ST 705 Linear models and variance components
Homework problem set 7

February 20, 2024

1. Monahan exercise 3.9.

2. Monahan exercise 3.24.

3. Consider the simple linear regression model $y_i = \beta_0 + \beta_1 x_i + u_i$ for $i \in \{1, \ldots, n\}$. Show that if the x_i are equally spaced (i.e., $x_i = s + ti$ for some scalars s and t), then $y_i = \gamma_0 + \gamma_1 i + u_i$ is a reparameterization.

4. Consider the vector space, $P_3(\mathbb{R})$, of polynomials over $\mathbb{R}$ with degree at most 3, and with the inner product $\langle f, g \rangle := \int_{-1}^{1} f(t)g(t)\ dt$. Beginning with the standard basis, $\{1, x, x^2, x^3\}$, construct an orthonormal basis for $P_3(\mathbb{R})$.

5. Suppose that $v_1, \ldots, v_p \in \mathbb{R}^n$ are a set of linearly independent vectors, and $w_1, \ldots, w_p \in \mathbb{R}^n$ are the orthogonal vectors obtained from $v_1, \ldots, v_p$ by the Gram-Schmidt process. Furthermore, denote by $u_1, \ldots, u_p$ the normalized vectors corresponding to $w_1, \ldots, w_p$, and define the matrix $R \in \mathbb{R}^{p \times p}$ by

$$R_{ij} := \begin{cases} \|w_j\| & \text{if } i = j \\ \langle v_j, u_i \rangle & \text{if } i < j \\ 0 & \text{if } i > j \end{cases} .$$

   Prove that $V = UR$, where V is the matrix with columns $v_1, \ldots, v_p$ and U is the matrix with columns $u_1, \ldots, u_p$.

6. In the notation of the previous problem, suppose that $p = n$. Further, assume that $V = U_1 R_1 = U_2 R_2$, where U_1 and U_2 are orthogonal matrices and R_1 and R_2 are upper triangular. Prove that the matrix $R_2 R_1^{-1}$ is orthogonal and diagonal.