force operator
in real space

$$\mathcal{F}_\alpha = \frac{\partial}{\partial s} D_\alpha^{ss}(s) \frac{\partial}{\partial s} + D_\alpha^{\theta\theta}(s) \frac{\partial^2}{\partial \theta^2} + D_\alpha^{\phi\phi}(s) \frac{\partial^2}{\partial \phi^2}$$

force operator
in Fourier space

$$\tilde{F}_\alpha^{mn} = \frac{\partial}{\partial s} D_\alpha^{ss} \frac{\partial}{\partial s} - m^2 D_\alpha^{\theta\theta} - n^2 D_\alpha^{\phi\phi}$$

defining eqn for $Y_{R/Z}$ (precond'd forces)

$$\tilde{F}_\alpha^{mn} Y_\alpha^{mn} = \boxed{-} F_\alpha^{mn}(\mathbf{X})$$

radial discretization leads to tri-diagonal equation: $M_{ij}^{mn} \cdot Y_\alpha^{mn}(s_j) = F_\alpha^{mn}(s_i)$

M_{ij} needs to include additional "-" sign!

=> M_{ij} is the discretized version of $-\tilde{F}_\alpha$

$$\tilde{F}_{\alpha}^{mn} = \frac{\partial}{\partial s} D_{\alpha}^{ss} \frac{\partial}{\partial s} - m^2 D_{\alpha}^{\theta\theta} - n^2 D_{\alpha}^{\phi\phi}$$

$$M_{ij}^{mn} = \begin{cases} \dfrac{D_{\alpha}^{ss}(s_{i-1/2})}{\Delta s^2}, & j=i-1, \\ -\dfrac{D_{\alpha}^{ss}(s_{i-1/2}) + D_{\alpha}^{ss}(s_{i+1/2})}{\Delta s^2} - m^2 D_{\alpha}^{\theta\theta}(s_i) - n^2 D_{\alpha}^{\phi\phi}(s_i), & j=i, \quad (6b) \\ \dfrac{D_{\alpha}^{ss}(s_{i+1/2})}{\Delta s^2}, & j=i+1. \end{cases}$$

=> tri-diagonal eqn: $b_i Y(s_{i-1}) + d_i Y(s_i) + a_i Y(s_{i+1}) \approx F(s_i)$

In order to understand M_{ij}, need to understand what is meant by $\frac{\partial}{\partial s} D_{\alpha}^{ss} \frac{\partial}{\partial s}$ applied to Y

a)

$$-\frac{\partial Y_i^{1,[s]}(s)}{\partial s} D_{ss}^{Y^1}(s) \frac{\partial Y_j^{1,[s]}(s)}{\partial s}$$

(GVEC)

b)

$$\frac{\partial}{\partial s}\left(D_{\alpha}^{ss}(s)\frac{\partial Y}{\partial s}\right) = \frac{\partial D_{\alpha}^{ss}}{\partial s}\frac{\partial Y}{\partial s} + D_{\alpha}^{ss}\frac{\partial^2 Y}{\partial s^2}$$

(J.S.)

These expressions identical, except boundary conditions!

$$\frac{\partial}{\partial s}\left(D_\alpha^{ss}(s)\frac{\partial Y}{\partial s}\right) = \frac{\partial D_\alpha^{ss}}{\partial s}\frac{\partial Y}{\partial s} + D_\alpha^{ss}\frac{\partial^2 Y}{\partial s^2}$$

1 2 3 4

- D^{ss}_α is on half-grid
- Y is on full-grid
- result should approximate $F_\alpha(s_i)$ on full grid

1) $$\frac{\partial D_\alpha^{ss}}{\partial s} \approx \frac{D_\alpha^{ss}(s_{i+1/2}) - D_\alpha^{ss}(s_{i-1/2})}{\Delta s}$$

2) $$\frac{\partial Y}{\partial s} \approx \frac{Y(s_{i+1}) - Y(s_{i-1})}{2\Delta s}$$

1st order central: $$f_x(x,y) \approx \frac{f(x+h,y) - f(x-h,y)}{2h}$$

3) $$D_\alpha^{ss}(s_i) \approx \frac{D_\alpha^{ss}(s_{i+1/2}) + D_\alpha^{ss}(s_{i-1/2})}{2}$$

interpolate to full-grid from neighboring half-grid points

4) $$\frac{\partial^2 Y}{\partial s^2} \approx \frac{Y(s_{i+1}) - 2Y(s_i) + Y(s_{i-1})}{\Delta s^2}$$

2nd order central: $$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

... putting it all together: $$b_i Y(s_{i-1}) + d_i Y(s_i) + a_i Y(s_{i+1}) \approx F(s_i)$$

$$\frac{\partial}{\partial s}\left(D_\alpha^{ss}(s)\frac{\partial Y}{\partial s}\right) \approx \underbrace{\frac{D_\alpha^{ss}(s_{i-1/2})}{\Delta s^2}}_{b_i} Y(s_{i-1}) - \underbrace{\frac{D_\alpha^{ss}(s_{i-1/2}) + D_\alpha^{ss}(s_{i+1/2})}{\Delta s^2}}_{d_i} Y(s_i) + \underbrace{\frac{D_\alpha^{ss}(s_{i+1/2})}{\Delta s^2}}_{a_i} Y(s_{i+1})$$

$$M_{ij}^{mn} = \begin{cases} \boxed{\dfrac{D_\alpha^{ss}(s_{i-1/2})}{\Delta s^2}}, & j = i-1, \\ \boxed{-\dfrac{D_\alpha^{ss}(s_{i-1/2}) + D_\alpha^{ss}(s_{i+1/2})}{\Delta s^2} - m^2 D_\alpha^{\theta\theta}(s_i) - n^2 D_\alpha^{\phi\phi}(s_i)}, & j = i, \\ \boxed{\dfrac{D_\alpha^{ss}(s_{i+1/2})}{\Delta s^2}}, & j = i+1. \end{cases} \quad (6b)$$

... putting it all together: $\boxed{b_i} Y(s_{i-1}) + \boxed{d_i} Y(s_i) + \boxed{a_i} Y(s_{i+1}) \approx F(s_i)$

$$\frac{\partial}{\partial s}\left(D_\alpha^{ss}(s)\frac{\partial Y}{\partial s}\right) \approx \underbrace{\frac{D_\alpha^{ss}(s_{i-1/2})}{\Delta s^2}}_{b_i} Y(s_{i-1}) - \underbrace{\frac{D_\alpha^{ss}(s_{i-1/2}) + D_\alpha^{ss}(s_{i+1/2})}{\Delta s^2}}_{d_i} Y(s_i) + \underbrace{\frac{D_\alpha^{ss}(s_{i+1/2})}{\Delta s^2}}_{a_i} Y(s_{i+1})$$

... plus additional diagonal terms for θ and Φ