ST 705 Linear models and variance components
Homework problem set 3

January 21, 2025

1. Let V be a convex subset of some vector space. Recall that a function $f : V \to \mathbb{R}$ is said to be *convex* if for every $x, y \in V$ and every $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

   Show, by definition, that the sum of squared errors function

$$Q(\beta) := \|Y - X\beta\|_2^2$$

   is convex.

2. Let $G : \mathbb{R}^p \to \mathbb{R}$ defined by $G(\beta) := (y - X\beta)'W(y - X\beta)$. Derive an expression for $\nabla_\beta G(\beta)$.

3. Let $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{p\times q}$. Show that if $\text{rank}(BC) = \text{rank}(B)$, then $\text{col}(BC) = \text{col}(B)$, where $\text{col}(\cdot)$ denotes the column space.

4. Let $A \in \mathbb{R}^{n\times p}$ with $\text{rank}(A) = p$. Further, suppose $X \in \mathbb{R}^{n\times q}$ with $\text{col}(X) = \text{col}(A)$. Show that there exists a unique matrix S so that $X = AS$.

5. Show that a $p \times p$ matrix A is symmetric and idempotent with rank s if and only if there exists a $p \times s$ matrix G with orthonormal columns such that $A = GG'$. Note that G is called a *semi-orthogonal* matrix.

6. The defining property of a projection matrix A is that $A^2 = A$ (recall the definition of the square of a matrix from your linear algebra course). Establish the following facts.
   (a) If A is a projection matrix, then all of its eigenvalues are either zero or one.
   (b) If $A \in \mathbb{R}^{p\times p}$ is a projection and symmetric (i.e., an orthogonal projection matrix), then for every vector v the projection Av is orthogonal to $v - Av$.
   (c) $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$.
   (d) $\text{tr}(AB) = \text{tr}(BA)$.