Spherical varieties, L-functions, and crystal bases

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Notes available at: `http://jonathanpwang.com/notes/sphL_RT_slides_handout.pdf`

Outline

1. What is a spherical variety?
2. Local geometric duality
3. Function-theoretic results
4. Geometry

- $F = \mathbb{F}_q(\!(t)\!)$, $O = \mathbb{F}_q[\![t]\!]$
- $k = \overline{\mathbb{F}}_q$
- G connected split reductive group $/\mathbb{F}_q$

What is a spherical variety?

Definition

A G-variety $X_{/\mathbb{F}_q}$ is called spherical if X_k is a normal variety with finitely many B_k orbits.

Finiteness condition gives good combinatorics (spherical root datum, rational cones, fans)

Examples:

- Toric varieties $G = T$
- Symmetric spaces $K\backslash G$
  - Group $X = G' \circlearrowleft G' \times G' = G$

Why are they relevant

Conjecture (Sakellaridis, Sakellaridis–Venkatesh)

Representation theory (harmonic analysis) of functions on an affine spherical variety X, in particular involving the "IC function" of $X(O)$, is related to an L-function

$$L(s, \pi, \rho_X)$$

where $\rho_X : \check{G}_X \to \mathrm{GL}(V_X)$ is a $\check{G}_X$-representation of a possibly different group $\check{G}_X$

There is a map $\check{G}_X \to \check{G}$, constructed (in most cases) by Gaitsgory–Nadler, Sakellaridis–Venkatesh, Knop–Schalke.

Relation to physics

- $T^*X \to \mathfrak{g}^*$ is a Hamiltonian G-space
- (Gaitto–Witten) Hamiltonian G-space $\rightsquigarrow$ boundary theory for super Yang–Mills TFT for G
- S-duality for boundary theories predicts:

$$\boxed{G \circlearrowright T^*X \to \mathfrak{g}^*} \longleftrightarrow \boxed{\check{G} \circlearrowright M^\vee \to \check{\mathfrak{g}}^*}$$

Prediction (Ben-Zvi–Sakellaridis–Venkatesh)

When X is a spherical variety, there exists $V_X \in \mathrm{Rep}(\check{G}_X)$ such that

$$M^\vee = V_X \times^{\check{G}_X} \check{G} := (V_X \times \check{G})/\check{G}_X$$

is a Hamiltonian $\check{G}$-space.

	$X \circlearrowleft G$	$\check{G}_X$	V_X
Usual Langlands	$G' \circlearrowleft G' \times G'$	$\check{G}'$	$\check{\mathfrak{g}}'$
Whittaker normalization	$(N, \psi)\backslash G$	$\check{G}$	0
Hecke	$\mathbb{G}_m \backslash \mathrm{PGL}_2$	$\check{G} = \mathrm{SL}_2$	$T^*\mathrm{std}$
Rankin–Selberg, Jacquet–Piatetski-Shapiro–Shalika	$\overline{H\backslash \mathrm{GL}_n \times \mathrm{GL}_n} = \mathrm{GL}_n \times \mathbb{A}^n$	$\check{G}$	$T^*(\mathrm{std} \otimes \mathrm{std})$
Gan–Gross–Prasad	$\mathrm{SO}_{2n} \backslash \mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n}$	$\check{G} = \mathrm{SO}_{2n} \times \mathrm{Sp}_{2n}$	$\mathrm{std} \otimes \mathrm{std}$

Local geometric duality

- $k = \overline{\mathbb{F}}_q$
- $X_F(k) = X(k((t)))$ formal loop space of X – this is an ind-scheme
- Let $X^\bullet$ denote the open G-orbit of X.
- $X_F^\bullet = X_F - (X - X^\bullet)_F$

We quantize the previous duality:

Conjecture (Ben-Zvi–Sakellaridis–Venkatesh)

There exists a monoidal equivalence

$$D^b_{G_O}(X_F^\bullet) \cong D^b_{\mathrm{perf}}(\mathbb{V}_X/\check{G}_X)$$

where $\mathbb{V}_X$ is a $\mathbb{Z}$-graded, super $\check{G}_X$-representation.

This is a generalization of derived Satake equivalence ($X = G \circlearrowleft G \times G$)

$$D^b_{G_O}(G_F/G_O) \cong D^b_{\mathrm{perf}}(\check{\mathfrak{g}}^*[2]/\check{G})$$

$\check{G}_X = \check{G}$

For this talk, assume $\check{G}_X = \check{G}$ (and X has no type N roots). ['N' is for normalizer]

Equivalent to:

(Base change to k)

- X has open B-orbit $X^\circ \cong B$
- $X^\circ P_\alpha / \mathcal{R}(P_\alpha) \cong \mathbb{G}_m \backslash \mathrm{PGL}_2$ for every simple α, $P_\alpha \supset B$

Plancherel formula

By functions-sheaves analogy, previous conjecture can be viewed as geometric realization of Plancherel formula for $L^2(X^\bullet(F))$:

$$L^2(X^\bullet(F))^{G(O)} = \int_{\chi \in \widehat{T}_X/W_X} \pi_\chi^{G(O)} d\chi$$

where $\widehat{T}_X$ is maximal compact in $\check{T}_X$ and π_χ is principal series.

In particular, we have a spectral decomposition

$$\|IC_{X(O)}\|^2 = \int_{\widehat{T}_X/W_X} \|IC_{X(O)}\|_\chi^2 d\chi$$

and conjecture predicts that

$$\|IC_{X(O)}\|_\chi^2 = \frac{L(s_0, \pi_\chi, V_X)}{L(1, \pi_\chi, \check{\mathfrak{g}}_X)}$$

up to known constant and zeta factors.

Theorem (Sakellaridis–Venkatesh á la Bernstein)

There exists a $G(F)$-equivariant map

$$\mathrm{Asymp} : C^{\infty}(X^{\bullet}(F)) \to C^{\infty}(X_0^{\bullet}(F))$$

where $X_0^{\bullet}$ "looks like" $N^{-}\backslash G$, such that

$$\|\Phi\|_{\chi}^{2} = \|\mathrm{Asymp}(\Phi)\|_{\chi}^{2}.$$

So function-theoretically, the problem amounts to computing $\mathrm{Asymp}(IC_{X(O)})$.

Theorem (Sakellaridis–W)

Assume that the open B-orbit $X^\circ = B$.
Then, Asymp is realized via the functions-sheaves dictionary as a nearby cycles functor on finite type models of $X_F^\bullet \rightsquigarrow (X_0^\bullet)_F$.

In this situation, $X_0^\bullet = N^-\backslash G$ so

$$\mathrm{Asymp}(IC_{X(O)}) \in C^\infty(N^-(F)\backslash G(F)/G(O)) = \mathrm{Fn}(\check{\Lambda}).$$

Conjecture 1 (Sakellaridis–Venkatesh)

Assume X affine spherical, $\check{G}_X = \check{G}$ and X has no type N roots.

- *There exists $M^\vee = V_X$ a symplectic $\check{G}$-representation with Hamiltonian structure, and $\mathbb{V}_X = V_X^{odd}[1]$.*

There exists a $\check{T}$ polarization $V_X = V_X^+ \oplus (V_X^+)^*$ such that

$$\mathrm{Asymp}(IC_{X(O)}) = \frac{\prod_{\check{\alpha}\in\check{\Phi}_G^+}(1 - q^{-1}e^{\check{\alpha}})}{\prod_{\check{\lambda}\in\mathrm{wt}(V_X^+)}(1 - q^{-\frac{1}{2}}e^{\check{\lambda}})} \in \mathrm{Fn}(\check{\Lambda})$$

where $e^{\check{\lambda}}$ is the indicator function of $\check{\lambda}$, $e^{\check{\lambda}}e^{\check{\mu}} = e^{\check{\lambda}+\check{\mu}}$

Mellin transform ($=$ spectral decomposition) gives

$$(\mathrm{Asymp}(IC_{X(O)})_\chi = \frac{L(\frac{1}{2}, \chi, V_X^+)}{L(1, \chi, \check{\mathfrak{n}})}, \text{ this is “half” of } \frac{L(\frac{1}{2}, \chi, V_X)}{L(1, \chi, \check{\mathfrak{g}}/\check{\mathfrak{t}})}$$

Theorem (Sakellaridis–W)

Assume X affine spherical, $\check{G}_X = \check{G}$ and X has no type N roots. Then the Mellin transform

$$(\mathrm{Asymp}(\mathrm{IC}_{X(O)}))_\chi = \frac{L(\frac{1}{2}, \chi, V_X^+)}{L(1, \chi, \check{\mathfrak{n}})}$$

for some $V_X^+ \in \mathrm{Rep}(\check{T})$ such that:

1. *$V'_X := V_X^+ \oplus (V_X^+)^*$ has action of $(\mathrm{SL}_2)_\alpha$ for every simple root α*
   - *We do not check Serre relations*
2. *Assuming V'_X satisfies Serre relations (so it is a $\check{G}$-representation), we determine its highest weights with multiplicities (in terms of X)*

- (2) gives recipe for conjectural (ρ_X, V_X) in terms of X
- If V_X is minuscule, then $V_X = V'_X$.
- We show H reductive implies minuscule assumption.

Previous work

Asymp($IC_{X(O)}$) was previously considered by:

- Sakellaridis ('08, '13):
  - $X = H\backslash G$ and H is reductive (iff $H\backslash G$ is affine), no assumption on $\check{G}_X$
  - doesn't consider $X \supsetneq H\backslash G$
- Braverman–Finkelberg–Gaitsgory–Mirković [BFGM] '02:
  - $X = \overline{N^-\backslash G}$, $\check{G}_X = \check{T}$, $V_X = \check{\mathfrak{n}}$
- S. Schieder '16:
  - $X = G'$ group case, $G = G' \times G'$, $V_X = \check{\mathfrak{g}}'$
- Bouthier–Ngô–Sakellaridis [BNS] '16:
  - $X \supset G'$ is L-monoid, $G = G' \times G'$, $\check{G}_X = \check{G}'$, $V_X = \check{\mathfrak{g}}' \oplus T^* V^{\check{\lambda}}$
- J. Campbell '17:
  - $X = (N, \psi)\backslash G$ Whittaker

Geometry

- Base change to $k = \overline{\mathbb{F}}_q$ (or $k = \mathbb{C}$)
- $X_O(k) = X(k[\![t]\!])$
- Problem: X_O is an infinite type scheme
- Bouthier–Ngô–Sakellaridis: IC function still makes sense by Grinberg–Kazhdan theorem – use finite type schemes to model X_O

Zastava space

Drinfeld's proof of Grinberg–Kazhdan theorem gives an explicit finite type model for X_O:

Definition

Let $C = \mathbb{A}^1$ the affine line. Define

$$\mathcal{Y} = \mathrm{Maps}_{\mathrm{gen}}(C, X/B \supset X^\circ/B)$$

Following Finkelberg–Mirković, we call this the **Zastava space** of X.

Fact: $\mathcal{Y}$ is an infinite disjoint union of finite type schemes.

$$\begin{array}{c} \mathcal{Y} \\ \Big\downarrow{\scriptstyle \pi} \\ \mathcal{A} \\ \cap \\ \{\check{\Lambda}\text{-valued divisors on } C\} \end{array}$$

Define the **central fiber** $\mathbb{Y}^{\check{\lambda}} = \pi^{-1}(\check{\lambda} \cdot v)$ for a single point $v \in C(k)$.

$$\begin{tikzcd} \mathcal{Y} \arrow[d] & \mathbb{Y}^{\check{\lambda}} \arrow[l] \arrow[d] \\ \mathcal{A} & \check{\lambda} \cdot v \arrow[l] \end{tikzcd}$$

Graded factorization property

The fiber $\pi^{-1}(\check{\lambda}_1 v_1 + \check{\lambda}_2 v_2)$ for distinct v_1, v_2 is isomorphic to $\mathbb{Y}^{\check{\lambda}_1} \times \mathbb{Y}^{\check{\lambda}_2}$.

Upshot

By Braden's contraction principle, computation of Asymp / nearby cycles amounts to computing $\pi_! \mathrm{IC}_{\mathcal{Y}}$.

Semi-small map

Can compactify π to proper map $\bar{\pi} : \overline{\mathcal{Y}} \to \mathcal{A}$.

Theorem (Sakellaridis–W)

Under previous assumptions, $\bar{\pi} : \overline{\mathcal{Y}} \to \mathcal{A}$ is stratified semi-small. In particular, $\bar{\pi}_! \mathrm{IC}_{\overline{\mathcal{Y}}}$ is perverse.

If $\overline{\mathcal{Y}}$ is smooth, then semi-smallness amounts to the inequality

$$\dim \overline{\mathbb{Y}}^{\check{\lambda}} \leq \mathrm{crit}(\check{\lambda})$$

Decomposition theorem + factorization property imply

Euler product

$$tr(\mathrm{Fr}, (\bar{\pi}_! \mathrm{IC}_{\overline{\mathcal{Y}}})|^*_{?\cdot v}) = \frac{1}{\prod_{\check{\lambda} \in \mathfrak{B}^+} (1 - q^{-\frac{1}{2}} e^{\check{\lambda}})}$$

$\mathfrak{B}^+$ = irred. components of $\overline{\mathbb{Y}}^{\check{\lambda}}$ of dim $= \mathrm{crit}(\check{\lambda})$ as $\check{\lambda}$ varies

- $\mathfrak{B}^+ =$ irred. components of $\overline{\mathbb{Y}}^{\check{\lambda}}$ of dim $= \mathrm{crit}(\check{\lambda})$ as $\check{\lambda}$ varies
- Define V_X^+ to have basis $\mathfrak{B}^+$
- Formally set $\mathfrak{B} = \mathfrak{B}^+ \sqcup (\mathfrak{B}^+)^*$, so $(\mathfrak{B}^+)^*$ is a basis of $(V_X^+)^*$

Theorem (Sakellaridis–W)

$\mathfrak{B}$ has the structure of a (Kashiwara) crystal, i.e., graph with weighted vertices and edges $\leftrightarrow$ raising/lowering operators $\tilde{e}_\alpha, \tilde{f}_\alpha$

Crystal basis is the (Lusztig) canonical basis at $q = 0$ of a f.d. $U_q(\check{\mathfrak{g}})$-module.

$$\text{f.d. } \check{G}\text{-representation} \rightsquigarrow \text{crystal basis} \in \{\text{crystals}\}$$

Conjecture 1

$\mathfrak{B}$ is the crystal basis for a finite dimensional $\check{G}$-representation V_X.

- Conjecture 1 implies $V'_X = V_X$ is a $\check{G}$-representation ($\mathfrak{B} \leftrightarrow V'_X$).
- Conjecture 1 resembles geometric constructions of crystal bases by Braverman–Gaitsgory using Mirković–Vilonen cycles
- $\mathbb{Y}^{\check{\lambda}}, \overline{\mathbb{Y}}^{\check{\lambda}} \subset \mathrm{Gr}_G$
- $\mathbb{Y}^{\check{\lambda},0} = H_F G_O \cap N_F t^{\check{\lambda}} G_O \subset \mathrm{Gr}_G$