# DAY 4: BEYOND TWO AND THREE

MUSIC: THE NUMBER THEORY OF SOUND J-LO MC2023

Throughout this sheet, we assume the base frequency of every scale is 1. This simplifies the property of being closed under composing and inverting intervals:

**Definition.** Suppose $1 \in S$. Then $S$ is *closed under composing and inverting intervals* if for all $r, s \in S$, we have $rs \in S$ and $\frac{1}{r} \in S$. In other words, $S$ is a *subgroup* of the positive reals under multiplication.

## 1. Recap

Yesterday we saw that it's impossible for a scale to satisfy the following all at once:

- octave equivalence;
- next note property;
- closed under composing and inverting intervals;
- contains any rational number that is not a power of 2.

If we want to build the scale out of rational numbers, then we have to give up being closed under composing and inverting intervals (resulting in *just intonation*). If we want scales to be closed under composing and inverting intervals, we have to give up using rational numbers other than powers of 2 (resulting in *equal temperament*). The goal is to get *as close as possible* to satisfying all of the conditions.

If $\frac{m}{n}$ is very close to $\log_2 3$, then $2^{m/n}$ is very close to 3, so the interval $\frac{3}{2}$ (a "perfect fifth") is very close to an equal-tempered interval obtained by dividing the octave into $n$ steps and taking $m - n$ of them.

The first few best rational approximations of $\log_2 3$ are the following:

$$\frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{11}{7}, \frac{19}{12}, \frac{46}{29}, \frac{65}{41}, \frac{84}{53}, \frac{317}{200}, \frac{401}{253}, \frac{485}{306}, \frac{569}{359}, \frac{1054}{665}, \frac{13133}{8286}, \ldots$$

If we divide the octave into 1, 2, or 3 steps, there aren't quite enough notes to be interesting. But after this, the next few rational approximations give us actual scales that have been used widely over the course of history:

- *pentatonic scale*: 5 steps in an octave, 3 of these to a perfect fifth.
- *diatonic scale*: 7 steps in an octave, 4 of these to a perfect fifth.
- *chromatic scale*: 12 steps in an octave, 7 of these to a perfect fifth.

## 2. Pythagorean scales and commas

A "Pythagorean scale" is one that contains pitches of the form $2^m 3^n$ for all $m \in \mathbb{Z}$ and some $n \in \mathbb{Z}$ (in other words, they are produced using octaves and fifths). Below is one example of a 12-note Pythagorean scale, together with labels for convenience.

Note that instead of using the powers of 3 from $3^0$ to $3^{11}$, this scale was created using the powers of 3 from $3^{-5}$ to $3^5$ (and octave equivalence).

| Label: | $C$ | $D\flat$ | $D$ | $E\flat$ | $E$ | $F$ | $G\flat$ | $G$ | $A\flat$ | $A$ | $B\flat$ | $B$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Equal-tempered: | 1 | $2^{\frac{1}{12}}$ | $2^{\frac{2}{12}}$ | $2^{\frac{3}{12}}$ | $2^{\frac{4}{12}}$ | $2^{\frac{5}{12}}$ | $2^{\frac{6}{12}}$ | $2^{\frac{7}{12}}$ | $2^{\frac{8}{12}}$ | $2^{\frac{9}{12}}$ | $2^{\frac{10}{12}}$ | $2^{\frac{11}{12}}$ |
| (as decimal): | 1.000 | 1.059 | 1.122 | 1.189 | 1.260 | 1.335 | 1.414 | 1.498 | 1.587 | 1.682 | 1.782 | 1.888 |
| Pythagorean: | 1 | $\frac{256}{243}$ | $\frac{9}{8}$ | $\frac{32}{27}$ | $\frac{81}{64}$ | $\frac{4}{3}$ | | $\frac{3}{2}$ | $\frac{128}{81}$ | $\frac{27}{16}$ | $\frac{16}{9}$ | $\frac{243}{128}$ |
| (as decimal): | 1.000 | 1.053 | 1.125 | 1.185 | 1.266 | 1.333 | | 1.500 | 1.580 | 1.688 | 1.778 | 1.898 |
| 5-limit: | 1 | | | | | | | | | | | |
| (as decimal): | | | | | | | | | | | | |

(1) We can reach a Pythagorean $G\flat$ in two ways, getting two different answers; fill in these answers in the table above. What is the ratio between these two answers?

If a scale $S$ is not closed under composing and inverting intervals, then it is sometimes possible to find four elements $f_1, f_2, g_1, g_2 \in S$ where $\frac{f_2}{f_1}$ is very close to, but not quite equal to, $\frac{g_2}{g_1}$. The ratio $\frac{f_2}{f_1} / \frac{g_2}{g_1}$ is called a *comma* of the scale $S$; commas measure the failure of a scale to be closed under combining and inverting intervals. For example, the answer to question (1) is called the *Pythagorean comma*.

## 3. THE INTERVAL 5

There are many ratios of small numbers that *aren't* built out of octaves and fifths, and we may want these intervals in our scales! For example, the "major third"[1] $\frac{5}{4}$ is a very mild interval that would be great to have in our scale, but does not appear in the Pythagorean scale.

(2) The Pythagorean scale *approximates* the true major third; what is the ratio between the Pythagorean major third and the true major third? (This ratio is called the "syntonic comma."[2])

(3) By combining octaves, perfect fifths, and major thirds in various ways, you can replace most of the Pythagorean scale with ratios of much smaller numbers. Fill in the row labeled "5-limit" above; you should be able to use denominators smaller than 10 for all but two of the entries.

[1] Try not to get too confused by the fact that the fifth uses a 3 and the third uses a 5.

[2] also known as the "chromatic diesis," the "comma of Didymus," the "Ptolemaic comma," and the "diatonic comma." Fun fact: the numerator and denominator of this interval are the largest pair of consecutive integers with the property that the only primes dividing each are 2, 3, or 5.

(4) The scale you've developed is not closed under combining and inverting intervals; how many commas can you find? For instance, try to find two notes where the interval between them is close to but not equal to $\frac{3}{2}$, and compute the ratio between this interval and $\frac{3}{2}$.

## MORE THAN TWO NOTES

**Definition.** A *chord* is a set of at least three distinct frequencies chosen from a scale. A chord with exactly three frequencies is called a *triad*.

(5) Suppose $r = \frac{a}{b} > 1$ is an interval. What is the period of $\sin(2\pi t) + \sin(2\pi r t)$?
(6) Suppose $r = \frac{a}{b} > 1$ and $s = \frac{c}{d} > 1$ are intervals. What is the period $\sin(2\pi t) + \sin(2\pi r t) + \sin(2\pi s t)$?
(7) What happens if $r$ and $s$ are integers? What is the period of

$$\sin(2\pi t) + \sin(4\pi t) + \sin(6\pi t) + \sin(8\pi t) + \sin(10\pi t) + \sin(12\pi t)?$$

(A note with frequency $nf$ is called a "harmonic" of a note with frequency $f$; this question is about adding harmonics to $\sin(2\pi t)$.)
(8) Using intervals from the 5-limit scale you came up with earlier, find chords such that the resulting wave has period $2, 3, 4, 5, 6, 8, 9, 10, 12$. Use the scale builder tool to play these chords. How spicy are they? If you have the appropriate music theory background, can you identify which chords you get in each case?