# Behavioral Expectations Equilibrium Toolkit: Conceptual Documentation

**Jonathan J Adams***

May 2, 2024

*VERY EARLY DRAFT!*

Current BEET version: 0.2

github link

*University of Florida. Website: www.jonathanjadams.com Email: adamsjonathan@ufl.edu.
This is an early draft, so please email me with comments and questions!

# 1 Introduction

This document describes the BEET toolkit for solving behavioral expectations.

# 2 Uhlig Form

I say a behavioral macroeconomic model is in "Uhlig" form if

$$0 = \mathbb{E}_t^k \left[ F x_{t+1} + G x_t + H x_{t-1} + L z_{t+1} + M z_t \right] \tag{1}$$

where $x_t$ is a $m \times 1$ vector of endogenous state variables chosen at time $t$, and $z_t$ is a vector of exogenous state variables given by the *actual law of motion* (ALM)

$$[ALM]: \qquad\qquad z_{t+1} = N z_t + \epsilon_{t+1}$$

with $\epsilon_{t+1}$ an iid stochastic shock satsifying $\mathbb{E}_t^k[\epsilon_{t+1}] = 0$.

We would like to find the matrices $P$ and $Q^k$ to solve the model (1) resursively by

$$x_t = P x_{t-1} + Q^k z_t \tag{2}$$

Uhlig (2001) Theorem 1 solves this model in the case of rational expectations. The behavioral expectations case is similar, except it utilizes an incorrect *perceived law of motion* (PLM) for the exogenous state, namely:

$$[PLM]: \qquad\qquad z_{t+1} = N_k z_t + \epsilon_{t+1} \tag{3}$$

which is related to the ALM by the behavioral expectation operator $\mathbb{E}^k$ such that

$$N_k z_t = \mathbb{E}^k[z_{t+1} | z_t]$$

Depending on the behavioral expectations process, some extra work may be required to write the exogenous states in this way. See Appendix B.

Then, following Uhlig's proof, the solution to the model is given by:

**Theorem 1** *A behavioral solution* (2) *is given by a matrix $P$ satisfying the quadratic*

$$0 = F P^2 + G P + H$$

*and the behavioral matrix $Q^k$ satisfying*

$$Q^k = -(V^k)^{-1} vec(LN_k + M)$$

*where*

$$V^k = (N_k)' \otimes F + I_k \otimes (FP + G)$$

**Proof:** Appendix A.1

The presumption here is that agents know how choices $x_t$ depend on the exogenous state $z_t$, and forecast the endogenous vector $x_{t+1}$ by forecasting the exogenous state $z_{t+1}$ with their behavioral expectations, and then mapping that back to $x_{t+1}$ outcomes. *This is equivalent to directly forecasting $x_{t+1}$ if the behavioral expectations operator is "series-agnostic"*, but may not hold more generally.

# 3 Mapping from Uhlig to GENSYS Form

Writing the model in Uhlig form is convenient for precisely characterizing the agnets' information set and expectations. But it may be desirable to solve the model with Chris Sims' GENSYS algorithm instead (Sims, 2002). This section describes how to map the behavioral model (1) to the GENSYS form.

Sims' form for a macroeconomic model is

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi z_t + \Pi \eta_t \tag{4}$$

where $y_t$ is a vector of endogenous choice variables, while $\eta_t$ is a vector of endogenously determined forecast errors. $z_t$ is as before: an exogenous stochastic vector that is observed by the agents.

Between the two forms, the variables are mapped by

$$y_t = \begin{pmatrix} x_t \\ \mathbb{E}_t^k[x_{t+1}] \end{pmatrix}$$

following this mapping, the model in Uhlig form (1) can be written in GENSYS form by

$$\begin{pmatrix} -G & -F \\ I & 0 \end{pmatrix} \begin{pmatrix} x_t \\ \mathbb{E}_t^k[x_{t+1}] \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x_{t-1} \\ \mathbb{E}_{t-1}^k[x_t] \end{pmatrix} + \begin{pmatrix} LN_k + M \\ 0 \end{pmatrix} z_t + \begin{pmatrix} 0 \\ I \end{pmatrix} \left(x_t - \mathbb{E}_{t-1}^k[x_t]\right)$$

so that Sims' matrices are given by

$$\Gamma_0 = \begin{pmatrix} -G & -F \\ I & 0 \end{pmatrix} \qquad \Gamma_1 = \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \qquad \Psi = \begin{pmatrix} LN_k + M \\ 0 \end{pmatrix} \qquad \Pi = \begin{pmatrix} 0 \\ I \end{pmatrix}$$

and $\eta = \left(x_t - \mathbb{E}_{t-1}^k[x_t]\right)$.

# 4 Solving Behavioral Models with GENSYS

When Chris Sims' GENSYS function is used to solve models, behavioral expectations must be treated carefully. The algorithm imposes rational expectations as a part of the solution method, so unlike the Uhlig method, applying to a behavioral model is not a simple as substituting a perceived law of motion into the solution equations.

To demonstrate how to properly handle behavioral models in GENSYS, this section rederives Sims' method when expectations are not necessarily rational, then describes how to interpret the GENSYS function output.

## 4.1 Solution to Models in GENSYS Form

This section largely follows the notation in Sims (2002), albeit using lag operator notation instead of time subscripts.

A model in GENSYS form is

$$\Gamma_0 y = \Gamma_1 L y + C + \Psi z + \Pi \eta_k \tag{5}$$

There main difference here from Sims (2002) is that the forecast error $\eta_k$ now has a subscript denoting the expectation type $k$ that applies to the model. When we write the behavioral expectation as a quasi-linear operator $\mathcal{E}_k$, we will have $\mathcal{E}_k \eta_k = 0$.

Analyzing the behavioral model requires one crucial assumption that is not needed in the rational case: **no variables in $y_t$ can be known with certainty at time**

$t-1$.[1] This rules out the inclusion of lagged variables in $y_t$, so the model can depend on state variables through matrix $\Gamma_1$, but additional lags are not allowed. Without this assumption, Theorem 2 will not hold. Additionally, I impose the same regularity conditions as Sims (e.g. no undefined generalized eigenvalues, certain matrices must be invertible, etc.) as well as the additional regularity conditions for behavioral models from Adams (2023) (e.g. no generalized eigenvalues that are both stable and unstable).

**Theorem 2** *A solution to the behavioral model* (5) *is*

$$y = \Theta_1 L y + \Theta_c + \Theta_0 z + \Theta_y (I - \Theta_f \mathcal{E}_k)^{-1} \Theta_z \mathcal{E}_k z$$

*where the coefficient matrices are as defined in Sims (2002):*

$$H = Z \begin{pmatrix} \Lambda_{11} & \Lambda_{12} - \Phi\Lambda_{22} \\ 0 & I \end{pmatrix}^{-1} \qquad \Theta_1 = Z_1 \Lambda_{11}^{-1} \begin{pmatrix} \Omega_{11} & -(\Omega_{12} - \Phi\Omega_{22}) \end{pmatrix} Z$$

$$\Theta_c = H \begin{pmatrix} Q_1 - \Phi Q_2 \\ (\Omega_{22} - \Lambda_{22})^{-1} Q_2 \end{pmatrix} C \qquad \Theta_0 = H \begin{pmatrix} Q_1 - \Phi Q_2 \\ 0 \end{pmatrix} \Psi$$

$$\Theta_y = -H_2 \qquad \Theta_f = M \qquad \Theta_z = \Omega_{22}^{-1} Q_2 \Psi$$

*with $Z_1$ denoting the first block column of $Z$ and $H_2$ denoting the second block column of $H$.*

**Proof:** Appendix A.2

## 4.2 Implications for Implementation

Theorem 2 implies that the behavioral expectations affect the GENSYS solution through only one channel: the dependence of $y_t$ on expectations of future exogenous terms $z_t$. Behavioral expectations do not affect the immediate impact of a shock (the matrix $\Theta_0$) which is unforecastable whether ones' expectations are rational or not.

[1] This assumption is not needed when expectations are rational, because the rational expectation operator is always linear. However, behavioral expectation operators are only linear when applied to future variables.

The theorem also implies that an appropriate way of handling behavioral expectations is simply solve the rational expectations version of the model, then adjust the $\Theta_y(I-\Theta_f\mathcal{E}_k)^{-1}\Theta_z\mathcal{E}_k$ term to accommodate alternative expectations.

When applied to a rational expectations model, the GENSYS function outputs:

$$\text{G1} = \Theta_1 \qquad \text{C} = \Theta_c \qquad \text{impact} = \Theta_0 \qquad \text{ywt} = \theta_y \qquad \text{fmat} = \Theta_f \qquad \text{fwt} = \Theta_z$$

How can the behavioral term $\Theta_y(I-\Theta_f\mathcal{E}_k)^{-1}\Theta_z\mathcal{E}_k z$ be constructed from this output?

Suppose the exogenous series $z$ can be represented with a PLM as in equation (3). With lag operator notation, this implies

$$\mathcal{E}_k z = N_k z$$

which means the unknown behavioral term is given by

$$\Theta_y(I-\Theta_f\mathcal{E}_k)^{-1}\Theta_z\mathcal{E}_k z = \Theta_y(\sum_{j=0}^{\infty}\Theta_f^j\mathcal{E}_k^{j+1})\Theta_z z$$

$$= \Theta_y(\sum_{j=0}^{\infty}\Theta_f^j\Theta_z\mathcal{E}_k^{j+1})z = \Theta_y(\sum_{j=0}^{\infty}\Theta_f^j\Theta_z N_k^{j+1})z$$

# References

ADAMS, J. J. (2023): "Equilibrium Determinacy With Behavioral Expectations," *University of Florida mimeo.*

SIMS, C. A. (2002): "Solving Linear Rational Expectations Models," *Computational Economics*, 20(1-2), 1–20, Num Pages: 1-20 Place: Dordrecht, Netherlands Publisher: Springer Nature B.V.

UHLIG, H. (2001): "A Toolkit for Analysing Nonlinear Dynamic Stochastic Models Easily," in *Computational Methods for the Study of Dynamic Economies*, ed. by R. Marimon, and A. Scott, p. 0. Oxford University Press.

# A Proofs

## A.1 Proof of Theorem 1

**Proof.** Substitute the solution (2) into the equilibrium condition (1):

$$0 = \mathbb{E}_t^k \left[ F(Px_t + Q^k z_{t+1}) + Gx_t + Hx_{t-1} + Lz_{t+1} + Mz_t \right]$$

Evaluate the behavioral expectation and collect terms

$$0 = (FP + G)x_t + Hx_{t-1} + ((FQ^k + L)N_k + M)z_t$$

then substitute again

$$0 = (FP + G)(Px_{t-1} + Q^k z_t) + Hx_{t-1} + ((FQ^k + L)N_k + M)z_t$$

Collecting terms on $x_{t-1}$ and $z_t$ gives the usual matrix quadratic equation:

$$0 = FP^2 + GP + H$$

and given $P$, the coefficient on $z_t$ must satisfy

$$0 = (FP + G)^k + (FQ^k + L)N_k + M$$

so $Q^k$ must satisfy

$$V^k Q^k = -vec(LN_k + M)$$

■

## A.2 Proof of Theorem 2

**Proof.** Take the generalized Schur ("QZ") decomposition such that

$$Q^* \Lambda Z^* = \Gamma_0 \qquad\qquad Q^* \Omega Z^* = \Gamma_1$$

where $Q$ and $Z$ are unitary, $\Lambda$ and $\Omega$ are upper triangular, and ordered so that the generalized eigenvalues ($\phi_i \equiv \omega_{ii}/\lambda_{ii}$) are in increasing order down the diagonals.

Multiply by $Q$ and define $w \equiv Z^* y$ to write equation (5) as

$$\Lambda w = \Omega L w + QC + Q\Psi z + Q\Pi \eta_k$$

The first block row of this matrix equation (corresponding to the generalized eigenvalues with magnitude $|\phi_i| < 1$) is the stable block, while the second block row (corresponding to the generalized eigenvalues with magnitude $|\phi_i| > r(\mathcal{E}_k)$) where $r(\mathcal{E}_k)$ is the spectral radius of the behavioral expectation operator. Divided into these blocks, the model becomes

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{pmatrix} \begin{pmatrix} L w_1 \\ L w_2 \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} (C + \Psi z + \Pi \eta_k) \tag{6}$$

The second block row is explosive and must be solved forwards:

$$\Omega_{22}^{-1} \Lambda_{22} w_2 = L w_2 + \Omega_{22}^{-1} x_2$$

where $x_2 \equiv Q_2 (C + \Psi z + \Pi \eta_k)$. Take expectations of both sides:[2]

$$\Omega_{22}^{-1} \Lambda_{22} \mathcal{E}_k w_2 = w_2 + \Omega_{22}^{-1} \mathcal{E}_k x_2$$

which assumes that agents perfectly forecast lagged variables so that $\mathcal{E}_k L = I$, assumes that expectations are XXXXXXX so that matrices commute with $\mathcal{E}_k$. Inversion gives $w_2$:

$$w_2 = -(I - \Omega_{22}^{-1} \Lambda_{22} \mathcal{E}_k)^{-1} \Omega_{22}^{-1} \mathcal{E}_k x_2 \tag{7}$$

$x_2$ is endogenous because it contains the forecast error, which remains to be solved for. To do so, apply $I - L\mathcal{E}_k$ (the "forecast error" operator) to the unstable block:

$$(I - L\mathcal{E}_k) \Omega_{22}^{-1} \Lambda_{22} w_2 = (I - L\mathcal{E}_k) \Omega_{22}^{-1} x_2$$

The $L w_2$ on the right-hand side has disappeared because there is no forecast error

[2] At this point, Sims iterates forward in time, which is innocuous under rational expectations because the law of iterated expectations holds. This is not always true for behavioral expectations, so it is crucial to be careful about applying operators appropriately. Specifically, in this algebra of singly-infinite vectors and operators, $L^{-1}$ does not exist. Instead, we must take expectations at this step.

over lagged variables.[3] Substituting for $w_2$ gives

$$-(I - L\mathcal{E}_k)\Lambda_{22}(I - \Omega_{22}^{-1}\Lambda_{22}\mathcal{E}_k)^{-1}\Omega_{22}^{-1}\mathcal{E}_k x_2 = (I - L\mathcal{E}_k)x_2$$

Eliminate terms that are known with certainty or zero in expectation:

$$-(I - L\mathcal{E}_k)\Lambda_{22}(I - \Omega_{22}^{-1}\Lambda_{22}\mathcal{E}_k)^{-1}\Omega_{22}^{-1}\mathcal{E}_k Q_2\Psi z = (I - L\mathcal{E}_k)(Q_2\Psi z + Q_2\Pi\eta_k) \tag{8}$$

Rearrange and use $(I - L\mathcal{E}_k)\eta_k = \eta_k$:

$$-(I - L\mathcal{E}_k)(I - \Lambda_{22}\Omega_{22}^{-1}\mathcal{E}_k)^{-1}Q_2\Psi z = Q_2\Pi\eta_k$$

so if $Q_2\Pi$ is invertible, then $\eta_k$ is uniquely given by

$$\eta_k = -(Q_2\Pi)^{-1}(I - L\mathcal{E}_k)(I - \Lambda_{22}\Omega_{22}^{-1}\mathcal{E}_k)^{-1}Q_2\Psi z$$

If $Q_2\Pi$ is not invertible, then there may be many potential $\eta_k$ satisfying equation (8) – in which case the model features a continuum of equilibria – or there may be no solution at all. Regardless, we assume that the row space of $Q_1\Pi$ is contained in that of $Q_2\Pi$ so that there is some matrix $\Phi$ such that

$$Q_1\Pi = \Phi q_2\Pi$$

This implies that the entire system can be rewritten as

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} - \Phi\Lambda_{22} \\ 0 & I \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} - \Phi\Omega_{22} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Lw_1 \\ Lw_2 \end{pmatrix} + \begin{pmatrix} Q_1 - \Phi Q_2 \\ (\Omega_{22} - \Lambda_{22})^{-1}Q_2 \end{pmatrix} C$$
$$+ \begin{pmatrix} Q_1 - \Phi Q_2 \\ 0 \end{pmatrix}\Psi z - \begin{pmatrix} 0 \\ (I - M\mathcal{E}_k)^{-1}\Omega_{22}^{-1}Q_2\Psi\mathcal{E}_k z \end{pmatrix}$$

where the first block row follows from pre-multiplying equation (6) by $\begin{pmatrix} I & -\Phi \end{pmatrix}$, the second block row encodes equation (7), and $M \equiv \Omega_{22}^{-1}\Lambda_{22}$. Finally, pre-multiplying by $Q$ translates the system into

$$y = \Theta_1 Ly + \Theta_c + \Theta_0 z + \Theta_y(I - \Theta_f\mathcal{E}_k)^{-1}\Theta_z\mathcal{E}_k z$$

[3] $(I - L\mathcal{E}_k)L = 0$ because $\mathcal{E}_k L = I$ even though $L\mathcal{E}_k \neq 0$.

■

# B Collapsing the Exogenous States

Suppose we have a model in Uhlig form (1) satisfying the ALM $z_{t+1} = Nz_t + \epsilon_{t+1}$, but which cannot be written with a corresponding PLM, i.e. there is no $N_k$ such that in general $N_k z_t = \mathbb{E}^k[z_{t+1}|z_t]$.

Consider behavioral expectations of the following "Subrational Diagonal" form:

$$\mathbb{E}_t^k[x_{t+1}] = \sum_{j=0}^{J} \phi_j \mathbb{E}_{t-j}^k[x_{t+1}]$$

In this case, we can write

$$\mathbb{E}_t^k[z_{t+1}] = \sum_{j=0}^{J} \phi_j (N_k)^{j+1} z_{t-j}$$

Expectations depend on $z_t$ plus up to $J$ additional lags. stack these as a single vector $\mathbf{z}_t$:

$$\mathbf{z}_t \equiv \begin{pmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-J} \end{pmatrix}$$

The ALM for $\mathbf{z}_t$ is

$$\mathbf{z}_{t+1} = \mathbf{N}\mathbf{z}_t + \vec{\epsilon}_{t+1}$$

where

$$\mathbf{N} \equiv \begin{pmatrix} N & 0 & 0 & ... & 0 \\ I & 0 & 0 & ... & 0 \\ 0 & I & 0 & ... & 0 \\ 0 & 0 & I & ... & 0 \\ \vdots & \ddots & \ddots & ... & \vdots \end{pmatrix} \qquad \vec{\epsilon}_t \equiv \begin{pmatrix} \epsilon_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with PLM $\mathbb{E}_t^k[\mathbf{z}_{t+1}] = N_k \mathbf{z}_t$ where

$$\mathbf{N}^k \equiv \begin{pmatrix} \phi_0 N & \phi_1 N^2 & \phi_2 N^3 & ... & \phi_J N^{J+1} \\ I & 0 & 0 & ... & 0 \\ 0 & I & 0 & ... & 0 \\ 0 & 0 & I & ... & 0 \\ \vdots & \ddots & \ddots & ... & \vdots \end{pmatrix}$$

Then, the model can be rewritten in Uhlig form albeit with $\mathbf{z}_t$ instead of $z_t$:

$$0 = \mathbb{E}_t^k \left[ F x_{t+1} + G x_t + H x_{t-1} + \mathbf{L} \mathbf{z}_{t+1} + \mathbf{M} \mathbf{z}_t \right]$$

where the matrices $\mathbf{L}$ and $\mathbf{M}$ are defined as

$$\mathbf{L} \equiv \begin{pmatrix} L & 0 & 0 & ... & 0 \end{pmatrix} \qquad \mathbf{M} \equiv \begin{pmatrix} M & 0 & 0 & ... & 0 \end{pmatrix}$$