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Notes on Algebraic Semigroups.

1. Generalities.

All our groups and varieties will be over an alegbraically close field k. Unless stated otherwise, the charactersitic of k is arbitrary.

(Definitions of a semigroup, identity element and zero element. Whenever we speak about the group of units we assume existence of the identity element).

Examples

(i) $Mat(n)$

(ii) $Mat^r(n) = \{x \in Mat(n) \mid \text{rank } x \leq r\}$

(iii) k^n with coordinatewise multiplication.

(iv) Let S be a semigroup, X, Y two arbitrary algebraic varieties and $\phi : Y \times X \to S$ an arbitrary map of algebraic varieties. Then $X \times S \times Y$ can be given a structure of a semigroup: $(x_1, s_1, y_1) \cdot (x_2, s_2, y_2) = (x_1, s_1\phi(y_1, x_2)s_2, y_2)$. Most often this semigroup has no zero and no identity element.

Theorem 1. *(a) For every affine algebraic semigroup S there exists a natural number n and a closed embedding $\psi : S \to Mat(n)$.*

(b) If S has a unit, the map ψ may be chosen in such a way that $1_S = \psi^{-1}(1_n)$ and $\psi^{-1}(GL(n))$ is equal to the group of units in S.

Corrolary 2. *(a) If S is irreducible and has an identity element then its group of units $G(S)$ is open in S.*

(b) the group of units $G(S)$ is an algebraic group.

From now on we will asume that the semigroup S has an identity element 1_S.

Consider the category $Rep_k(S)$ of all finite dimensional representations of S over k. The objects of $Rep_k(S)$ are pairs (V, ρ_V), where V is a k-vector space and $\rho_V : S \to End_k\ V$ is a semigroup homomorphism sending 1_S to the identity. The morphisms in $Rep_k(S)$ are given by linear maps commuting with the S-action.

One has the natural forgetful functor $\mu_S : Rep_k(S) \to Vect_k$ to the category of k-vactor spaces $Vect_k$, which sends (V, ρ_V) to $V \in Vect_k$.

Consider the endomorphisms of μ_S as a tensor functor. Any such endomorphism is defined by giving an element $\lambda_V \in End_k\ V$ for every representation V of S. The collection $\{\lambda_V\}$ should satisfy the following properties

(1) $\lambda_k = 1 \in k = End_k\ \mathbf{1}_k$ where $\mathbf{1}_k$ is the trivial one-dimensional representation;
(2) $\lambda_{V_1 \otimes V_2} = \lambda_{V_1} \otimes \lambda_{V_2}$;
(3) if $\alpha : V_1 \to V_2$ is a linear map which commutes with the S-action then $\lambda_{V_2} \circ \alpha = \alpha \circ \lambda_{V_1}$.

The set S' of such endomorphisms has a semigroup structure with an identity element, and one can show, cf [??], that S' is an affie algebraic semigroup.

Theorem 3. *The natural homomorphim $S \to S'$ which sends $s \in S$ to the collection $\{\rho_V(s) \in End_k\ V\}$, is an isomorphism of affine algebraic semigroups.*

Finally, we consider an affine algebraic group G and the set of all semigroups G_+ which contain G as the group of units.

Theorem 4. *There exists a bijection between*

(i) the set of affine algebraic semigroups G_+ which contain G as the group of units, and

(ii) the set of full subcategories $Rep_+ \subset Rep_k(G)$ which satisfy

(a) $\mathbf{1}_k \in Rep_+$,
(b) Rep_+ is closed with respect to direct sums, tensor products and passing to a subquotient,
(c) Rep_+ contains an exact representation of G.

2. Reductive algebraic semigroups.

We consider the pair $G \subset G_+$ consisting of a semigroup G and its group of units G. We impose the following conditions on (G, G_+):

- G is reductive
- G_+ is irreducible
- G_+ is normal

The last condition is not too restrictive, since the normalization of a semigroup has a unique semigroup structure which agrees with the normalization morphism (follows from the universal property of normalizations).

As usual, we choose a maximal torus $T \subset G$ and a Borel subgroup $B \supset T$. Let W denote the Weyl group, $\alpha_1, \ldots, \alpha_l$ the simple roots,

$P = Hom(T, \mathbb{C}^*)$ the weight lattice and $Q^\vee = Hom(\mathbb{C}^*, T)$ the dual coroot lattice.

Our goal is to prove the following result

Theorem 5. *(Classification of Reductive Semigroups)*

There exists a bijection between

(i) the set of normal affine irreducible semigroups G_+ containing G as its group of units, and

(ii) the set of W-invariant rational polyhedral convex cones $K \subset P \otimes_{\mathbb{Z}} \mathbb{R}$ which contain zero and are non-degenerate, i.e. not contained in a hyperplane

Remark. The above theorem implies that if G is semisimple then there is only one semigroup G_+ containing G as the group of units, namely G itself.

Comments on proof in charactersitic zero:

Note that the algebra $k[G_+]$ of regular functions on G_+ is naturally a subalgebra of $k[G]$ of functions on the group G.

We have a natural $G \times G$-action on G_+ (given by $(g_1, g_2) \cdot s = g_1 s g_2^{-1}$) and $k[G_+]$ is $G \times G$-invariant with respect to the induced action on $k[G]$.

In the case when $char\ k = 0$ the algebra $k[G]$ has the following description as a $G \times G$-module. Let V_χ be the irreducible representation of G with highest weight $\chi \in P_+$. Since G is mapped into $GL(V_\chi) \subset End_k\ V_\chi$, any linear function on $End_k\ V_\chi$ is naturally a function of G. We have the following decomposition

$$k[G] = \sum_{\chi \in P_+} \Big[End_k\ V_\chi \Big]^*$$

into direct sum of irreducible $G \times G$-modules. The algebra structure on $k[G]$ satisfies

$$\big[End_k\ V_\chi\big]^* \cdot \big[End_k\ V_\mu\big]^* = \bigoplus_{\psi \in \Lambda(\chi,\mu)} \big[End_k\ V_\psi\big]^*$$

where $\Lambda(\chi, \mu)$ is the set of dominant weight of the tensor product $V_\chi \otimes V_\mu$. Hence, giving a $G \times G$-invariant subalgebra in $k[G]$ amounts to giving a subset $K_+ \subset P_+$ which satisfies

$$\chi, \mu \in K_+ \text{ and } \psi \in \Lambda(\chi, \mu) \Rightarrow \psi \in K_+.$$

In particular, since $\chi + \mu \in \Lambda(\chi, \mu)$, K_+ is a subsemigroup of P_+. The semigroup algebra $k[K_+]$ can be recovered from $k[G_+]$ as the algebra of $U_- \times U_+$-invariants.

It is known then certain properties (finitely generated, has no nilpotents, normal, has no zero divisors) are satisfied for an algebra A ($= k[G_+]$) with an action of a reductive group G' ($= G \times G$) if and only if they are satisfied for the algebra $A^{U'}$ of invariants with respect to the unipotent subgroup U' ($= U_- \times U_+$).

In particular, the normality of G_+ implies that the cone K_+ is saturated (i.e. is an intersection with a real cone). Since $k[G_+]$ is finitely generated, so is K_+, and since $k[G_+]$ and $k[G]$ have the same field of functions, K_+ is non-degenerate.

Now the cone K of the theorem can be recovered as $W \cdot K_+$.

3. Proof on the Classification Theorem.

When k is of arbitrary charactersitc, one can suggest an alternative proof. Let K (resp. $\mathcal{O}$) be the field $k((t))$ of formal Lauraunt series (resp. the ring $k[[t]]$ of formal Taylor series) in variable t. Let also $G(K)$ (resp. $G(\mathcal{O})$ be the group of K (resp. $\mathcal{O}$)-valued points of G. For any semigroup G_+ containing G as the group of units, we consider the intersection

$$A(G_+) = G_+(\mathcal{O}) \cap G(K)$$

In other words, the intesesection $A(G_+)$ parametrizes all commutative diagrams of homomorphisms:

$$\begin{array}{ccc} k[G] & \longrightarrow & K \\ \uparrow & & \uparrow \\ k[G_+] & \longrightarrow & \mathcal{O} \end{array}$$

where the vertical arrows denote the natural embeddings. Note that this defininition does not use the groups structure on G In our case we can also say that the subset $A(G_+) \subset G(K)$ is invariant with respect to left and right mulitplication by $G(\mathcal{O})$.

Lemma 6. *Let G_+, G'_+ be two normal affine vairieties containing an affine variety G as a dense open subset. If $G_+(\mathcal{O}) \cap G(K) = G'_+(\mathcal{O}) \cap G(K)$ then $G_+ = G'_+$, i.e. $k[G_+] = k[G'_+]$ as subalgebras of $k[G]$.*

Proof. We will show how to reconstruct G_+ from $A(G_+)$.

By definition, every element $\phi \in A(G_+)$ is a homomorphism $\phi : k[G] \to K$. Consider

$$V(G_+) = \bigcap_{\phi \in A(G_+)} \phi^{-1}(\mathcal{O}).$$

We want to show that $k[G_+] = V(G_+)$. The inclusion $k[G_+] \subset V(G_+)$ follows from definitions. Suppose $f \in V(G_+) \setminus k[G_+]$. Then f gives a rational function on G_+ which by normality of G_+ can be extended to a map $f : G_+^\circ \to \mathbb{P}^1_k$ defined on an open subset $G_+^\circ \subset G_+$ with complement of codimension ≥ 2.

If f is regular on G_+°, we can extend it to a regular function on G_+ by normality of G, which implies $f \in k[G_+]$. Otherwise, let $D \subset G_+^\circ$ be the divisor of poles of f and let $x \in D$ be a smooth point (which exists by normality of G_+). We can choose $\phi : A(G_+)$ inducing a map $Spec(\mathcal{O}) \to G_+$, such that the closed point of $Spec(\mathcal{O})$ maps to x. Then $f \notin \phi^{-1}(\mathcal{O})$ (since f has a pole along D), hence $f \notin V(G_+)$. Contradiction $\square$

Let T_+ be the closure of T in G_+. We define the following subgroups $C^* \subset P$, $C_* \subset Q^\vee$:

$$C_* \subset Q^\vee = \{\text{all } t^\mu : k^* \to T \text{ which extend to a regular map } k \to T_+\},$$

$$C^* \subset P = \{\text{all } e^\lambda : T \to k^* \text{ which extend to a regular map } T_+ \to k\}.$$

Proposition 7.
(a) $C_* = \{\mu \in P \mid \langle \mu, \lambda \rangle \geq 0 \quad \forall \lambda \in C^*\}$;
(b) C_* *is a* W*-invariant saturated subsemigroup of* $Q^\vee$;
(c) if $\alpha \in C_*$ *and* $-\alpha \in C_*$ *then* $\alpha = 0$.

Proof. It follows from definitions that $t^\mu \in C_*$ iff for all $fink[T_+]$ the composition $f(t^\mu)$, which is *apriori* an element of $k[t, t^{-1}]$, is in fact an element of $k[t]$. But any $f \in k[T_+]$ is a linear combination of $e^\lambda \in C^* \subset P$. Since $e^\lambda(t^\mu) = t^{\langle \lambda, \mu \rangle}$, we have proved (a).

(b)

(c) $\square$

C^* is finitely generated by Gordan's lemma. Consider the cone $\bar{C}^*$ dual to C_*. *Apriori* $\bar{C}^*$ is the saturation of C^* (though eventually we will prove that the two cones coincide). By part (c) of the above proposition $\bar{C}^*$ is not contained in a hyperplane.

Consider $C_+ = \bar{C}^* \cap P_+$ (the weights which are positive on simple coroots).

C_+ is generated by $\lambda_1, \ldots, \lambda_n$. Take a representation V_i whith highest weight λ_i, such that the weights of V_i belong to $\bar{C}^*$ (for example, the Weyl module would do). Let G'_+ be the closure of the image of G in $\prod_{i=1}^n GL(V_i)$. In characteristic > 0, G'_+ may not be normal (in *char* $k = 0$ we have normality due to invariants on unipotent subgroups). Hence, we define $\widetilde{G}_+$ to be the normalization of G'_+ and $\widetilde{T}_+$ be the closure of T in $\widetilde{G}_+$.

If T'_+ is the closure of the T in G'_+ then T'_+ is the toric variety associated to $\bar{C}^*$, hence T'_* is normal. The morphism $\widetilde{T}_+ \to T'_+$ is finite and birational, hence it is an isomorphism. Therefore, the cone $\widetilde{C}_*$ obtained from $\widetilde{G}_+$, coincides with C_*. By Lemma ?.?, we have $G_+ \simeq \widetilde{G}_+$, and in particular $C^* = \bar{C}^*$. $\square$

We want to formulate one corollary of the Classification Theorem. Let $\mathcal{G}r = G(K)/G(\mathcal{O})$ be the affine grassmanian of G, and define $\mathcal{G}r_+ = A(G_+)/G(\mathcal{O})$. By its definition, $\mathcal{G}r_+ \subset \mathcal{G}r$ is a union of $G(\mathcal{O})$-orbits. These orbits can be understood via the following theorem due to Iwahori and Matsumoto

Theorem 8. *(i) Every double $G(\mathcal{O})$-coset in $G(K)$ contains at least one element of the type t^λ : Spec $K \to T \subset G$ where t^λ is a group homomorphism obtained from $\lambda \in Q^\vee = Hom_{alg\ groups}(k^*, T)$.*

(ii) Two elements t^λ and t^μ belong to the same double coset if and only if $\lambda = w(\mu)$ for some $w \in W$

In particular, the $G(\mathcal{O})$-orbits on $\mathcal{G}r$ are parametrized by $Q^\vee/W$.

Corrolary 9. *The subvariety $\mathcal{G}r_+ \subset \mathcal{G}r$, defined above, is given by the subset $C_*/W \subset Q^\vee/W$, where $C_* \subset Q^\vee$ is the cone defined in ...*

4. $G \times G$-ORBITS ON G_+.

In this section G_+ is the semigroup corresponding to a cone $C_* \subset Q^\vee$ as in the Classification Theorem. We want to give an explicit description of $G \times G$-orbits on G_+.

Example. Consider $GL(3) \subset Mat(3)$. In this case $Q^\vee$ is a free abelian group of rank 3, and C_* can be identified with the positive coordinate octant. $GL(3) \times GL(3)$-orbits on $Mat(3)$ are parametrized by rank of a matrix, and the closure structure is also clear.

In the general case an important part is played by idempotents in G_+. Suppose $t^\lambda : k^* \to T$ is in C_*, i.e. it extend to a regular map $k \to T_+$. Denote by $0^\lambda \in T_+$ the image of $0 \in k$. Then $(0^\lambda)^2 = 0^\lambda$, i.e. 0^λ is an idempotent.

Proposition 10. *The following statements hold*

(i) $0^\lambda = 0^\mu$ if and only if λ and μ belong to the same face of the cone C_*;

(ii) each T-orbit on T_+ contains exactly one point of the type 0^λ

Proof. Follows from the classical theory of toric varieties, see ???. □

Hence, when we use the notation 0^λ we may assume that λ stands for a face of the cone C_*.

Example. In the above example $GL(3) \subset Mat(3)$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_i \geq 0$, and t^λ is given by the diagonal matryx $diag(t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3})$. Hence $0^\lambda = diag(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ where $\varepsilon_i = 0$ in $\lambda_i > 0$ and $\varepsilon_i = 1$ if $\lambda_i = 0$.

Proposition 11. $G_+ = \bigcup_{\lambda \in C_*} G\, 0^\lambda\, G$

Proof. Let $g \in G_+$ and considet $\varphi(t) \in G_+(\mathcal{O}) \cap G(K)$ such that $\varphi(0) = g$. By Iwahori-Matsumoto Theorem we know that $\varphi(t) = g_1(t)\, t^\lambda\, g_2(t)$ where $g_i(t) \in G(\mathcal{O})$ and t^λ is automatically in C_*. Now we take $\lim_{t\to 0}$ and get $g = g_1(0) \cdot 0^\lambda \cdot g_2(0)$. □

It is also clear that for a lifting $n_w \in N_G(T)$ of an element $w \in W = N_G(T)/T$, we have

$$n_w 0^\lambda n_w^{-1} = 0^{w(\lambda)}$$

Denote by Λ the finite set of all faces of C_*. Then Λ has a natural partial order: $\lambda_1 \geq \lambda_2$ if the face λ_1 contains the face λ_2 in its closure. Consider the quotient set Λ/W with the induced partial order and for any $\lambda \in \Lambda$ let $\bar{\lambda}$ be its image in Λ/W

Theorem 12. *$G\, 0^\lambda G = G\, 0^\mu G$ if and only if $\bar{\lambda} = \bar{\mu}$.*

Proof. To appear soon... □.

Corrolary 13. *The orbits of $G \times G$ on G_+ are parametrized by the quotient set Λ/W and the natural partial order on the set of orbits coincides with the natural partial order on Λ/W.*

Example. In the case of $GL(3) \subset Mat(3)$ the cone C_* is the positive octant in $\mathbb{Z}^3$. The three two-dimensional faces are permuted by $W = S_3$, and the same is true about the one-dimensional faces. Hence the quotient set Λ/W has four elements which naturally correspond to matrices of ranks 3, 2, 1 and 0, respectively.

5. Vinberg Semigroup.

In this section we consider a semisimple simply-connected group G_0. It is a consequence of the Classification Theorem that the only affine algebraic semigroup containing G_0 as the group of units, is G_0 itself. Still, we may consider the semigroups G_+ for which the group of units

G is a reductive group satisfying $[G,G]=G_0$. In section we construct a semigroup in this class which is universal in a certain sense.

Let T_0 be its maximal torus and Z_0 be the center, a finite subgroup of T_0. We are going to construct a canonical semigroup $Env(G_0)$, called the *Vinberg semigroup* or *enveloping semigroup* of G_0, which contains as the group of units the reductive group $G=(T_0\times G_0)/Z_0$, where Z_0 acts on $T_0\times G_0$ by $z\cdot(t,g)=(tz,z^{-1}g)$. Thus, in the quotient the pair (tz,g) gets identified with (t,zg).

Since the maximal torus T of G is nothing but the quotient $(T_0\times T_0)/Z_0$, the weight lattice $P(G)$ of the group G is identified with the sublattice in $P(G_0)\times P(G_0)$ formed by all pairs of characters (λ,μ), $\lambda,\mu\in Hom_{alg\,group}(T_0,k^*)$, such that the $\lambda|_{Z_0}=\mu|_{Z_0}$. By (??) this is equivalent to saying that $\lambda-\mu$ belongs to the root sublattice $Q(G_0)\subset P(G_0)$ generated by the roots of G_0.

We fix a system $\alpha_1,\dots,\alpha_l$ of positive roots of G_0 and give

The first definition of $Env(G_0)$. Let $C^*\subset P(G)$ be the cone formed by the pairs $(\lambda,\mu)\in P(G)$, such that $\lambda\geq\mu$, i.e. $\lambda-w(\mu)$ is a positive integral linear combination of the simple roots $\alpha_1,\dots,\alpha_l$, for all elements $w\in W$ of the Weyl group W of G_0. Then $Env(G_0)$ is the semigroup corresponding to the cone C^*.

Remarks.

(i) There exists exactly one $w'\in W$ such that $w(\mu)$ is dominant. Then the inequality $\lambda\geq w'(\mu)$ implies $\lambda\geq w(\mu)$ for any other $w\in W$. In particular, λ itself is dominant.

(ii) One can check that the intersection $C_+=C^*\cap P_+(G)$ is generated by the vectors $(\alpha_1,0),\dots,(\alpha_l,0)$, and $(\omega_1,\omega_1),\dots,(\omega_l,\omega_l)$, where $\omega_i\in P_+(G_0)$ are simple dominant weights. Hence, in view of (??) the above definition amounts to the following construction. Let k_{α_i} be the one-dinesional representation of G which is trivial on $G_0\subset G$, and is given by the character α_i on the central subgroup $T_0\subset G$. Let V_{ω_i} be the Weyl module of G_0 made into a G-module by letting the center $T_0\subset G$ act by the character ω_i. Then the space

$$V=V_{\omega_1}\oplus V_{\omega_2}\oplus\ldots\oplus V_{\omega_l}\oplus k_{\alpha_1}\oplus\ldots\oplus k_{\alpha_l}$$

is naturally an exact representation of G. In characterisitc zero we let $Env(G_0)$ the the closure of G in $\displaystyle\prod_{i=1}^{l} End_k\,V_{\omega_i}\times\prod_{j=1}^{l}k_{\alpha_j}$. In characteristic p this closure may not be normal and we define $Env(G_0)$ to be its normalization.

(iii) One can show that the natural morphism $Env(G_0)\to k^l$ is flat.

Examples.

(i) When $G_0 = SL(2)$ we have $G = GL(2)$ and $Env(SL(2))$ is the closure of $GL(2)$ in $Mat(2) \times k$ with respect to the embedding $g \mapsto (g, det\ g)$. This closure can be defined as the set of pairs $(A, a) \in Mat(2) \times k$ such that $det\ A = a$. Hence the second component is uniquely defined by the first, and we have $Env(SL(2)) = Mat(2)$.

(ii) When $G_0 = SL(3)$ we have to consider the closure of the image of the map $k^* \times k^* \times SL(3) \to GL(3) \times GL(3) \times k^* \times k^*$ given by

$$(t_1, t_2, A) \mapsto \Big(t_1 A,\ t_2 (A^t)^{-1},\ \frac{t_1^2}{t_2},\ \frac{t_2^2}{t_1}\Big)$$

To write the equations for the closure of $SL(3)$, recall that for any $(n \times n)$-matrix u there exists an *adjugate* matrix, to be denoted by $\Lambda^{n-1}(u)$, the entries of which are certain polynomial functions in the entries of u satisfying $\Lambda^{n-1}(u) \cdot u^t = Det(u) \cdot E$.

In this notation, the enveloping semigroup $Env(SL(3)) \subset k \times k \times Mat(3) \times Mat(3)$ is the set of all points $(\alpha_1, \alpha_2, u_1, u_2)$ which satisfy the equations

$$\begin{aligned} \Lambda^2 u_1 =& \alpha_1 u_2; \qquad \Lambda^2 u_2 = \alpha_2 u_1 \\ u_1 u_2^t =& u_2^t u_1 = \alpha_1 \alpha_2 E \end{aligned}$$

If $\alpha_1 = \alpha_2 = 1$ then $u_2^t = u_1^{-1}$ and $det\ u_1 = 1$ hence the fiber of the natural projection $Env(SL(3)) \to k \times k$ over the point $(1, 1)$, is naturally isomorphic to $SL(3)$. One the other hand, if $\alpha_1 = \alpha_2 = 0$, then the fiber is formed by all pairs of rank 1 matrices (u_1, u_2), such that $u_1 u_2^t = u_2^t u_1 = 0$. Note that the space of such matrices is also naturally a semigroup. Therefore, we can think of $Env(SL(3)) \to k \times k$ as a multi-parameter degeneration of the group structure on $SL(3)$, into the semigroup structure on the fiber over $(0, 0)$

Question: What is the set of smooth points of $Env(SL(3))$?

Now we want to give another definition of the semigroup $Env(G_0)$ (valid only in characteristic zero) which generalizes the degeneration picture of the previous example. To that end, recall the definition of the Rees algebra.

Let A be a commutative algebra with an identity element 1_A over the field k, and S be a finitely generated abelian semigroup with zero. Consider an S-filtration of A given by subspaces $A_s \subset A$, $s \in S$, such that $A_{s_1} \cdot A_{s_2} \subset A_{s_1+s_2}$, and $A = \cap_{s \in S} A_s$. Then the *Rees* algebra associated to the filtration $\{A_s\}$ is an S-graded commutative algebra

$$Rees(A) = \bigoplus_{s\in S} t^s A_s,$$

where t^s are formal symbols (thus, in A_s, we may have $A_{s_1} \subset A_{s_2}$ while in $Rees(A)$ these two subspaces belong to different graded pieces). The product structure on $Rees(A)$ is defined via the obvious map $t^{s_1}A_{s_1} \times t^{s_2}A_{s_2} \to t^{s_1+s_2}A_{s_1+s_2}$.

Let $Q \subset P$ be the lattice generated by the simple roots $\alpha_1, \ldots, \alpha_l$, and let Q_+ be the subset in Q formed by non-negative *integral* linear combinations of simple roots. Denote also by $\widetilde{Q}_+$ the set of all non-negative *rational* linear combinations which belong to P. It is known that $P_+ \subset \widetilde{Q}_+$ (cf. ??).

Denote by $k[G]^\lambda$ the subspace $\left[End_k\ V_\chi\right]^* \subset k[G_0]$ defined in the end of Section 2. Then

$$k[G_0] = \bigoplus_{\chi\in P_+} k[G_0]^\chi$$

For any $\lambda \in \widetilde{Q}_+$ let

$$k[G_0]_\lambda = \bigoplus_{\mu\leq\lambda} k[G_0]^\mu$$

where, as before, $\mu \leq \lambda$ means that $\lambda - \mu \in Q_+$. In particular, we have $1 \in k[G_0]^\lambda$ iff $\lambda \in Q_+ \subset \widetilde{Q}_+$. Moreover, dim $k[G_0]^\lambda \leq 1$ if $\lambda \in \widetilde{Q}_+ \setminus P_+$.

The second definition of $Env(G_0)$. We define $Env(G_0)$ to be the spectrum $Spec\ Rees(k[G_0])$ of the Rees algebra associated with the filtration $\{k[G_0]_\lambda\}_{\lambda\in\widetilde{Q}_+}$.

Exercise. Prove that, in this definition, $Env(G_0)$ has a structure of a semigroup.

The semigroup algebra $k[Q_+]$ is embedded into $Rees(k[G_0])$ via the assigmenet $c \cdot [\alpha] \mapsto c \cdot t^\alpha \in k[G_0]_\alpha \subset Rees(k[G_0])$. This induces a map $\pi : Env(G_0) \to k^l \simeq Spec\ k[Q_+]$. Let $\mathbf{1} \in k^l$ be the point defined by the equations $t^\alpha = 1$, $\alpha \in Q_+$ and, similarly, let $\mathbf{0} \in k^l$ be the point given by $t^\alpha = 0$, $\alpha \in Q_+ \setminus \{0\}$.

Theorem 14.
(i) The morphism $\pi : Env(G_0) \to k^l$ is flat;
(ii) the fiber $\pi^{-1}(\mathbf{1})$ is isomorphic to G_0;
(iii) the fiber $\pi^{-1}(\mathbf{0})$ is isomorphic to the spectrum Spec $gr(G_0)$ of the graded algebra $gr(G_0)$ associated to the filtration $k[G_0]_\lambda$.

Exercise. Prove that the fiber $\pi^{-1}(\mathbf{0})$ has a structure of a semigroup. This semigroup is called the *asymptotic semigroup* of G_0.

Thus, the map $Env(G_0) \to k^l$ can be viewed as a multi-parameter deformation of the group G_0 into the asymptotic semigroup $As(G_0)$.

6. A semigroup associated to a parabolic subgroup.

The following construction was used (implicitly) by Braverman and Gaitsgory. Let $P \subset G$ be a parabolic subgroup of a reductive group G, and let $U(P)$ be the unipotent radical of P. Consider also the Levi quotient $M = P/U(P)$.

The reductive group M acts from the right on the quasi-affine homogeneous space $G/U(P)$.

Definition. Let M_+ be the affine algebraic semigroup containing M, defined by any of the following equivalent conditions

(i) For any G-module V, the natural action of M on the subspace of invariants $V^{U(P)}$ extends to the action of M_+, and M_+ is universal with such property;

(ii) M_+ is the closure of $e \cdot M$ in the affine closure $\overline{G/U(P)}$ of $G/U(P)$;

(iii) the right action of M on $G/U(P)$ extends to the right action of M_+ on $\overline{G/U(P)}$, and M_+ is universal with such property.

Exercise. Check that the three conditions above are indeed equivalent.

Theorem 15. *(i) If char $k = 0$ then M_+ is normal.*

(ii) In arbitrary characteristic, the normalization of M_+ is the semigroup associated with the rational cone $W_M \cdot C_M$, where $W_M \subset W$ is the Weyl group of M, and C_M is the intersection

$$(P_M)_+ \cap (P_G)_+ \subset P_G = P_M$$

of the cones of dominant weights for M and G, respectively.