ST 705 Linear models and variance components
Homework problem set 4

January 31, 2023

1. Monahan exercise A.34.
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3. The defining property of a projection matrix A is that $A^2 = A$ (recall the definition of the square of a matrix from your linear algebra course). Establish the following facts.
   (a) If A is a projection matrix, then all of its eigenvalues are either zero or one.
   (b) If $A \in \mathbb{R}^{p\times p}$ is a projection and symmetric (i.e., an orthogonal projection matrix), then for every vector v the projection Av is orthogonal to $v - Av$.
   (c) $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$.
   (d) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.
4. In lecture, we proved a lemma that $(X'X)^g X'$ is a generalized inverse of X.
   (a) Verify that $X(X'X)^g$ is a generalized inverse of X'.
   (b) We proved that $P_X := X(X'X)^g X'$ is the unique projection onto column(X). Is $(X'X)^g X'$ the unique generalized inverse of X?
5. Let $X = QR$ where Q has orthonormal columns. Prove that if $\operatorname{rank}(X) = \operatorname{rank}(Q)$, then $P_X = QQ'$.
6. Let $A \in \mathbb{R}^{n\times p}$.
   (a) Prove that if A^g is a generalized inverse of A (i.e., only satisfying $AA^gA = A$), then $(A^g)'$ is a generalized inverse of A'. Conclude from this fact that $P_X := X(X'X)^g X'$ is symmetric.
   (b) Prove the existence **and** uniqueness of the Moore-Penrose generalized inverse, usually denoted A^+, of A.
   (c) Show that if A has full column rank, then $A^+ = (A'A)^{-1}A'$.
   (d) Show that if A has full row rank, then $A^+ = A'(AA')^{-1}$.