Behavioral Expectations Equilibrium Toolkit: Conceptual Documentation

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This is an early draft, so please email me with comments and questions!

1　Introduction

This document describes the BEET toolkit for solving behavioral expectations.

2　Uhlig Form

I say a behavioral macroeconomic model is in "Uhlig" form if

$$0 = \mathbb{E}_t^k \left[F x_{t+1} \right] + G x_t + H x_{t-1} + L z_{t+1} + M z_t \tag{1}$$

where x_t is a $m \times 1$ vector of endogenous state variables chosen at time t, and z_t is a vector of exogenous state variables given by the *actual law of motion* (ALM)

$$[ALM]: \qquad z_{t+1} = N z_t + \epsilon_{t+1}$$

with ϵ_{t+1} an iid stochastic shock satsifying $\mathbb{E}_t^k[\epsilon_{t+1}] = 0$.

We would like to find the matrices P and Q^k to solve the model (1) resursively by

$$x_t = P x_{t-1} + Q^k z_t \tag{2}$$

Uhlig (2001) Theorem 1 solves this model in the case of rational expectations. The behavioral expectations case is similar, except it utilizes an incorrect *perceived law of motion* (PLM) for the exogenous state, namely:

$$[PLM]: \qquad z_{t+1} = N_k z_t + \epsilon_{t+1} \tag{3}$$

which is related to the ALM by the behavioral expectation operator $\mathbb{E}^k$ such that

$$N_k^h z_t = \mathbb{E}^k[z_{t+h}|z_t] \qquad \forall h \geq 0$$

Depending on the form of behavioral expectations, some extra work may be required to write the exogenous states in this way. See Appendix B.

Then, following Uhlig's proof, the solution to the model is given by:

Theorem 1 *A behavioral solution (2) is given by a matrix P satisfying the quadratic*

$$0 = F P^2 + G P + H$$

and the behavioral matrix Q^k satisfying

$$Q^k = -(V^k)^{-1} vec(LN_k + M)$$

where

$$V^k = (N_k)' \otimes F + I_k \otimes (FP + G)$$

Proof: Appendix A.1

The presumption here is that agents know how choices x_t depend on the exogenous state z_t, and forecast the endogenous vector x_{t+1} by forecasting the exogenous state z_{t+1} with their behavioral expectations, and then mapping that back to x_{t+1} outcomes. *This is equivalent to directly forecasting x_{t+1} if the behavioral expectations operator is "series-agnostic"*, but may not hold more generally.

3 Mapping from Uhlig to GENSYS Form

Writing the model in Uhlig form is convenient for precisely characterizing the agnets' information set and expectations. But it may be desirable to solve the model with Chris Sims' GENSYS algorithm instead (Sims, 2002). This section describes how to map the behavioral model (1) to the GENSYS form.

Sims' form for a macroeconomic model is

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi z_t + \Pi \eta_t \tag{4}$$

where y_t is a vector of endogenous choice variables, while η_t is a vector of endogenously determined forecast errors. z_t is as before: an exogenous stochastic vector that is observed by the agents.

Between the two forms, the variables are mapped by

$$y_t = \begin{pmatrix} x_t \\ \mathbb{E}_t^k[x_{t+1}] \end{pmatrix}$$

following this mapping, the model in Uhlig form (1) can be written in GENSYS form

by

$$\begin{pmatrix} -G & -F \\ I & 0 \end{pmatrix} \begin{pmatrix} x_t \\ \mathbb{E}_t^k[x_{t+1}] \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x_{t-1} \\ \mathbb{E}_{t-1}^k[x_t] \end{pmatrix} + \begin{pmatrix} LN_k + M \\ 0 \end{pmatrix} z_t + \begin{pmatrix} 0 \\ I \end{pmatrix} \left(x_t - \mathbb{E}_{t-1}^k[x_t] \right)$$

so that Sims' matrices are given by

$$\Gamma_0 = \begin{pmatrix} -G & -F \\ I & 0 \end{pmatrix} \qquad \Gamma_1 = \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \qquad \Psi = \begin{pmatrix} LN_k + M \\ 0 \end{pmatrix} \qquad \Pi = \begin{pmatrix} 0 \\ I \end{pmatrix}$$

and $\eta = \left(x_t - \mathbb{E}_{t-1}^k[x_t] \right)$.

4   Solving Behavioral Models with GENSYS

When Chris Sims' GENSYS function is used to solve models, behavioral expectations must be treated carefully. The algorithm imposes rational expectations as a part of the solution method, so unlike the Uhlig method, applying to a behavioral model is not a simple as substituting a perceived law of motion into the solution equations.

To demonstrate how to properly handle behavioral models in GENSYS, this section rederives Sims' method when expectations are not necessarily rational, then describes how to interpret the GENSYS function output.

4.1   Solution to Models in GENSYS Form

This section largely follows the notation in Sims (2002), albeit using lag operator notation instead of time subscripts.

A model in GENSYS form is

$$\Gamma_0 y = \Gamma_1 Ly + C + \Psi z + \Pi \eta_k \tag{5}$$

There main difference here from Sims (2002) is that the forecast error η_k now has a subscript denoting the expectation type k that applies to the model. When we write the behavioral expectation as a quasi-linear operator $\mathcal{E}_k$, we will have $\mathcal{E}_k \eta_k = 0$.

Analyzing the behavioral model requires one crucial assumption that is not needed in the rational case: **no variables in y_t can be known with certainty at time**

$t - 1$.[1] This rules out the inclusion of lagged variables in y_t, so the model can depend on state variables through matrix Γ_1, but additional lags are not allowed. Without this assumption, Theorem 2 will not hold. Additionally, I impose the same regularity conditions as Sims (e.g. no undefined generalized eigenvalues, certain matrices must be invertible, etc.) as well as the additional regularity conditions for behavioral models from Adams (2023) (e.g. no generalized eigenvalues that are both stable and unstable).

Theorem 2 *A solution to the behavioral model* (5) *is*

$$y = \Theta_1 L y + \Theta_c + \Theta_0 z + \Theta_y (I - \Theta_f \mathcal{E}_k)^{-1} \Theta_z \mathcal{E}_k z$$

where the coefficient matrices are as defined in Sims (2002):

$$H = Z \begin{pmatrix} \Lambda_{11} & \Lambda_{12} - \Phi\Lambda_{22} \\ 0 & I \end{pmatrix}^{-1} \qquad \Theta_1 = Z_1 \Lambda_{11}^{-1} \begin{pmatrix} \Omega_{11} & -(\Omega_{12} - \Phi\Omega_{22}) \end{pmatrix} Z$$

$$\Theta_c = H \begin{pmatrix} Q_1 - \Phi Q_2 \\ (\Omega_{22} - \Lambda_{22})^{-1} Q_2 \end{pmatrix} C \qquad \Theta_0 = H \begin{pmatrix} Q_1 - \Phi Q_2 \\ 0 \end{pmatrix} \Psi$$

$$\Theta_y = -H_2 \qquad \Theta_f = M \qquad \Theta_z = \Omega_{22}^{-1} Q_2 \Psi$$

with Z_1 denoting the first block column of Z and H_2 denoting the second block column of H.

Proof: Appendix A.2

4.2 Implications for Implementation

Theorem 2 implies that the behavioral expectations affect the GENSYS solution through only one channel: the dependence of y_t on expectations of future exogenous terms z_t. Behavioral expectations do not affect the immediate impact of a shock (the matrix Θ_0) which is unforecastable whether ones' expectations are rational or not.

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[1] This assumption is not needed when expectations are rational, because the rational expectation operator is always linear. However, behavioral expectation operators are only linear when applied to future variables.

The theorem also implies that an appropriate way of handling behavioral expectations is simply solve the rational expectations version of the model, then adjust the $\Theta_y(I - \Theta_f\mathcal{E}_k)^{-1}\Theta_z\mathcal{E}_k$ term to accommodate alternative expectations.

When applied to a rational expectations model, the GENSYS function outputs:

$$\text{G1} = \Theta_1 \qquad \text{C} = \Theta_c \qquad \text{impact} = \Theta_0 \qquad \text{ywt} = \theta_y \qquad \text{fmat} = \Theta_f \qquad \text{fwt} = \Theta_z$$

How can the behavioral term $\Theta_y(I - \Theta_f\mathcal{E}_k)^{-1}\Theta_z\mathcal{E}_k z$ be constructed from this output?

Suppose the exogenous series z can be represented with a PLM as in equation (3). With lag operator notation, this implies

$$\mathcal{E}_k z = N_k z$$

which means the unknown behavioral term is given by

$$\Theta_y(I - \Theta_f\mathcal{E}_k)^{-1}\Theta_z\mathcal{E}_k z = \Theta_y(\sum_{j=0}^{\infty}\Theta_f^j\mathcal{E}_k^{j+1})\Theta_z z$$

$$= \Theta_y(\sum_{j=0}^{\infty}\Theta_f^j\Theta_z\mathcal{E}_k^{j+1})z = \Theta_y(\sum_{j=0}^{\infty}\Theta_f^j\Theta_z N_k^{j+1})z$$

4.3 Example: Correct Application of GENSYS to a Behavioral Asset Pricing Model

Consider the following behavioral asset pricing model:

$$p_t = z_t + \mathbb{E}_t^k[p_{t+1}]$$

where z_t is an AR(1) dividend given by

$$z_t = \rho z_{t-1} + \epsilon_t$$

In Uhlig form, this model can be written as:

$$0 = \mathbb{E}_t^k[\underbrace{-\beta}_{F} p_{t+1} + \underbrace{1}_{G} p_t + \underbrace{-1}_{M} z_t]$$

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In GENSYS form, this model can be written

$$
\underbrace{\begin{pmatrix} 1 & -\beta \\ 1 & 0 \end{pmatrix}}_{\Gamma_0} \begin{pmatrix} p_t \\ \mathbb{E}_t^k[p_{t+1}] \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\Gamma_1} \begin{pmatrix} p_{t-1} \\ \mathbb{E}_{t-1}^k[p_t] \end{pmatrix} + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\Psi} z_t + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\Pi} \eta_{k,t}
$$

to which Theorem 2 describes the solution.

However, there is another way of applying GENSYS, which may appear reasonable at first glance, but can lead to incorrect conclusions. The practitioner might decide to rewrite the behavioral model as a raitonal expectations model, by hardcoding the behavioral expectations as a function of raitonal expectations in the model's equilibrium conditions. Written as a rational expectations model, can GENSYS be applied without modification?

No. As the following examples demonstrate, this method is not reliable. Next, we consider two examples of behavioral expectations: misextrapolation and diagnostic expectations.

4.3.1   Behavioral Asset Pricing with Misextrapolation

Suppose that agents have *misextrapolation* expectations, i.e. for some $\theta^{ME} \in (0, \beta^{-1})$ they forecast by

$$
\mathbb{E}_t^{ME}[p_{t+1}] = \theta^{ME}\mathbb{E}_t[p_{t+1}]
$$

Then the PLM is

$$
[\text{ME PLM:}] \qquad z_t = \theta^{ME}\rho z_{t-1} + \epsilon_t
$$

and per Theorem 2, the solution can be constructed from GENSYS output by

$$
p_t = \theta_1^{ME} p_{t-1} + \theta_0^{ME} z_t + \theta_y^{ME}(I - \theta_f^{ME}\theta^{ME}\rho)^{-1}\theta_z^{ME}\theta^{ME}\rho z_t
$$

$$
= \text{G1}p_{t-1} + \text{impact}z_t + \text{ymt}(I - \text{fmat}\theta^{ME}\rho)^{-1}\text{fwt}\theta^{ME}\rho z_t]
$$

This model is simple enough that it is easy to check the numerical solution against the analytic solution which is given by

$$
p_t = z_t + \beta\mathbb{E}_t^{ME}[z_{t+1}] + \beta^2\mathbb{E}_t^{ME}[\mathbb{E}_{t+1}^{ME}[z_{t+2}]] + \dots
$$

$$
= z_t + \beta\theta^{ME}\mathbb{E}_t[z_{t+1}] + \beta^2(\theta^{ME})^2\mathbb{E}_t[\mathbb{E}_{t+1}[z_{t+2}]] + \dots
$$

$$= \frac{z_t}{1 - \beta\theta^{ME}\rho}$$

Figure 1a gives the impulse response to a dividend shock as calculated by both the Uhlig (solid black line) and GENSYS (dashed blue line) methods. They both match the analytical solution. The figure also plots results from GENSYS is rewritten as a rational expectations model (dotted red line). In this case, all solutions are consistent with one another.

In general, it is not correct to rewrite the behavioral model as a rational expectations model. But in this case, the two approaches give consistent solutions. The rational expectations model encodes the behavioral expectation directly into the equilibrium condition:

$$\underbrace{\begin{pmatrix} 1 & -\beta\theta^{ME} \\ 1 & 0 \end{pmatrix}}_{\Gamma_0} \begin{pmatrix} p_t \\ \mathbb{E}_t[p_{t+1}] \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\Gamma_1} \begin{pmatrix} p_{t-1} \\ \mathbb{E}_{t-1}[p_t] \end{pmatrix} + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\Psi} z_t + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\Pi} \eta_t$$

where η_t denotes the rational forecast error. As a rational expectations model, this can now be solved with GENSYS as usual. But it will not *always* generate the correct solution to the behavioral model, as the next section demonstrates.
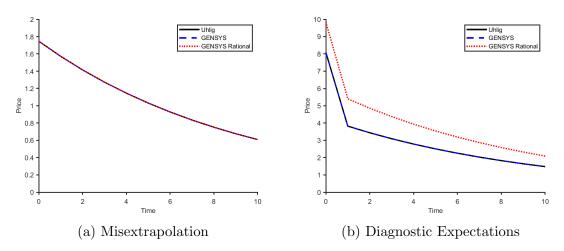

(a) Misextrapolation   (b) Diagnostic Expectations

Figure 1: IRFs to a Dividend Shock in the Behavioral Asset Pricing Model

Notes: In all cases, $\beta = 0.95$, $\rho = 0.90$. The behavioral parameters are $\theta^{ME} = \theta^{DE} = 0.5$. The solutions are labeled as: "Uhlig" was solved using the Uhlig toolkit, "GENSYS" was solved using GENSYS when the model is written in the form of equation (5), and "GENSYS Rational" was solved by rewriting the behavioral models as rational expectations models.

4.3.2 Behavioral Asset Pricing with Diagnostic Expectations

Suppose that agents have *diagnostic* expectations, i.e. for some θ^{DE} they forecast by

$$\mathbb{E}_t^{ME}[p_{t+1}] = (1 + \theta^{DE})\mathbb{E}_t[p_{t+1}] - \theta^{DE}\mathbb{E}_{t-1}[p_{t+1}]$$

Then the PLM is

$$[\text{DE PLM:}] \qquad z_t = (1 + \theta^{DE})\rho z_{t-1} - \theta^{DE}\rho z_{t-2} + \epsilon_t$$

and per Theorem 2, the solution can be constructed from GENSYS output by

$$p_t = \Theta_1 p_{t-1} + \Theta_0 z_t + \Theta_y(I - \Theta_f\rho)^{-1}\Theta_z\rho z_t + \Theta_y\Theta_z\theta^{DE}(\rho z_t - \rho^2 z_{t-1})$$

$$= \text{G1}p_{t-1} + \text{impact}z_t + \text{ymt}(I - \text{fmat}\rho)^{-1}\text{fwt}\rho z_t + \text{ymtfwt}\theta^{DE}(\rho z_t - \rho^2 z_{t-1})]$$

As before, an incorrect application of GENSYS is to rewrite the behavioral model as a rational expectations model:

$$
\underbrace{\begin{pmatrix} 1 & -\beta(1+\theta^{DE}) & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\Gamma_0}
\begin{pmatrix} p_t \\ \mathbb{E}_t[p_{t+1}] \\ \mathbb{E}_t[p_{t+2}] \end{pmatrix}
$$

$$
= \underbrace{\begin{pmatrix} 0 & 0 & -\beta\theta^{DE} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\Gamma_1}
\begin{pmatrix} p_{t-1} \\ \mathbb{E}_{t-1}[p_t] \\ \mathbb{E}_{t-1}[p_{t+1}] \end{pmatrix}
+ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\Psi} z_t
+ \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\Pi} \eta_t
$$

where the first dimension of η_t is the rational forecast error of p_t, and the second dimension is the rational forecast error of $\mathbb{E}_t[p_{t+1}]$.

Figure REFERENCE gives the impulse response to a dividend shock as calculated by both the Uhlig (solid black line) and GENSYS (dashed blue line) methods. They both match the analytical solution. However, the figure also plots a false solution when GENSYS is not applied correctly (dotted red line).

References

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A   Proofs

A.1   Proof of Theorem 1

Proof. Substitute the solution (2) into the equilibrium condition (1):

$$0 = \mathbb{E}_t^k \left[F(Px_t + Q^k z_{t+1}) + Gx_t + Hx_{t-1} + Lz_{t+1} + Mz_t \right]$$

Evaluate the behavioral expectation and collect terms

$$0 = (FP + G)x_t + Hx_{t-1} + ((FQ^k + L)N_k + M)z_t$$

then substitute again

$$0 = (FP + G)(Px_{t-1} + Q^k z_t) + Hx_{t-1} + ((FQ^k + L)N_k + M)z_t$$

Collecting terms on x_{t-1} and z_t gives the usual matrix quadratic equation:

$$0 = FP^2 + GP + H$$

and given P, the coefficient on z_t must satisfy

$$0 = (FP + G)^k + (FQ^k + L)N_k + M$$

so Q^k must satisfy

$$V^k Q^k = -vec(LN_k + M)$$

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A.2   Proof of Theorem 2

Proof. Take the generalized Schur ("QZ") decomposition such that

$$Q^* \Lambda Z^* = \Gamma_0 \qquad Q^* \Omega Z^* = \Gamma_1$$

where Q and Z are unitary, Λ and Ω are upper triangular, and ordered so that the generalized eigenvalues ($\phi_i \equiv \omega_{ii}/\lambda_{ii}$) are in increasing order down the diagonals.

Multiply by Q and define $w \equiv Z^*y$ to write equation (5) as

$$\Lambda w = \Omega L w + QC + Q\Psi z + Q\Pi\eta_k$$

The first block row of this matrix equation (corresponding to the generalized eigenvalues with magnitude $|\phi_i| < 1$) is the stable block, while the second block row (corresponding to the generalized eigenvalues with magnitude $|\phi_i| > r(\mathcal{E}_k)$) where $r(\mathcal{E}_k)$ is the spectral radius of the behavioral expectation operator. Divided into these blocks, the model becomes

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{pmatrix} \begin{pmatrix} Lw_1 \\ Lw_2 \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} (C + \Psi z + \Pi\eta_k) \quad (6)$$

The second block row is explosive and must be solved forwards:

$$\Omega_{22}^{-1}\Lambda_{22} w_2 = Lw_2 + \Omega_{22}^{-1}x_2$$

where $x_2 \equiv Q_2(C + \Psi z + \Pi\eta_k)$. Take expectations of both sides:[2]

$$\Omega_{22}^{-1}\Lambda_{22}\mathcal{E}_k w_2 = w_2 + \Omega_{22}^{-1}\mathcal{E}_k x_2$$

which assumes that agents perfectly forecast lagged variables so that $\mathcal{E}_k L = I$, assumes that expectations are XXXXXXX so that matrices commute with $\mathcal{E}_k$. Inversion gives w_2:

$$w_2 = -(I - \Omega_{22}^{-1}\Lambda_{22}\mathcal{E}_k)^{-1}\Omega_{22}^{-1}\mathcal{E}_k x_2 \quad (7)$$

x_2 is endogenous because it contains the forecast error, which remains to be solved for. To do so, apply $I - L\mathcal{E}_k$ (the "forecast error" operator) to the unstable block:

$$(I - L\mathcal{E}_k)\Omega_{22}^{-1}\Lambda_{22} w_2 = (I - L\mathcal{E}_k)\Omega_{22}^{-1}x_2$$

The Lw_2 on the right-hand side has disappeared because there is no forecast error

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[2]At this point, Sims iterates forward in time, which is innocuous under rational expectations because the law of iterated expectations holds. This is not always true for behavioral expectations, so it is crucial to be careful about applying operators appropriately. Specifically, in this algebra of singly-infinite vectors and operators, L^{-1} does not exist. Instead, we must take expectations at this step.

over lagged variables.[3] Substituting for w_2 gives

$$-(I - L\mathcal{E}_k)\Lambda_{22}(I - \Omega_{22}^{-1}\Lambda_{22}\mathcal{E}_k)^{-1}\Omega_{22}^{-1}\mathcal{E}_k x_2 = (I - L\mathcal{E}_k)x_2$$

Eliminate terms that are known with certainty or zero in expectation:

$$-(I - L\mathcal{E}_k)\Lambda_{22}(I - \Omega_{22}^{-1}\Lambda_{22}\mathcal{E}_k)^{-1}\Omega_{22}^{-1}\mathcal{E}_k Q_2\Psi z = (I - L\mathcal{E}_k)(Q_2\Psi z + Q_2\Pi\eta_k) \quad (8)$$

Rearrange and use $(I - L\mathcal{E}_k)\eta_k = \eta_k$:

$$-(I - L\mathcal{E}_k)(I - \Lambda_{22}\Omega_{22}^{-1}\mathcal{E}_k)^{-1}Q_2\Psi z = Q_2\Pi\eta_k$$

so if $Q_2\Pi$ is invertible, then η_k is uniquely given by

$$\eta_k = -(Q_2\Pi)^{-1}(I - L\mathcal{E}_k)(I - \Lambda_{22}\Omega_{22}^{-1}\mathcal{E}_k)^{-1}Q_2\Psi z$$

If $Q_2\Pi$ is not invertible, then there may be many potential η_k satisfying equation (8) – in which case the model features a continuum of equilibria – or there may be no solution at all. Regardless, we assume that the row space of $Q_1\Pi$ is contained in that of $Q_2\Pi$ so that there is some matrix Φ such that

$$Q_1\Pi = \Phi q_2\Pi$$

This implies that the entire system can be rewritten as

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} - \Phi\Lambda_{22} \\ 0 & I \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} - \Phi\Omega_{22} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Lw_1 \\ Lw_2 \end{pmatrix} + \begin{pmatrix} Q_1 - \Phi Q_2 \\ (\Omega_{22} - \Lambda_{22})^{-1}Q_2 \end{pmatrix}C$$
$$+ \begin{pmatrix} Q_1 - \Phi Q_2 \\ 0 \end{pmatrix}\Psi z - \begin{pmatrix} 0 \\ (I - M\mathcal{E}_k)^{-1}\Omega_{22}^{-1}Q_2\Psi\mathcal{E}_k z \end{pmatrix}$$

where the first block row follows from pre-multiplying equation (6) by $\begin{pmatrix} I & -\Phi \end{pmatrix}$, the second block row encodes equation (7), and $M \equiv \Omega_{22}^{-1}\Lambda_{22}$. Finally, pre-multiplying by Q translates the system into

$$y = \Theta_1 Ly + \Theta_c + \Theta_0 z + \Theta_y(I - \Theta_f\mathcal{E}_k)^{-1}\Theta_z\mathcal{E}_k z$$

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[3] $(I - L\mathcal{E}_k)L = 0$ because $\mathcal{E}_k L = I$ even though $L\mathcal{E}_k \neq 0$.

■

B Collapsing the Exogenous States

Suppose we have a model in Uhlig form (1) satisfying the ALM $z_{t+1} = Nz_t + \epsilon_{t+1}$, but which cannot be written with a corresponding PLM, i.e. there is no N_k such that in general $N_k z_t = \mathbb{E}^k[z_{t+1}|z_t]$.

Consider behavioral expectations of the following "Subrational Diagonal" form:

$$\mathbb{E}_t^k[x_{t+1}] = \sum_{j=0}^{J} \phi_j \mathbb{E}_{t-j}^k[x_{t+1}]$$

In this case, we can write

$$\mathbb{E}_t^k[z_{t+1}] = \sum_{j=0}^{J} \phi_j (N_k)^{j+1} z_{t-j}$$

Expectations depend on z_t plus up to J additional lags. stack these as a single vector $\mathbf{z}_t$:

$$\mathbf{z}_t \equiv \begin{pmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-J} \end{pmatrix}$$

The ALM for $\mathbf{z}_t$ is

$$\mathbf{z}_{t+1} = \mathbf{N}\mathbf{z}_t + \vec{\epsilon}_{t+1}$$

where

$$\mathbf{N} \equiv \begin{pmatrix} N & 0 & 0 & \dots & 0 \\ I & 0 & 0 & \dots & 0 \\ 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \ddots & \ddots & \dots & \vdots \end{pmatrix} \qquad \vec{\epsilon}_t \equiv \begin{pmatrix} \epsilon_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with PLM $\mathbb{E}_t^k[\mathbf{z}_{t+1}] = N_k \mathbf{z}_t$ where

$$\mathbf{N}^k \equiv \begin{pmatrix} \phi_0 N & \phi_1 N^2 & \phi_2 N^3 & \dots & \phi_J N^{J+1} \\ I & 0 & 0 & \dots & 0 \\ 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \ddots & \ddots & \dots & \vdots \end{pmatrix}$$

Then, the model can be rewritten in Uhlig form albeit with $\mathbf{z}_t$ instead of z_t:

$$0 = \mathbb{E}_t^k \left[F x_{t+1} + G x_t + H x_{t-1} + \mathbf{L} \mathbf{z}_{t+1} + \mathbf{M} \mathbf{z}_t \right]$$

where the matrices $\mathbf{L}$ and $\mathbf{M}$ are defined as

$$\mathbf{L} \equiv \begin{pmatrix} L & 0 & 0 & \dots & 0 \end{pmatrix} \qquad \mathbf{M} \equiv \begin{pmatrix} M & 0 & 0 & \dots & 0 \end{pmatrix}$$