Optimal Policy Without Rational Expectations: A Sufficient Statistic Solution

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EARLY DRAFT

Abstract

This document describes the BEET code.

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This is an early draft, so please email me with comments and questions!

1 Introduction

2 Uhlig Form

I say a behavioral macroeconomic model is in "Uhlig" form if

$$0 = \mathbb{E}_t^k \left[Fx_{t+1} + Gx_t + Hx_{t-1} + Lz_{t+1} + Mz_t \right] \tag{1}$$

where x_t is a $m \times 1$ vector of endogenous state variables chosen at time t, and z_t is a vector of exogenous state variables given by the *actual law of motion* (ALM)

$$[ALM]: \qquad z_{t+1} = Nz_t + \epsilon_{t+1}$$

with ϵ_{t+1} an iid stochastic shock satsifying $\mathbb{E}_t^k[\epsilon_{t+1}] = 0$.

We would like to find the matrices P and Q^k to solve the model (1) resursively by

$$x_t = Px_{t-1} + Q^k z_t \tag{2}$$

Uhlig (2001) Theorem 1 solves this model in the case of rational expectations. The behavioral expectations case is similar, except it utilizes an incorrect *perceived law of motion* (PLM) for the exogenous state, namely:

$$[PLM]: \qquad z_{t+1} = N^k z_t + \epsilon_{t+1}$$

which is related to the ALM by the behavioral expectation operator $\mathbb{E}^k$ such that

$$N^k z_t = \mathbb{E}^k[z_{t+1} | z_t]$$

Depending on the behavioral expectations process, some extra work may be required to write the exogenous states in this way. See Appendix A.

Then, following Uhlig's proof, the solution to the model is given by:

Theorem 1 *A behavioral solution* (2) *is given by a matrix P satisfying the quadratic*

$$0 = FP^2 + GP + H$$

and the behavioral matrix Q^k satisfying

$$Q^k = -(V^k)^{-1} vec(LN^k + M)$$

where

$$V^k = (N^k)' \otimes F + I_k \otimes (FP + G)$$

Proof. Substitute the solution (2) into the equilibrium condition (1):

$$0 = \mathbb{E}_t^k \left[F(Px_t + Q^k z_{t+1}) + Gx_t + Hx_{t-1} + Lz_{t+1} + Mz_t \right]$$

Evaluate the behavioral expectation and collect terms

$$0 = (FP + G)x_t + Hx_{t-1} + ((FQ^k + L)N^k + M)z_t$$

then substitute again

$$0 = (FP + G)(Px_{t-1} + Q^k z_t) + Hx_{t-1} + ((FQ^k + L)N^k + M)z_t$$

Collecting terms on x_{t-1} and z_t gives the usual matrix quadratic equation:

$$0 = FP^2 + GP + H$$

and given P, the coefficient on z_t must satisfy

$$0 = (FP + G)^k + (FQ^k + L)N^k + M$$

so Q^k must satisfy

$$V^k Q^k = -vec(LN^k + M)$$

■

The presumption here is that agents know how choices x_t depend on the exogenous state z_t, and forecast the endogenous vector x_{t+1} by forecasting the exogenous state z_{t+1} with their behavioral expectations, and then mapping that back to x_{t+1} outcomes. *This is equivalent to directly forecasting x_{t+1} if the behavioral expectations operator is "series-agnostic"*, but may not hold more generally.

3 Mapping from Uhlig to GENSYS Form

Writing the model in Uhlig form is convenient for precisely characterizing the agnets' information set and expectations. But it may be desirable to solve the model with Chris Sims' GENSYS algorithm instead. This section describes how to map the behavioral model (1) to the GENSYS form.

Sims' form for a macroeconomic model is

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi z_t + \Pi \eta_t \tag{3}$$

where y_t is a vector of endogenous choice variables, while η_t is a vector of endogenously determined forecast errors. z_t is as before: an exogenous stochastic vector that is observed by the agents.

Between the two forms, the variables are mapped by

$$y_t = \begin{pmatrix} x_t \\ \mathbb{E}_t^k[x_{t+1}] \end{pmatrix}$$

following this mapping, the model in Uhlig form (1) can be written in GENSYS form by

$$\begin{pmatrix} -G & -F \\ I & 0 \end{pmatrix} \begin{pmatrix} x_t \\ \mathbb{E}_t^k[x_{t+1}] \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x_{t-1} \\ \mathbb{E}_{t-1}^k[x_t] \end{pmatrix} + \begin{pmatrix} LN^k + M \\ 0 \end{pmatrix} z_t + \begin{pmatrix} 0 \\ I \end{pmatrix} \left(x_t - \mathbb{E}_{t-1}^k[x_t]\right)$$

so that Sims' matrices are given by

$$\Gamma_0 = \begin{pmatrix} -G & -F \\ I & 0 \end{pmatrix} \qquad \Gamma_1 = \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \qquad \Psi = \begin{pmatrix} LN^k + M \\ 0 \end{pmatrix} \qquad \Pi = \begin{pmatrix} 0 \\ I \end{pmatrix}$$

and $\eta = \left(x_t - \mathbb{E}_{t-1}^k[x_t]\right)$.

Now here's the issue: with behavioral expectations, η is *not* white noise. Can this model be solved with Sims' GENSYS program? If so, how?

I believe the answer is yes, but to be honest, we should prove it.

Output is:

$$y_t = \mathrm{G1} y_{t-1} + \mathrm{C} + \mathrm{impact} + \mathrm{ymt}(I - \mathrm{fmat}\mathbb{E}_t^k L^{-1})^{-1} \mathrm{fwt}\mathbb{E}_t^k[z_{t+1}]$$

References

UHLIG, H. (2001): “A Toolkit for Analysing Nonlinear Dynamic Stochastic Models Easily,” in *Computational Methods for the Study of Dynamic Economies*, ed. by R. Marimon, and A. Scott, p. 0. Oxford University Press.

A Collapsing the Exogenous States

Suppose we have a model in Uhlig form (1) satisfying the ALM $z_{t+1} = Nz_t + \epsilon_{t+1}$, but which cannot be written with a corresponding PLM, i.e. there is no N^k such that in general $N^k z_t = \mathbb{E}^k[z_{t+1}|z_t]$.

Consider behavioral expectations of the following "Subrational Diagonal" form:

$$\mathbb{E}_t^k[x_{t+1}] = \sum_{j=0}^{J} \phi_j \mathbb{E}_{t-j}^k[x_{t+1}]$$

In this case, we can write

$$\mathbb{E}_t^k[z_{t+1}] = \sum_{j=0}^{J} \phi_j (N^k)^{j+1} z_{t-j}$$

Expectations depend on z_t plus up to J additional lags. stack these as a single vector $\mathbf{z}_t$:

$$\mathbf{z}_t \equiv \begin{pmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-J} \end{pmatrix}$$

The ALM for $\mathbf{z}_t$ is

$$\mathbf{z}_{t+1} = \mathbf{N}\mathbf{z}_t + \vec{\epsilon}_{t+1}$$

where

$$\mathbf{N} \equiv \begin{pmatrix} N & 0 & 0 & ... & 0 \\ I & 0 & 0 & ... & 0 \\ 0 & I & 0 & ... & 0 \\ 0 & 0 & I & ... & 0 \\ \vdots & \ddots & \ddots & ... & \vdots \end{pmatrix} \qquad \vec{\epsilon}_t \equiv \begin{pmatrix} \epsilon_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with PLM $\mathbb{E}_t^k[\mathbf{z}_{t+1}] = N^k \mathbf{z}_t$ where

$$\mathbf{N}^k \equiv \begin{pmatrix} \phi_0 N & \phi_1 N^2 & \phi_2 N^3 & ... & \phi_J N^{J+1} \\ I & 0 & 0 & ... & 0 \\ 0 & I & 0 & ... & 0 \\ 0 & 0 & I & ... & 0 \\ \vdots & \ddots & \ddots & ... & \vdots \end{pmatrix}$$

DO ITERATED EXPECTATIONS WORK THE WAY I BELIEVE THEY SHOULD WITH THIS? DOE SIT MATTER?

Then, the model can be rewritten in Uhlig form albeit with $\mathbf{z}_t$ instead of z_t:

$$0 = \mathbb{E}_t^k \left[F x_{t+1} + G x_t + H x_{t-1} + \mathbf{L}\mathbf{z}_{t+1} + \mathbf{M}\mathbf{z}_t \right]$$

where the matrices $\mathbf{L}$ and $\mathbf{M}$ are defined as

$$\mathbf{L} \equiv \begin{pmatrix} L & 0 & 0 & ... & 0 \end{pmatrix} \qquad \mathbf{M} \equiv \begin{pmatrix} M & 0 & 0 & ... & 0 \end{pmatrix}$$