DAY 1: DEFINING THE DICTIONARY AND DRAWING LINES

CURVES THAT CLASSIFY GEOMETRY PROBLEMS   J-LO   MC2022

Goal

Suppose we have a collection of geometric objects that we want to study. For example:

(a) Pythagorean triples (right triangles with integer side lengths).
(b) Right triangles where the legs are integers and the hypotenuse is $\sqrt{2}$ times an integer.
(c) Triangles with integer side lengths and one $60°$ angle.
(d) Pythagorean triples with area equal to a perfect square.
(e) Triangles with integer side lengths and integer area, where two of the side lengths are in a ratio of 3 to 4.
(f) Pairs of triangles with integer side lengths, one right and one isosceles, that have equal area and equal perimeter.

Each of these collections is *scale-invariant*: if you take one solution and multiply all the lengths by any fixed whole number, you get another solution (e.g. $(3, 4, 5)$, $(6, 8, 10)$, $(30, 40, 50)$ are all Pythagorean triples). So to make the question more interesting we will only consider **primitive solutions** in each collection: solutions that are not scaled copies of a smaller solution.

(1) For each of the six collections above, make a guess: how many primitive solutions do you think there are? Is there a finite list, are there infinitely many, or are there none at all?

The goal of this class is to develop a method that will allow us to find as many primitive solutions as possible.

The Dictionary: from solutions to rational points

To illustrate the method, let's classify Pythagorean triples. If we divide the equation $a^2 + b^2 = c^2$ by c^2, and set $x = \frac{a}{c}$, $y = \frac{b}{c}$, we obtain a rational point (x, y) satisfying the equation

$$x^2 + y^2 = 1.$$

We can also trace through this process in the other direction: given a point (x, y) on the unit circle, we can clear denominators to obtain a Pythagorean triple. For example, the point $(\frac{3}{5}, \frac{4}{5})$ on the unit circle satisfies $\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1$, and we have to multiply by 5^2 to clear the denominators; this gives us the Pythagorean triple $(3, 4, 5)$.

To summarize, we have a "dictionary": two functions that allow us to translate back and forth between describing Pythagorean triples using side lengths, and describing them using rational points.

$$\{(a, b, c) \in \mathbb{Z}^3 : a^2 + b^2 = c^2;\ a, b, c > 0\} \Leftrightarrow \{(x, y) \in \mathbb{Q}^2 : x^2 + y^2 = 1\}$$
$$(a, b, c) \mapsto \left(\tfrac{a}{c}, \tfrac{b}{c}\right)$$
$$(dx, dy, d) \leftarrow\!\shortmid (x, y),$$

where d is the least common multiple of the denominators of x and y.

(2) Unfortunately, the second function does not quite give us what we want: for example, $(x, y) = (-1, 0)$ is sent to $(a, b, c) = (-1, 0, 1)$, which is not a Pythagorean triple! What extra constraints do x, y need to satisfy in order to ensure that only Pythagorean triples are produced?

(3) Start with $(a, b, c) = (6, 8, 10)$, translate to a rational point, then translate back to a Pythagorean triple. Do you end up back where you started?

(4) Prove that there is a bijection between *primitive* Pythagorean triples and rational points on the unit circle satisfying the constraint from problem (2).

(5) Primitive solutions to each of the six collections from page 1 correspond to rational points on some curve. What curve do you get for (b) and (c)? (There can be more than one answer; it depends on how you define the new variables.)

FINDING RATIONAL POINTS

By changing variables, we've turned the problem *how to generate primitive Pythagorean triples* into a new problem: *how to generate rational points on the unit circle?*

There are lots of ways to parametrize points on the circle, for example:

- By angle: Given θ, produce the point $(\cos\theta, \sin\theta)$
- By x-coordinate: Given x, produce the point $(x, \sqrt{1 - x^2})$
- By slope: Given m, produce a point (x, y) satisfying $\frac{y}{x} = m$

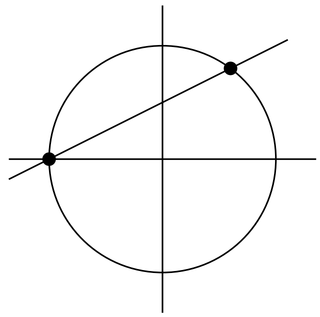
(6) All three of the methods above involve finding an intersection point with the unit circle and a line. What is the line in each case?

(7) For each of the above methods, discuss whether it would be useful in finding *rational* points on the circle.

(8) Can you come up with a new method that may be useful in finding rational points on the circle?

USE A PROJECTOR

Idea: To generate new rational points, draw lines through *existing* rational points.



(9) Given a rational number t, write down the equation for the line through $(-1, 0)$ with slope t. Where does this line intersect the unit circle?

(10) Prove every rational point on the unit circle (other than $(-1, 0)$ itself) can be produced in this way by some choice of t.

(11) Use problems (4) and (10) to find a formula for primitive Pythagorean triples.

FURTHER EXPLORATION (OPTIONAL)

Read through the problems below and try the ones that look interesting!

(12) Complete problem (5): for the collections (d), (e), and (f), solutions correspond to rational points on what curve? ((f) is a challenge)

(13) Let's explore what happens if we project from a different point in problem (9). If you replace $(-1, 0)$ with $(\frac{3}{5}, \frac{4}{5})$, what is the resulting formula for rational points in terms of t? Find a value of t that produces the point $(\frac{3}{5}, \frac{4}{5})$ itself (this does not happen with $(-1, 0)$!); what does this mean geometrically? Are there any rational points that are not produced by the formula?

(14) Try using stereographic projection on some of the curves classifying the remaining collections (the solutions to problems (5) and (12)). Each of the curves should have at least one rational point that isn't too hard to find. What happens if you try to use stereographic projection from that point?

(15) Prove that there are no right triangles with legs that are integers and hypotenuse $\sqrt{3}$ times an integer.

(16) A **rational conic section** is a curve defined by an equation of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

where $a, b, c, d, e, f \in \mathbb{Q}$, and at least one of a, b, c is nonzero. Prove that a rational conic section has either 0, 1, or ∞ rational points.