ST 705 Linear models and variance components
Homework problem set 7

March 3, 2022

1. In the simple linear regression model $y_i = \beta_0 + x_i\beta_1 + u_i$ for $i \in \{1, \ldots, n\}$, show that β_0 is estimable **by finding** a vector a and scalar c such that $E(c + a'y) = \beta_0$.

2. Prove that if $\lambda^{(1)\prime}\beta, \ldots, \lambda^{(k)\prime}\beta$ are estimable, then so is
$$\sum_{j=1}^{k} d_j \lambda^{(j)\prime}\beta,$$
for any scalar constants $d_1, \ldots, d_k$.

3. Show that if the least squares estimator $\lambda'\widehat{\beta}$ is the same for all solutions $\widehat{\beta}$ to the normal equations, then $\lambda'\beta$ is estimable.

4. Give an example using the one-way ANOVA model from Section 3.4 to show that if $\lambda'\beta$ is not estimable, then $\lambda'\widehat{\beta}$ is not unbiased.

5. Consider the model $Y_{ij} = \mu + \alpha_i + \beta_i x_{ij} + U_{ij}$, for $i \in \{1, \ldots, n\}$ and $j \in \{1 \ldots, m\}$. Further, assume that $\sum_{j=1}^{m}(x_{ij} - \bar{x}_{i\cdot})^2 > 0$ for all $i \in \{1, \ldots, n\}$. Derive the necessary and sufficient conditions for an estimable function $\lambda'\gamma$, where $\gamma := (\mu, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)'$.

6. Consider the model $Y_{ijk} = \mu + \alpha_i + \beta_j + \theta_{ij} + U_{ijk}$, with $k \in \{1, \ldots, m_{ij}\}$, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$, and $E(U_{ijk}) = 0$. Find necessary and sufficient conditions for which $\lambda'\gamma$ is estimable for
$$\gamma = \begin{pmatrix} \mu & \alpha_1 & \cdots & \alpha_n & \beta_1 & \cdots & \beta_m & \theta_{11} & \cdots & \theta_{nm} \end{pmatrix}'.$$

7. Exercise 3.6 from Monahan.

8. Exercise 3.7 from Monahan.