The Dynamic Distribution in the Fixed Cost Model: An Analytical Solution *

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Abstract

I derive an analytical solution to the Kolmogorov forward equation for fixed cost models. This is a challenging PDE, because the dynamic distribution depends on the flow of resetting agents, which is endogenously determined by the distribution itself. I show there is a shortcut that allows the flow function to be derived without first finding the entire distribution of agents. This shortcut is also valuable because many aggregate variables can be written in terms of the flow function alone. As an example, I solve the canonical menu cost model. In it, the analytical solution uncovers effects that are inconsistent with local approximation methods. Specifically, the effects of shocks are both size- and state-dependent. These nonlinearities are substantial: if a monetary shock is sufficiently large, it can even reverse the sign of the effect on output. The method also allows for drift; changes to trend inflation in the menu cost model cause large swings in short-run inflation.

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1 Introduction

Many economic decisions require paying fixed costs. But the macroeconomics of fixed costs is challenging to study, because it often requires keeping track of an infinite-dimensional state variable: the dynamic distribution of agents. The evolution of the distribution is governed by a partial differential equation (PDE), the *Kolmogorov Forward Equation* (KFE).[1] This equation has proven difficult to solve, because it depends on the flow of resetting agents, which itself is determined endogenously from the dynamic distribution. As a result of this nonlinear feedback, theorists rely on perturbation methods or other approximations to characterize aggregate behavior.

To address these challenges, I derive an analytical solution for the dynamic distribution in a canonical fixed cost model. The key insight is that the endogenous flow of resetting agents can be determined without first solving for the dynamic distribution of states. This shortcuts the endogenous feedback that prevented solving the model. Once the time path of the "reset flow" is known, the dynamic distribution is found using a standard PDE solution. But the reset flow is even more valuable: many macroeconomic variables that depend on the distribution can be calculated using the reset flow *alone*, circumventing the need to calculate the dynamic distribution at all.

The results in this paper are useful for studying aggregate shocks, because the KFE determines how the distribution of agents responds over time, and macroeconomic variables such as output or inflation are functions of this distribution. The general model describes economic decisions subject to a wide variety of frictions. This type of fixed cost model – with aggregate shocks and a dynamic distribution governed by a KFE – appears in many examples, including: investment adjustment costs (Baley and Blanco, 2021), hiring and firing costs (Elsby and Michaels, 2019), information acquisition costs (Alvarez, Lippi, and Paciello, 2018), wage renegotiation costs (Blanco and Drenik, 2023), and most famously menu costs (Golosov and Lucas Jr., 2007; Midrigan, 2011).

The analytical solution will be particularly useful in settings where endogenous "reinjection" cannot be ignored, such as models featuring drift or aggregates that depend on higher-order moments. To demonstrate, I solve an example with both features: a classic menu cost model. The solution provides three immediate lessons.

[1] In most settings it would be more informative to use the term "Fokker-Planck equation", which is a specific type of KFE. But the Kolmogorov terminology is most common in economics, so I use it as well.

First, it reveals that the effects of aggregate shocks are *size-dependent*. The output response does not scale with shock size, which affects all features of the impulse response function (IRF), including the shape, immediate impact, and cumulative impulse response (CIR).[2] Indeed, the size-dependence is not even monotonic, and is so strong that a large enough shock will *reverse* the sign of the effect on output. Second, the effects of aggregate shocks are *state-dependent*. A shock will imply different impulse responses if it follows a previous shock immediately or with long delay.[3] Output effects are largest after the longest delays, suggesting that conclusions drawn from perturbations to the stationary distribution may not generalize. Third, the solution reveals that even shocks that have small effects on the inaction region can lead to substantial short-run dynamics. In the menu cost example, I analyze an increase in trend inflation, and show how solving for the entire IRF reveals large effects that are missed when comparing steady states or calculating the CIR. Finally, all three of these lessons can be derived using the reset flow shortcut, i.e. without ever calculating the full distribution

This is an improvement over existing methods. Historically, fixed cost models employed clever assumptions such that the distribution was not a necessary aggregate state variable (Caplin and Spulber, 1987; Caplin and Leahy, 1997). In recent years, theorists have made considerable progress understanding the macroeconomics of fixed cost models by utilizing a variety of approximations.[4] Many researchers employ perturbation methods around the steady state distribution, which yields valid conclusions for small, rare shocks. This approach is convenient to characterize linear relationships between aggregate variables (e.g. Gertler and Leahy, 2008; Auclert, Rigato, Rognlie, and Straub, 2024) or to approximate their dependence on the distribution (e.g. Alvarez and Lippi, 2014; Alvarez, Lippi, and Souganidis, 2023). Without explicit linearization, Alvarez, Le Bihan, and Lippi (2016) derive a sufficient statistic for the cumulative effect of small one-off shocks in the nonlinear model. However, numerical solutions to the nonlinear model demonstrate that conclusions regarding

[2] Empirically, the effects of cost shocks on price-setting exhibit strong size-dependence, which Cavallo, Lippi, and Miyahara (2024) document using granular pricing data in the food and beverage industry.

[3] This echoes a classic result from Caplin and Leahy (1997).

[4] The approximations discussed in this section have been useful for theoretical analysis of fixed cost models. Further approximations are used for quantitative analysis, e.g. Midrigan (2011) and many other papers use the Krusell and Smith (1998) method to encode the infinite dimensional distribution.

small rare shocks will not necessarily hold for large or frequent shocks (Golosov and Lucas Jr., 2007; Cavallo, Lippi, and Miyahara, 2024). My analysis of the analytical solution agrees.

The literature is aware of the limitations of local approximations to the nonlinear dynamics featured in fixed cost models. To address these issues, Alvarez and Lippi (2022) make substantial progress by considering an alternative approximation: they assume there is no "reinjection", i.e. agents leave the distribution after resetting. This method is useful for calculating certain IRFs in some models. Specifically, if the aggregate variable of interest is calculated by integrating an anti-symmetric function over the distribution (e.g. odd moments) and if the model features no drift (e.g. no trend inflation or productivity growth) then the aggregate variable's IRF without reinjection is equivalent to the true IRF. Alvarez and Lippi go on to show that when these conditions hold, the IRF shape is invariant to shock size. In contrast, the approach in Section 3 develops the full analytical solution by finding and incorporating the equilibrium reinjection behavior (the reset flow). Among other results, the analytical solution gives the IRF for variables whose aggregating functions are not strictly odd, and allows for models with drift.

The next section describes the canonical fixed cost model. Section 3 derives the solution. Section 4 demonstrates how to express the dynamics for aggregate variables in terms of the reset flow. Section 5 computes the solution in a menu cost model, and demonstrates the size and state dependent effects of shocks. Section 6 concludes.

2 The Fixed Cost Model

This section introduces the canonical fixed cost model, which describes the distribution of agents who must pay a fixed cost to adjust some state variable. The model abstracts from the specific microfoundations that generate this behavior, but it applies to a wide variety of economic decisions including the menu cost model presented in Section 5. Solving agents' optimal decisions is both well understood and also specific to the setting at hand. This paper focuses on the general problem of solving the dynamic distribution.[5]

There is a continuum of agents; at any time t, each agent is characterized by a

[5]See Stokey (2008) for a textbook treatment.

state variable x. Each agent's state variable follows a diffusion:

$$dx = \mu dt + \sigma dW \tag{1}$$

where W is an independent Brownian motion, and μ captures a common trend.

When an agent's state is sufficiently low ($x \leq 0$) or sufficiently high ($x \geq \bar{x}$) it is willing to pay a fixed cost and reset its state to $x = x^*$. The interval $[0, \bar{x}]$ with $0 < x^* < \bar{x}$ is assumed to be constant. Agents also randomly reset at rate $\eta \geq 0$, leaving the distribution and reentering at $x = x^*$.

The distribution of agents' states is $h(x,t)$, a function on $x \in [0, \bar{x}]$ and $t \geq 0$. Solving the fixed cost model entails finding the function $h(x,t)$ that satisfies a number of conditions:

1. The distribution satisfies the KFE:

$$\partial_t h(x,t) = \gamma \partial_x^2 h(x,t) - \mu \partial_x h(x,t) - \eta h(x,t) \tag{2}$$

   on the interval $[0, x^*) \cup (x^*, \bar{x}]$ where $\gamma \equiv \frac{\sigma^2}{2}$.

2. The *continuity boundary condition*: while $h(x,t)$ might not be differentiable at x^*, it must be continuous.

3. The bounds 0 and $\bar{x}$ are absorbing barriers, implying the Dirchlet boundary conditions

$$h(0,t) = 0 \qquad h(\bar{x},t) = 0$$

4. The distribution is consistent with the initial condition

$$h(x,0) = \phi(x)$$

5. Probability is conserved, i.e. for all t,

$$\int_0^{\bar{x}} h(x,t)dx = 1$$

This is a very simple model, but has been used to study economic inaction in a wide variety of settings.

3 Solution

The solution approach is express the model as a standard PDE problem, albeit with an additional unknown function, the *reset flow* of probability $F(t)$, which will capture the rate at which agents hit the barriers, reset, and reenter the distribution. It is necessary to jointly solve for the functions describing the probability flow and the distribution of agents.

3.1 Expression as a Standard Problem

Most of this PDE problem is standard textbook material; this KFE is simply the usual *heat equation*, except at $x = x^*$. But the unusual boundary conditions prevent application of Sturm-Louiville theory to easily solve the problem. How must the function behave at $x = x^*$ in order to satisfy probability conservation?

I rewrite the model into a standard form, plus an additional unknown function $F(t)$. Define the *reset flow* $F(t)$ as

$$F(t) \equiv \gamma \partial_x h(0,t) - \mu h(0,t) - (\gamma \partial_x h(\bar{x},t) - \mu h(\bar{x},t)) + \eta$$

This object represents the flow of probability out of the interval $[0, \bar{x}]$; the first terms capture the flow out of the boundaries, and the final term captures the rate η random flow affecting the entire interval. These agents reset their state, and reenter the distribution at x^*, i.e. there is a point-like source at x^* where probability enters at rate $F(t)$.

Lemma 1 formalizes how this property is implied by the conservation assumption $\int_0^{\bar{x}} h(x,t)dx = 1$.

Lemma 1. *The following* non-homogeneous heat equation *holds for* $x \in [0, \bar{x}]$ *and* $t \geq 0$*:*

$$\partial_t h(x,t) = \gamma \partial_x^2 h(x,t) - \mu \partial_x h(x,t) - \eta h(x,t) + \delta(x - x^*) F(t)$$

where $\delta(x - x^*)$ *is the Dirac delta and* $F(t)$ *is the reset flow.*

Proof. The KFE (2) holds everywhere on the interval $[0, \bar{x}]$ except at x^*:

$$\partial_t h(x,t) - \left(\gamma \partial_x^2 h(x,t) - \mu \partial_x h(x,t) - \eta h(x,t)\right) = \begin{cases} 0 & x \neq x^* \\ R(t) & x = x^* \end{cases}$$

for some residual unknown function $R(t)$. Therefore, we can extend the KFE to the entire interval by writing

$$\partial_t h(x,t) = \gamma \partial_x^2 h(x,t) + \delta(x - x^*) R(t)$$

the Dirac delta $\delta(x - x^*)$ indicates that $R(t)$ enters the equation only at $x = x^*$.

The conservation assumption implies that the total density is unchanging:

$$0 = \partial_t \int_0^{\bar{x}} h(x,t) dx$$

$$= \int_0^{\bar{x}} \left(\gamma \partial_x^2 h(x,t) - \mu \partial_x h(x,t) - \eta h(x,t) + \delta(x - x^*) R(t) \right) dx$$

$$= (\gamma \partial_x h(b,t) - \mu h(b,t)) - (\gamma \partial_x h(0,t) - \mu h(0,t)) - \eta + R(t)$$

$$\implies F(t) = R(t)$$

□

Lemma 1 allows the model to be rewritten as a standard PDE problem, conditional on $F(t)$:

Problem 1.

$$\partial_t h(x,t) = \gamma \partial_x^2 h(x,t) - \mu \partial_x h(x,t) - \eta h(x,t) + \delta(x - x^*) F(t)$$

$$h(0,t) = 0 \qquad h(\bar{x},t) = 0$$

$$h(x,0) = \phi(x)$$

for $x \in [0, \bar{x}]$, $t \geq 0$

3.2 Useful Functions and the Conditional Solution

Define the $T_n(t)$ eigenfunction for $n = 1, 2, 3...$ by

$$T_n(t) \equiv e^{-\lambda_n t} \quad \lambda_n \equiv \frac{\gamma n^2 \pi^2}{\bar{x}^2} + \frac{\mu^2}{4\gamma} + \eta$$

$T_n(t)$ is an eigenfunction of the ∂_t operator: $\partial_t T_n(t) = -\lambda_n T_n(t)$.

Similarly, define the $X_n(x)$ eigenfunction for $n = 1, 2, 3...$ by

$$X_n(x) \equiv \sqrt{\frac{2}{\bar{x}}} e^{\frac{\mu}{2\gamma}x} \sin\left(\frac{n\pi x}{\bar{x}}\right) \tag{3}$$

$X_n(x)$ is an eigenfunction of the *Kolmogorov* operator $(\gamma\partial_x^2 - \mu\partial_x - \eta)$ with eigenvalue $-\lambda_n$. Observe that $T_n(t)X_n(x)$ solves the *homogeneous* PDE $\partial_t T_n(t)X_n(x) = \gamma\partial_x^2 T_n(t)X_n(x) - \mu\partial_x T_n(t)X_n(x) - \eta T_n(t)X_n(x)$ and satisfies the boundary conditions $h(0,t) = 0$ and $h(\bar{x},t) = 0$.

This problem's Green's Function $G(x, y, t)$ can be concisely written in terms of these eigenfunctions as

$$G(x, y, t) \equiv \sum_{n=1}^{\infty} -X_n(x)X_n(-y)T_n(t) \tag{4}$$

Moreover, any sufficiently regular function on the interval $[0, \bar{x}]$ can be written in "Fourier space" by expressing it as an infinite sum of the $X_n(x)$ eigenfunctions.

With this notation, Property 1 gives the textbook solution to the KFE *given* the unknown reset flow $F(t)$:

Property 1. *Given $F(t)$, the standard solution to the PDE Problem 1 is*

$$h(x,t) = \int_0^{\bar{x}} \phi(y)G(x, y, t)dy + \int_0^t F(\tau)G(x, x^*, t-\tau)d\tau \tag{5}$$

See for reference Polyanin (2001, Sec. 1.1.1) or for an equivalent expression without the Green's function, see a standard PDE textbook such as Evans (2022, Sec. 2.3.1).[6]

3.3 The Reset Flow

The previous section gives the model solution conditional on the reset flow function $F(t)$. But the reset flow is itself determined by $h(x, t)$. This section describes how, and then demonstrates how to determine the reset flow in isolation, i.e. without first knowing the solution for $h(x, t)$.

[6] Alvarez, Lippi, and Souganidis (2023) also apply a standard heat equation solution to the KFE, albeit without the endogenous reset flow.

In continuous time, it is often useful to work in the frequency space instead of traditional time. The following lemma gives the *Laplace transform* $\hat{F}(s)$ of the flow function. As usual, the Laplace transform of a function is defined by

$$\mathcal{L}\{F\}(s) \equiv \int_0^\infty F(t)e^{-st}dt$$

Lemma 2. *The Laplace transform of the reset flow* $\hat{F}(s) \equiv \mathcal{L}\{F\}(s)$ *satisfies*

$$\hat{F}(s) = \frac{\alpha(s)}{1-\beta(s)}$$

$\alpha(s)$ *and* $\beta(s)$ *denote the functions*

$$\alpha(s) = \frac{\eta}{s} + \sum_{n=1}^{\infty} \frac{\theta_n a_n}{s+\lambda_n} \qquad\qquad \beta(s) = \sum_{n=1}^{\infty} \frac{\theta_n b_n}{s+\lambda_n}$$

where the coefficient series a_n*,* b_n*, and* θ_n *are defined*

$$a_n \equiv -\int_0^{\bar{x}} X_n(-y)\phi(y)dy \qquad b_n \equiv -X_n(-x^*) \tag{6}$$

$$\theta_n \equiv \left(\gamma\left(X_n'(0) - X_n'(\bar{x})\right) - \mu\left(X_n(0) - X_n(\bar{x})\right)\right)$$

Proof. Define the operator $\mathcal{J} \equiv \gamma\partial_x - \mu$. Then the flow function is given by

$$F(t) = \eta + \mathcal{J}\left(h(0,t) - h(b,t)\right)$$

$$= \eta + \int_0^{\bar{x}} \phi(y)\mathcal{J}\left(G(0,y,t) - G(\bar{x},y,t)\right)dy + \int_0^t F(\tau)\mathcal{J}\left(G(0,x^*,t-\tau) - G(\bar{x},x^*,t-\tau)\right)d\tau$$

by Property 1. Decompose the Green's function using equation (4):

$$= \eta - \sum_{n=1}^{\infty}\int_0^{\bar{x}} \phi(y)\left(\mathcal{J}X_n(0) - \mathcal{J}X_n(\bar{x})\right)X_n(-y)T_n(t)dy$$

$$- \sum_{n=1}^{\infty}\int_0^{t} F(\tau)\left(\mathcal{J}X_n(0) - \mathcal{J}X_n(\bar{x})\right)X_n(-x^*)T_n(t-\tau)d\tau$$

Substitute using $\theta_n = \mathcal{J}\left(X_n(0) - X_n(\bar{x})\right)$:

$$= \eta - \sum_{n=1}^{\infty} \int_0^{\bar{x}} \phi(y)\theta_n X_n(-y) T_n(t) dy - \sum_{n=1}^{\infty} \int_0^{t} F(\tau)\theta_n X_n(-x^*) T_n(t-\tau) d\tau$$

$$= \eta + \sum_{n=1}^{\infty} \alpha_n T_n(t) + \int_0^t \left(\sum_{n=1}^{\infty} \beta_n T_n(t-\tau) \right) F(\tau) d\tau$$

Take the Laplace transform, so that convolution becomes multiplication:

$$\hat{F}(s) = \frac{\eta}{s} + \sum_{n=1}^{\infty} \frac{\theta_n a_n}{s+\lambda_n} + \left(\sum_{n=1}^{\infty} \frac{\theta_n b_n}{s+\lambda_n} \right) \hat{F}(s)$$

$$= \hat{\alpha}(s) + \hat{\beta}(s)\hat{F}(s)$$

Invert $1 - \hat{\beta}(s)$ to isolate $\hat{F}(s)$. □

Lemma 2 allows the reset flow $F(t)$ to be calculated without first knowing the distribution $h(x,t)$. Then Theorem 1 in the next section easily gives the solution for $h(x,t)$.

But the Lemma has further value. In many models, aggregate outcomes of interest depend on integrating some function over the distribution $h(x,t)$. Section 4 shows that the behavior of such aggregates over time can be calculated directly from the flow without needing to find $h(x,t)$ or evaluate any integrals.

3.4 Solution

Theorem 1 presents the analytical solution for the distribution of agents $h(x,t)$, in terms of $\phi(x)$ and the known functions $\alpha(s)$ and $\beta(s)$

Theorem 1 (Solution)**.** *The unique function solving Problem 1 is*

$$h(x,t) = \sum_{n=1}^{\infty} X_n(x) S_n(t) \qquad\qquad S_n(t) \equiv \sum_{m=1}^{\infty} \xi^h_{n,m} e^{-\rho^h_{n,m} t}$$

where $\xi^h_{n,m}$ and $-\rho^h_{n,m}$ are the coefficients and poles of the partial fraction expansion

$$\sum_{m=1}^{\infty} \frac{\xi^h_{n,m}}{s+\rho^h_{n,m}} = \frac{\alpha(s) b_n + (1-\beta(s)) a_n}{(1-\beta(s))(s+\lambda_n)}$$

Proof. The homogeneous solution (the first term in equation (5)) is

$$\int_0^{\bar{x}} \phi(y)G(x,y,t)dy = -\int_0^{\bar{x}} \phi_y(y)\sum_{n=1}^{\infty} X_n(x)X_n(-y)T_n(t)dy$$

$$= \sum_{n=1}^{\infty} a_n X_n(x)T_n(t)$$

which follows from the definitions of the a_n sequence (equation (6)) and Greens function (equation (4)).

The non-homogeneous component (the second term in equation (5)) is

$$\int_0^t F(s)G(x,x^*,t-s)ds = \sum_{n=1}^{\infty} b_n X_n(x)\int_0^t F(s)T_n(t-s)ds$$

The Laplace transform of the integral term is

$$\mathcal{L}\left\{\int_0^t F(s)T_n(t-s)ds\right\} = \hat{F}(s)\hat{T}_n(s) = \hat{F}(s)\frac{1}{s+\lambda_n}$$

so Lemma 2 implies that the non-homogeneous component can be written as

$$\sum_{n=1}^{\infty} b_n X_n(x)\int_0^t F(s)T_n(t-s)ds = \sum_{n=1}^{\infty} b_n X_n(x)\mathcal{L}^{-1}\left\{\frac{\alpha(s)}{1-\beta(s)}\frac{1}{s+\lambda_n}\right\}(t)$$

Combining the two components gives

$$h(x,t) = \sum_{n=1}^{\infty} X_n(x)\left(a_n T_n(t) + \mathcal{L}^{-1}\left\{\frac{\alpha(s)}{1-\beta(s)}\frac{b_n}{s+\lambda_n}\right\}(t)\right)$$

$$= \sum_{n=1}^{\infty} X_n(x)\left(\mathcal{L}^{-1}\left\{\frac{a_n}{s+\lambda_n} + \frac{\alpha(s)}{1-\beta(s)}\frac{b_n}{s+\lambda_n}\right\}(t)\right)$$

$$= \sum_{n=1}^{\infty} X_n(x)\left(\mathcal{L}^{-1}\left\{\sum_{m=1}^{\infty}\frac{\xi^h_{n,m}}{s+\rho^h_{n,m}}\right\}(t)\right) = \sum_{n=1}^{\infty} X_n(x)\sum_{m=1}^{\infty}\xi^h_{n,m}e^{-\rho^h_{n,m}t}$$

Finally, Problem 1 is a standard non-homogeneous heat equation problem, so its solution must be unique given $F(t)$ (Evans, 2022, Sec. 2.3 Thm. 5), and $F(t)$ must be unique per Lemma 2 because $\beta(s) \neq 1$. □

The solution $h(x,t)$ is easy to evaluate using the partial fraction expansion of $\frac{\alpha(s)b_n+(1-\beta(s))a_n}{(1-\beta(s))(s+\lambda_n)}$, a rational function. This is easily computed in any mathematical computing environment, and allows for the $h(x,t)$ distribution to be tractably represented by vectors encoding the $\xi^h_{n,m}$ and $\rho^h_{n,m}$ series. Still, it can be cleaner to simply write the solution in terms of the Laplace transform of the dynamic distribution $\hat{h}(x,s)$:

Corollary 1. *The Laplace transform of the model solution* $\hat{h}(x,s) \equiv \mathcal{L}\{h(x,t)\}(s)$ *is*

$$\hat{h}(x,s) = \sum_{n=1}^{\infty} X_n(x) \frac{\alpha(s)b_n + (1-\beta(s))a_n}{(1-\beta(s))(s+\lambda_n)}$$

This section gave an analytical solution for the dynamic distribution $h(x,t)$. The distribution may be intrinsically interesting, but as the next section shows, is not always necessary to describe the macroeconomy.

4 Using the Flow Function to Calculate Aggregate Dynamics

The dynamics of aggregate variables depend on the distribution $h(x,t)$. In many cases, an aggregate variable $Z(t)$ (or some transformation thereof) requires integrating over the distribution by

$$Z(t) = \int_0^{\bar{x}} f_Z(x) h(x,t) dx \tag{7}$$

for some function $f_Z(x)$. How easily the integral can be evaluated depends on the functional form. This section will discuss some common examples.[7]

Without knowing anything about $h(x,t)$, this integral could be challenging to evaluate. But the examples in this section share a fortunate feature: the reset flow function $F(t)$ gives a shortcut for evaluating the integral without first finding the distribution $h(x,t)$.

[7] The representation in equation (7) also reveals the relationship with Alvarez and Lippi (2022). When $\bar{x} = 2x^*$ and $f_Z(x)$ is an odd function, the IRF can be found easily by solving the model without "reinjection". Without reinjection, the reset flow is $F(t) = 0$, and the PDE reduces to a standard homogeneous heat equation.

4.1 Frequency-Domain Macroeconomic Aggregates

It is not necessary to find the dynamic distribution $h(x,t)$ in order to calculate aggregate dynamics. The typical approach in macroeconomics is to calculate $h(x,t)$ numerically, then evaluate the integral in equation (7) to find $Z(t)$. But there is a valuable shortcut: to prove Theorem 1, I first wrote $h(x,t)$ in terms of the reset flow $F(t)$. This means that we can simply calculate $Z(t)$ by using the information in $F(t)$, which avoids constructing and integrating the $h(x,t)$ distribution entirely.

This shortcut is cleanest in the frequency-domain, where any Laplace-transformed aggregate variable $\hat{Z}(s)$ is *linear* in the frequency-domain reset function $\hat{F}(s)$. Theorem 2 gives this result. Then Corollary 2 gives the time-domain expression for $Z(t)$.

Theorem 2 (Linearity)**.** *The Laplace transform $\hat{Z}(s)$ of an aggregate variable $Z(t)$ given by equation* (7) *is linear in the Laplace transform of the reset flow $\hat{F}(s)$:*

$$\hat{Z}(s) = \alpha^Z(s) + \beta^Z(s)\hat{F}(s)$$

for

$$\alpha^Z(s) \equiv \sum_{n=1}^{\infty} \frac{\theta_n^Z a_n}{s+\lambda_n} \qquad \beta^Z(s) \equiv \sum_{n=1}^{\infty} \frac{\theta_n^Z b_n}{s+\lambda_n}$$

where a_n and b_n are given by Lemma 2 and

$$\theta_n^Z \equiv \int_0^{\bar{x}} f_Z(x) X_n(x) dx$$

Proof. Equations (4) and (5) imply

$$h(x,t) = -\sum_{n=1}^{\infty} \left(\int_0^{\bar{x}} \phi(y) X_n(x) X_n(-y) T_n(t) dy + \int_0^t F(\tau) X_n(x) X_n(-x^*) T_n(t-\tau) d\tau \right)$$

$$= \sum_{n=1}^{\infty} \left(a_n X_n(x) T_n(t) + \int_0^t F(\tau) b_n X_n(x) T_n(t-\tau) d\tau \right)$$

Take the Laplace transform, using $\mathcal{L}(T_n)(s) = \frac{1}{s+\lambda_n}$:

$$\hat{h}(x,s) = \sum_{n=1}^{\infty} \left(\frac{a_n X_n(x)}{s+\lambda_n} + \hat{F}(s) \frac{b_n X_n(x)}{s+\lambda_n} \right) \tag{8}$$

The Laplace transform of equation (7) gives

$$\hat{Z}(s) = \int_0^{\bar{x}} f_Z(x)\hat{h}(x,s)dx$$

Combine this with equation (8):

$$\hat{Z}(s) = \int_0^{\bar{x}} f_Z(x)\sum_{n=1}^{\infty} X_n(x)\left(\frac{a_n}{s+\lambda_n} + \hat{F}(s)\frac{b_n}{s+\lambda_n}\right)dx$$

then the definition of θ_n^Z implies

$$\hat{Z}(s) = \sum_{n=1}^{\infty}\left(\frac{\theta_n^Z a_n}{s+\lambda_n} + F(s)\frac{\theta_n^Z b_n}{s+\lambda_n}\right) = \alpha^Z(s) + \beta^Z(s)\hat{F}(s)$$

□

4.2 Time-Domain Macroeconomic Aggregates

Theorem 2 showed that macroeconomic aggregates are easily calculated in the frequency-domain, where $\hat{Z}(s)$ is linear in the reset flow. This section demonstrates how to convert $\hat{Z}(s)$ into macroeconomic objects that are typically reported: the impulse response function (IRF) and cumulative impulse response (CIR).

A convenient property of the frequency-domain is that $\hat{Z}(s)$ is a rational function (i.e. a polynomial fraction). Therefore, it is easily invertible to the time domain. Denote its partial fraction expansion by

$$\hat{Z}(s) = \sum_{j=0}^{\infty}\frac{\xi_j^Z}{s+\rho_j^Z} \tag{9}$$

As a convention, let index $j=0$ denote the zero pole, i.e. $\rho_0^Z = 0$. Therefore $\xi_0 = \overline{Z}$, the long-run value of the aggregate variable $Z(t)$.

Corollary 2 gives the IRF in terms of the partial fraction expansion. It also gives the CIR, which many economists use to summarize the effects of shocks in continuous time models,

Corollary 2 (Impulse Response Function)**.** *The impulse response function* $IRF^Z(t) \equiv$

$Z(t) - \overline{Z}$ is

$$Z(t) - \overline{Z} = \sum_{j=1}^{\infty} \xi_j^Z e^{-\rho_j^Z t}$$

and the cumulative impulse response CIR^Z is

$$CIR^Z = \sum_{j=1}^{\infty} \frac{\xi_j^Z}{\rho_j^Z}$$

where ξ_j^Z and ρ_j^Z correspond to the partial fraction expansion in equation (9).

Proof. Use the inverse Laplace transform:

$$Z(t) = \sum_{j=0}^{\infty} \mathcal{L}^{-1}\{\frac{\xi_j^Z}{s+\rho_j^Z}\}(t) = \sum_{j=0}^{\infty} \xi_j^Z e^{-\rho_j^Z t}$$

$\rho_0^Z = 0$, so the $j = 0$ term is simply $\xi_0^Z = \overline{Z}$. Therefore $Z(t) - \overline{Z} = \sum_{j=1}^{\infty} \xi_j^Z e^{-\rho_j^Z t}$.

To find the CIR, integrate the IRF over time:

$$CIR^Z = \int_0^{\infty} Z(t)dt - \overline{Z} = \int_0^{\infty} \sum_{j=1}^{\infty} \xi_j^Z e^{-\rho_j^Z t} dt = \sum_{j=1}^{\infty} \frac{\xi_j^Z}{\rho_j^Z}$$

□

Corollary 2 is immediately useful in settings where $Z(t)$ itself is relevant (e.g. investment models). At other times, the aggregate variable of interest first requires a transformation to be written as $Z(t)$ in a form satisfying equation (10) (e.g. the Golosov-Lucas output gap).

4.3 Examples of Aggregating Functions

In this section, I discuss common examples of $f_Z(x)$ aggregating functions. Appendix A derives their associated θ_n^Z series.

The Average Exponential State: Many aggregate variables $Z(t)$ depend on an average exponential function of the state:

$$Z(t) = \int_0^{\bar{x}} e^{\psi x} h(x,t) dx \qquad (10)$$

for some ψ. For example, in Golosov and Lucas Jr. (2007) x is the log markup gap, and evaluating this integral gives an output gap (raised to some power). This is also the case in the example I explore in Section 5. In investment models, x is the log capital-productivity ratio, and evaluating this integral gives a measure of aggregate capital.

The Average State: In some cases, aggregate variables $Z(t)$ depend on the average state ($f_Z(x) = x$):

$$Z(t) = \int_0^{\bar{x}} xh(x,t)dx \tag{11}$$

It is also common to use the average state as an approximation of a nonlinear function. For example, in the menu cost model studied by Alvarez, Ferrara, Gautier, Le Bihan, and Lippi (2024), the aggregate output gap is calculated as proportional to the average markup gap.

The Squared State: Dynamics of higher order moments may be valuable to calculate, for example to study the dynamics of misallocation. Many cases depend on the average squared state ($f_Z(x) = x^2$):

$$Z(t) = \int_0^{\bar{x}} x^2 h(x,t)dx \tag{12}$$

which determines the time-varying variance of x. For example, in Baley and Blanco (2024) and Adams, Chen, Fang, Hattori, and Rojas (2025), the average squared state measures capital misallocation. In Cavallo, Lippi, and Miyahara (2023) it captures welfare costs of inflation due to misallocation.

5 Example: Monetary Shocks in a Menu Cost Model

An advantage of the analytical solution over existing methods that ignore reinjection is the ability to handle models with drift and macroeconomic aggregates that depend on even moments. This section presents an example featuring both: a menu cost model resembling Golosov and Lucas Jr. (2007). The setting is entirely standard, so I forgo description of the model's microfoundations.[8] I show that the analytical solution is valuable for analyzing scenarios such as large shocks, state-dependence,

[8] Interested readers are referred to Alvarez, Ferrara, Gautier, Le Bihan, and Lippi (2024), who offer a concise description, albeit with an approximation to the aggregation equation (14).

and a change to trend inflation that affects the inaction region.

5.1 Environment

Firms must pay a fixed cost to change prices. The firm's markup μ is the difference between its log price p and log marginal cost $w - z$:

$$\mu = p - w + z$$

where w is the constant economy-wide log nominal wage and z represents a firm-specific log quality term. Firms face CES demand with constant optimal markup μ^*. Therefore the state variable for the firm is its *markup gap* g:

$$g \equiv p - w + z - \mu^*$$

Firms reset when their markup gap exits the interval $[\underline{g}, \bar{g}]$. To fit the solution framework where $x \in [0, \bar{x}]$, I normalize the markup gap as x:

$$x = g - \underline{g}$$

Quality z follows a Brownian motion:

$$dz = \sigma dW$$

where W is a Wiener process, independent across firms.

At first, I consider a model without trend inflation. Therefore, in the inaction region, this Brownian motion also describes the markup gap x.

Because firms face a fixed menu cost, their optimal behavior is to leave prices unchanged for normalized markup gaps in the interval $x \in [0, \bar{x}]$. For values outside this interval, firms immediately reset markups to the optimal markup μ^*, which implies that the normalized markup gap x resets to $x^* = -\underline{g}$.

Finally, the price level in the Golosov-Lucas economy is determined by

$$(P(t)/W(t))^{1-\epsilon} e^{(\epsilon-1)(\mu^*+\underline{g})} = \int_0^{\bar{x}} e^{(1-\epsilon)x} h(x,t) dx \tag{13}$$

which follows from the CES price aggregator and the definition of the markup gap x.

$W(t)$ denotes the nominal wage, ϵ is the elasticity of substitution across firms' output, and α is the marginal disutility of labor. Then, aggregate output is determined from a labor supply equation: $\alpha Y(t)^{\nu} = W(t)/P(t)$. Therefore $Y(t)$ can also be found by aggregating over the distribution $h(x,t)$:

$$Y(t)^{\nu(\epsilon-1)}\alpha^{\epsilon-1}e^{(\epsilon-1)(\mu^*+\underline{g})} = \int_0^{\bar{x}} e^{(1-\epsilon)x}h(x,t)dx \tag{14}$$

where $1/\nu$ is the intertemporal elasticity of substitution. $1-\epsilon < 0$ so if all firms reduce their markups, output increases and prices fall.

Equations (13) and (14) imply that Proposition 1 can be used to find the output function $Y(t)$ and price level $P(t)$.[9]

5.2 Large Monetary Shocks

In this section, I study a permanent unanticipated monetary shock to an economy in the stationary distribution $\bar{h}(x)$.[10] The monetary shock permanently increases the nominal marginal cost of all firms by Δ. Accordingly, the shock decreases the markup gap x by Δ for all firms.[11]

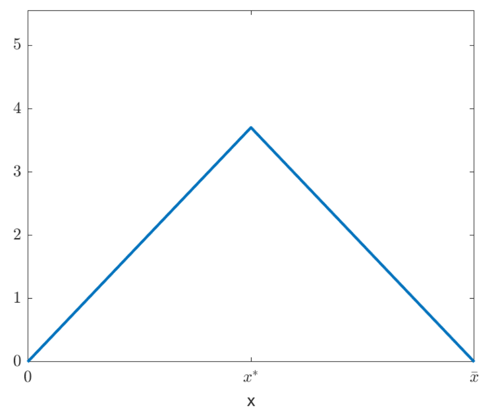
I parameterize the model following the approach in Alvarez, Le Bihan, and Lippi (2016): in the driftless Golsov-Lucas model, the standard deviation of price changes is $SD(dp) = \bar{x} - x^*$, and the average number of price changes is $N(dp) = 2\gamma/SD(dp)^2$. Cavallo, Lippi, and Miyahara (2024) measure $N(dp)$ and $SD(dp)$ for food and beverages across rich countries before the 2022 inflation; their estimates imply $\gamma = 0.069$ and $\bar{x} = 0.54$. $t = 1$ corresponds to one year. When calculating aggregate output, I also follow their choice for the elasticity of substitution between goods, setting $\epsilon = 6$. Then I set $\nu = 1$ so that households have log utility, and set the marginal disutility of labor at $\alpha = 1$.

The $\Delta = x^*$ shock pushes the $[0, x^*]$ portion of the stationary distribution (Figure 1a) to the lower bound at $x = 0$. These firms immediately increase prices and reset

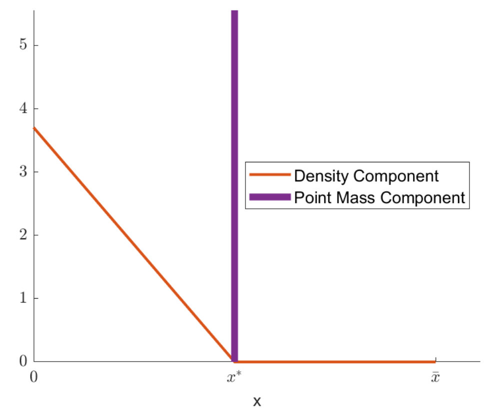
[9] In this model, the aggregating function $e^{(1-\epsilon)x}$ is not odd, so the approximation without reinjection cannot be used to calculate the aggregate IRF.

[10] Appendix B derives the stationary distribution, which is a triangle.

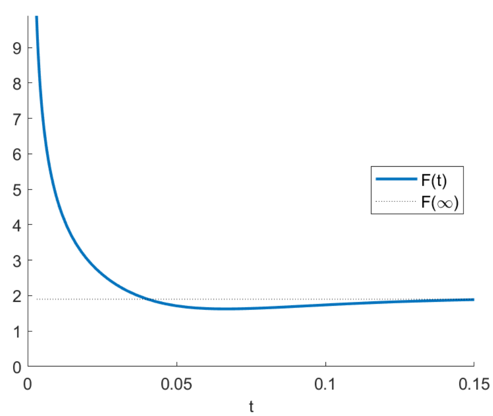
[11] It is assumed that the shock does not affect the inaction region. This assumption applies exactly in some fixed cost models. More generally, it abstracts from possible general equilibrium effects, although Alvarez and Lippi (2014) and Cavallo, Lippi, and Miyahara (2024) show that these effects are negligible in the menu cost model.



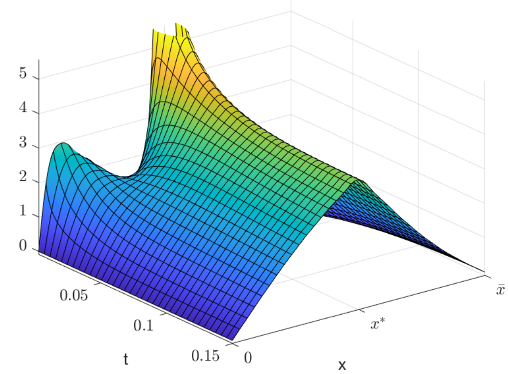
(a) Stationary distribution $h(x, \infty)$



(b) Initial condition $\phi(x)$



(c) Flow of price-resetting firms



(d) Dynamic distribution $h(x, t)$

Figure 1: Effects of a Large Shock

Notes: The results are calculated for an annual calibration with $\gamma = 0.069$, $x^* = 0.27$, and $\bar{x} = 0.54$. The figures plot the response to a $\Delta = x^*$ shock to the stationary distribution. The dynamic plots for $F(t)$ and $h(x, t)$ begin at time $t = 0.002$.

their markup gap to x^*. Thus the initial condition has a point mass of 0.5 at the origin. Additionally, the shock shifts the $[x^*, \bar{x}]$ positive half of the stationary distribution to the $[0, x^*]$ interval. Figure 1b plots these point mass and density components, which together make up the initial condition $\phi(x) = h(x, 0)$.

Lemma 2 implies that the initial condition is sufficient to solve for $F(t)$, the flow of resetting firms. Figure 1c plots this function. In the initial condition $\phi(x)$, many firms are near the boundary at $x = 0$, so the flow remains high for a while before falling below $\overline{F}$, and then asymptoting back to this long-run value.

Figure 1d plots the entire distribution, again beginning at $t = 0.02$ because there is a discontinuity at $(x, t) = (0, 0)$. The evolution explains why the reset flow is non-monotonic. After the shock, many agents are near the boundary at $x = 0$, so the reset flow remains high. But they quickly diffuse over the boundary and the reset flow falls. At the same time, the firms that reset on impact are slow to diffuse to the upper and lower boundaries, so the flow actually falls below the limiting value because most mass is far from the boundaries. Then in the long run, the distribution $h(x, t)$ approaches the triangular stationary distribution, and the flow approaches its limit too.

Seeing the distribution in Figure 1d is helpful to understand how the model behaves. But to analyze aggregates such as output or inflation, Theorem 2 says there is a shortcut: only the flow function in Figure 1c is needed. Therefore, in the remaining sections I explore these macroeconomic aggregates without ever calculating $h(x, t)$.

5.3 Macroeconomic Effects of Aggregate Shocks: Size Dependence

One lesson revealed by the analytical solution is that aggregate shocks have *size-dependent* effects, because the flow of price-resetting firms is endogenously determined by the shock size. Shock size has a straightforward piecewise-linear effect on the initial condition $h(x, 0)$. But shocks that shift the initial condition also distort the flow of resetting firms. Moreover, the size of the shift affects the flow function $F(t)$ non-linearly: larger shocks (up to some threshold) raise the density of firms near a price-reset boundary. All non-negligible shocks have an immediate large effect ($F(0)$ is always infinite) but small shocks will lead to quick returns to $\overline{F}$, while large shocks will have slower convergence, as in Figure 1c.

To study size dependence, I consider permanent monetary shocks of arbitrary size to the stationary distribution $\bar{h}(x)$. A shock of size Δ reduces the markup gaps of all firms by Δ, shifting the distribution to the left and causing a mass of firms to immediately reset prices.

Therefore, the initial condition associated with a Δ size shock is for $x \in [0, \bar{x}]$:

$$\phi_\Delta(x) = \overline{h}(x + \Delta) + \delta(x) \int_0^\Delta \overline{h}(x)dx$$

The density component $\overline{h}(x + \Delta)$ is written by defining $\overline{h}(x) = 0$ for $x \notin [0, \bar{x}]$. $\int_0^\Delta \overline{h}(x)dx$ is the mass of firms that would be shifted left of the lower bound, except they reset prices and reappear at x^*. To analyze the effects of the shock, I calculate the impulse response function (IRF) for log output relative to the steady state:

$$IRF^Y(t) = \log Y(t) - \log Y(\infty) \tag{15}$$

Figure 2a demonstrates how shock size affects the output dynamics. The figure plots the IRFs for permanent monetary shocks to the stationary distribution of firms. The smallest shock is size $\Delta = 0.001$, whose effects are well understood with current approximation methods. The monetary shock lowers markups, raising aggregate output. Convergence after a small shock is relatively swift, because the initial distribution $\phi_{0.001}(x)$ is not far from the stationary distribution $\overline{h}(x)$. The "large" shock ($\Delta = x^*$) has a much larger impact because more prices reset, and the distribution $\phi_{x^*}(x)$ is far from $\overline{h}(x)$, so there are large distortions to output during the long time that it takes to converge.

However, the largest shock ($\Delta = 0.9\bar{x}$) is very dissimilar: the effect on output is *negative*. Why? After a shock, some firms shift left by Δ, while the remaining firms reset prices, shifting right by b. When Δ is small, the leftward shift is small, even though it affects most firms. So the rightward movement of resetting firms dominates, and average x increases. But when Δ is large, the leftward shift is large, and the rightward shift of resetting firms is relatively small because they still only increase their markup gap x by b, so the leftward shift dominates.

For this sign-reversal, it is crucial that the integral determining output is the average value of $e^{(1-\epsilon)x}$ (equation (14)). Average x always moves in the same direction

(Alvarez and Lippi, 2022) but average $e^{(1-\epsilon)x}$ does not.[12] To see why, consider the extreme case: if the shock is $\Delta \geq \bar{x}$, then the entire distribution collapses to x^*. This is a mean-preserving reduction in the variance of x, which must decrease the average value of the convex function $e^{(1-\epsilon)x}$ by Jensen's inequality. The sign-reversal is further demonstrated in Figure 2b, which plots two summary statistics for a range of shock sizes: the instantaneous effect and the cumulative deviation from the steady state (CIR). Shock size Δ has large effects on these statistics, and the relationships are not even monotonic, let alone consistently signed. Moreover, these effects are not scaled by shock size, as is sometimes reported. Rescaling by shock size would cause the effects to fall off rapidly with Δ.

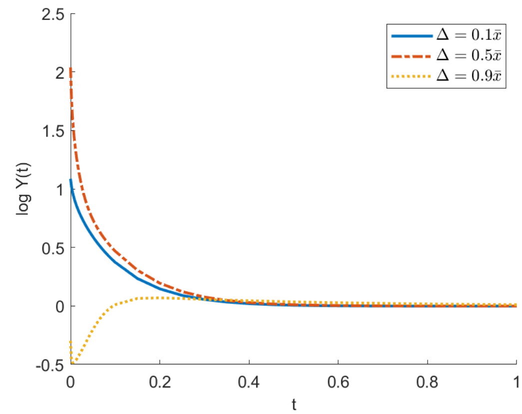
5.4 Macroeconomic Effects of Aggregate Shocks: State Dependence

A second lesson from the analytical solution is that effects of aggregate shocks are *state-dependent*. It is easy to see why: consider two consecutive shocks of size Δ, as studied in Section 5.2. If the second shocks follows immediately after the first shock, it will be as if there is a large shock of size 2Δ. But if the second shocks occurs much later once $h(x,t)$ has nearly converged to the stationary distribution, then the shock's effect will closely resemble the original size Δ shock. And Figure 2b demonstrated that these two cases imply different IRF shapes.

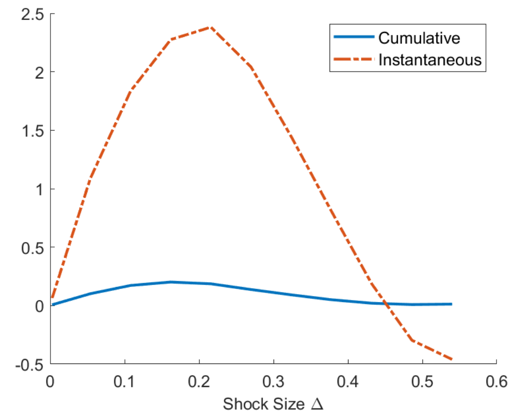
This form of state-dependence may be surprising because the KFE is a linear PDE, which is typically separable into an initial condition component and a forcing term. But in the fixed cost model, this separability is broken, because the forcing term $\delta(x - x^*)F(t)$ (the re-entry of resetting firms) is endogenously determined. This is clear in Lemma 2: the initial condition determines the a_n sequence, which affects the flow $F(t)$. As a result, a shock perturbing the stationary distribution will not have the same effects as a shock following a sequence of earlier shocks.

To demonstrate the state-dependence, I examine the effects of a pair of permanent monetary shocks. Again, I calculate the output IRFs, which are now relative to the counterfactual in which only the first shock occurs. The first shock has size x^* as in Figure 1; then, it is followed by a small second shock of size $\Delta = 0.01$. The second

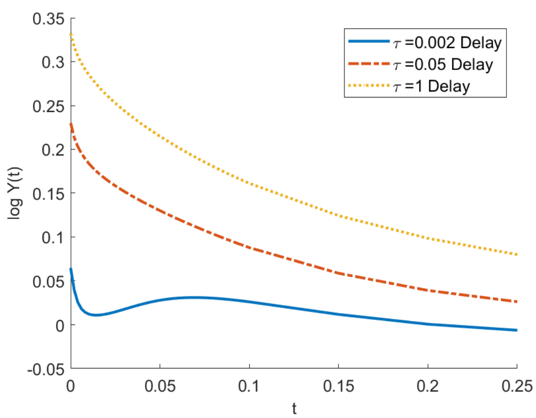
[12] This explains the difference with the size-dependence studied by Cavallo, Lippi, and Miyahara (2023). In their menu cost model, output is proportional to average x, so increasing the shock size decreases the marginal effect, but not enough to reverse the sign.



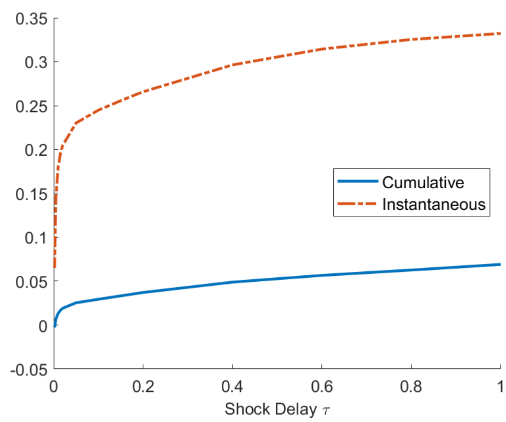
(a) IRFs to Shocks of Different Sizes



(b) Instantaneous Effects and CIRs to Shocks of Different Sizes



(c) IRFs to Shocks with Different Delays



(d) Instantaneous Effects and CIRs to Shocks with Different Delays

Figure 2: Size- and State-Dependent Effects of Monetary Shocks on Output

Notes: The results are calculated for an annual calibration with $\gamma = 0.069$, $x^* = 0.27$, and $\bar{x} = 0.54$. The figures plot the response to a $\Delta = x^*$ shock to the stationary distribution. To calculate aggregate output, $\nu = 1$, $\epsilon = 6$, and $\alpha = 1$. The IRFs are given by equation (15). Size-dependent figures plot Δ shocks to the stationary distribution.

shock arrives τ time after the initial shock.

Figure 2c demonstrates state-dependence by plotting how the output response to the second shock depends on the delay τ since the first shock. When the delay is long ($\tau = 1$) the IRF is nearly a shock to the stationary distribution. Accordingly, the impulse response function closely resembles the result in Figure 2a. But as the delay changes, the shape of the IRF changes. Figure 2d documents how the summary statistics change with the delay. The pattern is non-monotonic, but longer delays eventually feature larger instantaneous and cumulative effects. As the delay gets very long, the initial distribution approaches the stationary distribution, and the values converge to the $\Delta = 0.01$ results from Figure 2b.

One quantitative conclusion is that the standard approach of perturbing the stationary distribution is an extreme assumption. A shock to the stationary distribution implies larger macroeconomic effects than if the economy was previously buffeted. If shocks are small, then approximations around the stationary distribution are appropriate (Alvarez, Lippi, and Souganidis, 2023). But if some shocks are large, then applying their entire history will be necessary in quantitative work. The analytical solution in this paper will be useful to do such analysis.

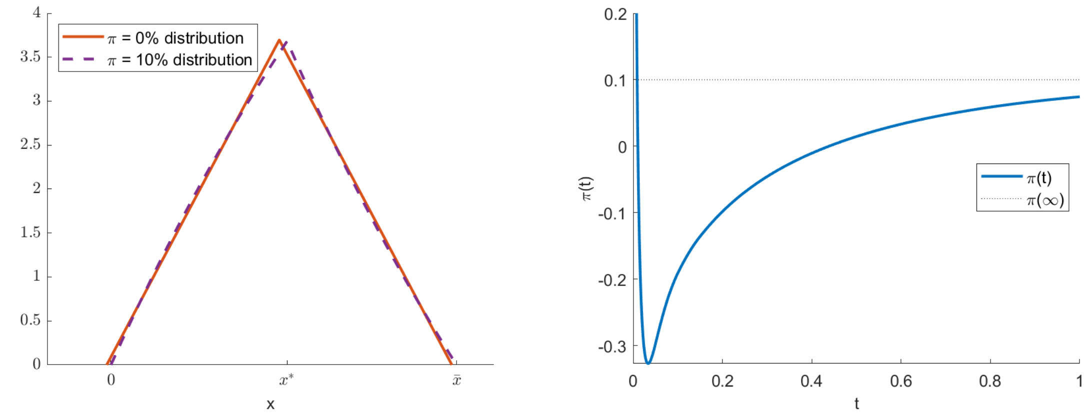
5.5 Macroeconomic Effects of Increasing Trend Inflation

A caveat of the analytical solution in this paper is that it applies to models where the inaction boundaries remain fixed. However, certain shocks can affect the boundaries in many models. In this section, I show that this paper's method can still be useful to analyze these cases.

Specifically, I study a permanent increase in the trend inflation rate $\bar{\pi}$. In the menu cost model, the trend $\bar{\pi}$ is modeled as the long-run growth rate of the money supply. With trend inflation, the Kolmogorov forward equation becomes

$$\partial_t h(x,t) = \gamma \partial_x^2 h(x,t) + \bar{\pi} \partial_x h(x,t) + \delta(x) F(t) \tag{16}$$

When $\bar{\pi}$ is higher, firms' markup gaps drift downwards over time. As a result, firms behave differently: they reset price gaps to a higher level, they increase both boundaries, and increase the width of the inaction region. As before, I follow Cavallo, Lippi, and Miyahara (2024) and disregard any general equilibrium feedback, assuming that firms adopt the new steady-state policy function.



(a) Stationary Distributions (b) Dynamic Inflation Rate

Figure 3: Effects of a Permanent Increase in Trend Inflation

Notes: The results are calculated for an annual calibration with $\gamma = 0.069$, $x^* = 0.27$, and $\bar{x} = 0.54$ when $\bar{\pi} = 0$. These values imply $\kappa = 0.006$ when $\rho = 0.05$. At time $t = 0$, trend inflation increases to $\bar{\pi} = 0.10$. The first panel plots how this changes affects the stationary distribution, and the second panel plots the response of inflation.

When the new inaction region is normalized with lower bound at zero, the $\bar{\pi} = 0$ stationary distribution has mass to the left of zero. These firms immediately change prices. Reassigning these firms to a mass point at the new x^* gives the initial condition $h(x, 0)$ for the post-shock economy. Numerically, I consider an increase in trend inflation from 0% to 10% annually. Figure 3b plots the stationary distributions before and after the shock. The change is very small, even though a 10% inflation increase is large. The inaction boundaries and reset point barely increase, and the shape of the new stationary distribution is still close to triangular.

Yet, even though the distribution is mostly unaffected, the shock causes substantial and slow-moving inflation dynamics. Because of the exponential aggregation, a small change to $h(x, t)$ can have large effects on the price level through equation (13). On impact, a small mass of firms immediately reset prices, and leave a remaining mass of firms very close to the lower boundary after $t = 0$. This is why there is an initial spike of inflation in the interval immediately after the shock. Then, inflation slows and falls below zero, the as the newly reset group of firms has space to drift downwards. Over time, the inflation rate slowly rises to the new 10% trend as the economy converges to the new stationary distribution.

This contrast between the small change to the stationary distribution and the large transition dynamics highlight one value of the analytical solution: it gives the exact IRF for macroeconomic aggregates. And because the inflation IRF features both positive and negative regions, the large and slow-moving response is partially masked when considering only the CIR, which was previously the limit of the literature's analytical characterization of dynamics in fixed cost models.

6 Conclusion

This paper presented the analytical solution to the fixed cost model's dynamic distribution. This solution is valuable for understanding the macroeconomics of fixed costs. It allows for theoretical characterization of the dynamic effects of aggregate shocks on macroeconomic variables, and provides an analytical shortcut when computing quantitative results.

While the simple fixed cost model applies to a variety of economic settings, the method will apply more generally. Whenever a KFE with endogenous resets needs solving – be it with additional state variables, complementarities, aggregate forcing terms, or other features – this paper's solution derivation provides a way forward. Use the Fourier representation and Laplace transforms to find the endogenous reset flow analytically, then use textbook PDE solutions to solve for the distribution.

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A Aggregating Function θ_n^Z Series

Proposition 1 gives the θ_n^Z sequence for each of the aggregating function examples described in Section 4.3. These series can be used with Theorem 2 to derive aggregate dynamics.

Proposition 1. *For a function $f_Z(x)$, the integrating sequence is*

$$\theta_n^Z = \int_0^{\bar{x}} f_Z(x)\sqrt{\frac{2}{\bar{x}}} e^{\frac{\mu}{2\gamma}x} \sin\left(\frac{\pi n x}{\bar{x}}\right) dx$$

Let $\chi = \sqrt{\frac{2}{\bar{x}}}$, $\zeta = \frac{\mu}{2\gamma}$, and $\omega_n = \zeta^2 + \left(\frac{\pi n}{\bar{x}}\right)^2$. Then:

1. *For the exponential case $f_Z(x) = e^{\psi x}$:*

$$\theta_n^Z = \frac{\chi \pi n \left(1 - (-1)^n e^{(\psi+\zeta)\bar{x}}\right)}{\bar{x}\left((\psi+\zeta)^2 + (\frac{\pi n}{\bar{x}})^2\right)}$$

2. *For the linear case $f_Z(x) = x$:*

$$\theta_n^Z = \chi \frac{\pi n}{\omega_n}\left[\frac{2\zeta}{\bar{x}\omega_n}\left((-1)^n e^{\zeta \bar{x}} - 1\right) - (-1)^n e^{\zeta\bar{x}}\right]$$

3. *For the quadratic case $f_Z(x) = x^2$:*

$$\theta_n^Z = \chi \frac{\pi n}{\bar{x}\omega_n^3}\left[(-1)^n e^{\zeta\bar{x}}\left(-\omega_n^2\bar{x}^2 + 4\zeta\omega_n\bar{x} - 2\left(3\zeta^2 - \left(\frac{\pi n}{\bar{x}}\right)^2\right)\right) + 2\left(3\zeta^2 - \left(\frac{\pi n}{\bar{x}}\right)^2\right)\right]$$

Proof. To compute these expressions, with the simplified notation and definition of $X_n(x)$, the desired sequences are given by:

$$\theta_n^Z = \int_0^{\bar{x}} f_Z(x)\chi e^{\zeta x} \sin\left(\frac{\pi n x}{\bar{x}}\right) dx$$

The following results are standard for exponential-sine integrals:

$$\int e^{px}\sin(qx)dx = \frac{e^{px}}{p^2+q^2}[p\sin(qx) - q\cos(qx)] \tag{17}$$

$$\begin{aligned}\int xe^{px}\sin(qx)dx = \frac{e^{px}}{(p^2+q^2)^2}\Big[&\sin(qx)\Big(p(p^2+q^2)x - (p^2-q^2)\Big)\\ &+\cos(qx)\Big(-q(p^2+q^2)x + 2pq\Big)\Big]\end{aligned} \tag{18}$$

$$\begin{aligned}\int x^2e^{px}\sin(qx)dx = \frac{e^{px}}{W^3}\Bigg[&\sin(qx)\Big(p(p^2+q^2)^2x^2 - 2(p^2-q^2)(p^2+q^2)x + 2p(p^2-3q^2)\Big)\\ &+\cos(qx)\Big(-q(p^2+q^2)^2x^2 + 4pq(p^2+q^2)x - 2q(3p^2-q^2)\Big)\Bigg]\end{aligned} \tag{19}$$

In the exponential case, $f_Z(x) = e^{\psi x}$ implies

$$\theta_n^Z = \chi\int_0^{\bar{x}} e^{(\psi+\zeta)x}\sin\left(\frac{\pi n x}{\bar{x}}\right)dx$$

Applying equation (17) with $p = \psi+\zeta$ and $q = \frac{\pi n}{\bar{x}}$, and evaluating at the limits with $\sin(\pi n) = 0$ and $\cos(\pi n) = (-1)^n$:

$$\theta_n^Z = \chi\frac{\pi n}{\bar{x}[(\psi+\zeta)^2 + (\frac{\pi n}{\bar{x}})^2]}\left(1 - (-1)^n e^{(\psi+\zeta)\bar{x}}\right)$$

In the linear case, $f_Z(x) = x$ implies

$$\theta_n^Z = \chi\int_0^{\bar{x}} xe^{\zeta x}\sin\left(\frac{\pi n x}{\bar{x}}\right)dx$$

Apply equation (18) with $p = \zeta$ and $q = \frac{\pi n}{\bar{x}}$, and evaluate at the limits:

$$\chi\int_0^{\bar{x}} xe^{\zeta x}\sin\left(\frac{\pi n x}{\bar{x}}\right)dx = \chi\frac{\pi n}{\bar{x}\omega_n^2}\left[(-1)^n e^{\zeta\bar{x}}\big(2\zeta - \omega_n\bar{x}\big) - 2\zeta\right] \tag{20}$$

In the quadratic case, $f_Z(x) = x^2$ implies

$$\theta_n^Z = \chi\int_0^{\bar{x}} x^2e^{\zeta x}\sin\left(\frac{\pi n x}{\bar{x}}\right)dx$$

Apply equation (19) and evaluate at the limits:

$$\chi \int_0^{\bar{x}} x^2 e^{\zeta x} \sin\left(\frac{\pi n x}{\bar{x}}\right) dx =$$
$$\chi \frac{\pi n}{\bar{x}\omega_n^3}\left[(-1)^n e^{\zeta \bar{x}}\left(-\omega_n^2 \bar{x}^2 + 4\zeta\omega_n \bar{x} - 2\left(3\zeta^2 - \left(\frac{\pi n}{\bar{x}}\right)^2\right)\right) + 2\left(3\zeta^2 - \left(\frac{\pi n}{\bar{x}}\right)^2\right)\right]$$

□

B The Stationary Distribution

This section derives the stationary distribution for the simple case where $\mu = \eta = 0$.

The stationary distribution $\overline{h}(x) \equiv \lim_{t\to\infty} h(x,t)$ solves the KFE with $\partial_t h(x,t) = 0$:

$$0 = \gamma \partial_x^2 \overline{h}(x) + \delta(x)\overline{F} \tag{21}$$

where $\overline{F} \equiv \lim_{t\to\infty} F(t)$ is the limiting flow. Corollary 3 gives the solution.

Corollary 3. *The stationary distribution $\overline{h}(x)$ is given by*

$$\overline{h}(x) = \begin{cases} \frac{2(x-a)}{-a(b-a)} & a \le x \le 0 \\ \frac{2(b-x)}{b(b-a)} & 0 \le x \le b \end{cases}$$

and the limiting flow is

$$\overline{F} = \frac{2}{-ab}$$

Proof. The stationary equation (21) implies that $\overline{h}(x)$ is linear for $x \neq 0$. This implies $\overline{h}(x)$ is of the form

$$\overline{h}(x) = \begin{cases} d_1 x + c_1 & a \le x < 0 \\ d_2 x + c_2 & 0 > x \le b \end{cases}$$

The continuity condition requires

$$c_1 = c_2$$

while the boundary conditions require

$$d_1 a + c_1 = 0 \qquad d_2 b + c_2 = 0$$

Combine these three equations to solve for the remaining coefficients in terms of c_1:

$$d_1 = -\frac{c_1}{a} \qquad d_2 = -\frac{c_1}{b}$$

The mass conservation condition says that $\overline{h}(x)$ must integrate to 1. $\overline{h}(x)$ is a triangle with height c_1, so the integral is given by:

$$1 = \frac{1}{2}(b-a)c_1$$

This implies that the linear terms are

$$c_1 = \frac{2}{b-a} = c_2 \qquad d_1 = -\frac{2}{(b-a)a} \qquad d_2 = -\frac{2}{(b-a)b}$$

which simplify by

$$d_1 x + c_1 = \left(-\frac{x}{a} + 1\right)\frac{2}{b-a} = \frac{2\,(x-a)}{-a(b-a)}$$

$$d_2 x + c_2 = \left(-\frac{x}{b} + 1\right)\frac{2}{b-a} = \frac{2\,(b-x)}{b(b-a)}$$

The limiting flow is

$$\begin{aligned}\overline{F} &= \partial_x \overline{h}(a) - \partial_x \overline{h}(b) \\ &= d_1 - d_2 = \frac{2}{-a(b-a)} + \frac{2}{b(b-a)} = \frac{2}{-ab}\end{aligned}$$

□

In the Section 5 model, the barriers are $b = 1$ and $a = -1$. Figure 1a plots this stationary distribution as implied by Corollary 3.

C The Price-Setting Problem in the Menu Cost Model

This appendix describes how the menu cost model ingredients determine the inaction region. For a concise description of the model's microfoundations, see Auclert, Rognlie, and Straub (2024), or Alvarez, Ferrara, Gautier, Le Bihan, and Lippi (2024) for a continuous time setting with additional features.

When firms face menu costs, their problem is to choose the critical points of the inaction region: the lower bound $\underline{x}$, the upper bound $\bar{x}$, and the optimal reset point x^*.

Inside the inaction region, the firm's value follows a Hamilton-Jacobi-Bellman equation. In the steady-state, it is given by

$$\rho v(x) = -x^2 - \bar{\pi} v'(x) + \gamma v''(x) \tag{22}$$

where $\bar{\pi}$ is the average inflation rate, 2γ is the variance of the productivity process. κ is the fixed cost of changing prices. The boundary conditions for this problem are:

- The value-matching conditions:

$$v(\underline{x}) = v(\bar{x}) = v(x^*) + \kappa$$

- The smooth-pasting and reset-optimality conditions:

$$v'(\underline{x}) = v'(\bar{x}) = v'(x^*) = 0$$

These conditions characterize the optimal choices of $\underline{x}$ $\bar{x}$, and x^*. When mapping to the standard fixed cost problem, I normalize these values so that the lower bound is at $\underline{x} = 0$.