DAY 5: BEYOND FIVE

MUSIC: THE NUMBER THEORY OF SOUND J-LO MC2023

Yesterday we tried to build a scale out of octaves (2), perfect fifths ($\frac{3}{2}$), and major thirds ($\frac{5}{4}$). Amazingly, we found that all of these intervals could be approximated by an equal-tempered scale: $\frac{3}{2} \approx 2^{7/12}$ and $\frac{5}{4} \approx 2^{4/12}$. This is strange, because we decided to use a 12-note scale only because it approximates $\frac{3}{2}$ very well. Why would we expect this scale to approximate $\frac{5}{4}$ as well? In general, how would we find equal-tempered scales that approximate both $\frac{3}{2}$ and $\frac{5}{4}$?

(1) Suppose we can find a positive real number $b > 1$ and whole numbers n_2, n_3, n_5 such that

$$b^{n_2} = 2, \qquad b^{n_3} \approx 3, \qquad b^{n_5} \approx 5.$$

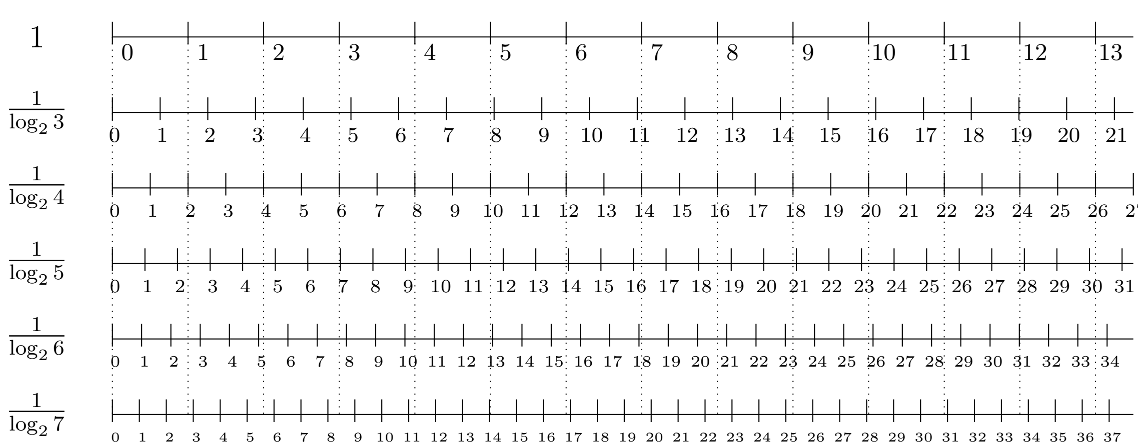
Explain how we could use this to build an equal-tempered scale that approximates the perfect fifth and major third.

(2) By rearranging the above expression, show that it's equivalent to look for whole numbers n_2, n_3, n_5 such that

$$n_2 \approx \frac{n_3}{\log_2 3} \approx \frac{n_5}{\log_2 5}.$$

If we only needed to satisfy one of the approximate equalities above, we could rearrange it into a best rational approximation problem: for instance, $n_2 \approx \frac{n_3}{\log_2 3}$ implies that $\frac{n_3}{n_2} \approx \log_2 3$. But to solve multiple of these at once will require different tools.

(3) In the figure below, each ruler is labeled with the unit distance between tick marks. How would you solve the approximate equalities from question (2) using this diagram?



(4) Find a solution to $n_2 \approx \frac{n_5}{\log_2 5}$. Interpret this result musically: what does this say about the relationship between the major third $\frac{5}{4}$ and the octave 2?

(5) Let
$$f_k(t) = \cos(2\pi t \log_2 k).$$
Note that $f_k(t) = 1$ exactly when t is an integer multiple of $\frac{1}{\log_2 k}$. What does it mean if $f_2(t) + f_3(t) + f_5(t)$ is very large?

(6) Using a graphing calculator like Desmos, graph $f_2(t) + f_3(t)$. Where are the highest points? Does this agree with what we discovered yesterday?

(7) Graph $f_2(t) + f_3(t) + f_5(t)$ and look for the highest peaks. Are the results different from the previous question? Why would this be?

(8) (Optional) Look for the *lowest* points of $f_2(t) + f_3(t) + f_5(t)$. What do these points represent? Try using the scale builder to test out one of these scales.

To infinity

The function f_k allows us to measure how good an equal-tempered scale is at approximating the interval k: if $f_k(n)$ is close to 1, then dividing an octave into n steps will allow us to approximate the interval k. By adding up $f_k(n)$ for multiple values of k, we can get a single quantity that measures how well a scale can approximate many values of k at once. Can we find a function that measures how well an equal-tempered scale approximates *all* whole number intervals at once?

(9) (Optional) Prove that $\sum_{k=1}^{\infty} f_k(t)$ does not converge.

The previous problem shouldn't disappoint us too much: we don't want every k to be measured with equal importance anyways (if a scale approximates 20011 extremely well but does not approximate 3 or 5 very well, it's a bad scale). So let's add coefficients to make the higher values of k matter less.

(10) (Optional) Let $a > 1$. Prove that $\sum_{k=1}^{\infty} \frac{f_k(t)}{k^a}$ converges.

(11) Euler's identity says that $e^{i\theta} = \cos\theta + i\sin\theta$; in particular, we can write $\cos\theta = \mathrm{Re}(e^{i\theta})$. Use this to express $\frac{f_k(t)}{k^a}$ as the real part of a single term of the form $\frac{1}{k^{\text{something}}}$.