ST 705 Linear models and variance components
Homework problem set 3

January 26, 2022

1. Let $G : \mathbb{R}^p \to \mathbb{R}$ defined by $G(\beta) := (y - X\beta)'W(y - X\beta)$. Derive an expression for $\nabla_\beta G(\beta)$.

2. Show that if $\text{rank}(BC) = \text{rank}(B)$, then $\text{col}(BC) = \text{col}(B)$, where $\text{col}(\cdot)$ denotes the column space.

3. Let $A \in \mathbb{R}^{n\times p}$ with $\text{rank}(A) = p$. Further, suppose $X \in \mathbb{R}^{n\times q}$ with $\text{col}(X) = \text{col}(A)$. Show that there exists a unique matrix S so that $X = AS$.

4. Let $X \in \mathbb{R}^{n\times p}$, $u \in \mathbb{R}^n$, and $v \in \mathbb{R}^p$.

   (a) Prove that

$$|u'Xv| \leq \left(\max_{1\leq j\leq p} \left\{ \sum_{i=1}^{n} |X_{i,j}| \right\} \right)^{\frac{1}{2}} \left(\max_{1\leq i\leq n} \left\{ \sum_{j=1}^{p} |X_{i,j}| \right\} \right)^{\frac{1}{2}} \cdot \|u\|_2 \cdot \|v\|_2.$$

   (b) Show that the Cauchy-Schwarz inequality is a special case of the inequality given in part (a).

5. Let A be an $m \times n$ matrix with rank m. Prove that there exists an $n \times m$ matrix B such that $AB = I_m$.

6. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Prove that AB can be written as a sum of n matrices of rank at most one. Hint: think about empirical covariance matrices.

7. Let S be a nonempty subset of an inner product space V. The orthogonal complement to the set S is defined as

$$S^\perp := \{x \in V : \langle x, y\rangle = 0 \text{ for every } y \in S\}.$$

   (a) Show that $S^\perp$ is a subspace of V for any $S \subseteq V$.

   (b) Let $W \subseteq V$ be a finite dimensional subspace, and let $y \in V$. Show that there exist **unique** vectors $u \in W$ and $z \in W^\perp$ such that $y = u + z$.

   (c) Let $X \in \mathbb{R}^{n\times p}$. Verify that $\text{col}(X)$ and $\text{null}(X')$ are orthogonal complements.