Derived Satake equivalence for Godement–Jacquet monoids

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Perimeter Institute

Periods, functoriality and L-functions
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Goal: geometrize Godement-Jacquet

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$\mathsf{M}_n = n \times n$ matrices

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More generally, study $C_c^\infty(\mathsf{M}_n(F))$.

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This talk: categorify $C_c^\infty(\mathsf{M}_n(F))^{\mathsf{GL}_n(\mathcal{O}) \times \mathsf{GL}_n(\mathcal{O})}$ to ℓ-adic sheaves on $\mathsf{M}_n(F)$ and then spectrally decompose using geometric techniques.

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Functions–sheaves dictionary:

$$D_c^b(\mathsf{GL}_n(\mathcal{O}) \backslash \mathsf{M}_n(F) / \mathsf{GL}_n(\mathcal{O})) = ?$$

General framework

G split reductive group

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Conjecture (SV)

(1) We have Plancherel formula: for $f_1, f_2 \in L^2(X(F))$,

$$\langle f_1, f_2 \rangle_{L^2(X(F))} = \int_{(\varphi:\Gamma_F \to \check{G}_X)/\sim} J_\varphi(f_1, f_2) d\mu(\varphi)$$

where J_φ picks up contribution of A-packet of $\Gamma_F \to \check{G}_X \to \check{G}$.

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(2) For Satake parameter $\varphi = \chi \in \check{T}_X / W_X$, we have

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This is the "definition" of ρ_X.

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Let X be affine, smooth, spherical G-variety. There is equivalence of derived categories

$$D_c^b(X_F/G_{\mathcal{O}}) \cong \mathrm{Perf}^{[]}(V_X^*/\check{G}_X)$$

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Heuristic: RHom in LHS recover relative characters J_χ after trace of Frobenius.

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X_F is ind-scheme over $\overline{\mathbb{F}}_q$ such that $X_F(\overline{\mathbb{F}}_q) = X(\overline{\mathbb{F}}_q(\!(t)\!))$.

Technical issue: X_F is union of infinite dim'l schemes – constructible ℓ-adic sheaves on such spaces not well studied.

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RHS means perfect complexes of $\check{G}_X$-equivariant $\mathrm{Sym}^{[]}(V_X)$-modules, up to quasi-isomorphism

$\mathrm{Sym}^{[]}(V_X)$ is dg algebra with zero differential and grading given by special values of L-function

Example: derived Satake equivalence

Theorem (Bezrukavnikov–Finkelberg, after Ginzburg)

For any reductive group G, there is equivalence of monoidal categories

$$D_c^b(G_{\mathcal{O}} \backslash G_F / G_{\mathcal{O}}) \cong \mathrm{Perf}(\check{\mathfrak{g}}^*[2]/\check{G})$$

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View this as geometrization of Harish-Chandra Plancherel formula:

$$\langle \mathbf{1}_{G(\mathcal{O})}, \mathbf{1}_{G(\mathcal{O})} \rangle_{L^2} = \int_{\chi \in \check{T}^{\mathrm{cpct}}/W} L(1, \pi_\chi, \mathrm{Ad}) \cdot \frac{d\chi}{L(0, \pi_\chi, \mathrm{Ad})}$$

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Normalize with respect to Whittaker Plancherel measure:

$$d\mu(\chi) = \frac{d\chi}{L(0, \pi_\chi, \mathrm{Ad})}.$$

Main equivalence, version 1

Theorem (Tsao-Hsien Chen–W)

Let $(X, G) = (\mathrm{M}_n, \mathrm{GL}_n \times \mathrm{GL}_n)$*. We have equivalence*

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Geometrization of L^2-version of unramified, local Godement-Jacquet theory:

$$\langle \mathbf{1}_{\mathsf{M}_n(\mathcal{O})}, \mathbf{1}_{\mathsf{M}_n(\mathcal{O})} \rangle_{L^2(\mathsf{M}_n(F))} = \int_{\chi \in (S^1)^n / S_n} L(\tfrac{-n+1}{2}, \pi_\chi, \mathrm{std}) L(\tfrac{n+1}{2}, \pi_\chi, \mathrm{std}^*) d\mu(\chi)$$

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In this case $\check{G}_X = \mathrm{GL}_n$ and $\rho_X = \mathrm{Ad} \times \mathrm{std} \times \mathrm{std}^*$.

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Note: $\mathrm{GL}_n \times \mathfrak{gl}_n^* \times \mathbb{A}^n \times \mathbb{A}^{n*} = T^*(\mathrm{GL}_n \times \mathbb{A}^n)$

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RHS is derived category of $\mathrm{Sym}^{[]}(\mathrm{GL}_n \times \mathfrak{gl}_n \times \mathbb{A}^n \times \mathbb{A}^{n*})$-modules set-theoretically supported in preimage of $\mathcal{N}ilp \subset \check{\mathfrak{g}}^*$ under moment map

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Theorem (Bouthier-Ngô-Sakellaridis)

The spectral decomposition of $IC_{M_{\rho,\mathcal{O}}}$ gives $L(s,\pi_\chi,\rho)$.

It is more natural to consider $M^\bullet_{\rho,F}$ rather than $M_{\rho,F}$.

Main equivalence, version 3

Theorem (Tsao-Hsien Chen–W)

There is equivalence

$$D(\mathrm{GL}_{n,\mathcal{O}} \backslash \mathrm{M}^{\bullet}_{n,F} / \mathrm{GL}_{n,\mathcal{O}}) \cong \mathrm{QCoh}^{[]}(\mathfrak{gl}_n^* \times \mathbb{A}^n \times \mathbb{A}^{n*} / \mathrm{GL}_n)_{\mathcal{N}_n \times \{0\} \times \mathbb{A}^{n*}}$$

where $\mathcal{N}_n \subset \mathfrak{gl}_n^*$ is nilpotent cone.

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Koszul duality implies

Corollary

There is equivalence of abelian categories

$$\mathrm{Perv}(\mathrm{GL}_{n,\mathcal{O}} \backslash \mathrm{M}^{\bullet}_{n,F} / \mathrm{GL}_{n,\mathcal{O}}) \cong \wedge(\mathrm{std} \oplus \mathrm{std}^*)\text{-}\mathrm{mod}^{\mathrm{GL}_n}_{\mathrm{fd}, \mathbb{A}^n \times \{0\}}$$

Proof sketch

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- Idea: apply equivariant cohomology to A.

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What do $p_!$ and p_* correspond to spectrally?

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and $\kappa^2(a, b) = (\mathrm{Id}, \kappa(a)^T, e_n, (b-a)^T)$.

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- **Key Fact:** the union of two images has complement of codimension 2

Thank you!