ST 705 Linear models and variance components
Homework problem set 8

March 21, 2023

1. Monahan exercise 4.1
2. Monahan exercise 4.2
3. Monahan exercise 4.3
4. Monahan exercise 4.9
5. Suppose that $Y_1, \ldots, Y_n \overset{\text{iid}}{\sim} \text{Uniform}(0, 2\theta)$, and define $U_i := Y_i - \theta$ for $i \in \{1, \ldots, n\}$.
   (a) Find the mean and variance of $U := (U_1, \ldots, U_n)'$.
   (b) Show that $Y := (Y_1, \ldots, Y_n)'$ is generated according to a linear model that satisfies the Gauss-Markov assumptions.
   (c) Find the BLUE of θ, and denote the BLUE by $\hat{\theta}_{\text{OLS}}$.
   (d) Find c so that the estimator $\hat{\theta} = cY_{(n)}$ is unbiased for θ, where $Y_{(i)}$ denotes the ith order statistic, and compute the variance of $\hat{\theta}$.
   (e) Compare the variances of $\hat{\theta}_{\text{OLS}}$ and $\hat{\theta}$, and provide intuition for your finding.
6. The problem of least squares regression can be understood as a special case of the more general problem of ridge regression. For an n-dimensional column vector y and an $n \times p$ design matrix X, the problem of ridge regression is to solve for the parameter vector b that minimizes
$$a\|b\|_2^2 + \|y - Xb\|_2^2,$$
where $a \geq 0$ is fixed.
   (a) Derive a closed-form expression of the ridge regression solution.
   (b) Assume that X has full column rank, and suppose that y is an observed instance of the random vector $Y = X\beta + U$, where $\beta \in \mathbb{R}^p$ is fixed and U satisfies the Gauss-Markov assumptions. Under what condition(s) is the ridge regression solution the best linear unbiased estimator (BLUE) for any β?