ST 705 Linear models and variance components
Homework problem set 7

March 2, 2023

1. Monahan exercise 3.6

2. Monahan exercise 3.7

3. Monahan exercise 3.9 – not necessary to do the “(More practice)” item

4. Assume that $Y = X\beta + U$, where X is an $n \times p$ matrix with $\text{rank}(X) = k < p$, and assume $\lambda'\beta$ is estimable.

   (a) Construct an argument to determine the rank of the matrix $\begin{pmatrix} X \\ \lambda' \end{pmatrix}$.

   (b) Construct an argument to determine the rank of the matrix $\begin{pmatrix} X \\ \lambda'(I - P_{X'}) \end{pmatrix}$.

5. Let X be an $n \times p$ matrix with $\text{rank}(X) = r$, and C be a $(p - r) \times p$ matrix with $\text{rank}(C) = p - r$, such that $\text{col}(X') \cap \text{col}(C') = \{0\}$. Show that $C(X'X + C'C)^{-1}C' = I_{p-r}$.

6. Let X be an $n \times p$ matrix with $\text{rank}(X) = r$, and C be a $(p - r) \times p$ matrix with $\text{rank}(C) = p - r$, such that $\text{col}(X') \cap \text{col}(C') = \{0\}$. Show that

$$\text{rank} \begin{pmatrix} X \\ C \end{pmatrix} = p.$$

7. Consider the restricted linear model $Y = X\beta + U$ over the constrained parameter space $\{P'\beta = \delta\}$, for some full-column rank matrix P. Set up the Langrangian function and derive the *restricted normal equations* (RNE),

$$\begin{pmatrix} X'X & P \\ P' & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \theta \end{pmatrix} = \begin{pmatrix} X'y \\ \delta \end{pmatrix}.$$