ST 705 Linear models and variance components
Homework problem set 4

February 2, 2022

1. Let V be a finite-dimensional inner product space over $\mathbb{C}$, and let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for V.

   (a) Show that for any $x, y \in V$,

   $$\langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}.$$

   This is called Parseval's identity.

   (b) For $V = \mathbb{R}^2$ use Parseval's identity to prove the Pythagorean theorem. Generalize this result to $\mathbb{R}^n$.

2. Let V be an inner product space over $\mathbb{C}$, and let $\{v_1, \ldots, v_n\} \subset V$ be orthonormal.

   (a) Prove that for any $x \in V$,

   $$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i \rangle|^2.$$

   This is called Bessel's inequality.

   (b) Show that Bessel's inequality is an equality if and only if $x \in \text{span}\{v_1, \ldots, v_n\}$.

3. The defining property of a projection matrix A is that $A^2 = A$ (recall the definition of the square of a matrix from your linear algebra course). Establish the following facts.

   (a) If A is a projection matrix, then all of its eigenvalues are either zero or one.

   (b) If $A \in \mathbb{R}^{p \times p}$ is a projection and symmetric (i.e., an orthogonal projection matrix), then for every vector v the projection Av is orthogonal to $v - Av$.

   (c) $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$.

   (d) $\text{tr}(AB) = \text{tr}(BA)$.

4. Show that a $p \times p$ matrix A is symmetric and idempotent with rank s if and only if there exists a $p \times s$ matrix G with orthonormal columns such that $A = GG'$. Note that G is called a *semi-orthogonal* matrix.

5. Exercise 2.8 from Monahan.

6. Exercise 2.9 from Monahan.

7. Exercise 2.11 from Monahan.