Accurate Biot-Savart Routines with Correct Asymptotic Behavior

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Abstract

A set of routines to compute the magnetic vector potential and magnetic field of two types of current carriers is presented. The (infinitely thin) current carrier types are a straight wire segment and a circular wire loop. The routines are highly accurate and exhibit the correct asymptotic behavior far away from and close to the current carrier. A suitable global set of test points is introduced and the methods presented in this work are tested against results obtained using arbitrary-precision arithmetic on all test points. The results are accurate to approximately 16 decimal digits of precision when computed using 64 bit floating point arithmetic, with few exceptions where accuracy drops to 13 digits. These primitive current carrier types can be used to make up more complex arrangements, such as a current along a polygon (by means of defining straight wire segments from point to point along the polygon) and a multi-winding coil with circular cross-section. Reference data is provided along with the code for benchmarks with other implementations.

Outline

- Introduction
- Methods
- Results
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Introduction

- Overall goal: Compute accurately
  - $\mathbf{A}_z$, $\mathbf{B}_\varphi$ (straight wire segment)
  - $\mathbf{A}_\varphi$, $\mathbf{B}_\rho$, $\mathbf{B}_z$ (circular wire loop)

Introduction

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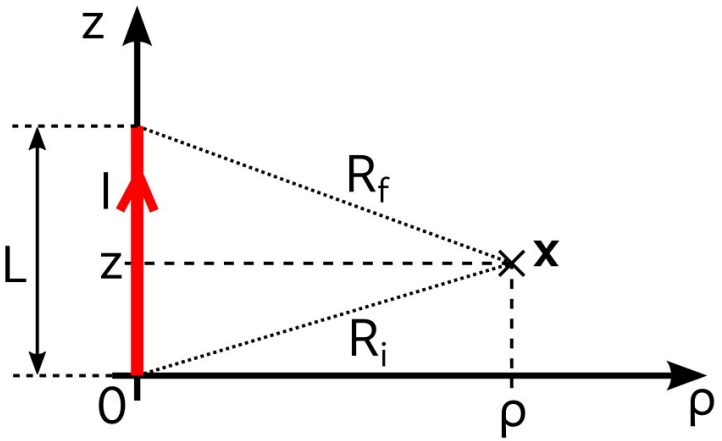

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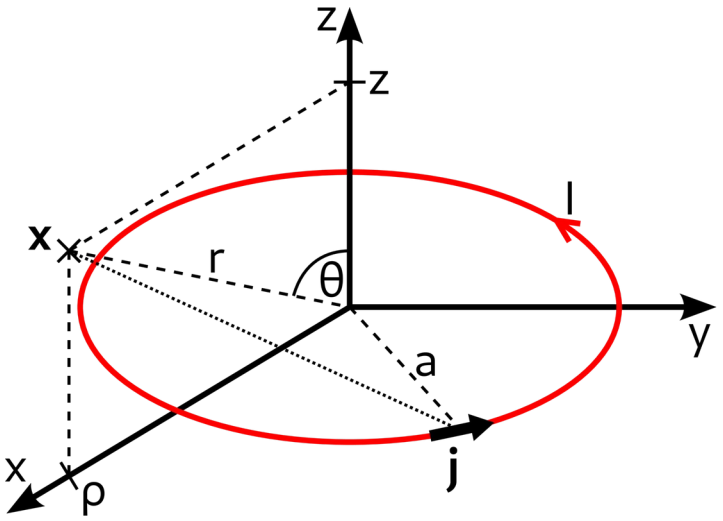


Introduction

- Overall goal: Compute accurately
  - $\mathbf{A}_z$, $\mathbf{B}_\varphi$ (straight wire segment)
  - $\mathbf{A}_\varphi$, $\mathbf{B}_\rho$, $\mathbf{B}_z$ (circular wire loop)
- Approach
  - perform evaluation in local cylindrical coordinate system of current carrier
  - decomposition of evaluation domain (ρ, z)
  - derive accurate special case formulations
  - dynamic switchover between formulations, depending on evaluation position
  - use of normalized coordinates and ratios only (no machine-precision-dependent constants)

Introduction

- But why all of this in the first place ... ?
  - consistency in code comparisons
  - finite differencing
  - field-line tracing
  - free-boundary VMEC during iterations

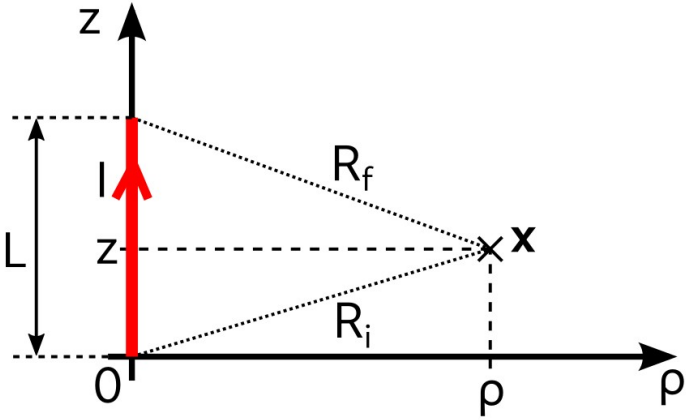
Straight Wire Segment

- Magnetic Vector Potential: $\mathbf{A} = A_z \hat{\mathbf{e}}_z$

$$A_z(\rho, z) = \frac{\mu_0 I}{4\pi} \ln\left(\frac{1+\epsilon}{1-\epsilon}\right) \quad \text{with} \quad \epsilon = \frac{L}{R_i + R_f} \quad \text{and} \quad \begin{aligned} R_i &= \sqrt{\rho^2 + z^2} \\ R_f &= \sqrt{\rho^2 + (L - z)^2} \end{aligned}$$

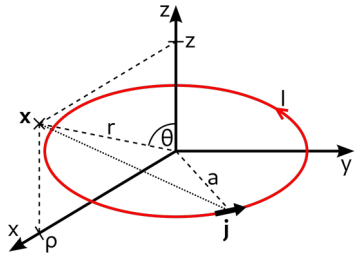
- Magnetic Field: $\mathbf{B} = B_\varphi \hat{\mathbf{e}}_\varphi$

$$B_\varphi(\rho, z) = \frac{\mu_0 I}{4\pi} \frac{\rho}{R_i} \frac{2L(R_i + R_f)}{R_f} \frac{1}{(R_i + R_f)^2 - L^2}$$



Hanson & Hirshman, Phys. Plasmas **9**, 4410 (2002)

Circular Wire Loop



- General Complete Elliptic Integral

$$\mathrm{cel}(k_c, p, a, b) = \int_0^{\pi/2} \frac{a\cos^2(\varphi) + b\sin^2(\varphi)}{\cos^2(\varphi) + p\sin^2(\varphi)} \frac{\mathrm{d}\varphi}{\sqrt{\cos^2(\varphi) + k_c^2\sin^2(\varphi)}}$$

- Magnetic Vector Potential: $\mathbf{A}(\mathbf{x}) = A_\varphi(\rho, z)\,\hat{\mathbf{e}}_\varphi$

$$A_\varphi(\rho', z') = \frac{\mu_0 I}{\pi} \frac{1}{\sqrt{z'^2 + (1+\rho')^2}} \mathrm{cel}(k_c, 1, -1, 1)$$

$$\rho' = \rho/a \qquad z' = z/a$$

- Magnetic Field: $\mathbf{B}(\mathbf{x}) = B_\rho \hat{\mathbf{e}}_\rho + B_z \hat{\mathbf{e}}_z$

$$k_c^2 = \frac{z'^2 + (1-\rho')^2}{z'^2 + (1+\rho')^2}$$

$$B_\rho(\rho', z') = \frac{\mu_0 I}{\pi a} \frac{z'}{\left[z'^2 + (1+\rho')^2\right]^{\frac{3}{2}}} \mathrm{cel}(k_c, k_c^2, -1, 1)$$

$$B_z(\rho', z') = \frac{\mu_0 I}{2\pi a} \frac{1}{\rho'\sqrt{z'^2 + (1+\rho')^2}} \left[\mathrm{cel}(k_c, 1, -1, 1) + \frac{1 + k_c^2 - (1-k_c^2)\,\rho'}{2} \mathrm{cel}(k_c, k_c^2, -1, 1)\right]$$

R. Bulirsch, Num. Math. **13**, 305-315 (1969)

Belcher, Olbert & Derby, *TEAL Physics and Mathematics Documentation*, MIT (2008)

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- Separate geometric parts of formulas
- Testing method
- Computation of reference data
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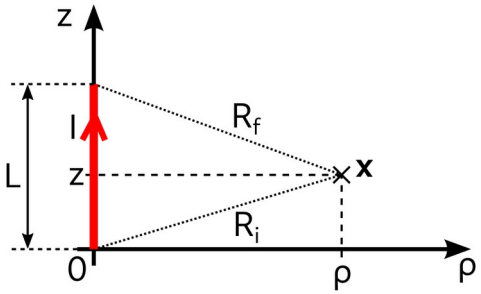
Methods

- Separate geometric parts of formulas
  - Straight Wire Segment
    - Magnetic Vector Potential



$$A_z(\rho, z) = \frac{\mu_0 I}{2\pi} \tilde{A}_z(\rho', z') \quad \text{with} \quad \tilde{A}_z(\rho', z') = \frac{1}{2} \ln\left(\frac{1+\epsilon}{1-\epsilon}\right) = \operatorname{atanh}(\epsilon)$$

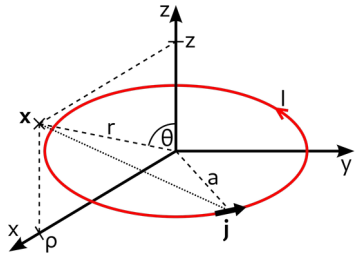
    - Magnetic Field

$$B_\varphi(\rho, z) = \frac{\mu_0 I}{4\pi L} \tilde{B}_\varphi(\rho', z') \quad \text{with} \quad \tilde{B}_\varphi(\rho', z') = \left(\frac{1}{r_f} + \frac{1}{r_i}\right) \frac{2\rho'}{(r_i + r_f)^2 - 1}$$

"physics"

"geometry"

Methods



- Separate geometric parts of formulas
  - Circular Wire Loop
    - Magnetic Vector Potential

$$A_\varphi(\rho, z) = \frac{\mu_0 I}{\pi} \tilde{A}_\varphi(\rho', z') \quad \text{with} \quad \tilde{A}_\varphi(\rho', z') = \frac{1}{\sqrt{z'^2 + (1+\rho')^2}} \operatorname{cel}(k_c, 1, -1, 1)$$

    - Magnetic Field (similarly for B_z)

$$B_\rho(\rho, z) = \frac{\mu_0 I}{\pi a} \tilde{B}_\rho(\rho', z') \quad \text{with} \quad \tilde{B}_\rho(\rho', z') = \frac{z'}{[z'^2 + (1+\rho')^2]^{\frac{3}{2}}} \operatorname{cel}(k_c, k_c^2, -1, 1)$$

"physics"

"geometry"

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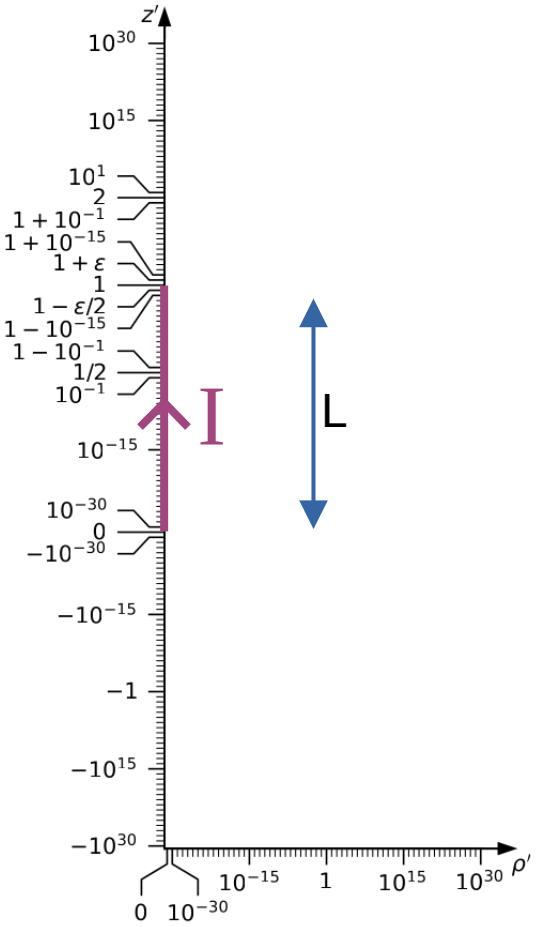
Testing method

- Select appropriate set of test points
- Compute reference data using arbitrary-precision software (`mpmath` Python package, Mathematica)
- "number of correct digits" (*a* to be tested, *b* is reference)

$$\delta(a,b) = \begin{cases} \log_{10}\left(\min\left(1, \left|\frac{a-b}{b}\right|\right)\right) & b \neq 0, a \neq b \\ 0 & b = 0, a \neq 0 \\ -16 & \text{else} \end{cases}$$

chosen for 64-bit IEE-754 `binary64`;
adjust for other types

Test Points: Straight Wire Segment



vertical grid points: z' = z/L

- -10^{30}, -10^{29}, ..., -10^{2},
  -10, -1, -0.1,
  -10^{-2}, ..., -10^{-29}, -10^{-30}
- 0
- 10^{-30}, 10^{-29}, ..., 10^{-2},
  0.1,
- 0.5,
- 0.9,
  $1-10^{-2}$, $1-10^{-3}$, ..., $1-10^{-15}$, $1-\varepsilon/2$,
- 1,
- $1+\varepsilon$, $1+10^{-15}$, $1+10^{-14}$, ..., $1+10^{-2}$,
  1.1,
- 2,
- 10,
  10^{2}, 10^{3}, ..., 10^{30}

radial grid points: ρ' = ρ/L

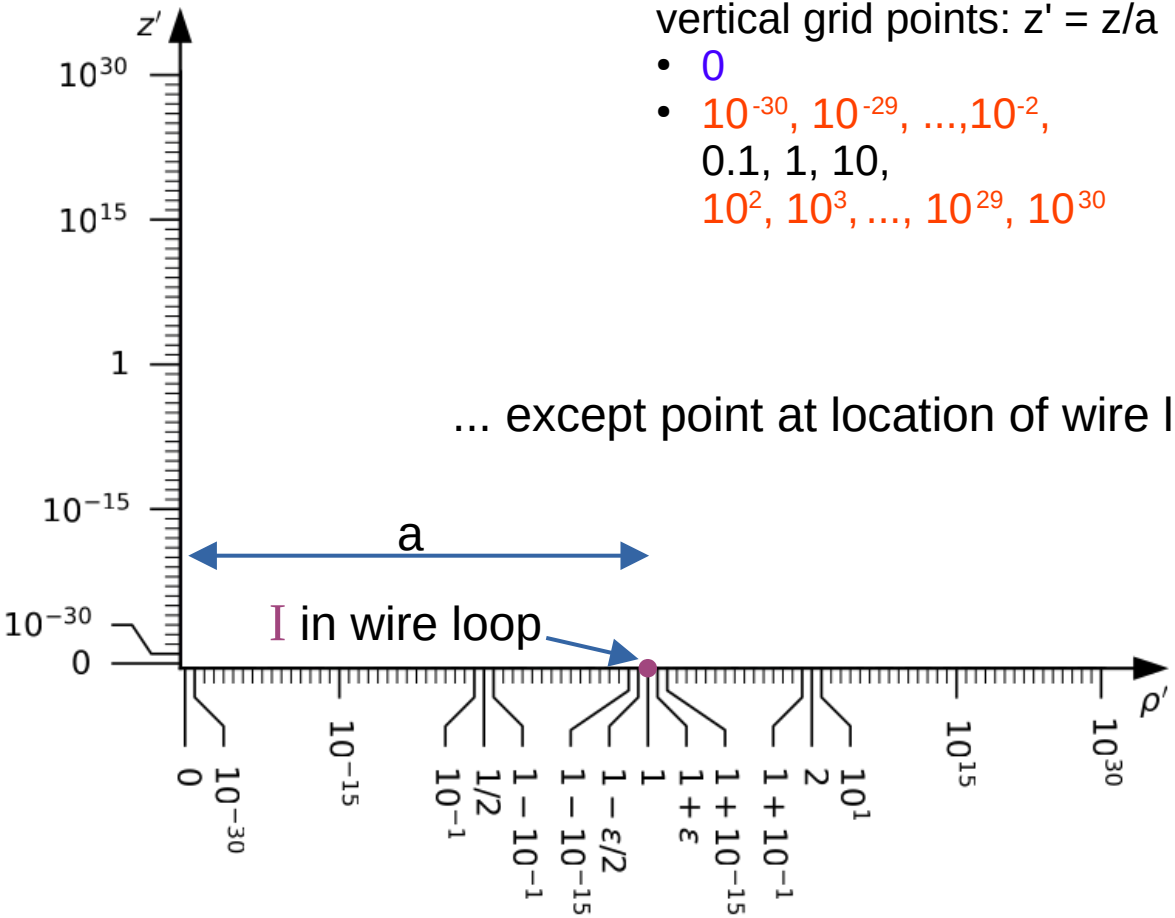
- 0
- 10^{-30}, 10^{-29}, ...,10^{-2},
  0.1, 1, 10,
  10^{2}, 10^{3}, ..., 10^{29}, 10^{30}

... except points on wire segment

here: ε = machine precision
(2.2 x 10^{-16} for binary64)

regular use cases
special case formulation
asymptotic behavior

Test Points: Circular Wire Loop



... except point at location of wire loop

vertical grid points: z' = z/a

- 0
- 10^{-30}, 10^{-29}, ...,10^{-2},
  0.1, 1, 10,
  10^{2}, 10^{3}, ..., 10^{29}, 10^{30}

radial grid points: ρ' = ρ/a

- 0
- 10^{-30}, 10^{-29}, ..., 10^{-2},
  0.1,
- 0.5,
- 0.9,
  $1-10^{-2}$, $1-10^{-3}$, ..., $1-10^{-15}$, $1-\varepsilon/2$,
- 1,
- $1+\varepsilon$, $1+10^{-15}$, $1+10^{-14}$, ..., $1+10^{-2}$,
  1.1,
- 2,
- 10,
  10^{2}, 10^{3}, ..., 10^{30}

here: ε = machine precision
(2.2 x 10^{-16} for binary64)

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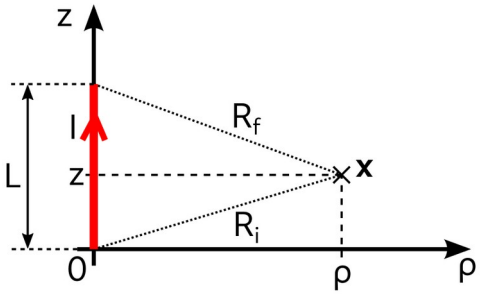
Computation of Reference Data

- Straight Wire Segment
  - direct evaluation to 300 digits in arbitrary-precision software:



$$r_i = \sqrt{\rho'^2 + z'^2}$$

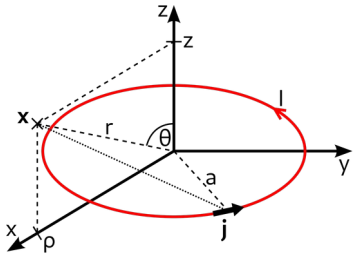
$$r_f = \sqrt{\rho'^2 + (1 - z')^2}$$

$$\epsilon = \frac{1}{r_i + r_f}$$

$$\tilde{A}_z(\rho', z') = \operatorname{atanh}(\epsilon)$$

$$\tilde{B}_\varphi(\rho', z') = \left(\frac{1}{r_f} + \frac{1}{r_i}\right) \frac{2\rho'}{(r_i + r_f)^2 - 1}$$

Computation of Reference Data



- Circular Wire Loop (1)
  - numerical quadrature to 200 digits in arbitrary-precision software:

$$k_c^2 \leftarrow \frac{z'^2 + (1-\rho')^2}{z'^2 + (1+\rho')^2}$$

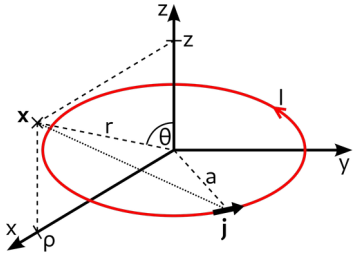
if $\rho' = 0 : \tilde{A}_\varphi = 0$; else:

$$\tilde{A}_\varphi \leftarrow \frac{1}{\sqrt{z'^2 + (1+\rho')^2}} \int_0^{\pi/2} \frac{\sin^2(\varphi) - \cos^2(\varphi)}{\sqrt{\cos^2(\varphi) + k_c^2 \sin^2(\varphi)}} \, \mathrm{d}\varphi$$

if $\rho' = 0$ or $z' = 0 : \tilde{B}_\rho = 0$; else:

$$\tilde{B}_\rho \leftarrow \frac{z'}{\left[z'^2 + (1+\rho')^2\right]^{3/2}} \int_0^{\pi/2} \frac{\sin^2(\varphi) - \cos^2(\varphi)}{\left[\cos^2(\varphi) + k_c^2 \sin^2(\varphi)\right]^{3/2}} \, \mathrm{d}\varphi$$

Computation of Reference Data



- Circular Wire Loop (2)
  - numerical quadrature to 200 digits in arbitrary-precision software:

$$k^2 \leftarrow \begin{cases} 4/\left(\frac{1}{\rho'} + 2 + \rho'\right) & z' = 0 \\ \frac{4\rho'}{z'^2 + (1+\rho')^2} & \text{else} \end{cases}$$

$$\tilde{B}_z \leftarrow \frac{1}{\left[z'^2 + (1+\rho')^2\right]^{3/2}} \int_0^{\pi/2} \frac{(1-\rho')\sin^2(\varphi) + (1+\rho')\cos^2(\varphi)}{\left[1 - k^2 \sin^2(\varphi)\right]^{3/2}} \, \mathrm{d}\varphi$$

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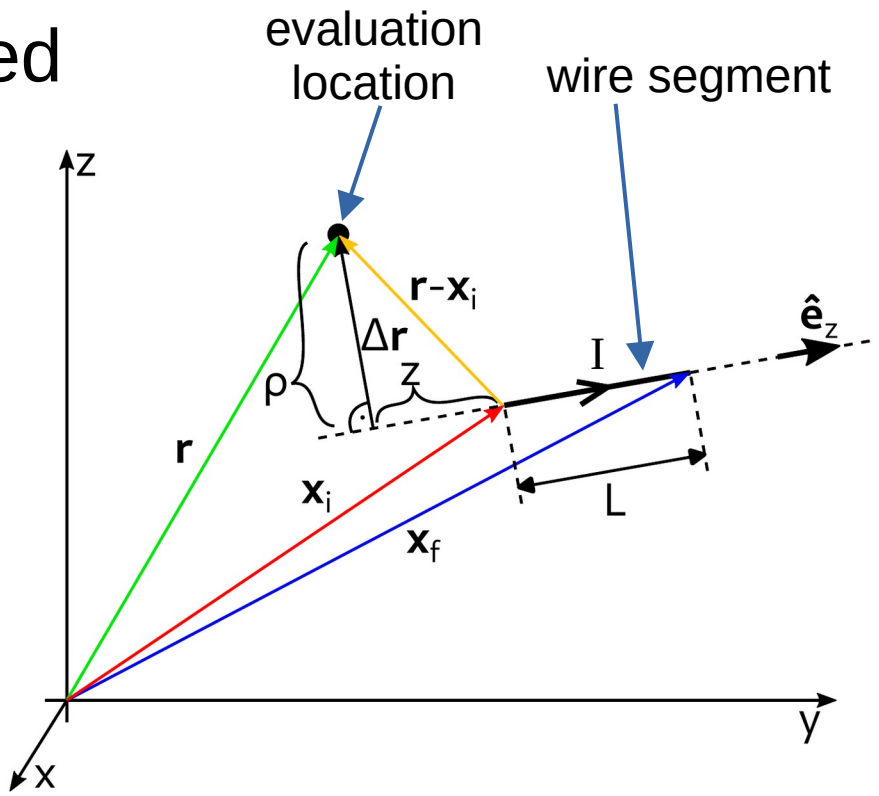
Cartesian Interface

- Straight Wire Segment specified via endpoints $\mathbf{x}_i$ and $\mathbf{x}_f$

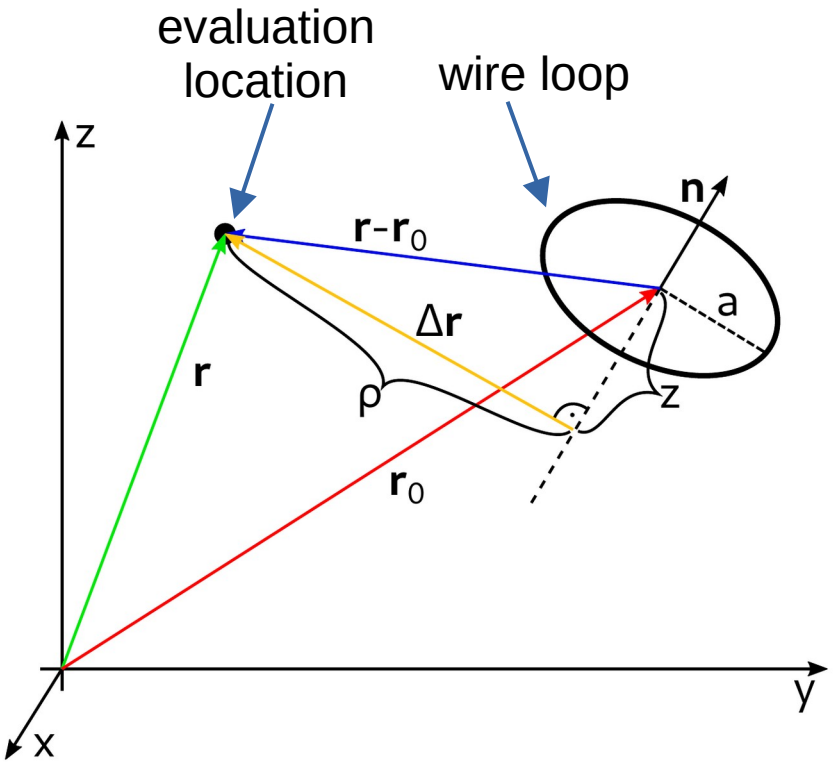
$$
\left.
\begin{aligned}
L &= |\mathbf{x}_f - \mathbf{x}_i| \\
\hat{\mathbf{e}}_z &= \frac{\mathbf{x}_f - \mathbf{x}_i}{L} \\
z &= (\mathbf{r} - \mathbf{x}_i) \cdot \hat{\mathbf{e}}_z \\
z' &= \frac{z}{L} = \frac{1}{L}(\mathbf{r} - \mathbf{x}_i) \cdot \hat{\mathbf{e}}_z \\
\Delta\mathbf{r} &= (\mathbf{r} - \mathbf{x}_i) - z\,\hat{\mathbf{e}}_z \\
\hat{\mathbf{e}}_\rho &= \frac{\Delta\mathbf{r}}{\rho} \\
\rho' &= \frac{\rho}{a} = \frac{1}{a}|(\mathbf{r} - \mathbf{x}_i) - z\,\hat{\mathbf{e}}_z| \\
\hat{\mathbf{e}}_\varphi &= \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_\rho
\end{aligned}
\right\}
\begin{aligned}
\mathbf{A}(\mathbf{r}) &= A_z \hat{\mathbf{e}}_z \\
\mathbf{B}(\mathbf{r}) &= B_\varphi \hat{\mathbf{e}}_\varphi
\end{aligned}
$$



Cartesian Interface

- Circular Wire Loop specified via origin $\mathbf{r}_0$, normal $\mathbf{n}$ and radius a

$$\left.\begin{aligned}
\hat{\mathbf{e}}_z &= \frac{\mathbf{n}}{|\mathbf{n}|} \\
z &= (\mathbf{r}-\mathbf{r}_0)\cdot\hat{\mathbf{e}}_z \\
z' &= \frac{z}{a} = \frac{1}{a}(\mathbf{r}-\mathbf{r}_0)\cdot\hat{\mathbf{e}}_z \\
\Delta\mathbf{r} &= (\mathbf{r}-\mathbf{r}_0) - z\,\hat{\mathbf{e}}_z \\
\rho &= |\Delta\mathbf{r}| \\
\hat{\mathbf{e}}_\rho &= \frac{\Delta\mathbf{r}}{\rho} \\
\rho' &= \frac{\rho}{a} = \frac{1}{a}|(\mathbf{r}-\mathbf{r}_0) - z\,\hat{\mathbf{e}}_z| \\
\hat{\mathbf{e}}_\varphi &= \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_\rho
\end{aligned}\right\}
\begin{aligned}
\mathbf{A}(\mathbf{r}) &= A_\varphi \hat{\mathbf{e}}_\varphi \\
\mathbf{B}(\mathbf{r}) &= B_\rho \hat{\mathbf{e}}_\rho + B_z \hat{\mathbf{e}}_z
\end{aligned}$$



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- Straight Wire Segment (SWS)
  - Magnetic Vector Potential: A_z
  - Magnetic Field: B_φ
- Circular Wire Loop (CWL)
  - Magnetic Vector Potential: A_φ
  - Magnetic Field: B_ρ
  - Magnetic Field: B_z
- Cartesian Interface

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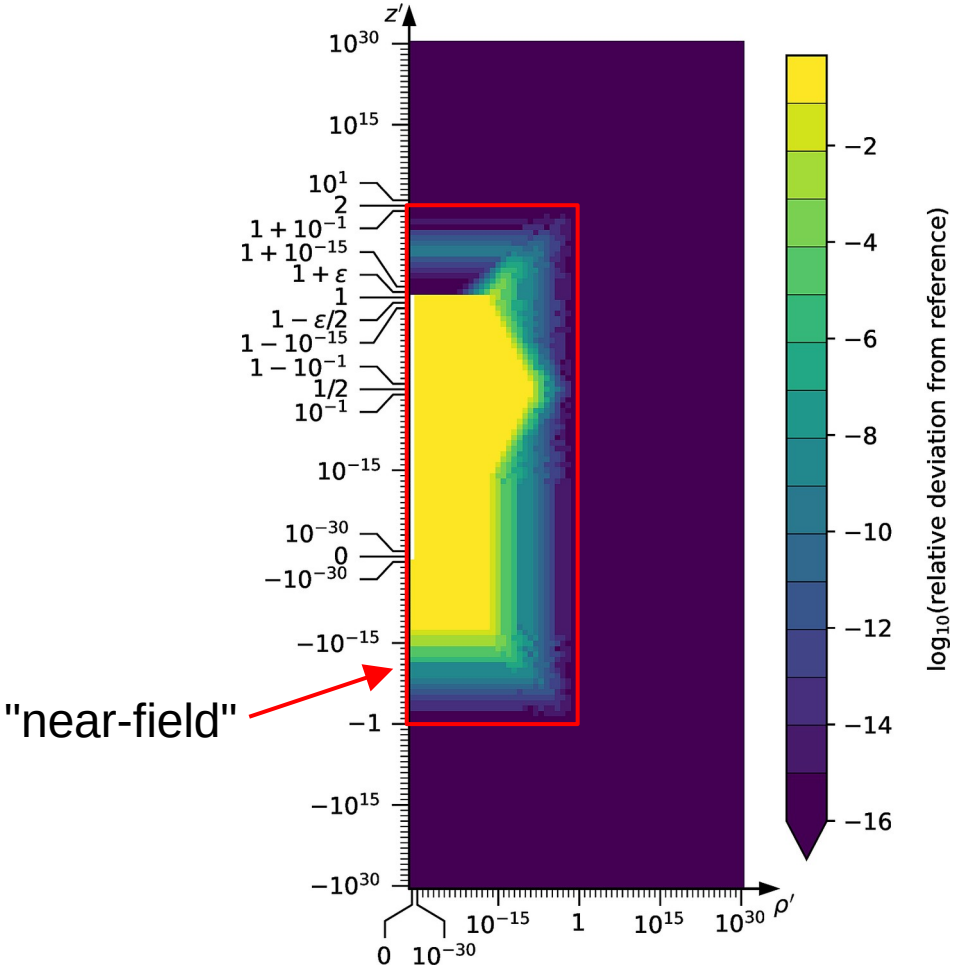
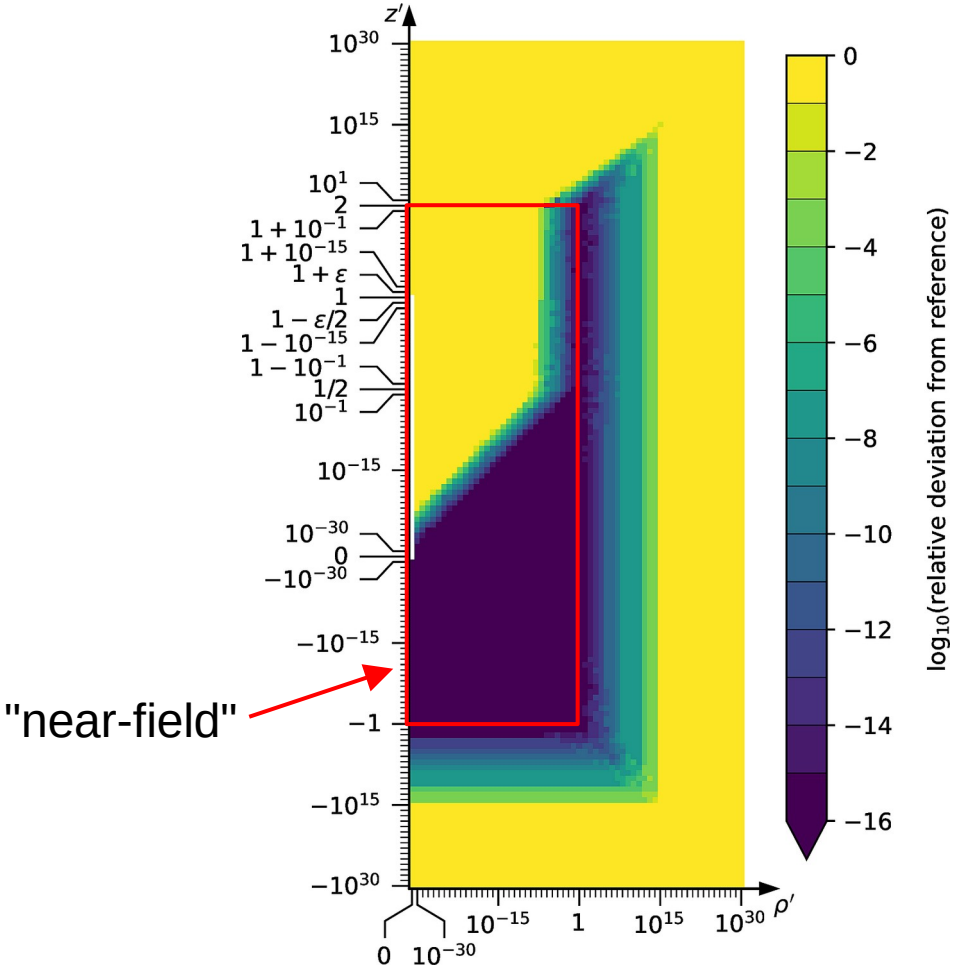
  - Magnetic Field: B_φ
- Circular Wire Loop (CWL)
  - Magnetic Vector Potential: A_φ
  - Magnetic Field: B_ρ
  - Magnetic Field: B_z
- Cartesian Interface

SWS, A_z, MAKEGRID



SWS, A_z, Mag3D



"near-field"

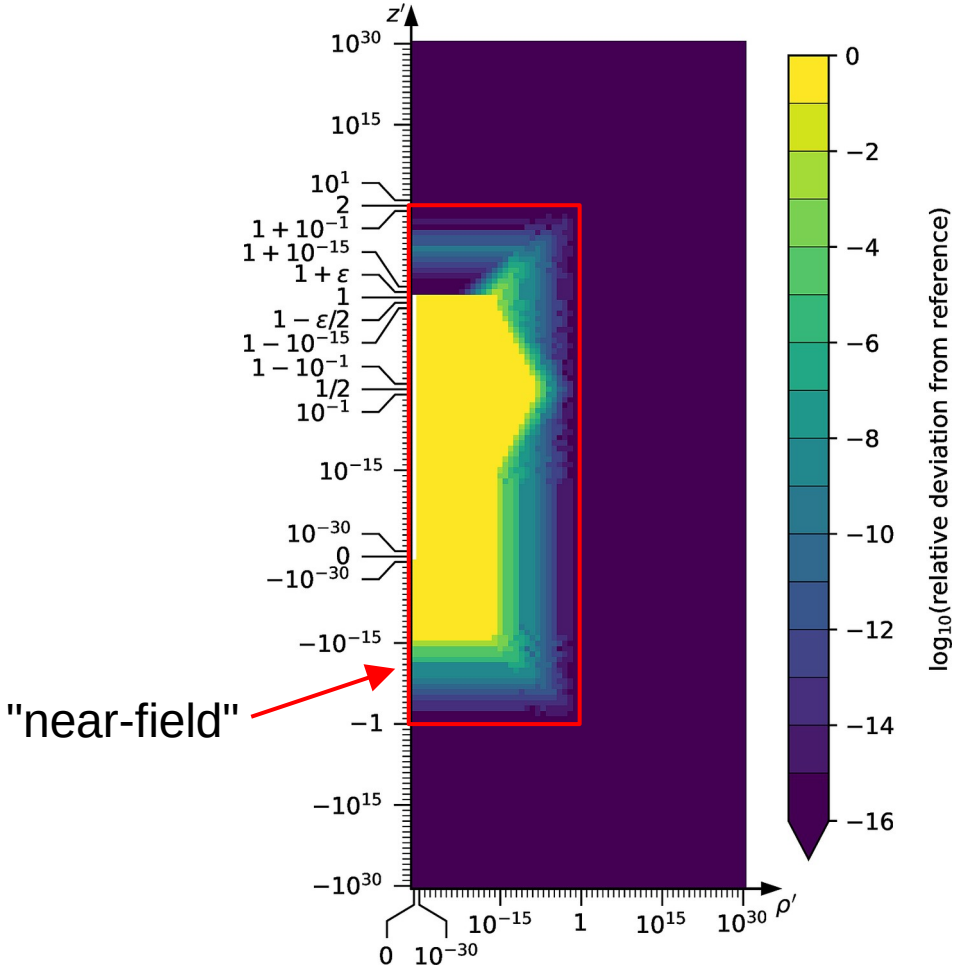
SWS, A_z, far-field method

$$\tilde{A}_{\mathrm{z,f}}(\rho', z') = \mathrm{atanh}(\epsilon)$$

with $$\epsilon = \frac{1}{r_\mathrm{i} + r_\mathrm{f}}$$

$$r_i = \sqrt{\rho'^2 + z'^2}$$

$$r_f = \sqrt{\rho'^2 + (1 - z')^2}$$



SWS, A_z, near-field method

Start from: $\tilde{A}_z(\rho', z') = \frac{1}{2}\ln\left(\frac{1+\epsilon}{1-\epsilon}\right)$

Note that:

$$1-\epsilon = 1 - \frac{1}{r_i + r_f} = \frac{r_i + r_f - 1}{r_i + r_f} = \frac{r_i + r_f - 1}{(r_i + r_f - 1) + 1} = \frac{n}{n+1}$$

This leads to:

$$\tilde{A}_{z,\mathrm{n}}(\rho', z') = \frac{1}{2}\left[\ln\left(2 - (1-\epsilon)\right) - \ln\left(1-\epsilon\right)\right]$$

$$= \frac{1}{2}\left[\ln\left(n+2\right) - \ln\left(n\right)\right]$$

with:

$$n = (r_\mathrm{i} - z') + (r_\mathrm{f} - (1 - z'))$$

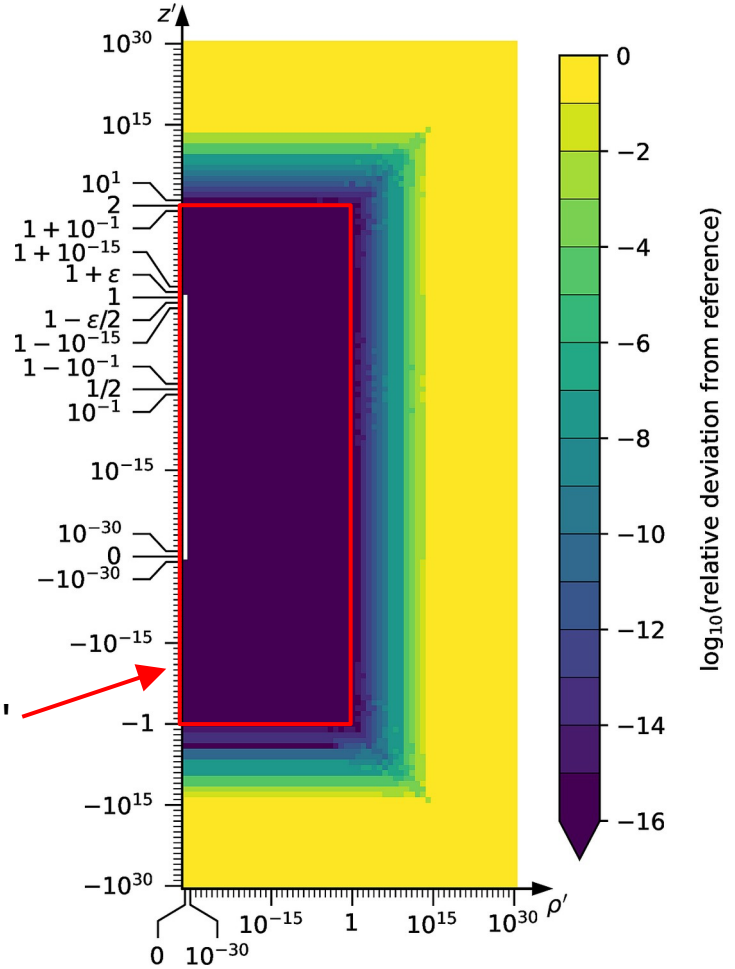
$$r_\mathrm{i} - z' = 2 r_\mathrm{i} \sin^2(\alpha/2)$$

$$r_\mathrm{f} - (1 - z') = 2 r_\mathrm{f} \sin^2(\beta/2)$$

$$\alpha = \mathtt{atan2}(\rho', z')$$

$$\beta = \mathtt{atan2}(\rho', 1 - z')$$

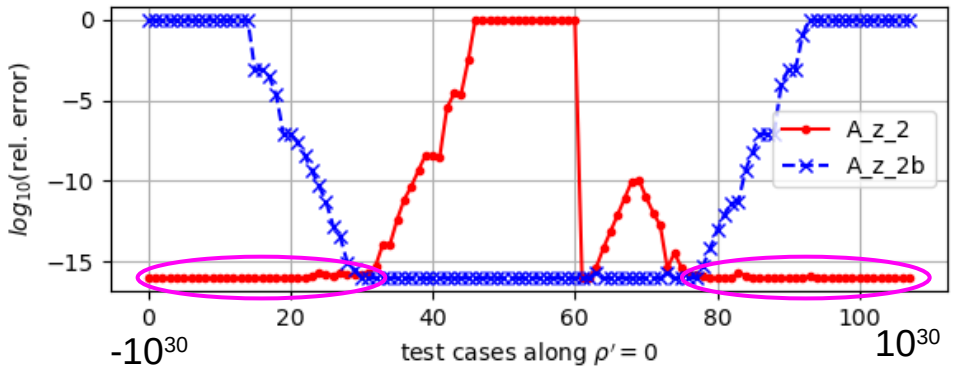
"near-field"

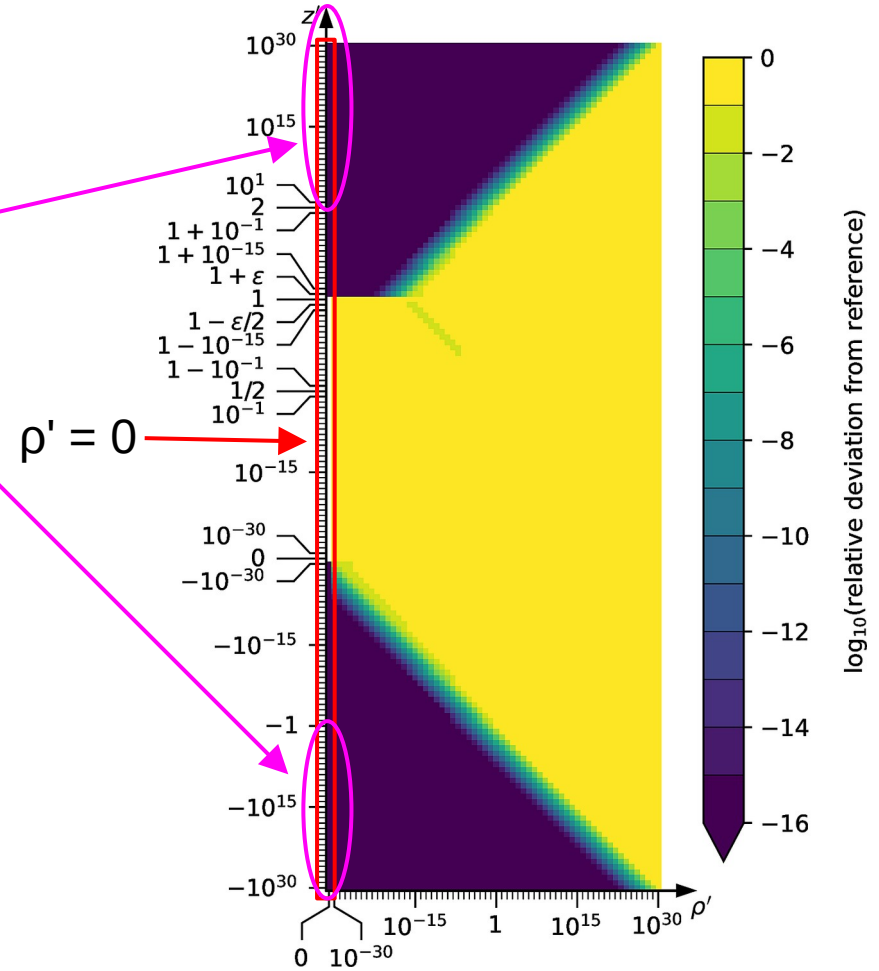


SWS, A_z, special case ρ' = 0

"far-field" along ρ' = 0:

$$\tilde{A}_{\mathrm{z,ax,f}}(z') = \operatorname{atanh}\left(\frac{1}{|z'| + |1 - z'|}\right)$$

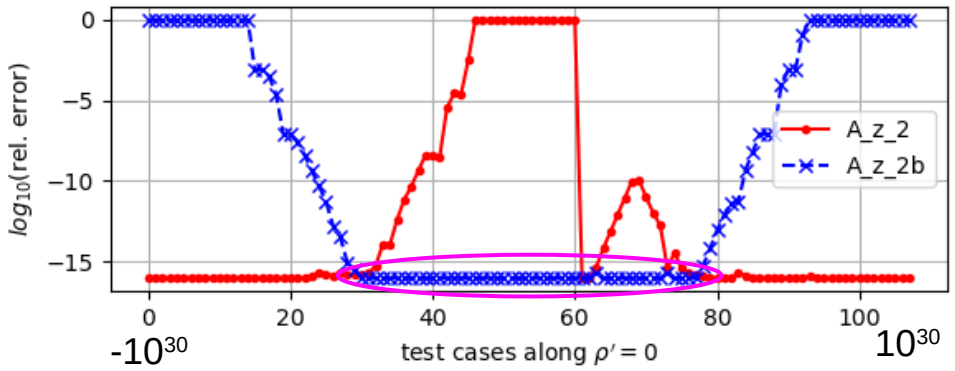


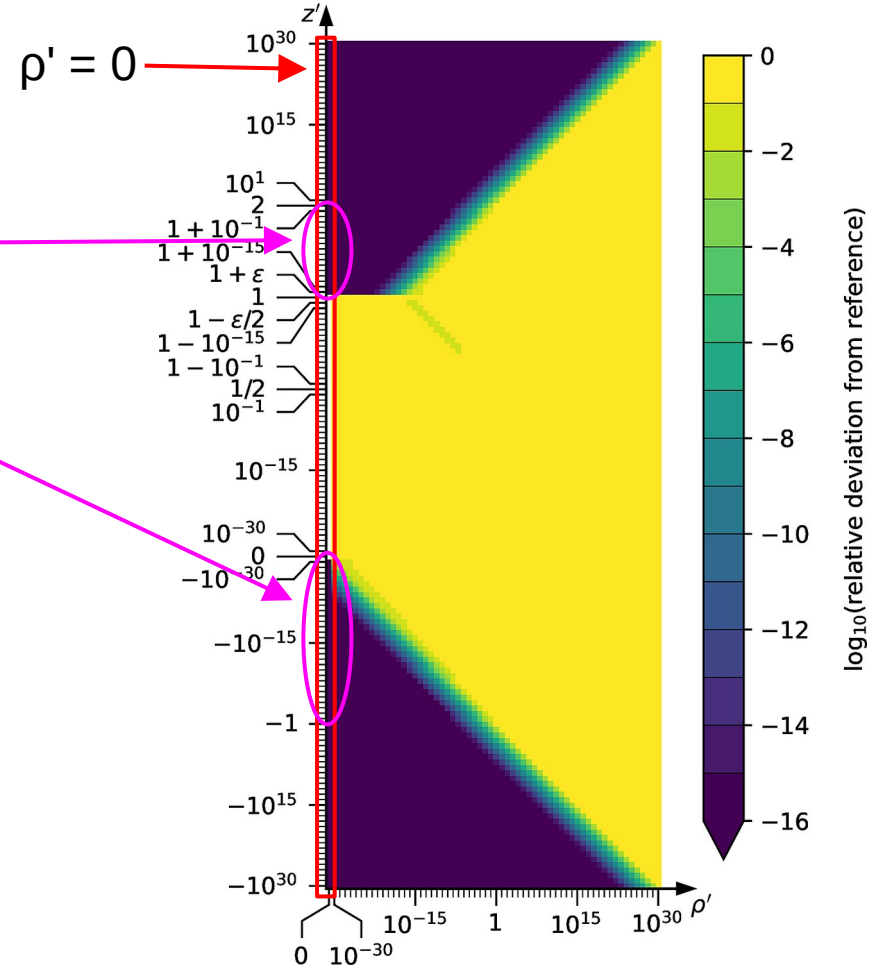


SWS, A_z, special case ρ' = 0

"near-field" along ρ' = 0:

$$\tilde{A}_{\mathrm{z,ax,n}}(z') = \frac{1}{2}\frac{z'}{|z'|}\ln\left(\left|\frac{z'}{1-z'}\right|\right)$$

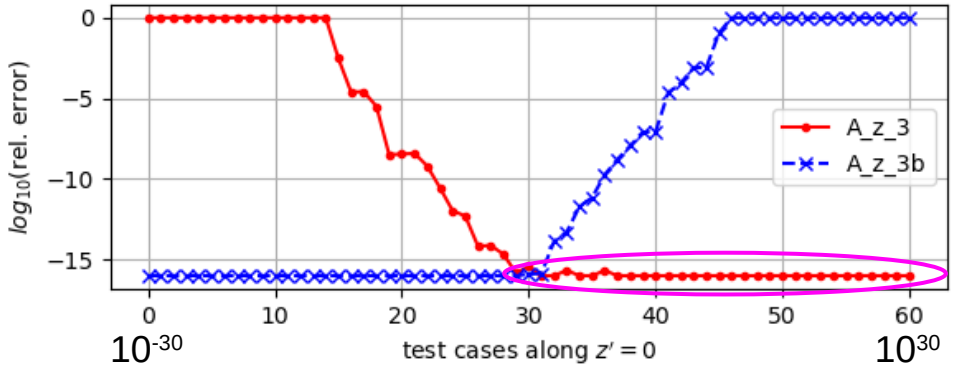


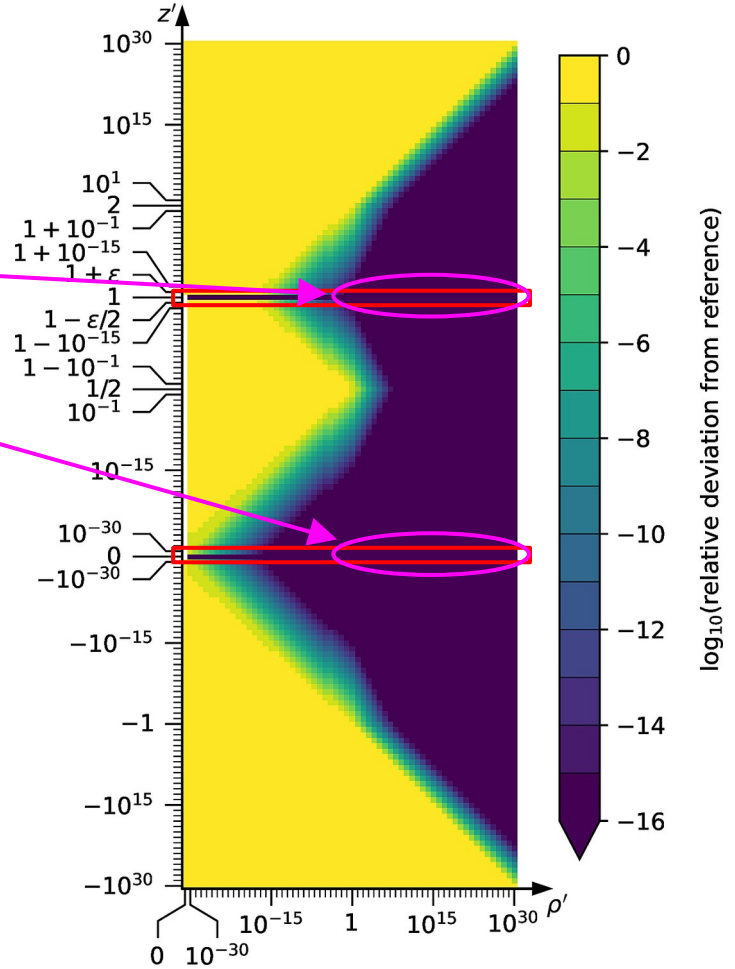


SWS, A_z, special case z' ∈ {0,1}

"far-field" along z' ∈{0, 1}:

$$\tilde{A}_{\mathrm{z,rad,f}}(\rho') = \operatorname{atanh}\left(\frac{1}{\rho' + \sqrt{\rho'^2 + 1}}\right)$$





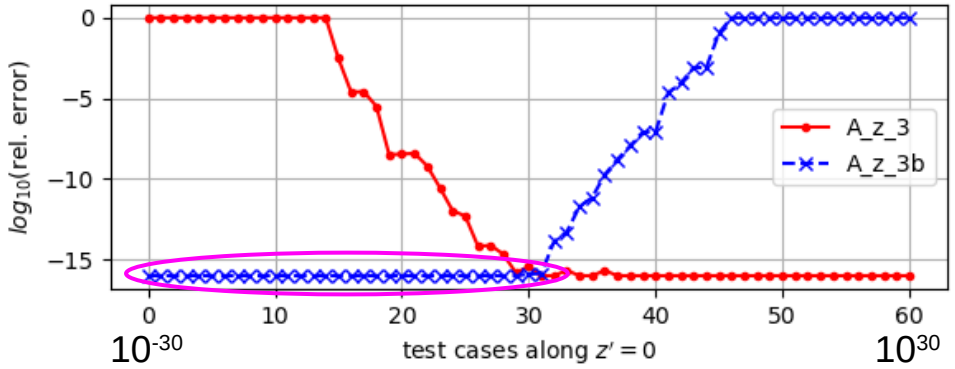
SWS, A_z, special case z' ∈ {0,1}

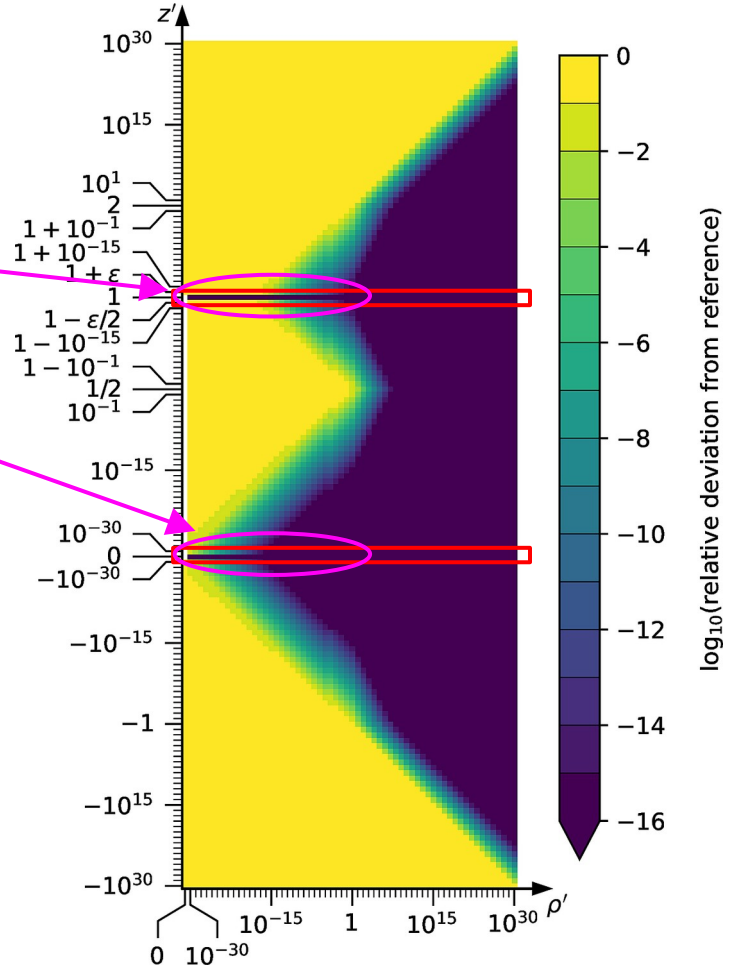
"near-field" along z'∈{0, 1}:

$$\tilde{A}_{\mathrm{z,rad,n}}(\rho') = \frac{1}{2}\ln\left(\frac{\rho' c + 1 + c}{\rho' c + 2s^2}\right)$$

with: $$c = \frac{1}{\sqrt{\rho'^2 + 1}}$$

$$s = \sin(\arctan(\rho')/2)$$





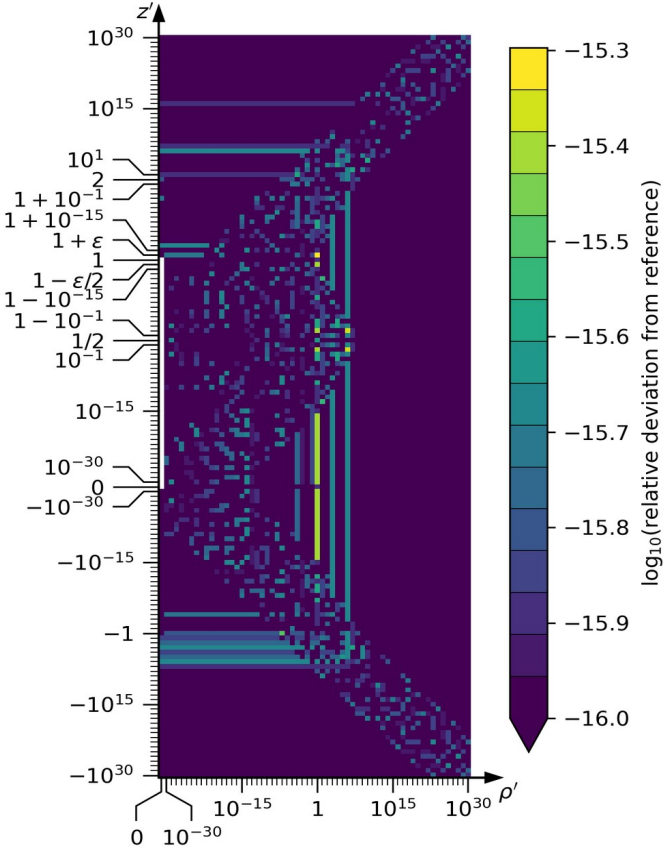
SWS, A_z, full solution

$$\tilde{A}_z(\rho', z') = \begin{cases} \tilde{A}_{\mathrm{z,ax}}(z') & : \rho' = 0, \text{any } z' \\ \tilde{A}_{\mathrm{z,rad}}(\rho') & : \rho' > 0, z' \in \{0, 1\} \\ \tilde{A}_{\mathrm{z,f}}(\rho', z') & : \rho' \geq 1 \text{ or } z' \leq 1 \text{ or } z' > 2 \\ \tilde{A}_{\mathrm{z,n}}(\rho', z') & : \text{else}\,. \end{cases}$$

with:

$$\tilde{A}_{\mathrm{z,ax}}(z') = \begin{cases} \tilde{A}_{\mathrm{z,ax,f}}(z') & : z' < -1 \text{ or } z' > 2 \\ \tilde{A}_{\mathrm{z,ax,n}}(z') & : \text{else} \end{cases}$$

$$\tilde{A}_{\mathrm{z,rad}}(\rho') = \begin{cases} \tilde{A}_{\mathrm{z,rad,f}}(\rho') & : \rho' > 1 \\ \tilde{A}_{\mathrm{z,rad,n}}(\rho') & : \text{else} \end{cases}$$



Relative errors are less than 10^{-15} for all test points!

Outline

- Introduction
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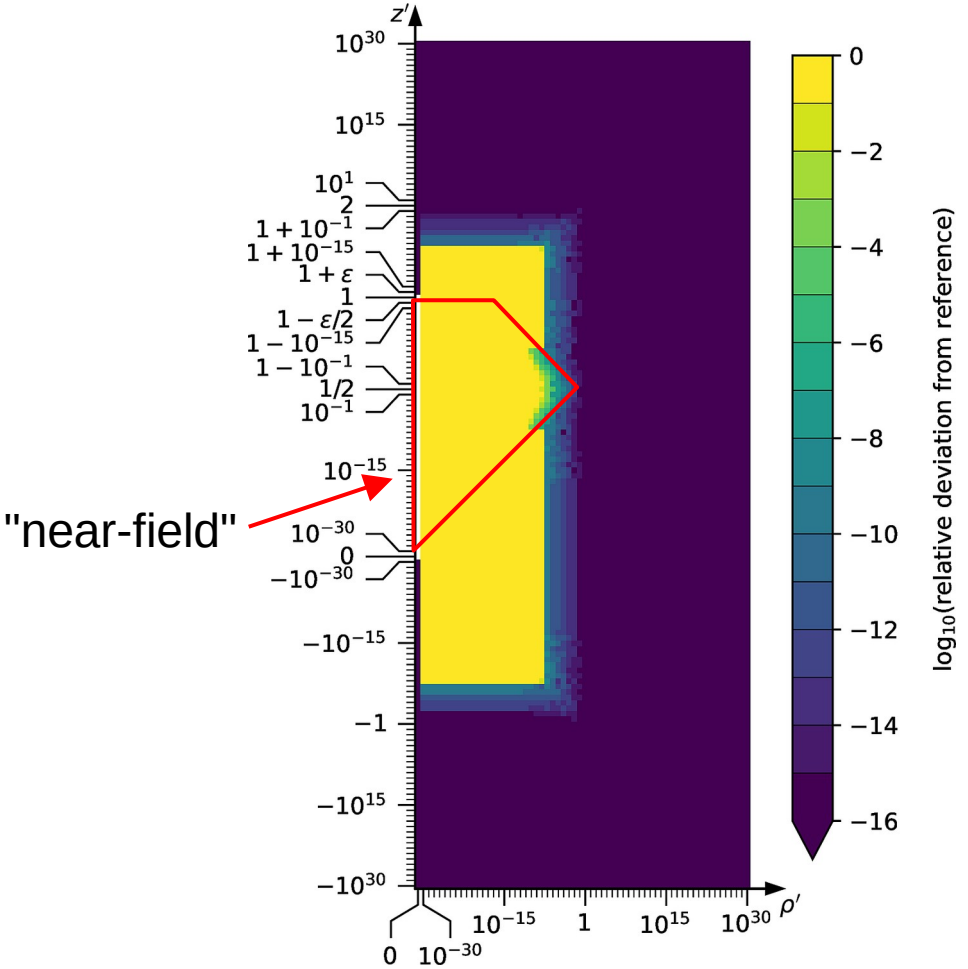
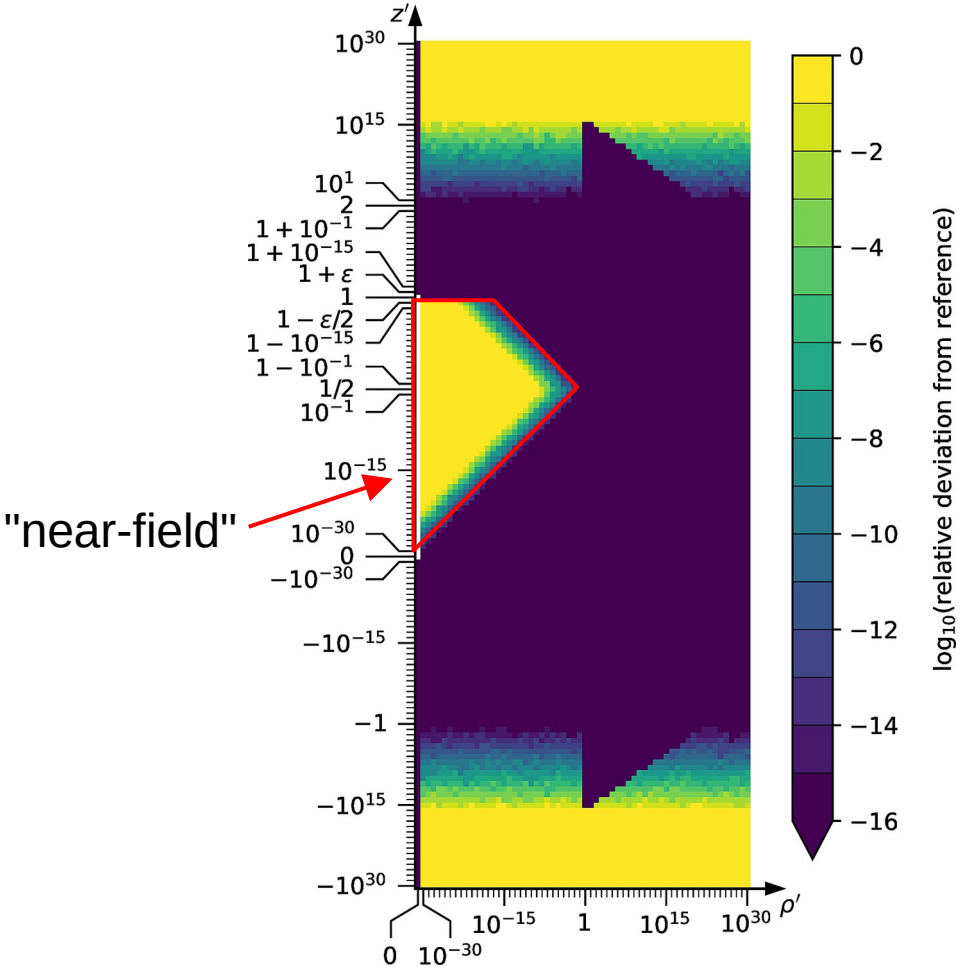
  - Magnetic Vector Potential: A_φ
  - Magnetic Field: B_ρ
  - Magnetic Field: B_z
- Cartesian Interface

SWS, B_φ, MAKEGRID



SWS, B_φ, Mag3D



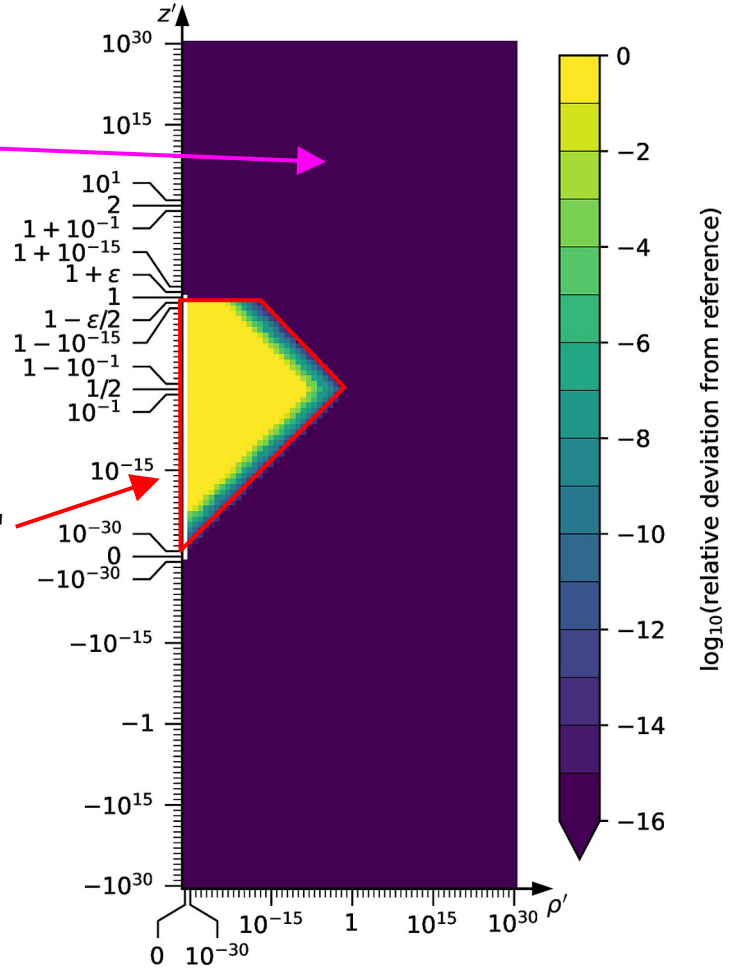
SWS, B_φ, far-field method

$$\tilde{B}_{\varphi,\mathrm{f}}(\rho', z') = \left(\frac{1}{r_\mathrm{f}} + \frac{1}{r_\mathrm{i}}\right) \frac{\rho'}{\rho'^2 + r_\mathrm{i} r_\mathrm{f} - z'(1 - z')}$$

with $r_i = \sqrt{\rho'^2 + z'^2}$

$$r_f = \sqrt{\rho'^2 + (1 - z')^2}$$

"near-field"



SWS, B_φ, near-field method

$$\tilde{B}_{\varphi,\mathrm{n}}(\rho', z') = \left(\frac{1}{r_\mathrm{f}} + \frac{1}{r_\mathrm{i}}\right) \frac{\rho'}{\rho'^2 + 2r_\mathrm{i}\left[r_\mathrm{f}\sin^2(\beta/2) + (1-z')\sin^2(\alpha/2)\right]}$$

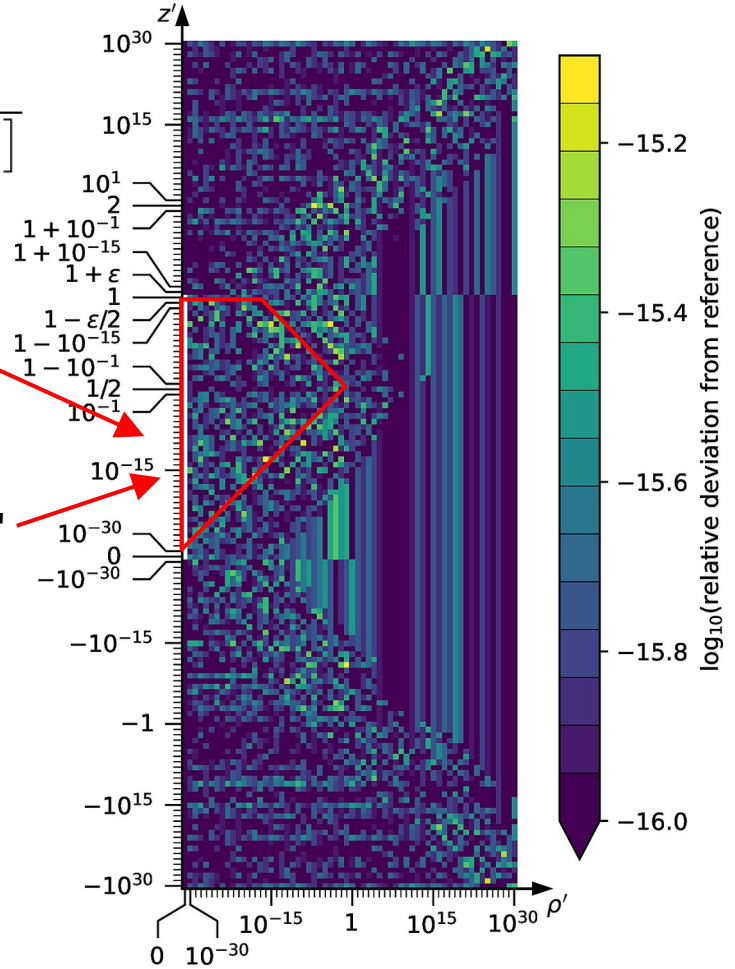
with $r_i = \sqrt{\rho'^2 + z'^2}$

$$r_f = \sqrt{\rho'^2 + (1-z')^2}$$

$$\alpha = \mathtt{atan2}(\rho', z')$$

$$\beta = \mathtt{atan2}(\rho', 1-z')$$

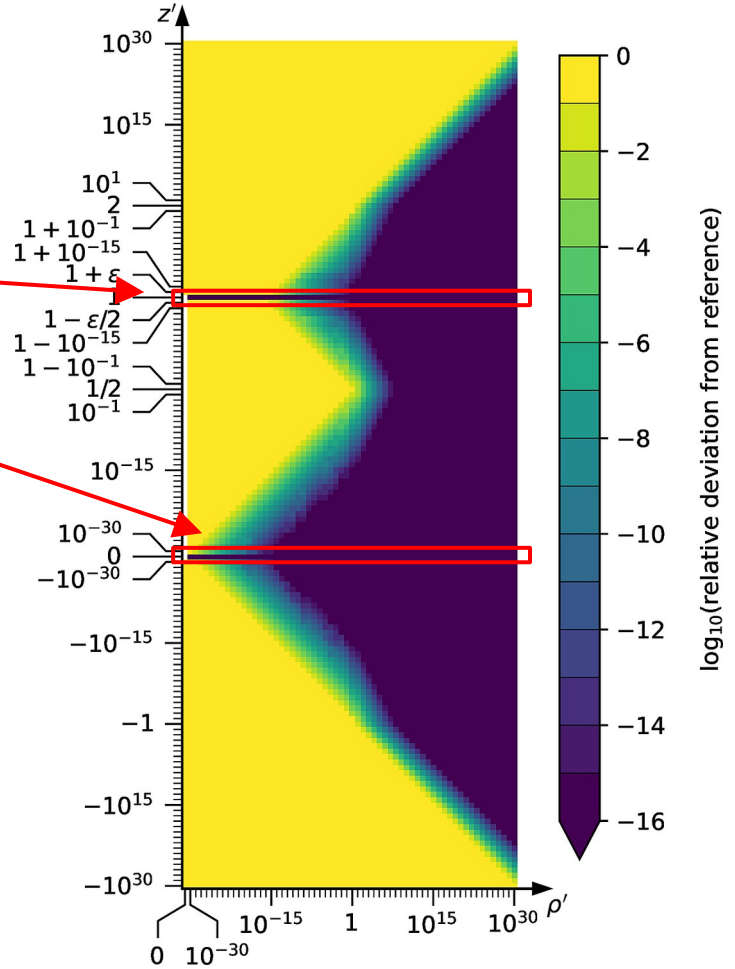
"near-field"



SWS, B_φ, special case $z' \in \{0,1\}$

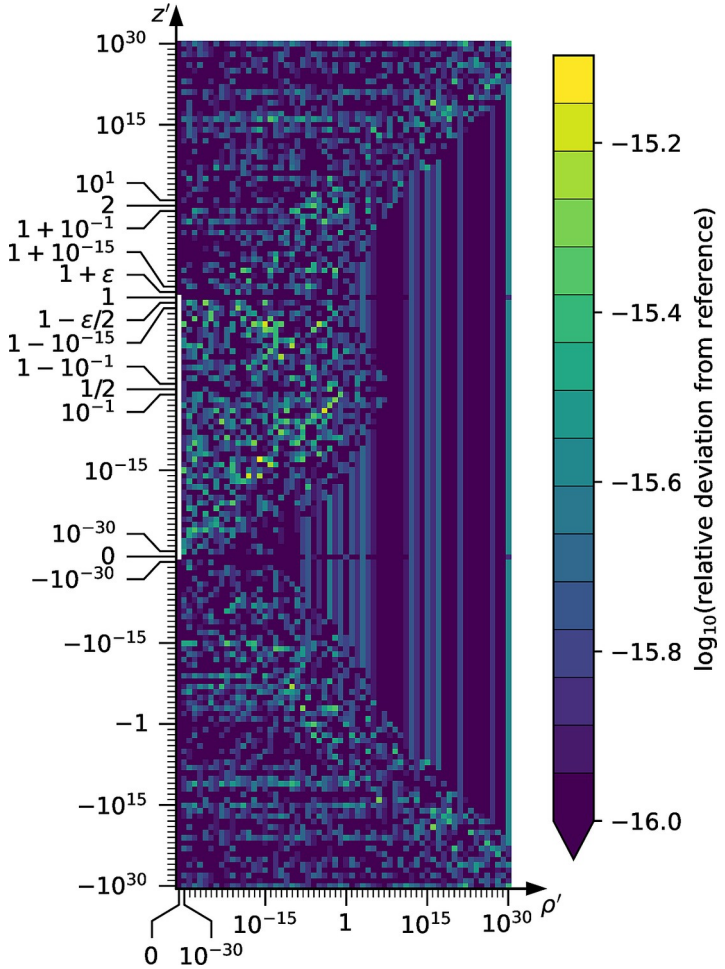
solution along $z' \in \{0, 1\}$:

$$\tilde{B}_{\varphi,\mathrm{rad}}(\rho') = \frac{1}{\rho'\sqrt{\rho'^2+1}}$$



SWS, B_φ, full solution

$$\tilde{B}_\varphi(\rho', z') = \begin{cases} 0 & : \rho' = 0, \text{any } z' \\ \tilde{B}_{\varphi,\mathrm{rad}}(\rho') & : \rho' > 0, z' \in \{0, 1\} \\ \tilde{B}_{\varphi,\mathrm{f}}(\rho', z') & : \rho' \geq 1 \text{ or } z' \leq 0 \text{ or } z' \geq 1 \\ & \quad \text{or } \rho'/(1 - z') \geq 1 \\ & \quad \text{or } \rho'/z' \geq 1 \\ \tilde{B}_{\varphi,\mathrm{n}}(\rho', z') & : \text{else}\,. \end{cases}$$



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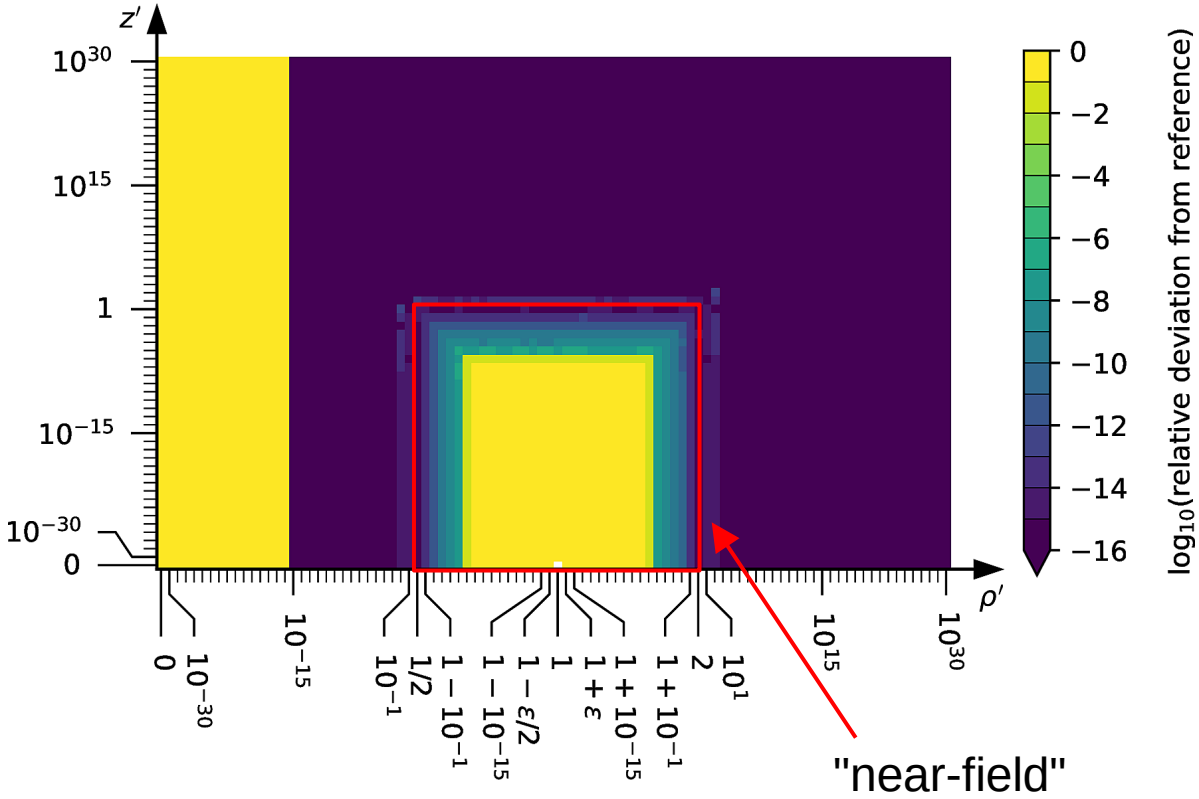
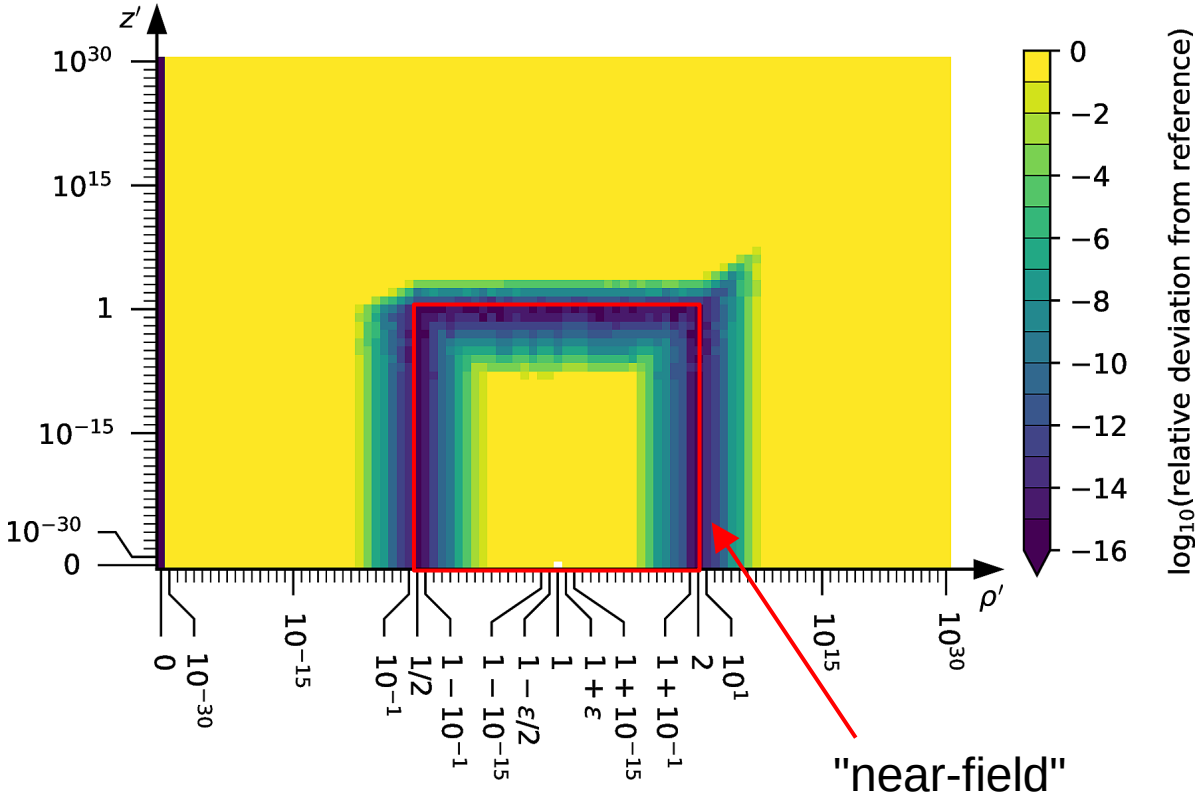
  - Magnetic Vector Potential: A_φ
  - Magnetic Field: B_ρ
  - Magnetic Field: B_z
- Cartesian Interface

CWL, A_φ, MAKEGRID



CWL, A_φ, Mag3D



CWL, A_φ, far-field

$$\tilde{A}_{\varphi,\mathrm{f}}(\rho', z') = \frac{k^2}{\sqrt{z'^2 + (1+\rho')^2}}\,\mathcal{C}(k_c)$$

with

$$\mathcal{C}(k_c) = \frac{2\,\mathcal{D}(k_c) - \mathcal{K}(k_c)}{k^2} = \mathrm{cel}\left(\frac{2\sqrt{k_c}}{1+k_c}, 1, 0, \frac{2}{(1+k_c)^3}\right)$$

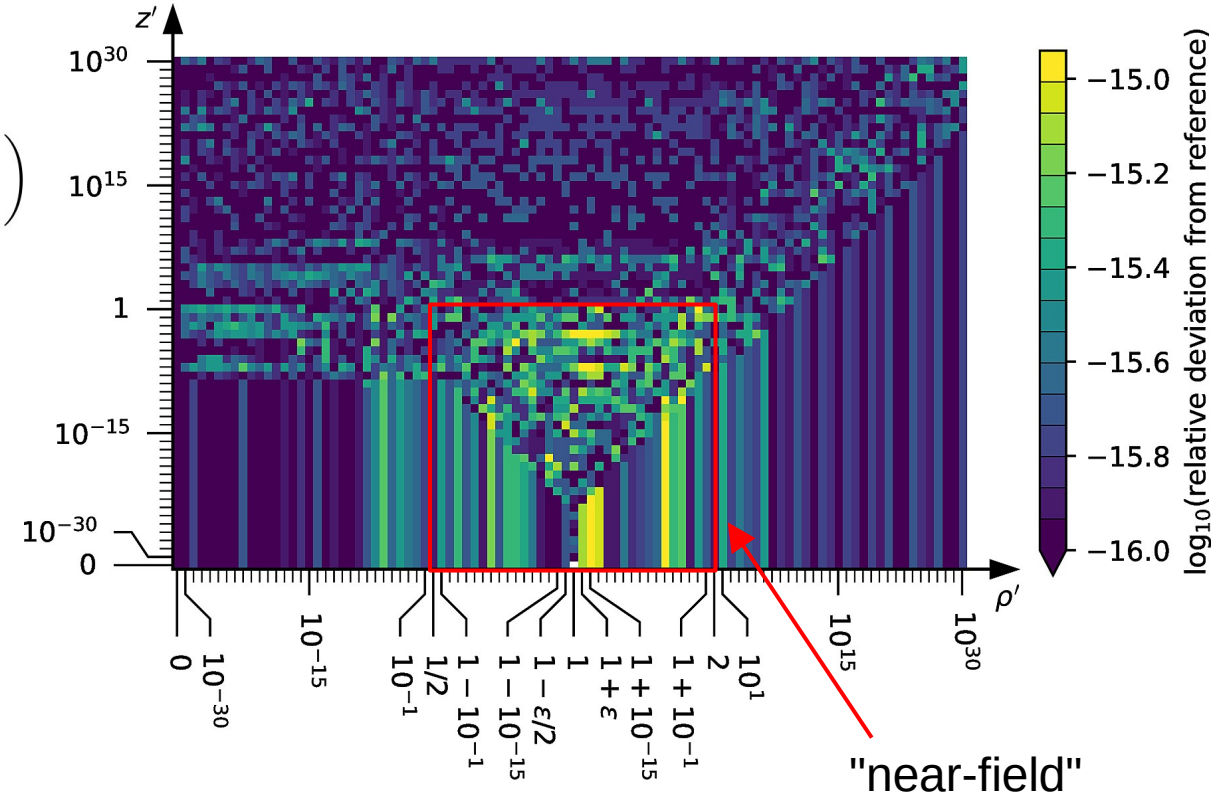
definitions (not used for computation):

$$\mathcal{D}(k_c) = \frac{\mathcal{K}(k_c) - \mathcal{E}(k_c)}{k^2} = \mathrm{cel}(k_c, 1, 0, 1)$$

$$\mathcal{K}(k_c) = \mathrm{cel}(k_c, 1, 1, 1)$$

$$\mathcal{E}(k_c) = \mathrm{cel}(k_c, 1, 1, k_c^2)$$

$$k^2 = \frac{4\rho'}{z'^2 + (1+\rho')^2}$$

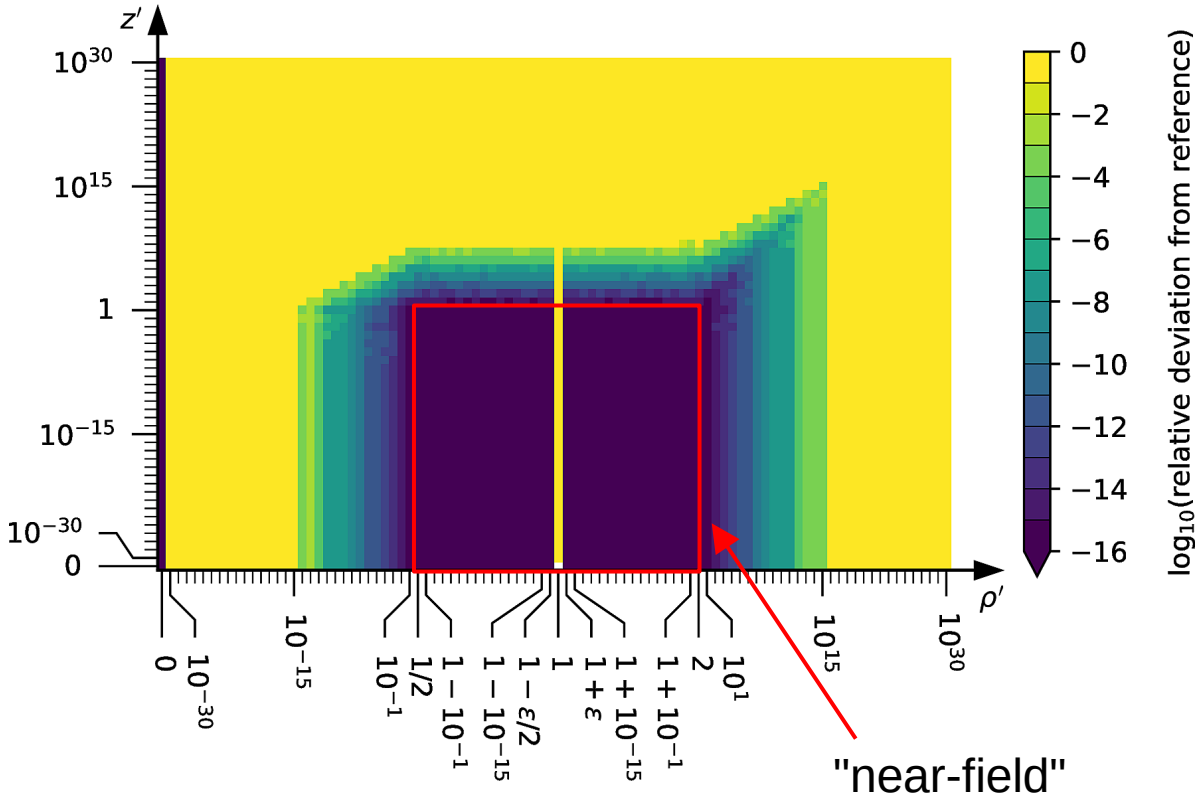


CWL, A_φ, near-field

$$\tilde{A}_{\varphi,\mathrm{n}}(\rho', z') = \frac{1}{|\rho'-1|\sqrt{\left(\frac{z'}{\rho'-1}\right)^2 + \left(1+\frac{2}{\rho'-1}\right)^2}}\,\mathrm{cel}(\sqrt{k_c^2}, 1, -1, 1)$$

with

$$k_c^2 = \frac{\left(\frac{z'}{\rho'-1}\right)^2 + 1}{\left(\frac{z'}{\rho'-1}\right)^2 + \left(1+\frac{2}{\rho'-1}\right)^2}$$

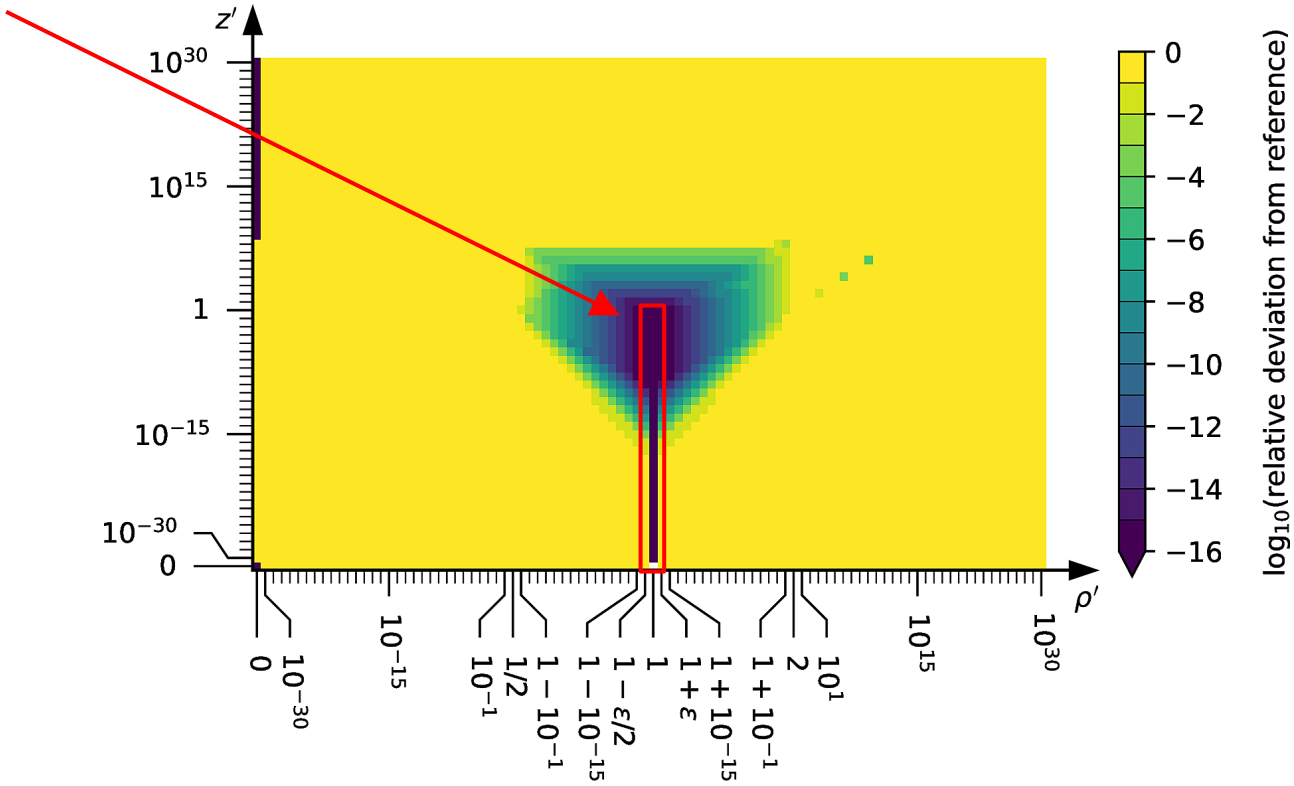


CWL, A_φ, special case ρ' = 1

$$\tilde{A}_{\varphi,\mathrm{v}}(z') = \frac{1}{|z'|}\,\mathrm{cel}\left(\frac{1}{k_c}, 1, 1, -1\right)$$

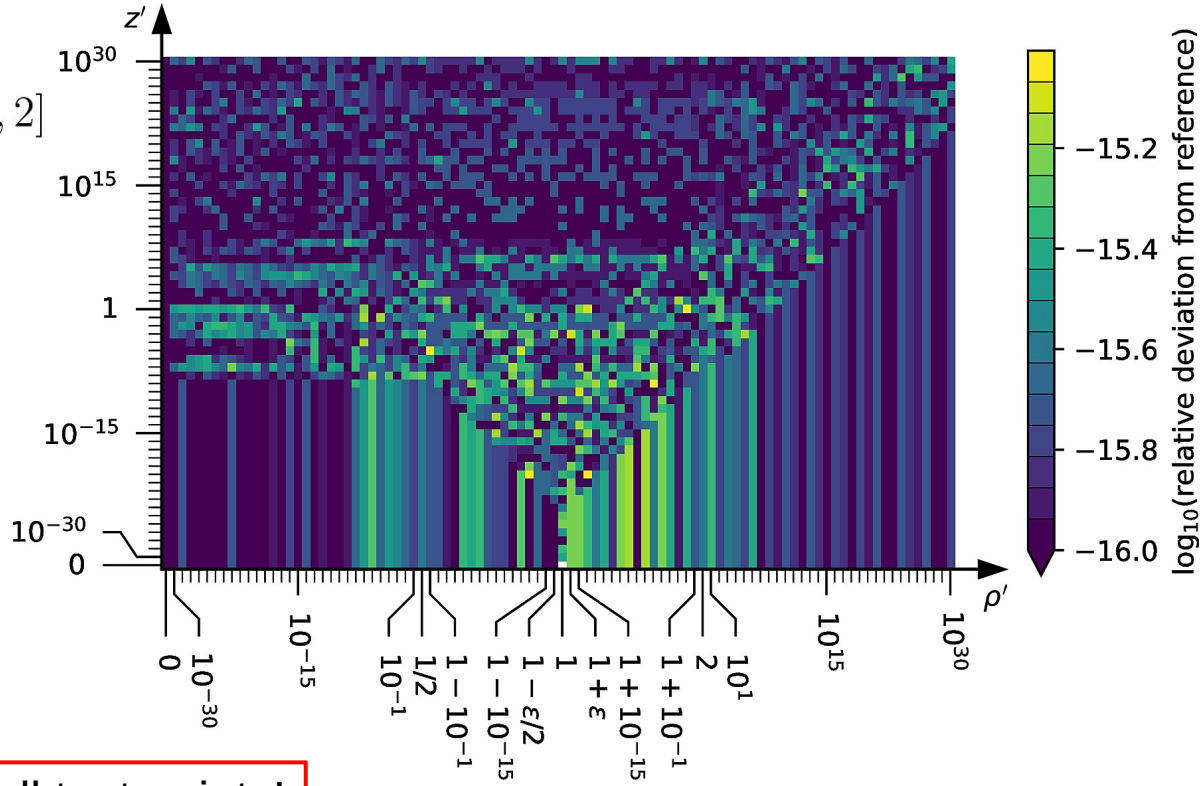
with

$$k_c = \frac{|z'|}{\sqrt{4 + z'^2}}$$



CWL, A_φ, full solution

$$\tilde{A}_\varphi(\rho', z') = \begin{cases} 0 & : \rho' = 0 \\ \tilde{A}_{\varphi,1}(\rho', z') & : z' \geq 1 \text{ or } \rho' \notin [1/2, 2] \\ \tilde{A}_{\varphi,6}(\rho', z') & : \rho' \neq 1 \\ \tilde{A}_{\varphi,5}(\rho', z') & : \text{else} \end{cases}$$



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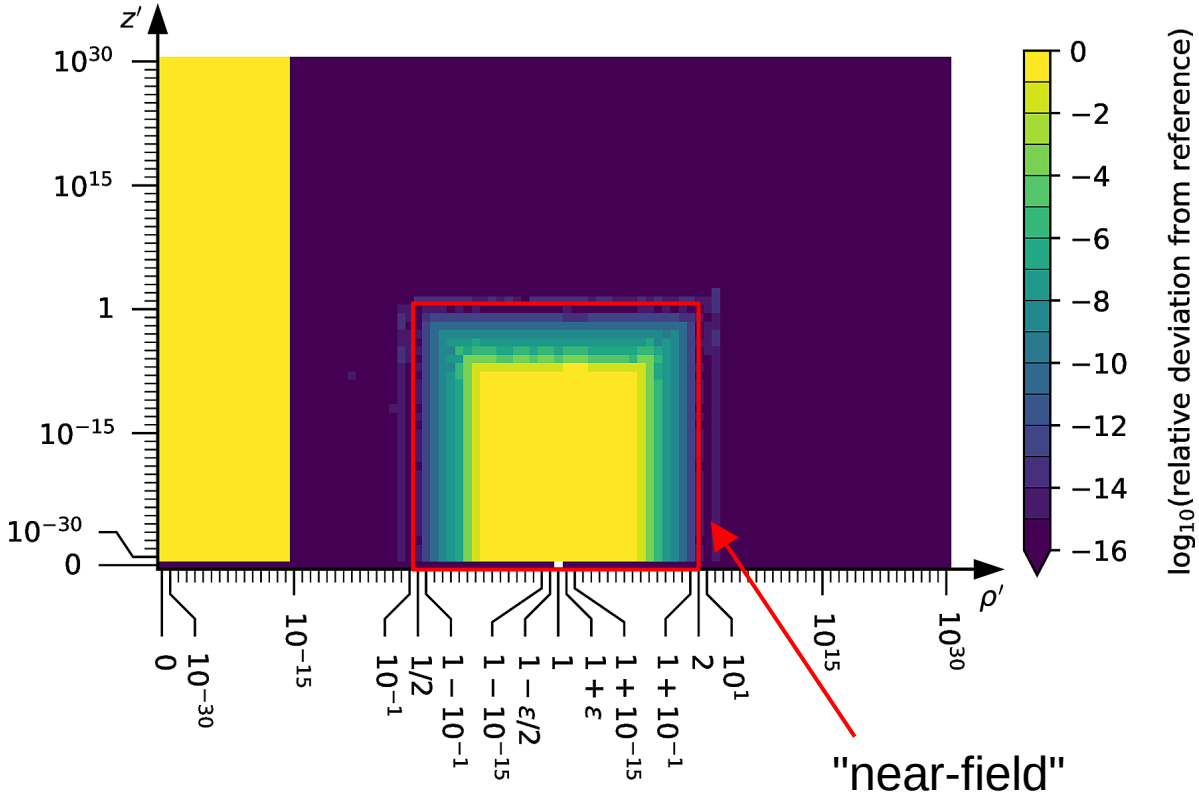
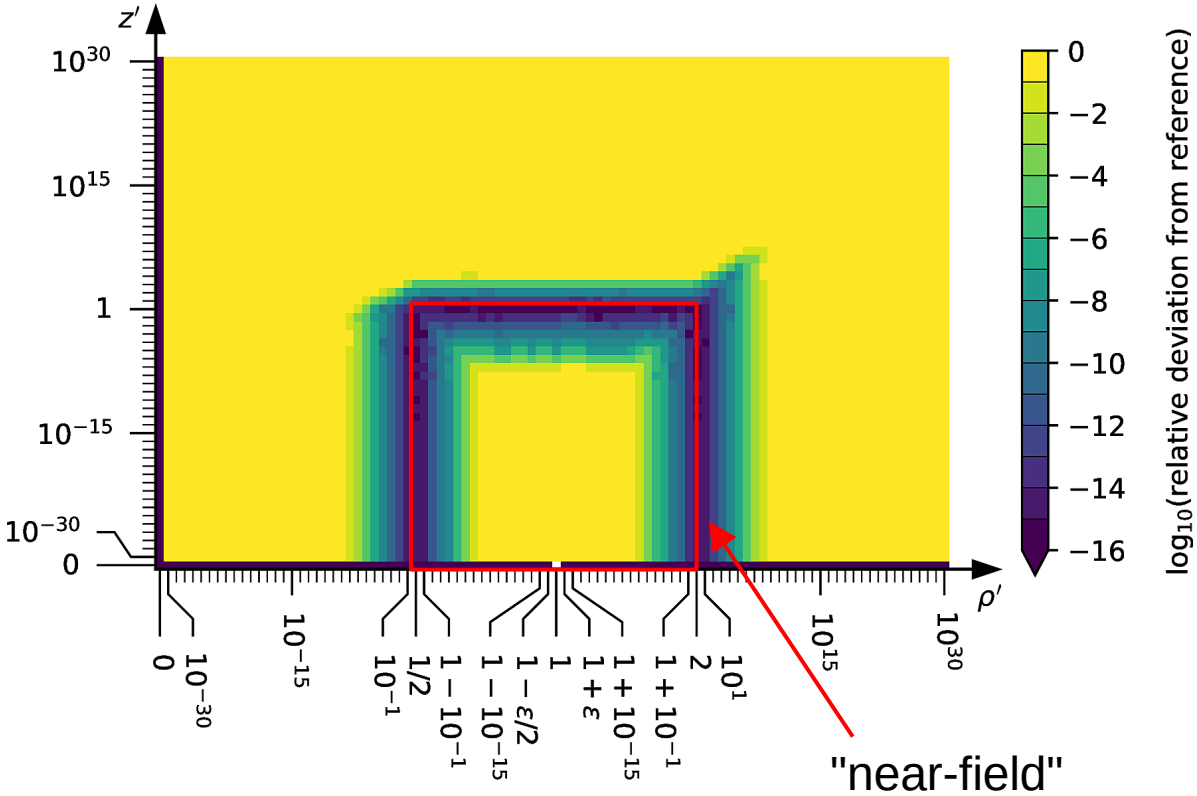
  - Magnetic Field: B_z
- Cartesian Interface

CWL, B_ρ, MAKEGRID



CWL, B_ρ, Mag3D

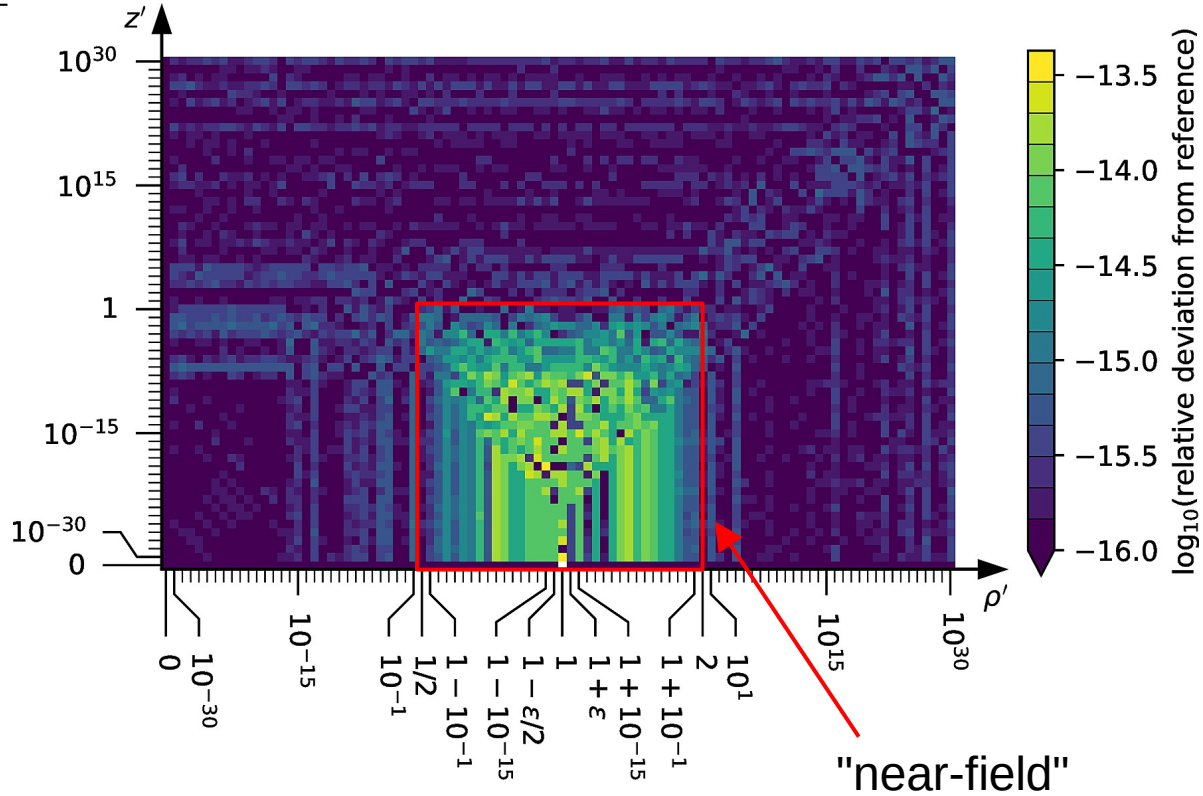


CWL, B_ρ, far-field

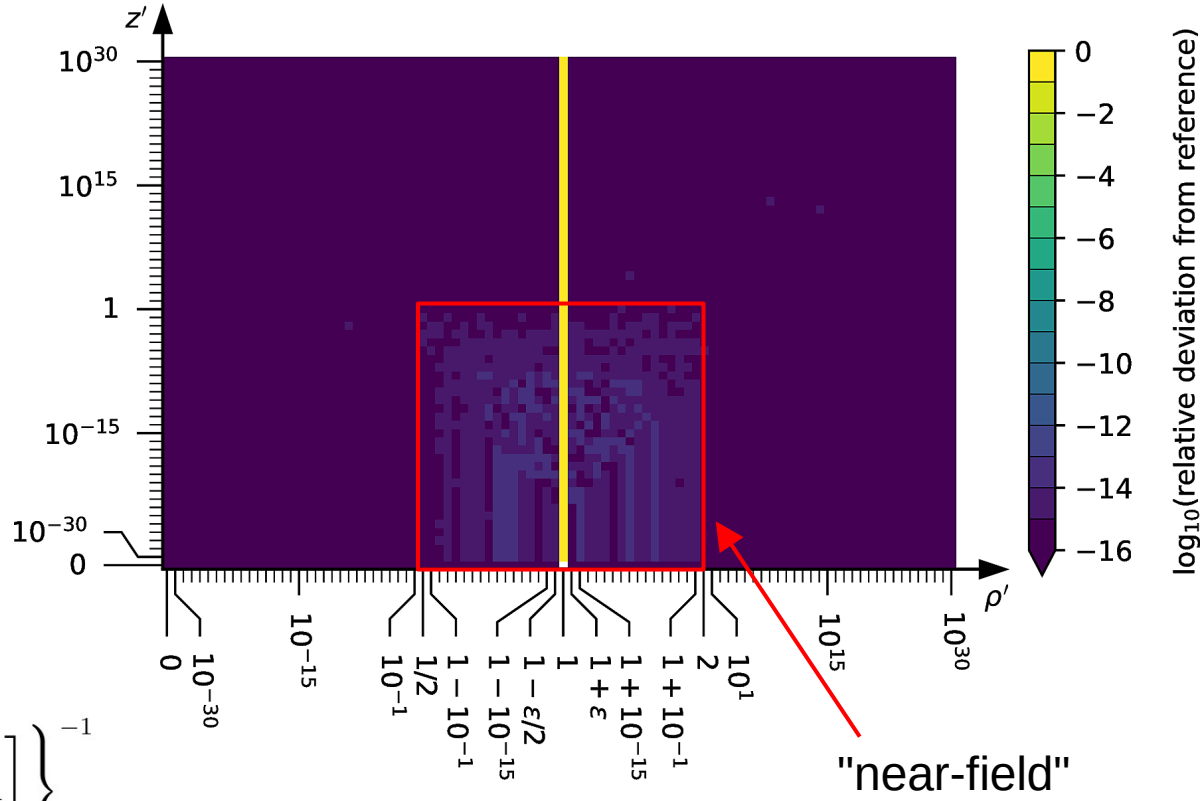
$$\tilde{B}_{\rho,\mathrm{f}}(\rho', z') = \frac{4\rho' z' \left[\mathcal{D}(k_c) - \mathcal{C}(k_c)\right]}{\left[z'^2 + (1+\rho')^2\right]^{3/2} \left[z'^2 + (1-\rho')^2\right]}$$

with

$$\mathcal{D}(k_c) = \frac{\mathcal{K}(k) - \mathcal{E}(k)}{k^2} = \mathrm{cel}(k_c, 1, 0, 1)$$



CWL, B_ρ, near-field except ρ' = 1



$$\tilde{B}_{\rho,\mathrm{n}}(\rho', z') = \frac{4\rho' \left|\frac{z'}{\rho'-1}\right| \left[\mathcal{D}(k_c) - \mathcal{C}(k_c)\right]}{(\rho'-1)^4} \left\{ \left[\left(\frac{z'}{\rho'-1}\right)^2 + \left(1 + \frac{2}{\rho'-1}\right)^2 \right]^{3/2} \left[\left(\frac{z'}{\rho'-1}\right)^2 + 1 \right] \right\}^{-1}$$

CWL, B_ρ, ρ' = 1 up to z' = 1

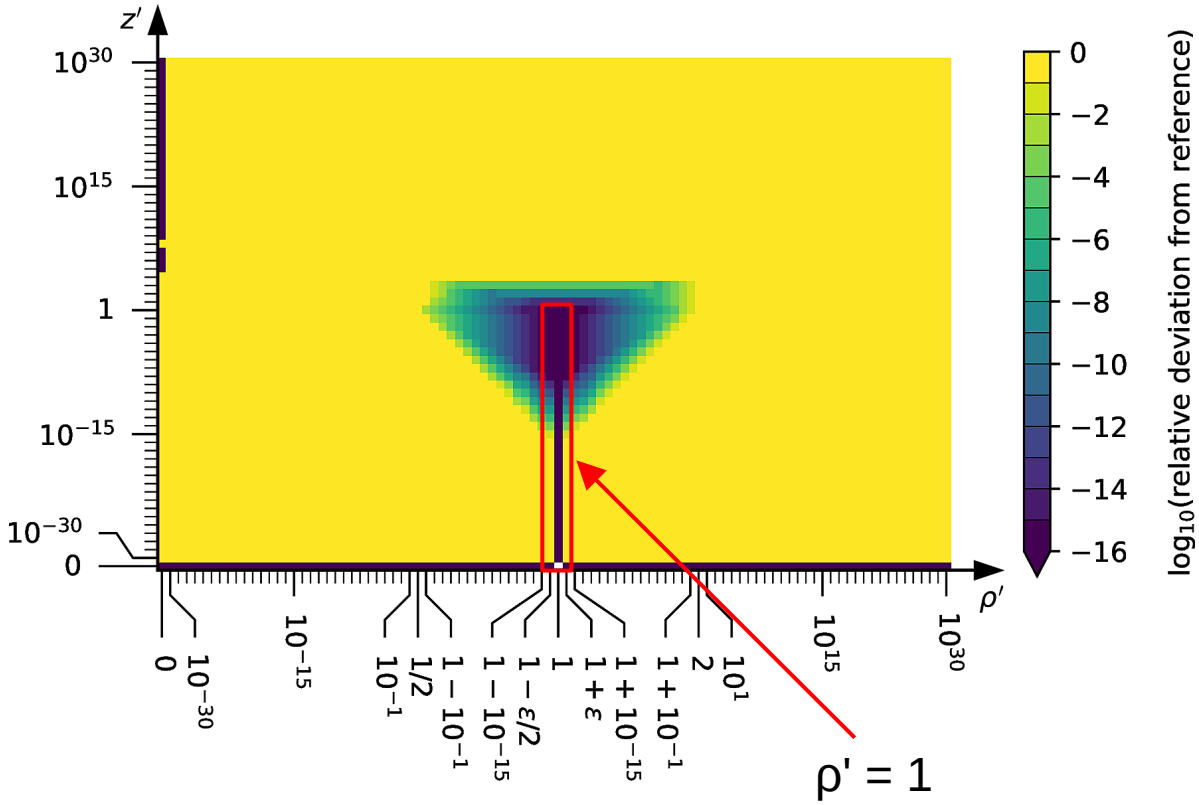
$$\tilde{B}_{\rho,\mathrm{v}}(z') = \frac{k_c}{2}\frac{z'}{|z'|}\left[\left(\frac{2}{z'^2}+1\right)\mathcal{E}(k_c) - \mathcal{K}(k_c)\right]$$

with

$$k_c^2 = \frac{1}{1+4/z'^2}$$

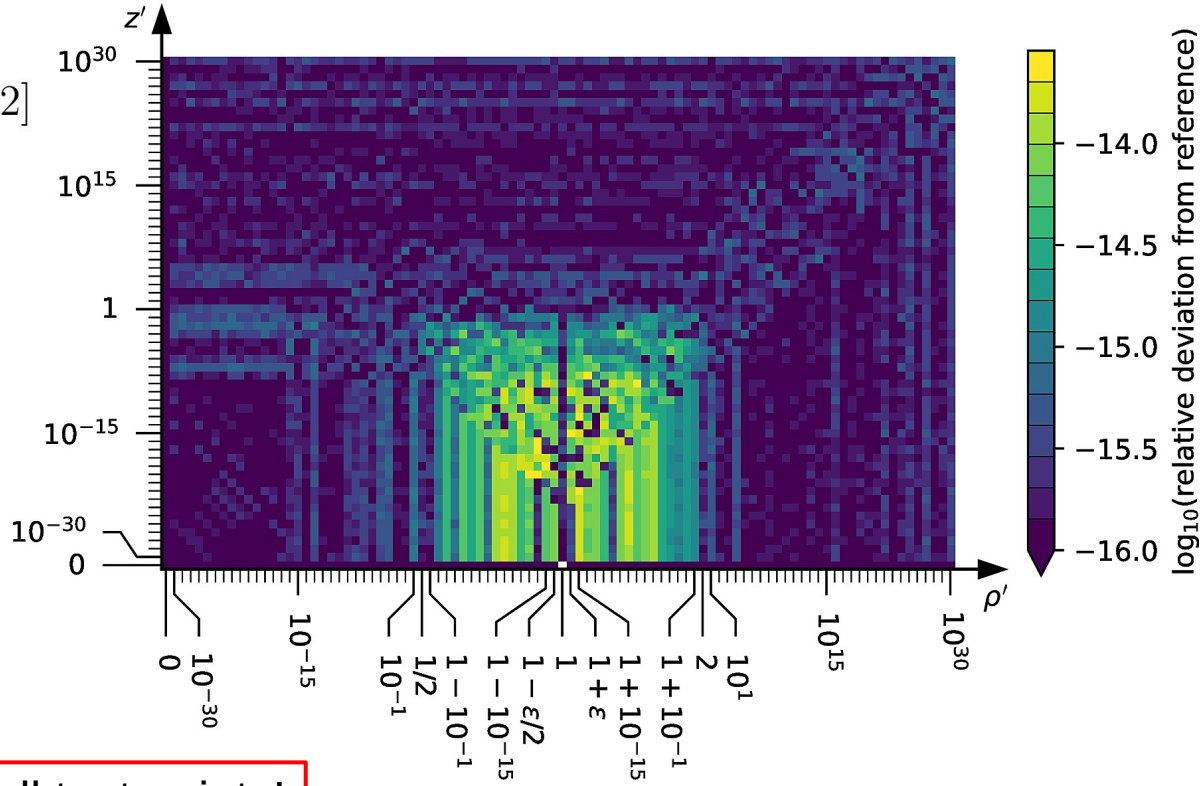
$$\mathcal{K}(k_c) = \mathrm{cel}(k_c, 1, 1, 1)$$

$$\mathcal{E}(k_c) = \mathrm{cel}(k_c, 1, 1, k_c^2)$$



CWL, B_ρ, full solution

$$\tilde{B}_\rho(\rho', z') = \begin{cases} 0 & : \rho' = 0 \text{ or } z' = 0 \\ \tilde{B}_{\rho,3}(\rho', z') & : z' \geq 1 \text{ or } \rho' \notin [1/2, 2] \\ \tilde{B}_{\rho,1}(\rho', z') & : \rho' \neq 1 \\ \tilde{B}_{\rho,4}(\rho', z') & : \text{else} \end{cases}$$



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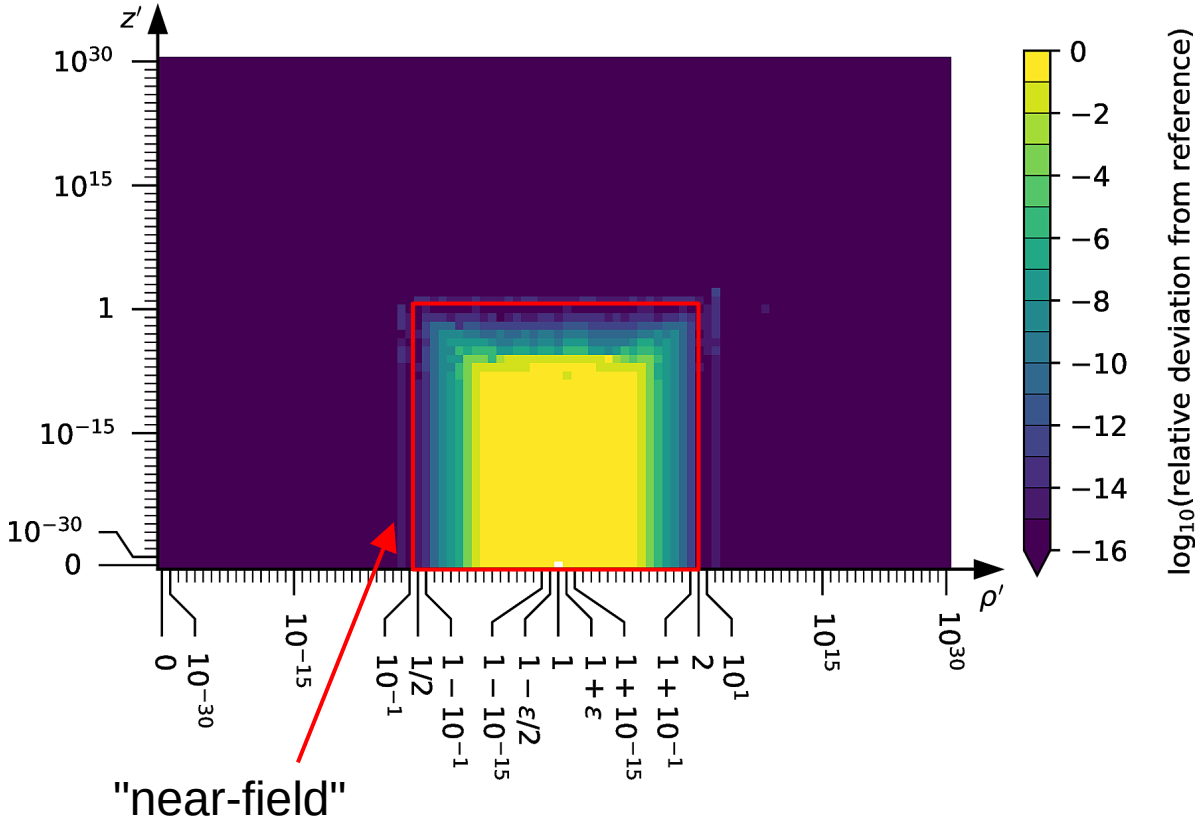
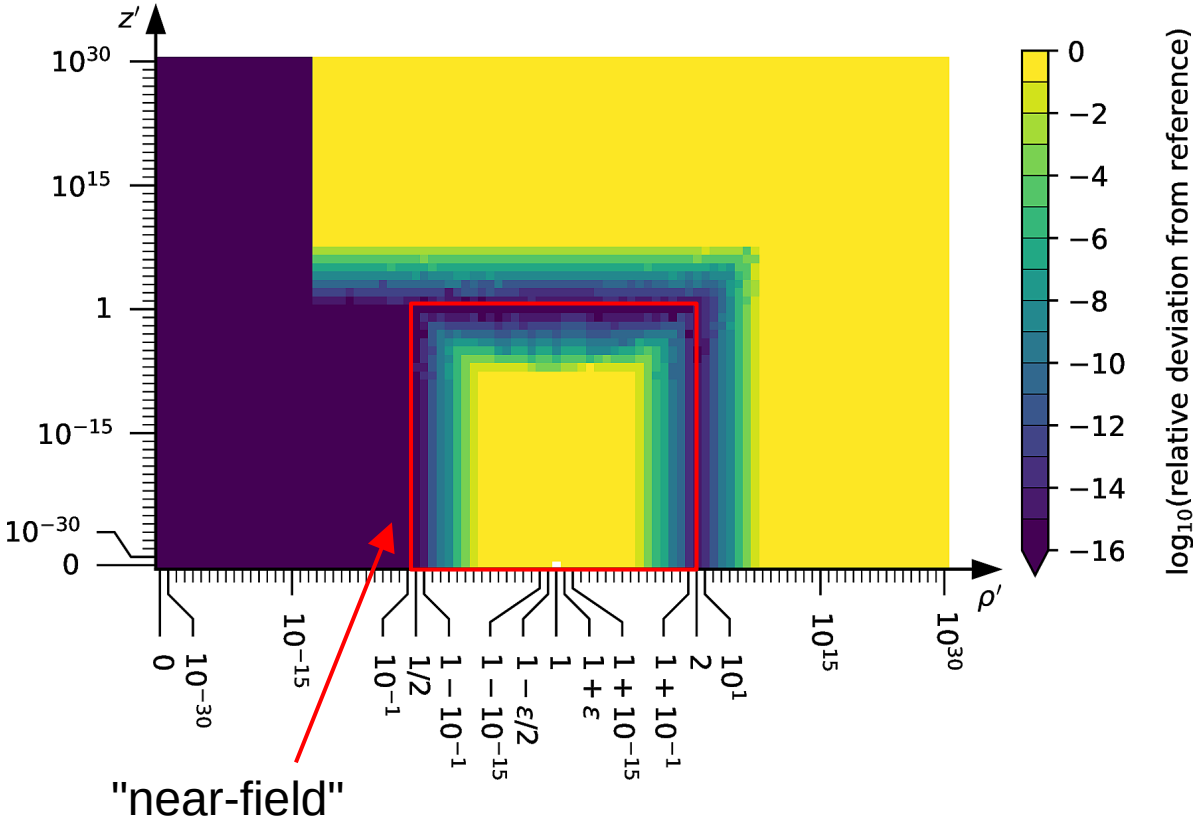
  - Magnetic Field: B_ρ
  - Magnetic Field: B_z
- Cartesian Interface

CWL, B_z, MAKEGRID

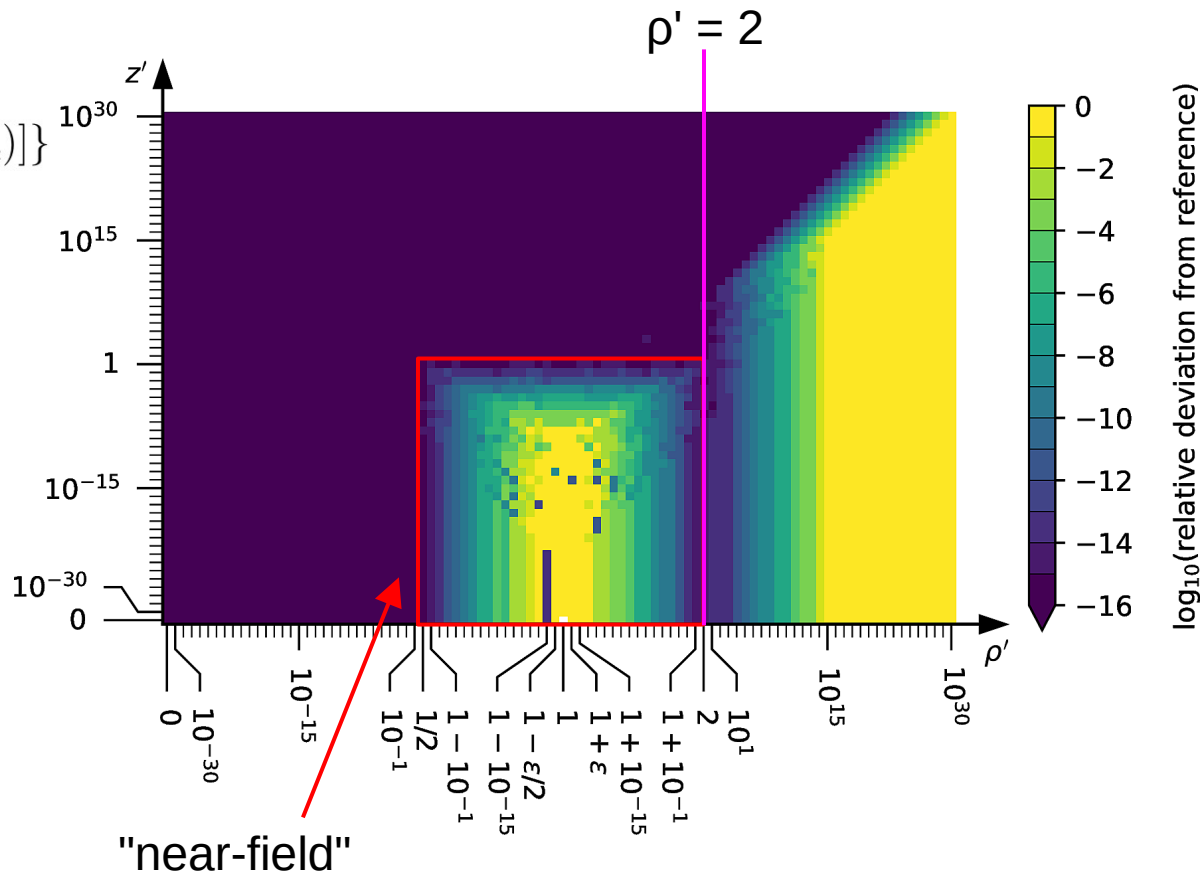


CWL, B_z, Mag3D


ρ'
10^{30}
10^{15}
1
10^{-15}
10^{-30}
0
10^{-30}
10^{-15}
10^{-1}
1/2
$1 - 10^{-1}$
$1 - 10^{-15}$
$1 - \varepsilon/2$
1
$1 + \varepsilon$
$1 + 10^{-15}$
$1 + 10^{-1}$
2
10^{1}
10^{15}
10^{30}
z'
0
−2
−4
−6
−8
−10
−12
−14
−16
$\log_{10}$(relative deviation from reference)
"near-field"

CWL, B_z, far-field (up to ρ' = 2)

$$\tilde{B}_{z,\mathrm{f1}}(\rho', z') = \frac{1}{\sqrt{z'^2 + (1+\rho')^2}\left[z'^2 + (1-\rho')^2\right]} \left\{\mathcal{E}(k_c) + \rho'\left[\mathcal{E}(k_c) - 2\mathcal{K}(k_c) + 2\mathcal{D}(k_c)\right]\right\}$$



CWL, B_z, far-field (for ρ' > 2)

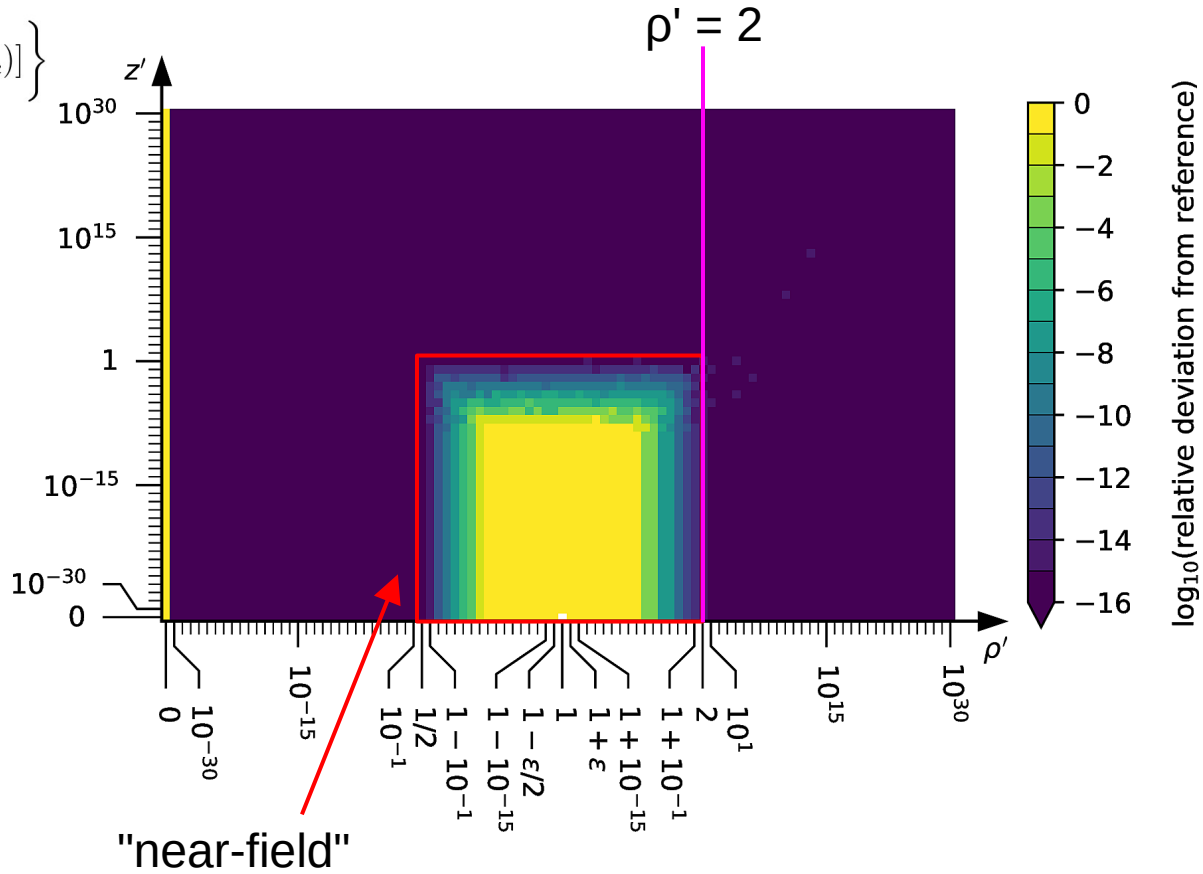
$$\tilde{B}_{z,\mathrm{f2}}(\rho', z') = \frac{1}{\sqrt{t_1 + t_2}(t_1 - t_2)\rho'^3}\left\{\mathcal{E}(k_c) + \frac{4}{\alpha_{\mathrm{cd}}}\left[\mathcal{C}(k_c) - \mathcal{D}(k_c)\right]\right\}$$

with

$$t_1 = \frac{z'^2 + 1}{\rho'^2} + 1$$

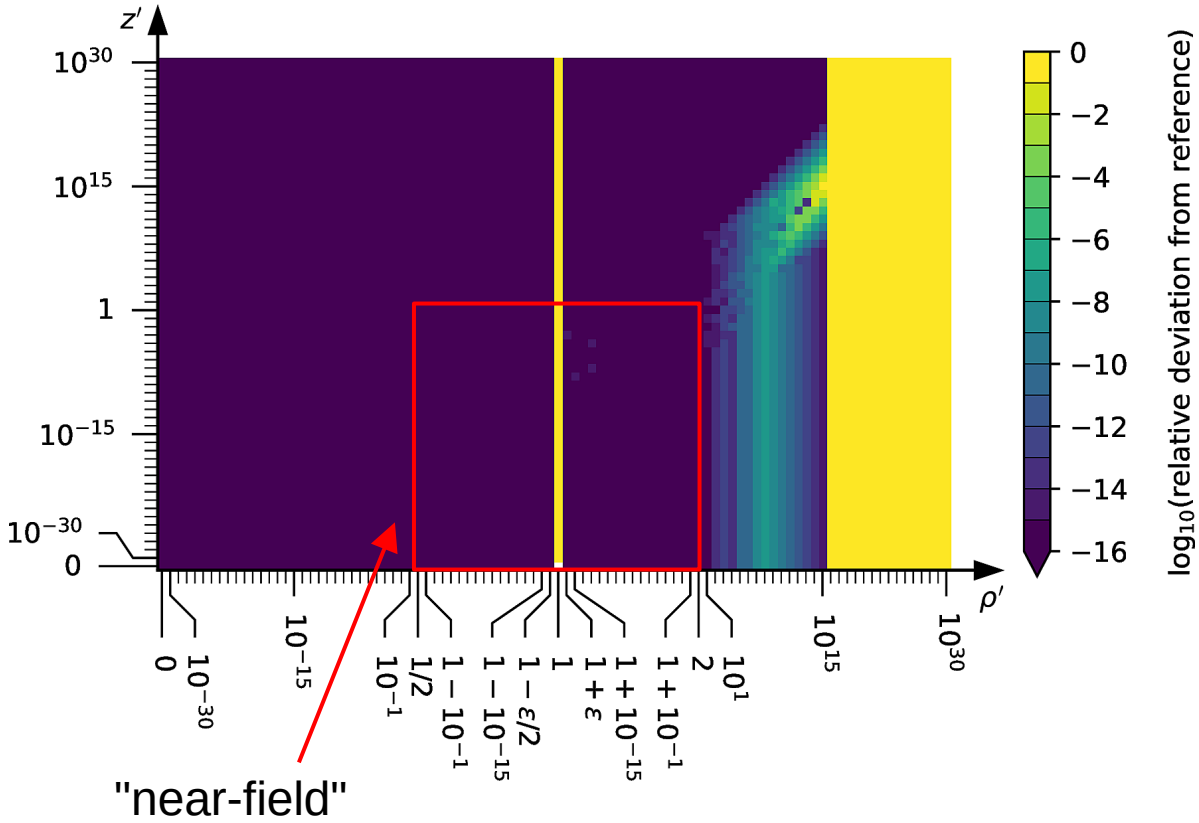
$$t_2 = \frac{2}{\rho'}$$

$$\alpha_{\mathrm{cd}} = 1 + \frac{1}{\rho'}\left[2 + \frac{1}{\rho'}\left(1 + z'^2\right)\right]$$



CWL, B_z, near-field except ρ' = 1

$$\tilde{B}_{z,\mathrm{n}}(\rho', z') = \frac{\mathrm{cel}\left(\sqrt{k_c^2}, k_c^2, 1+\rho', 1-\rho'\right)}{|\rho'-1|^3 \left[\left(\frac{z'}{\rho'-1}\right)^2 + \left(1+\frac{2}{\rho'-1}\right)^2\right]^{3/2}}$$

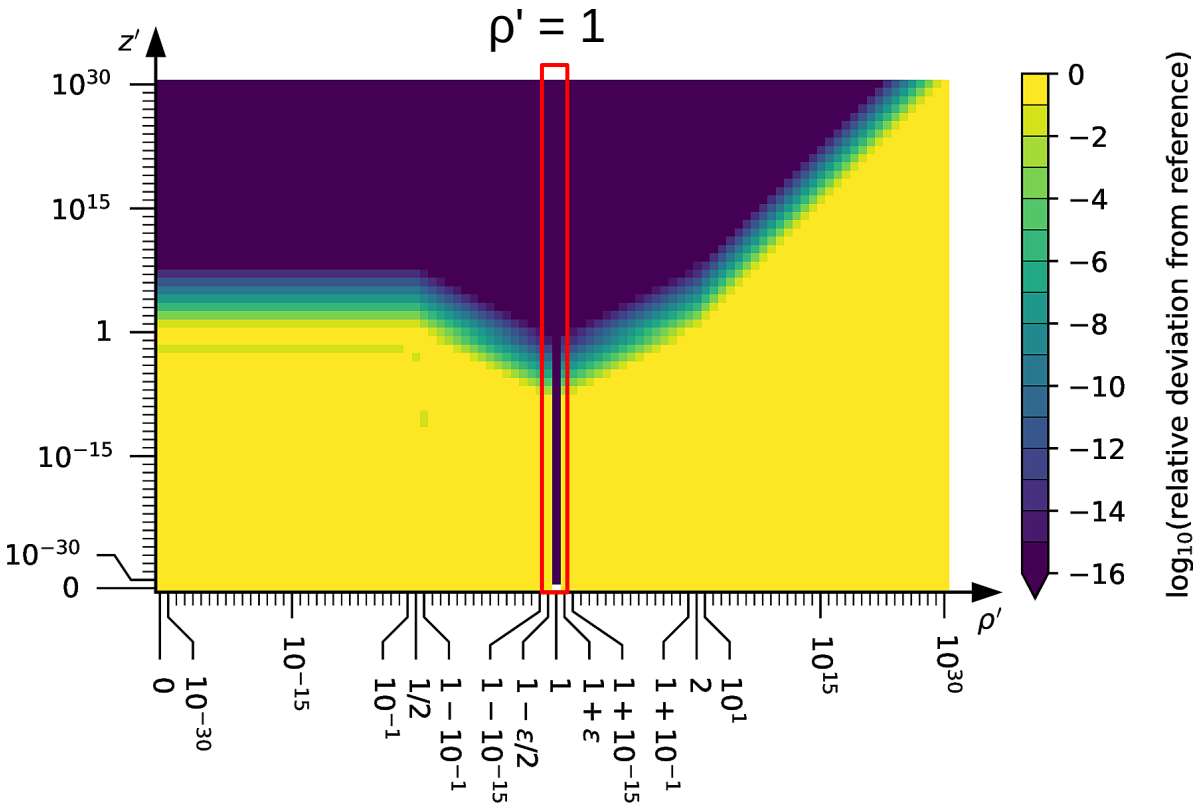


CWL, B_z, special case ρ' = 1

$$\tilde{B}_{z,\mathrm{v}}(z') = \frac{1}{\left[z'^2+4\right]^{3/2}} \operatorname{cel}\left(\sqrt{k_c^2}, k_c^2, 2, 0\right)$$

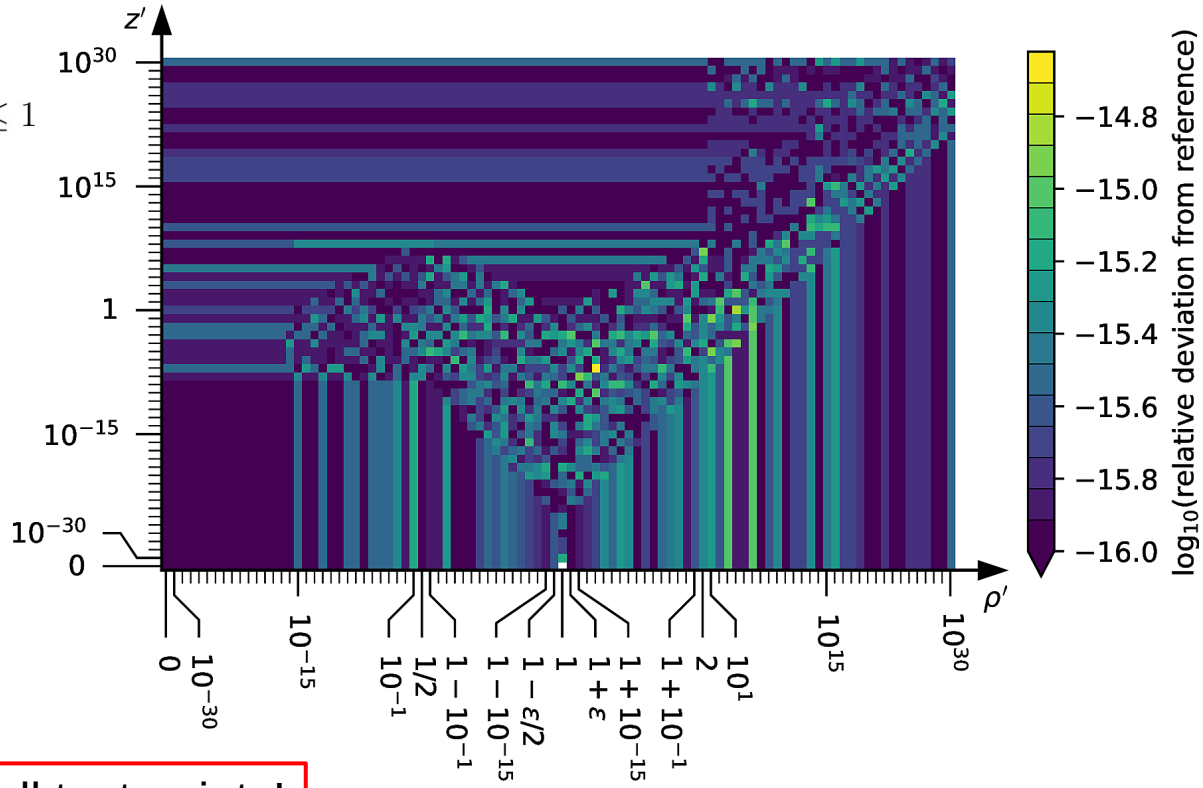
with

$$k_c^2 = \frac{z'^2}{z'^2+4}$$



CWL, B_z, full solution

$$\tilde{B}_z(\rho', z') = \begin{cases} \tilde{B}_{z,\mathrm{f1}}(\rho', z') & : \rho' < 1/2, \text{ any } z' \\ & \quad \text{or } \rho' \leq 2, |z'| > 1 \\ \tilde{B}_{z,\mathrm{f2}}(\rho', z') & : \rho' > 2, \text{ any } z' \\ \tilde{B}_{z,\mathrm{n}}(\rho', z') & : 1/2 \leq \rho' \leq 2 \text{ but } \rho' \neq 1, |z'| \leq 1 \\ \tilde{B}_{z,\mathrm{v}}(z') & : \text{else } (\rho' = 1, |z'| \leq 1)\,. \end{cases}$$



Relative errors are less than 10^{-14} for all test points!

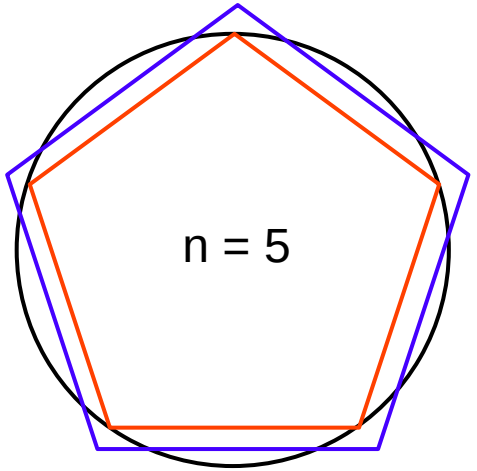
Outline

- Introduction
- Methods
- Results
- Summary

- Straight Wire Segment (SWS)
  - Magnetic Vector Potential: A_z
  - Magnetic Field: B_φ
- Circular Wire Loop (CWL)
  - Magnetic Vector Potential: A_φ
  - Magnetic Field: B_ρ
  - Magnetic Field: B_z
- Cartesian Interface

Cartesian Interface: test setup

- approximation of wire loop by n-point polygon
  a) points on wire loop
  b) slightly shifted outwards[1]
     => can cancel 2nd-order error
     => 4th-order convergence

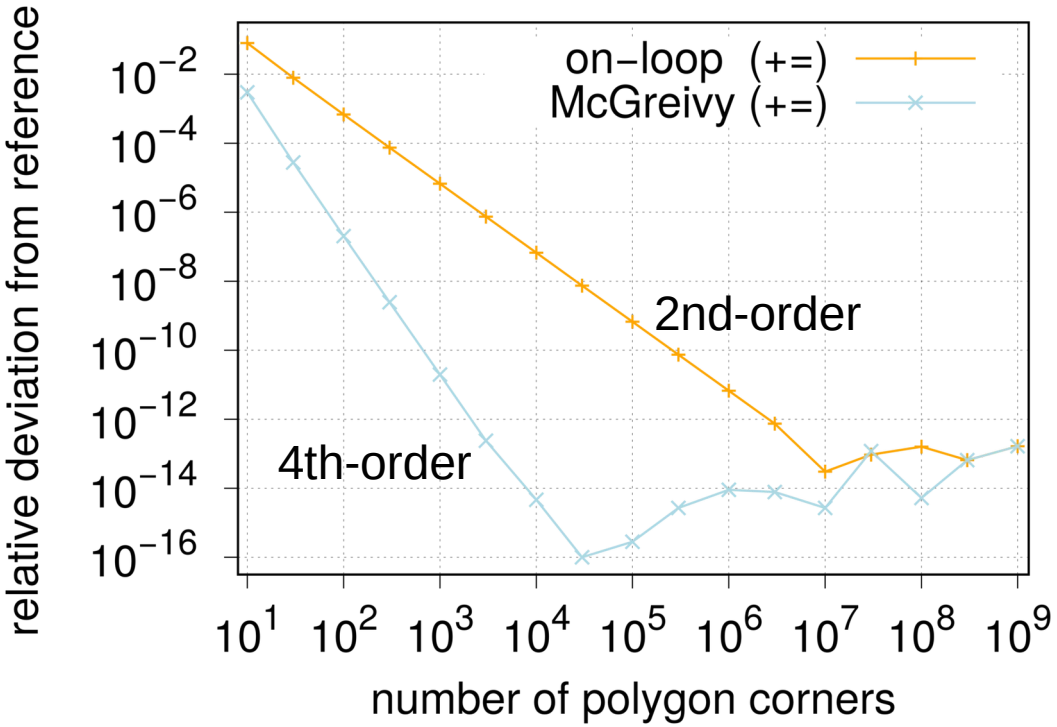


- for n→∞, polygon approximation should converge against circular wire loop solution

[1] N. McGreivy et al, Phys. Plasmas **28**, 082111 (2021)

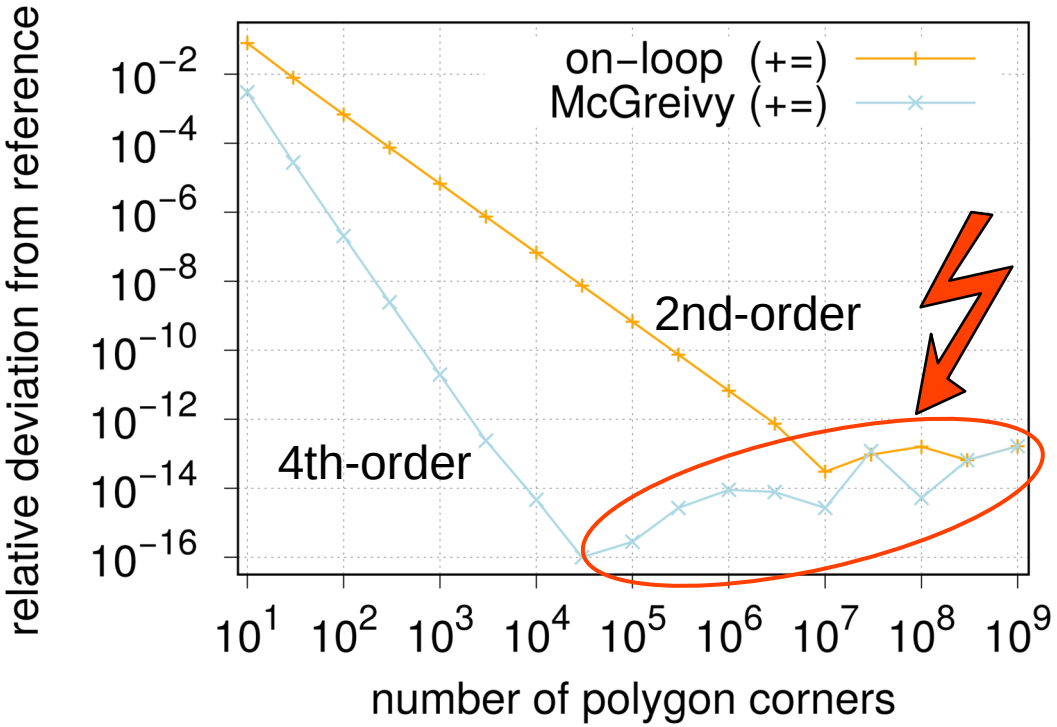
Cartesian Interface: results

- correct convergence is observed



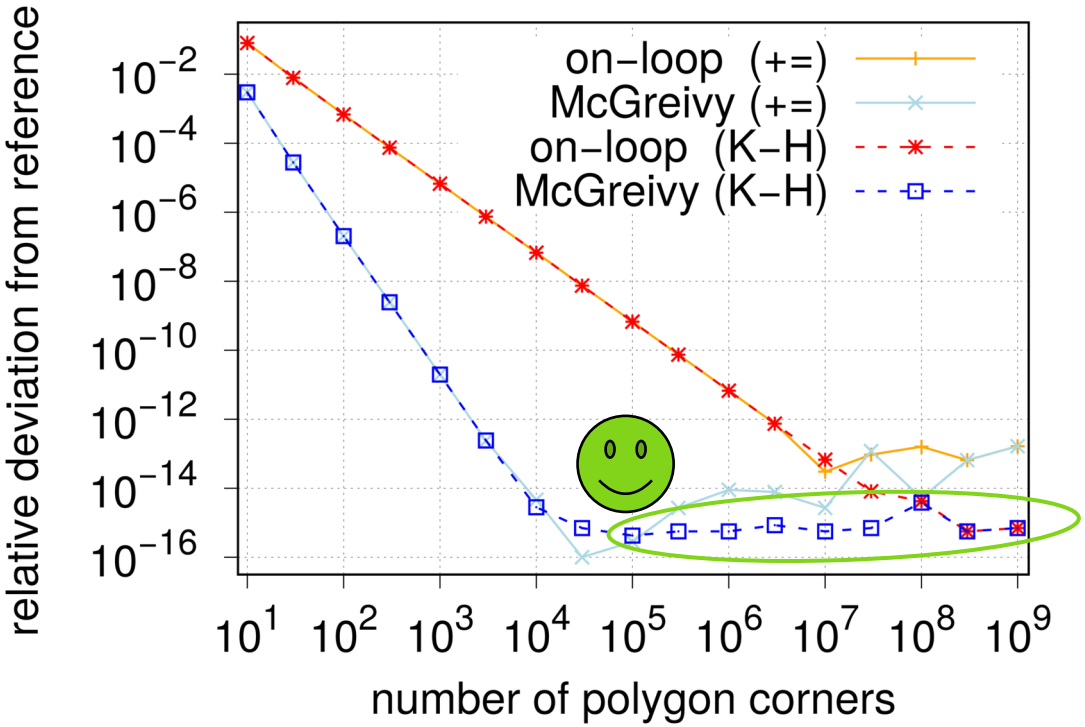
Cartesian Interface: results

- correct convergence is observed
- error accumulation for large number of polygon corners



Cartesian Interface: results

- correct convergence is observed
- error accumulation for large number of polygon corners
- compensated summation (Kahan-Babuška[1]) leads to correct result



[1] A. Klein, Computing **76**, 279 - 293 (2006)

Outline

- Introduction
- Methods
- Results
- Summary

Summary

- special-case formulations for **A** and **B** of
  - Straight Wire Segment
  - Circular Wire Loop
- test setup to yield accurate results
  - accurate to 13 ... 15 digits using `binary64`
  - correct asymptotic behavior
- application to polygon approximation
- intended to replace current Mag3D implementation