ST 705 Linear models and variance components
Homework problem set 6

February 16, 2022

1. Let $A \in \mathbb{R}^{n\times p}$.

   (a) Prove the existence **and** uniqueness of the Moore-Penrose generalized inverse, usually denoted A^+, of A.

   (b) Show that if A has full column rank, then $A^+ = (A'A)^{-1}A'$.

   (c) Show that if A has full row rank, then $A^+ = A'(AA')^{-1}$.

2. Let $X \in \mathbb{R}^{n\times p}$ with full column rank, $\Phi \in \mathbb{R}^{p\times m}$, and $U \in \mathbb{R}^{p\times p}$ be positive definite. Show that,

$$\|\Phi' X'(P_X - X(X'X + U)^{-1}X')^2 X\Phi\| \le \|\Phi' X'(P_X - X(X'X + U)^{-1}X')X\Phi\|,$$

   where $P_X = X(X'X)^{-1}X'$.

3. Suppose that $v_1, \ldots, v_p \in \mathbb{R}^n$ are a set of linearly independent vectors, and $w_1, \ldots, w_p \in \mathbb{R}^n$ are the orthogonal vectors obtained from $v_1, \ldots, v_p$ by the Gram-Schmidt process. Furthermore, denote by $u_1, \ldots, u_p$ the normalized vectors corresponding to $w_1, \ldots, w_p$, and define the matrix $R \in \mathbb{R}^{p\times p}$ by

$$R_{ij} := \begin{cases} \|w_j\| & \text{if } i = j \\ \langle v_j, u_i\rangle & \text{if } i < j \\ 0 & \text{if } i > j \end{cases}.$$

   Prove that $V = UR$, where V is the matrix with columns $v_1, \ldots, v_p$ and U is the matrix with columns $u_1, \ldots, u_p$.

4. In the notation of the previous problem, suppose that $p = n$. Further, assume that $V = U_1R_1 = U_2R_2$, where U_1 and U_2 are orthogonal matrices and R_1 and R_2 are upper triangular. Prove that the matrix $R_2R_1^{-1}$ is orthogonal and diagonal.

5. Exercise 2.22 from Monahan.

6. Exercise 2.23 from Monahan.
7. Exercise 2.24 from Monahan.