GINZBURG LECTURE SERIES

1. (Unofficial) solutions to Problem Set 1

1.1. (i) We know that Π is Poisson iff the Schouten bracket $[\Pi, \Pi] = 0$ vanishes. This can be checked locally, and since X is smooth, we can choose local coordinates x, y, z so that $\langle \mathsf{vol}, dx \wedge dy \wedge dz \rangle = 1$. Then you only need to check the Jacobi identity for $\{x, \{y, z\}\}$.

(ii) $(\partial_x \phi dx + \partial_y \phi dy + \partial_z \phi dz) \wedge dx \wedge dy = \partial_z \phi dx \wedge dy \wedge dz$, so $\{x, y\}_{d\phi} = \partial_z \phi$.

(iii) Since $d\phi \wedge d\phi = 0$, we have $\{\phi, \bullet\}_{d\phi} = 0$.

(iv) For a general level set, $d\phi$ is nowhere vanishing on the level set. Then the tangent space at a point of the level set is the subspace perpendicular to $d\phi$. This subspace is two-dimensional, and it is easy to see that it is spanned by the Hamiltonian vector fields. This shows that the connected components of the level set are symplectic leaves.

(v) Assume $\phi \neq 0$. Let B be the integral closure of $\mathbb{C}[\phi]$ inside $\mathbb{C}[x, y, z]$. Then the fibers of $\operatorname{Spec} B \to \operatorname{Spec} \mathbb{C}[\phi]$ are finite, and in particular discrete, so they correspond to distinct connected components of the level sets of ϕ. If f is in the Poisson kernel, then f must be constant on each symplectic leaf. By (iv), we deduce that $\operatorname{Spec} B[f] \to \operatorname{Spec} B$ is birational, surjective, quasi-finite. By Zariski's main theorem, this implies $B[f] = B$, i.e., $f \in B$.

(vi) $\operatorname{Spec} A_\phi$ is the level set $\phi = 0$. At a point $p \in \mathbb{C}^3$, the span of the Hamiltonian vector fields at p is either 2-dimensional if $d\phi_p \neq 0$ or 0 if p is a critical point. We deduce that the symplectic leaves are the connected components of the non-critical points in the level set, plus the isolated critical points.

1.2. (i) The Poisson bracket on $\operatorname{Sym} \mathfrak{g}$ is defined by extending the Lie bracket. The Hamiltonian vector fields are given by $\xi_a(b) = [a, b]$ for $a, b \in \mathfrak{g}$. So $\xi_a = \operatorname{ad}^*(a)$ as vector fields on $\mathfrak{g}^*$. Since $d \operatorname{Ad}^* = \operatorname{ad}^*$, where Ad^* is the coadjoint action of G on $\mathfrak{g}^*$, we deduce that the symplectic leaves in $\mathfrak{g}^*$ are the coadjoint G-orbits.

(ii) Let G_λ be the centralizer of λ. Then $O = G/G_\lambda$. The anchor map $\mathfrak{g} = T^*_\lambda \mathfrak{g}^* \to T_\lambda \mathfrak{g}^*$ sends $x \mapsto \lambda([x, \bullet]) = \operatorname{ad}^* x(\lambda)$, so it factors through $\mathfrak{g}/\mathfrak{g}_\lambda \simeq T_\lambda O$. It follows that $\omega(\operatorname{ad}^* x(\lambda), \operatorname{ad}^* y(\lambda)) = \lambda([x, y])$.

1.3. (i) [CG, Proposition 1.4.11] Let $\pi : T^*(G/H) \to G/H$ and

$$\pi_* : T_\lambda(T^* G/H) \to T_1(G/H) = \mathfrak{g}/\mathfrak{h}.$$

Define the 1-form ν by $\nu_\lambda(v) = \lambda(\pi_* v)$ for $v \in T_\lambda(T^* G/H)$. The symplectic form ω is defined as $d\nu$. Let $\tilde{\alpha} \in T(T^*(G/H))$ be the vertical vector field whose restriction to any fiber of $d\pi$ is α. By definition, ν vanishes on vertical vector fields. Note that $[\tilde{\alpha}, \tilde{\beta}]$ is still vertical. Therefore $\nu(\tilde{\alpha}) = \nu(\tilde{\beta}) = \nu([\tilde{\alpha}, \tilde{\beta}])$. Hence

$$\omega(\alpha, \beta) = d\nu(\tilde{\alpha}, \tilde{\beta})_\lambda = 0. \tag{1}$$

Let $\mathsf{act} : \mathfrak{g} \to T(T^*G/H)$ denote the vector fields defined by the infinitesimal action of G on $T^*(G/H) = G \times_H \mathfrak{h}^\perp$. Since the action is Hamiltonian, act factors through $\mathsf{H} : \mathfrak{g} \to \mathcal{O}(T^*G/H)$, and $\mathsf{H}_x(g,\gamma) = \nu(\mathsf{act}(x))(g,\gamma) = \gamma(gxg^{-1})$ for $g \in G, \gamma \in \mathfrak{h}^\perp$. We deduce that

$$\omega(x(\lambda), y(\lambda)) = \omega(\mathsf{act}(x), \mathsf{act}(y))_\lambda = \{\mathsf{H}_x, \mathsf{H}_y\}(\lambda) = \lambda([x,y]). \tag{2}$$

Lastly, we compute $\omega(x(\lambda), \alpha) = \omega(\mathsf{act}(x), \tilde{\alpha})_\lambda$. Since ν vanishes on vertical vector fields,

$$d\nu(\mathsf{act}(x), \tilde{\alpha}) = -\tilde{\alpha}(\mathsf{H}_x) - \nu([\mathsf{act}(x), \tilde{\alpha}]).$$

Observe that $\widetilde{\alpha}(\mathsf{H}_x) = \alpha(x)$ the constant function. Moreover $i_{[\mathsf{act}(x),\tilde{\alpha}]}\nu = -i_{\tilde{\alpha}} L_{\mathsf{act}(x)}\nu$ and $L_{\mathsf{act}(x)}\nu = i_{\mathsf{act}(x)}\omega + d(i_{\mathsf{act}(x)}\nu) = -d\mathsf{H}_x + d\mathsf{H}_x = 0$. In conclusion,

$$\omega(x(\lambda), \alpha) = -\alpha(x) \tag{3}$$

(ii) For $g \in G, \alpha \in \mathfrak{h}^\perp, x \in \mathfrak{g}$, the moment map is given by $\mu(g,\alpha)(x) = \mathsf{H}_x(g,\alpha) = \alpha(gxg^{-1})$. Thus $\mu(g,\alpha) = \mathrm{Ad}^*(g)\alpha$.

1.4. (i) For each subgroup $G \subset \Gamma$, let $U^G = \{v \in V \mid \Gamma_v = G\}$ where Γ_v is the stabilizer of v. We have the stratification by stabilizers $V = \bigsqcup_{G \subset \Gamma} U^G$. Let $p : V \to V/\Gamma$ be the quotient, and define $U_G = p(U^G)$. Then $U_G = U_H$ iff H and G are conjugate, and $V/\Gamma = \bigsqcup_{[G] \subset \Gamma} U_G$. The quotient $U^G \to U_G$ is Galois, so ω descends to a 2-form on U_G. Note that for $U^G \neq 0$, we have U^G is open in V^G since the complement is a union of lower dimensional subspaces. Since the isotypic components of V as a G-representation are symplectically orthogonal, V^G is a symplectic subspace. Hence U_G is symplectic. Therefore

$$V/\Gamma = \bigsqcup_{[G] \subset \Gamma} U_G$$

is a partition into symplectic leaves.

(ii) $U = U_1$ is the unique open dense symplectic leaf on V/Γ. Let $\pi : X \to V/\Gamma$ be a resolution of singularities. We consider the 2-form $\pi^*\omega$ on $\pi^{-1}(U)$ as a rational 2-form on X. By taking a resolution of singularities $Y \to (X \times_{V/\Gamma} V)_{\mathrm{red}}$, we get a commutative square

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\tilde{\pi}"] & X \arrow[d, "\pi"] \\ V \arrow[r, "p"] & V/\Gamma \end{tikzcd}$$

where $\tilde{\pi}$ is a resolution of V, and f is a proper, generically finite morphism of smooth varieties. We know that $f^*\pi^*\omega$ is regular since it equals the pullback of the symplectic form on V. Since X is normal, it suffices to show that $\pi^*\omega$ is regular away from a subset of codimension ≥ 2. Fact: the non-finite locus of f has codimension ≥ 2 in X. Therefore we shrink X to assume f is a finite map. Now one can check locally that if $f^*\omega'$ is regular given a rational form ω' on X, then ω' must be regular on X. Hence $\pi^*\omega$ is regular on X.