Behavioral Expectations Equilibrium Toolkit: Conceptual Documentation

Jonathan J Adams*

June 5, 2024

VERY EARLY DRAFT!

Current BEET version: 0.21

github link

*University of Florida. Website: www.jonathanjadams.com Email: adamsjonathan@ufl.edu.
This is an early draft, so please email me with comments and questions!

1 Introduction

This note documents how the BEET toolkit solves dynamic models with behavioral expectations.

Section 2 describes the how to write a behavioral model in a form that can be solved with the Uhlig (2001) approach. Then Section 3 describes how to incorporate stochastic belief distortions to behavioral models in a way that can be solved with the Uhlig toolkit.

Section 4 describes the how to write a behavioral model in a form that can be solved with the Sims (2002) approach. Then Section 5 describes how to derive sunspot equilibria when models in this form are not uniquely determined.

2 Solving Behavioral Models with the Uhlig Toolkit

I say a behavioral macroeconomic model is in "Uhlig" form if

$$0 = \mathbb{E}_t^k \left[F x_{t+1} \right] + G x_t + H x_{t-1} + L z_{t+1} + M z_t \tag{1}$$

where x_t is a $m \times 1$ vector of endogenous state variables chosen at time t, and z_t is a vector of exogenous state variables given by the *actual law of motion* (ALM)

$$[ALM] : \qquad z_{t+1} = N z_t + \epsilon_{t+1}$$

with ϵ_{t+1} an iid stochastic shock satsifying $\mathbb{E}_t^k[\epsilon_{t+1}] = 0$.

We would like to find the matrices P and Q^k to solve the model (1) resursively by

$$x_t = P x_{t-1} + Q^k z_t \tag{2}$$

Uhlig (2001) Theorem 1 solves this model in the case of rational expectations. The behavioral expectations case is similar, except it utilizes an incorrect *perceived law of motion* (PLM) for the exogenous state, namely:

$$[PLM] : \qquad z_{t+1} = N_k z_t + \epsilon_{t+1} \tag{3}$$

which is related to the ALM by the behavioral expectation operator $\mathbb{E}^k$ such that

$$N_k^h z_t = \mathbb{E}^k[z_{t+h}|z_t] \qquad \forall h \geq 0$$

Depending on the form of behavioral expectations, some extra work may be required to write the exogenous states in this way. See Appendix B.

Then, following Uhlig's proof, the solution to the model is given by:

Theorem 1 *A behavioral solution* (2) *is given by a matrix P satisfying the quadratic*

$$0 = FP^2 + GP + H$$

and the behavioral matrix Q^k satisfying

$$Q^k = -(V^k)^{-1} vec(LN_k + M)$$

where

$$V^k = (N_k)' \otimes F + I_k \otimes (FP + G)$$

Proof: Appendix A.1

The presumption here is that agents know how choices x_t depend on the exogenous state z_t, and forecast the endogenous vector x_{t+1} by forecasting the exogenous state z_{t+1} with their behavioral expectations, and then mapping that back to x_{t+1} outcomes. *This is equivalent to directly forecasting x_{t+1} if the behavioral expectations operator is "series-agnostic"*, but may not hold more generally.

3 Stochastic Behavioral Expectations

This section describes how BEET solves models with exogenous stochastic belief distortions, using the Uhlig toolkit. These distortions can take two forms, which must be handled differently: distortions about endogenous variables and distortions about exogenous variables. Adams and Barrett (2022) apply this approach to solve a New Keynesian model with shocks to both types of belief distortions.

3.1 Belief Distortions Regarding Endogenous Variables

In some models, agents have stochastic distortions that only affect forecasts of particular endogenous variables. For example, in Ascari, Fasani, Grazzini, and Rossi (2023), agents have rational expectations for all variables except for inflation, and their inflation forecasts are a stochastic perturbation of the rational expectation.

These types of distortions are easy to address. To do so, introduce a vector of endogenous forecasts f_t, which are given by

$$f_t = \mathbb{E}_t^k[x_{t+1}] + Zz_t \tag{4}$$

where Z is some matrix mapping exogenous states to belief distortions.

Next, consider a model without stochastic expectations expressed in Uhlig form (1). To introduce the stochastic component, replace expectations $\mathbb{E}_t^k[x_{t+1}]$ with the forecast vector f_t in the usual equilibrium conditions. The model becomes

$$0 = \mathbb{E}_t^k\left[\begin{pmatrix} 0 & 0 \\ -I & 0 \end{pmatrix}\begin{pmatrix} x_{t+1} \\ f_{t+1} \end{pmatrix}\right] + \begin{pmatrix} G & F \\ 0 & I \end{pmatrix}\begin{pmatrix} x_t \\ f_t \end{pmatrix} + \begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_{t-1} \\ f_{t-1} \end{pmatrix} + \begin{pmatrix} L \\ 0 \end{pmatrix} z_{t+1} + \begin{pmatrix} M \\ Z \end{pmatrix} z_t$$

which is a model in Uhlig form that can be solved with BEET. The first block row encodes the original equilibrium conditions (1). The second block row encodes teh forecast definitions (4).

3.2 Belief Distortions Regarding Exogenous Variables

In some models, agents have stochastic distortions that only affect forecasts of exogenous state variables. For example, in Adams and Barrett (2022), agents have rational expectations regarding some state variables, but their productivity forecasts are a stochastic perturbation of the rational expectation.

For any behavioral expectation $\mathbb{E}_t^k$, I define a subjective expectation $\tilde{\mathbb{E}}_t^k$, given by:

$$\tilde{\mathbb{E}}_t^k[z_{t+1}] = \mathbb{E}_t^k[z_{t+1}] + \zeta_t$$

where ζ_t is the exogenous belief distortion satisfying

$$\zeta_{t+1} = N^{\zeta}\zeta_t + \epsilon^{\zeta}_{t+1}$$

Introducing the subjective expectation implies a new perceived law of motion modifying equation (3):

$$[PLM]: \qquad z_{t+1} = N_k z_t + \epsilon_{t+1} + \zeta_t \tag{5}$$

and the distortion ζ_t itself has a perceived law of motion:

$$[PLM]: \qquad \zeta_{t+1} = N^{\zeta}_k \zeta_t + \epsilon^{\zeta}_{t+1}$$

with the matrix N^{ζ}_k constructed such that

$$\mathbb{E}^k_t[\zeta_{t+1}] = N^{\zeta}_k \zeta_t$$

With this from, the states z_t and ζ_t can be stacked into a single state vector with the

$$[PLM]: \qquad \begin{pmatrix} z_{t+1} \\ \zeta_{t+1} \end{pmatrix} = \begin{pmatrix} N_k & I \\ 0 & N^{\zeta}_k \end{pmatrix} \begin{pmatrix} z_t \\ \zeta_t \end{pmatrix} + \begin{pmatrix} \epsilon_{t+1} \\ \epsilon^{\zeta}_{t+1} \end{pmatrix}$$

and ALM

$$[ALM]: \qquad \begin{pmatrix} z_{t+1} \\ \zeta_{t+1} \end{pmatrix} = \begin{pmatrix} N & 0 \\ 0 & N^{\zeta} \end{pmatrix} \begin{pmatrix} z_t \\ \zeta_t \end{pmatrix} + \begin{pmatrix} \epsilon_{t+1} \\ \epsilon^{\zeta}_{t+1} \end{pmatrix}$$

Finally, the model is rewritten in Uhlig form in terms of this new state vector:

$$0 = \mathbb{E}^k_t\left[Fx_{t+1}\right] + Gx_t + Hx_{t-1} + \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_{t+1} \\ \zeta_{t+1} \end{pmatrix} + \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_t \\ \zeta_t \end{pmatrix}$$

which can be solved with BEET.

4 Solving Behavioral Models with GENSYS

When Chris Sims' GENSYS function is used to solve models, behavioral expectations must be treated carefully. The algorithm imposes rational expectations as a part of the solution method, so unlike the Uhlig method, applying to a behavioral model is not a simple as substituting a perceived law of motion into the solution equations.

To demonstrate how to properly handle behavioral models in GENSYS, this section rederives Sims' method when expectations are not necessarily rational, then describes how to interpret the GENSYS function output. But first, it demonstrates how to map between Uhlig and GENSYS forms of writing the same model.

4.1 Mapping from Uhlig to GENSYS Form

Writing the model in Uhlig form is convenient for precisely characterizing the agents' information set and expectations. But it may be desirable to solve the model with Chris Sims' GENSYS algorithm instead (Sims, 2002). This section describes how to map the behavioral model (1) to the GENSYS form.

Sims' form for a macroeconomic model is

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi z_t + \Pi \eta_t \tag{6}$$

where y_t is a vector of endogenous choice variables, while η_t is a vector of endogenously determined forecast errors. z_t is as before: an exogenous stochastic vector that is observed by the agents.

Between the two forms, the variables are mapped by

$$y_t = \begin{pmatrix} x_t \\ \mathbb{E}_t^k[x_{t+1}] \end{pmatrix}$$

following this mapping, the model in Uhlig form (1) can be written in GENSYS form by

$$\begin{pmatrix} -G & -F \\ I & 0 \end{pmatrix} \begin{pmatrix} x_t \\ \mathbb{E}_t^k[x_{t+1}] \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x_{t-1} \\ \mathbb{E}_{t-1}^k[x_t] \end{pmatrix} + \begin{pmatrix} LN_k + M \\ 0 \end{pmatrix} z_t + \begin{pmatrix} 0 \\ I \end{pmatrix} \left(x_t - \mathbb{E}_{t-1}^k[x_t]\right)$$

so that Sims' matrices are given by

$$\Gamma_0 = \begin{pmatrix} -G & -F \\ I & 0 \end{pmatrix} \qquad \Gamma_1 = \begin{pmatrix} H & 0 \\ 0 & I \end{pmatrix} \qquad \Psi = \begin{pmatrix} LN_k + M \\ 0 \end{pmatrix} \qquad \Pi = \begin{pmatrix} 0 \\ I \end{pmatrix}$$

and $\eta = \left(x_t - \mathbb{E}_{t-1}^k[x_t]\right)$.

4.2 Solution to Models in GENSYS Form

This section largely follows the notation in Sims (2002), albeit using lag operator notation instead of time subscripts.

A model in GENSYS form is

$$\Gamma_0 y = \Gamma_1 L y + C + \Psi z + \Pi \eta_k \tag{7}$$

There main difference here from Sims (2002) is that the forecast error η_k now has a subscript denoting the expectation type k that applies to the model. When we write the behavioral expectation as a quasi-linear operator $\mathcal{E}_k$, we will have $\mathcal{E}_k \eta_k = 0$.

Analyzing the behavioral model requires one crucial assumption that is not needed in the rational case: **no variables in y_t can be known with certainty at time $t-1$.**[1] This rules out the inclusion of lagged variables in y_t, so the model can depend on state variables through matrix Γ_1, but additional lags are not allowed. Without this assumption, Theorem 2 will not hold. Additionally, I impose the same regularity conditions as Sims (e.g. no undefined generalized eigenvalues, certain matrices must be invertible, etc.) as well as the additional regularity conditions for behavioral models from Adams (2023) (e.g. no generalized eigenvalues that are both stable and unstable).

Theorem 2 *A solution to the behavioral model* (7) *is*

$$y = \Theta_1 L y + \Theta_c + \Theta_0 z + \Theta_y (I - \Theta_f \mathcal{E}_k)^{-1} \Theta_z \mathcal{E}_k z$$

where the coefficient matrices are as defined in Sims (2002):

$$H = Z \begin{pmatrix} \Lambda_{11} & \Lambda_{12} - \Phi\Lambda_{22} \\ 0 & I \end{pmatrix}^{-1} \qquad \Theta_1 = Z_1 \Lambda_{11}^{-1} \begin{pmatrix} \Omega_{11} & -(\Omega_{12} - \Phi\Omega_{22}) \end{pmatrix} Z$$

$$\Theta_c = H \begin{pmatrix} Q_1 - \Phi Q_2 \\ (\Omega_{22} - \Lambda_{22})^{-1} Q_2 \end{pmatrix} C \qquad \Theta_0 = H \begin{pmatrix} Q_1 - \Phi Q_2 \\ 0 \end{pmatrix} \Psi$$

$$\Theta_y = -H_2 \qquad \Theta_f = M \qquad \Theta_z = \Omega_{22}^{-1} Q_2 \Psi$$

[1] This assumption is not needed when expectations are rational, because the rational expectation operator is always linear. However, behavioral expectation operators are only linear when applied to future variables.

with Z_1 denoting the first block column of Z and H_2 denoting the second block column of H.

Proof: Appendix A.2

4.3 Implications for Implementation

Theorem 2 implies that the behavioral expectations affect the GENSYS solution through only one channel: the dependence of y_t on expectations of future exogenous terms z_t. Behavioral expectations do not affect the immediate impact of a shock (the matrix Θ_0) which is unforecastable whether ones' expectations are rational or not.

The theorem also implies that an appropriate way of handling behavioral expectations is simply solve the rational expectations version of the model, then adjust the $\Theta_y(I-\Theta_f\mathcal{E}_k)^{-1}\Theta_z\mathcal{E}_k$ term to accommodate alternative expectations.

When applied to a rational expectations model, the GENSYS function outputs:

$$\text{G1} = \Theta_1 \qquad \text{C} = \Theta_c \qquad \text{impact} = \Theta_0 \qquad \text{ywt} = \theta_y \qquad \text{fmat} = \Theta_f \qquad \text{fwt} = \Theta_z$$

How can the behavioral term $\Theta_y(I-\Theta_f\mathcal{E}_k)^{-1}\Theta_z\mathcal{E}_k z$ be constructed from this output?

Suppose the exogenous series z can be represented with a PLM as in equation (3). With lag operator notation, this implies

$$\mathcal{E}_k z = N_k z$$

which means the unknown behavioral term is given by

$$\Theta_y(I-\Theta_f\mathcal{E}_k)^{-1}\Theta_z\mathcal{E}_k z = \Theta_y(\sum_{j=0}^{\infty}\Theta_f^j\mathcal{E}_k^{j+1})\Theta_z z$$

$$= \Theta_y(\sum_{j=0}^{\infty}\Theta_f^j\Theta_z\mathcal{E}_k^{j+1})z = \Theta_y(\sum_{j=0}^{\infty}\Theta_f^j\Theta_z N_k^{j+1})z$$

4.4 Example: Correct Application of GENSYS to a Behavioral Asset Pricing Model

Consider the following behavioral asset pricing model:

$$p_t = z_t + \mathbb{E}_t^k[p_{t+1}]$$

where z_t is an AR(1) dividend given by

$$z_t = \rho z_{t-1} + \epsilon_t$$

In Uhlig form, this model can be written as:

$$0 = \mathbb{E}_t^k[\underbrace{-\beta}_{F} p_{t+1} + \underbrace{1}_{G} p_t + \underbrace{-1}_{M} z_t]$$

In GENSYS form, this model can be written

$$\underbrace{\begin{pmatrix} 1 & -\beta \\ 1 & 0 \end{pmatrix}}_{\Gamma_0} \begin{pmatrix} p_t \\ \mathbb{E}_t^k[p_{t+1}] \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\Gamma_1} \begin{pmatrix} p_{t-1} \\ \mathbb{E}_{t-1}^k[p_t] \end{pmatrix} + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\Psi} z_t + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\Pi} \eta_{k,t}$$

to which Theorem 2 describes the solution.

However, there is another way of applying GENSYS, which may appear reasonable at first glance, but can lead to incorrect conclusions. The practitioner might decide to rewrite the behavioral model as a raitonal expectations model, by hardcoding the behavioral expectations as a function of raitonal expectations in the model's equilibrium conditions. Written as a rational expectations model, can GENSYS be applied without modification?

No. As the following examples demonstrate, this method is not reliable. Next, we consider two examples of behavioral expectations: misextrapolation and diagnostic expectations.

4.4.1 Behavioral Asset Pricing with Misextrapolation

Suppose that agents have *misextrapolation* expectations, i.e. for some $\theta^{ME} \in (0, \beta^{-1})$ they forecast by

$$\mathbb{E}_t^{ME}[p_{t+1}] = \theta^{ME}\mathbb{E}_t[p_{t+1}]$$

Then the PLM is

$$\text{[ME PLM:]} \qquad z_t = \theta^{ME}\rho z_{t-1} + \epsilon_t$$

and per Theorem 2, the solution can be constructed from GENSYS output by

$$p_t = \theta_1^{ME}p_{t-1} + \theta_0^{ME}z_t + \theta_y^{ME}(I - \theta_f^{ME}\theta^{ME}\rho)^{-1}\theta_z^{ME}\theta^{ME}\rho z_t$$

$$= \text{G1}p_{t-1} + \text{impact}z_t + \text{ymt}(I - \text{fmat}\theta^{ME}\rho)^{-1}\text{fwt}\theta^{ME}\rho z_t]$$

This model is simple enough that it is easy to check the numerical solution against the analytic solution which is given by

$$p_t = z_t + \beta\mathbb{E}_t^{ME}[z_{t+1}] + \beta^2\mathbb{E}_t^{ME}[\mathbb{E}_{t+1}^{ME}[z_{t+2}]] + ...$$

$$= z_t + \beta\theta^{ME}\mathbb{E}_t[z_{t+1}] + \beta^2(\theta^{ME})^2\mathbb{E}_t[\mathbb{E}_{t+1}[z_{t+2}]] + ...$$

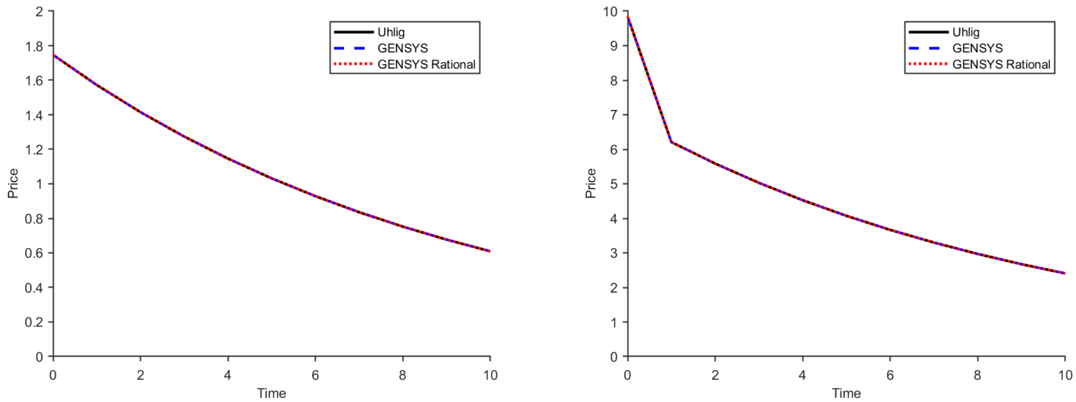
$$= \frac{z_t}{1 - \beta\theta^{ME}\rho}$$

Figure 1a gives the impulse response to a dividend shock as calculated by both the Uhlig (solid black line) and GENSYS (dashed blue line) methods. They both match the analytical solution. The figure also plots results from GENSYS is rewritten as a rational expectations model (dotted red line). In this case, all solutions are consistent with one another.

In general, it should be possible to rewrite the behavioral model as a rational expectations model, if care is taken to do so correctly. The rational expectations model encodes the behavioral expectation directly into the equilibrium condition:

$$\underbrace{\begin{pmatrix} 1 & -\beta\theta^{ME} \\ 1 & 0 \end{pmatrix}}_{\Gamma_0}\begin{pmatrix} p_t \\ \mathbb{E}_t[p_{t+1}] \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\Gamma_1}\begin{pmatrix} p_{t-1} \\ \mathbb{E}_{t-1}[p_t] \end{pmatrix} + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\Psi} z_t + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\Pi} \eta_t$$

where η_t denotes the rational forecast error. As a rational expectations model, this can now be solved with GENSYS as usual.



(a) Misextrapolation (b) Diagnostic Expectations

Figure 1: IRFs to a Dividend Shock in the Behavioral Asset Pricing Model

Notes: In all cases, $\beta = 0.95$, $\rho = 0.90$. The behavioral parameters are $\theta^{ME} = \theta^{DE} = 0.5$. The solutions are labeled as: "Uhlig" was solved using the Uhlig toolkit, "GENSYS" was solved using GENSYS when the model is written in the form of equation (7), and "GENSYS Rational" was solved by rewriting the behavioral models as rational expectations models.

4.4.2 Behavioral Asset Pricing with Diagnostic Expectations

Suppose that agents have *diagnostic* expectations, i.e. for some θ^{DE} they forecast by

$$\mathbb{E}_t^{ME}[p_{t+1}] = (1+\theta^{DE})\mathbb{E}_t[p_{t+1}] - \theta^{DE}\mathbb{E}_{t-1}[p_{t+1}]$$

Then the PLM is

$$\text{[DE PLM:]} \qquad z_t = (1+\theta^{DE})\rho z_{t-1} - \theta^{DE}\rho^2 z_{t-2} + \epsilon_t$$

and per Theorem 2, the solution can be constructed from GENSYS output by

$$p_t = \Theta_1 p_{t-1} + \Theta_0 z_t + \Theta_y (I - \Theta_f \rho)^{-1}\Theta_z \rho z_t + \Theta_y \Theta_z \theta^{DE}(\rho z_t - \rho^2 z_{t-1})$$

$$= \text{G1} p_{t-1} + \text{impact} z_t + \text{ymt}(I - \text{fmat}\rho)^{-1}\text{fwt}\rho z_t + \text{ymtfwt}\theta^{DE}(\rho z_t - \rho^2 z_{t-1})]$$

As before, an incorrect application of GENSYS is to rewrite the behavioral model as a rational expectations model:

$$\underbrace{\begin{pmatrix} 1 & -\beta(1+\theta^{DE}) & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\Gamma_0} \begin{pmatrix} p_t \\ \mathbb{E}_t[p_{t+1}] \\ \mathbb{E}_t[p_{t+2}] \end{pmatrix}$$
$$= \underbrace{\begin{pmatrix} 0 & 0 & -\beta\theta^{DE} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\Gamma_1} \begin{pmatrix} p_{t-1} \\ \mathbb{E}_{t-1}[p_t] \\ \mathbb{E}_{t-1}[p_{t+1}] \end{pmatrix} + \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\Psi} z_t + \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\Pi} \eta_t$$

where the first dimension of η_t is the rational forecast error of p_t, and the second dimension is the rational forecast error of $\mathbb{E}_t[p_{t+1}]$.

Figure 1b gives the impulse response to a dividend shock as calculated by both the Uhlig (solid black line) and GENSYS (dashed blue line) methods. They both match the solution where the model is rewritten and solved as a rational expectations model (dotted red line).

5 Sunspot Equilibria with GENSYS

With rational expectations, Sims (2002) shows that the necessary and sufficient condition for an equilibrium to be unique is that the columns of $Q_1\Pi$ are contained in the row space of $Q_2\Pi$, i.e. there exists some matrix Φ such that

$$Q_1\Pi = \Phi Q_2\Pi \tag{8}$$

If this condition fails, then the component of the forecast errors determined by the explosive block $Q_2\Pi\eta_t$ are insufficient to pin down the remaining component of the forecast errors $Q_1\Pi\eta_t$. To be concrete, let $\Phi \equiv Q_1\Pi(Q_2\Pi)^+$ where $(Q_2\Pi)^+$ is the pseudo inverse of $Q_2\Pi$.

This principle holds without rational expectations, albeit with an additional caveat: it is not enough that the explosive equations do not pin down all forecast errors. It must also be the case that it is possible to construct new forecast errors that are compatible with sunspot equilibria. Adams (2023) demonstrates that this is not generally possible: some behavioral expectations do not admit sunspot equilibria even when the Behavioral Blanchard Kahn condition fails.

Theorem 3 formalizes these conditions. The regularity conditions are still assumed to hold, so that a solution must exist. Some additional notation is required: diagonalize $\Lambda_{11}^{-1}\Omega_{11} = Q_S D_S Q_S^{-1}$ so that D_S is a diagonal matrix containing the stable eigenvalues.

Theorem 3 *If $Q_1\Pi(I-(Q_2\Pi)^{+}Q_2\Pi)$ is non-zero, and at least one stable eigenvalue is less than the spectral radius of the behavioral expectation operator $r(\mathcal{E}_k)$, then there are multiple solutions and given any solution y_t, $y_t + \hat{y}_t$ is also a solution for*

$$\hat{y}_t = Z \begin{pmatrix} Q_S e_{\odot} \\ 0 \end{pmatrix} \tilde{w}_t^{\odot}$$

where $\tilde{w}_t^{\odot}$ is any eigenseries of the behavioral expectation operator $\mathcal{E}_k$ associated with a stable eigenvalue identified by basis vector $e_{\odot}$.

Proof: Appendix A.3

Theorem 3 shows that behavioral expectations complicate a model's solutions when it features multiplicity. When expectations are rational, forecast errors are simply white noise, and sunspot equilibria can be derived directly from the matrix outputs of the GENSYS program; this is the Lubik and Schorfheide (2004) method. However, when expectations are non-rational, sunspot equilibria can still be constructed using the GENSYS matrices (equation 16 demonstrates this explicitly) but the forecast errors are no longer white noise.

Instead, finding sunspot equilibria requires finding appropriate eigenseries of the behavioral expectation operator. Because these eigenseries have more complicated time series properties than for rational expectations (where eigenseries are always AR(1)), behavioral expectations can make for richer dynamics in sunspot equilibria.

6 Calculating Behavioral Eigenseries

In general, finding the eigenseries y of a behavioral expectations operator $\mathcal{E}_k$ can be a challenging problem. But for some types of behavioral expectations, the problem is tractable. The BEET toolkit can do so in these cases.

6.1 Subrational Diagonal Expectations

"Subrational" expectations are forward looking, but perhaps do not weigh different forecasts correctly. The "diagonal" class is a further refinement: these expectations are a linear combination of rational forecasts. Specifically, an expectation operator $\mathbb{E}^k$ is subrational diagonal if there is a square-summable series ϕ_j such that

$$\mathbb{E}_t^k X_{t+1} = \sum_{j=0}^{\infty} \phi_j \mathbb{E}_{t-j}[X_{t+1}]$$

where $\mathbb{E}_t$ is the rational expectation.

This definition may seem limited, but it includes many popular forms, including:

- Rational expectations: $\phi_j = 1$ for all j
- Cognitive discounting: $\phi_j = \theta < 1$ for all j
- Diagnostic expectations: $\phi_j = 1$ for all $j \geq J$, and $\phi_j = 1 + \theta$ for $j < J$
- Sticky information: $\phi_j = (1 - \theta)\theta^j$ for all j

BEET encodes subrational diagonal expectations as a vector "BE_phivec":

$$\vec{\phi} \equiv \begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_J \end{pmatrix}$$

with the assumption that $\phi_j = 0$ for all $j \geq J$. J is referred to as the "order" of the subrational diagonal expectation, and BEET can solve for any eigenseries with finite order. As a result, it can only solve the eigenseries for infinite-order expectations such as sticky information up to a high order approximation (fortunately, the sticky information eigenseries is known analytically).

Consider an eigenseries Y_t in moving average form: $Y_t = \sum_{j=0}^{\infty} y_j \nu_{t-j}$ where ν_t are white shocks. With this form, the eigenseries equation $\lambda Y_t = \mathbb{E}_t^k[Y_{t+1}]$ becomes

$$\lambda \sum_{j=0}^{\infty} y_j \nu_{t-j} = \phi_0 y_1 \nu_t + (\phi_0 + \phi_1) y_2 \nu_{t-1} + (\phi_0 + \phi_1 + \phi_2) y_3 \nu_{t-2} + ...$$

which implies the following restrictions:

$$\lambda y_j = \sum_{k=0}^{j} \phi_k y_{j+1}$$

This problem simplifies because these restrictions are homogeneous for $j \geq J$:

$$\bar{\rho} y_j = y_{j+1} \qquad \forall j \geq J \tag{9}$$

where $\bar{\rho} = \frac{\lambda}{\sum_{k=0}^{J} \phi_k}$.

Equation (9) implies that an eigenseries $Y_t = \sum_{j=0}^{\infty} y_j L^j \nu_t$ must be ARMA(1,J) of the form

$$Y_t = \frac{\sum_{j=0}^{J} a_j L^j}{1 - \bar{\rho} L} \nu_t \tag{10}$$

assuming $\bar{\rho} < 1$, otherwise λ is not an eigenvalue. The initial coefficient can be normalized to $a_0 = 1$, so finding the eigenseries comes down to solving J equations for the remaining J coefficients.

The $\ell + 1$ period-ahead rational expectations of an ARMA(1,J) satisfying form (10) are given by

$$\mathbb{E}_t[Y_{t+\ell+1}] = \frac{\sum_{j=0}^{\ell} a_j \bar{\rho}^{\ell+1-j} + \sum_{j=\ell+1}^{J} a_j L^{j-(\ell+1)}}{1 - \bar{\rho} L} \nu_t$$

for $\ell + 1 \leq J$. Using lags of these forecasts to construct the behavioral expectation, the eigenseries equation becomes

$$\lambda \frac{\sum_{j=0}^{J} a_j L^j}{1 - \bar{\rho} L} \nu_t = \sum_{\ell=0}^{J} \phi_\ell L^\ell \left(\frac{\sum_{j=0}^{\ell} a_j \bar{\rho}^{\ell+1-j} + \sum_{j=\ell+1}^{J} a_j L^{j-(\ell+1)}}{1 - \bar{\rho} L} \right) \nu_t$$

Considering only the lag operator polynomials, the denominator cancels from both sides, leaving the numerators related by:

$$\lambda \sum_{j=0}^{J} a_j L^j = \sum_{\ell=0}^{J} \phi_\ell L^\ell \left(\sum_{j=0}^{\ell} a_j \bar{\rho}^{\ell+1-j} + \sum_{j=\ell+1}^{J} a_j L^{j-(\ell+1)} \right)$$

Collecting coefficients on lag operators gives the restrictions:

$$\lambda a_0 = \phi_0 \bar{\rho} a_0 + \phi_0 a_1$$

$$\lambda a_1 = \phi_1(\bar{\rho}^2 a_0 + \bar{\rho} a_1) + (\phi_1 + \phi_0) a_2$$

$$\lambda a_2 = \phi_2(\bar{\rho}^3 a_0 + \bar{\rho}^2 a_1 + \bar{\rho} a_2) + (\phi_2 + \phi_1 + \phi_0) a_3$$

$$\vdots$$

or in matrix form:

$$\lambda \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_J \end{pmatrix} = \begin{pmatrix} \phi_0 \overline{\rho} & \phi_0 & 0 & 0 & \cdots & 0 \\ \phi_1 \overline{\rho}^2 & \phi_1 \overline{\rho} & \phi_1 + \phi_0 & 0 & \ddots & 0 \\ \phi_2 \overline{\rho}^3 & \phi_2 \overline{\rho}^2 & \phi_2 \overline{\rho} & \phi_2 + \phi_1 + \phi_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \phi_J \overline{\rho}^{J+1} & \phi_J \overline{\rho}^J & \phi_J \overline{\rho}^{J-1} & \phi_J \overline{\rho}^{J-2} & \cdots & \phi_J \overline{\rho} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_J \end{pmatrix}$$

which we can write more concisely as

$$\lambda \vec{a} = B_{\bar{\rho}} \vec{a}$$

Thus $\vec{a}$ is the eigenvector of matrix $B_{\bar{\rho}}$ associated with λ.

6.2 BEET Calculation: Eigenseries in VAR(1) Form

The BEET toolkit reports an eigenseries $\tilde{w}_t^{\odot}$ in VAR(1) form. In lag operator notation, $\tilde{w}_t^{\odot}$ is given by

$$\tilde{w}_t^{\odot} = \frac{\sum_{j=0}^{J} a_j L^j}{1 - \bar{\rho} L} \nu_t$$

where ν_t is any white noise process. BEET calculates this eigenseries for subrational expectations, and reports it in VAR(1) form:

$$\vec{\mathbf{w}}_t^{\odot} \equiv \begin{pmatrix} \tilde{w}_t^{\odot} \\ \nu_t \\ \nu_{t-1} \\ \vdots \\ \nu_{t-J} \end{pmatrix}$$

$$= \begin{pmatrix} \overline{\rho} & a_0 & a_1 & a_2 & \cdots & a_J \\ 0 & 0 & 1 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 \end{pmatrix} \vec{\mathbf{w}}_{t-1}^{\odot} \equiv \mathbf{B_w} \vec{\mathbf{w}}_{t-1}^{\odot}$$

Then per Theorem 3, the equilibrium difference $\hat{y}_t$ is constructed by:

$$\hat{y}_t = \Theta_{\odot} \vec{\mathbf{w}}_t^{\odot}$$

where

$$\Theta_{\odot} \equiv Z \begin{pmatrix} Q_S e_{\odot} \\ 0 \end{pmatrix} e_w$$

and the basis row vector e_w selects the first entry in $\vec{\mathbf{w}}_t^{\odot}$. Because this works for any white noise process ν_t, a rescaling $\gamma \hat{y}_t$ also gives an equilibrium for any γ.

In the toolkit, BEET reports the matrices $\Theta_{\odot}$ and $\mathbf{B_w}$ so that a continuum of equilibria may be easily constructed.

References

Adams, J. J. (2023): “Equilibrium Determinacy With Behavioral Expectations,” *University of Florida mimeo.*

Adams, J. J., and P. Barrett (2022): “Shocks to Inflation Expectations,” *IMF Working Papers*, 2022(072), ISBN: 9798400206313 Publisher: International Monetary Fund Section: IMF Working Papers.

Ascari, G., S. Fasani, J. Grazzini, and L. Rossi (2023): “Endogenous uncertainty and the macroeconomic impact of shocks to inflation expectations,” *Journal of Monetary Economics*.

Lubik, T. A., and F. Schorfheide (2004): “Testing for Indeterminacy: An Application to U.S. Monetary Policy,” *American Economic Review*, 94(1), 190–217.

Sims, C. A. (2002): “Solving Linear Rational Expectations Models,” *Computational Economics*, 20(1-2), 1–20, Num Pages: 1-20 Place: Dordrecht, Netherlands Publisher: Springer Nature B.V.

Uhlig, H. (2001): “A Toolkit for Analysing Nonlinear Dynamic Stochastic Models Easily,” in *Computational Methods for the Study of Dynamic Economies*, ed. by R. Marimon, and A. Scott, p. 0. Oxford University Press.

A Proofs

A.1 Proof of Theorem 1

Proof. Substitute the solution (2) into the equilibrium condition (1):

$$0 = \mathbb{E}_t^k \left[F(Px_t + Q^k z_{t+1}) + Gx_t + Hx_{t-1} + Lz_{t+1} + Mz_t\right]$$

Evaluate the behavioral expectation and collect terms

$$0 = (FP + G)x_t + Hx_{t-1} + ((FQ^k + L)N_k + M)z_t$$

then substitute again

$$0 = (FP + G)(Px_{t-1} + Q^k z_t) + Hx_{t-1} + ((FQ^k + L)N_k + M)z_t$$

Collecting terms on x_{t-1} and z_t gives the usual matrix quadratic equation:

$$0 = FP^2 + GP + H$$

and given P, the coefficient on z_t must satisfy

$$0 = (FP + G)^k + (FQ^k + L)N_k + M$$

so Q^k must satisfy

$$V^k Q^k = -vec(LN_k + M)$$

■

A.2 Proof of Theorem 2

Proof. Take the generalized Schur ("QZ") decomposition such that

$$Q^*\Lambda Z^* = \Gamma_0 \qquad Q^*\Omega Z^* = \Gamma_1$$

where Q and Z are unitary, Λ and Ω are upper triangular, and ordered so that the generalized eigenvalues ($\phi_i \equiv \omega_{ii}/\lambda_{ii}$) are in increasing order down the diagonals.

Multiply by Q and define $w \equiv Z^* y$ to write equation (7) as

$$\Lambda w = \Omega L w + QC + Q\Psi z + Q\Pi\eta_k$$

The first block row of this matrix equation (corresponding to the generalized eigenvalues with magnitude $|\phi_i| < 1$) is the stable block, while the second block row (corresponding to the generalized eigenvalues with magnitude $|\phi_i| > r(\mathcal{E}_k)$) where $r(\mathcal{E}_k)$ is the spectral radius of the behavioral expectation operator. Divided into these blocks, the model becomes

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{pmatrix} \begin{pmatrix} Lw_1 \\ Lw_2 \end{pmatrix} + \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} (C + \Psi z + \Pi\eta_k) \quad (11)$$

The second block row is explosive and must be solved forwards:

$$\Omega_{22}^{-1}\Lambda_{22} w_2 = L w_2 + \Omega_{22}^{-1} x_2$$

where $x_2 \equiv Q_2 (C + \Psi z + \Pi\eta_k)$. Take expectations of both sides:[2]

$$\Omega_{22}^{-1}\Lambda_{22}\mathcal{E}_k w_2 = w_2 + \Omega_{22}^{-1}\mathcal{E}_k x_2$$

which assumes that agents perfectly forecast lagged variables so that $\mathcal{E}_k L = I$, assumes that expectations are "series-agnostic" so that matrices commute with $\mathcal{E}_k$. Inversion gives w_2:

$$w_2 = -(I - \Omega_{22}^{-1}\Lambda_{22}\mathcal{E}_k)^{-1}\Omega_{22}^{-1}\mathcal{E}_k x_2 \quad (12)$$

x_2 is endogenous because it contains the forecast error, which remains to be solved for. To do so, apply $I - L\mathcal{E}_k$ (the "forecast error" operator) to the unstable block:

$$(I - L\mathcal{E}_k)\Omega_{22}^{-1}\Lambda_{22} w_2 = (I - L\mathcal{E}_k)\Omega_{22}^{-1} x_2$$

The Lw_2 on the right-hand side has disappeared because there is no forecast error

[2]At this point, Sims iterates forward in time, which is innocuous under rational expectations because the law of iterated expectations holds. This is not always true for behavioral expectations, so it is crucial to be careful about applying operators appropriately. Specifically, in this algebra of singly-infinite vectors and operators, L^{-1} does not exist. Instead, we must take expectations at this step.

over lagged variables.[3] Substituting for w_2 gives

$$-(I - L\mathcal{E}_k)\Lambda_{22}(I - \Omega_{22}^{-1}\Lambda_{22}\mathcal{E}_k)^{-1}\Omega_{22}^{-1}\mathcal{E}_k x_2 = (I - L\mathcal{E}_k)x_2$$

Eliminate terms that are known with certainty or zero in expectation:

$$-(I - L\mathcal{E}_k)\Lambda_{22}(I - \Omega_{22}^{-1}\Lambda_{22}\mathcal{E}_k)^{-1}\Omega_{22}^{-1}\mathcal{E}_k Q_2\Psi z = (I - L\mathcal{E}_k)(Q_2\Psi z + Q_2\Pi\eta_k) \quad (13)$$

Rearrange and use $(I - L\mathcal{E}_k)\eta_k = \eta_k$:

$$-(I - L\mathcal{E}_k)(I - \Lambda_{22}\Omega_{22}^{-1}\mathcal{E}_k)^{-1}Q_2\Psi z = Q_2\Pi\eta_k \quad (14)$$

so if $Q_2\Pi$ is invertible, then η_k is uniquely given by

$$\eta_k = -(Q_2\Pi)^{-1}(I - L\mathcal{E}_k)(I - \Lambda_{22}\Omega_{22}^{-1}\mathcal{E}_k)^{-1}Q_2\Psi z$$

If $Q_2\Pi$ is not invertible, then there may be many potential η_k satisfying equation (13) – in which case the model features a continuum of equilibria – or there may be no solution at all. Regardless, we assume that the columns of $Q_1\Pi$ are contained in the row space of $Q_2\Pi$ so that there is some matrix Φ such that

$$Q_1\Pi = \Phi Q_2\Pi$$

This implies that the entire system can be rewritten as

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} - \Phi\Lambda_{22} \\ 0 & I \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} - \Phi\Omega_{22} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Lw_1 \\ Lw_2 \end{pmatrix} + \begin{pmatrix} Q_1 - \Phi Q_2 \\ (\Omega_{22} - \Lambda_{22})^{-1}Q_2 \end{pmatrix} C$$
$$+ \begin{pmatrix} Q_1 - \Phi Q_2 \\ 0 \end{pmatrix}\Psi z - \begin{pmatrix} 0 \\ (I - M\mathcal{E}_k)^{-1}\Omega_{22}^{-1}Q_2\Psi\mathcal{E}_k z \end{pmatrix}$$

where the first block row follows from pre-multiplying equation (11) by $\begin{pmatrix} I & -\Phi \end{pmatrix}$, the second block row encodes equation (12), and $M \equiv \Omega_{22}^{-1}\Lambda_{22}$. Finally, pre-multiplying

[3] $(I - L\mathcal{E}_k)L = 0$ because $\mathcal{E}_k L = I$ even though $L\mathcal{E}_k \neq 0$.

by $Z\begin{pmatrix} \Lambda_{11} & \Lambda_{12}-\Phi\Lambda_{22} \\ 0 & I \end{pmatrix}^{-1}$ translates the system into

$$y = \Theta_1 Ly + \Theta_c + \Theta_0 z + \Theta_y (I - \Theta_f \mathcal{E}_k)^{-1} \Theta_z \mathcal{E}_k z$$

■

A.3 Proof of Theorem 3

Proof. As in the proof of Theorem 2, we can transform the dynamic system (11) by left-multiplying by $\begin{pmatrix} I & -\Phi \end{pmatrix}$ to create a new stable block, and encoding equation (12) in the second block:

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12}-\Phi\Lambda_{22} \\ 0 & I \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12}-\Phi\Omega_{22} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Lw_1 \\ Lw_2 \end{pmatrix} + \begin{pmatrix} Q_1-\Phi Q_2 \\ (\Omega_{22}-\Lambda_{22})^{-1}Q_2 \end{pmatrix} C$$
$$+ \begin{pmatrix} Q_1-\Phi Q_2 \\ 0 \end{pmatrix}\Psi z - \begin{pmatrix} 0 \\ (I-M\mathcal{E}_k)^{-1}\Omega_{22}^{-1}Q_2\Psi\mathcal{E}_k z \end{pmatrix}$$
$$+ \begin{pmatrix} Q_1-\Phi Q_2 \\ 0 \end{pmatrix}\Pi\eta_k$$

The only difference from the proof of Theorem 2 is the final term: $(Q_1\Pi - \Phi Q_2\Pi)\eta_k$ may be non-zero.

To simplify, consider differences between any two equilibria, denoted with hats; the system becomes:

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12}-\Phi\Lambda_{22} \\ 0 & I \end{pmatrix}\begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12}-\Phi\Omega_{22} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} L\hat{w}_1 \\ L\hat{w}_2 \end{pmatrix}$$
$$+ \begin{pmatrix} Q_1-\Phi Q_2 \\ 0 \end{pmatrix}\Pi\hat{\eta}_k \quad (15)$$

Pre-multiplying by $Z\begin{pmatrix} \Lambda_{11} & \Lambda_{12}-\Phi\Lambda_{22} \\ 0 & I \end{pmatrix}^{-1}$ translates this into

$$\hat{y} = \Theta_1 L\hat{y} + \Theta_\odot \hat{\eta}_k \quad (16)$$

where

$$\Theta_\odot \equiv H \begin{pmatrix} Q_1 - \Phi Q_2 \\ 0 \end{pmatrix} \Pi$$

Thus far, the proof has shown how forecast errors can lead to sunspot equilibria if $Q_1 \Pi(I - (Q_2\Pi)^+ Q_2 \Pi)$ is non-zero. It remains to be shown that such forecast errors may exist.

The system (15) can be reduced to the first block, because $\hat{w}_2 = 0$:

$$\hat{w}_1 = \Lambda_{11}^{-1}\Omega_{11} L \hat{w}_1 + \Lambda_{11}^{-1}(Q_1 - \Phi Q_2)\Pi \hat{\eta}_k$$

$$\left(I - \Lambda_{11}^{-1}\Omega_{11} L\right) \hat{w}_1 = \Lambda_{11}^{-1}(Q_1 - \Phi Q_2)\Pi \hat{\eta}_k$$

Substitute with the diagonalization $\Lambda_{11}^{-1}\Omega_{11} = Q_S D_S Q_S^{-1}$ and let $\tilde{Q} \equiv Q_S^{-1}\Lambda_{11}^{-1}(Q_1 - \Phi Q_2)\Pi$ so that

$$\left(I - D_S L\right) Q_S^{-1} \hat{w}_1 = \tilde{Q} \hat{\eta}_k \tag{17}$$

Let ϕ_i denote the ith eigenvalue in D_S, and let $\tilde{w}_i$ denote the ith row of $Q_S^{-1}\hat{w}_1$. There exists a non-zero $\tilde{w}_i$ and associated forecast error $\tilde{\eta}_i = (I - L\mathcal{E}_k)\tilde{w}_i$ satisfying equation (17) if

$$(I - \phi_i L)\tilde{w}_i = \tilde{\eta}_i$$

$$\implies \phi_i \tilde{w}_i = \mathcal{E}_k \tilde{w}_i$$

thus $\tilde{w}_i$ must be an eigenseries of the diagnostic expectation operator $\mathcal{E}_k$ with eigenvalue ϕ_i. The assumption that at least one stable eigenvalue is less than the spectral radius $r(\mathcal{E}_k)$ guarantees that such a $\tilde{w}_i$ exists.

Let $e_\odot$ denote the standard basis vector identifying any such "sunspot dimension" with eigenseries $\tilde{w}_\odot$ and eigenvalue $\phi_\odot$ appearing in D_S. A valid solution difference can be constructed by

$$Q_S^{-1}\hat{w}_1 = e_\odot \tilde{w}_\odot \qquad\qquad \tilde{Q}\hat{\eta}_k = e_\odot \tilde{\eta}_\odot$$

which maps to the sunspot solution (16) by

$$\hat{y} = Z \begin{pmatrix} Q_S e_\odot \\ 0 \end{pmatrix} \tilde{w}_\odot \qquad\qquad \Theta_\odot \hat{\eta}_k = Z \begin{pmatrix} Q_S e_\odot \\ 0 \end{pmatrix} \tilde{\eta}_\odot$$

Rewriting from operator notation to time series notation gives the expression stated in the Theorem. ■

B Collapsing the Exogenous States

Suppose we have a model in Uhlig form (1) satisfying the ALM $z_{t+1} = Nz_t + \epsilon_{t+1}$, but which cannot be written with a corresponding PLM, i.e. there is no N_k such that in general $N_k z_t = \mathbb{E}^k[z_{t+1}|z_t]$.

Consider behavioral expectations of the following "Subrational Diagonal" form:

$$\mathbb{E}_t^k[x_{t+1}] = \sum_{j=0}^{J} \phi_j \mathbb{E}_{t-j}^k[x_{t+1}]$$

and define

$$\psi_j \equiv \sum_{i=0}^{J} \phi_i$$

It is possible to give the exogenous state the required PLM form by rewriting it as a large vector including shock lags:

$$\mathbf{z}_t \equiv \begin{pmatrix} z_t \\ u_t \end{pmatrix} \qquad u_t \equiv \begin{pmatrix} \epsilon_t \\ N\epsilon_{t-1} \\ N^2\epsilon_{t-2} \\ \vdots \\ N^{J-1}\epsilon_{t-(J-1)} \end{pmatrix}$$

The PLM for u_t is derived first. The one-period-ahead rational expectation of the vector u_t is given by

$$\mathbb{E}_t[u_{t+1}] = \begin{pmatrix} 0 & 0 & 0 & ... & 0 \\ N & 0 & 0 & ... & 0 \\ 0 & N & 0 & ... & 0 \\ 0 & 0 & N & ... & 0 \\ \vdots & \ddots & \ddots & ... & \vdots \end{pmatrix} u_t \equiv \mathbf{N}_U u_t$$

and the horizon-h rational expectation of the vector u_t is given by

$$\mathbb{E}_{t-h}[u_t] = \mathbf{N}_U^h u_{t-h}$$
$$= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ N^h \epsilon_{t-h} \\ N^{h+1}\epsilon_{t-h-1} \\ \vdots \\ N^{J-1}\epsilon_{t-(J-1)} \end{pmatrix} = D_{N,h} u_t$$

where $D_{N,h}$ is a block diagonal matrix with h 0 blocks followed by $J-h$ N blocks.

Therefore the behavioral expectation of u_{t+1} is

$$\mathbb{E}_t^k[u_{t+1}] = \sum_{j=0}^{J} \phi_j \mathbf{N}_U^{j+1} u_{t-j} = \sum_{j=0}^{J} \phi_j \mathbf{N}_U D_{N,h} u_t$$

$$= \begin{pmatrix} 0 & 0 & 0 & ... & 0 \\ \psi_0 N & 0 & 0 & ... & 0 \\ 0 & \psi_1 N & 0 & ... & 0 \\ 0 & 0 & \psi_2 N & ... & 0 \\ \vdots & \ddots & \ddots & ... & \vdots \end{pmatrix} u_t \equiv \mathbf{E}_U u_t$$

Next, consider the lag $z_{t-J} = \epsilon_{t-J} + N\epsilon_{t-J-1} + ...$ z_t can be constructed from this lag and u_t by

$$z_t = \vec{\mathbf{1}} u_t + N^J z_{t-J} \tag{18}$$

We can write the behavioral expectation using the infinite history of shock vectors

$$\mathbb{E}_t^k[z_{t+1}] = \sum_{j=0}^{\infty} N^{j+1} \psi_j \epsilon_{t-j}$$

where $\psi_j = \psi_J$ for all $j \geq J$. This reduces to a finite state space:

$$\mathbb{E}_t^k[z_{t+1}] = N(\vec{\psi} u_t + \psi_J N^J z_{t-j})$$

where $\vec{\psi}$ is the row vector of ψ_j from 0 to $J-1$. Substitute using equation (18):

$$\mathbb{E}_t^k[z_{t+1}] = N(\vec{\psi} - \psi_J \vec{\mathbf{1}})u_t + N\psi_J z_t$$

Thus we can construct the PLM $\mathbb{E}_t^k[\mathbf{z}_{t+1}] = \mathbf{N}_k \mathbf{z}_t$ by

$$\mathbf{N}^k \mathbf{z}_t \equiv \begin{pmatrix} N\psi_J & N(\vec{\psi} - \psi_J \vec{\mathbf{1}}) \\ \vec{\mathbf{0}} & \mathbf{E}_u \end{pmatrix} \begin{pmatrix} z_t \\ u_t \end{pmatrix}$$

The ALM for $\mathbf{z}_t$ is

$$\mathbf{z}_{t+1} = \mathbf{N}\mathbf{z}_t + \vec{\epsilon}_{t+1}$$

where

$$\mathbf{N} \equiv \begin{pmatrix} N & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ 0 & I & 0 & ... & 0 \\ 0 & 0 & I & ... & 0 \\ \vdots & \ddots & \ddots & ... & \vdots \end{pmatrix} \qquad \vec{\epsilon}_t \equiv \begin{pmatrix} \epsilon_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Then, the model can be rewritten in Uhlig form albeit with $\mathbf{z}_t$ instead of z_t:

$$0 = \mathbb{E}_t^k \left[F x_{t+1} + G x_t + H x_{t-1} + \mathbf{L}\mathbf{z}_{t+1} + \mathbf{M}\mathbf{z}_t \right]$$

where the matrices $\mathbf{L}$ and $\mathbf{M}$ are defined as

$$\mathbf{L} \equiv \begin{pmatrix} L & 0 & 0 & ... & 0 \end{pmatrix} \qquad \mathbf{M} \equiv \begin{pmatrix} M & 0 & 0 & ... & 0 \end{pmatrix}$$