The Dynamic Distribution in the Fixed Cost Model: An Analytical Solution

Jonathan J. Adams*

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Abstract

I derive an analytical solution to the Kolmogorov forward equation for a fixed cost model. This is a challenging PDE, because the dynamic distribution depends on the flow of resetting agents, which is endogenously determined by the distribution itself. I show there is a shortcut that allows the flow function to be derived without first finding the entire distribution of agents. This shortcut is also valuable because many aggregate variables can be written in terms of the flow function alone. As an example, I solve the canonical menu cost model. In it, the analytical solution uncovers effects that are inconsistent with local approximation methods. Specifically, the effects of shocks are both size and state dependent. These nonlinearities are substantial; if a monetary shock is sufficiently large, it can even reverse the sign of the effect on output.

JEL-Codes: C60, E31, E32, E51

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*University of Florida. Website: www.jonathanjadams.com Email: adamsjonathanj@gmail.com.
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1 Introduction

Many economic decisions require paying fixed costs. But the macroeconomics of fixed costs is challenging to study, because it often requires keeping track of an infinite-dimensional state variable: the dynamic distribution of agents. The evolution of the distribution is governed by a partial differential equation (PDE), the *Kolmogorov Forward Equation* (KFE).[1] This equation has proven difficult to solve, because it depends on the flow of resetting agents, which itself is determined endogenously from the dynamic distribution. As a result of this nonlinear feedback, theorists rely on perturbation methods or other approximations to characterize aggregate behavior.

To address these challenges, I derive an analytical solution for the dynamic distribution in a canonical fixed cost model. The key insight is that the endogenous flow of resetting agents can be determined without first solving for the dynamic distribution of states. This shortcuts the endogenous feedback that prevented solving the model. Once the time path of the "reset flow" is known, the dynamic distribution is found using a standard PDE solution. But the reset flow is even more valuable: many macroeconomic variables that depend on the distribution can be calculated using the reset flow alone, circumventing the need to calculate the dynamic distribution at all.

The results in this paper are useful for studying the effects of *aggregate* shocks in the model, because the KFE determines how the distribution of agents responds over time, and macroeconomic variables such as output or inflation are typically functions of this distribution. The general model describes economic decisions subject to a wide variety of frictions. This type of fixed cost model – with aggregate shocks and a dynamic distribution governed by a KFE – appears in many examples, including: investment adjustment costs (Baley and Blanco, 2021), hiring and firing costs (Elsby and Michaels, 2019), information acquisition costs (Alvarez, Lippi, and Paciello, 2018), wage renegotiation costs (Blanco and Drenik, 2023), and most famously menu costs (Golosov and Lucas Jr., 2007; Midrigan, 2011). I demonstrate the analytical solution by applying it to a menu cost model, and explore the effects of monetary shocks. And while the example is a simple symmetric model, I show how extensions with asymmetry, drift, and random resets can be transformed into the simple model and solved in the same fashion.

[1] In most settings it would be more informative to use the term "Fokker-Planck equation", which is a specific type of KFE. But the Kolmogorov terminology is most common in economics, so I use it as well.

The analytical solution will be useful for addressing many theoretical and quantitative questions. But it also provides some immediate lessons. First, the solution reveals that the effects of aggregate shocks are *size-dependent*. The impulse response of a macroeconomic variable does not scale with shock size. Instead, the shock size affects all features of the impulse response function, including the shape, immediate impact, and cumulative impulse response (CIR).[2] Second, the effectsof aggregate shocks are *state-dependent*. A shock will imply different impulse responses if it follows a previous shock immediately or with long delay; this echoes a classic result from Caplin and Leahy (1997). The menu cost example in Section 5 demonstrates that the size and state dependencies are nontrival. They are not even monotonic. And the size-dependence is so strong that a large enough shock will change the sign of the effect on output.

This is an improvement over existing methods. Historically, fixed cost models employed clever assumptions such that the distribution was not a necessary aggregate state variable (Caplin and Spulber, 1987; Caplin and Leahy, 1997). In recent years, theorists have made considerable progress understanding the macroeconomics of fixed cost models by utilizing a variety of approximations.[3] Many researchers employ perturbation methods around the steady state distribution, which yields valid conclusions for small, rare shocks. This approach is convenient to characterize linear relationships between aggregate variables (e.g. Gertler and Leahy, 2008; Auclert, Rigato, Rognlie, and Straub, 2024) or to approximate their dependence on the distribution (e.g. Alvarez and Lippi, 2014; Alvarez, Lippi, and Souganidis, 2023). Without explicit linearization, Alvarez, Le Bihan, and Lippi (2016) derive a sufficient statistic for the cumulative effect of small one-off shocks in the nonlinear model. However, numerical solutions to the nonlinear model demonstrate that conclusions regarding small rare shocks will not necessarily hold for large or frequent shocks (Golosov and Lucas Jr., 2007; Cavallo, Lippi, and Miyahara, 2024). My analysis of the analytical solution agrees.

The literature is aware of the limitations of local approximations to the nonlinear

[2]Empirically, the effects of cost shocks on price-setting exhibit strong size-dependence, which Cavallo, Lippi, and Miyahara (2024) document using granular pricing data in the food and beverage industry.

[3]The approximations discussed in this section have been useful for theoretical analysis of fixed cost models. Further approximations are used for quantitative analysis, e.g. Midrigan (2011) and many other papers use the Krusell and Smith (1998) method to encode the infinite dimensional distribution.

dynamics featured in fixed cost models. To address these issues, Alvarez and Lippi (2022) make substantial progress by considering an alternative approximation: they assume there is no "reinjection", i.e. agents leave the distribution after resetting. This method is useful for calculating certain impulse response functions (IRFs) in some models. Specifically, if the aggregate variable of interest is calculated by integrating an *odd* function over the distribution, and if the model is symmetric (e.g. there is no drift in inflation or productivity) then the aggregate variable's IRF without reinjection is equivalent to the true IRF. Alvarez and Lippi go on to show that when these conditions hold, the IRF shape is invariant to shock size. In contrast, the approach in Section 3 develops the full analytical solution by finding and incorporating the equilibrium reinjection behavior (the reset flow). Among other results, the analytical solution gives the IRF for variables whose aggregating functions are not strictly odd, and allows for models with drift.

The next section describes the canonical fixed cost model. Section 3 derives the solution. Section 4 demonstrates how to express the dynamics for aggregate variables in terms of the reset flow. Section 5 computes the solution in a menu cost model, and demonstrates the size and state dependent effects of shocks. Section 6 concludes.

2 The Fixed Cost Model

This section introduces the canonical fixed cost model, which describes the distribution of agents who must pay a fixed cost to adjust some state variable. The model abstracts from the specific microfoundations that generate this behavior, but it applies to a wide variety of economic decisions including the menu cost model presented in Section 5. Solving agents' optimal decisions is both well understood and also specific to the setting at hand; I focus on the general problem of solving the dynamic distribution.[4]

There is a continuum of agents; at any time t, each agent is characterized by a state variable x. Each agent's state variable follows an independent Brownian motion. When an agent's state is sufficiently low ($x \leq a$) or sufficiently high ($x \geq b$) it is willing to pay a fixed cost and reset its state to $x = 0$. The interval $[a, b]$ with $a < 0 < b$ is assumed to be constant.

The distribution of agents' states is $h(x, t)$, a function on $x \in [a, b]$ and $t \geq 0$.

[4] See Stokey (2008) for a textbook treatment.

Solving the fixed cost model entails finding the function $h(x,t)$ that satisfies a number of conditions:

1. The distribution satisfies the KFE:

$$\partial_t h(x,t) = \gamma \partial_x^2 h(x,t) \tag{1}$$

   on the interval $[a, 0) \cup (0, b]$ where $2\gamma t$ is the variance of the Brownian motion over t units of time.

2. The *continuity boundary condition*: while $h(x,t)$ might not be differentiable at $x = 0$, it must be continuous.

3. The bounds a and b are absorbing barriers, implying the Dirchlet boundary conditions

$$h(a,t) = 0 \qquad h(b,t) = 0$$

4. The distribution is consistent with the initial condition

$$h(x,0) = \phi(x)$$

5. Probability is conserved, i.e. for all t,

$$\int_a^b h(x,t)dx = 1$$

This is a very simple fixed cost model, but many models with more interesting features including drift or random resets can be rewritten in this form with an appropriate transformation (Section 5.4).

3 Solution

The solution approach is express the model as a standard PDE problem, albeit with an additional unknown function, the *reset flow* of probability $F(t)$, which will capture the rate at which agents hit the barriers, reset, and reenter the distribution. It is necessary to jointly solve for the functions describing the probability flow and the distribution of agents.

3.1 Expression as a Standard Problem

Most of this PDE problem is standard textbook material; this KFE is simply the usual *heat equation*, except at $x = 0$. But the unusual boundary conditions prevent application of Sturm-Louiville theory to easily solve the problem. How must the function behave at $x = 0$ in order to satisfy probability conservation?

A variety of standard PDE problems have known solutions. I begin by rewriting the model into a standard form, albeit with the inclusion of an additional unknown function $F(t)$. I define the *reset flow* $F(t)$ as

$$F(t) \equiv \gamma \partial_x h(a, t) - \gamma \partial_x h(b, t)$$

This object represents the flow of probability out of the interval $[a, b]$. These agents reset their state, and reenter the distribution at 0, i.e. there is a point-like source at 0 where probability enters at rate $F(t)$.

Lemma 1 formalizes how this property is implied by the conservation assumption $\int_a^b h(x, t)dx = 1$.

Lemma 1. *The following* non-homogeneous heat equation *holds for* $x \in [a, b]$ *and* $t \geq 0$*:*

$$\partial_t h(x, t) = \gamma \partial_x^2 h(x, t) + \delta(x) F(t)$$

where $\delta(x)$ *is the Dirac delta and* $F(t)$ *is the reset flow.*

Proof. The KFE (1) holds everywhere on the interval $[a, b]$ except at 0:

$$\partial_t h(x, t) - \gamma \partial_x^2 h(x, t) = \begin{cases} 0 & x \neq 0 \\ R(t) & x = 0 \end{cases}$$

for some residual function $R(t)$. Therefore, we can extend the KFE to the entire interval by writing

$$\partial_t h(x, t) = \gamma \partial_x^2 h(x, t) + \delta(x) R(t)$$

The conservation assumption implies that the total density is unchanging:

$$0 = \partial_t \int_a^b h(x, t)dx$$

$$= \int_a^b \left(\gamma\partial_x^2 h(x,t) + \delta(x)R(t)\right) dx = \gamma\partial_x h(b,t)dx - \gamma\partial_x h(a,t)dx + R(t)$$

$$\implies F(t) = R(t)$$

$\square$

Lemma 1 allows the model to be rewritten as a standard PDE problem, conditional on $F(t)$:

Problem 1.

$$\partial_t h(x,t) = \gamma\partial_x^2 h(x,t) + \delta(x)F(t)$$

$$h(a,t) = 0 \qquad h(b,t) = 0$$

$$h(x,0) = \phi(x)$$

for $x \in [a,b]$, $t \geq 0$

3.2 Useful Functions and the Conditional Solution

Define the ∂_x^2 eigenfunction $X_n(x)$ by

$$X_n(x) \equiv \cos(\frac{\pi n}{b-a}a)\sin(\frac{\pi n}{b-a}x) - \sin(\frac{\pi n}{b-a}a)\cos(\frac{\pi n}{b-a}x) \tag{2}$$

and the ∂_t eigenfunction $T_n(x)$ by

$$T_n(x) \equiv e^{-\lambda_n t} \quad \lambda_n \equiv \gamma\left(\frac{\pi n}{b-a}\right)^2$$

Observe that $T_n(t)X_n(x)$ solves the *homogeneous* PDE $\partial_t h(x,t) = \gamma\partial_x^2 h(x,t)$ and satisfies the boundary conditions $h(a,t) = 0$ and $h(b,t) = 0$ for $n = 1, 2, 3...$

This problem's Green's Function $G(x,y,t)$ can be concisely written in terms of these eigenfunctions as

$$G(x,y,t) \equiv \sum_{n=1}^{\infty} X_n(x)X_n(y)T_n(t) \tag{3}$$

Moreover, any sufficiently regular function on the interval $[a,b]$ can be written in "Fourier space" by expressing it as an infinite sum of the $X_n(x)$ eigenfunctions. In particular, the PDE solution will be written in terms of the Fourier basis as

$$h(x,t) = \sum_{n=1}^{\infty} S_n(t) X_n(x)$$

Thus solving the model is equivalent to finding the series of $S_n(t)$ functions.

With this notation in hand, Property 1 gives the textbook solution to the KFE *given* the unknown reset flow $F(t)$:

Property 1. *Given $F(t)$, the standard solution to the PDE Problem 1 is*

$$h(x,t) = \int_a^b \phi(y) G(x,y,t) dy + \int_0^t \int_a^b \delta(y) F(s) G(x,y,t-s) dy ds \tag{4}$$

See for reference Polyanin (2001, Sec. 1.1.1) or for an equivalent expression without the Green's function, see a standard PDE textbook such as Evans (2022, Sec. 2.3.1).[5]

3.3 The Reset Flow

The previous section gives the model solution conditional on the reset flow function $F(t) = \gamma \partial_x h(a,t) - \gamma \partial_x h(b,t)$. But the reset flow is itself determined by $h(x,t)$. This section describes how, and then demonstrates how to determine the reset flow in isolation, i.e. without first knowing the solution for $h(x,t)$.

Lemma 2. *The reset flow $F(t)$ is determined from the $S_n(t)$ functions by*

$$F(t) = \sum_{n=1}^{\infty} \theta_n S_n(t)$$

where

$$\theta_n \equiv \begin{cases} 2\gamma \frac{\pi n}{b-a} & n \,\text{odd} \\ 0 & n \,\text{even} \end{cases}$$

Proof. In terms of the distribution $h(x,t)$, the reset flow is given by

$$F(t) = \gamma \partial_x h(a,t) - \gamma \partial_x h(b,t)$$

[5] Alvarez, Lippi, and Souganidis (2023) also apply a standard heat equation solution to the KFE, albeit without the endogenous reset flow.

$$= \gamma \sum_{n=1}^{\infty} S_n(t) \left(\partial_x X_n(a) - \partial_x X_n(b)\right)$$

The eigenfunction derivative is given by

$$\partial_x X_n(x) = \frac{\pi n}{b-a} \left(\cos(\frac{\pi n}{b-a} a) \cos(\frac{\pi n}{b-a} x) + \sin(\frac{\pi n}{b-a} a) \sin(\frac{\pi n}{b-a} x) \right)$$

which implies

$$\partial_x X_n(a) = \frac{\pi n}{b-a} \left(\cos^2(\frac{\pi n}{b-a} a) + \sin^2(\frac{\pi n}{b-a} a) \right) = \frac{\pi n}{b-a}$$

$$\partial_x X_n(b) = \frac{\pi n}{b-a} \left(\cos(\frac{\pi n}{b-a} a) \cos(\frac{\pi n}{b-a} b) + \sin(\frac{\pi n}{b-a} a) \sin(\frac{\pi n}{b-a} b) \right)$$

$$= \frac{\pi n}{b-a} \left(\cos(\frac{\pi n}{b-a} a) \cos(\frac{\pi n}{b-a} a + \pi n) + \sin(\frac{\pi n}{b-a} a) \sin(\frac{\pi n}{b-a} a + \pi n) \right)$$

$$= \begin{cases} -\frac{\pi n}{b-a} & n \,\text{odd} \\ \frac{\pi n}{b-a} & n \,\text{even} \end{cases}$$

Therefore:

$$F(t) = 2\gamma \sum_{n \,\text{odd}} \frac{\pi n}{b-a} S_n(t)$$

□

Lemma 2 demonstrates precisely how the flow $F(t)$ depends on $S_n(t)$, the eigenfunction coefficients of the solution. This is valuable, because it allows for $F(t)$ to be determined by the next result.

Lemma 3. *The Laplace transform $\mathcal{L}$ of the reset flow $F(t)$ satisfies*

$$\mathcal{L}\{F\} = \frac{\hat{\alpha}(s)}{1 - \hat{\beta}(s)}$$

where $\hat{\alpha}$ and $\hat{\beta}$ denote Laplace transforms of the functions

$$\alpha(t) \equiv \sum_{n=1}^{\infty} \theta_n a_n T_n(t) \qquad \beta(t) \equiv \sum_{n=1}^{\infty} \theta_n b_n T_n(t)$$

and the coefficient series a_n and b_n are defined

$$a_n \equiv \int_a^b \phi(y) X_n(y) \qquad b_n \equiv \sin(\frac{a\pi n}{b-a}) \tag{5}$$

Proof. Substitute for the Green's function in the conditional solution (4):

$$h(x,t) = \int_a^b \phi(y) \sum_{n=1}^{\infty} X_n(x) X_n(y) T_n(t) dy + \int_0^t \int_a^b \delta(y) F(s) \sum_{n=1}^{\infty} X_n(x) X_n(y) T_n(t-s) dy ds$$

$$= \sum_{n=1}^{\infty} X_n(x) T_n(t) \int_a^b \phi(y) X_n(y) dy + \sum_{n=1}^{\infty} X_n(x) X_n(0) \int_0^t F(s) T_n(t-s) ds$$

$$= \sum_{n=1}^{\infty} X_n(x) a_n T_n(t) + \sum_{n=1}^{\infty} X_n(x) b_n \int_0^t F(s) T_n(t-s) ds$$

Collect coefficients on $X_n(x)$:

$$S_n(t) = a_n T_n(t) + b_n \int_0^t F(s) T_n(t-s) ds \tag{6}$$

Then apply the weighted sum $F(t) = \sum_{n=1}^{\infty} \theta_n S_n(t)$ (Lemma 3):

$$F(t) = \sum_{n=1}^{\infty} \theta_n a_n T_n(t) + \sum_{n=1}^{\infty} \theta_n b_n \int_0^t F(s) T_n(t-s) ds$$

$$= \alpha(t) + \int_0^t F(s) \beta(t-s) ds$$

Take the Laplace transform:

$$\hat{F}(s) = \hat{\alpha}(s) + \hat{F}(s)\hat{\beta}(s) = \frac{\hat{\alpha}(s)}{1-\hat{\beta}(s)}$$

□

Lemma 3 allows the reset flow $F(t)$ to be calculated without first knowing the distribution $h(x,t)$. Then Theorem 1 in the next section easily gives the solution for $h(x,t)$.

But the Lemma has further value. In many models, aggregate outcomes of interest depend on integrating some function over the distribution $h(x,t)$. Section 4 shows that the behavior of such aggregates over time can be calculated directly from the flow without needing to find $h(x,t)$ or evaluate any integrals.

3.4 Solution

Theorem 1 presents the analytical solution for the distribution of agents $h(x,t)$, in terms of the initial condition $\phi(x)$ and the known functions $\hat{\alpha}(s)$ and $\hat{\beta}(s)$

Theorem 1. *The unique function solving Problem 1 is*

$$h(x,t) = \sum_{n=1}^{\infty} X_n(x) \left(a_n T_n(t) + \mathcal{L}^{-1} \left\{ \frac{\hat{\alpha}(s)}{1-\hat{\beta}(s)} \frac{b_n}{s+\lambda_n} \right\} \right)$$

Proof. The homogeneous solution (the first term in equation (4)) is

$$\int_a^b \phi(y) G(x,y,t) dy = \int_a^b \phi_y(y) \sum_{n=1}^{\infty} X_n(x) X_n(y) T_n(t) dy$$

$$= \sum_{n=1}^{\infty} a_n X_n(x) T_n(t)$$

which follows from the definitions of the a_n sequence (equation (5)) and Greens function (equation (3)).

The non-homogeneous component (the second term in equation (4)) is

$$\int_0^t \int_a^b \delta(y) F(s) G(x,y,t-s) dy ds = \int_0^t F(s) G(x,0,t-s) ds$$

$$= \sum_{n=1}^{\infty} b_n X_n(x) \int_0^t F(s) T_n(t-s) ds$$

The Laplace transform of the integral term is

$$\mathcal{L} \left\{ \int_0^t F(s) T_n(t-s) ds \right\} = \hat{F}(s) \hat{T}_n(s) = \hat{F}(s) \frac{1}{s+\lambda_n}$$

So Lemma 3 implies that the non-homogeneous component can be written as

$$\sum_{n=1}^{\infty} b_n X_n(x) \int_0^t F(s) T_n(t-s) ds = \sum_{n=1}^{\infty} b_n X_n(x) \mathcal{L}^{-1} \left\{ \frac{\hat{\alpha}(s)}{1-\hat{\beta}(s)} \frac{1}{s+\lambda_n} \right\}$$

Finally, Problem 1 is a standard non-homogeneous heat equation problem, so its solution must be unique given $F(t)$ (Evans, 2022, Sec. 2.3 Thm. 5), and $F(t)$ must be unique per Lemma 3 because $\hat{\beta}(s) \neq 1$. □

4 Using the Flow Function to Calculate Aggregate Dynamics

The dynamics of aggregate variables depend on the distribution $h(x,t)$. In many cases, an aggregate variable $Z(t)$ (or some transformation thereof) requires integrating over the distribution by

$$Z(t) = \int_a^b f_Z(x) h(x,t) dx \tag{7}$$

for some function $f_Z(x)$. How easily the integral can be evaluated depends on the functional form. This section works through some common examples.[6]

Without knowing anything about $h(x,t)$, this integral could be challenging to evaluate. But the examples in this section share a fortunate feature: the reset flow function $F(t)$ gives a shortcut for evaluating the integral without first finding the distribution $h(x,t)$.

4.1 Preliminaries

For some $f_Z(x)$ functions, the integral in equation (7) is simple to evaluate. Lemma 4 says that this is the case when $\int_a^b f_Z(x) X_n(x) dx$ is known. Then the next sections give examples when this is true.

Lemma 4. *If the integrals $\theta_n^Z \equiv \int_a^b f_Z(x) X_n(x) dx$ can be evaluated for all n, then given the reset flow $F(t)$, the aggregate variable $Z(t)$ satisfies*

[6]The representation in equation (7) also reveals the relationship with Alvarez and Lippi (2022). When $a = -b$, and $f_Z(x)$ is an odd function, the IRF can be found easily by solving the model without "reinjection". Without reinjection, the reset flow is $F(t) = 0$, and the PDE reduces to a standard homogeneous heat equation.

$$Z(t) = \alpha^Z(t) + \int_0^t F(s)\beta^Z(t-s)ds$$

with functions

$$\alpha^Z(t) \equiv \sum_{n=1}^{\infty} \theta_n^Z a_n T_n(t) \qquad \beta^Z(t) \equiv \sum_{n=1}^{\infty} \theta_n^Z b_n T_n(t)$$

Proof. Substitute $h(x,t) = \sum_{n=1}^{\infty} S_n(t)X_n(x)$ into equation (7):

$$Z(t) = \int_a^b f_Z(x) \left(\sum_{n=1}^{\infty} S_n(t)X_n(x) \right) dx$$

$$= \sum_{n=1}^{\infty} S_n(t) \int_a^b f_Z(x)X_n(x)dx = \sum_{n=1}^{\infty} \theta_n^Z S_n(t)$$

Equation (6) implies

$$Z(t) = \sum_{n=1}^{\infty} \theta_n^Z a_n T_n(t) + \sum_{n=1}^{\infty} \theta_n^Z b_n \int_0^t F(s)T_n(t-s)ds$$

$$= \alpha^Z(t) + \int_0^t F(s)\beta^Z(t-s)ds$$

□

4.2 Aggregates as Functions of Average Exponentials

This section considers aggregate variables $Z(t)$ that depend on an average exponential function of the state:

$$Z(t) = \int_a^b e^{\psi x} h(x,t)dx \tag{8}$$

for some ψ. For example, in Golosov and Lucas Jr. (2007) x is the log markup gap, and evaluating this integral gives an output gap (raised to some power). In investment models, x is the log capital-productivity ratio, and evaluating this integral gives a measure of aggregate capital. Applying Lemma 4 will do so, using the sequence given by Proposition 1:

Proposition 1. *Given the reset flow $F(t)$, the aggregate variable $Z(t)$ defined by equation* (8) *is given by Lemma 4 for the coefficients*

$$\theta_n^Z \equiv \frac{\pi n}{b-a} \frac{e^{\psi a} - e^{\psi b}(-1)^n}{\psi^2 + \left(\frac{\pi n}{b-a}\right)^2}$$

Proof: Appendix A.1

4.3 Aggregates as Functions of the Average State

This section considers aggregate variables $Z(t)$ that depend on the average state ($f_Z(x) = x$):

$$Z(t) = \int_a^b x h(x,t) dx \tag{9}$$

Transforming the average state $Z(t)$ yields an expression for many aggregate variables of interest. For example, in the menu cost model studied by Alvarez, Ferrara, Gautier, Le Bihan, and Lippi (2024), the aggregate output gap is proportional to the average markup gap. Proposition 2 shows that the average state is simple to calculate from the reset function:

Proposition 2. *Given the reset flow $F(t)$, the aggregate variable $Z(t)$ defined by equation* (9) *is given by Lemma 4 for the coefficients*

$$\theta_n^Z \equiv \frac{a - b(-1)^n}{\frac{\pi n}{b-a}}$$

Proof: Appendix A.2

4.4 Aggregates as Functions of the Squared State

Dynamics of higher order moments may be valuable to calculate, for example to study the dynamics of misallocation. This section considers the average squared state ($f_Z(x) = x^2$):

$$Z(t) = \int_a^b x^2 h(x,t) dx \tag{10}$$

which can be combined with Proposition 2 to calculate the time-varying variance.

Proposition 3. *Given the reset flow $F(t)$, the aggregate variable $Z(t)$ defined by equation* (10) *is given by Lemma 4 for the coefficients*

$$\theta_n^Z \equiv \left(\frac{a^2}{\left(\frac{\pi n}{b-a}\right)} + \frac{2}{\left(\frac{\pi n}{b-a}\right)^3} \right) - \left(\frac{b^2}{\left(\frac{\pi n}{b-a}\right)} + \frac{2}{\left(\frac{\pi n}{b-a}\right)^3} \right) (-1)^n$$

Proof: Appendix A.3

4.5 The Laplace Transform and Cumulative Impulse Response

The solutions from Propositions 1, 2, and 3 have a common form; they only differ in the appropriate $\alpha^Z(t)$ and $\beta^Z(t)$ functions. This section describes some results that are independent of the particular $f_Z(x)$ function.

It is often useful to work with the Laplace transforms directly, in which case $\hat{Z}(s) \equiv \mathcal{L}\{Z\}(s)$ is neatly given by

$$\hat{Z}(s) = \hat{\alpha}^Z + \hat{\beta}^Z(s)\hat{F}(s) = \hat{\alpha}^Z + \hat{\beta}^Z(s)\frac{\hat{\alpha}(s)}{1 - \hat{\beta}(s)}$$

For example, the Laplace transform allows for quick calculation of the cumulative impulse response function. $\hat{Z}(s)$ is a polynomial fraction; denote its partial fraction expansion by

$$\hat{Z}(s) = \sum_{j=0}^{\infty} \frac{\xi_j^Z}{s + \rho_j^Z}$$

As a convention, let index $j = 0$ denote the zero pole, i.e. $\rho_0^Z = 0$.

Corollary 1 gives the *cumulative impulse response* from the partial fraction terms.

Corollary 1. *The cumulative impulse response* $CIR^Z = \int_0^\infty Z(t)dt - \overline{Z}$ *is*

$$CIR^Z = \sum_{j=1}^{\infty} \frac{\xi_j^Z}{\rho_j^Z}$$

where $\overline{Z} \equiv \lim_{t\to\infty} Z(t)$ *denotes the steady state value.*

Proof. Use the inverse Laplace transform:

$$Z(t) = \sum_{j=0}^{\infty} \mathcal{L}^{-1}\{\frac{\xi_j^Z}{s + \rho_j^Z}\} = \sum_{j=0}^{\infty} \xi_j^Z e^{-\rho_j^Z t}$$

Then integrate, noting that $\overline{Z} = \xi_0$:

$$\int_0^\infty Z(t)dt - \overline{Z} = \int_0^\infty \sum_{j=1}^\infty \xi_j^Z e^{-\rho_j^Z t} dt = \sum_{j=1}^\infty \frac{\xi_j^Z}{\rho_j^Z}$$

$\square$

Corollary 1 is useful in settings where $Z(t)$ itself is relevant (e.g. investment models). At other times, the aggregate variable of interest first requires a transformation to be written as $Z(t)$ in a form satisfying equation (8) (e.g. the Golosov-Lucas output gap). In such a setting, $Z(t)$ will have to be untransformed first, and then integrated directly.

5 Example: Monetary Shocks in a Menu Cost Model

This section presents a menu cost model resembling Golosov and Lucas Jr. (2007). The setting is entirely standard, so I forgo description of the model's microfoundations.[7]

Firms must pay a fixed cost to change prices. The firm's markup μ is the difference between its log price p and log marginal cost $w - z$:

$$\mu = p - w + z$$

where w is the constant economy-wide log nominal wage and z represents a firm-specific log quality term. Firms face CES demand with constant optimal markup μ^*. Therefore the state variable for the firm is its *markup gap* x:

$$x \equiv p - w + z - \mu^*$$

Quality z follows a Brownian motion:

$$dz = \sigma dW$$

where W is a Wiener process, independent across firms.

[7] Interested readers are referred to Alvarez, Ferrara, Gautier, Le Bihan, and Lippi (2024), who offer a concise description, albeit with an approximation to the aggregation equation (11).

Because firms face a fixed menu cost, their optimal behavior is to leave prices unchanged for markup gaps in the interval $x \in [-b, b]$. For markup gaps outside this interval, firms immediately reset to the optimal markup μ^*, which implies that the markup gap x resets to 0. Thus, this menu-cost model is already in the form for which Theorem 1 gives the solution. If the model featured trend inflation or random price resets (the "Calvo-plus" class of models) a change of variables would first be necessary to apply the theorem (Section 5.4).

Finally, aggregate output $Y(t)$ in the Golosov-Lucas economy is determined from the distribution by

$$Y(t)^{\eta(\epsilon-1)}\alpha^{\epsilon-1}e^{(\epsilon-1)\mu^*} = \int_a^b e^{(1-\epsilon)x}h(x,t)dx \tag{11}$$

$1/\eta$ is the intertemporal elasticity of substitution, ϵ is the elasticity of substitution across firms' output, and α denotes the marginal disutility of labor. $1-\epsilon < 0$ so if all firms reduce their markups, output increases. The aggregate price level is determined by

$$\left(P(t)/W(t)\right)^{1-\epsilon} e^{(\epsilon-1)\mu^*} = \int_a^b e^{(1-\epsilon)x}h(x,t)dx$$

so Proposition 1 can be used to find the output function $Y(t)$ and price level $P(t)$.[8]

5.1 Medium Sized Monetary Shocks

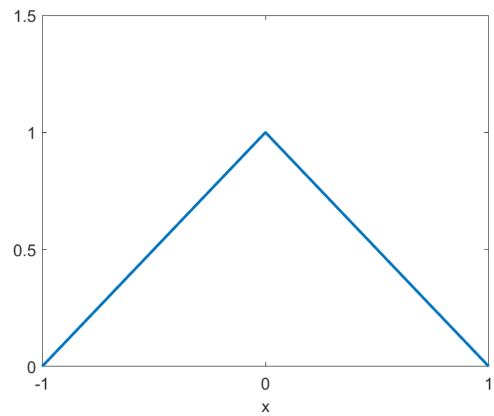
In this section, I study a permanent unanticipated monetary shock to an economy in the stationary distribution $\bar{h}(x)$.[9] The monetary shock permanently increases the nominal marginal cost of all firms by b. Accordingly, the shock decreases the markup gap x by b for all firms.[10] In the results that follow, I parameterize the model so that $b = 1$. I also let $\sigma^2 = 1$, implying that a firm's productivity diffusion is standard normal over one unit of time.

This shock pushes the $[-1, 0]$ half of the stationary distribution (Figure 1a) to

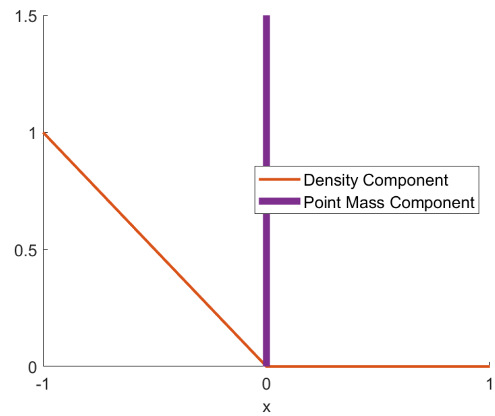
[8] In this model, the aggregating function $e^{(1-\epsilon)x}$ is not odd, so the approximation without reinjection cannot be used to calculate the aggregate IRF.

[9] Appendix B derives the stationary distribution, which is a triangle.

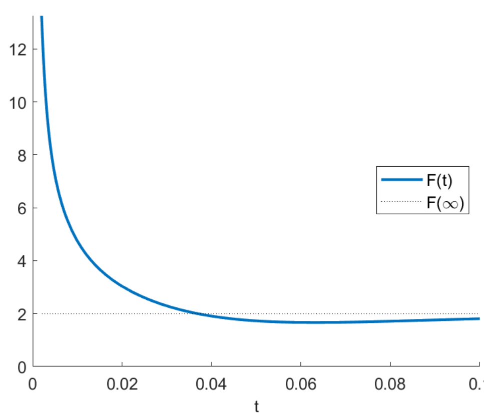
[10] It is assumed that the shock does not affect the inaction region. This assumption applies exactly in some fixed cost models. More generally, it abstracts from possible general equilibrium effects, although Alvarez and Lippi (2014) and Cavallo, Lippi, and Miyahara (2024) show that these effects are negligible in the menu cost model.



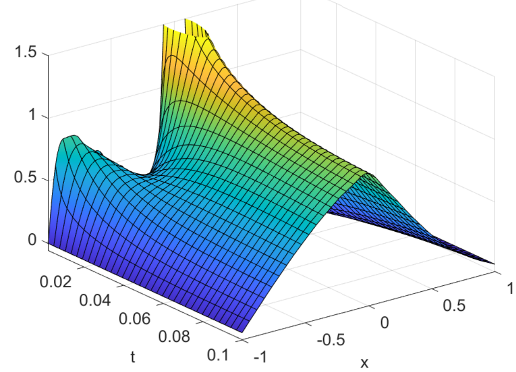
(a) Stationary distribution $h(x, \infty)$



(b) Initial condition $\phi(x)$



(c) Flow of price-resetting firms



(d) Dynamic distribution $h(x, t)$

Figure 1: Effects of a Medium-Sized Shock

Notes: The results are calculated for an economy parameterized by $-a = b = 1$, $\sigma = 1$, $\eta = 1$, $\epsilon = 1$, and $\alpha = 1$. The figures plot the response to a $\Delta = 1$ shock to the stationary distribution. The dynamic plots for $F(t)$ and $h(x, t)$ begin at time $t = 0.002$.

the lower bound -1. These firms immediately increase prices and reset their markup gap to 0. Thus the initial condition has a 0.5 point mass at the origin. Additionally, the shock shifts the $[0,1]$ positive half of the stationary distribution to the $[-1,0]$ interval. Figure 1b plots these point mass and density components, which together make up the initial condition $\phi(x) = h(x,0)$.

Lemma 3 implies that the initial condition is sufficient to solve for $F(t)$, the flow of resetting firms. Figure 1c plots this function. The plot begins at $t = 0.002$, because at $t = 0$ the flow is infinite. In the initial condition $\phi(x)$, many firms are near the boundary $a = -1$, so the flow remains high for a while before falling below $2 = \overline{F}$, and then asymptoting back to this long-run value.

Figure 1d plots the entire distribution, again beginning at $t = 0.002$ because there is a discontinuity at $(x,t) = (-1,0)$. The evolution explains why the reset flow is non-monotonic. After the shock, many agents are near the boundary $a = -1$, so the reset flow remains high. But they quickly diffuse over the boundary and the reset flow falls. At the same time, the firms that reset on impact are slow to diffuse to the upper and lower boundaries, so the flow actually falls below the limiting value because most mass is far from the boundaries. Then in the long run, the distribution $h(x,t)$ approaches the triangular stationary distribution, and the flow approaches its limit too.

5.2 Macroeconomic Effects of Aggregate Shocks: Size Dependence

One lesson to be learned from the analytical solution is that the effects of aggregate shocks are *size-dependent*. This is because the flow of price-resetting firms is endogenously determined by the size of the shock. Shock size has a straightforward piecewise-linear effect on the initial condition $h(x,0)$. But shocks that shift the initial condition also distort the flow of resetting firms. Moreover, the size of the shift affects the flow function $F(t)$ non-linearly: larger shocks (up to some threshold) raise the density of firms near a price-reset boundary. All non-negligible shocks have an immediate large effect ($F(0)$ is always infinite) but small shocks will lead to quick returns to $\overline{F}$, while large shocks will have slower convergence, as in Figure 1c.

To study size dependence, I consider permanent monetary shocks of arbitrary size to the stationary distribution $\bar{h}(x)$. A shock of size Δ reduces the markup gaps of all firms by Δ, shifting the distribution to the left and causing a mass of firms to immediately reset prices.

Therefore, the initial condition associated with a Δ size shock is for $x \in [a, b]$:

$$\phi_\Delta(x) = \overline{h}(x + \Delta) + \delta(x) \int_a^{a+\Delta} \overline{h}(x)dx$$

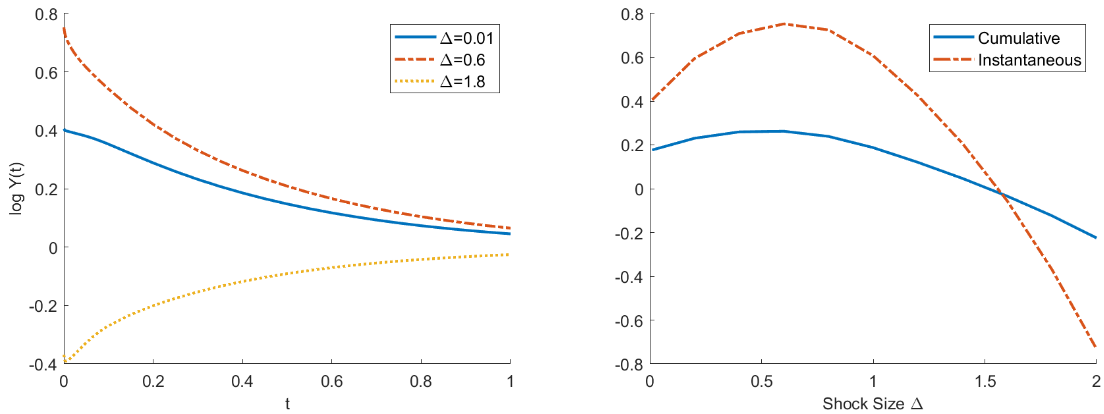
The density component $\overline{h}(x + \Delta)$ is written by defining $\overline{h}(x) = 0$ for $x \notin [a, b]$. $\int_a^{a+\Delta} \overline{h}(x)dx$ is the mass of firms that would be shifted left of the lower bound a, except they reset prices and reappear at 0. To analyze the effects of the shock, I calculate the impulse response function (IRF) for log output relative to the steady state:

$$IRF^Y(t) = \log Y(t) - \log Y(\infty) \tag{12}$$

Figure 2a demonstrates how shock size affects the shape of output's dynamic behavior. The figure plots the IRFs for permanent monetary shocks to the stationary distribution of firms. The smallest shock is size $\Delta = 0.01$, whose effects are well understood with current approximation methods. The monetary shock lowers markups, raising aggregate output. Convergence after a small shock is relatively swift, because the initial distribution $\phi_{0.01}(x)$ is not far from the stationary distribution $\overline{h}(x)$. The medium-sized shock ($\Delta = 0.6$) has a much larger impact because more prices reset, and the distribution $\phi_{0.6}(x)$ is far from $\overline{h}(x)$, so there are large distortions to output during the long time that it takes to converge.

However, the largest shock ($\Delta = 1.8$) is very disimilar: the effect on output is *negative*. Why? After a shock, some firms shift left by Δ, while the remaining firms reset prices, shifting right by b. When Δ is small, the leftward shift is small, even though it affects most firms. So the rightward movement of resetting firms dominates, and average x increases. But when Δ is large, the leftward shift is large, and the rightward shift of resetting firms is relatively small because they still only increase their markup gap x by b, so the leftward shift dominates.

For this sign-reversal, it is crucial that the integral determining output is the average value of $e^{(1-\epsilon)x}$ (equation (11)). Average x always moves in the same direction (Alvarez and Lippi, 2022) but average $e^{(1-\epsilon)x}$ does not. To see why, consider the extreme case: if the shock is $\Delta \geq 2$, then the entire distribution collapses to 0. This is a mean-preserving reduction in the variance of x, which must decrease the average



(a) IRFs to Shocks of Different Sizes

(b) Instantaneous Effects and CIRs

Figure 2: Size-Dependent Effects of Monetary Shocks on Output

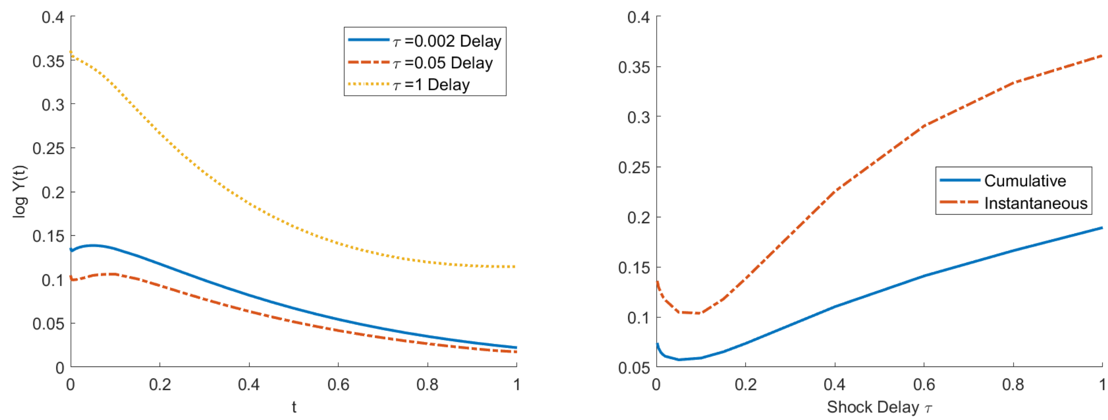
Notes: The results are calculated for an economy parameterized by $-a = b = 1$, $\sigma = 1$, $\eta = 1$, $\epsilon = 1$, and $\alpha = 1$. The IRFs are given by equation (12). Size-dependent figures plot Δ shocks to the stationary distribution.

value of the convex function $e^{(1-\epsilon)x}$ by Jensen's inequality. The sign-reversal is further demonstrated in Figure 2b, which plots two summary statistics for a range of shock sizes: the instantaneous effect and the cumulative deviation from the steady state (CIR). Shock size Δ has large effects on these statistics, and the relationships are not even monotonic, let alone consistently signed. Moreover, these effects are not scaled by shock size, as is sometimes reported. Rescaling by shock size would cause the effects to fall off rapidly with Δ.

5.3 Macroeconomic Effects of Aggregate Shocks: State Dependence

A second lesson from the analytical solution is that effects of aggregate shocks are *state-dependent*. It is easy to see why: consider two consecutive shocks of size b, as studied in Section 5.1. If the second shocks follows immediately after the first shock, it will be as if there is a large shock of size $2b$. But if the second shocks occurs much later once $h(x,t)$ has nearly converged to the stationary distribution, then the shock's effect will closely resemble the original size b shock. And Figure 2b demonstrated that these two cases imply different IRF shapes.

This form of state-dependence is somewhat unusual. Typically when working with



(a) IRFs to Shocks with Different Delays (b) Instantaneous Effects and CIRs

Figure 3: State-Dependent Effects of Monetary Shocks

Notes: The results are calculated for an economy parameterized by $-a = b = 1$, $\sigma = 1$, $\eta = 1$, $\epsilon = 1$, and $\alpha = 1$. The IRFs are given by equation (12). State-dependent figures plot small shocks τ time after a $\Delta = 1$ shock to the stationary distribution.

PDEs, the solution can be neatly separated into a component that depends on the initial condition and a component that depends on any forcing term. But this is not the case in the fixed cost model, because the forcing term $\delta(x)F(t)$ (the re-entry of resetting firms) is endogenously determined. This state-dependence can be seen from the function $\alpha(t)$, which depends on the initial condition $\phi(x)$, and yet appears in the solution for the reset flow $F(t)$ (Lemma 3). As a result, a shock perturbing the stationary distribution will not have the same effects as a shock following a sequence of earlier shocks.

To demonstrate the state-dependence, I examine the effects of a pair of permanent monetary shocks. Again, I calculate the output IRFs, which are now relative to the counterfactual in which only the first shock occurs. The first shock has size b as in Figure 1; then, it is followed by a small second shock of size $\Delta = 0.01$. The second shock arrives τ time after the initial shock.

Figure 3a demonstrates state-dependence by plotting how the output response to the second shock depends on the delay τ since the first shock. When the delay is long ($\tau = 1$) the IRF is nearly a shock to the stationary distribution. Accordingly, the impulse response function closely resembles the result in Figure 2a. But as the delay changes, the shape of the IRF changes. Figure 3b documents how the summary statistics change with the delay. The pattern is non-monotonic, but longer delays eventually feature larger instantaneous and cumulative effects. As the delay gets very long, the initial distribution approaches the stationary distribution, and the values converge to the $\Delta = 0.01$ results from Figure 2b.

5.4 Model Extensions

Does the solution method for the simple fixed cost model apply more generally? Yes. This section demonstrates how several common extensions can be rewritten as the simple fixed cost model, albeit with a minor modification of the incoming flow of firms.

Lemma 5. *For a fixed point model with KFE*

$$\partial_t h(x,t) = \gamma \partial_x^2 h(x,t) + \delta(x) e^{\varphi_1 t} \left(F(t) + \varphi_2\right)$$

the reset flow function is given by

$$F(t) = \mathcal{L}^{-1}\left\{\frac{\hat{\alpha}^{\varphi}(s)}{1-\hat{\beta}^{\varphi}(s)}\right\}$$

where

$$\alpha^{\varphi}(t) \equiv \sum_{n=1}^{\infty} \theta_n \left(a_n T_n(t) + \varphi_2 b_n \int_0^t e^{\varphi_1 s} T_n(t-s) ds\right) \qquad \beta^{\varphi}(t) \equiv \sum_{n=1}^{\infty} \theta_n b_n e^{\varphi_1 t} T_n(t)$$

Proof: Appendix A.4

Lemma 5 describes how to solve the modified problem (the proof closely follows that of Lemma 3, which is a special case for $\varphi_1 = 1$ and $\varphi_2 = 0$) to find the reset flow $F(t)$. Property 1 can be applied and then untransformed to recover the solution to the original problem.

Next, I show how three model extensions can be rewritten in the form of Problem 1 with a standard change of variables.

1. **Non-zero resets** Suppose x has inaction interval $[\underline{x}, \overline{x}]$ with reset point $x^* \neq 0$, where $h(x,t)$ follows the KFE (1). Define $\tilde{x} \equiv x - x^*$, and define $\tilde{h}(\tilde{x},t) \equiv h(\tilde{x} + x^*, t)$. Then $\tilde{h}(\tilde{x},t)$ is in the standard form, described by Problem 1.

2. **Trend inflation** If the money supply grows at a constant rate $\bar{\pi}$, then the KFE is given by

$$\partial_t h(x,t) = \gamma \partial_x^2 h(x,t) + \bar{\pi}\partial_x h(x,t) + \delta(x)F(t)$$

with $F(t)$ defined as usual. Define $\tilde{h}(x,t) \equiv e^{\frac{\bar{\pi}^2}{4\gamma}t + \frac{\bar{\pi}}{2\gamma}x} h(x,t)$; its time derivative satisfies

$$\partial_t \tilde{h}(x,t) = \gamma \partial_x^2 \tilde{h}(x,t) + e^{\frac{\bar{\pi}^2}{4\gamma}t}\delta(x)F(t)$$

which uses that $e^{\frac{\bar{\pi}}{2\gamma}x}\delta(x) = \delta(x)$. Lemma 5 gives the solution, using $\varphi_1 = \frac{\bar{\pi}^2}{4\gamma}$. Typically, models with trend inflation feature non-zero resets, in which case this transformation can be combined with **(1)**.

3. **Random resets** In "Calvo-Plus" models, firms stochastically receive opportunities to reset prices costlessly. If firms receive these options at rate ξ, this adds a constant decay to the distribution, whose KFE is now given by:

$$\partial_t h(x,t) = \gamma \partial_x^2 h(x,t) - \xi h(x,t) + \delta(x)(F(t) + \xi)$$

The additional ξ in the non-homogeneous term is because the inflow of firms at $x = 0$ includes both the usual $F(t)$ outflow paying menu costs, as well as the outflow receiving free resets $\int_a^b \xi h(x,t)dx = \xi$. Define $\tilde{h}(x,t) \equiv e^{\xi t} h(x,t)$. This distribution satisfies

$$\partial_t \tilde{h}(x,t) = \gamma \partial_x^2 \tilde{h}(x,t) + \delta(x)e^{\xi t}\left(F(t) + \xi\right)$$

Lemma 5 gives the solution for $\varphi_1 = \varphi_2 = \xi$.

6 Conclusions and Next Steps

This paper presented the analytical solution to the fixed cost model's dynamic distribution. This solution is valuable for understanding the macroeconomics of fixed costs. It allows for theoretical characterization of the dynamic effects of aggregate shocks on macroeconomic variables. And it provides an analytical shortcut that can be used when computing quantitative results.

While the simple fixed cost model applies to a variety of economic settings, and can be easily augmented with simple extensions, the method for deriving the solution will apply more generally. Whenever a KFE with endogenous resets needs solving – be it with additional state variables, complementarities, aggregate forcing terms, or other features – this paper's method provides a way forward. Use the Fourier representation and Laplace transforms to find the endogenous reset flow analytically, then use textbook PDE solutions to solve for the distribution.

Finally, application to the menu cost model emphasized that size and state dependence affect the transmission of aggregate shocks to macroeconomic outcomes. These lessons suggest that further application of the theory developed in this paper may help understand nonlinear or state-dependent effects of many other types of macroeconomic shocks.

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A Additional Proofs

A.1 Proof of Proposition 1

Proof. derivative simplifies:

$$\partial_x \left(\frac{\psi e^{\psi x} X_n(x) - e^{\psi x} \partial_x X_n(x)}{\psi^2 + \left(\frac{\pi n}{b-a}\right)^2} \right) = \frac{\psi^2 e^{\psi x} X_n(x) + \psi e^{\psi x} \partial_x X_n(x) - \psi e^{\psi x} \partial_x X_n(x) - e^{\psi x} \partial_x^2 X_n(x)}{\psi^2 + \left(\frac{\pi n}{b-a}\right)^2}$$

$$= \frac{\psi^2 e^{\psi x} X_n(x) + \left(\frac{\pi n}{b-a}\right)^2 e^{\psi x} X_n(x)}{\psi^2 + \left(\frac{\pi n}{b-a}\right)^2} = e^{\psi x} X_n(x)$$

which uses that $\partial_x^2 X_n(x) = -\left(\frac{\pi n}{b-a}\right)^2 X_n(x)$.

Therefore

$$\int_a^b e^{\psi x} X_n(x) dx = \frac{\psi e^{\psi b} X_n(b) - e^{\psi b} \partial_x X_n(b)}{\psi^2 + \left(\frac{\pi n}{b-a}\right)^2} - \frac{\psi e^{\psi a} X_n(a) - e^{\psi a} \partial_x X_n(a)}{\psi^2 + \left(\frac{\pi n}{b-a}\right)^2}$$

By construction, $X_n(b) = X_n(a) = 0$:

$$= \frac{e^{\psi a} \partial_x X_n(a) - e^{\psi b} \partial_x X_n(b)}{\psi^2 + \left(\frac{\pi n}{b-a}\right)^2} = \frac{e^{\psi a} \frac{\pi n}{b-a} - e^{\psi b} (-1)^n \frac{\pi n}{b-a}}{\psi^2 + \left(\frac{\pi n}{b-a}\right)^2} = \theta_n^Z$$

□

A.2 Proof of Proposition 2

Proof. Observe that the following derivative simplifies:

$$\partial_x \left(\frac{X_n(x) - x \partial_x X_n(x)}{\left(\frac{\pi n}{b-a}\right)^2} \right) = \frac{\partial_x X_n(x) - x \partial_x^2 X_n(x) - \partial_x X_n(x)}{\left(\frac{\pi n}{b-a}\right)^2} = x X_n(x)$$

which uses that $\partial_x^2 X_n(x) = -\left(\frac{\pi n}{b-a}\right)^2 X_n(x)$.

Therefore

$$\int_a^b x X_n(x) dx = \frac{X_n(b) - b \partial_x X_n(b)}{\left(\frac{\pi n}{b-a}\right)^2} - \frac{X_n(a) - a \partial_x X_n(a)}{\left(\frac{\pi n}{b-a}\right)^2}$$

By construction, $X_n(b) = X_n(a) = 0$:

$$= \frac{a\partial_x X_n(a) - b\partial_x X_n(b)}{\left(\frac{\pi n}{b-a}\right)^2} = \frac{a\frac{\pi n}{b-a} - b\frac{\pi n}{b-a}(-1)^n}{\left(\frac{\pi n}{b-a}\right)^2} = \theta_n^Z$$

□

A.3 Proof of Proposition 3

Observe that the following derivative simplifies:

$$\partial_x \left(\frac{-x^2 \partial_x X_n(x) + 2x X_n(x)}{\left(\frac{\pi n}{b-a}\right)^2} - \frac{2\partial_x X_n(x)}{\left(\frac{\pi n}{b-a}\right)^4} \right)$$
$$= \frac{-2x\partial_x X_n(x) - x^2 \partial_x^2 X_n(x) + 2X_n(x) + 2x\partial_x X_n(x)}{\left(\frac{\pi n}{b-a}\right)^2} - \frac{2\partial_x^2 X_n(x)}{\left(\frac{\pi n}{b-a}\right)^4} = x^2 X_n(x)$$

which uses that $\partial_x^2 X_n(x) = -\left(\frac{\pi n}{b-a}\right)^2 X_n(x)$.

Therefore

$$\int_a^b x^2 X_n(x) dx = -\frac{b^2 \partial_x X_n(b)}{\left(\frac{\pi n}{b-a}\right)^2} - \frac{2\partial_x X_n(b)}{\left(\frac{\pi n}{b-a}\right)^4} + \frac{a^2 \partial_x X_n(a)}{\left(\frac{\pi n}{b-a}\right)^2} + \frac{2\partial_x X_n(a)}{\left(\frac{\pi n}{b-a}\right)^4}$$

because $X_n(b) = X_n(a) = 0$ by construction.

$$= \left(\frac{a^2}{\left(\frac{\pi n}{b-a}\right)^2} + \frac{2}{\left(\frac{\pi n}{b-a}\right)^4} \right) \partial_x X_n(a) - \left(\frac{b^2}{\left(\frac{\pi n}{b-a}\right)^2} + \frac{2}{\left(\frac{\pi n}{b-a}\right)^4} \right) \partial_x X_n(b)$$

$$= \left(\frac{a^2}{\left(\frac{\pi n}{b-a}\right)^2} + \frac{2}{\left(\frac{\pi n}{b-a}\right)^4} \right) \frac{\pi n}{b-a} - \left(\frac{b^2}{\left(\frac{\pi n}{b-a}\right)^2} + \frac{2}{\left(\frac{\pi n}{b-a}\right)^4} \right) \frac{\pi n}{b-a}(-1)^n$$

A.4 Proof of Lemma 5

Proof. With the modified non-homogeneous term, the conditional solution (4) becomes

$$h(x,t) = \int_a^b \phi(y) G(x,y,t) dy + \int_0^t \int_a^b \delta(y) e^{\varphi_1 s} (F(s) + \varphi_2) G(x,y,t-s) dy ds \quad (13)$$

substitute with the Green's function:

$$h(x,t)=\int_a^b \phi(y)\sum_{n=1}^{\infty}X_n(x)X_n(y)T_n(t)dy+\int_0^t\int_a^b \delta(y)e^{\varphi_1 s}(F(s)+\varphi_2)\sum_{n=1}^{\infty}X_n(x)X_n(y)T_n(t-s)dyds$$

$$=\sum_{n=1}^{\infty}X_n(x)a_nT_n(t)+\sum_{n=1}^{\infty}X_n(x)b_n\int_0^t e^{\varphi_1 s}(F(s)+\varphi_2)T_n(t-s)ds$$

Collect coefficients on $X_n(x)$:

$$S_n(t)=a_nT_n(t)+b_n\int_0^t e^{\varphi_1 s}(F(s)+\varphi_2)T_n(t-s)ds$$

Then apply the weighted sum $F(t)=\sum_{n=1}^{\infty}\theta_n S_n(t)$ (Lemma 3):

$$F(t)=\sum_{n=1}^{\infty}\theta_n a_nT_n(t)+\sum_{n=1}^{\infty}\varphi_2\theta_n b_n\int_0^t e^{\varphi_1 s}T_n(t-s)ds+\sum_{n=1}^{\infty}\theta_n b_n\int_0^t e^{\varphi_1 s}F(s)T_n(t-s)ds$$

$$=\alpha^{\varphi}(t)+\int_0^t F(s)\beta^{\varphi}(t-s)ds$$

Take the Laplace transform:

$$\hat{F}(s)=\hat{\alpha}(s)+\hat{F}(s)\hat{\beta}^{\varphi}(s)=\frac{\hat{\alpha}(s)}{1-\hat{\beta}^{\varphi}(s)}$$

□

B The Stationary Distribution

The stationary distribution $\bar{h}(x)\equiv\lim_{t\to\infty}h(x,t)$ solves the KFE with $\partial_t h(x,t)=0$:

$$0=\gamma\partial_x^2\bar{h}(x)+\delta(x)\overline{F} \tag{14}$$

where $\overline{F}\equiv\lim_{t\to\infty}F(t)$ is the limiting flow. Corollary 2 gives the solution.

Corollary 2. *The stationary distribution $\bar{h}(x)$ is given by*

$$\bar{h}(x)=\begin{cases}\frac{2(x-a)}{-a(b-a)} & a\leq x\leq 0\\ \frac{2(b-x)}{b(b-a)} & 0\leq x\leq b\end{cases}$$

and the limiting flow is

$$\overline{F} = \frac{2}{-ab}$$

Proof. The stationary equation (14) implies that $\overline{h}(x)$ is linear for $x \neq 0$. This implies $\overline{h}(x)$ is of the form

$$\overline{h}(x) = \begin{cases} d_1 x + c_1 & a \leq x < 0 \\ d_2 x + c_2 & 0 > x \leq b \end{cases}$$

The continuity condition requires

$$c_1 = c_2$$

while the boundary conditions require

$$d_1 a + c_1 = 0 \qquad d_2 b + c_2 = 0$$

Combine these three equations to solve for the remaining coefficients in terms of c_1:

$$d_1 = -\frac{c_1}{a} \qquad d_2 = -\frac{c_1}{b}$$

The mass conservation condition says that $\overline{h}(x)$ must integrate to 1. $\overline{h}(x)$ is a triangle with height c_1, so the integral is given by:

$$1 = \frac{1}{2}(b-a)c_1$$

This implies that the linear terms are

$$c_1 = \frac{2}{b-a} = c_2 \qquad d_1 = -\frac{2}{(b-a)a} \qquad d_2 = -\frac{2}{(b-a)b}$$

which simplify by

$$d_1 x + c_1 = \left(-\frac{x}{a} + 1\right)\frac{2}{b-a} = \frac{2\,(x-a)}{-a(b-a)}$$

$$d_2 x + c_2 = \left(-\frac{x}{b} + 1\right)\frac{2}{b-a} = \frac{2\,(b-x)}{b(b-a)}$$

The limiting flow is

$$\overline{F} = \partial_x \overline{h}(a) - \partial_x \overline{h}(b)$$

$$= d_1 - d_2 = \frac{2}{-a(b-a)} + \frac{2}{b(b-a)} = \frac{2}{-ab}$$

$\square$

In the Section 5 model, the barriers are $b = 1$ and $a = -1$. Figure 1a plots this stationary distribution as implied by Corollary 2.