ST 705 Linear models and variance components
Homework problem set 11

April 4, 2022

1. Exercise 4.28 from Monahan.

2. Exercise 5.2 from Monahan.

3. Exercise 5.6 from Monahan.

4. Exercise 5.9 from Monahan.

5. Exercise 5.10 from Monahan.

6. Let U and V be independent $\mathrm{N}(0,1)$ random variables, and define $Y := V$ and

$$X := \begin{cases} U & \text{if } UV \geq 0 \\ -U & \text{if } UV < 0 \end{cases}$$

   (a) Show that X and Y each follow the standard normal distribution, but that (X, Y) is not bivariate normal.

   (b) Show that X^2 and Y^2 are independent.

7. Suppose that $Y = X\beta + \varepsilon$, where $\varepsilon \sim \mathrm{N}_n(0, I_n)$, I_n is the $n \times n$ identity matrix, and X is an $n \times p$ matrix with full column rank. Further, let W be an $n \times q$ matrix with $\mathrm{Col}(W) \supseteq \mathrm{Col}(X)$ and $\mathrm{rank}(W) = q > p$, where $\mathrm{Col}(\cdot)$ denotes the column space of a matrix argument.

   (a) For a full column rank matrix A, denote $P_A := A(A'A)^{-1}A'$. Is $P_W - P_X$ an orthogonal projection matrix? If so, show. If not, then demonstrate why not.

   (b) Derive the Chernoff bound for a χ^2_k random variable V. That is, show that for any $s > 0$ and $t \in (0, \frac{1}{2})$,
$$P(V > s) \leq e^{-ts}(1-2t)^{-\frac{k}{2}}.$$
   Recall that the moment generating function of a χ^2_k random variable is $m(t) = (1-2t)^{-\frac{k}{2}}$ for $t < \frac{1}{2}$.

(c) Show that for any $s > 0$,

$$P\left(\left|\frac{Y'(I-P_X)Y}{Y'(I-P_W)Y}-1\right|>s\right)=P\left(\frac{Y'(P_W-P_X)Y}{Y'(I-P_W)Y}>s\right)<c(s)e^{-g(n,s)},$$

for fixed q and p, for some function, $g(n,s)$, that is increasing in n, and for some nonnegative function, $c(s)$, that does not depend on n. Recall that the density function of a χ^2_k random variable is

$$f(x)=\frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})}x^{\frac{k}{2}-1}e^{-\frac{x}{2}},$$

for $x > 0$, and the Stirling inequality for the gamma function gives,

$$\Gamma(x)\geq\sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x},$$

for $x > 0$.