ST 705 Linear models and variance components
Homework problem set 2

January 14, 2025

1. Monahan problem A.50.

2. Let
$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$
Is A diagonalizable? If so, find the eigenvalues and eigenvectors of A.

3. Let $A \in \mathbb{R}^{p \times p}$ be symmetric. Use the spectral decomposition of A to show that
$$\sup_{x \in \mathbb{R}^p \setminus \{0\}} \frac{x'Ax}{x'x} = \lambda_{\max},$$
where $\lambda_{\max}$ is the largest eigenvalue of A. Observe that this is a special case of the Courant-Fischer theorem (see `https://en.wikipedia.org/wiki/Min-max_theorem`).

4. Construct an $n \times n$ matrix A such that $\lambda_{\max}(A) \neq \sup_{v \neq 0} \left\{ \frac{v'Av}{v'v} \right\}$, where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of its argument. Why does your counter example not violate the Courant-Fischer theorem?

5. For matrices $A \in \mathbb{R}^{p \times q}$, the *spectral* norm is defined as,
$$\|A\|_2 := \sqrt{\sup_{x \neq 0} \frac{x'A'Ax}{x'x}}.$$
Further, the eigenvalues of $A'A$ are the squares of the *singular values* of A, so sometimes the definition of the spectral norm is expressed as
$$\|A\|_2 := \sigma_{\max}(A),$$
where $\sigma_{\max}$ denotes the largest singular value of A.

   (a) Verify that the spectral norm is a norm. Recall that a norm must satisfy the following axioms for any $A, B, C \in \mathbb{R}^{p \times q}$ and any $\alpha \in \mathbb{R}$.

i. $\|\alpha A\| = |\alpha|\|A\|$

ii. $\|A + B\| \le \|A\| + \|B\|$

iii. $\|A\| \ge 0$ with equality if and only if $A = 0$.

(b) Show that the spectral norm is sub-multiplicative for square matrices. That is, for $A, B \in \mathbb{R}^{p\times p}$, $\|AB\|_2 \le \|A\|_2\|B\|_2$.

6. Let A be a positive definite matrix, and show that

$$\operatorname{tr}(I - A^{-1}) \le \log\det(A) \le \operatorname{tr}(A - I).$$