MAE 143B Equation Sheet

P1.10 Galileo's Inclined Plane Problem

h, fall height (ft)
d, horizontal travel distance (in)*,
l, length of inclined plane (ft),
y, vertical distance*.

MATLAB commands:

Fitting linear model	$f =$ `fit(d, h, 'poly1')`.
Fitting quadratic model	$f =$ `fit(d, h, 'poly2')`, for $\mathtt{h} = l sin(\theta)$.

Estimating y:

Conservation of energy	$\frac{1}{2}mv_x^2 = mgh$, $\Rightarrow v_x = \sqrt{2gh}$.
Vertical motion	$y = \frac{1}{2}gt^2$, $\Rightarrow t = \sqrt{2y/g}$.
Horizontal motion	$d = v_x t$, $d = \sqrt{2gh}\sqrt{2y/g}$, $\Rightarrow d^2/4 = yh$.

* units converted to feet.

P2.10 Elevator Problem

Wheels	$J_1\dot{\omega} + b_1\omega = \tau + (f_1 - f_2)r$, $J_2\dot{\omega} + b_2\omega = \tau + (f_4 - f_3)r$.
Forces from J_1	$f_1 + m_1\dot{v_1} = m_1 g + f_3$.
Forces from J_2	$f_2 + m_2\dot{v_2} = m_2 g + f_4$.
Angular velocity	$v_1 = r\omega, v_2 = -r\omega$.

$$(J_1+J_2+r^2(m_1+m_2))\dot{\omega}+(b_1+b_2)\omega = \tau+gr(m_1-m_2) \quad (1)$$

ODE solution:

$$v_1(t) = \tilde{v_1}(1-e^{\lambda t}) + v_1(0)e^{\lambda t}, \quad (2)$$

Furthermore:

$$\bar{v_1} = v_1(t)_{t\to\infty}, TC = -\frac{1}{\lambda}.$$

Steady state error	$e_{SS} = \bar{v_1} - v_{SS} = \bar{v_1} - \tilde{v_1}$.
Open loop	$\lambda_{OL} = -\frac{b}{a}$, $\bar{v_1} = \frac{r}{b}[\tau + gr(m_1 - m_2)]$, $TC_{OL} = -\frac{1}{\lambda_{OL}}$.
Closed loop	$\tau(t) = K(\bar{v_1} - v_1(t))$ *controller* $\lambda_{CL} = -\frac{(b+Kr)}{a}$, $\tilde{v_1} = \frac{r}{b+Kr}[K\bar{v_1} + gr(m_1 - m_2)]$, $TC_{OL} = -\frac{1}{\lambda_{OL}}$, $K = \frac{a-5b}{5r}$.

3.1 Laplace Transform

$L[f](s) = \int_0^\infty e^{-sx} f(x)dx$

$f(t) = t^n, n \geq 0$	$F(s) = \frac{n!}{s^{n+1}}, s > 0$
$f(t) = e^{at}, a$ *constant*	$F(s) = \frac{1}{s-a}, s > a$
$f(t) = \sin bt, b$ *constant*	$F(s) = \frac{b}{s^2+b^2}, s > 0$
$f(t) = \cos bt, b$ *constant*	$F(s) = \frac{s}{s^2+b^2}, s > 0$
$f(t) = t^{-1/2}$	$F(s) = \frac{\pi}{s^{1/2}}, s > 0$
$f(t) = \delta(t-a)$	$F(s) = e^{-as}$
f'	$L[f'] = sL[f] - f(0)$
f''	$L[f''] = s^2L[f] - sf(0) - f'(0)$
$L[e^{at}f(t)]$	$L[f](s-a)$
$L[u_a(t)f(t-a)]$	$L[f]e^{-as}$

3.8 Frequency Response

Input	$u(t) = A\cos\omega t + \phi$,
Response	$y_{ss}(t) = A\lvert G(j\omega)\rvert cos(\omega t + \phi + \angle G(j\omega))$.
Phase angle	$\angle G(j\omega) = \arctan\left(\frac{\Im G(j\omega)}{\Re G(j\omega)}\right)$, $\angle G(j\omega) = \angle N\{G(j\omega)\} - \angle D\{G(j\omega)\}$.

For a LTI system without poles on the imaginary axis, i.e., $G_0(s) = 0$.

4.1 Tracking, Sensitivity and Integral Control

Lemma 4.1 — Asymptotic tracking

Let $S(s)$ be the asymptotically stable transfer function $S(0) = 0$. The system asymptotic tracks a constant reference input if the product GK has a pole at the origin.

Lemma 4.3 — Internal stability

The closed-loop system is internally stable if and only if $S(s)$ is asymptotically stable and any pole-zero cancellations of the product GK satisfy $\Re\{z\} < 0$.

Asymptotic stability	If $G(s)$ converges and is bounded for all $\Re(s) \geq 0$, i.e., $G(s)$ does not have poles on the imaginary axis or on the right-hand side of the complex plane.
Asymptotic tracking	If $S(s)$ has a zero at $s = 0$ and the product GK has a pole at the origin.
Asymptotic disturbance rejection	If $D(s)$ has a zero at $s = 0$, or the controller $K(s)$ has a pole at the origin.
Asymptotic tracking	If $S(s)$ has a zero at $s = 0$.

4.4 Feedback with Disturbances

Sensitivity	i.e., from $\bar{v_1}$ to e $S = \frac{1}{1+GK}$
Disturbance	i.e., from w to e $D = GS = \frac{G}{1+GK}$
Complementary sensitivity	i.e., from $\bar{v_1}$ to v_1 $H = GKS = \frac{GK}{1+GK}$.
Design	$Q = KS = \frac{K}{1+GK}$.

Closed-loop system	*For inputs*
$y = G(u+w)$,	$y = H\bar{y} - Hv + Dw$,
$u = K\tilde{e}$,	$u = Q\bar{y} - Qv - Hw$.
$\tilde{e} = \bar{y} - (y+v)$.	with measurement noise v.
Tracking error	$e = \bar{y} - y = S\bar{y} + Hv - Dw$.

4.7 Pole-Zero Cancellations and Stability

Sensitivity	The poles of $S(s)$ are the zeros of the characteristic equation of $S(s)$. Furthermore, the zeros of $S(s)$ are the poles of the product GK.
GK	$\tilde{G}\tilde{K} = \frac{N_{\tilde{G}}N_{\tilde{K}}}{D_{\tilde{G}}D_{\tilde{K}}}$ which have polynomial roots $N_{\tilde{G}}N_{\tilde{K}} + D_{\tilde{G}}D_{\tilde{K}} = 0$.
S	If all the roots of $\tilde{G}\tilde{K}$ have negative real parts, then $S(s) = \frac{D_{\tilde{G}}D_{\tilde{K}}}{N_{\tilde{G}}N_{\tilde{K}} + D_{\tilde{G}}D_{\tilde{K}}}$ and $H = 1 - S$ are asymptotically stable.
SG and SK	$SG = \frac{(s-z)}{(s-p)}\frac{N_{\tilde{G}}N_{\tilde{K}}}{N_{\tilde{G}}N_{\tilde{K}} + D_{\tilde{G}}D_{\tilde{K}}}$, $SK = \frac{(s-p)}{(s-z)}\frac{N_{\tilde{K}}D_{\tilde{G}}}{N_{\tilde{G}}N_{\tilde{K}} + D_{\tilde{G}}D_{\tilde{K}}}$ are stable only if p and z are both negative.

5.2 State-Space Models

State-space form	$\dot{x} = Ax + Bu, y = Cx + Du$.
A	The state matrix of size $n\times n$, where n is the number of state variables.
B	The input matrix of size $n \times m$, where m is the number of outputs.
C	The output matrix of size $m \times n$.
D	The transition matrix of size $m\times m$.
Equilibrium points	Set partial derivatives of system equations to zero, i.e., $\frac{\partial y}{\partial x_1} = 0$.

6.1 Second-Order Systems

Characteristic equation	$s^2 + 2\zeta\omega_n s + \omega_n^2, \quad \omega_n > 0.$
Canonical form	$(1 - \omega^2/\omega_n^2 + j2\zeta\omega/\omega_n)^k, \quad \lvert\zeta\rvert < 1.$
Poles	$p = -\omega_n(\zeta \pm \sqrt{(\zeta^2-1)}), \Re\{p\} = -\omega_n, \quad \lvert p\rvert = \omega_n \left\lvert -\zeta \pm j\sqrt{1-\zeta^2}\right\rvert$
Inverse Laplace Transform	$\mathscr{L}^{-1} = \left\{\frac{k}{s+\zeta\omega_n - j\omega_d} + \frac{k^*}{s+\zeta\omega_n + j\omega_d}\right\} = 2\lvert k\rvert e^{-\zeta\omega_n t}\cos(w_d t + \angle k)$
Parameters	$\omega_d = \omega_n\sqrt{1-\zeta^2}, \quad 0 < \zeta < 1.$ $\phi_d = \arcsin\zeta = \arctan\left(\frac{\zeta\omega_n}{\omega_d}\right).$
Step response	$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}, \quad \omega_n > 0.$ $y(t) = L^{-1}\left\{\frac{G(s)}{s}\right\},$ $= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t + \pi/2 - \phi_d)$

where ζ is the *damping ratio* and ω_n is the *natural frequency.*

6.4 Root-Locus

Characteristic equation	$1 + \alpha L(s), \quad \alpha \geq 0,$ $L(s) = G(s)K(s)F(s).$
Branches	Equal to the order of the characteristic equation, i.e., the denominator of the loop gain transfer function $L(s)$.
Asymptotes	$N = n - m$, asymptotes/closed-loop poles for n poles and m zeros in the loop-gain transfer function. The asymptotes intersect the real axis at the point $c = \frac{\sum_{k=1}^n p_k - \sum_{i=1}^m z_i}{n-m}$ with angle $\angle\phi_k = \frac{\pi+2\pi k}{n-m}$ for $k = 0, ..., n-m-1$.
Root-locus on $\Re$	A point s on the real axis will only be on the root-locus if it is to the left of an odd number of poles and zeros.
Break-away points	$\text{CE} + k = 0, \quad k = -\text{CE}.$ s is a break-away point if $\frac{dk}{ds} = 0$ is real and positive.
Angle of departure	Root-locus leaves a complex pole p_j at $\theta_d = 180° + \sum\angle(p_j - z_i) - \sum\angle p_j - p_i$.
Angle of arrival	Root-locus arrives a complex zero z_j at $\theta_a = 180° - \sum\angle(z_j - z_i) - \sum\angle z_j - p_i$.

7.1 Bode Plot

Magnitude	$20\log_{10}\lvert G(j\omega)\rvert$, in units of dB.
Gain offset	$20\log_{10}\lvert G(0)\rvert$ for constant gain, $-20dB$ for pole at origin, $+20dB$ for zero at origin.
Gain margin	$GM = 0 - A_m dB$, The corresponding point on the magnitude curve at the phase crossover frequency, i.e., the vertical axis intersection point where $\angle G(j\omega) = \lvert -180°\rvert$.
Poles	$(1+\frac{s}{\tau})^k \to -20 \times k\frac{dB}{dec}$ at $\omega = \tau$ if stable, else $+20 \times k\frac{dB}{dec}$. If second-order, $(\frac{1}{(1+2\zeta/\omega_n)}j\omega + \frac{1}{\omega_n^2}(j\omega)^2) \to -40\frac{dB}{dec}$ at $\omega = \omega_n$.
Zeros	$(1+\frac{s}{\tau})^k \to +20 \times k\frac{dB}{dec}$ at $\omega = \tau$ if stable, else $-20\times k\frac{dB}{dec}$. If second-order, i.e., $(\frac{1}{(1+2\zeta/\omega_n)}j\omega + \frac{1}{\omega_n^2}(j\omega)^2) \to +40\frac{dB}{dec}$ at $\omega = \omega_n$.
Slope	Ends with $-20 \times (n-m)\frac{dB}{dec}$.
Unit gain	β, such that $\beta\lvert G(0)\rvert = 1$, i.e., substitute $s = j\omega = 0$ for poles/zeros of $G(s)$ not at the origin, and $\beta\lvert G(j1) = 1$, i.e., $s = j1$ for poles/zeros at the origin, then solve for the value of β that makes the expression of magnitude $\lvert G(j\omega)\rvert$ true.
Phase	$\angle G(j\omega)$, in units of *degrees.*
Gain offset	$+180°$ if negative, $+90°$ for each pole, $-90°$ zero at origin.
Phase margin	$PM = \phi_m - (-180°)$, the corresponding point on the phase curve at the gain crossover frequency, i.e., the vertical axis intersection point where $\lvert G(j\omega)\rvert = 0dB$.
Poles	$(1+\frac{s}{\tau})^k \to -45° \times k$ at $\omega = 0.1\tau$ if stable, else $+45° \times k$. If second-order, i.e., $\left(\frac{1}{(1+2\zeta/\omega_n)}j\omega + \frac{1}{\omega_n^2}(j\omega)^2\right) \to -\tan^{-1}\left(\frac{\frac{2\zeta\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)\frac{dB}{dec}$ at $\omega = 0.1\omega_n$.
Zeros	$(1+\frac{s}{\tau})^k = +45° \times k$ at $\omega = 0.1\tau$ if stable, else $-45° \times k$. If second-order, i.e., $\left(\frac{1}{(1+2\zeta/\omega_n)}j\omega + \frac{1}{\omega_n^2}(j\omega)^2\right) \to +\tan^{-1}\left(\frac{\frac{2\zeta\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)\frac{dB}{dec}$ at $\omega = 0.1\omega_n$.
Slope	ends flat at $\angle 90° \times (m_- + m_0 + n_+) - 90° \times (m_+ + n_- + n_0)$,

where m_+/n_+ are number of unstable poles/zeros, m_-/n_- stable poles/zeros, m_0/n_0 poles/zeros on imaginary axis.

7.6 Nyquist Stability Criterion

The curve of $G(j\omega)$ parameterized by the value $\omega \in \Re$ describes the polar plot of the system, i.e., the phase and magnitude on the real and complex plane. The polar plot has a contour Γ covering the entire right-hand side of the complex plane from the limit $\rho \to \infty$.

Theorem 7.1	If a function f is analytical inside and on the the positively oriented simple closed contour C except at a finite number of poles inside C and f has no zeros on C then $\frac{1}{2\pi}\Delta_C^0 \arg f(s) = \frac{1}{2\pi j}\int_C \frac{f'(s)}{f(s)}ds = Z_C - P_c$, where Z_C is the number of zeros, P_C poles inside contour C counting their multiplicities.
Theorem 7.2	Assume that a transfer function $L(s)$ has no poles on the imaginary axis. For any given $\alpha > 0$ the loop transfer function in negative feedback with static gain controller α is asymptotically stable if and only if the number of counter-clockwise encirclements of the image contour Γ around the point $-1/\alpha$ is equal to the number of poles of $L(s)$ on the right-hand side of the complex plane.
Closed-loop stability	The closed-loop system $L(s)$ is asymptotically stable if and only if $Z_\Gamma = P_\Gamma - \frac{1}{2\pi}\Delta_\Gamma^{-\frac{1}{\alpha}}\arg L(s) = 0.$

where Z_Γ is the number of closed-loop poles and P_Γ the number of open-loop poles of $L(s)$ inside the contour Γ, i.e., on the RHP.

Complex Numbers

For a complex number in the form $a + jb$,

Amplitude	$\lvert A\rvert = \sqrt{a^2 - (jb)^2} = \sqrt{a^2+b^2}$, where $j^2 = -1$.
Phase	$\phi = \arctan(\frac{b}{a})$, where $\arctan(0) = 0°, \arctan(1) = 45°, \arctan(\infty) = 90°$.