INVARIANT DIFFERENTIAL OPERATORS

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The goal of this talk is to go over Knop's paper [5] in the context of a spherical variety[1] X with a right action by a connective reductive group G over an algebraically closed field k of characteristic 0. We fix a maximal split torus $A \subset G$.

Recall that Harish-Chandra showed that the center $\mathfrak{Z}(\mathfrak{g})$ of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is isomorphic to the polynomial algebra $k[\mathfrak{a}^*]^W$, and this algebra controls the spectral decomposition of $L^2(G)$. To study the spectral decomposition of $L^2(X)$, the role of $\mathfrak{Z}(\mathfrak{g})$ is replaced by the algebra $\mathcal{D}(X)^G$ of invariant differential operators on X. One of the main results of [5] is the following:

Theorem 0.1 ([5, Corollary 6.3, Theorem 6.5])**.** *The algebra $\mathcal{D}(X)^G$ is commutative and there is a canonical isomorphism*

$$\mathcal{D}(X)^G \cong k[\rho + \mathfrak{a}_X^*]^{W_X}.$$

The outline is as follows: Knop first defines the algebra $\mathfrak{U}(X) \subset \mathcal{D}(X)$ of completely regular differential operators and defines $\mathfrak{Z}(X) := \mathfrak{U}(X)^G$. It turns out that $\mathfrak{Z}(X)$ is exactly the center of $\mathfrak{U}(X)$, and for X spherical, $\mathfrak{Z}(X) = \mathcal{D}(X)^G$. These algebras of differential operators are all related to the non-abelian analog of the moment map, and the idea is to reduce to known properties of the moment map by applying a vanishing theorem of Kollár [2].

Historically, Knop proved properties about the moment map first in [3] and then considered the quantized version in [5]. But in this talk I will discuss the properties of the quantized and graded versions simultaneously.

1. Moment map and the sheaf $\mathfrak{U}_X$

We do not need X to be spherical in this section.

1.1. Localized moment map.

Assume for now that X is smooth.

I will use subscripts to denote sheaves and parentheses to denote global sections, e.g., $\mathcal{D}_X$ is the sheaf of differential operators and $\mathcal{D}(X) = \Gamma(X, \mathcal{D}_X)$ is the algebra of global sections.

The G-action on X extends to a map of algebras

$$\mathfrak{U}(\mathfrak{g}) \to \mathcal{D}(X). \tag{1.1}$$

This extends to a map of sheaves of algebras

$$\mathcal{O}_X \underset{k}{\otimes} \mathfrak{U}(\mathfrak{g}) \to \mathcal{D}_X. \tag{1.2}$$

Define the sheaf of algebras $\mathfrak{U}_X$ to be the image of this map.

If we take associated graded of (1.2), we get the map of commutative algebras

$$\mathcal{O}_X \otimes k[\mathfrak{g}^*] \to \mathrm{Sym}^\bullet_{\mathcal{O}_X}(\Omega_X^\vee),$$

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[1] We will assume X is quasi-affine whenever convenient. Knop in fact works in a much more general setting of not necessarily spherical G-varieties!

with $\Omega_X^\vee$ the tangent sheaf. Applying relative Spec gives the map of varieties

$$\pi \times \Phi : T_X^* \to X \times \mathfrak{g}^*$$

where T_X^* is the cotangent bundle, $\pi : T_X^* \to X$ is the projection, and $\Phi : T_X^* \to \mathfrak{g}^*$ is the moment map. Define $T_X^{\mathfrak{g}}$ to be the closure of the image of $\pi \times \Phi$ (Knop denotes $T_X^{\mathfrak{g}}$ by $\widetilde{T}_X$, but to avoid confusion with previous notation we use the notation of [9, 8.3]). The moment map factors as

$$\Phi : T_X^* \xrightarrow{\kappa} T_X^{\mathfrak{g}} \xrightarrow{\Phi} \mathfrak{g}^*.$$

(Recall that if we took associated graded of (1.1), we get $k[\mathfrak{g}^*] = \operatorname{gr} \mathfrak{U}(\mathfrak{g}) \to \operatorname{gr} \mathcal{D}(X) \hookrightarrow k[T_X^*]$, where the last inclusion is the symbol map.) The morphism $\Phi : T_X^{\mathfrak{g}} \to \mathfrak{g}^*$ is called the *localized moment map*.

If we also use $\pi : T_X^{\mathfrak{g}} \to X$ to denote the projection, then $\pi_*(\mathcal{O}_{T_X^{\mathfrak{g}}}) \subset \operatorname{Sym}^\bullet(\Omega_X^\vee)$ is the image of $\mathcal{O}_X \otimes k[\mathfrak{g}^*] \to \operatorname{Sym}^\bullet(\Omega_X^\vee)$. So $T_X^{\mathfrak{g}}$ is the abelian analog of the sheaf of algebras $\mathfrak{U}_X$. Observe that

(1) The projection $\pi : T_X^{\mathfrak{g}} \to X$ is affine.
(2) The map $\Phi : T_X^{\mathfrak{g}} \to \mathfrak{g}^*$ is proper if X is complete.

Example 1.2. Let $X^\bullet = H\backslash G$ denote the open G-orbit in X. Then $T_{H\backslash G}^* = \mathfrak{h}^\perp \times^H G$ embeds into $H\backslash G \times \mathfrak{g}^*$ and $T_{H\backslash G}^* = T_{H\backslash G}^{\mathfrak{g}}$. Since $\mathcal{D}_{H\backslash G}$ is generated by elements in degrees ≤ 1, we deduce that $\mathfrak{U}_{H\backslash G} = \mathcal{D}_{H\backslash G}$.

We have shown that $T_{X^\bullet}^* = T_{X^\bullet}^{\mathfrak{g}} \subset T_X^{\mathfrak{g}}$. If X is complete, then $\Phi : T_X^{\mathfrak{g}} \to \mathfrak{g}^*$ is a proper map extending the moment map $T_{H\backslash G}^* \to \mathfrak{g}^*$. This properness is the primary reason we constructed the localized moment map.

Example 1.3. Let $X = \mathbb{A}^1$ and $G = \mathbb{G}_m$. Then $\mathcal{D}(X)$ has a basis formed by $x^a(\frac{d}{dx})^b$ for $a, b \geq 0$. On the other hand, $\mathfrak{U}(\mathfrak{g}) \subset \mathcal{D}(X)$ has a basis formed by $(x\frac{d}{dx})^b$ for $b \geq 0$. So $\Gamma(X, \mathfrak{U}_X) \subsetneq \mathcal{D}(X)$ has a basis formed by $x^a(x\frac{d}{dx})^b$ for $a, b \geq 0$.

The moment map $\Phi : T_X^* = \mathbb{A}^2 \to \mathfrak{g}^* = \mathbb{A}^1$ sends $(y, x) \mapsto xy$. So we have $T_X^{\mathfrak{g}} = \mathfrak{g}^* \times X = \mathbb{A}^2$ but the map $\kappa : T_X^* \to T_X^{\mathfrak{g}}$ corresponds to the map $\mathbb{A}^2 \to \mathbb{A}^2 : (y, x) \mapsto (xy, x)$. In particular, the fiber over $(0, 0)$ is $\mathbb{A}^1$.

Below we study various properties of $T_X^{\mathfrak{g}}$.

Lemma 1.4 ([5, Lemma 2.1])**.** *The fiber $\kappa^{-1}(y)$ of $y \in T_X^{\mathfrak{g}}$ under $T_X^* \xrightarrow{\kappa} T_X^{\mathfrak{g}}$ is either empty or isomorphic to $(T_x(Gx))^\perp \subset T_x^*X$ for $x = \pi(y)$. In particular, every fiber is irreducible.*

Corollary 1.5. *The map $\kappa : T_X^* \to T_X^{\mathfrak{g}}$ is an isomorphism iff G acts transitively on X.*

We say that X is *pseudo-free* if $T_X^{\mathfrak{g}} \to X$ is a vector bundle (this is true for example when $X = H\backslash G$ is smooth homogeneous). If we let $n = \dim X$ then from Example 1.2 we have $T_{X^\bullet}^* \subset T_X^{\mathfrak{g}}$ is a vector bundle of rank n over $X^\bullet$, and we have a map

$$\sigma : X^\bullet \to \operatorname{Gr}_n(\mathfrak{g}^*) : x \mapsto \mathfrak{g}_x^\perp,$$

where $\operatorname{Gr}_n(\mathfrak{g}^*)$ denotes the usual Grassmannian of n-dimensional subspaces of $\mathfrak{g}^*$, and $\mathfrak{g}_x$ denotes the stabilizer of x in $\mathfrak{g}$.

Lemma 1.6 ([5, Lemma 2.4])**.** *X is pseudo-free iff σ extends to a morphism $\sigma : X \to \operatorname{Gr}_n(\mathfrak{g}^*)$ on all of X.*

In other words, pseudo-freeness means that the general subspaces $\mathfrak{g}_x^\perp$ degenerate at the boundary to specific limits.

Corollary 1.7. *For arbitrary X, there exists a smooth pseudo-free G-variety $\widetilde{X}$ together with a projective, birational, equivariant map $\widetilde{X} \to X$.*

Proof. Take the closure of $X^\bullet$ in $X \times \mathrm{Gr}_n(\mathfrak{g}^*)$ and choose an equivariant resolution of singularities $\widetilde{X}$. Then the map $\widetilde{X} \to X$ will be projective because $\mathrm{Gr}_n(\mathfrak{g}^*)$ is, and it has the stated properties. □

Example 1.8. Let $X = \mathbb{A}^2$ with the standard action by $G = \mathrm{SL}_2$. Then $X^\bullet = \mathbb{A}^2 \setminus \{0\} = N^- \backslash G$ and X is horospherical. The moment map $\Phi : T^*_X = \mathbb{A}^2 \times \mathbb{A}^2 \to \mathfrak{g}^*$ sends (v, ξ) to the functional $(A \in \mathfrak{g} \mapsto \langle \xi, Av \rangle) \in \mathfrak{g}^*$. We see that while $T^*_0 X = \mathbb{A}^2$, we have $T^{\mathfrak{g}}_0 X = \{0\} \times \mathfrak{g}^*$ has dimension 3. In particular $T^*_X \to T^{\mathfrak{g}}_X$ is not surjective, and X is not pseudo-free.

If we consider the blowup $\widetilde{X}$ of $\mathbb{A}^2$ at 0, then $\widetilde{X}$ will be a smooth pseudo-free resolution of X.

1.9. **Filtrations.** The sheaf $\mathfrak{U}_X$ carries two filtrations: one induced by $\mathfrak{U}(\mathfrak{g})^{(n)}$ and the other induced by $\mathcal{D}^{(n)}_X$. Let $\mathcal{F}_X := \pi_*(\mathcal{O}_{T^{\mathfrak{g}}_X})$. We have the following commutative diagram of sheaves of graded algebras:

$$\begin{tikzcd} \mathrm{gr}_{\mathfrak{U}} \mathfrak{U}_X \arrow[rr] \arrow[d] & & \mathcal{F}_X \arrow[d, hook] \\ \mathrm{gr}_{\mathcal{D}} \mathfrak{U}_X \arrow[r, hook] & \mathrm{gr}\, \mathcal{D}_X \arrow[r, phantom, "="] & \mathrm{Sym}^\bullet(\Omega^\vee_X) \end{tikzcd} \tag{1.3}$$

Theorem 1.10. *Let X be smooth and pseudo-free. Then the $\mathfrak{U}$-filtration and the $\mathcal{D}$-filtration of $\mathfrak{U}_X$ coincide and the canonical map*

$$\mathrm{gr}\, \mathfrak{U}_X \to \pi_*(\mathcal{O}_{T^{\mathfrak{g}}_X})$$

is an isomorphism.

Proof. Denote the n^{th} graded component of $\mathcal{F}_X$ by $\mathcal{F}^n_X$. Since $T^{\mathfrak{g}}_X$ is a vector bundle, $\mathcal{F}^1_X$ is locally free and $\mathcal{F}^n_X = \mathrm{Sym}^n_{\mathcal{O}_X}(\mathcal{F}^1_X)$. The map $\bar{\varphi} : \mathrm{gr}_{\mathfrak{U}} \mathfrak{U}_X \to \mathcal{F}_X$ is by construction an isomorphism in degrees 0 and 1. Thus there is a unique homogeneous map $\psi : \mathcal{F}_X = \mathrm{Sym}^\bullet_{\mathcal{O}_X}(\mathcal{F}^1_X) \to \mathrm{gr}_{\mathfrak{U}} \mathfrak{U}_X$ such that $\bar{\varphi} \circ \psi = \mathrm{id}$. By construction, $\mathrm{gr}_{\mathfrak{U}} \mathfrak{U}_X$ is generated by elements of degree ≤ 1, so $\psi \circ \bar{\varphi}$ is surjective. Since $\psi \circ \bar{\varphi}$ is an endomorphism, it is an isomorphism.

To see that the filtrations $F^\bullet_{\mathfrak{U}}$ and $F^\bullet_{\mathcal{D}}$ coincide, observe from diagram (1.3) that $\mathrm{gr}_{\mathfrak{U}} \mathfrak{U}_X \to \mathrm{gr}_{\mathcal{D}} \mathfrak{U}_X$ is injective. Therefore

$$F^n_{\mathfrak{U}} \mathfrak{U}_X = F^{n+1}_{\mathfrak{U}} \mathfrak{U}_X \cap F^n_{\mathcal{D}} \mathfrak{U}_X = F^{n+2}_{\mathfrak{U}} \mathfrak{U}_X \cap F^n_{\mathcal{D}} \mathfrak{U}_X = \cdots = F^n_{\mathcal{D}} \mathfrak{U}_X. \qquad \square$$

Corollary 1.11. *Let X be smooth and pseudo-free. Then for any $n \geq 0$ there is a short exact sequence*

$$0 \to \mathfrak{U}^{(n-1)}_X \to \mathfrak{U}^{(n)}_X \to \mathrm{Sym}^n \mathcal{F}^1_X = (\pi_* \mathcal{O}_{T^{\mathfrak{g}}_X})^n \to 0.$$

In particular, the sheaves $\mathfrak{U}^{(n)}_X$ are locally free as left or right $\mathcal{O}_X$-modules.

1.12. **Functoriality.** We will later define the algebra $\mathfrak{U}(X)$ from the sheaf $\mathfrak{U}_X$, but we want $\mathfrak{U}(X)$ to be an equivariant birational invariant of X. For this we will need some functoriality properties of the sheaves $\mathfrak{U}_X$.

Lemma 1.13 ([5, Corollary 3.2])**.** *Let $\varphi : \widetilde{X} \to X$ be an equivariant proper birational morphism between smooth pseudo-free G-varieties. Then there is a canonical algebra isomorphism*

$$\mathfrak{U}_X \xrightarrow{\sim} \varphi_* \mathfrak{U}_{\widetilde{X}}.$$

2. The algebra $\mathfrak{U}(X)$

Let X be an arbitrary spherical G-variety (dropping the smoothness assumption).

2.1. **Definition of $\mathfrak{U}(X)$.** Let $\varphi : \widetilde{X} \to X$ be equivariant, birational, and proper, with $\widetilde{X}$ smooth and pseudo-free, which exists by Corollary 1.7. Let

$$\overline{\mathfrak{U}}_X := \varphi_* \mathfrak{U}_{\widetilde{X}} \subset \mathcal{D}_X.$$

By Lemma 1.13, the sheaf $\overline{\mathfrak{U}}_X$ does not depend on the choice of resolution $\widetilde{X}$.

We define[2] the algebra of *completely regular* differential operators $\mathfrak{U}(X)$ as follows: take any equivariant completion $X \hookrightarrow \overline{X}$ and let

$$\mathfrak{U}(X) := \Gamma(\overline{X}, \overline{\mathfrak{U}}_{\overline{X}}).$$

Since for any two completions $X \hookrightarrow \overline{X}_1, X \hookrightarrow \overline{X}_2$ the closure of the diagonal $X \hookrightarrow \overline{X}_1 \times \overline{X}_2$ dominates $\overline{X}_i$, Lemma 1.13 again implies that the definition is independent of the choice of compactification.

It is evident from the construction that $\mathfrak{U}(X)$ is an equivariant birational invariant of X.

2.2. **Normalized moment map.** Before proceeding further, let us discuss the abelian analog of $\mathfrak{U}(X)$. Assume first that X is smooth. We can factor the localized moment map $\Phi : T^{\mathfrak{g}}_X \to \mathfrak{g}^*$ into

$$T^{\mathfrak{g}}_X \xrightarrow{\widetilde{\Phi}} \mathfrak{g}^*_X \to \mathfrak{g}^*$$

where $\mathfrak{g}^*_X$ is the normalization (i.e., integral closure) of $\overline{\Phi(T^{\mathfrak{g}}_X)}$ in $T^{\mathfrak{g}}_X$. Since the fibers of $T^*_X \to T^{\mathfrak{g}}_X$ are irreducible by Lemma 1.4 and T^*_X is smooth hence normal, we can equivalently define $\mathfrak{g}^*_X$ as the normalization of the closure of the image of the moment map $T^*_X \to \mathfrak{g}^*$ in $k(T^*_X)$.

This factorization has the defining properties that $\mathfrak{g}^*_X \to \mathfrak{g}^*$ is finite and the *normalized moment map* $\widetilde{\Phi} : T^{\mathfrak{g}}_X \to \mathfrak{g}^*_X$ has irreducible general fibers. When X is complete, this coincides with the Stein factorization of Φ.

Example 1.2 shows that $\overline{\Phi(T^{\mathfrak{g}}_X)} = \overline{\Phi(T^*_{X^\bullet})}$ and hence $\mathfrak{g}^*_X$ is an equivariant birational invariant of X. This allows us to define $\mathfrak{g}^*_X$ also for non-smooth X.

We will use a vanishing theorem to show that $\mathfrak{g}^*_X$ is the commutative version of $\mathfrak{U}(X)$. Let X once again be arbitrary spherical.

Theorem 2.3 ([5, Theorem 6.1])**.** *There are canonical isomorphisms*

$$\operatorname{gr} \mathfrak{U}(X) \xrightarrow{\sim} k[\mathfrak{g}^*_X], \qquad \operatorname{gr} \mathfrak{U}(X)^G \xrightarrow{\sim} k[\mathfrak{g}^*_X]^G.$$

Example 2.4. Take $X = G$ with G acting on X by left translations (this is not a spherical variety, but the above definitions still make sense). Then $T^*_X = T^{\mathfrak{g}}_X = G \times \mathfrak{g}^*$ is the trivial bundle and $\mathfrak{g}^*_X = \mathfrak{g}^*$. In this case $\overline{\mathfrak{U}}_{\overline{X}} = \mathcal{O}_{\overline{X}} \otimes \mathfrak{U}(\mathfrak{g})$ for any completion $\overline{X}$ of X. Therefore $\mathfrak{U}(X) = \mathfrak{U}(\mathfrak{g})$.

Example 2.5. Let $X = H$ and $G = H \times H$, so X is G-spherical. Then $T^*_X = T^{\mathfrak{g}}_X = \mathfrak{h}^* \times H$, where one copy of H acts by left translation on H and coadjoint action on $\mathfrak{h}^*$ while the other copy of H acts by right translation on H and trivial action. The moment map $\Phi : \mathfrak{h}^* \times H \to \mathfrak{g}^* = \mathfrak{h}^* \times \mathfrak{h}^*$ sends $(\xi, h) \mapsto (\xi, -\operatorname{Ad}(h)\xi)$. Then $\overline{\Phi(T^*_X)} = \mathfrak{h}^* \times_{\mathfrak{h}^*/\!\!/H} \mathfrak{h}^*$ because it is true on the regular semisimple locus. So $\mathfrak{g}^*_X$ is the normalization of $\mathfrak{h}^* \times_{\mathfrak{h}^*/\!\!/H} \mathfrak{h}^*$ in $\mathfrak{h}^* \times H$.

[2] The definition of $\mathfrak{U}(X)$ does not require X to be spherical.

Example 2.6. Consider $X = H\backslash G$ horospherical. Let $P^- = N_G(H)$ and $A_X = P^-/H$. The moment map factors as

$$\Phi : \mathfrak{h}^\perp \overset{H}{\times} G \twoheadrightarrow \mathfrak{h}^\perp \overset{P^-}{\times} G \xrightarrow{\varphi} \mathfrak{g}^*.$$

Since $P^-\backslash G$ is complete, φ is proper and $\overline{\Phi(T_X^*)} = G\mathfrak{h}^\perp$. We identify $\mathfrak{g} \cong \mathfrak{g}^*$ via the Killing form. Then $\mathfrak{h}^\perp$ identifies with $\mathfrak{a}_X \bigoplus \mathfrak{u}(\mathfrak{p}^-)$. By [6, Theorem 5.1] of Richardson, the adjoint action map $\mathfrak{u}(\mathfrak{p}^-) \times^{P^-} G \to G\mathfrak{u}(\mathfrak{p}^-)$ has finite general fibers. It follows that φ has finite general fibers. Therefore there is a proper birational map

$$\mathfrak{h}^\perp \overset{P^-}{\times} G \to \mathfrak{g}_X^*.$$

We have checked all but the last sentence of the following:

Lemma 2.7 ([3, Lemma 4.1])**.** *There exists a proper, birational map $Z = \mathfrak{h}^\perp \times^{P^-} G \to \mathfrak{g}_X^*$ with finite general fibers. We have $H^i(Z, \mathcal{O}_Z) = 0$ for $i > 0$.*

The last statement is deduced from the Grauert–Riemenschneider vanishing theorem.

Example 2.8. Let $X = A\backslash \mathrm{SL}_2$ with $G = \mathrm{SL}_2$. Here $T_X^* = (\mathfrak{n} + \mathfrak{n}^-) \times^A G$ where we identify $\mathfrak{g} \cong \mathfrak{g}^*$ via Killing form. The moment map is surjective. Let $e \in \mathfrak{n}, f \in \mathfrak{n}^-, h \in \mathfrak{a}$ be a standard $\mathfrak{sl}_2$-triple. Then $k \to \mathfrak{g}/\!\!/G : c \mapsto e + cf$ is an isomorphism. This gives a section of the finite map $k \to \mathfrak{g}_X^*/\!\!/G \to \mathfrak{g}^*/\!\!/G$. Thus $\mathfrak{g}_X^*/\!\!/G = \mathfrak{g}^*/\!\!/G$ and hence $\mathfrak{g}_X^* = \mathfrak{g}^*$. From this and Theorem 2.3 we can deduce that $\mathfrak{U}(X) = \mathfrak{U}(\mathfrak{g})$.

2.9. **Functoriality.** We mention some functoriality properties of $\mathfrak{U}(X)$.

Lemma 2.10 ([5, Corollary 3.4])**.** *Let $\varphi : X \to Y$ be an equivariant map.*

1. *If φ is dominant with irreducible general fibers, then it induces a map of algebras $\mathfrak{U}(X) \to \mathfrak{U}(Y)$.*
2. *If φ is generically injective, then it induces an algebra homomorphism $\mathfrak{U}(Y) \to \mathfrak{U}(X)$.*

2.11. **The vanishing theorem.** We now present the vanishing theorem [5, Theorem 4.1], which is the heart of the paper [5].

Theorem 2.12. *Let X be a smooth, pseudo-free spherical variety. Then the following hold for all $i > 0$:*

1. $H^i(X, \mathfrak{U}_X^{(n)}) = 0$ *for all* $n \geq 0$,
2. $H^i(X, (\pi_* \mathcal{O}_{T_X^{\mathfrak{g}}})^n) = 0$ *for all* $n \geq 0$,
3. $H^i(T_X^{\mathfrak{g}}, \mathcal{O}_{T_X^{\mathfrak{g}}}) = 0$,
4. $R^i \widetilde{\Phi}_* \mathcal{O}_{T_X^{\mathfrak{g}}} = 0$.

The proof will use the following theorem of Kollár:

Theorem 2.13 ([2, Theorem 7.1], [5, Theorem 4.2])**.** *Let $\varphi : Y \to Z$ be a proper morphism where Y is smooth and Z has rational singularities. Assume that the general fiber F of φ is connected with $H^i(F, \mathcal{O}_F) = 0$ for $i > 0$. Then $R^i\varphi_* \mathcal{O}_Y = 0$ for all $i > 0$.*

By definition, Z has rational singularities if it is normal, of finite type, and there exists a proper birational map $f : \widetilde{Z} \to Z$ from a regular scheme $\widetilde{Z}$ such that $R^i f_* \mathcal{O}_{\widetilde{Z}} = 0$ for $i > 0$.

Proof of Theorem 2.12. First we reduce all assertions to (4). (2) implies (1) by the long exact sequence in cohomology associated to the short exact sequence in Corollary 1.11. (3) implies (2) by the Leray spectral sequence for the affine morphism $\pi : T_X^{\mathfrak{g}} \to X$, which gives

$$\bigoplus_{n=0}^{\infty} H^i(X, (\pi_* \mathcal{O}_{T_X^{\mathfrak{g}}})^n) = H^i(X, \pi_* \mathcal{O}_{T_X^{\mathfrak{g}}}) = H^i(T_X^{\mathfrak{g}}, \mathcal{O}_{T_X^{\mathfrak{g}}}) = 0.$$

Finally (4) implies (3) since $\mathfrak{g}_X^*$ is affine.

To prove (4), we want to apply Theorem 2.13 to the map $\widetilde{\Phi} : T_X^{\mathfrak{g}} \to \mathfrak{g}_X^*$. So the remainder of the proof is to check that $\widetilde{\Phi}$ satisfies the conditions of Theorem 2.13.

Lemma 2.14 ([5, Lemma 4.3])**.** *Let X be smooth. Then $\mathfrak{g}_X^*$ has rational singularities. In particular, it is Cohen–Macaulay.*

Proof. Let X_0 be the horospherical degeneration of X. Then there is an action of W_X on $\mathfrak{g}_{X_0}^*$ with $\mathfrak{g}_X^* = \mathfrak{g}_{X_0}^* /\!\!/ W_X$ by [3, 6.4]. Using this, one can reduce (cf. [1]) to checking that $\mathfrak{g}_{X_0}^*$ has rational singularities, which is Lemma 2.7. □

Lastly, it remains to check that the general fiber of $\widetilde{\Phi} : T_X^{\mathfrak{g}} \to \mathfrak{g}_X^*$ has vanishing higher cohomologies. Now we use the fact [8, Lemma 1] that the higher cohomology of the structure sheaf of a smooth unirational[3] variety vanishes. We will show that the general fiber of $\widetilde{\Phi}$ is unirational. Since this is a generic property, we may assume $X = H\backslash G$. Then $T_X^{\mathfrak{g}} = T_X^*$ (Example 1.2).

Theorem 2.15 ([5, Theorem 5.1])**.** *Let $X = H\backslash G$ be homogeneous. Then the general fiber of $\widetilde{\Phi} : T_X^* \to \mathfrak{g}_X^*$ is unirational.*

The proof of this theorem is much harder for X not spherical, and that proof is the crux of Knop's paper [5]. For spherical varieties, the theorem follows from the next lemma.

Lemma 2.16 ([3, 7.1, 8.1])**.** *Suppose X is smooth and let $\Psi : T_X^* \to \mathfrak{g}_X^* /\!\!/ G$ denote the composition of $\widetilde{\Phi}$ with the projection $\mathfrak{g}_X^* \to \mathfrak{g}_X^* /\!\!/ G$. Let $\alpha \in T_X^*$ be a general point. Then $G\alpha$ is dense in $\Psi^{-1}\Psi(\alpha)$ with codimension $\dim A_X$ in T_X^*. Moreover, $\widetilde{\Phi}^{-1}\widetilde{\Phi}(\alpha)$ contains $G_{\widetilde{\Phi}(\alpha)}\alpha \cong G_{\widetilde{\Phi}(\alpha)}/G_\alpha$ as a dense open, and this orbit is non-canonically isomorphic to A_X.*

Proof. The assertions can all be deduced from the local structure theorem (cf. [4, §4]) for quasi-affine (hence non-degenerate) X. □

Proof of Theorem 2.15. Lemma 2.15 implies that the general fiber of $\widetilde{\Phi}$ contains a torus as a dense open and is hence unirational. □

We have shown that $\widetilde{\Phi} : T_X^{\mathfrak{g}} \to \mathfrak{g}_X^*$ satisfies the conditions of Theorem 2.13, which proves (4) and hence Theorem 2.12. □

3. The algebra $\mathfrak{Z}(X)$

Recall that by construction $\mathfrak{U}(X)$ has a canonical filtration (induced by the filtration on $\mathfrak{U}(\mathfrak{g})$ *or* $\mathcal{D}(X)$). We apply Theorem 2.12 to prove the previously stated Theorem 2.3, which says that $\mathfrak{U}(X), \mathfrak{U}(X)^G$ are the non-commutative versions of $\mathfrak{g}_X^*$ and $\mathfrak{g}_X^* /\!\!/ G$, respectively.

Proof of Theorem 2.3. The second isomorphism

$$\operatorname{gr} \mathfrak{U}(X)^G = \operatorname{gr} k[\mathfrak{g}_X^*]^G$$

follows from the first isomorphism $\operatorname{gr} \mathfrak{U}(X) = \operatorname{gr} k[\mathfrak{g}_X^*]$ by linear reductivity of G (i.e., taking G-invariance is exact).

We prove the first isomorphism. Since both sides are G-birational invariants, we may assume X is smooth, complete, and pseudo-free. Then $\mathfrak{U}(X) = H^0(X, \mathfrak{U}_X)$. Since $\widetilde{\Phi} : T_X^{\mathfrak{g}} \to \mathfrak{g}_X^*$ is proper with irreducible general fibers, $\widetilde{\Phi}_*(\mathcal{O}_{T_X^{\mathfrak{g}}}) = \mathcal{O}_{\mathfrak{g}_X^*}$ and $k[\mathfrak{g}_X^*] = H^0(X, \pi_* \mathcal{O}_{T_X^{\mathfrak{g}}})$. Hence the

[3]A variety is unirational if it is dominated by a rational variety (i.e., a variety birationally equivalent to $\mathbb{P}^n$ for some n).

first isomorphism follows from the long exact sequence associated to the short exact sequence in Corollary 1.11 and the Vanishing Theorem 2.12. ∎

Define $\mathfrak{Z}(X) := \mathfrak{U}(X)^G$, which is a filtered algebra.

3.1. **Chevalley isomorphism.** Theorem 2.3 implies that $\operatorname{gr}\mathfrak{Z}(X) = k[\mathfrak{g}_X^*]^G = k[\mathfrak{g}_X^*/\!\!/G]$. In this graded (commutative) case, we have a "Chevalley isomorphism" $\mathfrak{g}_X^*/\!\!/G \cong \mathfrak{a}_X^*/\!\!/W_X$.

For any subset $\mathfrak{s}$ of $\mathfrak{a}^*$ let $N_W(\mathfrak{s}) := \{w \in W \mid w\mathfrak{s} = \mathfrak{s}\}$, $C_W(\mathfrak{s}) := \{w \in W \mid w|_{\mathfrak{s}} = \mathrm{id}_{\mathfrak{s}}\}$, and $W(\mathfrak{s}) := N_W(\mathfrak{s})/C_W(\mathfrak{s})$.

Lemma 3.2 (Definition of W_X in [3])**.** *There is a canonical isomorphism*

$$\mathfrak{g}_X^*/\!\!/G \cong \mathfrak{a}_X^*/\!\!/W_X.$$

Proof. In [3], the little Weyl group W_X is essentially defined by Galois theory so that the above isomorphism holds. Let us briefly explain, and also explain why this agrees with the definition of W_X given in the previous talk. Assume that X is smooth and non-degenerate.

Consider the composition $T_X^* \to \mathfrak{g}^* \to \mathfrak{g}^*/\!\!/G \cong \mathfrak{a}^*/\!\!/W$. Recall from the previous talk [4, §3] that $Z := T_X^* \times_{\mathfrak{a}^*/\!\!/W} \mathfrak{a}_X^*$ can have several irreducible components, and we singled out one of them, denoted $\widehat{T}_X^*$. By considering the regular locus, we see that the normalization of the image of $\mathfrak{a}_X^* \to \mathfrak{a}^*/\!\!/W$ equals $\mathfrak{a}_X^*/\!\!/W(\mathfrak{a}_X^*)$. The group $W(\mathfrak{a}_X^*)$ acts on Z and permutes the irreducible components transitively. We defined $W_X \subset W(\mathfrak{a}_X^*)$ to be the subgroup of elements which map $\widehat{T}_X^*$ to itself. It acts freely on $\widehat{T}_X^*$ and the map $\widehat{T}_X^*/\!\!/W_X \to T_X^*$ is birational.

By the local structure theorem (cf. [4, Lemma 3.4]), we can choose a section $\widehat{\sigma} : \mathfrak{a}_X^* \to \widehat{T}_X^*$ of $\widehat{\Psi} : \widehat{T}_X^* \to \mathfrak{a}_X^*$. This implies that the generic fibers of $\widehat{\Psi}$ are irreducible (contains a dense G-orbit). Hence the generic fibers of $\widehat{T}_X^*/\!\!/W_X \to \mathfrak{a}_X^*/\!\!/W_X$ are also irreducible. This implies that $k[\mathfrak{a}_X^*]^{W_X}$ is integrally closed in $k[T_X^*]$. We have the following diagram:

$$\begin{tikzcd}
\mathfrak{a}_X^* \arrow[r, "\widehat{\sigma}"] \arrow[drr] & \widehat{T}_X^* \arrow[r, hook] & Z \arrow[rrr] \arrow[d] & & & \mathfrak{a}_X^* \arrow[d] \\
& & T_X^* \arrow[r] & \mathfrak{g}_X^*/\!\!/G \arrow[r] & \mathfrak{a}_X^*/\!\!/W(\mathfrak{a}_X^*) \arrow[r] & \mathfrak{a}^*/\!\!/W
\end{tikzcd}$$

where the map $\mathfrak{a}_X^* \to \mathfrak{g}_X^*/\!\!/G$ is surjective. We have finite surjective maps $\mathfrak{a}_X^* \to \mathfrak{g}_X^*/\!\!/G \to \mathfrak{a}_X^*/\!\!/W(\mathfrak{a}_X^*)$, so by Galois theory, $\mathfrak{g}_X^*/\!\!/G \cong \mathfrak{a}_X^*/\!\!/W_X'$ for some $W_X' \subset W(\mathfrak{a}_X^*)$. This W_X' is the little Weyl group defined in [3]. By construction, $k[\mathfrak{g}_X^*]^G = k[\mathfrak{a}_X^*]^{W_X'}$ is integrally closed in $k[T_X^*]$. Therefore $k[\mathfrak{a}_X^*]^{W_X'} = k[\mathfrak{a}_X^*]^{W_X}$ and consequently $W_X' = W_X$. In the process, we have also shown the Chevalley isomorphism. ∎

Remark 3.3. Since we have seen that $T_X^* \to \mathfrak{a}_X^*/\!\!/W_X$ has irreducible general fibers, it follows that $T_X^* \times_{\mathfrak{a}_X^*/\!\!/W_X} \mathfrak{a}_X^*$ is irreducible. Thus we get that

$$\widehat{T}_X^* = T_X^* \underset{\mathfrak{a}_X^*/\!\!/W_X}{\times} \mathfrak{a}_X^*.$$

Lemma 3.4 ([5, Corollary 6.3])**.** *We have an equality* $\mathfrak{Z}(X) = \mathfrak{D}(X)^G$.

Proof. Since $\mathfrak{Z}(X) = \mathfrak{Z}(X^\bullet)$ and $\mathcal{D}(X) \subset \mathcal{D}(X^\bullet)$, we may assume $X = X^\bullet$ is homogeneous (hence smooth). By Theorem 2.3, we have the diagram of associated graded algebras

$$\begin{tikzcd} \mathrm{gr}\,\mathfrak{Z}(X) \arrow[r, hook] \arrow[d, "\sim"] & \mathrm{gr}\,\mathcal{D}(X)^G \arrow[d, hook] \\ k[\mathfrak{g}_X^*]^G \arrow[r] & k[T_X^*]^G \end{tikzcd}$$

The bottom arrow is an isomorphism by the following lemma, so we deduce that $\mathrm{gr}\,\mathfrak{Z}(X) = \mathrm{gr}\,\mathcal{D}(X)^G$, which proves the claim. $\square$

Lemma 3.5. *For X smooth, the canonical map $k[\mathfrak{g}_X^*]^G \to k[T_X^*]^G$ is an isomorphism.*

Proof. Lemma 2.16 says that the codimension of a general G-orbit in T_X^* equals $r := \dim A_X$. This implies that $\mathrm{tr.deg}_k\, k(T_X^*)^G = r = \dim \mathfrak{g}_X^* /\!\!/ G = r$. Hence $k(T_X^*)^G$ is an algebraic extension over the field of rational functions on $\mathfrak{g}_X^* /\!\!/ G$. By definition, $k[\mathfrak{g}_X^*]$ is integrally closed in $k[T_X^*]$. Therefore $k[\mathfrak{g}_X^*]^G$ is also integrally closed in $k[T_X^*]^G$, which implies that $k[\mathfrak{g}_X^*]^G = k[T_X^*]^G$. $\square$

3.6. **Harish-Chandra isomorphism.** First we recall the classical Harish-Chandra isomorphism for $\mathfrak{Z}(\mathfrak{g})$. By PBW, we have $\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{a}) \oplus (\mathfrak{n}\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{n}^-)$. We identify $\mathfrak{U}(\mathfrak{a})$ with $k[\mathfrak{a}^*]$. Then Harish-Chandra's Theorem says that

$$\mathfrak{Z}(\mathfrak{g}) \hookrightarrow \mathfrak{U}(\mathfrak{g}) \twoheadrightarrow \mathfrak{U}(\mathfrak{a}) = k[\mathfrak{a}^*]$$

induces an isomorphism of $\mathfrak{Z}(\mathfrak{g})$ with $k[\mathfrak{a}^*]^{W\cdot}$, where $W\cdot$ is the shifted dot action $w \cdot \lambda = w(\lambda + \rho) - \rho$. We have an isomorphism $k[\mathfrak{a}^*]^{W\cdot} \cong k[\rho + \mathfrak{a}^*]^W$ by ρ-shift. We now want to generalize this to the setting of spherical varieties, i.e., prove Theorem 0.1.

3.6.1. *Horospherical case.* Now consider $X_0 = H\backslash G$ horospherical. Then we have a right action of A_X on X_0 commuting with the left G-action. Therefore we get a map

$$k[\mathfrak{a}_X^*] = \mathfrak{U}(\mathfrak{a}_X) \to \mathcal{D}(X_0)^G = \mathfrak{Z}(X_0).$$

By taking associated graded and using Theorem 2.3, we see that this map is an isomorphism. Let us identify $\mathfrak{Z}(X_0)$ with $k[\rho + \mathfrak{a}_X^*]$ by ρ-shift.

This gives us another description of the Harish-Chandra isomorphism for $\mathfrak{Z}(\mathfrak{g})$: let $X_0 = N\backslash G$. The G-action on X_0 gives a map $\mathfrak{Z}(\mathfrak{g}) \to \mathfrak{Z}(N\backslash G)$, and we just saw that $\mathfrak{Z}(N\backslash G) = k[\rho + \mathfrak{a}^*]$. One can check ([5, Lemma 6.4]) that this map is compatible with the Harish-Chandra isomorphism, i.e., the following diagram commutes:

$$\begin{tikzcd} \mathfrak{Z}(\mathfrak{g}) \arrow[r] \arrow[d, "\mathrm{HC}"', "\sim"] & \mathfrak{Z}(N\backslash G) \arrow[d, "\sim"] \\ k[\rho + \mathfrak{a}^*]^W \arrow[r, hook] & k[\rho + \mathfrak{a}^*] \end{tikzcd} \tag{3.1}$$

3.6.2. *Affine degenerations.* For a quasi-affine spherical variety X, it is possible to construct a flat family $\Delta : \mathcal{X} \to \mathbb{A}^1$ where $\mathcal{X}$ is a $G \times \mathbb{G}_m$-variety such that $X_t := \Delta^{-1}(t)$ is G-isomorphic to X for $t \neq 0$ and X_0 is horospherical (cf. [7, §2.5], [3, 2.7]). The construction is an easy Rees algebra construction, but we will skip it and just give an example:

Example 3.7. Let $X = H = \mathrm{SL}_2$ and $G = H \times H$. Then $\mathcal{X} = \mathrm{Mat}_2$ and $\Delta : \mathcal{X} \to \mathbb{A}^1$ is the determinant map. The horospherical variety X_0 is the affine variety of matrices with rank ≤ 1. Observe that X_0 is the canonical affine closure of the space of rank 1 matrices, which is isomorphic to $H \times H/A_H^{\mathrm{diag}}(N_H \times N_H^-)$.

3.7.1. *General case.* For any subset $\mathfrak{s}$ of $\mathfrak{a}^*$ let $N_W(\mathfrak{s}) := \{w \in W \mid w\mathfrak{s} = \mathfrak{s}\}$, $C_W(\mathfrak{s}) := \{w \in W \mid w|_{\mathfrak{s}} = \mathrm{id}_{\mathfrak{s}}\}$, and $W(\mathfrak{s}) := N_W(\mathfrak{s})/C_W(\mathfrak{s})$. We have $N_W(\rho + \mathfrak{a}_X^*) \cap C_W(\mathfrak{a}_X^*) = C_W(\rho + \mathfrak{a}_X^*)$. The RHS is trivial since the isotropy group of ρ is trivial. Therefore we have inclusions

$$N_W(\rho + \mathfrak{a}_X^*) \subset W(\mathfrak{a}_X^*) \supset W_X.$$

Theorem 3.8 ([5, Theorem 6.5]). *We have $W_X \subset N_W(\rho + \mathfrak{a}_X^*)$ and there is a canonical isomorphism i_0^G such that the following diagram commutes:*

$$\begin{tikzcd} \mathfrak{Z}(\mathfrak{g}) \arrow[r, "\sim"] \arrow[d] & k[\rho + \mathfrak{a}^*]^W \arrow[d, "res"] \\ \mathfrak{Z}(X) \arrow[r, "i_0^G"] & k[\rho + \mathfrak{a}_X^*]^{W_X} \end{tikzcd}$$

Proof. Let $\mathfrak{X} \to \mathbb{A}^1$ be the affine degeneration of X. We consider $\mathfrak{X}$ as a G-variety. While $\mathfrak{X}$ is not G-spherical, the definition of $\mathfrak{U}(\mathfrak{X})$ still makes sense. It is straightforward to check that $\mathfrak{U}(X) = \mathfrak{U}(X \times \mathbb{A}^1)$, where G acts trivially on $\mathbb{A}^1$ (cf. [5, Lemma 3.5]). Since $X \times \mathbb{A}^1$ and $\mathfrak{X}$ are birationally G-equivalent, we have $\mathfrak{U}(X) = \mathfrak{U}(X \times \mathbb{A}^1) = \mathfrak{U}(\mathfrak{X})$. By functoriality (Lemma 2.10), we get a map

$$i_0 : \mathfrak{U}(X) = \mathfrak{U}(\mathfrak{X}) \to \mathfrak{U}(X_0),$$

which induces a map

$$i_0^G : \mathfrak{Z}(X) \to \mathfrak{Z}(X_0) \cong k[\rho + \mathfrak{a}_X^*].$$

The associated graded of i_0^G corresponds (by Theorem 2.3) to the surjective finite projection $\mathfrak{a}_X^* \to \mathfrak{a}_X^* /\!\!/ W_X$. This implies that i_0^G is injective. In particular, $\mathfrak{Z}(X)$ is commutative and i_0^G is an integral extension. From (3.1), we deduce that the diagram

$$\begin{tikzcd} \mathfrak{Z}(\mathfrak{g}) \cong k[\rho + \mathfrak{a}^*]^W \arrow[r, hook] \arrow[d] & k[\mathfrak{a}^*] \arrow[d] \\ \mathfrak{Z}(X) \arrow[r, "i_0^G"] & k[\rho + \mathfrak{a}_X^*] \end{tikzcd}$$

commutes, where $\mathfrak{Z}(\mathfrak{g}) \to \mathfrak{Z}(X)$ is induced by the G-action on X. Since $\mathfrak{Z}(X)$ is contained in a polynomial ring and its associated graded is integrally closed, it is itself integrally closed. Therefore by Galois theory, there is a subgroup $W_0 \subset N_W(\rho + \mathfrak{a}_X^*)$ such that

$$\mathfrak{Z}(X) = k[\rho + \mathfrak{a}_X^*]^{W_0}.$$

On the other hand, $\mathrm{gr}\, \mathfrak{Z}(X) = k[\mathfrak{a}_X^*]^{W_X}$, and it follows that $W_0 = W_X$.

Finally, the map i_0^G is independent of the degeneration $\mathfrak{X}$ because it factorizes $k[\rho + \mathfrak{a}^*]^W \to k[\rho + \mathfrak{a}_X^*]$ and $\mathrm{gr}\, i_0^G : k[\mathfrak{a}_X^*]^{W_X} \hookrightarrow k[\mathfrak{a}_X^*]$ is canonically defined: hence $i_0^G : \mathfrak{Z}(X) \to k[\rho + \mathfrak{a}_X^*]$ is a filtered map of finite $k[\rho + \mathfrak{a}^*]^W$-modules whose associated graded is uniquely determined. By picking finitely many generators, we can deduce that i_0^G is canonical. $\square$

Remark 3.9. *A priori*, the little Weyl group $W_X \subset W(\mathfrak{a}_X^*)$ is defined as a subquotient of W. The theorem shows that $W_X \subset N_W(\rho + \mathfrak{a}_X^*)$, so it is in fact canonically a subgroup of W.

Lastly, we remark that as the notation suggests, $\mathfrak{Z}(X)$ is indeed the center of $\mathfrak{U}(X)$. Furthermore, $\mathfrak{U}(X)$ is a free $\mathfrak{Z}(X)$-module. See [5, Corollary 7.5] for the (short) proof.

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