ST 705 Linear models and variance components
Homework problem set 5

February 7, 2023

1. Let S be a nonempty subset of an inner product space V. The orthogonal complement to the set S is defined as

$$S^{\perp} := \{x \in V : \langle x, y \rangle = 0 \text{ for every } y \in S\}.$$

   (a) Show that $S^{\perp}$ is a subspace of V for any $S \subseteq V$.

   (b) Let $W \subseteq V$ be a finite dimensional subspace, and let $y \in V$. Show that there exist **unique** vectors $u \in W$ and $z \in W^{\perp}$ such that $y = u + z$.

   (c) Let $X \in \mathbb{R}^{n \times p}$. Verify that $\text{col}(X)$ and $\text{null}(X')$ are orthogonal complements.

2. Denote by W a matrix with $\text{col}(W) = \text{null}(P')$. Show that $\text{null}(W') = \text{col}(P)$.

3. Let V be a finite-dimensional inner product space over $\mathbb{C}$, and let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for V.

   (a) Show that for any $x, y \in V$,

$$\langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}.$$

   This is called Parseval's identity.

   (b) For $V = \mathbb{R}^2$ use Parseval's identity to prove the Pythagorean theorem. Generalize this result to $\mathbb{R}^n$.

4. Let V be an inner product space over $\mathbb{C}$, and let $\{v_1, \ldots, v_n\} \subset V$ be orthonormal.

   (a) Prove that for any $x \in V$,

$$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i \rangle|^2.$$

   This is called Bessel's inequality.

   (b) Show that Bessel's inequality is an equality if and only if $x \in \text{span}\{v_1, \ldots, v_n\}$.

5. Show that a $p \times p$ matrix A is symmetric and idempotent with rank s if and only if there exists a $p \times s$ matrix G with orthonormal columns such that $A = GG'$. Note that G is called a *semi-orthogonal* matrix.

6. Suppose that $v_1, \ldots, v_p \in \mathbb{R}^n$ are a set of linearly independent vectors, and $w_1, \ldots, w_p \in \mathbb{R}^n$ are the orthogonal vectors obtained from $v_1, \ldots, v_p$ by the Gram-Schmidt process. Furthermore, denote by $u_1, \ldots, u_p$ the normalized vectors corresponding to $w_1, \ldots, w_p$, and define the matrix $R \in \mathbb{R}^{p \times p}$ by

$$R_{ij} := \begin{cases} \|w_j\| & \text{if } i = j \\ \langle v_j, u_i \rangle & \text{if } i < j \\ 0 & \text{if } i > j \end{cases}.$$

Prove that $V = UR$, where V is the matrix with columns $v_1, \ldots, v_p$ and U is the matrix with columns $u_1, \ldots, u_p$.