Introduction to Bayesian Concepts

Lecture 1

https://jonathantemplin.com/bayesian-psychometric-modeling-fall-2024/

Today’s Lecture Objectives

1. Bayesian Statistics: A Definition
2. Posterior Distributions
3. Bayesian Updating

...but, before we formally begin...

A Personal Pilgramage (Summer 2022)

Thomas Bayes (1701-1761)

Class Discussion: What is Bayesian?

Bayesian from Birth?

- Thomas Bayes was born in 1701 in London
- His father was a nonconformist minister
  - Nonconformists were Protestant Christians who did not conform to the doctrines of the Church of England
- Bayes was a Presbyterian minister
  - Presbyterians are a Protestant denomination that follows a democratic system of church government
- Bayes was a mathematician and philosopher
  - He was a fellow of the Royal Society
  - He was a friend of Richard Price, a famous moral philosopher
- Also…Bayes' methods make for a great children's book!:
  - https://www.amazon.com/Bayesian-Probability-Babies-Chris-Ferrie/dp/1492680796

The Basics of Bayesian Analyses

- Bayesian statistical analysis refers to the use of models where some or all of the parameters are treated as random components
  - Each parameter comes from some type of distribution
- The likelihood function of the data is then augmented with an additional term that represents the likelihood of the prior distribution for each parameter
  - Think of this as saying each parameter has a certain likelihood – the height of the prior distribution
- The final estimates are then considered summaries of the posterior distribution of the parameter, conditional on the data
  - In practice, we use these estimates to make inferences, just as is done when using non-Bayesian approaches (e.g., maximum likelihood/least squares)

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Why are Bayesian Methods Used?

- Bayesian methods get used because of the *relative* accessibility of one method of estimation (MCMC – to be discussed shortly)
- There are four main reasons why people use MCMC:

1. Missing data
2. Lack of software capable of handling large sized analyses
3. New models/generalizations of models not available in software
4. Philosoplyical Reasons (e.g., membership in the cult of Bayes)

Perceptions and Issues with Bayesian Methods

- The use of Bayesian statistics has been controversial, historically (but less so today)
  - The use of certain prior distributions can produce results that are biased or reflect subjective judgment rather than objective science
- Most MCMC estimation methods are computationally intensive
  - Until very recently, very few methods available for those who aren’t into programming in Fortran, C, or C++
- Understanding of what Bayesian methods had been very limited outside the field of mathematical statistics (but that is changing now)
- Over the past 20 years, Bayesian methods have become widespread – making new models estimable and becoming standard in some social science fields (quantitative psychology and educational measurement)

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How Bayesian Statistics Work

Bayesian methods rely on Bayes' Theorem

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} \propto P(B \mid A)P(A)$$

Here:

- $P(A)$ is the prior distribution (pdf) of A (i.e., WHY THINGS ARE BAYESIAN)
- $P(B)$ is the marginal distribution (pdf) of B
- $P(B \mid A)$ is the conditional distribution (pdf) of B, given A
- $P(A \mid B)$ is the posterior distribution (pdf) of A, given B

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A Live Bayesian Example

- Suppose we wanted to assess the probability of rolling a one on a six-sided die:

$$p_1 = P(D = 1)$$

- We then collect a sample of data $\boldsymbol{X} = \{0, 1, 0, 1, 1\}$
    - These are independent tosses of the die
- The posterior distribution of the probability of a one conditional on the data is:

$$P(p_1 \mid \boldsymbol{X})$$

- We can determine this via Bayes theorem:

$$P(p_1 \mid \boldsymbol{X}) = \frac{P(\boldsymbol{X} \mid p_1)P(p_1)}{P(\boldsymbol{X})} \propto P(\boldsymbol{X} \mid p_1)P(p_1)$$

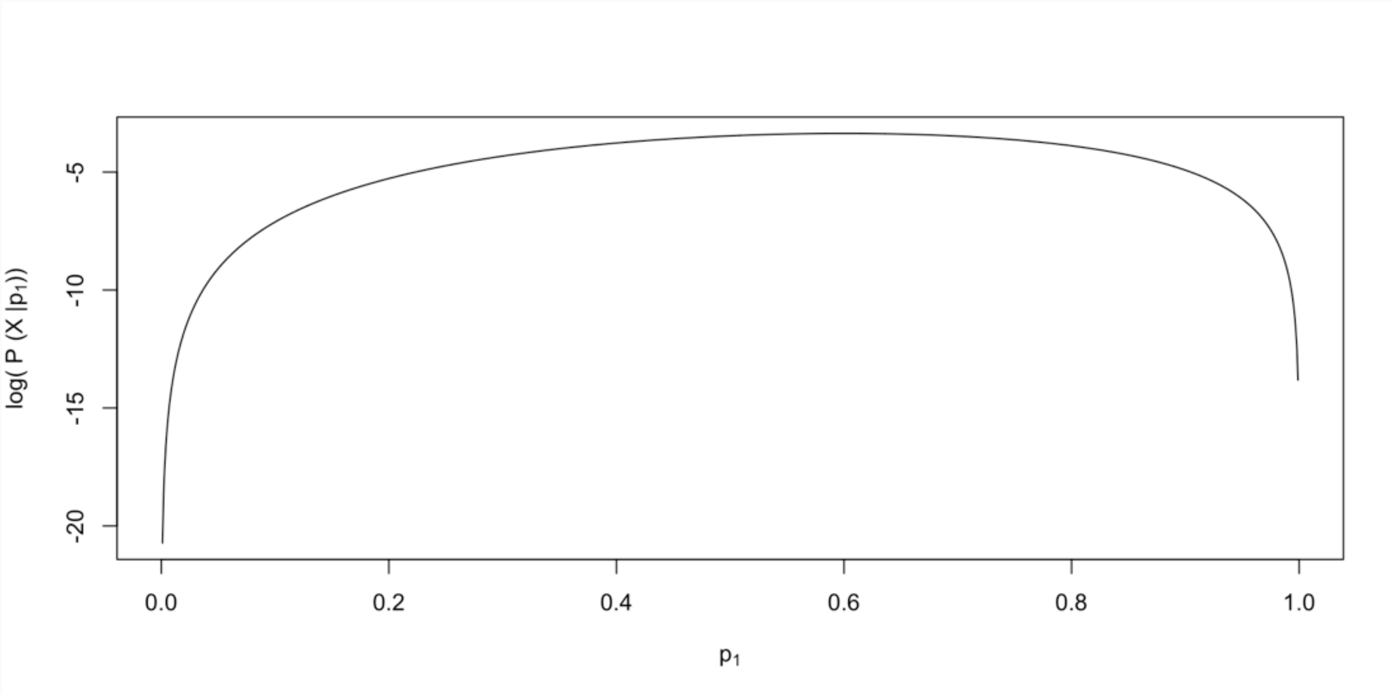
Defining the Likelihood Function $P(\boldsymbol{X} \mid p_1)$

The likelihood of the data given the parameter:

$$P(\boldsymbol{X} \mid p_1) = \prod_{i=1}^{N} p_1^{X_i} (1 - p_1)^{(1 - X_i)}$$

- Any given roll of the dice X_i is a Bernoulli variable $X_i \sim B(p_1)$
  - A "success" is defined by rolling a one
- The product in the likelihood function comes from each roll being independent
  - The outcome of a roll does not depend on previous or future rolls

Visualizing the Likelihood Function



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Choosing the Prior Distribution for p_1

We must now pick the prior distribution of p_1 :

$$P(p_1)$$

- Our choice is subjective: Many distributions to choose from
- What we know is that for a “fair” die, the probability of rolling a one is $\frac{1}{6}$
  - But…probability is not a distribution
- Instead, let’s consider a Beta distribution $p_1 \sim Beta(\alpha, \beta)$

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The Beta Distribution

For parameters that range between zero and one (or two finite end points), the Beta distribution makes a good choice for a prior:

$$P(p_1) = \frac{(p_1)^{\alpha-1}(1-p_1)^{\beta 1-1}}{B(\alpha,\beta)},$$

where:

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

and,

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$

More Beta Distribution

The Beta distribution has a mean of $\frac{\alpha}{\alpha+\beta}$

- The parameters α and β are called hyperparameters
  - Hyperparameters are parameters of prior distributions
- We can pick values of α and β to correspond to $\frac{1}{6}$
  - Many choices: $\alpha = 1$ and $\beta = 5$ have the same mean as $\alpha = 100$ and $\beta = 500$
- What is the difference?
  - How strongly we feel in our beliefs…as quantified by…

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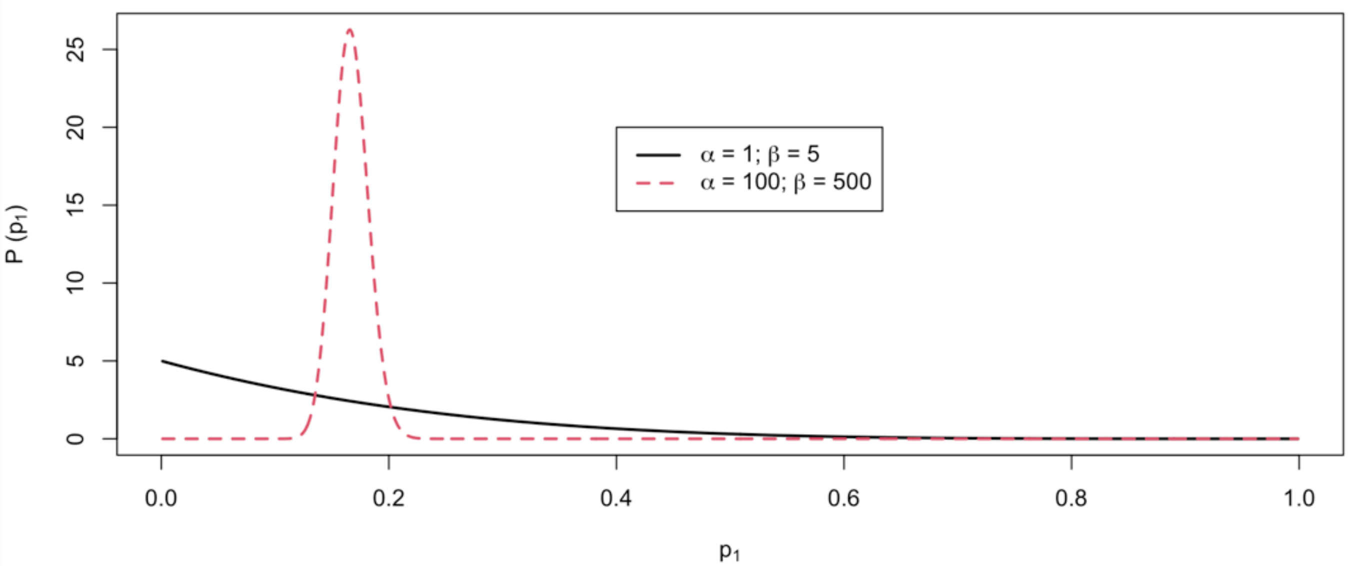
More More Beta Distribution

The Beta distribution has a variance of $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

- Choosing $\alpha = 1$ and $\beta = 5$ yields a prior with mean $\frac{1}{6}$ and variance 0.02
- Choosing $\alpha = 100$ and $\beta = 500$ yields a prior with mean $\frac{1}{6}$ and variance 0.0002
- The smaller prior variance means the prior is more informative
  - Informative priors are those that have relatively small variances
  - Uninformative priors are those that have relatively large variances

Visualizing $P(p_1)$



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The Posterior Distribution

Choosing a Beta distribution for a prior for p_1 is *very* convenient

- When combined with Bernoulli (Binomial) data likelihood the posterior distribution can be derived analytically
- The posterior distribution is also a Beta distribution
  - $\alpha = a + \sum_{i=1}^{N} X_i$ (a is the hyperparameter of the prior distribution)
  - $\beta = b + N - \sum_{i=1}^{N} X_i$ (b is the hyperparameter of the posterior distribution)
- The Beta prior is said to be a conjugate prior: A prior distribution that leads to a posterior distribution of the same family
  - Here, prior == Beta and posterior == Beta

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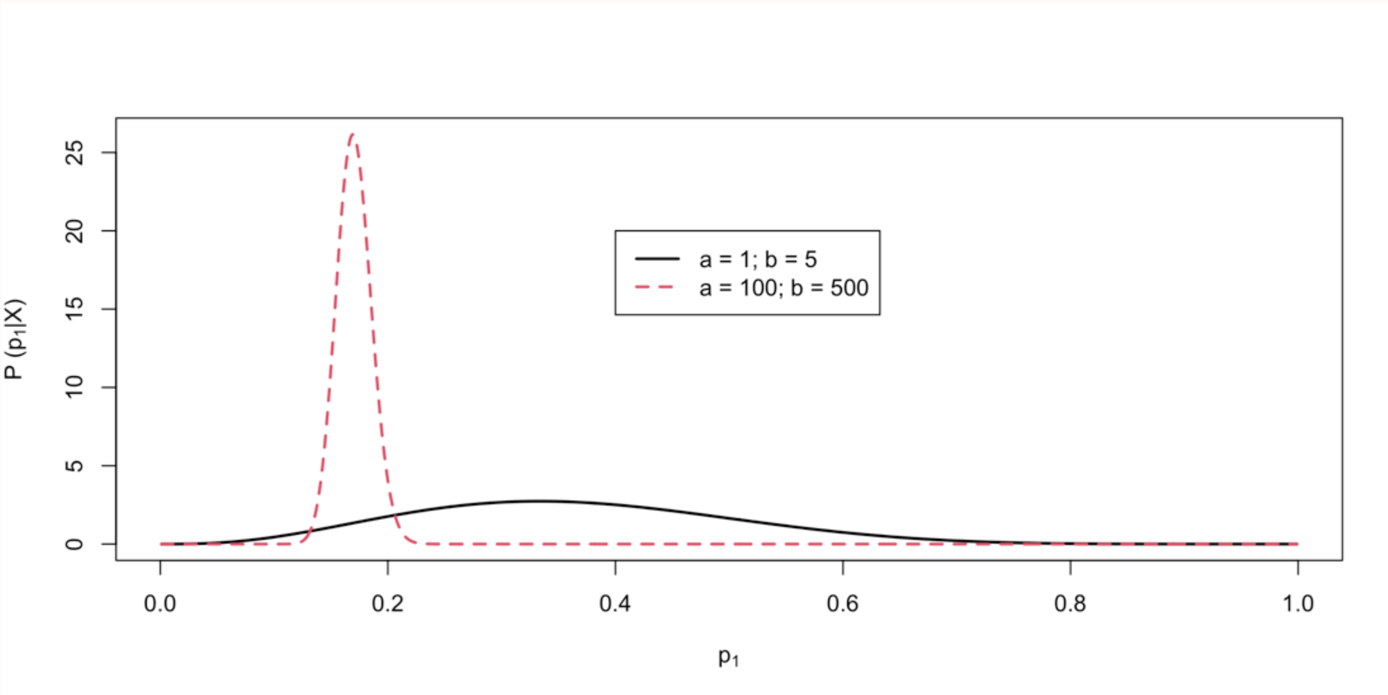
Visualizing The Posterior Distribution



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Bayesian Estimates are Summaries of the Posterior Distribution

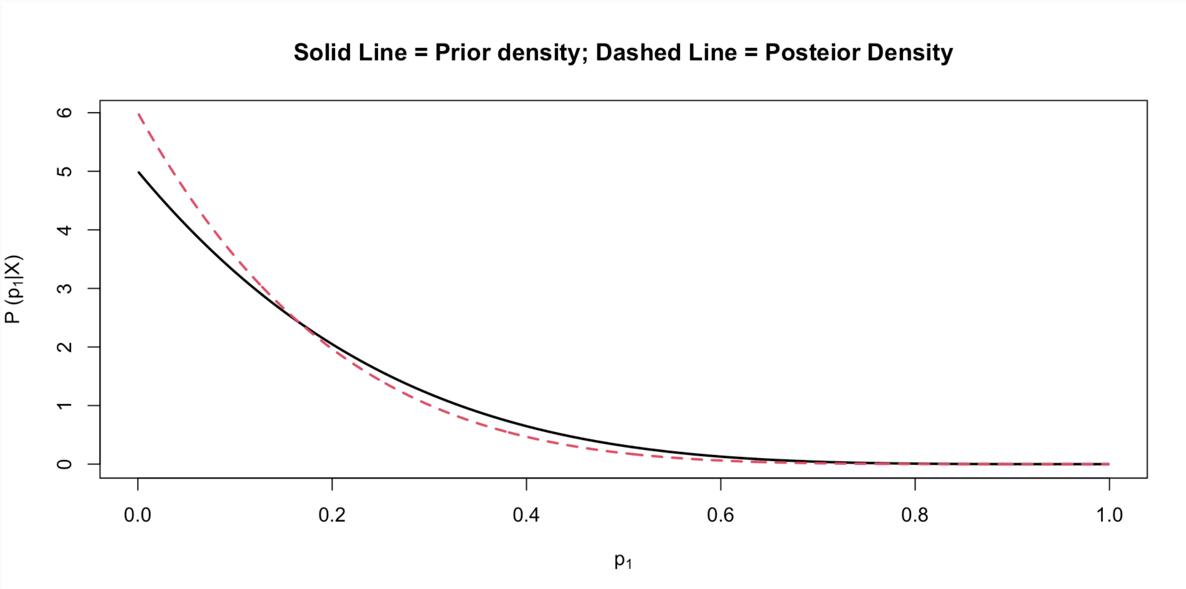
To determine the estimate of p_1 , we use summaries of the posterior distribution:

- With prior hyperparameters $a = 1$ and $b = 5$
  - $\hat{p_1} = \frac{1+3}{1+3+5+2} = \frac{4}{11} = .36$
  - SD = `0.1388659`
- With prior hyperparameters $a = 100$ and $b = 500$
  - $\hat{p_1} = \frac{100+3}{(100+3)+(500+2)} = \frac{103}{605} = .17$
  - SD = `0.0152679`
- The standard deviation (SD) of the posterior distribution is analogous to the standard error in frequentist statistics

Bayesian Updating

We can use the posterior distribution as a prior!

Let's roll a die to find out how...



Wrapping Up

Today was a very quick introduction to Bayesian concepts:

- prior distribution
  - hyperparameters
  - informative/uninformative
  - conjugate prior
- data likelihood
- posterior distribution
- Next we will discuss psychometric models and how they fit into Bayesian methods