP^2 + GP + H$$

and the behavioral matrix Q^k satisfying

$$Q^k = -(V^k)^{-1} vec(LN^k + M)$$

where

$$V^k = (N^k)' \otimes F + I_k \otimes (FP + G)$$

Proof. Substitute the solution (2) into the equilibrium condition (1):

$$0 = \mathbb{E}_t^k \left[F(Px_t + Q^k z_{t+1}) + Gx_t + Hx_{t-1} + Lz_{t+1} + Mz_t \right]$$

Evaluate the behavioral expectation and collect terms

$$0 = (FP + G)x_t + Hx_{t-1} + ((FQ^k + L)N^k + M)z_t$$

then substitute again

$$0 = (FP + G)(Px_{t-1} + Q^k z_t) + Hx_{t-1} + ((FQ^k + L)N^k + M)z_t$$

Collecting terms on x_{t-1} and z_t gives the usual matrix quadratic equation:

$$0 = FP^2 + GP + H$$

and given P, the coefficient on z_t must satisfy

$$0 = (FP + G)^k + (FQ^k + L)N^k + M$$

so Q^k must satisfy

$$V^k Q^k = -vec(LN^k + M)$$

■

The presumption here is that agents know how choices x_t depend on the exogenous state z_t, and forecast the endogenous vector x_{t+1} by forecasting the exogenous state z_{t+1} with their behavioral expectations, and then mapping that back to x_{t+1} outcomes. *This is equivalent to directly forecasting x_{t+1} if the behavioral expectations operator is "series-agnostic"*, but may not hold more generally.

3 Mapping from Uhlig to GENSYS Form

Writing the model in Uhlig form is convenient for precisely characterizing the agnets' information set and expectations. But it may be desirable to solve the model with Chris Sims' GENSYS algorithm instead (Sims, 2002). This section describes how to map the behavioral model (1) to the GENSYS form.

Sims' form for a macroeconomic model is

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi z_t + \Pi \eta_t \tag{3}$$

where y_t is a vector of endogenous choice variables, while η_t is a vector of endogenously determined forecast errors. z_t is as before: an exogenous stochastic vector that is observed by the agents.

Between the two forms, the variables are mapped by

$$y_t = \begin{pmatrix} x_t \\ \mathbb{E}_t^k[x_{t+1}] \end{pmatrix}$$

following this mapping, the model in Uhlig form (1) can be written in GENSYS form by

$$\begin{pmatrix} -G & -F \\ I & 0 \end{pmatrix} \begin{pmatrix} x_t \\ \mathbb{E}_t^k[x_{t+1}] \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x_{t-1} \\ \mathbb{E}_{t-1}^k[x_t] \end{pmatrix} + \begin{pmatrix} LN^k + M \\ 0 \end{pmatrix} z_t + \begin{pmatrix} 0 \\ I \end{pmatrix} \left(x_t - \mathbb{E}_{t-1}^k[x_t]\right)$$

so that Sims' matrices are given by

$$\Gamma_0 = \begin{pmatrix} -G & -F \\ I & 0 \end{pmatrix} \qquad \Gamma_1 = \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \qquad \Psi = \begin{pmatrix} LN^k + M \\ 0 \end{pmatrix} \qquad \Pi = \begin{pmatrix} 0 \\ I \end{pmatrix}$$

and $\eta = \left(x_t - \mathbb{E}_{t-1}^k[x_t]\right)$.

4 Calculating Behavioral Eigenseries

In general, finding the eigenseries y of a behavioral expectations operator $\mathcal{E}^k$ can be a challenging problem. But for some types of behavioral expectations, the problem is tractable. The BEET toolkit can do so in these cases.

4.1 Subrational Diagonal Expectations

"Subrational" expectations are forward looking, but perhaps do not weigh different forecasts correctly. The "diagonal" class is a further refinement: these expectations are a linear combination of rational forecasts. Specifically, an expectation operator $\mathbb{E}^k$ is subrational diagonal if there is a square-summable series ϕ_j such that

$$\mathbb{E}_t^k X_{t+1} = \sum_{j=0}^{\infty} \phi_j \mathbb{E}_{t-j}[X_{t+1}]$$

where $\mathbb{E}_t$ is the rational expectation.

This definition may seem limited, but it includes many popular forms, including:

- Rational expectations: $\phi_j = 1$ for all j
- Cognitive discounting: $\phi_j = \theta < 1$ for all j
- Diagnostic expectations: $\phi_j = 1$ for all $j \geq J$, and $\phi_j = 1 + \theta$ for $j < J$
- Sticky information: $\phi_j = (1-\theta)\theta^j$ for all j

BEET encodes subrational diagonal expectations as a vector "BE_phivec":

$$\vec{\phi} \equiv \begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_J \end{pmatrix}$$

with the assumption that $\phi_j = 0$ for all $j \geq J$. J is referred to as the "order" of the subrational diagonal expectation, and BEET can solve for any eigenseries with finite order. As a result, it can only solve the eigenseries for infinite-order expectations such as sticky information up to a high order approximation (fortunately, the sticky information eigenseries is known analytically).

Consider an eigenseries Y_t in moving average form: $Y_t = \sum_{j=0}^{\infty} y_j \nu_{t-j}$ where ν_t are white shocks. With this form, the eigenseries equation $\lambda Y_t = \mathbb{E}_t^k[Y_{t+1}]$ becomes

$$\lambda \sum_{j=0}^{\infty} y_j \nu_{t-j} = \phi_0 y_1 \nu_t + (\phi_0 + \phi_1) y_2 \nu_{t-1} + (\phi_0 + \phi_1 + \phi_2) y_3 \nu_{t-2} + ...$$

which implies the following restrictions:

$$\lambda y_j = \sum_{k=0}^{j} \phi_k y_{j+1}$$

This problem simplifies because these restrictions are homogeneous for $j \geq J$:

$$y_j = \bar{\rho} y_{j+1} \qquad \forall j \geq J \tag{4}$$

where $\bar{\rho} = \frac{\sum_{k=0}^{J} \phi_k}{\lambda}$.

Equation (4) implies that an eigenseries $Y_t = \sum_{j=0}^{\infty} y_j L^j \nu_t$ must be ARMA(1,J) of the form

$$Y_t = \frac{\sum_{j=0}^{J} a_j L^j}{1 - \bar{\rho} L} \nu_t \tag{5}$$

assuming $\bar{\rho} < 1$, otherwise λ is not an eigenvalue. The initial coefficient can be normalized to $a_0 = 1$, so finding the eigenseries comes down to solving J equations for the remaining J coefficients.

The $\ell + 1$ period-ahead rational expectations of an ARMA(1,J) satisfying form (5) are given by

$$\mathbb{E}_t[Y_{t+\ell+1}] = \frac{\sum_{j=0}^{\ell} a_j \bar{\rho}^{\ell+1-j} + \sum_{j=\ell+1}^{J} a_j L^{j-(\ell+1)}}{1 - \bar{\rho} L} \nu_t$$

for $\ell + 1 \leq J$. Using lags of these forecasts to construct the behavioral expectation, the eigenseries equation becomes

$$\lambda \frac{\sum_{j=0}^{J} a_j L^j}{1 - \bar{\rho} L} \nu_t = \sum_{\ell=0}^{J} \phi_\ell L^\ell \left(\frac{\sum_{j=0}^{\ell} a_j \bar{\rho}^{\ell+1-j} + \sum_{j=\ell+1}^{J} a_j L^{j-(\ell+1)}}{1 - \bar{\rho} L} \right) \nu_t$$

Considering only the lag operator polynomials, the denominator cancels from both sides, leaving the numerators related by:

$$\lambda \sum_{j=0}^{J} a_j L^j = \sum_{\ell=0}^{J} \phi_\ell L^\ell \left(\sum_{j=0}^{\ell} a_j \bar{\rho}^{\ell+1-j} + \sum_{j=\ell+1}^{J} a_j L^{j-(\ell+1)} \right)$$

Collecting coefficients on lag operators gives the restrictions:

$$\lambda a_0 = \phi_0 \bar{\rho} a_0 + \phi_0 a_1$$

$$\lambda a_1 = \phi_1(\bar{\rho}^2 a_0 + \bar{\rho} a_1) + (\phi_1 + \phi_0) a_2$$

$$\lambda a_2 = \phi_2(\bar{\rho}^3 a_0 + \bar{\rho}^2 a_1 + \bar{\rho} a_2) + (\phi_2 + \phi_1 + \phi_0) a_3$$

$$\vdots$$

or in matrix form:

$$\lambda \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_J \end{pmatrix} = \begin{pmatrix} \phi_0\overline{\rho} & \phi_0 & 0 & 0 & \cdots & 0 \\ \phi_1\overline{\rho}^2 & \phi_1\overline{\rho} & \phi_1 + \phi_0 & 0 & \ddots & 0 \\ \phi_2\overline{\rho}^3 & \phi_2\overline{\rho}^2 & \phi_2\overline{\rho} & \phi_2 + \phi_1 + \phi_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \phi_J\overline{\rho}^{J+1} & \phi_J\overline{\rho}^J & \phi_J\overline{\rho}^{J-1} & \phi_J\overline{\rho}^{J-2} & \cdots & \phi_J\overline{\rho} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_J \end{pmatrix}$$

which we can write more concisely as

$$\lambda \vec{a} = B_{\bar{\rho}} \vec{a}$$

Thus $\vec{a}$ is the eigenvector of matrix $B_{\bar{\rho}}$ associated with λ.

Finally, λ itself is unknown. $B_{\bar{\rho}}$ is determined by $\bar{\rho}$ which depends on λ. So the BEET toolkit searches for a value of λ whose associated matrix $B_{\bar{\rho}}$ has λ as an eigenvalue.

References

SIMS, C. A. (2002): "Solving Linear Rational Expectations Models," *Computational Economics*, 20(1-2), 1–20, Num Pages: 1-20 Place: Dordrecht, Netherlands Publisher: Springer Nature B.V.

UHLIG, H. (2001): "A Toolkit for Analysing Nonlinear Dynamic Stochastic Models Easily," in *Computational Methods for the Study of Dynamic Economies*, ed. by R. Marimon, and A. Scott, p. 0. Oxford University Press.

A Collapsing the Exogenous States

Suppose we have a model in Uhlig form (1) satisfying the ALM $z_{t+1} = Nz_t + \epsilon_{t+1}$, but which cannot be written with a corresponding PLM, i.e. there is no N^k such that in general $N^k z_t = \mathbb{E}^k[z_{t+1}|z_t]$.

Consider behavioral expectations of the following "Subrational Diagonal" form:

$$\mathbb{E}_t^k[x_{t+1}] = \sum_{j=0}^{J} \phi_j \mathbb{E}_{t-j}^k[x_{t+1}]$$

In this case, we can write

$$\mathbb{E}_t^k[z_{t+1}] = \sum_{j=0}^{J} \phi_j (N^k)^{j+1} z_{t-j}$$

Expectations depend on z_t plus up to J additional lags. stack these as a single vector $\mathbf{z}_t$:

$$\mathbf{z}_t \equiv \begin{pmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-J} \end{pmatrix}$$

The ALM for $\mathbf{z}_t$ is

$$\mathbf{z}_{t+1} = \mathbf{N}\mathbf{z}_t + \vec{\epsilon}_{t+1}$$

where

$$\mathbf{N} \equiv \begin{pmatrix} N & 0 & 0 & ... & 0 \\ I & 0 & 0 & ... & 0 \\ 0 & I & 0 & ... & 0 \\ 0 & 0 & I & ... & 0 \\ \vdots & \ddots & \ddots & ... & \vdots \end{pmatrix} \qquad \vec{\epsilon}_t \equiv \begin{pmatrix} \epsilon_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with PLM $\mathbb{E}_t^k[\mathbf{z}_{t+1}] = N^k \mathbf{z}_t$ where

$$\mathbf{N}^k \equiv \begin{pmatrix} \phi_0 N & \phi_1 N^2 & \phi_2 N^3 & ... & \phi_J N^{J+1} \\ I & 0 & 0 & ... & 0 \\ 0 & I & 0 & ... & 0 \\ 0 & 0 & I & ... & 0 \\ \vdots & \ddots & \ddots & ... & \vdots \end{pmatrix}$$

Then, the model can be rewritten in Uhlig form albeit with $\mathbf{z}_t$ instead of z_t:

$$0 = \mathbb{E}_t^k \left[F x_{t+1} + G x_t + H x_{t-1} + \mathbf{L}\mathbf{z}_{t+1} + \mathbf{M}\mathbf{z}_t \right]$$

where the matrices $\mathbf{L}$ and $\mathbf{M}$ are defined as

$$\mathbf{L} \equiv \begin{pmatrix} L & 0 & 0 & ... & 0 \end{pmatrix} \qquad \mathbf{M} \equiv \begin{pmatrix} M & 0 & 0 & ... & 0 \end{pmatrix}$$