ST 705 MIDTERM

March 2, 2022

NAME:

STUDENT ID:

- You have **75 minutes** to complete this exam.
- This is a **closed book, closed notes** exam.

1. (3 points) Let $X \in \mathbb{R}^{n \times p}$ and $u \in \text{col}(X)$, where $\text{col}(\cdot)$ denotes the column space of a matrix argument. Show that

$$\{\beta : X\beta = u\} = \{\beta : \beta = (X'X)^g X'u + (I_p - (X'X)^g X'X)z \text{ for some } z \in \mathbb{R}^p\}.$$

2. (3 points) Consider the simple linear regression model $y_i = \beta_0 + \beta_1 x_i + u_i$ for $i \in \{1, \ldots, n\}$. Show that if the x_i are equally spaced (i.e., $x_i = s + ti$ for some scalars s and t), then $y_i = \gamma_0 + \gamma_1 i + u_i$ is a reparameterization.

3. Let Σ be a $p \times p$ symmetric non-negative definite matrix, and consider the partition

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

where Σ_{11} is $k \times k$ and Σ_{22} is $(p-k) \times (p-k)$. Prove the following statements.

   (a) (3 points) $\text{null}(\Sigma_{22}) \subseteq \text{null}(\Sigma_{12})$, where $\text{null}(\cdot)$ denotes the null space of a matrix argument.

   (b) (3 points) $\text{col}(\Sigma_{21}) \subseteq \text{col}(\Sigma_{22})$, where $\text{col}(\cdot)$ denotes the column space of a matrix argument.

4. Define $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $f(x, y) := \|x\|_2 \|y\|_2 \cos(\theta)$, where θ is the angle between x and y.

   (a) (3 points) Show that $f(\cdot, \cdot)$ defines an inner product over $\mathbb{R}^n$.

   (b) (3 points) Assuming $n = 2$, draw a picture to verify that $f(x, e_i) = x_i$ for $i \in \{1, 2\}$, where e_i is the i-th standard basis vector (WLOG it suffices to only consider $i = 1$).

   (c) (3 points) Assuming that the relation in (b) holds for arbitrary n, prove that $f(x, y) = x'y$ (i.e., prove that $f(x, y)$ is the Euclidean dot product).

   (d) (3 points) Using the result of part (c), provide an explicit proof of the Cauchy-Schwarz inequality for the Euclidean dot product.