ST 705 Linear models and variance components

Homework problem set 8

March 9, 2022

1. Let X be an $n \times p$ matrix with $\text{rank}(X) = r$, and C be a $(p-r) \times p$ matrix with $\text{rank}(C) = p-r$, such that $\text{col}(X') \cap \text{col}(C') = \{0\}$. Show that $C(X'X + C'C)^{-1}C' = I_{p-r}$.

2. Let X be an $n \times p$ matrix with $\text{rank}(X) = r$, and C be a $(p-r) \times p$ matrix with $\text{rank}(C) = p-r$, such that $\text{col}(X') \cap \text{col}(C') = \{0\}$. Show that

$$\text{rank}\begin{pmatrix} X \\ C \end{pmatrix} = p.$$

3. Assume that $Y = X\beta + U$, where X is an $n \times p$ matrix with $\text{rank}(X) = k < p$, and assume $\lambda'\beta$ is estimable.

   (a) Construct an argument to determine the rank of the matrix $\begin{pmatrix} X \\ \lambda' \end{pmatrix}$.

   (b) Construct an argument to determine the rank of the matrix $\begin{pmatrix} X \\ \lambda'(I - P_{X'}) \end{pmatrix}$.

4. Consider the restricted linear model $Y = X\beta + U$ over the constrained parameter space $\{P'\beta = \delta\}$, for some full-column rank matrix P. Set up the Langrangian function and derive the *restricted normal equations* (RNE),

$$\begin{pmatrix} X'X & P \\ P' & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \theta \end{pmatrix} = \begin{pmatrix} X'y \\ \delta \end{pmatrix}.$$

5. Prove that there exists a solution to the RNE.

6. Exercise 3.20 from Monahan.

7. Exercise 3.24 from Monahan.

8. Exercise 3.26 from Monahan.