July 23, 2020 3:10 PM

The Riemann Zeta Function Predicts Pianos

$\zeta\left(\frac{1}{2}+\frac{24\pi i}{\ln 2}\right) =$ 

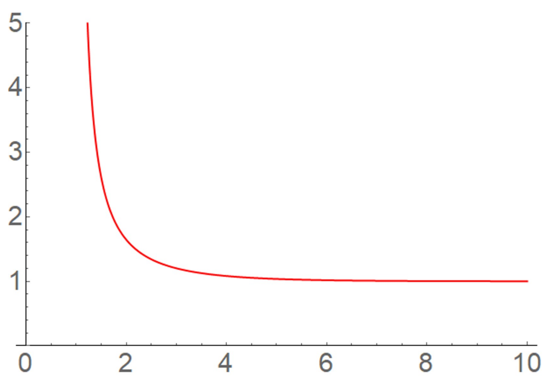
(and why you will never play in tune)

Music, Fractions, Logarithms, Riemann Zeta, ...

Riemann Zeta Function

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} \cdots$$

$\zeta(1) = \infty$ (harmonic series)



Values of $\zeta(s)$ tell stories.

$\zeta(1) = \infty \Rightarrow$ Infinite primes!

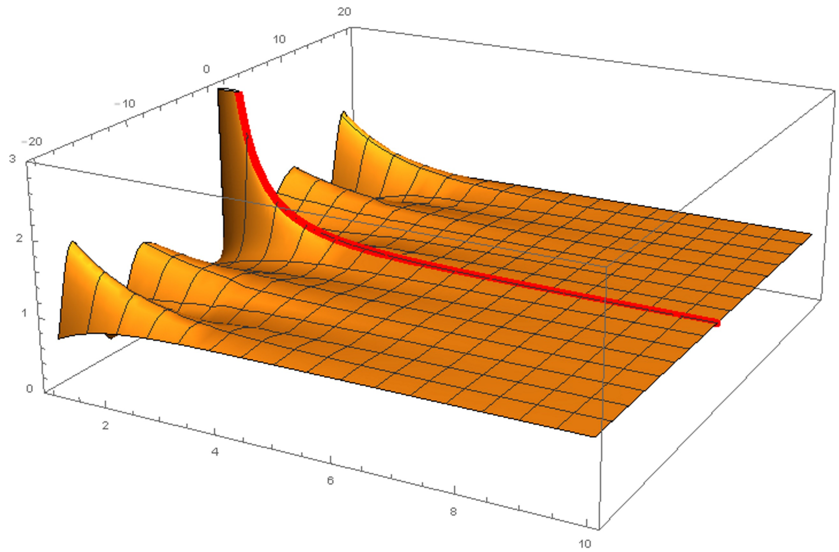
$\zeta(n) \Rightarrow \frac{1}{\zeta(n)} =$ probability that a random integer is n^{th}-power-free.

Eric's class on Zeta values

... about $\zeta(a+bi)$?

Eric's class on

But what about $\zeta(a+bi)$?



$\mathrm{Re}(\zeta(s))$,

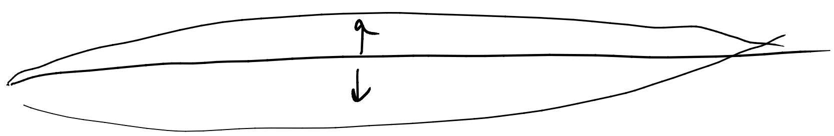
$s = a+bi$

Riemann hypothesis:

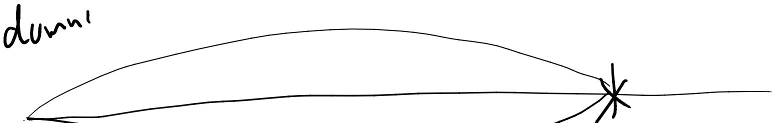
$\zeta(\frac{1}{2}+it)$.

what does this tell us?

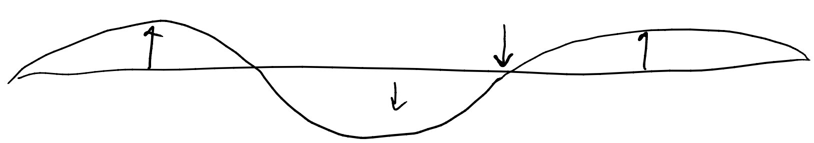
<u>Pitches</u>



hold down:



Touch lightly:



only works if the point I touch is a <u>rational number</u>.

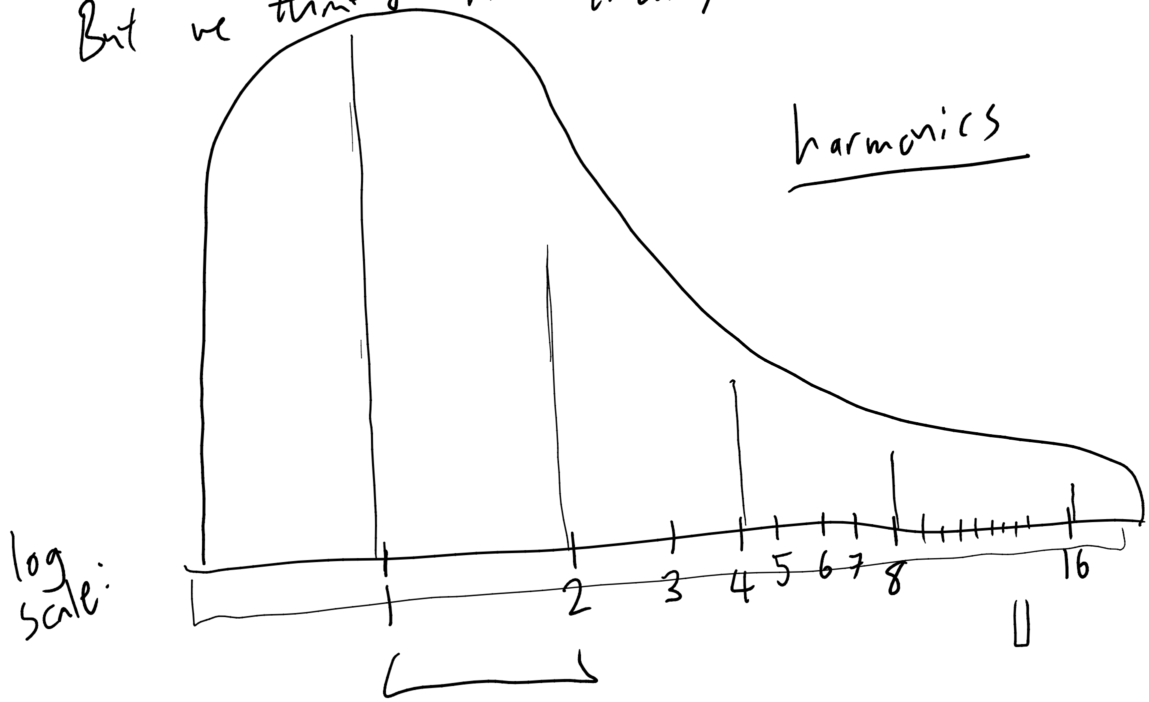
wavelengths: $\frac{1}{n}$. "<u>harmonics</u>" "overtone"

frequency: n

Pitch is multiplicative

But we think of it linearly.

harmonics



A scale is a set of frequencies.

A scale has harmony if it contains $\mathbb{N}$.
$\{1, 2, 3, 4, 5, \ldots\}$

A scale has evenness if it can be produced by a single interval.

Base interval b, and all frequencies are powers of b.

In particular, a good scale should contain 2 (octave) and 3 (octave + fifth).

$$b^n = 2 \qquad b^m = 3$$

$$\underbrace{b = 2}_{b^{nm} = 2^m} \qquad \underbrace{b \qquad}_{b^{nm} = 3^n}.$$

$\Longrightarrow$ find integer solution to

$$\underbrace{2^m}_{\text{even}} = \underbrace{3^n}_{\text{odd.}} \qquad \checkmark$$

Perfect Scales are impossible!

Approximations?

$$2^m \approx 3^n \ ?$$

$$\Leftrightarrow \quad m \approx n \log_2 3$$

$$\Leftrightarrow \quad \frac{m}{n} \approx \log_2 3$$: irrational. BUT it has rational approximations!

$\log_2 3 \approx 1.58\ldots$

$$\underbrace{\frac{8}{5}}, \quad \frac{11}{7}, \quad \frac{19}{12}, \ \ldots$$

Work back....

$b^n \approx 2$, $b^m \approx 3$ has solutions.

$\underbrace{b^5 \approx 2}$, $b^8 \approx 3 \Longrightarrow$ pentatonic.

$b^7 \approx 2$, $b^{11} \approx 3 \Longrightarrow$ diatonic

$b^{12} \approx 2$, $b^{19} \approx 3 \Longrightarrow$ chromatic

---

$\log_2 3$ contains history of Western music

... approximations

$\log_2 3$ contains history of Western music in its rational approximations

$$b^n \approx 2, \quad b^m \approx 3, \quad b^{2n} \approx 4, \quad b^r \approx 5$$

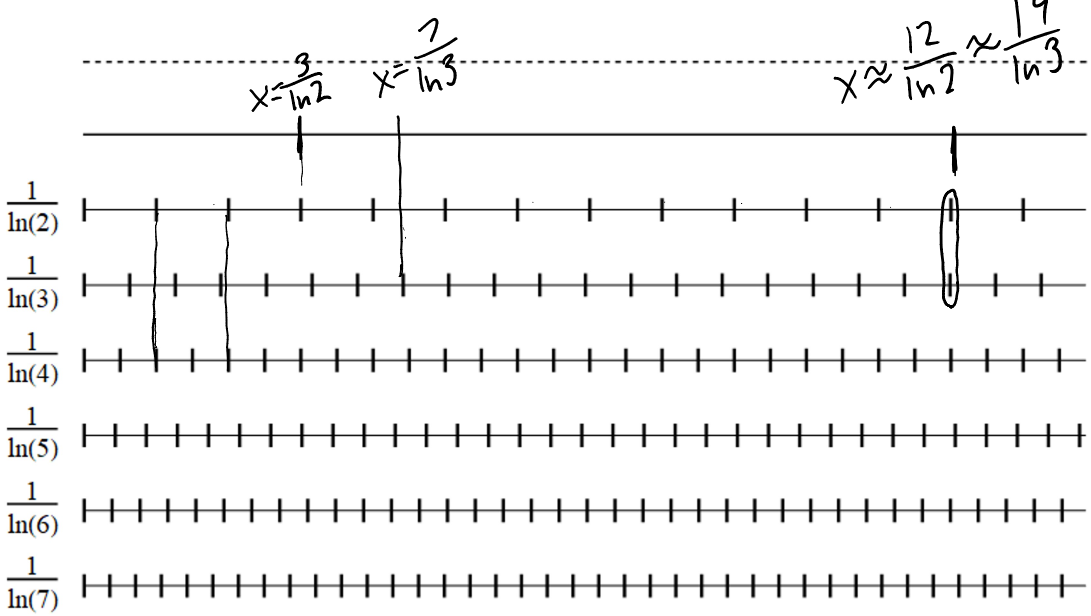
Rational approximation doesn't help if we want to compare 3 things

$$b^{n_2} \approx 2, \quad b^{n_3} \approx 3, \quad b^{n_4} \approx 4, \dots, b^{n_k} \approx k \quad ?$$

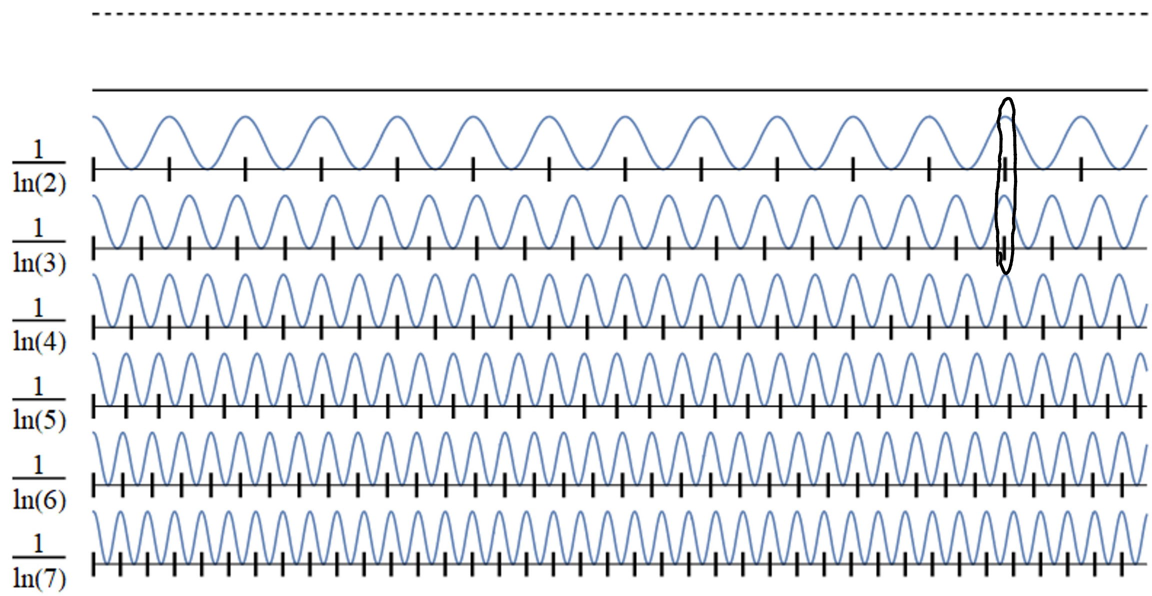
$$n_2 \ln b \approx \ln 2, \quad n_3 \ln b \approx \ln 3, \dots, \quad n_k \ln b \approx \ln k.$$

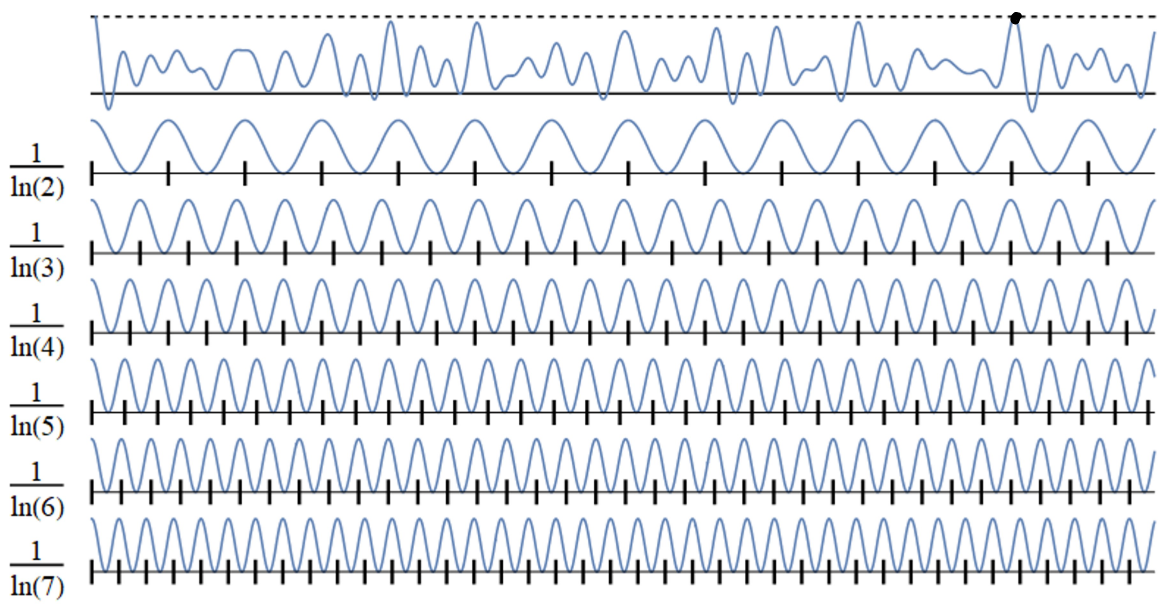
$$n_2\left(\frac{1}{\ln 2}\right) \approx \frac{1}{\ln b}, \dots \quad n_k\left(\frac{1}{\ln k}\right) \approx \frac{1}{\ln b}$$

If we set $x = \frac{1}{\ln b}$ ... means we are looking for x which is an integer multiple of $\frac{1}{\ln 2}, \frac{1}{\ln 3}, \dots, \frac{1}{\ln k}$.



Can we find a point next to as many tick marks as possible?





← adding all the functions below

$\cos(2\pi x \ln 2)$

$\cos(2\pi x \ln 3)$

$\cos(2\pi x \ln 4)$

$\vdots$

Summarize:

- want x close to integer multiples of $\frac{1}{\ln k}$
- choose a function which is largest at these multiples
- Add them together.
- Sum maximized $\Rightarrow$ many of the component functions are almost maximized.

$\cos(2\pi x)$ maximized at integers.

Note: $\cos(2\pi x)$ maximized at integers.

$\cos(2\pi\alpha \ln k)$ maximized at integer multiples of $\frac{1}{\ln k}$.

So let's add them up!

$$\sum_{k=1}^{\infty} \cos(2\pi x \ln k).$$

Problems!

- This does not converge.
- This gives every integer equal importance.

Fix: introduce scaling factor.

$$P(x) = \sum_{k=1}^{\infty} \frac{1}{k^\alpha} \cos(2\pi x \ln k)$$

$\boxed{\alpha \text{ a fixed constant}}$

what is this function?

$P(x)$ is maximized at values of $x \left(= \frac{1}{\ln b}\right)$ which give scales approximating many harmonics.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$P(x) = \sum_{k=1}^{\infty} \frac{1}{k^\alpha} \cos(-2\pi x \ln k)$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^\alpha} \mathrm{Re}\left[e^{i(-2\pi x \ln k)}\right]$$

$$= \mathrm{Re}\left[\sum_{k=1}^{\infty} \frac{1}{k^\alpha} e^{-2\pi i x \ln k}\right]$$

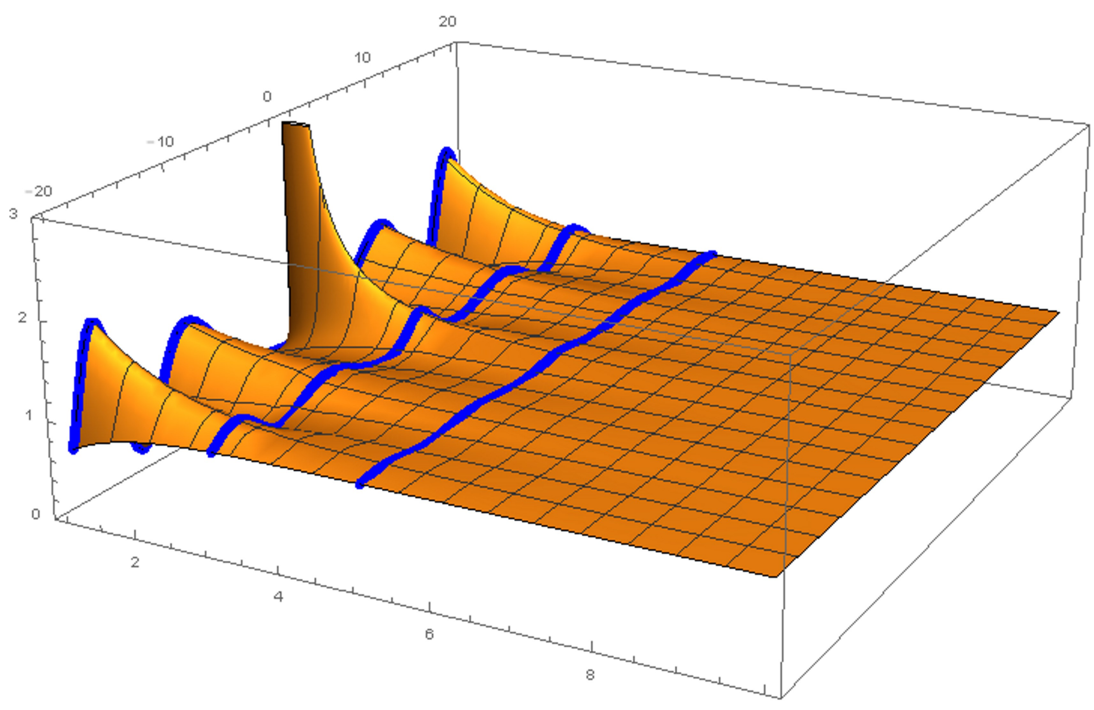
$$= \operatorname{Re}\left[\sum_{k=1}^{\infty} \frac{1}{k^{\alpha}}\left(e^{\ln k}\right)^{-2\pi i x}\right]$$

$$= \operatorname{Re}\left[\sum_{k=1}^{\infty} \frac{1}{k^{\alpha}} \cdot k^{-2\pi i x}\right]$$

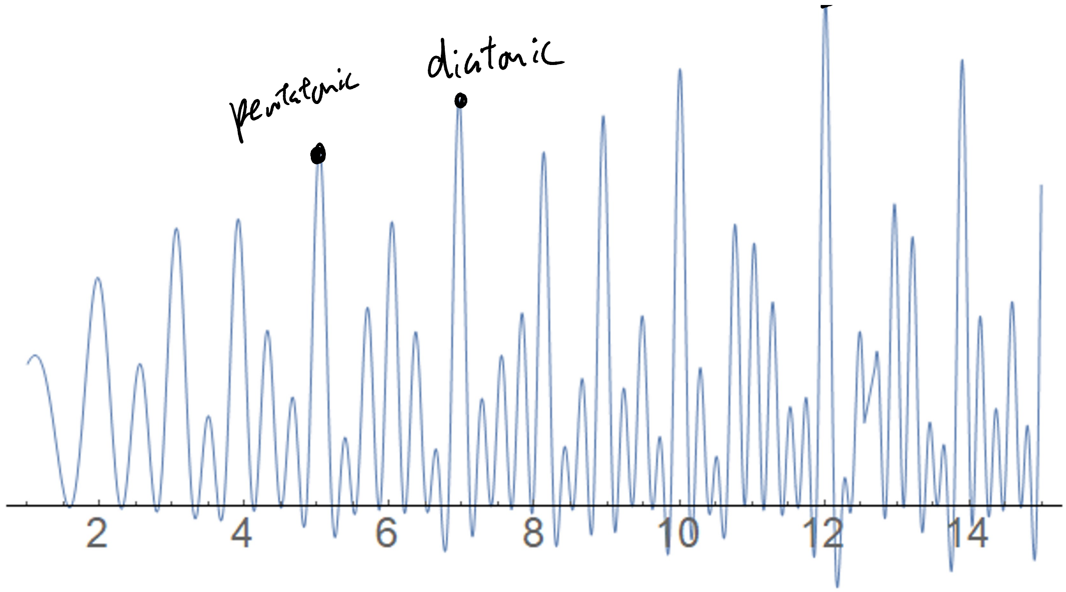
$$= \operatorname{Re}\left[\sum_{k=1}^{\infty} \frac{1}{k^{\alpha+2\pi i x}}\right]$$

$$= \operatorname{Re}\left[\zeta(\alpha + 2\pi i x)\right]$$

When $\zeta(\alpha+2\pi i x)$ has large real part, then this gives us a piano (a good scale: evenness, harmonics)



$\operatorname{Re}\left(\zeta\left(\alpha + \frac{2\pi i x}{\ln 2}\right)\right)$



Why you can never play in tune?

$b^n = 2, \quad b^m = 3.$ octave / fifth must be out of tune.

(In a piano, octave + fifth is actually $2^{19/12} \approx \cancel{2.98\ldots}$ 2.997 (corrected after talk))

Why Riemann Zeta predicts pianos:

$\zeta\left(\frac{1}{2} + \frac{24 i \pi}{\ln 2}\right)$ is large. Hence, pianos. $\square$
