Chapter 1

Analysis

Analysis is the study of infinite processes. This includes numbers, variables, functions, infinities, vectors, and calculus.

1.1 Pre-Calculus

This section covers the basic building blocks of calculus through common defintions and theorems.

1.1.1 Common Definitions

Notations and values frequently referenced throughout calculus.

Absolute Value

$$\begin{aligned} |x| &= \begin{cases} x & : \ x \geq 0 \\ -x & : \ x < 0 \end{cases} \\ &= \sqrt{x^2} \end{aligned}$$

Floor and Ceiling Functions

$$\begin{aligned} \lfloor x \rfloor &= \text{greatest integer less than or equal to x} \\ \lceil x \rceil &= \text{smallest integer greater than or equal to x} \end{aligned}$$

Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x}$$
$$\text{csch } x = \frac{1}{\sinh x}$$
$$\text{sech } x = \frac{1}{\cosh x}$$
$$\coth x = \frac{\cosh x}{\sinh x}$$

$$\sinh(-x) = -\sinh x$$
$$\cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1$$
$$\tanh^2 x + \text{sech}^2\ x = 1$$

Extrema

Absolute Maximum x_0 $(\forall x \in D)(f(x_0) \geq f(x))$
Absolute Minimum x_0 $(\forall x \in D)(f(x_0) \leq f(x))$
Relative Maximum x_0 $(\exists \varepsilon > 0)(\forall x \in D)(x_0 - x < \varepsilon \implies f(x_0) \geq f(x))$
Relative Minimum x_0 $(\exists \varepsilon > 0)(\forall x \in D)(x_0 - x < \varepsilon \implies f(x_0) \leq f(x))$

1.1.2 Trigonometric Identities

Essential in many advanced anti-derivatives and useful for complex analysis.

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$

Double-Angle Formulae

$$\sin(2x) = 2 \sin x \cos x$$
$$\cos(2x) = \cos^2 x - \sin^2 x$$

Inverse Trigonometric Intervals

$$
\begin{aligned}
\sin^{-1}(\sin x) &= x \text{ for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\
\cos^{-1}(\cos x) &= x \text{ for } 0 \leq x \leq \pi \\
\tan^{-1}(\tan x) &= x \text{ for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}
\end{aligned}
$$

Euler's and de Moivre's Formulae

$$
\begin{aligned}
e^{ix} &= \cos x + i \sin x \\
\cos nx + i \sin nx &= (\cos x + i \sin x)^n
\end{aligned}
$$

1.1.3 Polar Coordinates

When each point on a plain is defined by a distance and an angle.

Relation to Cartesian Coordinate System

$$
\begin{aligned}
x &= r \cos \varphi \\
y &= r \cos \varphi \\
r &= \sqrt{x^2 + y^2} \\
\varphi &= \begin{cases} \arccos(\frac{x}{r}) & \text{if } y \geq 0 \\ -\arccos(\frac{x}{r}) & \text{if } y < 0 \end{cases}
\end{aligned}
$$

1.2 Single Variable Calculus

1.2.1 Limits and Continuity

Definition of a Finite Limit

$\lim_{x \to a} f(x) = L$ (The limit of f, as x approaches a, is L.) OR
$f(x) \to L$ as $x \to a$ (f tends to L as x tends to a.)

$$(\forall \varepsilon > 0)(\exists \delta > 0)(0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon)$$

Definition of an Infinite Limit

$$
\begin{aligned}
&\lim_{x \to a} f(x) = \infty \\
&(\forall M > 0)(\exists \delta > 0)(0 < |x - a| < \delta \implies f(x) > M)
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{x \to a} f(x) = -\infty \\
&(\forall N < 0)(\exists \delta > 0)(0 < |x - a| < \delta \implies f(x) < N)
\end{aligned}
$$

Definition of a Limit at Infinity

$$\begin{aligned}
&\lim_{x\to\infty} f(x) = L \\
&(\forall\varepsilon > 0)(\exists n)(x > n \implies |f(x) - L| < \varepsilon)
\end{aligned}$$

One-Sided Limits

$\lim_{x\to a^+} f(x) = L$ (The limit of f as x approaches a from above is L.)

$(\forall\varepsilon > 0)(\exists\delta > 0)(0 < x - a < \delta \implies |f(x) - L| < \varepsilon)$

$\lim_{x\to a^-} f(x) = L$ (The limit of f as x approaches a from below is L.)

$(\forall\varepsilon > 0)(\exists\delta > 0)(0 < a - x < \delta \implies |f(x) - L| < \varepsilon)$

$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$ if and only if $\lim_{x\to a} f(x) = L$

Limit Properties

$$\begin{aligned}
&\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x) \\
&\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) \\
&\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) \\
&\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \\
&\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n
\end{aligned}$$

$$\lim_{x\to a} g(x) = L \land \lim_{x\to L} f(x) = f(L) \implies \lim_{x\to a}[f(g(x))] = f(\lim_{x\to a} g(x))$$

Continuity

A function f is continuous at a if $\lim_{x\to a} f(x) = f(a)$

A function f is continuous over an interval A if $(\forall a \in A)(\lim_{x\to a} f(x) = f(a))$

Discontinuity

A discontinuity on f at a is removable if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

A discontinuity on f at a is a jump if $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$

A discontinuity on f at a is infinite if $\lim_{x\to a} f(x) = \pm\infty$

Asymptotes

$$\text{Horizontal Asymptote } \lim_{x\to\infty} f(x) = b \implies y_a = b$$
$$\text{Oblique Asymptote } \lim_{x\to\infty} [f(x) - (mx+b)] = 0 \implies y_a = mx + b$$

Euler's Number

$$e = \lim_{n\to 0} \left(1 + \frac{1}{n}\right)^n$$
$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

Squeeze Theorem

$$(f(x) \leq g(x) \leq h(x) \wedge \lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L) \implies \lim_{x\to a} g(x) = L$$

Intermediate Value Theorem

$$f \text{ continuous on } [a,b] \implies (\exists c \in [a,b])(f(c) \in \pm[f(a), f(b)])$$

1.2.2 Differential Calculus

Definition of Derivative

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x) - f(h)}{x - h} \end{aligned}$$

Differentiability

A function f is differentiable at a if $f'(a) \in \mathbb{R}$

A function f is differentiable over an interval $A = (a,b)$ if $(\forall a \in A)(f'(a) \in \mathbb{R})$

$$f'(a) \in \mathbb{R} \implies \lim_{x\to a} f(x) = f(a)$$

Derivative Properties

$$\frac{d}{dx} c = 0$$
$$\frac{d}{dx} cf(x) = c\frac{df}{dx}$$
$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{df}{dx} \pm \frac{dg}{dx}$$

Power Rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

Product Rule

$$\frac{d}{dx}fg = f'g + g'f$$

Quotient Rule

$$\frac{d}{dx}\frac{f}{g} = \frac{f'g - g'f}{g^2}$$

Chain Rule

$$\frac{d}{dx}f(g) = f'(g)g'$$

Trigonometric Derivatives

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}\csc x = -\csc x \cot x$$
$$\frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\cot x = -\csc^2 x$$

Inverse Trigonometric Derivatives

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1} x = \frac{1}{x^2+1}$$
$$\frac{d}{dx}\csc^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}\cot^{-1} x = -\frac{1}{x^2+1}$$

Hyperbolic Derivatives

$$\begin{aligned}
\frac{d}{dx}\sinh x &= \cosh x \\
\frac{d}{dx}\cosh x &= \sin x \\
\frac{d}{dx}\tanh x &= \operatorname{sech}^2 x \\
\frac{d}{dx}\text{csch } x &= -\text{csch } x \coth x \\
\frac{d}{dx}\text{sech } x &= -\text{sech } x \tanh x \\
\frac{d}{dx}\coth x &= -\operatorname{csch}^2 x
\end{aligned}$$

Common Derivatives

$$\begin{aligned}
\frac{d}{dx}e^x &= e^x \\
\frac{d}{dx}b^x &= b^x \ln x \\
\frac{d}{dx}\ln x &= \frac{1}{x} \\
\frac{d}{dx}\ln|x| &= \frac{1}{x} \\
\frac{d}{dx}\log_a x &= \frac{1}{x \ln a} \\
\frac{d}{dx}|x| &= \frac{|x|}{x}
\end{aligned}$$

Logarithmic Differentiation

$$\begin{aligned}
y &= f(x) \\
\ln y &= \ln(f(x)) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}\ln(f(x)) \\
\frac{dy}{dx} &= f(x)\frac{d}{dx}\ln(f(x))
\end{aligned}$$

Linear Approximation

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$
$$L \text{ represents the linearization of } f \text{ at } a$$

Extrema

Extreme Value Theorem f is continuous on $[a,b] \implies f$ attains max and min
Fermat's Theorem $f(c \in (a,b))$ is a local extrema.$f'(c) \in \mathbb{R} \implies f'(c) = 0$
Critical Point $f(c \in (a,b))$ s.t. $f'(c) = 0 \vee f'(c) \notin \mathbb{R}$

Graphical Analysis

$$\text{Rolle's Theorem } f \text{ is continuous on } [a,b], \text{ differentiable on } (a,b)$$
$$\text{and } f(a) = f(b) \implies \exists c \in (a,b) f'(c) = 0$$
$$\text{Mean Value Theorem } f \text{ is continuous on } [a,b], \text{ differentiable on } (a,b)$$
$$\implies \exists c \in (a,b) f'(c) = \frac{f(b) - f(a)}{b - a}$$

Exponential Growth and Decay

$$\text{Law of Natural Growth } \frac{dy}{dt} = ky \text{ and } k > 0$$
$$\text{Law of Natural Decay } \frac{dy}{dt} = ky \text{ and } k < 0$$
$$\text{Solution } y(t) = y(0)e^{kt}$$

Monotonicity

$$f'(x) > 0 \implies f \text{ is increasing}$$
$$f'(x) < 0 \implies f \text{ is decreasing}$$

First Derivative Test

$$f' \text{ changes from positive to negative at } c \implies f(c) \text{ is local max}$$
$$f' \text{ changes from negative to positive at } c \implies f(c) \text{ is local min}$$

Concavity

$$f''(x) > 0 \implies f \text{ is concave up}$$
$$f''(x) < 0 \implies f \text{ is concave down}$$

Inflection point p on f s.t. f has a change in concavity and is continuous at p

Second Derivative Test

$$f'(c) = 0 \land f'' > 0 \implies f(c) \text{ is local min}$$
$$f'(c) = 0 \land f'' < 0 \implies f(c) \text{ is local max}$$

L'Hospital's Rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \text{ Indeterminate form } \frac{0}{0} \text{ or } \frac{\infty}{\infty} \implies \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Newton-Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

1.2.3 Integral Calculus

Definition of Antiderivative

$$(\forall x \in (a, b))(F'(x) = f(x) \implies \text{ F is the antiderivative of f})$$

Riemann Sums

$$\Delta x = \frac{b - a}{n} \text{ and } x_i = a + i\Delta x$$

$$\text{Right } \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Underestimate if decreasing, overestimate if increasing

$$\text{Left } \lim_{n \to \infty} L_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

Underestimate if increasing, overestimate if decreasing

$$\text{Midpoint } \lim_{n \to \infty} M_n = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x$$

Underestimate if concave up, overestimate if concave down

$$\text{Trapezoidal } \lim_{n \to \infty} T_n = \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{f(x_i) + f(x_{i-1})}{2}\right)\Delta x = \frac{1}{2} \lim_{n \to \infty} [R_n + L_n]$$

Underestimate if concave down, overestimate if concave up

Definition of Definite Integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Or more precisely,

$$\begin{gathered}(\forall \varepsilon > 0)(\exists N \in \mathbb{Z})(\forall n > N)(\forall x_i^* \in [x_{i-1}, x_i]) \\ \left(\left|\int_a^b f(x)\,dx - \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x\right| < \varepsilon\right)\end{gathered}$$

Fundamental Theorem of Calculus

$\forall x \in [a,b]\ F(x) = \displaystyle\int_a^x f(t)\,dt$ and f is continuous on $[a,b] \implies F'(x) = f(x)$

$\forall x \in [a,b]\ F'(x) = f(x)$ and f is real on $[a,b] \implies \displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$

Integral Properties

$$\begin{aligned}
&\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \\
&\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx \\
&\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \\
&\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx
\end{aligned}$$

Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad n \neq 1$$

Common Antiderivatives

$$\int \frac{1}{x}\,dx = \ln|x| + C$$
$$\int e^x\,dx = e^x + C$$
$$\int b^x\,dx = \frac{b^x}{\ln b} + C$$
$$\int \ln x\,dx = x\ln x - x + C$$
$$\int \log_b x\,dx = x\log_b x - \frac{x}{\ln b} + C$$
$$\int |x|\,dx = \frac{x|x|}{2} + C$$

Trigonometric Antiderivatives

$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \tan x\,dx = \ln|\sec x| + C$$
$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$
$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$
$$\int \cot x\,dx = \ln|sinx| + C$$

1.3 Series

1.4 Vector Calculus

Chapter 2

Discrete Mathematics

Chapter 3

Statistics