# DAY 3: PRECISION IMPOSSIBLE

MUSIC: THE NUMBER THEORY OF SOUND    J-LO    MC2023

## 1. Recap and Setup

Yesterday we defined a scale as just being a set of *pitches*. Equivalently, we can describe a scale using a base frequency $f$ and a set of *intervals* $r$ (so the scale is the set of all pitches of the form $fr$). But there is really one extra condition that all scales should have.[1]

**Definition.** A scale $S$ satisfies the "next note" property if for any $f \in S$, there exists a pitch $\text{next}(f) \in S$ such that $\text{next}(f) > f$ and there are no elements $g \in S$ with $f < g < \text{next}(f)$.

Yesterday we began to discover a tension underlying scales. Roughly speaking, there are two kinds of properties that you might want a scale to have:

(A) "Mildness" properties: you should be able to play ratios of small whole numbers.

(B) "Well-spaced" properties: you should be able to move up and down the scale in steps.

Octave equivalence is a property of type (A), and so is "contains two notes in an interval of $\frac{3}{2}$." Being closed under combining and inverting intervals is type (B).

(1) Suppose a scale $S$ satisfies octave equivalence and the next note property, and is closed under combining and inverting intervals. Prove that the only rational numbers in $S$ are powers of 2.

In other words, a scale satisfying strong conditions of type (B) will fail most conditions of type (A), and a scale satisfying strong conditions of type (A) will fail most conditions of type (B)! So we're going to have to compromise. There are two major approaches to doing this:

- **Just intonation:** keep (A), mess up (B). Design a scale using rational numbers. Allow composing and inverting intervals as much as possible, but accept the fact that it doesn't always work.
- **Equal temperament:** keep (B), mess up (A). Design a scale using powers of a root of 2, to ensure composing and inverting intervals is always possible. Approximate some rational numbers as closely as possible.

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[1] As far as I'm aware, every scale in history has this property. The only exception would be music which allows *all* frequencies in a certain range, for example in glissandos or vibrato. But I think it's better to think of these examples as happening *outside* of the framework of scales; scales are not the right tool to describe these musical effects.

Equal temperament is the most common solution used today, but wasn't widely adopted until the late 18th century.[2] Before then, just intonation was the common approach to music.

## 2. CHOOSE YOUR OWN ADVENTURE

We'll only consider scales with octave equivalence and next notes — but now you get to choose whether to build a scale with just intonation or equal temperament! In both cases we'll ask the same question: *how many pitch classes should you include in your scale?* It turns out that the answer depends on how strongly you want both (A) and (B) to hold.

| **Just intonation** | **Equal temperament** |
|---|---|
| Fix some positive integer $n$ and let $S$ contain the intervals $1, 3, 3^2, \ldots, 3^{n-1}$.<br>(2) Prove that inverting and composing intervals (as on page 3 of the day 2 handout) always yields a pitch that is either 1, $3^n$, or $\frac{1}{3^n}$ times a pitch in $S$. If $3^n$ is very close to $2^m$ for some integer $m$, conclude that inverting and composing intervals in $S$ always yields a pitch very close to an element of $S$.<br>(If you have music theory background: how does this relate to the circle of fifths?) | Let $f \in S$ be the base frequency, and $b = \frac{\text{next}(f)}{f}$ the smallest interval greater than 1 that our scale can play.<br>(2) Prove that every interval in $S$ is a power of $b$.<br>In particular, $b^n = 2$ for some positive integer $n$ by octave equivalence. If we want our scale to be able to play an interval close to 3, we also need some integer $m$ with $b^m$ close to 3. |

(3) Regardless of which scale type you chose, show that you get a good scale if you can find a rational number close to $\log_2 3$ (and the closer it is, the better the scale). What do the numerator and denominator of this rational number each correspond to in the resulting scale?

**Definition.** Given a real number $\alpha$, a rational number $\frac{p}{q}$ with $q > 0$ is a *best rational approximation* for $\alpha$ if $|\alpha - \frac{p}{q}| \leq |\alpha - \frac{a}{b}|$ for all rationals $\frac{a}{b}$ with $0 < b \leq q$.

(4) The first two best rational approximations for $\log_2 3$ are $\frac{2}{1}$ and $\frac{3}{2}$. What are the next five? What scales can you produce using each of these rational numbers?

(5) Suppose that instead of wanting to include 3 in our scale, we wanted to include 5. How do your answers to questions (2)–(4) change?

(6) Suppose you want a scale to contain both 3 AND 5 (either exactly if you're using just intonation, or approximately if you're using equal temperament). How would you look for scales that are decent at satisfying both (A) and (B)?

[2]See https://www.britannica.com/art/equal-temperament for more on the history of equal temperament.

## 3. Crash course in continued fractions

Best rational approximations to $\alpha$ can be found by brute force: for each $q = 1, 2, 3, \ldots$, let $p$ denote the closest integer to $q\alpha$ (we can take $p = \lfloor q\alpha + \frac{1}{2} \rfloor$), and check if the result is closer to $\alpha$ than anything you've found previously. But there is a much faster algorithm for finding best rational approximations using *continued fractions*.

To compute the continued fraction of a number $\alpha$, you perform the following steps. Set $\alpha_0 := \alpha$. For each $k = 0, 1, 2, \ldots$ do the following:

a. Write $\alpha_k = a_k + r_k$, where $a_k = \lfloor \alpha_k \rfloor$ and $r_k \in [0, 1)$.
b. If $r_k = 0$, halt, otherwise, set $\alpha_{k+1} = \frac{1}{r_k}$ and repeat.

Then we write

$$\alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},$$

which we write as $\alpha = [a_0; a_1, a_2, a_3, \ldots]$. For example, $\frac{5}{3} = [1; 1, 2]$, and $\pi = [3; 7, 15, 1, 292, 1, \ldots]$. The algorithm halts (the continued fraction expansion is finite) if and only if $\alpha$ is a rational number.

(7) Compute the continued fraction expansion of $\log_2 3$ up through $a_{10}$.

Here is one fact from the theory of continued fractions that will be helpful to us.

**Theorem.** Let $\alpha = [a_0; a_1, a_2, \ldots]$. If $\frac{p}{q}$ is a best rational approximation to $\alpha$ with $q > 1$, then for some $k \geq 1$ and some integer $b$ with $\frac{a_k}{2} \leq b \leq a_k$, we have $\frac{p}{q} = [a_0; a_1, a_2, \ldots, a_{k-1}, b]$. Conversely, for any $k \geq 1$ and any integer $b$ with $\frac{a_k}{2} < b \leq a_k$, the rational number $[a_0; a_1, a_2, \ldots, a_{k-1}, b]$ is a best rational approximation.

(Notice that the statement isn't quite an if-and-only-if: if $b = \frac{a_k}{2}$, then $[a_0; a_1, a_2, \ldots, a_{k-1}, b]$ is sometimes a best rational approximation, but sometimes it isn't.)

For an example of this result in action, see the next page for a list of the best rational approximations of $\pi$. Want to understand the theory behind continued fractions and figure out how to prove this? Go to Ben's week 4 class on continued fractions!

(8) Use the theorem above to compute the first few best rational approximations of $\log_2 3$.
(9) You get extremely good rational approximations if you cut off the continued fraction expansion right before a large term (for example, $\frac{355}{113} = [3; 7, 15, 1]$ is closer to $\pi$ than any other rational number with denominator smaller than 15000). Use this to explain why dividing the octave into 12 is a particularly good choice. What are some other good options for the number of steps in an octave?

Here is an ordered list of the first 15 best rational approximations to $\pi$.

$$
\begin{aligned}
[3] &= & \tfrac{3}{1} & = \pi - 0.1415926536\ldots \\
[3;4] &= & \tfrac{13}{4} & = \pi + 0.1084073464\ldots \\
[3;5] &= & \tfrac{16}{5} & = \pi + 0.0584073464\ldots \\
[3;6] &= & \tfrac{19}{6} & = \pi + 0.0250740131\ldots \\
[3;7] &= & \tfrac{22}{7} & = \pi + 0.0012644893\ldots \\
[3;7,8] &= & \tfrac{179}{57} & = \pi - 0.0012417764\ldots \\
[3;7,9] &= & \tfrac{201}{64} & = \pi - 0.0009676536\ldots \\
[3;7,10] &= & \tfrac{223}{71} & = \pi - 0.0007475832\ldots \\
[3;7,11] &= & \tfrac{245}{78} & = \pi - 0.0005670126\ldots \\
[3;7,12] &= & \tfrac{267}{85} & = \pi - 0.0004161830\ldots \\
[3;7,13] &= & \tfrac{289}{92} & = \pi - 0.0002883058\ldots \\
[3;7,14] &= & \tfrac{311}{99} & = \pi - 0.0001785122\ldots \\
[3;7,15] &= & \tfrac{333}{106} & = \pi - 0.0000832196\ldots \\
[3;7,15,1] &= & \tfrac{355}{113} & = \pi + 0.0000002667\ldots \\
[3;7,15,1,146] &= & \tfrac{48307}{15377} & = \pi - 0.0000002662\ldots
\end{aligned}
$$