MAE 143B Equation Sheet

P1.10 Galileo's Inclined Plane Problem

h, fall height (ft)
d, horizontal travel distance (in)*,
l, length of inclined plane (ft),
y, vertical distance*.

MATLAB commands:

Fitting linear model	f = `fit(d, h, 'poly1')`.
Fitting quadratic model	f = `fit(d, h, 'poly2')`, for `h` $= lsin(\theta)$.

Estimating y:

Conservation of energy	$\frac{1}{2}mv_x^2 = mgh$, $\Rightarrow v_x = \sqrt{2gh}$.
Vertical motion	$y = \frac{1}{2}gt^2$, $\Rightarrow t = \sqrt{2y/g}$.
Horizontal motion	$d = v_x t$, $d = \sqrt{2gh}\sqrt{2y/g}$, $\Rightarrow d^2/4 = yh$.

* units converted to feet.

P2.10 Elevator Problem

Wheels	$J_1\dot{\omega} + b_1\omega = \tau + (f_1 - f_2)r$, $J_2\dot{\omega} + b_2\omega = \tau + (f_4 - f_3)r$.
Forces from J_1	$f_1 + m_1\dot{v}_1 = m_1 g + f_3$.
Forces from J_2	$f_2 + m_2\dot{v}_2 = m_2 g + f_4$.
Angular velocity	$v_1 = r\omega, v_2 = -r\omega$.

$$(J_1+J_2+r^2(m_1+m_2))\dot{\omega}+(b_1+b_2)\omega = \tau+gr(m_1-m_2) \quad (1)$$

ODE solution:

$$v_1(t) = \tilde{v_1}(1-e^{\lambda t}) + v_1(0)e^{\lambda t}, \quad (2)$$

Furthermore:

$$\bar{v_1} = v_1(t)_{t\to\infty}, TC = -\frac{1}{\lambda}.$$

Steady state error	$e_{SS} = \bar{v_1} - v_{SS} = \bar{v_1} - \tilde{v_1}$.
Open loop	$\lambda_{OL} = -\frac{b}{a}$, $\bar{v_1} = \frac{r}{b}[\tau + gr(m_1 - m_2)]$, $TC_{OL} = -\frac{1}{\lambda_{OL}}$.
Closed loop	$\tau(t) = K(\bar{v_1} - v_1(t))$ *controller* $\lambda_{CL} = -\frac{(b+Kr)}{a}$, $\tilde{v_1} = \frac{r}{b+Kr}[K\bar{v_1} + gr(m_1 - m_2)]$, $TC_{OL} = -\frac{1}{\lambda_{OL}}$, $K = \frac{a-5b}{5r}$.

3.8 Frequency Response

Input	$u(t) = A\cos\omega t + \phi$,
Response	$y_{ss}(t) = A\lvert G(j\omega)\rvert cos(\omega t + \phi + \angle G(j\omega))$.
Phase angle	$\angle G(j\omega) = \arctan\left(\frac{\Im G(j\omega)}{\Re G(j\omega)}\right)$, $\angle G(j\omega) = \angle N\{G(j\omega)\} - \angle D\{G(j\omega)\}$.

For a LTI system without poles on the imaginary axis, i.e., $G_0(s) = 0$.

4.1 Tracking, Sensitivity and Integral Control

Lemma 4.1	Asymptotic tracking

Let $S(s)$ be the asymptotically stable transfer function $S(0) = 0$. The system asymptotic tracks a constant reference input if the product GK has a pole at the origin.

Lemma 4.3	Internal stability

The closed-loop system is internally stable if and only if $S(s)$ is asymptotically stable and any pole-zero cancellations of the product GK satisfy $\Re\{z\} < 0$.

Asymptotic stability	if $G(s)$ converges and is bounded for all $\Re(s) \geq 0$, i.e., $G(s)$ does not have poles on the imaginary axis or on the right-hand side of the complex plane.
Asymptotic tracking	if $S(s)$ has a zero at $s = 0$.
Asymptotic disturbance rejection	if $D(s)$ has a zero at $s = 0$, or the controller $K(s)$ has a pole at the origin, i.e., $s = j\omega$.
Asymptotic tracking	if $S(s)$ has a zero at $s = 0$.

4.4 Feedback with Disturbances

Sensitivity	i.e., from $\bar{v_1}$ to e $S = \frac{1}{1+GK}$
Disturbance	i.e., from w to e $D = GS = \frac{G}{1+GK}$
Output-input	i.e., from $\bar{v_1}$ to v_1 $H = GKS = \frac{GK}{1+GK}$.
Design	$Q = GS = \frac{K}{1+GK}$.

Closed-loop system	*For inputs*
$y = G(u+w)$,	$y = H\bar{y} - Hv + Dw$,
$u = K\tilde{e}$,	$u = Q\bar{y} - Qv - Hw$.
$\tilde{e} = \tilde{y} - (y+v)$.	with measurement noise v.
Tracking error	$e = \bar{y} - y = S\bar{y} + Hv - Dw$.

Laplace Transforms

$L[f](s) = \int_0^\infty e^{-sx} f(x)dx$

$f(t) = t^n, n \geq 0$	$F(s) = \frac{n!}{s^{n+1}}, s > 0$
$f(t) = e^{at}, a$ *constant*	$F(s) = \frac{1}{s-a}, s > a$
$f(t) = \sin bt, b$ *constant*	$F(s) = \frac{b}{s^2+b^2}, s > 0$
$f(t) = \cos bt, b$ *constant*	$F(s) = \frac{s}{s^2+b^2}, s > 0$
$f(t) = t^{-1/2}$	$F(s) = \frac{\pi}{s^{1/2}}, s > 0$
$f(t) = \delta(t-a)$	$F(s) = e^{-as}$
f'	$L[f'] = sL[f] - f(0)$
f''	$L[f''] = s^2L[f] - sf(0) - f'(0)$
$L[e^{at}f(t)]$	$L[f](s-a)$
$L[u_a(t)f(t-a)]$	$L[f]e^{-as}$

4.7 Pole-Zero Cancellations and Stability

Sensitivity	The poles of $S(s)$ are the zeros of the characteristic equation of $S(s)$. Furthermore, the zeros of $S(s)$ are the poles of the product GK.
GK	$\tilde{G}\tilde{K} = \frac{N_{\tilde{G}}N_{\tilde{K}}}{D_{\tilde{G}}D_{\tilde{K}}}$ which have polynomial roots $N_{\tilde{G}}N_{\tilde{K}} + D_{\tilde{G}}D_{\tilde{K}} = 0$.
S	If all the roots of $\tilde{G}\tilde{K}$ have negative real parts, then $S(s) = \frac{D_{\tilde{G}}D_{\tilde{K}}}{N_{\tilde{G}}N_{\tilde{K}}+D_{\tilde{G}}D_{\tilde{K}}}$ and $H = 1 - S$ are asymptotically stable.
SG and SK	$SG = \frac{(s-z)}{(s-p)}\frac{N_{\tilde{G}}N_{\tilde{K}}}{N_{\tilde{G}}N_{\tilde{K}}+D_{\tilde{G}}D_{\tilde{K}}}$, $SK = \frac{(s-p)}{(s-z)}\frac{N_{\tilde{K}}D_{\tilde{G}}}{N_{\tilde{G}}N_{\tilde{K}}+D_{\tilde{G}}D_{\tilde{K}}}$ are stable only if p and z are both negative.

6.1 Second-Order Systems

Characteristic equation	$s^2 + 2\zeta\omega_n s + \omega_n^2, \quad \omega_n > 0$.
Inverse Laplace Transform	$\mathscr{L}^{-1} = \left\{\frac{k}{s+\zeta\omega_n - j\omega_d} + \frac{k^*}{s+\zeta\omega_n + j\omega_d}\right\}$ $= 2\lvert k\rvert e^{-\zeta\omega_n t}\cos(\omega_d t + \angle k)$
Parameters	$\omega_d = \omega_n\sqrt{1-\zeta^2}, \quad 0 < \zeta < 1$. $\phi_d = \arcsin\zeta = \arctan\left(\frac{\zeta\omega_n}{\omega_d}\right)$.
Step response	$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}, \quad \omega_n > 0$. $y(t) = L^{-1}\left\{\frac{G(s)}{s}\right\}$, $= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t + \pi/2 - \phi_d)$

where ζ is the *damping ratio* and ω_n is the *natural frequency*.