ST 705 Linear models and variance components
Lab practice problem set 3

January 26, 2022

1. For matrices $A \in \mathbb{R}^{p \times q}$, the *spectral* norm is defined as,

$$\|A\|_2 := \sqrt{\sup_{x \neq 0} \frac{x'A'Ax}{x'x}}.$$

Further, the eigenvalues of $A'A$ are the squares of the *singular values* of A, so sometimes the definition of the spectral norm is expressed as

$$\|A\|_2 := \sigma_{\max}(A),$$

where $\sigma_{\max}$ denotes the largest singular value of A.

   (a) Verify that the spectral norm is a norm. Recall that a norm must satisfy the following axioms for any $A, B, C \in \mathbb{R}^{p \times q}$ and any $\alpha \in \mathbb{R}$.

      i. $\|\alpha A\| = |\alpha|\|A\|$

      ii. $\|A + B\| \leq \|A\| + \|B\|$

      iii. $\|A\| \geq 0$ with equality if and only if $A = 0$.

   (b) Show that the spectral norm is sub-multiplicative for square matrices. That is, for $A, B \in \mathbb{R}^{p \times p}$, $\|AB\|_2 \leq \|A\|_2\|B\|_2$.

2. Show that the R^2 value for a simple linear regression can never achieve 1 if the observed data consists of repeated (different) observations of the response, y, at the same value of the predictor, x.