Metropolis-Hastings algorithm: simple example

Suppose $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Exp}(\lambda)$. Then the likelihood function is

$$\ell(x_1, \ldots, x_n|\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}.$$

Assume the prior $\lambda \sim \text{Gamma}(\lambda \mid a, b)$.

The posterior density is then given by

$$\begin{aligned}
\pi(\lambda|x_1, \ldots, x_n) &= \frac{\ell(x_1, \ldots, x_n|\lambda) \cdot \pi(\lambda)}{\int \ell(x_1, \ldots, x_n|\lambda) \cdot \pi(\lambda)\ d\lambda} \\
&\propto \underbrace{\ell(x_1, \ldots, x_n|\lambda) \cdot \pi(\lambda)}_{=:\ f(\lambda)}.
\end{aligned}$$

Outline of a random walk Metropolis-Hastings algorithm:

Step 1. Given current $\lambda^{(t)}$, propose a new $\lambda^\star \sim \mathsf{N}(\cdot|\lambda^{(t)}, \sigma^2)$

Step 2. Set

$$\lambda^{(t+1)} = \begin{cases} \lambda^\star & \text{w.p. } \rho(\lambda^\star, \lambda^{(t)}) \\ \lambda^{(t)} & \text{w.p. } 1 - \rho(\lambda^\star, \lambda^{(t)}) \end{cases}$$

where

$$\begin{aligned} \rho(\lambda^\star, \lambda^{(t)}) &= \min\left\{ \frac{\pi(\lambda^\star|x_1, \ldots, x_n) \cdot \mathsf{N}(\lambda^{(t)}|\lambda^\star, \sigma^2)}{\pi(\lambda^{(t)}|x_1, \ldots, x_n) \cdot \mathsf{N}(\lambda^\star|\lambda^{(t)}, \sigma^2)}, 1 \right\} \\ &= \min\left\{ \frac{f(\lambda^\star)}{f(\lambda^{(t)})}, 1 \right\}. \end{aligned}$$

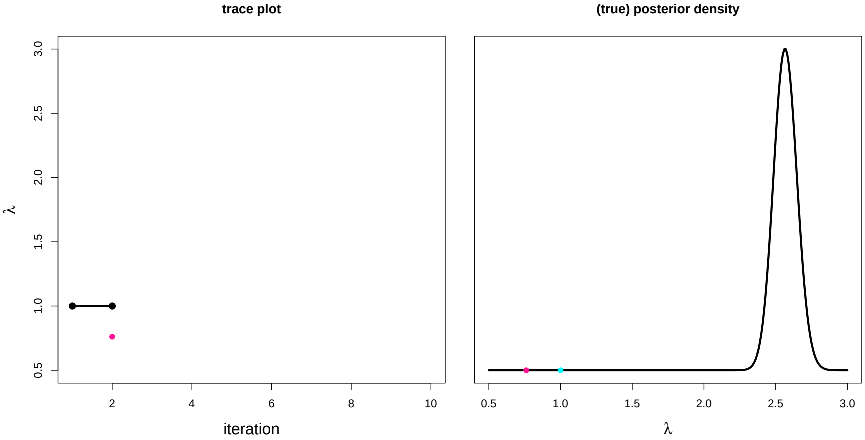
This is called the Metropolis-Hastings acceptance ratio.

Current $\lambda = 1$

Proposed λ $(\sim 1 + \mathrm{N}(0, 0.5^2)) = 0.7613$

MH Ratio $= 1e - 78$

Coin-flip $(\sim \mathrm{U}(0,1)) = 0.2788 \implies$ Reject

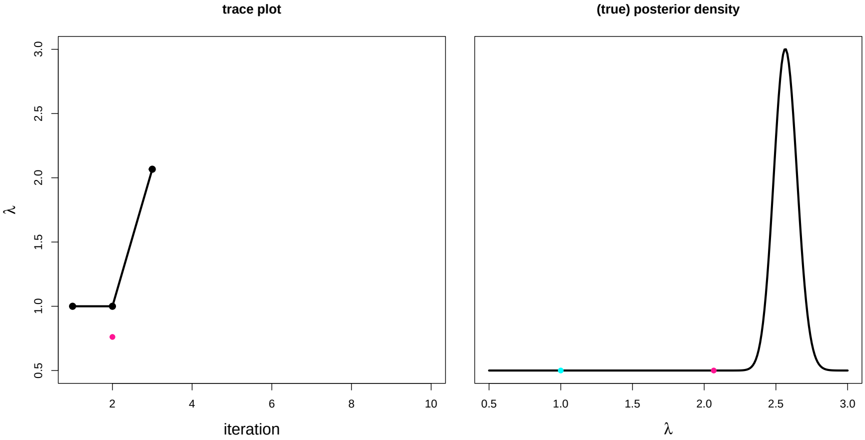


Current $\lambda = 1$

Proposed λ $(\sim 1 + \mathsf{N}(0, 0.5^2)) = 2.0667$

MH Ratio $= 4e + 134$

Coin-flip $(\sim \mathsf{U}(0,1)) = 0.5027 \implies$ Accept

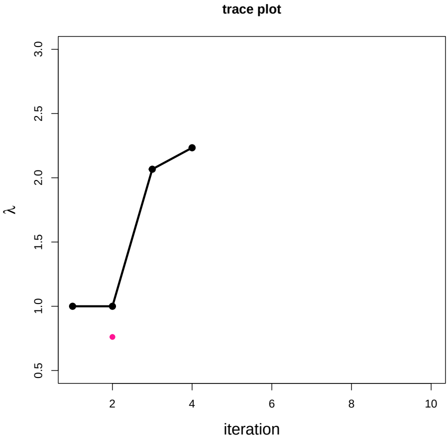


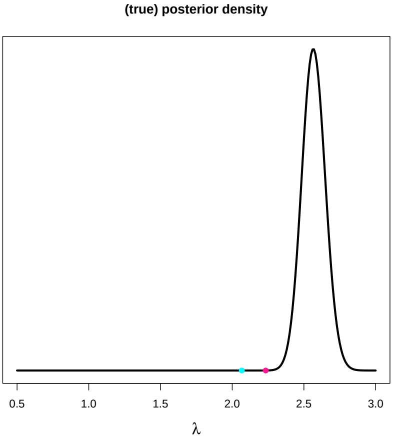
Current $\lambda = 2.0667$

Proposed λ $(\sim 2.0667 + \mathrm{N}(0, 0.5^2)) = 2.2337$

MH Ratio $= 3e + 05$

Coin-flip $(\sim \mathrm{U}(0,1)) = 0.3707 \implies$ Accept



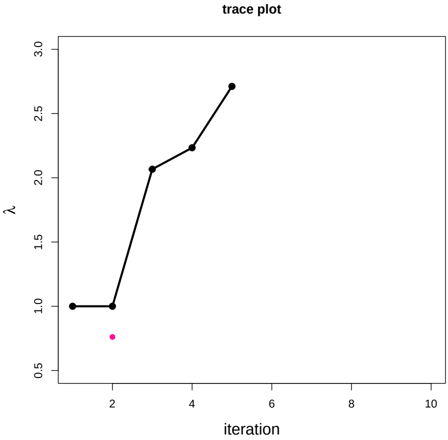


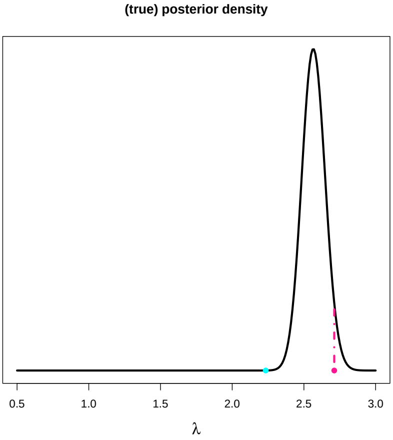
Current λ $= 2.2337$

Proposed λ $(\sim 2.2337 + \mathrm{N}(0, 0.5^2)) = 2.7115$

MH Ratio $= 1964$

Coin-flip $(\sim \mathrm{U}(0,1)) = 0.2875 \implies$ Accept



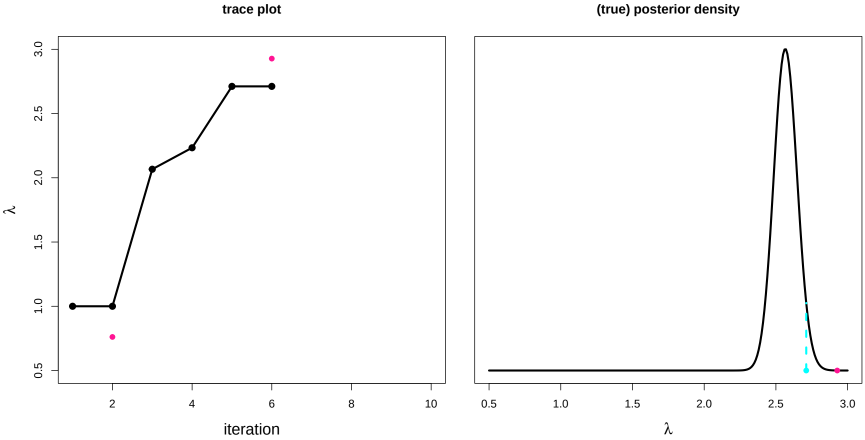


Current $\lambda = 2.7115$

Proposed λ $(\sim 2.7115 + \mathrm{N}(0, 0.5^2)) = 2.9276$

MH Ratio $= 0.0005$

Coin-flip $(\sim \mathrm{U}(0,1)) = 0.1298 \implies$ Reject

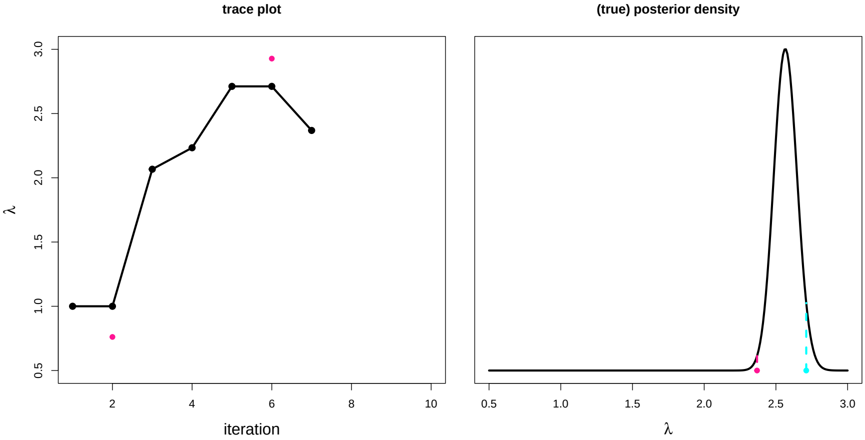


Current $\lambda = 2.7115$

Proposed λ $(\sim 2.7115 + \mathsf{N}(0, 0.5^2)) = 2.3685$

MH Ratio $= 0.2142$

Coin-flip $(\sim \mathsf{U}(0,1)) = 0.1653 \implies$ Accept

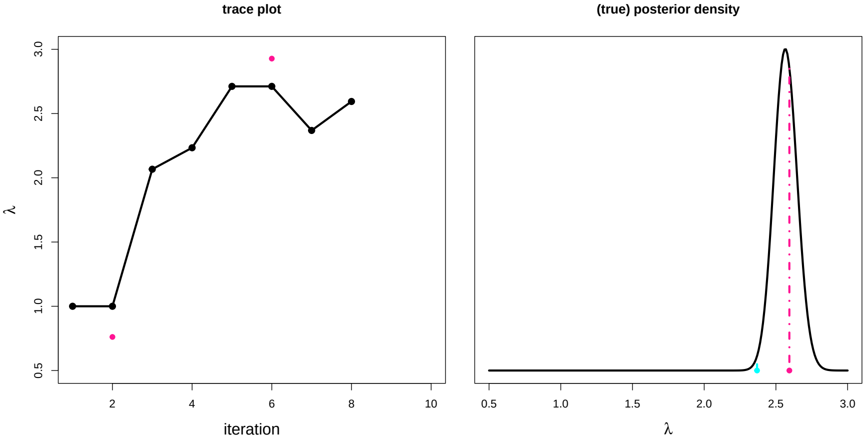


Current $\lambda = 2.3685$

Proposed $\lambda \ (\sim 2.3685 + \mathrm{N}(0, 0.5^2)) = 2.5939$

MH Ratio $= 21$

Coin-flip $(\sim \mathrm{U}(0,1)) = 0.0457 \implies$ Accept

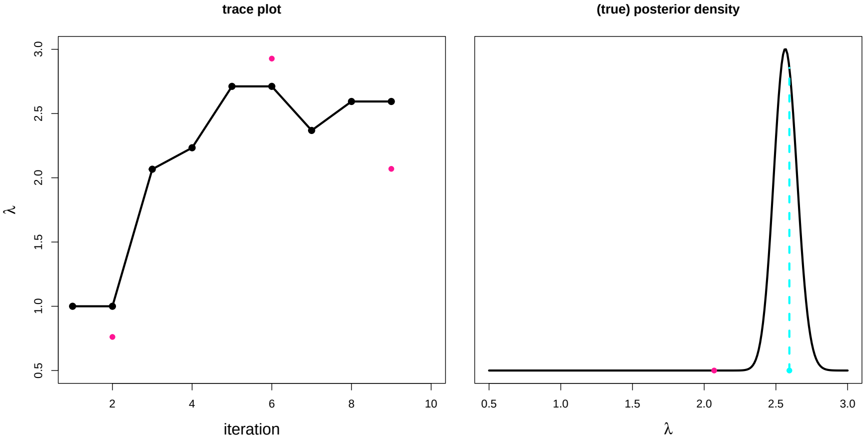


Current $\lambda = 2.5939$

Proposed λ $(\sim 2.5939 + \mathrm{N}(0, 0.5^2)) = 2.0695$

MH Ratio $= 5e - 10$

Coin-flip $(\sim \mathrm{U}(0, 1)) = 0.8348 \implies$ Reject

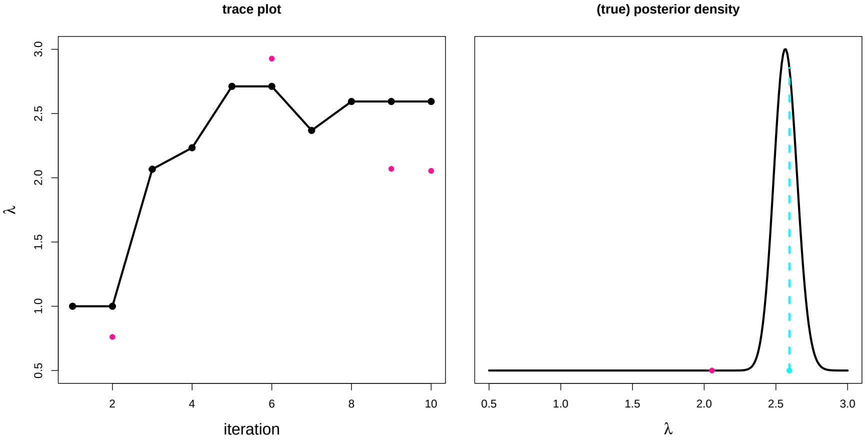


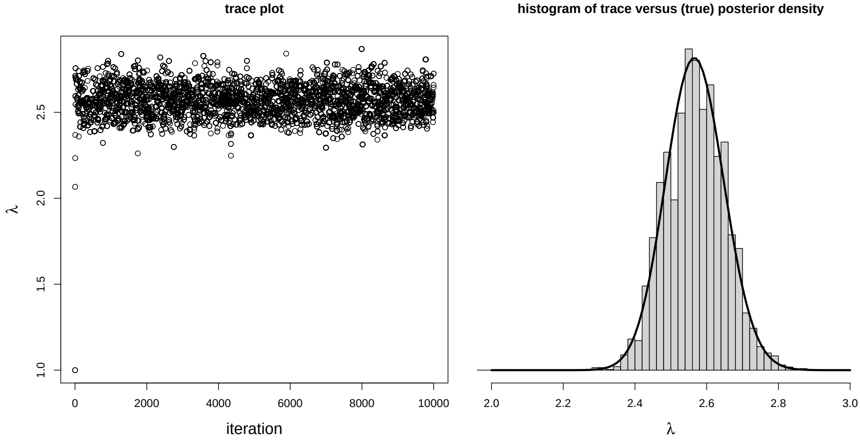
Current λ $= 2.5939$

Proposed λ $(\sim 2.5939 + \mathrm{N}(0, 0.5^2)) = 2.0542$

MH Ratio $= 1e - 10$

Coin-flip $(\sim \mathrm{U}(0,1)) = 0.3117 \implies$ Reject




trace plot
histogram of trace versus (true) posterior density
λ
1.0
1.5
2.0
2.5
0
2000
4000
6000
8000
10000
iteration
2.0
2.2
2.4
2.6
2.8
3.0
λ