ST 705 Linear models and variance components
Homework problem set 10

April 8, 2025

1. Monahan problem 5.11.
2. Monahan problem 5.12.
3. Monahan problem 5.14.
4. Monahan problem 5.19.
5. Construct two random variables that have zero correlation, but are *not* independent.
6. Let U and V be independent $\mathrm{N}(0,1)$ random variables, and define $Y := V$ and
$$X := \begin{cases} U & \text{if } UV \geq 0 \\ -U & \text{if } UV < 0 \end{cases}$$
   (a) Show that X and Y each follow the standard normal distribution, but that (X, Y) is not bivariate normal.
   (b) Show that X^2 and Y^2 are independent.
7. Let $X \sim \mathrm{N}_p(\mu, \Sigma)$. Show that for any partition of components, i.e.,
$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix}, \qquad \mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \Sigma_{11} & \cdots & \Sigma_{1m} \\ \vdots & \ddots & \vdots \\ \Sigma_{m1} & \cdots & \Sigma_{mm} \end{pmatrix},$$
   $X_1, \ldots, X_m$ are mutually independent if and only if $\Sigma_{ij} = 0$ for every $i \neq j$.
8. Recall the definition of the multivariate normal distribution from class:

   Definition 1 *The p-dimensional random vector Y is said to follow the multivariate normal distribution with mean μ and covariance matrix Σ if for every p-dimensional vector v such that $v'\Sigma v \neq 0$,*
$$v'Y \sim N(v'\mu, v'\Sigma v).$$
   Denote $Y \sim N_p(\mu, \Sigma)$. ■

Prove that if Σ is nonsingular, then $Y \sim \mathrm{N}_p(\mu, \Sigma)$ if and only if Y has density,

$$f(y) = \det(2\pi\Sigma)^{-\frac{1}{2}} e^{-\frac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)}.$$