DAY 2: BUILDING SCALES

MUSIC: THE NUMBER THEORY OF SOUND J-LO MC2023

Recap and Setup

Yesterday we experimented with pairs of frequencies, f_1 and f_2 (positive real numbers), and you made many exciting observations. Here are a few that I want to point out:

(A) The way two frequencies sound together depends primarily on the *ratio* between them.

(B) Most ratios are pretty spicy, but *ratios of small whole numbers* (e.g. 2, $\frac{3}{2}$, $\frac{5}{4}$) sound more mild.

Let's use these observations to establish some definitions.

- *Pitch*: another word for frequency (used more often in a music context, while frequency is used for arbitrary waves).
- *Scale*: A set of pitches.
- *Note*: a sound wave with a fixed pitch.
- *Interval*: a ratio of two pitches.
- *Unison*: The interval 1.
- *Octave*: The interval 2 or $\frac{1}{2}$ (we can distinguish between the two using directional words, like "octave above" or "down an octave").

Here is a sample sentence. "If one note has a pitch of 440 Hz and another note has a pitch of 880 Hz, then the two notes are an octave apart; the first note is an octave below the second."

In some sense, scales are the building blocks of music.[1] Most musical pieces are written by first fixing a scale, and then combining frequencies chosen from that scale.

If two notes are an octave apart, they fit together so well that most people just consider them to be essentially the same note!

Definition. Two pitches are in the same *pitch class* if the interval between them is 2^n for some integer n. A scale S satisfies *octave equivalence* if whenever $f \in S$, the entire pitch class of f is contained in S.

The vast majority of scales used throughout history satisfy octave equivalence.[2] What other properties might we want a scale to have?

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[1] Of course there's a lot more to music than just picking notes: there's also rhythm, timbre, dynamics, ornamentation, lyrics, and more.

[2] Octave equivalence is not universal: see https://en.wikipedia.org/wiki/Octave#Equivalence for a discussion of the history and neuroscience behind octave equivalence.

EXPLORATIONS

Open the "Scale Builder" (available either using the QR code or at rb.gy/p4ju5). Choose a base frequency (called the *key* or the *tonic* of the scale), and a collection of ratios. You will get an instrument that can play the tonic, as well as all multiples of the tonic by the provided ratios. For example, if you provide a base frequency of 200 Hz and a ratio 1.2, then your instrument will be able to play a note with a pitch of 240 Hz.

The scale builder constructs scales with octave equivalence. For example, if you include the ratio 3, it will automatically also include $\frac{3}{2}$.

Exploration 1. In your group, build a scale! Build many scales! Play songs with your scales! This is extremely open-ended, but your main goal is to answer the following question: *what properties might one want a scale to have?*

Try to keep these properties as simple as possible.[3] Once you've settled on some properties, try to construct a scale that satisfies these properties.

Additional questions:

(1) Observe that the scale builder displays notes in the scale *logarithmically* (the distance from 1 to 2 is equal to the distance from 2 to 4). What's the advantage of displaying the information this way?
(2) Can you produce a single scale in multiple ways? That is, starting with a different base frequency and different intervals from the base frequency, produce the same set of notes (up to octave equivalence).
(3) Think of how to *compose* two intervals and how to *invert* an interval. Write down a mathematical statement that describes scales that are closed under composing and inverting intervals.
(4) A scale S is *tonic-invariant* if for any choice of frequency $f \in S$, the set of intervals $\{\frac{g}{f} : g \in S\}$ is always the same set. Prove that a scale is tonic-invariant if and only if it is closed under composing and inverting intervals.
(5) Try to build a scale that contains $\frac{3}{2}$, satisfies octave equivalence, and is closed under composing and inverting intervals. What seems to happen? Make a conjecture and try to prove it.

[3]The reminder to keep things simple is especially important if you already know some music theory. "Can play a C#m7♭5" or "allows for Picardy thirds" are both perfectly valid properties, but perhaps wait until after class to classify the set of scales with these properties.

More additional questions:

There's more than one way to answer question (3) correctly, but here's one definition that works.

Definition. A scale S is *closed under inverting intervals* if whenever f and fr are in S, we also have $\frac{f}{r} \in S$. A scale S is *closed under composing intervals* if whenever $f, fr, fs \in S$, we also have $frs \in S$.

(6) Suppose a scale S satisfies octave equivalence, contains two pitches with an interval of 3 between them, and is closed under composing and inverting intervals. Then S contains infinitely many pitches between 1 and 2.

(7) If we include all powers of 3 up to 3^{11}, the resulting scale looks fairly evenly spaced out; but including 3^{12} suddenly makes two pitches extremely close to each other. Can you explain this mathematically?

Hint: Count the *number of octaves* in each interval. For instance, an interval of 2 means "up one octave" and an interval of 8 means "up three octaves." How many octaves are there in an interval of 3?

(8) By our observations from yesterday, a good way to create scales with many "mild" sounds is to use ratios of small whole numbers. Here's one way to do this systematically. Fix a set T of small positive integers, and build a scale using all intervals with numerators and denominators in T. You can exclude powers of 2 from T, because they will be included automatically by octave equivalence. For example:

- $T = \{1, 3\}$: only use the intervals $1, \frac{1}{3}, 3$. (By octave equivalence, this produces the intervals $\frac{4}{3}$ and $\frac{3}{2}$ between 1 and 2.)
- $T = \{1, 3, 5\}$: only use the intervals $1, \frac{1}{5}, \frac{1}{3}, \frac{3}{5}, \frac{5}{3}, 3, 5$.
- $T = \{1, 3, 5, 9\}$: what intervals should you use?
- $T = \{1, 3, 5, 9, 15\}$: what intervals should you use?

In each case, consider the resulting scale. What notes do you get? How spread out are they? Are there any gaps or clumps? Are there any combinations of *three* or more notes that sound nice together?

(Also: what's the advantage of including 9 and 15, but not 7, 11, or 13?)

(9) If we want to divide up an octave into n equal intervals, we will need to use the intervals $2^{1/n}, 2^{2/n}, \ldots, 2^{(n-1)/n}$. Can you find numbers of this form that are very close to ratios of small whole numbers?

More precisely: the average human ear can't distinguish between two pitches at an interval of $r < 1.003$ when the notes are played separately.[4] For example, $\frac{7/6}{2^{2/9}} \approx 1.0001$, and so most people can't tell apart the intervals $\frac{7}{6}$ and $2^{2/9}$. Can you find other pairs of intervals like this?

[4] This number, called the "just-noticeable difference" (JND), is different from person to person, but also depends on the pitch; for instance, above 1000 Hz the average JND is closer to 1.006.