ST 705 Linear models and variance components
Homework problem set 5

February 18, 2025

1. Let V be a finite-dimensional inner product space over $\mathbb{C}$, and let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for V.

   (a) Show that for any $x, y \in V$,

   $$\langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}.$$

   This is called Parseval's identity.

   (b) For $V = \mathbb{R}^2$ use Parseval's identity to prove the Pythagorean theorem. Generalize this result to $\mathbb{R}^n$.

2. Let V be an inner product space over $\mathbb{C}$, and let $\{v_1, \ldots, v_n\} \subset V$ be orthonormal.

   (a) Prove that for any $x \in V$,

   $$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i \rangle|^2.$$

   This is called Bessel's inequality.

   (b) Show that Bessel's inequality is an equality if and only if $x \in \text{span}\{v_1, \ldots, v_n\}$.

3. By hand, find an orthonormal basis of vectors for the subspace spanned by the set

   $$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

4. Write the Gram-Schmidt orthonormalization process as a computer program. Append your computer program with the upper triangular matrix in the QR decomposition, and verify that you can recover the matrix X formed by concatenating the columns in the previous problem.

5. Consider the vector space, $P_3(\mathbb{R})$, of polynomials over $\mathbb{R}$ with degree at most 3, and with the inner product $\langle f, g \rangle := \int_{-1}^{1} f(t)g(t)\ dt$. Beginning with the standard basis, $\{1, x, x^2, x^3\}$, construct an orthonormal basis for $P_3(\mathbb{R})$.

6. Suppose that $v_1, \ldots, v_p \in \mathbb{R}^n$ are a set of linearly independent vectors, and $w_1, \ldots, w_p \in \mathbb{R}^n$ are the orthogonal vectors obtained from $v_1, \ldots, v_p$ by the Gram-Schmidt process. Furthermore, denote by $u_1, \ldots, u_p$ the normalized vectors corresponding to $w_1, \ldots, w_p$, and define the matrix $R \in \mathbb{R}^{p \times p}$ by

$$R_{ij} := \begin{cases} \|w_j\| & \text{if } i = j \\ \langle v_j, u_i \rangle & \text{if } i < j \\ 0 & \text{if } i > j \end{cases}.$$

Prove that $V = UR$, where V is the matrix with columns $v_1, \ldots, v_p$ and U is the matrix with columns $u_1, \ldots, u_p$.

7. In the notation of the previous problem, suppose that $p = n$. Further, assume that $V = U_1 R_1 = U_2 R_2$, where U_1 and U_2 are orthogonal matrices and R_1 and R_2 are upper triangular. Prove that the matrix $R_2 R_1^{-1}$ is orthogonal and diagonal.