ST 705 Linear models and variance components
Homework problem set 3

January 24, 2023

1. Let V be a convex subset of some vector space. Recall that a function $f : V \to \mathbb{R}$ is said to be *convex* if for every $x, y \in V$ and every $\lambda \in [0, 1]$,
$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$
Show, by definition, that the sum of squared errors function
$$Q(\beta) := \|Y - X\beta\|_2^2$$
is convex.

2. Show that if $\text{rank}(BC) = \text{rank}(B)$, then $\text{col}(BC) = \text{col}(B)$, where $\text{col}(\cdot)$ denotes the column space.

3. Let $A \in \mathbb{R}^{n\times p}$ with $\text{rank}(A) = p$. Further, suppose $X \in \mathbb{R}^{n\times q}$ with $\text{col}(X) = \text{col}(A)$. Show that there exists a unique matrix S so that $X = AS$.

4. Show that the R^2 value for a simple linear regression can never achieve 1 if the observed data consists of repeated (different) observations of the response, y, at the same value of the predictor, x.

5. Suppose that the $m \times n$ matrix A has the form
$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$
where A_1 is an $n \times n$ nonsingular matrix, and $m > n$. Define $A^+ := (A'A)^{-1}A'$, and prove that $\|A^+\|_2 \leq \|A_1^{-1}\|_2$.

6. Let $X \in \mathbb{R}^{n\times p}$, $u \in \mathbb{R}^n$, and $v \in \mathbb{R}^p$.

   (a) Prove that
$$|u'Xv| \leq \left(\max_{1\leq j\leq p} \left\{ \sum_{i=1}^{n} |X_{i,j}| \right\} \right)^{\frac{1}{2}} \left(\max_{1\leq i\leq n} \left\{ \sum_{j=1}^{p} |X_{i,j}| \right\} \right)^{\frac{1}{2}} \cdot \|u\|_2 \cdot \|v\|_2.$$

(b) Show that the Cauchy-Schwarz inequality is a special case of the inequality given in part (a).