# NOTES ON INTERSECTION COMPLEXES AND $L$-FUNCTIONS

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Abstract. This are notes from informal lectures on *Intersection complexes and unramified $L$-factors* (joint paper with Yiannis Sakellaridis). Most of the notes are concerned with background material not covered in the paper – I claim no originality in these parts (except for errors) and appropriate references are given.

## Contents



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## 1. Integral representations of $L$-functions

In this section we give an overview of the program that proposes to associate to any affine spherical variety an integral representation of a certain $L$-function.

1.1. **Schwartz space.** For now, let $\Bbbk$ be any global field, and $G$ a connected split[1] reductive group over $\Bbbk$. Let $X$ be an affine spherical variety over $\Bbbk$. We assume:

- $X$ has good integral models away from a finite set of places (including archimedean). If $\Bbbk$ is a function field we just assume $X$ is defined over $\mathbb{F}_q$. For number fields, such good integral models exist by [Sa12, Proposition 2.3.5].
- If $\Bbbk$ is a function field, assume $X$ behaves "like a characteristic 0" variety.

Let $\mathbb{A}$ denote the adeles of $\Bbbk$. For a place $v$ of $\Bbbk$, let $F_v$ denote the local field with ring of integers $\mathcal{O}_v$. Let $[G] = G(\Bbbk)\backslash G(\mathbb{A})$.

1.1.1. Let $X^\bullet \subset X$ denote the open $G$-orbit[2]. In [Sa12, §3], Sakellaridis conjectures that there exists a nice Schwartz space

$$\mathcal{S}_X(X^\bullet(\mathbb{A})) := \bigotimes_v{}' \mathcal{S}_X(X^\bullet(F_v)) \subset C^\infty(X^\bullet(\mathbb{A}))$$

which is a restricted tensor product of local Schwartz spaces with respect to a certain *basic function* $\Phi_v^0 \in C^\infty(X^\bullet(F_v))$. The Schwartz space consists of functions on $X^\bullet(\mathbb{A})$, but its definition will depend on the affine embedding $X^\bullet \hookrightarrow X$. For example, the basis function $\Phi_v^0$ should have support contained in $X(\mathcal{O}_v) \cap X^\bullet(F_v)$.

We will return to the basis function in more detail later.

1.2. **Sakellaridis' conjecture.** Starting from a Schwartz function $\Phi \in \mathcal{S}_X(X^\bullet(\mathbb{A}))$, we can form the $X$-*theta series* $\Theta_X(\Phi, g) \in C^\infty([G])$.

$$\Theta_X(\Phi, g) = \sum_{\gamma \in X^\bullet(\Bbbk)} g \cdot \Phi(\gamma) \tag{1.1}$$

As long as $\Phi$ has some reasonable growth conditions, this sum will be absolutely convergent. In fact $\Theta_X(\Phi, g)$ will be of moderate growth in $g$ (see [Sa12, Proposition 3.1.3], [BP, Proposition A.1.1(ix)].

*Remark* 1.2.1. The formula (1.1) looks similar to the formula for the theta function in the theta correspondence: there we summed over $E^*$ where $E^*$ is a Lagrangian vector space in the $G = \mathrm{Sp}(E \oplus E^*)$-Hamiltonian variety $T^*(E^*) = E \oplus E^*$. Here we are summing over $X^\bullet$, which is a Lagrangian variety in the $G$-Hamiltonian variety $T^*X^\bullet$. These two construction should fall within the same unifying framework of [BZSV]. However the $G$-action on $\mathcal{S}(E^*)$ is more complicated because $E^*$ is not a $G$-stable Lagrangian while $X^\bullet$ is. Slogan: it's harder to quantize a $G$-Hamiltonian variety $M$ if you don't have a $G$-stable Lagrangian.

**Definition 1.2.2.** We define a "zeta integral"

$$\mathcal{Z}(\chi, \Phi, f) := \mathcal{P}_\Phi(\chi \cdot f) = \int_{[G]} \chi(g) \cdot \Theta_X(\Phi, g) \cdot f(g) dg$$

where $\chi$ is an idele class character of $G$ and $f \in \mathcal{A}_0(G)$ is a cusp form.

[1] There should be generalizations to non-split groups, but there may be subtleties with the Galois group.

[2] $X_{\bar{\Bbbk}}$ has a unique $G_{\bar{\Bbbk}}$-orbit by sphericity, which descends to a unique $G$-orbit of $X$.

This zeta integral *a priori* converges absolutely when $\log \chi$ lies in a translate of a rational cone prescribed by $X$ (these are called sufficiently $X$-positive characters). (Even though $\Theta_X(\Phi, g)$ has moderate growth, the integral may not converge when $Z(G)^0$ is non-trivial.)

Naively we would like to say that the "gcd" of $\mathcal{Z}(\chi, \Phi, f)$ over all $\Phi$ should give us an $L$-function. This is sometimes the case, but the general expectation is more nuanced – we refer to §1.4 for the precise expected relation to $L$-functions.

*Remark* 1.2.3. In order to replace $\chi$ with $s \in \mathbb{C}$ (for $L$-function comparisons) we would need to have a canonical family of idele class characters. This requires that $Z(G)^0$ be non-trivial. We can always reduce to this assumption when $X$ is not homogeneous: We can replace $G$ by $G \times^{Z(G)^0} Z(X^\bullet)$ so that $Z(G)^0 = Z(X^\bullet)$, where $Z(X^\bullet) := \mathrm{Aut}^G(X^\bullet)^0$ and $\mathrm{Aut}^G(X^\bullet) = N_G(H)/H$ is the group of $G$-automorphisms of $X^\bullet = H\backslash G$. It follows from some structural results on spherical varieties (cf. [Sa12, Proposition 2.2.6]) that if $X^\bullet$ is quasi-affine but not affine, then $Z(X^\bullet)$ is a non-trivial torus. (It is a non-trivial theorem that $\mathrm{Aut}^G(X^\bullet)$ is always diagonalizable.)

**Conjecture 1.2.4** (Weak form, [Sa12, Conjecture 3.2.4])**.** *For any $\Phi \in \mathcal{S}_X(X^\bullet(\mathbb{A}))$ and $f \in \mathcal{A}_{\mathrm{cusp}}(G)$, the integral $\mathcal{Z}(\chi, \Phi, f)$ admits meromorphic continuation to the space of all idele class characters $\chi$ on $G$. One can further hope for a "Fourier transform" and functional equation.*

*Remark* 1.2.5. This conjecture is a generalization of the conjecture of Braverman–Kazhdan [BrK] seeking to generalize Godement–Jacquet theory. In the Braverman–Kazhdan conjecture, you start with some $G$ with a "determinant" map $G \to \mathbb{G}_m$ and an irreducible representation $\rho$ of $\check{G}$ (satisfying some compatibility condition). Then to $\rho$ you can attach a reductive monoid $M_\rho \supset G$, which is an affine spherical variety for $G \times G$. Then Braverman–Kazhdan's Schwartz space $\mathcal{S}_\rho(G(\mathbb{A}))$ is just $\mathcal{S}_{M_\rho}(G(\mathbb{A}))$ in our notation, with the same conjectural properties. In that case we hope that the "gcd" of $\mathcal{Z}_{M_\rho}(|\det|^s, \Phi, f)$ equals the $L$-function associated to $\rho$.

There is also the following strong form of the conjecture, which seems far out of reach since it is not even known for parabolic Eisenstein series, i.e., the case $X = \overline{N_P\backslash G}^{\mathrm{aff}}$, where $N_P$ is the unipotent radical of a parabolic $P$ which is not a Borel.

**Conjecture 1.2.6** (Strong form, [Sa12, Conjecture 3.2.2])**.** *For every $\Phi \in \mathcal{S}_X(X^\bullet(\mathbb{A}))$, the $X$-Eisenstein series*

$$\mathcal{E}_X(\chi, \Phi, g) = \int_{Z(X^\bullet)(\mathbb{A})} \Theta_X(z \cdot \Phi, g)\chi(z) dz$$

*originally defined for sufficiently $X$-positive idele class characters, admits meromorphic continuation everywhere.*

For the problem of meromorphic continuation, it is not really important what the local Schwartz spaces $\mathcal{S}_X(X^\bullet(F_v))$ are: the integral can have arbitrary behavior at a finite set of places. Therefore what really matters is the choice of the basic function $\Phi_v^0 \in \mathcal{S}_X(X^\bullet(F_v))$.

1.3. **Local conjecture of Sakellaridis–Venkatesh.** In order to state the conjectural relation between $X$-period integrals and $L$-functions, we need to take a brief interlude into the local Plancherel theory of $L^2(X^\bullet(F_v))$, assuming that $X^\bullet(F_v)$ has a $G(F_v)$-eigenmeasure.

Let $\check{G}_X$ denote the spherical dual group of $X$. Let $G_X$ denote the split Langlands dual group of $\check{G}_X$. The Whittaker–Plancherel theorem, which can be deduced from the usual Harish-Chandra Plancherel theorem [SV, 6.3], gives a direct integral decomposition

$$L^2((N_X(F_v), \psi)\backslash G_X(F_v)) \cong \int_{\mathrm{Temp}(G_X)} \sigma^{\oplus m(\sigma)} d\mu_{G_X}(\sigma)$$

where $\mathrm{Temp}(G_X)$ denotes tempered $G_X(F_v)$-representations.

Now a paraphrase of the local conjecture is:

**Conjecture 1.3.1** ([SV, Conjecture 16.5.1])**.** *One expects a map* $\iota_* : \mathrm{Temp}(G_X) \to \mathrm{Irr}(G)$ [3] *such that there is a unitary isomorphism*

$$L^2(X^\bullet(F_v)) \cong \int_{\mathrm{Temp}(G_X)} \iota_*(\sigma)^{\oplus m(\sigma)} d\mu_{G_X}(\sigma). \tag{1.2}$$

Here $\iota_*$ sends tempered $G_X(F_v)$-representations to (not necessarily tempered) unitary representations of $G(F_v)$. The Arthur packet is specified by the map $\check{G}_X \times \mathrm{SL}_2 \to \check{G}$.

For $\pi = \iota_*(\sigma)$ the decomposition (1.2) induces a canonical embedding[4]

$$M_\pi : \pi \otimes \overline{\pi} \to C^\infty(X^\bullet(F_v) \times X^\bullet(F_v))$$

At least formally, we can pair this with $\Phi_{1,v} \otimes \Phi_{2,v}$ for Schwartz functions $\Phi_1, \Phi_2 \in \mathcal{S}_X(X^\bullet(F_v))$ to get a canonical Hermitian pairing

$$\alpha_{\Phi_1,\Phi_2,\pi} : \pi \otimes \overline{\pi} \to C^\infty(X^\bullet(F_v) \times X^\bullet(F_v)) \overset{\Phi_1 \otimes \Phi_2}{\to} \mathbb{C}. \tag{1.3}$$

It's not clear this pairing converges, but it's expected to almost everywhere.

*Remark* 1.3.2. We can write $\alpha_{\Phi_1,\Phi_2,\pi} = \ell_{\Phi_1,\pi} \otimes \ell_{\Phi_2,\overline{\pi}}$ where $\ell_{\Phi,\pi} : \pi \to \mathbb{C}$ is a linear functional

$$\ell_{\Phi,\pi} : \pi \hookrightarrow L^2(X^\bullet(F_v)) \overset{\Phi}{\to} \mathbb{C}$$

induced by (1.2), but the embedding $\pi \hookrightarrow L^2(X^\bullet(F_v))$ is not quite canonical (we can modify by an isometry). In the fortuitous circumstance where $X^\bullet$ "unfolds" to Whittaker model (cf. [SV, Theorem 9.5.9]), there is a canonical choice of $\ell_{\Phi,\pi}$, and it becomes reasonable to compare the $X$-period to $\ell_{\Phi,\pi}$ without "squaring".

1.3.3. When $X$ is *strongly tempered* (cf. [SV, 6.2]), a condition which implies that $\check{G}_X = \check{G}$, then $M_\pi : \pi \otimes \overline{\pi} \to C^\infty(X^\bullet(F_v) \times X^\bullet(F_v))$ is defined by extension by zero from the $(G \times G)(F_v)$-orbit of $\Delta X^\bullet(F_v) \subset X^\bullet(F_v) \times X^\bullet(F_v)$ by

$$M_\pi(u \otimes w)(x, x) = \int_{G_x} \langle \pi(h)u, w \rangle dh.$$

where $G_x$ is the $G(F_v)$-stabilizer of $x$. The fact that this integral converges and agrees with the Plancherel theorem description above is one of the main theorems [SV, Theorem 6.2.1] of Sakellaridis–Venkatesh. In particular, they show that a Plancherel decomposition like (1.2) exists when $X$ is strongly tempered (but they don't precisely determine the multiplicities).

**Example 1.3.4.** Suppose $X = H\backslash G$ is affine homogeneous and strongly tempered. (This is the case for the Gross–Prasad variety $\mathrm{SO}_n\backslash\mathrm{SO}_n \times \mathrm{SO}_{n+1}$.) Then $\Phi_v$ can be taken to be an appropriate smoothening of the delta function $\mathbf{1}_{x_0}$ for some base point $x_0 \in X(F_v)$. Then $\alpha_{\mathbf{1}_{x_0},\mathbf{1}_{x_0},\pi}(u \otimes w)$ is just evaluation of $M_\pi(u \otimes w)$ at $(x_0, x_0)$, so we get

$$\alpha_{\mathbf{1}_{x_0},\mathbf{1}_{x_0},\pi}(u \otimes w) = \int_{H(F_v)} \langle \pi(h)u, w \rangle dh.$$

This is the familiar pairing that shows up in the original Ichino–Ikeda conjecture.

[3] To be precise, $\mathrm{Irr}(G)$ and $L^2(X^\bullet(F_v))$ should be replaced by direct sum over all pure inner forms of $X^\bullet$, and $\mathrm{Temp}(G_X)$ should be replaced by generic representations ranging over Whittaker data for $G_X(F_v)$.

[4] If $\iota_*$ is not injective, which means $\check{G}_X \to \check{G}$ is not injective, this embedding depends on $\sigma$ not just $\pi$.

1.4. **Generalized Ichino-Ikeda conjecture.** We now explain the conjecture of Sakellaridis–Venkatesh on the relation between $X$-period integrals and special values of $L$-functions. They only state the conjecture when $X$ is affine homogeneous, so I am taking some liberties here.

There are probably hidden assumptions needed for these conjectures to make sense: for example, $X$ should not have type N roots[5].

1.4.1. Let $\pi = \otimes_v \pi_v$ be a tempered (i.e., $\pi_v$ is tempered at every place) and cuspidal[6] $G(\mathbb{A})$-representation (with an embedding into $\mathcal{A}_0(G)$).

**Definition 1.4.2.** For $f \in \pi$, we can formally write the $X$-period integral

$$\mathcal{P}_\Phi(f) := \int_{[G]} \Theta_X(\Phi, g) \cdot f(g) dg. \tag{1.4}$$

However when $X$ is not affine homogeneous, this integral may diverge. So we *conjecturally assume* that $\mathcal{Z}(\chi, \Phi, f)$ has meromorphic continuation to $\chi = 1$ and understand $\mathcal{P}_\Phi(f)$ to mean $\mathcal{Z}(1, \Phi, f)$ in the sense of meromorphic continuation in what follows.

Ignoring convergence issues, we formally have $\mathcal{Z}(\chi, \Phi, f) = \mathcal{P}_\Phi(\chi \cdot f)$, so relating the zeta integral to an $L$-function is roughly equivalent to relating the $X$-period to a special value of an $L$-function. Hence in what follows we will only discuss the period and not the zeta integral, but the reader should think of the two as equivalent.

1.4.3. *Multiplicity one assumption.* For psychological purposes, from now on we will assume that for all place $v$, we have

$$\dim \mathrm{Hom}_{G(F_v)}(\pi_v, C^\infty(X^\bullet(F_v))) \leq 1.$$

[7] In practice this is usually a hard theorem. Jacquet [J] also showed that even when the multiplicity one condition above fails, e.g., in the case $X = U_n \backslash \mathrm{GL}_n$, the period integral can still admit Euler product.

A consequence of the multiplicity $\leq 1$ assumption is that $\check{G}_X$ is a subgroup of $\check{G}$ (*a priori* $\check{G}_X \to \check{G}$ is finite).

1.4.4. Let $\Phi_i = \otimes_v \Phi_{i,v} \in \mathcal{S}_X(X^\bullet(\mathbb{A}))$ be Schwartz functions for $i = 1, 2$. We can consider the period $\mathcal{P}_\Phi|_\pi : \pi \to \mathbb{C}$ as a linear functional on $\pi$.

Assume that at each place $v$, the $G(F_v)$-representation $\pi_v$ occurs in the Plancherel decomposition of $L^2(X^\bullet(F_v))$. Here is where the local $L^2$-theory comes in: (1.3) gives a canonical pairing $\alpha_{\Phi_1,\Phi_2,\pi_v} : \pi_v \otimes \overline{\pi}_v \to \mathbb{C}$.

**Conjecture 1.4.5** (Global conjecture, [SV, Conjecture 17.4.1])**.** *Let $\pi$ be a tempered and cuspidal automorphic representation. For $\Phi_1, \Phi_2 \in \mathcal{S}_X(X^\bullet(\mathbb{A}))$ we have*

$$(\mathcal{P}_{\Phi_1} \otimes \mathcal{P}_{\Phi_2})|_{\pi \otimes \overline{\pi}} = \frac{1}{|S_\phi|} \prod_v^* \alpha_{\Phi_1,\Phi_2,\pi_v}$$

*where $S_\phi$ is the centralizer in $\check{G}_X$ of the "global Langlands parameter" $\phi$ associated to $\sigma$, where vaguely $\pi = \iota_*(\sigma)$.*

[5]'N' is for normalizer. We want to avoid examples like $\mathrm{O}_n \backslash \mathrm{GL}_n$, which Jacquet, Mao have shown has some metaplectic behavior which is not expected to be related to $L$-functions.

[6]If $\pi$ is a tempered representation that is not cuspidal, one can hope that there exists a regularization of the period integral in good cases.

[7]When $X \neq X^\bullet$, the space $C^\infty(X^\bullet(F_v))$ may need to be replaced by something else.

*Remark* 1.4.6. The horopsherical variety $X = \overline{N\backslash G}$ fails the multiplicity one assumption, and the $X$-period corresponds to integrating against the constant term of an automorphic form. The constant term does *not* factorize into an Euler product, so this is an indication that at least some kind of assumption is necessary for the global conjecture to hold.

Perhaps the most reasonable general class of spherical varieties $X$ to consider first are affine varieties that are strongly tempered and wavefront.

1.5. **Understanding the Euler product.** The Euler product $\prod_v^* \alpha_{\Phi_1,\Phi_2,\pi_v}$ does not converge above, so to make sense of it we need to normalize the $\alpha_{\Phi_1,\Phi_2,\pi_v}$ by certain $L$-factors so get the product to converge.

This is where the connection to $L$-functions comes in: at all but finitely many places, $\Phi_v = \Phi_v^0$ is the basic function in $\mathcal{S}_X(X^\bullet(F_v))$ and $\pi_v$ is an unramified principal series representation.

**Conjecture 1.5.1.** *There exists a $\mathbb{Z}$-graded finite-dimensional representation $\rho_X = \bigoplus_d \rho_X^d$ of the spherical $L$-group*[8] ${}^L G_X := \check{G}_X \rtimes W_{F_v}$ *(where $W_{F_v}$ is the Weil group) such that if*

- $\Phi_v^0$ *is the "IC function" associated to $X(\mathcal{O}_v)$,*
- $\pi_v = \iota_*(\sigma_v)$ *where $\sigma_v$ (and $\pi_v$) are unramified principal series,*
- $u \in \pi_v$ *is $G(\mathcal{O}_v)$-invariant vector normalized by $\|u_v\|^2 = 1$;*

*then one has*

$$\alpha_{\Phi^0,\Phi^0,\pi_v}(u,u) = L^\#_{X,v}(1/2,\sigma_v) := \Delta_v(0) \cdot \frac{L_{X,v}(1/2,\sigma_v)}{L(1,\sigma_v,\mathrm{Ad}_{\check{G}_X})}$$

*where $\Delta_v(s)$ is a product of local zeta factors which depends only on $X$ and not the representation $\sigma_v$, and*

$$L_{X,v}(s,\sigma_v) := \prod_d L_v(s + \frac{d-1}{2}, \sigma_v, \rho_X^d).$$

Given this conjecture, we can normalize $\alpha_{\Phi_1,\Phi_2,\pi_v}$ at every place by:

$$\alpha^\flat_{\Phi_1,\Phi_2,\pi_v} = \frac{1}{L^\#_{X,v}(1/2,\sigma_v)} \cdot \alpha_{\Phi_1,\Phi_2,\pi_v}.$$

Now part of Conjecture 1.4.5 is the requirement that the global $L$-function $L^\#_X(s,\sigma) = \prod_v L^\#_{X,v}(s,\sigma_v)$, which is *a priori* defined for $\mathrm{Re}(s) \gg 0$, can be meromorphically continued so that one can evaluate at $s = 1/2$. Given this, Conjecture 1.4.5 says that

$$(\mathcal{P}_{\Phi_1} \otimes \mathcal{P}_{\Phi_2})|_{\pi\otimes\overline{\pi}} = \frac{1}{|S_\phi|} \cdot L^\#_X(1/2,\sigma) \cdot \prod_v \alpha^\flat_{\Phi_1,\Phi_2,\pi_v}.$$

*Remark* 1.5.2. The conjecture of [BZSV] says that we should further expect the representation $(\rho_X, V_X)$ to satisfy the property that $\check{G} \times^{\check{G}_X} V_X$ is a $\check{G}$-Hamiltonian variety.

## 2. Examples

Before stating our results, we summarize some old and new examples of how this correspondence $X \rightsquigarrow \rho_X$ looks like.

[8]There is currently no general definition of the Weil group action on $\check{G}_X$, so we only state the $L$-group for ideological completeness.

2.1. **Classical examples.** In the table below, $T^*V = V \oplus V^*$. The names signify who discovered the corresponding integrals and determined what $\rho_X$ should be.

| | $X \circlearrowleft G$ | $\check{G}_X$ | $\rho_X$ |
|---|---|---|---|
| Usual Langlands | Group $G' \circlearrowleft G' \times G' = G$ | $\check{G}'$ | $\check{\mathfrak{g}}'$ |
| Whittaker normalization | $(N, \psi)\backslash G$ | $\check{G}$ | 0 |
| Tate's thesis | $\mathbb{A}^1 \circlearrowleft \mathbb{G}_m$ | $\mathbb{G}_m$ | $T^*\mathbb{C}$ |
| Hecke | $\mathbb{G}_m\backslash \mathrm{PGL}_2$ | $\check{G} = \mathrm{SL}_2$ | $T^*\mathrm{std}$ |
| Rankin–Selberg, Jacquet–Piatetski-Shapiro–Shalika | $\mathrm{GL}_n \times \mathbb{A}^n \circlearrowleft \mathrm{GL}_n \times \mathrm{GL}_n$, $H =$ diagonal mirabolic | $\check{G}$ | $T^*(\mathrm{std} \otimes \mathrm{std})$ |
| *loc cit.* | $\mathrm{GL}_n\backslash \mathrm{GL}_{n+1} \times \mathrm{GL}_n$ | $\check{G}$ | $T^*(\mathrm{std} \otimes \mathrm{std})$ |
| Gan–Gross–Prasad | $\mathrm{SO}_{2n}\backslash \mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n}$ | $\check{G} = \mathrm{SO}_{2n} \times \mathrm{Sp}_{2n}$ | $\mathrm{std} \otimes \mathrm{std}$ |
| Jacquet, Ichino | $\mathrm{PGL}_2^{\mathrm{diag}}\backslash \mathrm{PGL}_2^{\times 3}$ | $\check{G} = \mathrm{SL}_2^{\times 3}$ | $\mathrm{std} \otimes \mathrm{std} \otimes \mathrm{std}$ |

A more extensive list of examples may be found at https://www.jonathanpwang.com/notes/RelativeDualitydb.html

2.2. **New examples.** Smooth affine spherical varieties over $\mathbb{C}$ have all been classified by [KSt]. It may still be fruitful to peruse their tables to find new examples of number theoretic interest. However, when one compares this classification with the full classification of spherical varieties, one finds that there are many more affine spherical varieties that are singular.

By considering these singular affine spherical varieties, one can find new examples number theorists haven't discovered before:

**Example 2.2.1** ([SW, Example 1.1.3])**.** A new family of examples was provided by Sakellaridis [Sa12] generalizing the Rankin–Selberg convolution to an integral representation of the $n$-fold tensor product $L$-function for $\mathrm{GL}_2$.

Let $G = (\mathbb{G}_m \times \mathrm{SL}_2^{\times n})/\mu_2^{\mathrm{diag}} = \mathrm{GL}_2 \times_{\det} \cdots \times_{\det} \mathrm{GL}_2$ acting on $X^\bullet = H_0\backslash \mathrm{SL}_2^{\times n}$ where

$$H_0 = \left\{ \begin{pmatrix} 1 & \\ x_1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & \\ x_2 & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & \\ x_n & 1 \end{pmatrix} \middle| x_1 + x_2 + \cdots + x_n = 0 \right\},$$

where $a \in \mathbb{G}_m$ acts as left multiplication by $\left(\begin{smallmatrix} a^{-1} & \\ & a \end{smallmatrix}\right)$. Let $X$ be the affine closure of $X^\bullet$.

In this case $\check{G}_X = \check{G} = \mathrm{GL}_2 \times_{\det} \cdots \times_{\det} \mathrm{GL}_2$ and it follows from our work that

$$\rho_X = T^*(\mathrm{std}_2^{\otimes n} \otimes \mathrm{std}_1).$$

In fact, this is a case where the $X$-period $\mathcal{P}_{X,\Phi^0}$ unfolds to the Whittaker period.

**Proposition 2.2.2** ([SW, Proposition 9.3.1])**.** *Let $\Phi^0 = \prod \Phi_v^0$ be the product of basis functions. If $f \in \pi$ is everywhere unramified and normalized so first Whittaker coefficient $W_{f,v}(1) = 1$ at all places, then*

$$\mathcal{Z}(|\det|^s, \Phi^0, f) = L(s + 1 - \frac{n}{2}, \pi, \mathrm{std}_2^{\otimes n}).$$

*for* $\mathrm{Re}(s) \gg 0$.

Our methods do not give any meromorphic continuation results about the zeta integral.

2.2.3. *Explicit description of $X$.* Let $\mathrm{SL}_2^n = \prod^n \mathrm{Sp}(V_i)$ where $V_i$ are 2-dimensional symplectic spaces with fixed symplectic forms $\omega_i$. Then the space $X$ can be described as the set of $(\underline{v}, \zeta)$ where

- $\underline{v} = (v_i)_i \in \prod V_i$,
- $\zeta \in \wedge^n(V_1 \oplus \cdots \oplus V_n)$, such that:

- $\zeta \wedge \eta = 0$ for any $\eta = \sum c_i v_i$ with $\sum c_i = 0$,
- $\zeta \wedge v_n = \sum_i (-1)^i v_1 \wedge v_2 \wedge \cdots \wedge \omega_i \wedge \cdots \wedge v_n$, where $\omega_i$ is the fixed symplectic form on $V_i$. (The condition is really symmetric in the $v_i$'s, up to sign.)

The open $X^\bullet$ is the subset where the $v_i$'s are all nonzero.

**Example 2.2.4.** Here is a conjectural attempt to generalize the previous Example 2.2.1. I have not checked the combinatorics, and the formulation is by observation from rank 2 cases in Wasserman's table [Was].

Let $G_0$ be a simply connected split semisimple group. Let $N_0$ denote the unipotent radical of a Borel, and define $H_0$ to be the kernel of a non-degenerate character $N_0 \to \mathbb{G}_a$. Define $X = \overline{H_0\backslash G_0}^{\mathrm{aff}}$. Let $G = \mathbb{G}_m \times^{\mu_2} G_0$ act on $X$ where $\mathbb{G}_m$ acts on the left by $2\check{\rho}_{G_0}$, the cocharacter corresponding to the sum of positive coroots. Here $\mu_2 \to G_0$ is trivial if $\check{\rho}_{G_0}$ is in the coweight lattice. (For example when $G_0 = \mathrm{SL}_3$ we have $G = \mathbb{G}_m \times \mathrm{SL}_3$ where $\mathbb{G}_m$ acts by $\check{\rho}_{\mathrm{SL}_3}$.) Then for this $(G, X)$, I predict that

$$\rho_X = T^*(V^{\check{\rho}_{G_0}})$$

where $V^{\check{\rho}_{G_0}}$ really means the highest weight representation of $\check{G}$ with highest weight corresponding to $\mathbb{G}_m \to G : a \mod \mu_2 \mapsto (a, (2\check{\rho}_{G_0})(a)) \mod \mu_2$.

## 3. Summary of results

A rough summary of the content of [SW] is that we "almost" prove Conjecture 1.5.1 over a local function field in the case where $\check{G}_X = \check{G}$ and $X$ has no spherical roots of type N.

### 3.1. Previous work.

When $X = H\backslash G$ is *affine homogeneous* (by [Lun73], [Ric77], this is equivalent to $H$ being reductive), Sakellaridis ([Sa08, Sa13]) proved[9] Conjecture 1.5.1 using function-theoretic techniques. In these cases, $X$ is smooth so $\Phi_0$ is just the indicator function of $X(\mathcal{O}_v)$ in $X(F_v)$.

However when $X$ is singular, geometric considerations are needed if we want to understand the relation between intersection complexes (perverse sheaves) and $L$-values.

Explicit formulas for the "IC function" have been previously established (as well as other geometric results) in the following cases:

- Braverman–Finkelberg–Gaitsgory–Mirković [BFGM]:
  - $X = \overline{N^-\backslash G}$, $\check{G}_X = \check{T}$, $V_X = \check{\mathfrak{g}}^*/\check{\mathfrak{t}}^*$. Note that $\check{G} \times^{\check{G}_X} V_X = T^*(\check{G}/\check{T})$, which has some global incarnation as geometric Eisenstein series
- Bouthier–Ngô–Sakellaridis [BNS]:
  - $X \supset G'$ is an $L$-monoid, so here the group is $G = G' \times G'$, $\check{G}_X = \check{G}'$, and $V_X = \check{\mathfrak{g}}' \oplus T^*V^{\check{\lambda}}$ where $\check{\lambda}$ is the coweight appearing in the definition of an $L$-monoid.

### 3.2. Main result on Plancherel measure.

From now on, we will work entirely in the local setting and drop the subscript $_v$. Let $F = \mathbb{F}_q(\!(t)\!)$ denote a local function field and $\mathcal{O} = \mathbb{F}_q[\![t]\!]$ its ring of integers. We allow $G$ to be a (not necessarily split) connected reductive group over $\mathbb{F}_q$. Let $X$ be an affine spherical variety over $\mathbb{F}_q$.

[9] Technically, his work gives $\rho_X$ as a *virtual* representation but didn't show it's a true representation.

3.2.1. In the unramified situation, the map $\iota_*$ is more transparent: an unramified $G_X(F)$-representation is a principal series, the induction of an unramified character on $T_X(F)$. Such a character corresponds to a point in $\check{T}_X(\mathbb{C})$. Likewise, an unramified $G(F)$-representation $\pi$ corresponds to a point in $\check{T}(\mathbb{C})$. The map $\iota_*$ on unramified representations is just the map of dual maximal tori $\check{T}_X \to \check{T}$ induced by $\check{G}_X \to \check{G}$.

3.2.2. Suppose $\check{G}_X = \check{G}$ and $\pi$ is an unramified $G(F)$-representation. Then $\pi$ is the induction of a character $\chi : T(F) \to \mathbb{C}$. Note that the local $L$-factor $L(s, \chi, \rho'_X)$ can be defined for any $\check{T}$-representation $\rho'_X$, without requiring $\rho'_X$ to be a $\check{G}$-representation. With this, we can state our main theorem:

**Theorem 3.2.3** (Sakellaridis–W)**.** *Assume $X$ affine spherical, $\check{G}_X = \check{G}$ and $X$ has no type N roots*[10]*. Then there is a $(\check{T} \rtimes \langle \mathrm{Fr} \rangle)$-representation $\rho'_X$ such that: for $u \in \pi_\chi$ a $G(\mathcal{O})$-invariant vector normalized by $\|u\|^2 = 1$, we have*

$$\alpha_{\Phi^0, \Phi^0, \pi_\chi}(u, u) = \Delta(0) \frac{L(1/2, \chi, \rho'_X)}{L(1, \chi, \mathrm{Ad})}$$

*The $\check{T}$-representation $\rho'_X$ has the properties:*

(i) *$\rho'_X$ has an action of $(\mathrm{SL}_2)_\alpha$ for every simple root $\alpha$*
   - *We do not check the Weyl/Serre relations, which would imply $\rho'_X$ is a $\check{G}$-representation.*

(ii) *Assuming $\rho'_X$ is a $\check{G}$-representation, we determine its highest weights with multiplicities (in terms of prime $B$-divisors of $X$).*

*The factor $\Delta(0)$ can be extracted from* [SV, (17.8)].

We are confident that $\rho'_X$ is indeed a $\check{G}$-representation, but for clarity let us call $\rho_X$ the $\check{G}$-representation that it would have to equal according to (ii) above. Below we explain the definition of the assignment $X \rightsquigarrow \rho_X$.

*Remark* 3.2.4. Note that for a given $X$, if $\rho_X$ is minuscule, then there is no question of weight multiplicities, so $\rho'_X$ must equal $\rho_X$, i.e., we prove Conjecture 1.5.1 in those cases.

We also show that if $X$ is affine homogeneous, then $\rho_X$ must be minuscule.

We can also reduce the checking of $\rho'_X = \rho_X$ to the cases where $X = \overline{H\backslash G}^{\mathrm{aff}}$ and $G$ has semisimple rank 2. There are only 5-6 such cases with $\rho_X$ non-minuscule (four of which come from Example 2.2.4 for types $\mathsf{A}_2, \mathsf{B}_2, \mathsf{C}_2, \mathsf{G}_2$), which can in principle be checked "by hand".

There is some hope that our techniques will generalize to any $X$ (no restriction on $\check{G}_X$) by combining the knowledge from [BFGM, BNS].

3.3. **Recipe for $\rho_X$.** We now explain the combinatorial recipe for how to determine $\rho_X$ from $X$. The pattern is expected to hold for any $X$ without type N roots but we only prove it so far when $\check{G}_X = \check{G}$. We base change $X$ from $\mathbb{F}_q$ to $k = \overline{\mathbb{F}}_q$ below.

The combinatorial data attached to $X$ has two parts:

(a) one comes from the open $G$-orbit $X^\bullet$ and
(b) the other comes from the affine embedding $X^\bullet \hookrightarrow X$.

Part (b) gives a finite collection of anti-dominant weights $\check{\theta}_i \in \Lambda^-_{\check{G}_X}$. Then

$$\rho_X = \rho_{X^\bullet} \oplus \bigoplus T^* V^{\check{\theta}_i}$$

where $V^{\check{\theta}_i}$ is the irreducible $\check{G}_X$-representation with lowest weight $\check{\theta}_i$ and $\rho_{X^\bullet}$ is a representation determined by the data from (a) that we now describe.

[10]We also need some further assumptions over $\mathbb{F}_q$ to ensure $X$ behaves like it does in characteristic 0

In part (a), we consider the prime $B$-stable divisors in $X^\bullet$. These are called *colors*. A color $D$ determines a valuation $v_D$ on $k(X^\bullet)$ and in particular on the $B$-eigenvectors $k(X^\bullet)^{(B)}$. Since $X$ is spherical, $k(X^\bullet)^{(B)}/k^\times = \check{\Lambda}_{\check{G}_X}$, the coweight lattice of $\check{T}_X$. Thus restricting $v_D$ to $k(X^\bullet)^{(B)}$ determines a weight $\check{\nu}_D \in \Lambda_{\check{G}_X}$. Now $\rho_{X^\bullet}$ is the unique finite dimensional $\check{G}_X$-representation such that

- The highest weights of $\rho_{X^\bullet}$ are precisely the set $\Lambda^+_{\check{G}_X} \cap W_X\{\check{\nu}_D\}_{\text{colors } D}$, where $W_X$ is Weyl group of $\check{G}_X$, with multiplicities determined by:
- The multiplicity of the weight space of weight $\check{\nu}_D$ in $\rho_{X^\bullet}$ is equal to the number of colors $D'$ such that $\check{\nu}_D = \check{\nu}_{D'}$ (this number is either 1 or 2).

The graded degree of $\rho_{X^\bullet}$ is 1 (so corresponds to central $L$-value at 1/2) while the degree of $T^*V^{\check{\theta}_i}$ is a constant determined by $\check{\theta}_i$ and the $G$-eigen-volume form on $X^\bullet(F)$.

*Remark* 3.3.1. The case when two colors $D, D'$ have $\check{\nu}_D = \check{\nu}_{D'}$ arises from situations where you have $\mathbb{G}_m\backslash\mathrm{PGL}_2$ instead of $\mathbb{G}_m\backslash\mathrm{GL}_2$. In this case $\check{\nu}_D = \check{\alpha}/2$ for a simple root $\check{\alpha}$ of $\check{G}_X$. In practice you can always replace $G$ by a central extension to remove this case.

3.4. **Assumption $\check{G}_X = \check{G}$.** The condition that $X$ has $\check{G}_X = \check{G}$ and no type N roots has an easily accessible description, which I now explain.

This is equivalent to the following (after base change to $\overline{\mathbb{F}}_q$):

- $X$ has an open $B$-orbit $X^\circ$ acted on simply transitively by $B$ (so after choosing a base point $x_0 \in X^\circ$ we get $X^\circ \cong B$),
- $X^\circ P_\alpha/\mathcal{R}(P_\alpha) \cong \mathbb{G}_m\backslash\mathrm{PGL}_2$ for every simple $\alpha$. Here $P_\alpha \supset B$ is the standard sub-minimal parabolic corresponding to $\alpha$.

So this says $X$ has open subvarieties which "look" like the Hecke case $\mathbb{G}_m\backslash\mathrm{GL}_2$, and the complement of these opens are certain $B$-divisors.

*Remark* 3.4.1. An important corollary of this assumption is that the generic stabilizer subgroup $H$ is *connected*. This assumption saves us from many headaches/complications at various places. Geometrically, one would need to consider moduli of $\pi_0(H)$-local systems at various places.

3.5. **Bernstein asymptotics.** The computation of $\alpha_{\Phi^0,\Phi^0,\pi_X}(u,u)$ is done using the theory of Bernstein asymptotics developed in [SV]. We give a very brief idea of how this goes, without any assumption on $\check{G}_X$.

3.5.1. Plancherel decomposition (1.2) gives a decomposition

$$\|\Phi^0\|^2 = \int_{\mathrm{Temp}(G_X)} \|\Phi^0\|^2_\sigma d\mu_{G_X}(\sigma).$$

Since $\Phi^0$ is $G(\mathcal{O})$-invariant, we will only get contributions on the right hand side from unramified $\sigma$. In this formulation we have

$$\alpha_{\Phi^0,\Phi^0,\pi}(u,u) = \|\Phi^0\|^2_\sigma,$$

where $\iota_*(\sigma) = \pi$ is unramified.

3.5.2. From the usual Harish-Chandra theory of "parabolic descent", we have a direct sum decomposition

$$L^2((N_X(F),\psi)\backslash G_X(F)) = \bigoplus_\Theta L^2((N_X(F),\psi)\backslash G_X(F))_{\Theta,\mathrm{disc}}$$

ranging over conjugacy classes of parabolic subgroups of $G_X$, or equivalently subsets of $\Delta_X$, the set of simple coroots of $\check{G}_X$ (= the set of spherical roots of $X$).

Sakellaridis–Venkatesh establish the parallel decomposition for $L^2(X^\bullet(F))$.

**Theorem 3.5.3** ([SV, Scattering Theorem 7.3.1])**.** *Suppose $X^\bullet$ is wavefront and satisfies some technical assumptions. Then there is a decomposition*

$$L^2(X^\bullet(F)) = \bigoplus_{\Theta} L^2(X^\bullet(F))_\Theta$$

*running over subsets $\Theta \subset \Delta_X$, where roughly speaking $L^2(X^\bullet(F))_\Theta$ is the image of a canonical $G$-equivariant "$L^2$ Bernstein morphism"*

$$\iota_\Theta : L^2(X^\bullet_\Theta(F)) \to L^2(X^\bullet(F))$$

*and $X_\Theta$ is a* boundary degeneration *of $X$ at "$\Theta$-infinity".*

Here $\iota_\Theta$ can be thought of as an analog of unitary Eisenstein series (or Harish-Chandra's Eisenstein integral).

Luckily for us, we can show $\|\Phi^0\|^2$ only has contributions from the most continuous spectrum, which corresponds to $\Theta = \emptyset$. Thus we have

$$\|\Phi^0\|^2 = \int_{\check{T}^1_X/W_X} \|\Phi^0\|^2_\chi d\chi.$$

where $\check{T}^1_X$ denotes unitary unramified characters of $T_X(F)$. Here $d\chi$ is slightly different from $d\mu_{G_X}(\sigma_\chi)$, which will contribute a normalization factor of $\frac{L(1,\chi,\mathrm{Ad})}{L(0,\chi,\mathrm{Ad})}$. From Theorem 3.5.3 it follows that $\|\Phi^0\|^2_\chi = \frac{1}{|W_X|}\|\iota^*_\emptyset \Phi^0\|^2_\chi$, where $\iota^*_\emptyset : L^2(X^\bullet(F)) \to L^2(X^\bullet_\emptyset(F))$ is the adjoint of $\iota_\emptyset$.

3.5.4. The map $\iota^*_\emptyset$ in fact comes from a "smooth Bernstein asymptotics" map

$$e^*_\emptyset : C^\infty(X^\bullet(F)) \to C^\infty(X^\bullet_\emptyset(F))$$

which is roughly characterized by being $G(F)$-equivariant and capturing the asymptotic behavior of the function on $X^\bullet(F)$. Then $\iota^*_\emptyset$ is defined from $e^*_\emptyset$ by throwing away some "exponents" that do not belong to the $L^2$-space.

A global analog of $e^*_\emptyset$ is the constant term functor, while the analog of $i^*_\emptyset$ is Harish-Chandra's ($L^2$-)constant term integral. See also the first paragraph of [SV, §8].

## 3.6. Main result in terms of asymptotics.

Our result is more precisely stated in terms of asymptotics. We return to assuming $\check{G}_X = \check{G}$. In this setting, $X^\bullet_\emptyset = N^-\backslash G$.

**Theorem 3.6.1** (Sakellaridis-W)**.** *Same assumptions as in Theorem 3.2.3.*

(i) *The asymptotics map $e^*_\emptyset$ corresponds, under the functions-sheaves dictionary, to the nearby cycles functor on algebro-geometric models of $X^\bullet(F), X^\bullet_\emptyset(F)$.*
(ii) *There is a $\check{T}$-polarization of $\rho'_X = T^*(V^+_X)$ where $V^+_X$ is a $\mathbb{Z}$-graded $\check{T} \rtimes \langle \mathrm{Fr} \rangle$-representation such that*

$$(\eta\delta)^{\frac{1}{2}}(t^{\check{\lambda}}) \cdot e^*_\emptyset \Phi^0(t^{\check{\lambda}}) = \frac{L(\frac{1}{2}, \chi, V^+_X)}{L(0, \chi, \check{\mathfrak{n}})} \tag{3.1}$$

Here $\eta$ is the eigencharacter of the eigenmeasure on $X^\bullet(F)$. The factor $(\eta\delta)^{\frac{1}{2}}$ is a normalization factor, cf. Remark 4.1.6. The $\frac{1}{2}$ special $L$-value should be shifted according to the $\mathbb{Z}$-grading on $V^+_X$, which is 0 for $V^+_{X^\bullet}$, so I suppress the notation.

3.7. **Main result in terms of Radon transform.** In practice, the definitions of both $e_\emptyset^*$ and nearby cycles are rather opaque and hard to compute with. We instead do all our computations by passing to the Radon transform.

From now on we will fix a base point $x_0 \in X^\circ(\mathbb{F}_q)$ in the open $B$-orbit (such $x_0$ exists by [Sa08, Proposition 3.2.1]).

**Definition 3.7.1.** The $X$-Radon transform

$$\pi_! : C_c^\infty(X^\bullet(F)) \to C^\infty((N\backslash G)(F))$$

is defined by integrating over generic horocycles

$$\pi_!\Phi(g) := \int_{N(F)} \Phi(x_0 ng)dn, \quad g \in G(F).$$

The relation between $e_\emptyset^*$ and $\pi_!$ is that there is a commutative diagram

$$\begin{tikzcd} C_c^\infty(X^\bullet(F)) \arrow[rr, "e_\emptyset^*"] \arrow[dr, "\pi_!"'] & & C^\infty(X_\emptyset^\bullet(F)) \arrow[dl, dashed, "\pi_{\emptyset !}"] \\ & C^\infty((N\backslash G)(F)) & \end{tikzcd} \tag{3.2}$$

where $\pi_{\emptyset !} : C_+^\infty((N^-\backslash G)(F)) \to C_-^\infty((N\backslash G)(F))$ is the standard long intertwining operator, which is *invertible* if one considers some spaces with suitable support conditions (cf. [BK, Proposition 7.5(b)], [W, Proposition 2.8.5]). Therefore assuming some convergence properties, we can think of $e_\emptyset^*$ and $\pi_!$ as "the same".

Our main result in terms of the Radon transform, from which we recover Theorem 3.6.1 is:

**Theorem 3.7.2** (Sakellaridis-W)**.** *Same assumptions and notation as in Theorem 3.6.1.*

(i) *We show a triangle analogous to (3.2) commutes in the geometric setting of nearby cycles using the contraction principle (i.e., Braden's theorem).*
(ii) *The Mellin transform of $\pi_!\Phi^0 \in C^\infty((N\backslash G)(F))$ equals*

$$\widehat{\pi_!\Phi^0}(\chi) := \int_{T(F)} (\pi_!\Phi^0)(t)\chi(t)dt = \frac{L(\frac{1}{2}, \chi, V_X^+)}{L(1, \chi, \check{\mathfrak{n}})} \tag{3.3}$$

*on unramified characters $\chi : T(F) \to \mathbb{C}$ where the Mellin transform is convergent.*

3.7.3. Here is a more explicit description of $\pi_!\Phi^0$.

Note that $\pi_!\Phi^0 \in C^\infty((N\backslash G)(F))^{G(\mathcal{O})} = \mathrm{Fn}(\check{\Lambda}_G)$ is a function on the set of coweights of $G$. Assume that all the prime $B$-divisors of $X_k$ are defined over $\mathbb{F}_q$ (I think this implies that the Frobenius action on $\check{G}_X$ is trivial, i.e., ${}^L G_X = \check{G} \times \langle \mathrm{Fr} \rangle$). Then (3.3) is equivalent to saying

$$\pi_!\Phi^0 = \frac{\prod_{\check{\alpha} \in \check{\Phi}_G^+}(1 - q^{-1}e^{\check{\alpha}})}{\prod_{\check{\lambda} \in \mathrm{wt}(V_X^+)}(1 - q^{-\frac{1}{2}}e^{\check{\lambda}})} \tag{3.4}$$

where $e^{\check{\lambda}}$ is the indicator function of $\check{\lambda}$ and $e^{\check{\lambda}}e^{\check{\mu}} = e^{\check{\lambda}+\check{\mu}}$. The Euler product on the right should be understood via a power series expansion:

$$\frac{1}{1 - q^{-\frac{1}{2}}e^{\check{\lambda}}} = \sum_{n \geq 0} (q^{-\frac{1}{2}}e^{\check{\lambda}})^n.$$

3.8. **Formula for $\Phi^0$.** The basic function $\Phi^0$ is a $G(\mathcal{O}_v)$-invariant function on $X^\bullet(F)$.

We have the following parametrization of the set of orbits $X^\bullet(F)/G(\mathcal{O})$, proved by [GN] for $\mathbb{C}(\!(t)\!)$ and extended to non-archimedean local fields $F$ by [Sa12].

**Theorem 3.8.1** ([Sa12, Theorem 2.3.8])**.** *Assume $X^\bullet$ has good integral model over $\mathcal{O}$. If $\Lambda_G/\Lambda_X$ is torsion-free then there is a bijection between the set $X^\bullet(F)/G(\mathcal{O})$ and $-\Lambda^+_{\check{G}_X} \subset \check{\Lambda}_X$.*

Our assumptions (3.4) obvious satisfy the above. Hence in our case $\Phi^0$ is just a function on the monoid $-\check{\Lambda}^+_G$. We can in fact give a formula for $\Phi^0$ based on a variant of "inverse Satake transform" for asymptotics ([Sa18, Corollary 5.5]) once we know what the representation $\rho'_X = T^*V^+_X$ is.

**Proposition 3.8.2.** *We consider $\Phi^0$ as a function on $-\check{\Lambda}^+_G$ and $e^*_\emptyset\Phi^0$ as a function on $\check{\Lambda}_G$. Then*

$$\Phi^0 = (e^*_\emptyset\Phi^0)|_{-\check{\Lambda}^+_G}. \tag{3.5}$$

Since we have a formula for $e^*_\emptyset\Phi^0$ in terms of $V^+_X$ (see for example (3.4)), we could just use (3.5) as the definition of $\Phi^0$ without talking about IC functions. Then you can probably establish the formula for $e^*_\emptyset\Phi^0$ *a posteriori* by classical methods. Our proof that $\Phi^0$ equals the IC function of $X(\mathcal{O})$ is focused on establishing the connection between geometry and number theory in the hopes that this will lead to new methods in studying $L$-functions.

## 4. Geometric techniques

Let $F = \mathbb{F}_q(\!(t)\!)$ and $\mathcal{O} = \mathbb{F}_q[\![t]\!]$ as above. Let $k = \overline{\mathbb{F}}_q$; for the geometric results below, we can also take $k = \mathbb{C}$. Let $\mathbf{F} = k(\!(t)\!)$ and $\mathbf{O} = k[\![t]\!]$.

4.1. **IC function.** We would like to say that the IC function $\Phi^0$ is the trace of geometric Frobenius acting on the IC complex of $X(\mathcal{O})$.

To define IC complexes you need a theory of perverse sheaves (perverse t-structure) on an algebraic variety. Fortunately, we can give $X(\mathcal{O})$ algebro-geometric structure as follows:

**Definition 4.1.1.** The *formal arc space* of $X$ is the scheme $\mathbf{X_O}$ over $\mathbb{F}_q$ with functor of points

$$\mathbf{X_O}(R) = X(R[\![t]\!])$$

for a test $\mathbb{F}_q$-algebra $R$.

By definition, the $\mathbb{F}_q$-points of $\mathbf{X_O}$ equal $X(\mathcal{O})$.

*Remark* 4.1.2. For any scheme $X$ of finite type, $\mathbf{X_O}$ is representable by a scheme. When $X$ is affine, so is $\mathbf{X_O}$. However the functor $\mathbf{X_F}(R) = X(R(\!(t)\!))$ for the *formal loop space* is only representable by an ind-scheme when $X$ is affine – it does not have good properties for non-affine $X$.

4.1.3. Now we run into a problem. The theory of perverse sheaves is developed in [BBDG] for schemes (locally) *of finite type*. On the other hand, the scheme $\mathbf{X_O}$ is very much of infinite type. So far there is no general way to define perverse sheaves on infinite type schemes.

In the situation of the affine Grassmannian $\mathrm{Gr}_G = \mathbf{G_F}/\mathbf{G_O}$, the $\mathbf{G_O}$-orbit closures are at least of finite type, which saves the day. However for general affine $X$, the stack quotient $\mathbf{X_O}/\mathbf{G_O}$ will still not have strata that look like finite type schemes.

Nevertheless, Bouthier–Ngo–Sakellaridis [BNS] show that the IC function of $\mathbf{X_O}$, which should equal the trace of geometric Frobenius of $\mathrm{IC}_{\mathbf{X_O}}$, is well-defined. They use a theorem of Grinberg–Kazhdan (characteristic 0) and Drinfeld (any characteristic):

**Theorem 4.1.4** (Grinberg–Kazhdan, Drinfeld)**.** *Let $\gamma \in X(\mathbb{F}_q[\![t]\!])$ be an arc that generically lands in the smooth locus of $X$. Then there exists a finite type scheme $Y$ and $y \in Y(\mathbb{F}_q)$ such that there is an isomorphism of formal neighborhoods*

$$(\widehat{\mathbf{X_O}})_\gamma \cong \widehat{Y}_y \times \widehat{\mathbb{A}}^\infty.$$

*I.e., near generic arcs $\mathbf{X_O}$ has finite-type singularities.*

We call $Y$ as above a *model* of $\mathbf{X_O}$.

**Definition 4.1.5.** The IC function $\Phi^0$ of $X(\mathcal{O})$ is defined by

$$\Phi^0(\gamma) := \mathrm{tr}(\mathrm{Fr}_y, \mathrm{IC}_Y|_y^*)$$

where $(Y, y)$ are as in Theorem 4.1.4. [BNS] show that this definition is independent of the choice of $Y$.

*Remark* 4.1.6. Here $\mathrm{IC}_Y$ is normalized *without the Tate twist*, so if $Y$ is smooth then $\mathrm{IC}_Y = \overline{\mathbb{Q}}_{\ell Y}$. It is hard to make sense of the Tate twist since the dimension becomes infinite. However this discrepancy of normalizations is what accounts for the factor $(\eta\delta)^{\frac{1}{2}}$ in (3.1).

4.2. **Models for the formal arc space.** We will use the fact that Drinfeld's proof [D] of this theorem gives us explicit models for $\mathbf{X_O}$. This phenomenon was first used by Finkelberg–Mirković to study $X = \overline{G/N}$ ($\check{G}_X = \check{T}$). The two models are:

- the Artin stack $\mathcal{M}_X = \mathrm{Maps}_{\mathrm{gen}}(C, X/G \supset X^\bullet/G)$, which we call the *global model*[11].
- the *Zastava space*[12] $\mathcal{Y}_X = \mathrm{Maps}_{\mathrm{gen}}(C, X/B \supset X^\circ/B)$.

Both models are important for different reasons:

- The global model $\mathcal{M}_X$ allows us to model the part of the Hecke action of $\mathbf{G_F}$ on $\mathbf{X_F}$ that stays in $\mathbf{X_O}$. (More generally, one can define an ind-stack $\mathcal{M}_X^{H\text{-gen}}$ of maps with poles that models the $\mathbf{G_F}$-action on $\mathbf{X_F} - (\mathbf{X} - \mathbf{X}^\bullet)_\mathbf{F}$.)
- The Zastava space $\mathcal{Y}_X$ has a graded *factorization property* that is key to making the connection with $L$-values.

Drinfeld's proof [D] of Theorem 4.1.4 directly shows that $\mathcal{Y}_X$ is a model for $\mathbf{X_O}$. There is a general yoga that passes from the Zastava model $\mathcal{Y}_X$ to the global model $\mathcal{M}_X$; or one can argue directly as follows:

Fix a point $v \in C(\mathbb{F}_q)$. Identify the completed local ring $\mathcal{O}_v$ at $C$ with $\mathcal{O}$. Now we have a map $\mathrm{Spec}\,\mathcal{O}_v \to C$. Restricting along this map gives a map

$$\mathcal{M}_X = \mathrm{Maps}_{\mathrm{gen}}(C, X/G \supset X^\bullet/G) \to \mathrm{Maps}(\mathrm{Spec}\,\mathcal{O}_v, X/G) \cong \mathbf{X_O}/\mathbf{G_O}. \tag{4.1}$$

Now one can show that if you restrict (4.1) to functions $f : C \to X/G$ such that $f(C - v) \subset X^\bullet/G$, then (4.1) is formally smooth.

4.2.1. Considerations of the partial Hecke action on $\mathcal{M}_X$ allow us to reduce the study of $X$ to that of just $\overline{X^\bullet}^{\mathrm{aff}}$. (This is a simplification of some technical results, whose proofs require inspecting the interactions between the global and Zastava models.)

4.3. **Zastava space and the graded factorization property.** From now on I will base change to $k$ while keeping the same notation.(Everything is canonical enough that the action of Frobenius is easy to keep track of.)

We fix a base point $x_0 \in X^\circ(k)$ and identify $X^\circ \cong B$ under assumption 3.4. Our assumptions imply that the stack $X/B$ contains $X^\circ/B = \mathsf{pt}$ as an open substack.

[11]In certain places in the literature this is called the space of quasi-maps

[12]Zastava is Croatian for flag

4.3.1. A point $y \in \mathcal{Y}_X(k)$ is a map $C \to X/B$ generically landing in $\mathsf{pt}$. So by Beauville–Laszlo's theorem

$$y \leftrightarrow \left\{ \begin{array}{c} \text{finite set } \{v_i\}_{i\in I} \subset C(k), \\ \hat{y}_i \in (X(O_{v_i}) \cap X^\circ(F_{v_i}))/B(O_{v_i}), \\ y(C - \{v_i\}) = \mathsf{pt} \end{array} \right\}$$

Recall we are using $x_0 \in X^\circ(k)$ to identify $X^\circ \cong B$. Then

$$X^\circ(F_{v_i})/B(O_{v_i}) \cong \mathbf{B}_{\mathbf{F}_{v_i}}/\mathbf{B}_{\mathbf{O}_{v_i}}(k) = \mathrm{Gr}_{B,v_i}(k)$$

Now recall that $\mathrm{Gr}_B$ has the same connected components as $\mathrm{Gr}_T$, which are indexed by the coweight lattice $\check{\Lambda}$. So to each $\hat{y}_i$ is attached a coweight $\check{\lambda}_i \in \check{\Lambda}$.

From this we see that $\mathcal{Y}$ lives over a *configuration space*

$$\left\{ \check{\Lambda}\text{-valued divisors} : \sum_{i\in I} \check{\lambda}_i \cdot v_i,\ v_i \in C(k) \text{ distinct} \right\}$$

If $\check{\lambda}_i$ could be any coweight then we would need something fancy like the Ran space to make sense of the above. However, since $\hat{y}_i \in X(O_{v_i})$ is an arc, all the $\check{\lambda}_i$ belong to a strictly convex cone. So there is a sense of "positive" grading. More specifically,

$$\pi : \mathcal{Y} \to \mathcal{A} = \mathrm{Maps}(C, X /\!\!/ N/T).$$

Let me assume for ultimate simplicity that $X/\!\!/N = \mathbb{A}^r \supset \mathbb{G}_m^r = T$ with a corresponding basis $\check{\nu}_1, \dots, \check{\nu}_r \in \check{\Lambda}$ for the cocharacters whose limit as $t \to 0$ lands in $X/\!\!/N$. Then

$$\mathcal{A} = \mathrm{Maps}(C, \mathbb{A}^r/\mathbb{G}_m^r) = (\mathrm{Sym}\, C)^r = \bigsqcup_{(n_i)\in\mathbb{N}^r} C^{(n_1)} \times \cdots \times C^{(n_r)} =: \bigsqcup \mathcal{A}^{n_1\check{\nu}_1+\cdots+n_r\check{\nu}_r}$$

is the scheme of $r$ divisors on $C$. Let the preimage of $\mathcal{A}^{\check{\lambda}}$ be $\mathcal{Y}^{\check{\lambda}}$.

Then $\mathcal{Y}^{\check{\lambda}}$ is a *finite type* scheme.

4.3.2. *Graded factorization.* Notice that the fiber over $\check{\lambda}_1 \cdot v_1 + \check{\lambda}_2 \cdot v_2 \in \mathcal{A}^{\check{\lambda}_1+\check{\lambda}_2}$ where $v_1, v_2$ are distinct, only depends on the independent fibers over $\check{\lambda}_1 \cdot v_1$ and $\check{\lambda}_2 \cdot v_2$. This is called a *graded factorization* property of (the collection of components of) $\mathcal{Y}$.

Aside: in the situation above the $\mathcal{Y}^{\check{\lambda}}$ are indeed irreducible components, but we could only prove this in a *very* roundabout way.

4.4. **Central fibers.** The graded factorization property essentially says the fiber of $\pi$ over $\check{\lambda} \cdot v$ at a single point $v \in C(k)$ is the most important. This fiber is isomorphic to

$$\mathsf{Y}^{\check{\lambda}} := \mathrm{Gr}_{B,v}^{\check{\lambda}} \underset{\mathbf{X_F}/\mathbf{B_O}}{\times} \mathbf{X_O}/\mathbf{B_O},$$

where $\mathbf{B_F} \to \mathbf{X_F}$ is the action on $x_0$. This fiber doesn't depend on $v$. Observe that

$$\mathrm{tr}(\mathrm{Fr}, \pi_!\mathrm{IC}_{\mathcal{Y}}|^*_{\check{\lambda}\cdot v}) = \mathrm{tr}(\mathrm{Fr}, H_c^*(\mathsf{Y}^{\check{\lambda}}, \mathrm{IC}_{\mathcal{Y}})) = \int_{N(F)} \Phi_0(x_0 n t^{\check{\lambda}}) = \pi_!\Phi_0(t^{\check{\lambda}})$$

is the Radon transform we wanted to calculate back in (3.4).

**Example 4.4.1.** Let $X = \mathbb{G}_m \backslash \mathrm{GL}_2$ where $\mathbb{G}_m = \left(\begin{smallmatrix} * & \\ & 1 \end{smallmatrix}\right)$. Then $\mathcal{Y} = \mathrm{Maps}_{\mathrm{gen}}(C, X/B) = \mathrm{Maps}_{\mathrm{gen}}(C, \mathbb{G}_m \backslash \mathbb{P}^1)$ parametrizes

$$\mathcal{A}, \mathcal{L} \in \mathrm{Pic}, \mathcal{L} \xrightarrow{(x,y)} \mathcal{A} \oplus \mathcal{O}.$$

Generically landing in $X^\circ$ means $x, y$ do not simultaneously vanish after taking fiber at any point. What this amounts to is two divisors with disjoint support:

$$\mathcal{Y} = \operatorname{Sym} C \overset{\circ}{\times} \operatorname{Sym} C$$

Meanwhile $X/\!\!/N = \mathbb{A}^2$ with basis $\check{\varepsilon}_1 = (1,0), -\check{\varepsilon}_2 = (0,-1)$. So

$$\pi : \mathcal{Y} = \operatorname{Sym} C \overset{\circ}{\times} \operatorname{Sym} C \to \operatorname{Sym} C \times \operatorname{Sym} C = \mathcal{A}$$

is an open embedding. The preimage of $(n_1\check{\varepsilon}_1 - n_2\check{\varepsilon}_2) \cdot v$ is empty if $n_1, n_2$ are both nonzero, and a point otherwise. So we see

$$\pi_! \Phi_0 = e^0 + \sum_{n \geq 1} (q^{-n/2} e^{n\check{\varepsilon}_1} + q^{-n/2} e^{-n\check{\varepsilon}_2}) = \frac{1 - q^{-1} e^{\check{\alpha}}}{(1 - q^{-1/2} e^{\check{\varepsilon}_1})(1 - q^{-1/2} e^{-\check{\varepsilon}_2})}$$

since $\check{\alpha} = \check{\varepsilon}_1 - \check{\varepsilon}_2$. Note that $|\widehat{\pi_! \Phi_0}(\chi)|^2 = \frac{L(\chi, \mathrm{std} \oplus \mathrm{std}^*, 1/2)}{L(\chi, \check{\mathfrak{g}}/\check{\mathfrak{t}}, 1)}$.

As we see above, $\pi$ is not proper, but we can compactify it to:

$$\overline{\mathcal{Y}} = \operatorname{Maps}(C, X \times \overline{G/N}/(G^{\mathrm{diag}} \times T))$$

and we still have $\bar{\pi} : \overline{\mathcal{Y}} \to \mathcal{A}$. Let $\overline{\mathcal{Y}}^{\check{\lambda}}$ be preimage of $\mathcal{A}^{\check{\lambda}}$; then $\overline{\mathcal{Y}}$ still has the graded factorization property.

**Theorem 4.4.2** (Sakellaridis–W)**.** *Under our assumptions on $X$, the map $\bar{\pi} : \overline{\mathcal{Y}} \to \mathcal{A}$ is stratified semi-small.*

We emphasize that this is extremely special to the $\check{G}_X = \check{G}$ case! The statement is definitely false for example when $X = \overline{N^- \backslash G}$.

Toy situation: if $\overline{\mathcal{Y}}$ were smooth, then semi-smallness for $\bar{\pi}$ amounts to (because of factorization):

$$\dim \overline{\mathsf{Y}}^{\check{\lambda}} \leq \operatorname{crit}(\check{\lambda}) \tag{4.2}$$

The general situation is more complicated because of $\mathbf{G_O}$-orbit strata, but using our results on the Hecke action on $\mathcal{M}_X$, we get roughly the same requirement on the central fibers of $X^\bullet$. Under some assumptions, $\operatorname{crit}(\check{\lambda}) = \frac{\operatorname{len}(\check{\lambda})-1}{2}$, where $\operatorname{len}(\check{\lambda})$ is the number of $\check{\nu}_D$, for non-distinct colors $D$, that sum to equal $\check{\lambda}$.

4.4.3. A fact that is presumably known to experts but not often stated is that in the above situation where you have a semi-small map, the decomposition theorem together with the graded factorization property immediately tell you that

$$\operatorname{tr}(\mathrm{Fr}, (\bar{\pi}_! \mathrm{IC}_{\overline{\mathcal{Y}}})|^*_{?\cdot v}) = \frac{1}{\prod_{\check{\lambda} \in \mathfrak{B}^+} (1 - q^{-\frac{1}{2}} e^{\check{\lambda}})} \tag{4.3}$$

has the desired Euler product format. Here $\mathfrak{B}^+$ corresponds to the *relevant strata* supported at a single point. More specifically, $\mathfrak{B}^+ =$ the irreducible components of $\overline{\mathsf{Y}}^{\check{\lambda}}$ of $\dim = \operatorname{crit}(\check{\lambda})$ as $\check{\lambda}$ varies. (This is an oversimplification but it's almost true.)

*Remark* 4.4.4. Note that the right hand side of (4.3) almost looks like (3.4). The difference between $\mathrm{IC}_{\overline{\mathcal{Y}}}$ and $\mathrm{IC}_{\mathcal{Y}}$ accounts for the missing numerator, which also corresponds to the factor of $\frac{1}{L(1, \chi, \check{\mathfrak{n}})}$ in the Mellin transform (3.3).

4.5. **Crystals.** To reconnect with Theorem 3.7.2, define $V_X^+$ to be the $\check{T}$-representation with basis in bijection with $\mathfrak{B}^+$. The crux of Conjecture 1.5.1 is getting half of a $\check{G}$-representation $\rho_X$.

Since we know this is what we want, formally set $\mathfrak{B} = \mathfrak{B}^+ \sqcup (\mathfrak{B}^+)^*$, where $(\mathfrak{B}^+)^*$ is defined to be in bijection with $\mathfrak{B}^+$ but the weights are replaced by their negatives. In this way, $(\mathfrak{B}^+)^*$ naturally corresponds to a basis of $(V_X^+)^*$.

**Theorem 4.5.1** (Sakellaridis–W)**.** *$\mathfrak{B}$ has the structure of a (Kashiwara) crystal, i.e., a graph with weighted vertices and edges corresponding to lowering operators $\tilde{f}_\alpha$.*

We use this abstract combinatorial notion of crystal as a bridge to hopefully getting a crystal basis. A crystal basis is the (Lusztig) canonical basis[13] at $q = 0$ of an integrable $U_q(\check{\mathfrak{g}})$-module in category $\mathcal{O}$. So the crystal basis is a way for us to access a $\check{G}$-representation.

$$\text{f.d. } \check{G}\text{-representation } \rightsquigarrow \text{ crystal basis } \in \{\text{crystals}\}$$

**Conjecture 4.5.2.** *$\mathfrak{B}$ is the crystal basis for a finite dimensional $\check{G}$-representation $\rho_X$.*

This conjecture implies Conjecture 1.5.1 (by construction, $\mathfrak{B}$ corresponds to a basis of $\rho'_X = T^*V_X^+$).

4.6. **Further details.** We can identify $(\mathrm{Gr}_B^{\check{\lambda}})_{red} = \mathbf{N_F} t^{\check{\lambda}} \mathbf{G_O}/\mathbf{G_O} =: S^{\check{\lambda}} \subset \mathrm{Gr}_G$, i.e., a semi-infinite orbit. Let $\overline{S}^{\check{\lambda}}$ denote its closure in $\mathrm{Gr}_G$. Then the fiber of $\overline{\mathcal{Y}} \to \mathcal{A}$ over $\check{\lambda} \cdot v$ is

$$\overline{\mathsf{Y}}^{\check{\lambda}} = \overline{S}^{\check{\lambda}} \underset{\mathbf{X_F}/\mathbf{G_O}}{\times} \mathbf{X_O}/\mathbf{G_O}.$$

**Proposition 4.6.1** ([MV])**.** *The boundary $\overline{S}^{\check{\lambda}} = \bigcup_{\check{\mu} \le \check{\lambda}} S^{\check{\mu}}$ is given by a hyperplane section in $\mathrm{Gr}_G$.*

We have $\overline{\mathsf{Y}}^{\check{\lambda}} \cap S^{\check{\lambda}} = S^{\check{\lambda}} \times_{\mathbf{X_F}/\mathbf{G_O}} \mathbf{X_O}/\mathbf{G_O}$, which further breaks up according to $\mathbf{G_O}$-orbits of $\mathbf{X_O}$ into pieces

$$(\mathbf{N_F} t^{\check{\lambda}} \mathbf{G_O} \cap H_{\mathbf{F}} t^{\check{\theta}} \mathbf{G_O})/\mathbf{G_O}$$

with $\check{\theta} \in -\check{\Lambda}_G^+$.

The lowering operator we define on $\mathfrak{B}$ is roughly given by

$$\overline{\mathsf{Y}}^{\check{\lambda}} \rightsquigarrow \overline{\mathsf{Y}}^{\check{\lambda}} \cap S^{\check{\lambda}-\check{\alpha}} \subset \mathsf{Y}^{\check{\lambda}-\check{\alpha}}.$$

This does not quite uniquely specify how to lower an irreducible component to another irreducible component, but a reduction to considering affine embeddings of $\mathbb{G}_m \backslash \mathrm{GL}_2 \times (\text{torus})$ gives us enough information to pick out the correct irreducible component in $\mathbb{Y}^{\check{\lambda}-\check{\alpha}}$.

For a summary of the proofs of our main results, I refer to [SW, §1.3-1.5] of our paper.

[13] Canonical bases were first discovered by Lusztig '90 in types $\mathsf{A}, \mathsf{D}, \mathsf{E}$, and subsequently by Kashiwara using different methods. The crystal basis at $q = 0$ in types $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$ was discovered independently by Kashiwara at around the same time in '90.

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