2. Problem set 2 solutions

2.1. (i) In general for a Lie algebra $\mathfrak{g}$ with basis $x_1, \ldots, x_n$ with $[x_i, x_j] = \sum c_{ij}^k x_k$, we have $\{a, b\} = \sum_{i,j,k} c_{ij}^k x_k \frac{\partial a}{\partial x_i} \frac{\partial b}{\partial x_j}$ for $a, b \in \operatorname{Sym} \mathfrak{g}$. Specializing to $\mathfrak{g} = \mathfrak{sl}_2$ we have given $a, b \in \mathbb{C}[e, h, f]$, the Poisson bracket is

$$\{a, b\} = 2f \frac{\partial a}{\partial f} \frac{\partial b}{\partial h} - h \frac{\partial a}{\partial f} \frac{\partial b}{\partial e} - 2f \frac{\partial a}{\partial h} \frac{\partial b}{\partial f} + 2e \frac{\partial a}{\partial h} \frac{\partial b}{\partial e} + h \frac{\partial a}{\partial e} \frac{\partial b}{\partial f} - 2e \frac{\partial a}{\partial e} \frac{\partial b}{\partial h}.$$

Let $P = \frac{1}{2}h^2 + 2ef$. Then $\{a, b\} = \{a, b\}_{dP}$ where $\{-, -\}_{dP}$ is the Poisson bracket on $\mathbb{C}[e, h, f]$ from Problem 1.1 with respect to $df \wedge dh \wedge de$.

(ii) The Poisson center of $\operatorname{Sym} \mathfrak{g}$ is equal to the $\operatorname{ad}(\mathfrak{g})$-invariants of $\operatorname{Sym} \mathfrak{g}$. Since G is connected, this is the same as $(\operatorname{Sym} \mathfrak{g})^G$ the SL(2)-invariant polynomials on $\mathfrak{sl}_2$, which is generated by the determinant. Hence the Poisson center equals $\mathbb{C}[P]$.

2.2. (i) [CG, Proposition 1.4.6] Let $A \in \mathfrak{sp}(V)$. Define $\mathsf{H}_A \in \mathbb{C}[V]$ by

$$\mathsf{H}_A(v) = \frac{1}{2}\omega(Av, v).$$

Let $d_v\mathsf{H}_A$ denote the differential at $v \in V$. One checks that $d_v\mathsf{H}_A(w) = \omega(Av, w)$ for $w \in V$, i.e., $\mathsf{H} : \mathfrak{sp}(V) \to \mathbb{C}[V]$ is indeed the Hamiltonian of the natural action of $\mathrm{Sp}(V, \omega)$ on V. Now $\mu : V \to \mathfrak{sp}(V)^*$ is given by $\mu(v)(A) = \mathsf{H}_A(v) = \frac{1}{2}\omega(Av, v) = \operatorname{tr}(B_v A)$ where $B_v(w) = \frac{1}{2}\omega(w, v)v$. Thus $\mu : V \to \mathfrak{sp}(V)$ sends $v \mapsto B_v$.

(ii) The Hamiltonian is given by $\mathsf{H}_x = \lambda(\mathsf{act}(x)) \in \mathcal{O}(T^*\mathfrak{g})$ where $x \in \mathfrak{g}$ and λ is the canonical 1-form on $T^*\mathfrak{g}$. For $(\xi, y) \in T^*\mathfrak{g} = \mathfrak{g}^* \times \mathfrak{g}$, we have

$$\mathsf{H}_x(\xi, y) = \lambda_{(\xi,y)}(\mathsf{act}(x)) = \xi([x, y]).$$

Then $\mu : \mathfrak{g}^* \times \mathfrak{g} \to \mathfrak{g}^*$ is given by $\mu(\xi, y)(x) = \xi([x, y])$. Let $\xi(b) = \langle a, b\rangle$ for $a, b \in \mathfrak{g}$. Then $\xi([x, y]) = \langle a, [x, y]\rangle = \langle [y, a], x\rangle$. Thus under the identifications $\mathfrak{g}^* \cong \mathfrak{g}$ via $\langle -, -\rangle$ we have that $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ sends $(a, y) \mapsto [y, a]$.

2.3. Put $x = u^n + v^n$, $y = i(u^n - v^n)$, $z = (-4)^{1/n}uv$. This gives the isomorphism $\mathbb{C}[x, y, z]/(x^2 + y^2 + z^n) \cong \mathbb{C}[u, v]^{\Gamma_n}$.