ST 705 Linear models and variance components
Homework problem set 2

January 17, 2023

1. Monahan exercise A.50.

2. Let $A \in \mathbb{R}^{p \times p}$ be symmetric. Use the spectral decomposition of A to show that

$$\sup_{x \in \mathbb{R}^p \setminus \{0\}} \frac{x' A x}{x' x} = \lambda_{\max},$$

   where $\lambda_{\max}$ is the largest eigenvalue of A. Observe that this is a special case of the Courant-Fischer theorem (see `https://en.wikipedia.org/wiki/Min-max_theorem`).

3. Construct an $n \times n$ matrix A such that $\lambda_{\max}(A) \neq \sup_{v \neq 0} \left\{ \frac{v' A v}{v' v} \right\}$, where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of its argument. Why does your counter example not violate the Courant-Fischer theorem?

4. Let A be a positive definite matrix, and show that

$$\operatorname{tr}(I - A^{-1}) \leq \log \det(A) \leq \operatorname{tr}(A - I).$$

5. Let A be an $m \times n$ matrix with rank m. Prove that there exists an $n \times m$ matrix B such that $AB = I_m$.

6. For matrices $A \in \mathbb{R}^{p \times q}$, the *spectral* norm is defined as,

$$\|A\|_2 := \sqrt{\sup_{x \neq 0} \frac{x' A' A x}{x' x}}.$$

   Further, the eigenvalues of $A'A$ are the squares of the *singular values* of A, so sometimes the definition of the spectral norm is expressed as

$$\|A\|_2 := \sigma_{\max}(A),$$

   where $\sigma_{\max}$ denotes the largest singular value of A.

(a) Verify that the spectral norm is a norm. Recall that a norm must satisfy the following axioms for any $A, B, C \in \mathbb{R}^{p \times q}$ and any $\alpha \in \mathbb{R}$.

i. $\|\alpha A\| = |\alpha| \|A\|$

ii. $\|A + B\| \leq \|A\| + \|B\|$

iii. $\|A\| \geq 0$ with equality if and only if $A = 0$.

(b) Show that the spectral norm is sub-multiplicative for square matrices. That is, for $A, B \in \mathbb{R}^{p \times p}$, $\|AB\|_2 \leq \|A\|_2 \|B\|_2$.