Introduction to Multilevel Models (for Clustered Data)

- Topics:
  - What do multilevel models do?
  - From single-level to multilevel empty means models
  - Intraclass correlation (ICC) and design effects
  - Level-2 predictors
  - Level-1 predictors
  - Random slopes and cross-level interactions

Multilevel Models (MLMs) for Clustered* Data

- **Clustering = Nesting = Grouping = Hierarchies*
  - Key idea: Outcomes with >1 dimension of sampling simultaneously ("**micro**" units are nested in one or more types of "**macro**" units)
  - Each sampling dimension is considered its own "**level**" → **MLM**
  - MLMs can be used to predict outcomes from two-level (or more-level) sampling designs that result in nested and/or crossed observations

- The term "Multilevel Model" (MLM) has many synonyms:
  - **General Linear Mixed-Effects Models** (Fixed + Random = Mixed)
  - **Random Coefficients Models** (Random effects = latent variables)
  - **Hierarchical Linear Models** (HLM, but not = hierarchical regression)
  - Most MLM software is "univariate" → predict 1 outcome at a time
  - Multivariate MLMs can be estimated as "multilevel structural equation models" to predict 2+ outcomes at once (+ address missing predictors)

Labels for Organizing Models

- Outcome type: General (normal) vs. General*ized* (not normal)
- Dimensions of sampling: One (so one variance term per outcome) vs. **Multiple** (so multiple variance terms per outcome) → **OUR WORLD**

- **General Linear Models:** conditionally normal outcome distribution, **fixed effects** (identity link; only one dimension of sampling)

Note: Ordinary Least Squares is only for GLM

- **General*ized* Linear Models:** **any conditional outcome distribution**, **fixed** effects through **link functions**, no random effects (only one dimension)
- **General Linear Mixed Models:** conditionally normal outcome distribution, **fixed and random effects** (identity link, but **multiple dimensions** of sampling)
- **General*ized* Linear Mixed Models:** **any conditional outcome distribution**, **fixed and random effects** through **link functions** (**multiple dimensions**)
  - Same concepts as for general*ized* or mixed separately, but with more complexity in estimation

- "**Linear**" → fixed effects predict the *link-transformed* conditional mean of outcome in a **linear combination**: (effect*predictor) + (effect*predictor)…

Levels of Analysis in Two-Level Nested Data

- Between-Cluster (BC) Variation:
  - **Level-2** = "**INTER**-cluster differences" = cluster characteristics
- Within-Cluster (WC) Variation:
  - **Level-1** = "**INTRA**-cluster differences" = person characteristics
- **Any variable measured per person** could have **both** between-cluster and within-cluster variation!
  - **BC** = some clusters are higher/lower on average than other clusters
  - **WC** = some people are higher/lower than the rest of their cluster
  - Btw, univariate MLMs must address this differently for level-1 predictors vs. level-1 outcomes, but multivariate MLMs treat both the same way
- **So how do MLMs "handle" multiple levels of sampling?**

The Two Sides of *Any* Model

- **Model for the Means:**
  - **Fixed Effects**, the "structural" part (= latent variables means)
  - What you are used to **caring about for testing hypotheses**
  - How the expected outcome for a given observation varies as a function of their values for the predictor variables

- **Model for the Variance:**
  - **Random Effects and Residuals**, the "stochastic" or "error" part
    - Btw, random effect variances = latent variable variances
  - What you are used to **making assumptions about** instead
  - How residuals are distributed and related across observations (persons, clusters, items, etc.) → these relationships are called "dependency" and ***this is the primary way that multilevel models differ from general linear models*** *(GLMs; "regression")*

Two Sides of a General Linear Model (GLM)

p = person

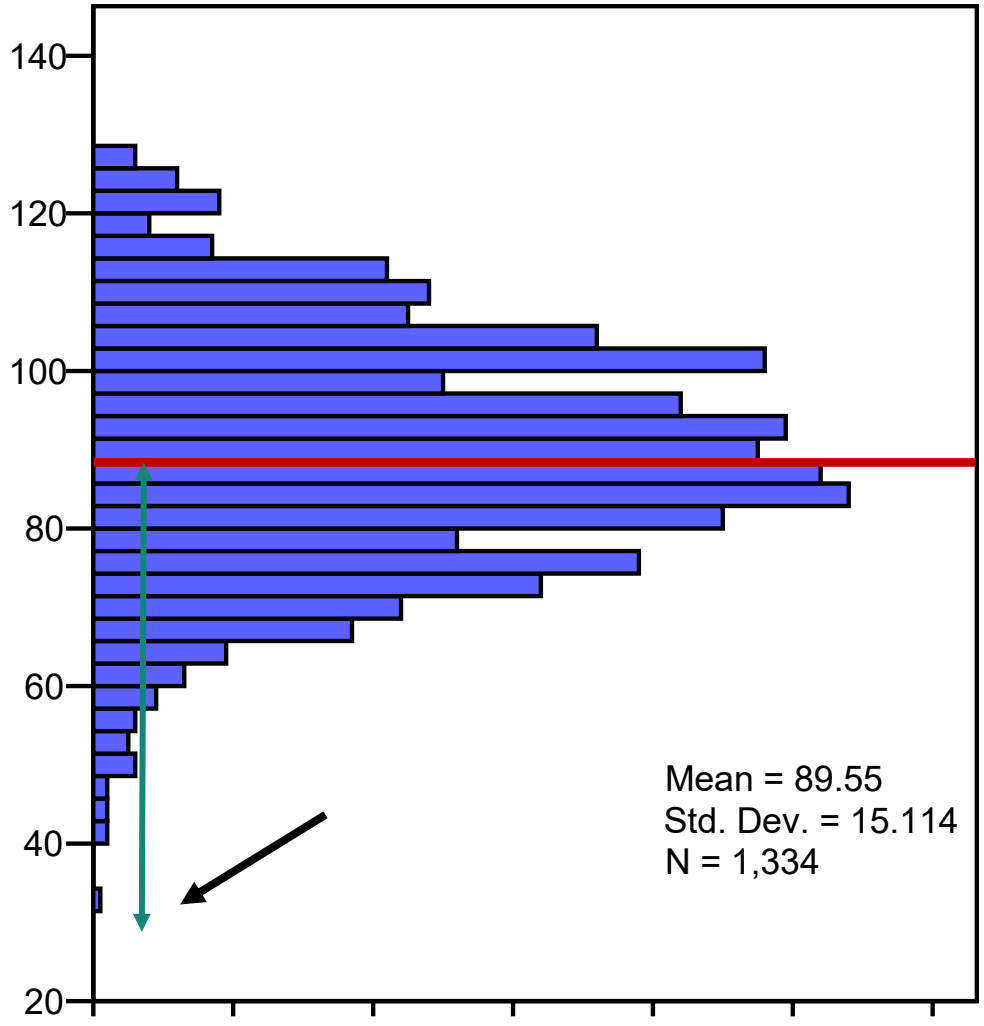
$$y_p = \beta_0 + \beta_1(x1_p) + \beta_2(x2_p) + \cdots + e_p$$

Our focus

- **Model for the Means (→ Predicted Values):**
  - Each person's expected (predicted) outcome is a weighted linear function of his/her values on $x1_p$ and $x2_p$ (and any other predictors); each variable is measured once per person (given by the p subscript)
  - **Estimated β constants** are called **fixed effects** (intercept or slopes)

- **Model for the Variance (→ "Piles" of Variance):**
  - $\boldsymbol{e_p \sim N(0, \sigma_e^2)}$ → ONE (between-person) source of unexplained variation
  - In GLMs, e_p has a mean of 0 with some estimated constant variance σ_e^2, is normally distributed, is unrelated to $x1_p$ and $x2_p$, and is **independent** across all observations (which is just one outcome per person here)
  - **There is only ONE source of residual variance in the above GLM because it was designed for only ONE dimension of sampling!**

An "Empty Means" General Linear Model → **Single-Level** Model *for the Variance*



$$y_p = \beta_0 + e_p$$

Filling in values:

$$32 = \underbrace{90}_{\hat{y}_p} + -58$$

$\hat{y}_p$ = "y-hat" model-predicted outcome

Model for the Means

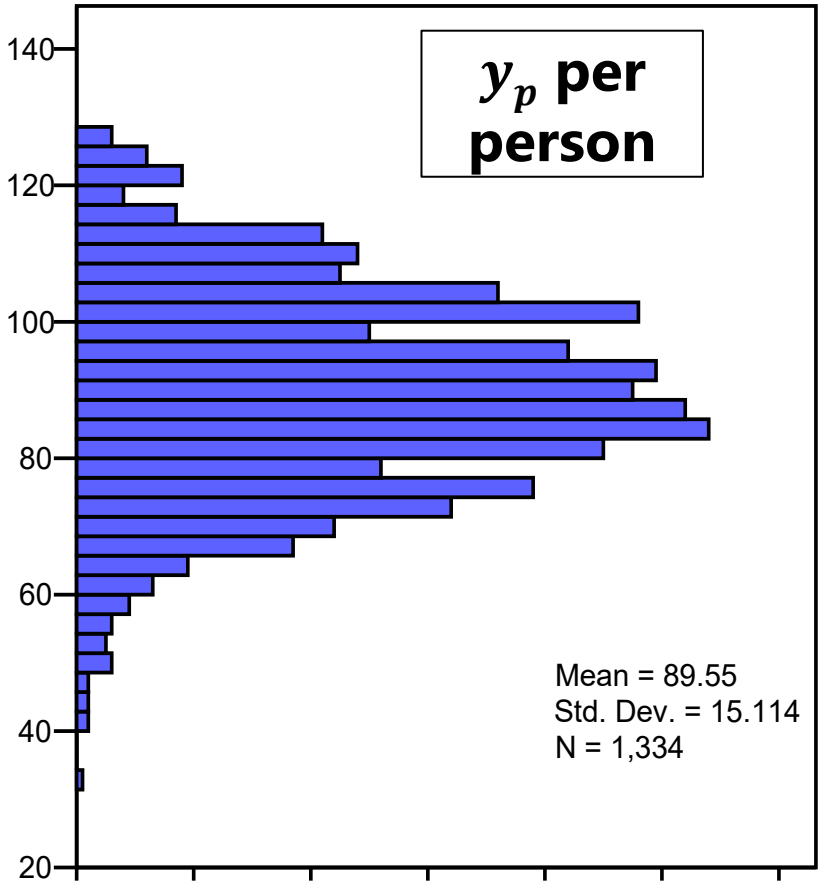
y_p residual variance:

$$\sigma_e^2 = \frac{\sum(y_p - \hat{y}_p)^2}{N-1}$$

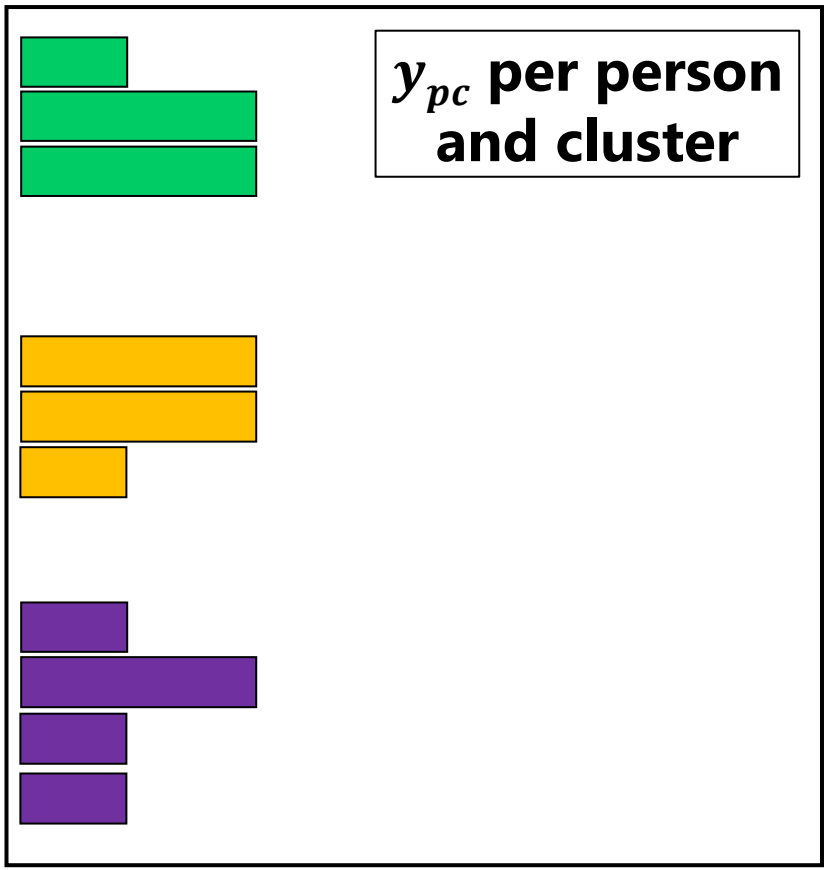
Without predictors, $\sigma_e^2 = \sigma_y^2$

Adding Multiple Persons per Cluster → **Two-Level** Model *for the Variance*

Full Sample Distribution



3 Clusters (c), 5 Persons (p)



Empty Means, Two-Level Model *for the Variance*

From a **one-level** to a **two-level model** for the variance:

Start off with the outcome's mean as a "best guess" for any outcome's value:

= Grand Mean

→ **Fixed Intercept**

Can make *better* guess by taking advantage of cluster-common information:

= Cluster Mean

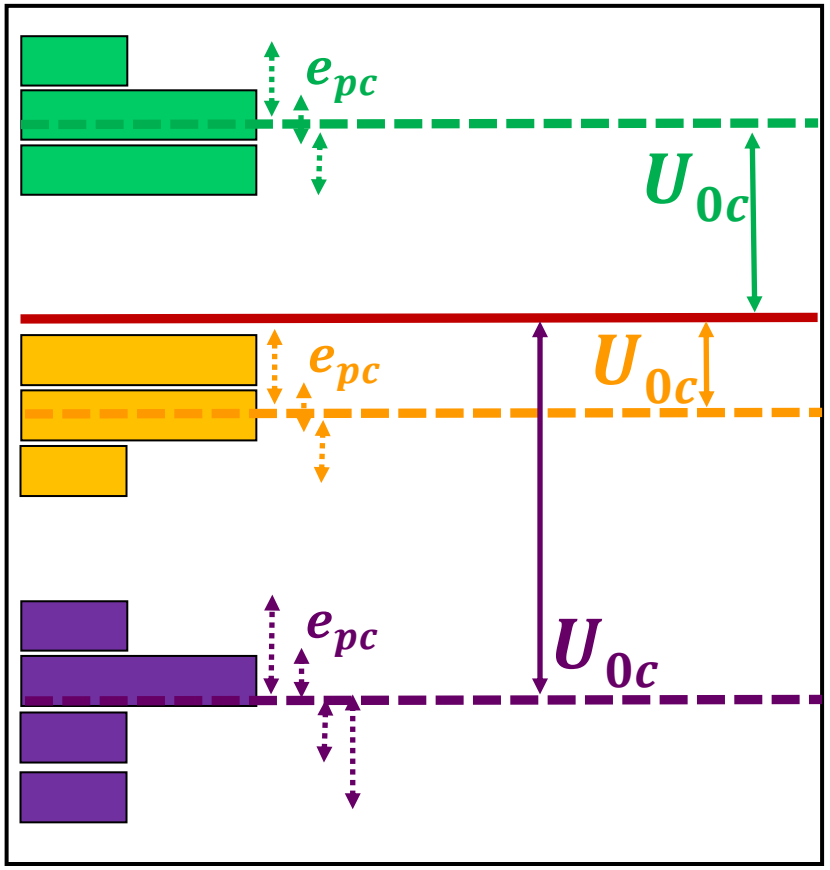
→ **Random Intercept**

Empty Means, Two-Level Model *for the Variance*

β_0 → **mean of cluster means**

y_{pc} **variance → 2 sources:**

$$y_{pc} = \beta_0 + U_{0c} + e_{pc}$$



Level-2 Random Intercept U_{0c} (with variance labeled $\tau^2_{U_0}$):

- **Between**-cluster (BC) variance
- **INTER**-cluster differences to be explained by cluster predictors

Level-1 Residual e_{pc} per person (with variance labeled σ^2_e):

- **Within**-cluster (WC) variance
- **INTRA**-cluster differences to be explained by person predictors

Two-Level Model Using Multilevel Notation: Empty Means, Random Intercept Model

GLM Empty Model:

$$y_p = \beta_0 + e_p$$

MLM Empty Model:

- Level 1:

$$y_{pc} = \beta_{0c} + e_{pc}$$

- Level 2:

$$\beta_{0c} = \gamma_{00} + U_{0c}$$

3 total parameters:

Model for the Means (1):

- Fixed Intercept γ_{00}

Model for the Variance (2):

- Level-1 **WC** Variance of e_{pc} → σ_e^2
- Level-2 **BC** Variance of U_{0c} → $\tau_{U_0}^2$

L1 Residual = person-specific deviation from cluster-predicted outcome

Fixed Intercept = mean of cluster means (because no predictors yet)

L2 Random Intercept = cluster-specific deviation from predicted intercept

Composite equation:

$$y_{pc} = \gamma_{00} + U_{0c} + e_{pc}$$

Intraclass Correlation (ICC)

$$\text{ICC} = \frac{\text{BC}}{\text{BC} + \text{WC}} = \frac{\text{L2 Intercept Var}}{\text{L2 Intercept Var} + \text{L1 Residual Var}}$$

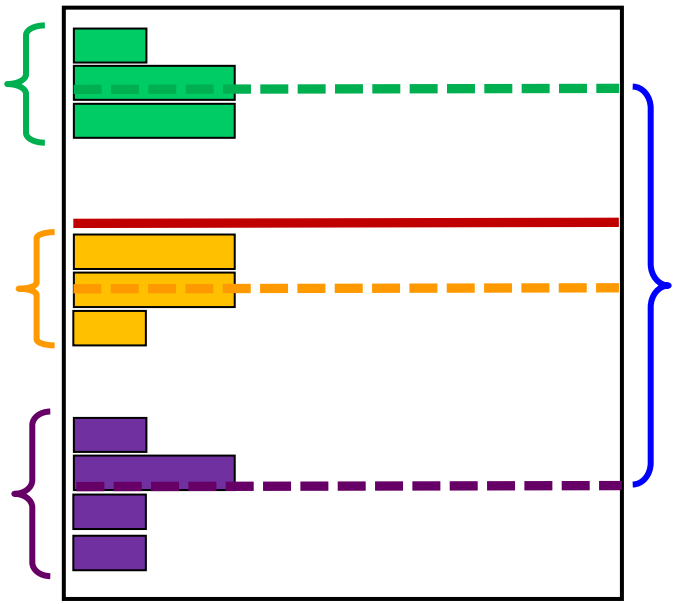
$$= \frac{\boldsymbol{\tau^2_{U_0}}}{\boldsymbol{\tau^2_{U_0} + \sigma^2_e}}$$

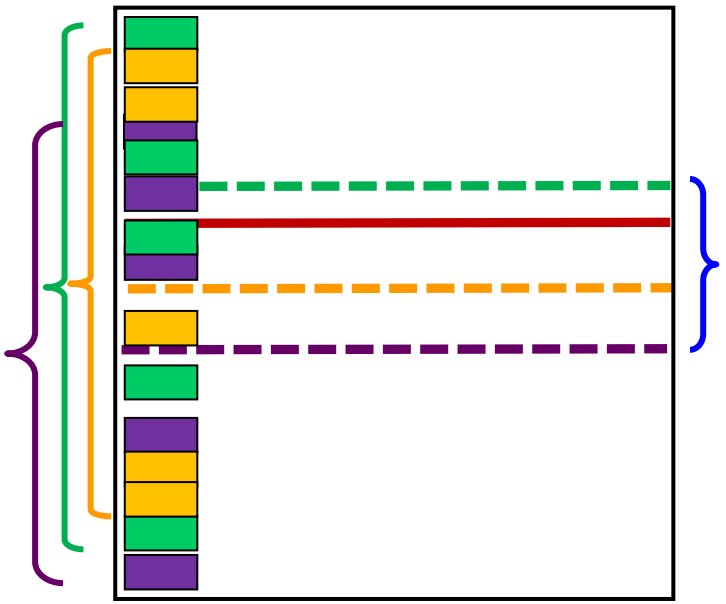
$\boldsymbol{\tau^2_{U_0}}$ → *Why don't all clusters have the same mean?*

$\boldsymbol{\sigma^2_e}$ → *Why don't all people from the same cluster have the same outcome?*

- ICC = Proportion of total variance that is between clusters
- ICC = Average correlation of persons from same cluster
- ICC is a standardized way of expressing how much *dependency (correlation)* there is due to cluster mean differences
  → **ICC is an effect size for *constant* cluster dependency**
  - Dependency of other kinds can still be created by differences across clusters in the slopes of person predictors (stay tuned!)
- Btw, no variance has been "explained" yet (just 2 kinds of "error")

Even though **between-cluster variance** is the numerator, **ICC = within-cluster** correlation!





ICC = BTW / BTW + within

→ Large ICC

→ Large correlation within clusters

$$\text{ICC} = \frac{\tau_{U_0}^2}{\tau_{U_0}^2 + \sigma_e^2}$$

ICC = btw / btw + WITHIN

→ Small ICC

→ Small correlation within clusters

Effects of Clustering on Effective N

- **Design Effect** expresses how much sample size needs to be adjusted due to clustering → "**effective sample size**"
- **Design Effect** = ratio of the variance using a given sampling design to the variance using a simple random sample from the same population, given the same total sample size either way

- Design Effect = $1 + ([L1n - 1] * \text{ICC})$ $\boxed{L1n = \#\text{ level-1 units}}$
- Effective sample size → $\text{Effective } N = \frac{\#\text{ Total Observations}}{\text{Design Effect}}$

- As ICC and cluster size go UP, effective N goes DOWN
  - See Snijders & Bosker (2012) for more info and for a modified formula that takes unequal group sizes into account

Demonstrating Two-Level Design Effects

- Design Effect = $1 + ([L1n - 1] * \text{ICC})$
- Effective sample size → Effective $N = \frac{\#\ \text{Total Observations}}{\text{Design Effect}}$

- $n = 5$ patients from each of 100 doctors, ICC = .30?
  - Patients Design Effect = 1 + ([5−1] * .30) = 2.20
  - **Effective N** = 500 / 2.20 = **227** (not 500)

- $n = 20$ students from each of 50 schools, ICC = .05?
  - Students Design Effect = 1 + ([20−1] * .05) = 1.95
  - **Effective N** = 1000 / 1.95 = **513** (not 1000)

Does a non-significant ICC mean you can ignore clustering and just do a regression?

- As ICC and cluster size go UP, effective N goes DOWN
  - So there is NO VALUE OF ICC that is "safe" to ignore, not even ~0!
  - An ICC=0 in an *empty means (unconditional)* model can become ICC>0 after adding person predictors because reducing the residual variance will then increase the random intercept variance (→ *conditional* ICC > 0)
    - Design effects can increase after including good person predictors!

- So just plan to do a multilevel analysis anyway…
  - Even if "that's not your question"… because people are in clusters, we still need to address **cluster dependency** (= **correlation**) because of:
    - Effect on person predictor fixed slope SEs → **biased SEs**
    - Potential for **contextual effects** of person predictors (stay tuned!)
      - A "clustered-sampling correction" to the SEs **will not fix this problem**!

2 Options for Cluster Differences

Represent Cluster Differences via Fixed Effects

- Include (#clusters−1) binary predictors for cluster membership in the **model for the means** → **so cluster is NOT a model "level"**
  - Main effects control for cluster mean differences only; interactions with person predictors are also needed to control for cluster slope differences
- Useful if #clusters < 10ish or you care about specific clusters, but then you cannot include cluster predictors → saturated mean diffs

Represent Cluster Differences via Random Effects

- Include a random intercept variance across clusters in the **model for the variance** → **then cluster IS a model "level"**
  - A random intercept controls for cluster mean differences only; a random slope variance is needed for cluster differences in person predictor slopes
- Better if #clusters > 10ish or you want to **predict** cluster differences
- Our examples will take this approach

Adding Level-2 Cluster Predictors

- **Level-2 predictors** are constant over persons from the same cluster—they are cluster-level characteristics
  - Example: Level-1 (L1) students (p) nested in level-2 (L2) schools (c)

- **Level-1:** $y_{pc} = \beta_{0c} + e_{pc}$

σ_e^2 → All possible L1 residual variance for within-school differences across students

- "**Unconditional**" **Level-2** (before cluster predictors):
  - $\beta_{0c} = \gamma_{00} + U_{0c}$

$\tau_{U_0}^2$ → **All possible L2 random intercept variance** due to school mean differences

- "**Conditional**" **Level-2** (after cluster predictors):
  - $\beta_{0c} = \gamma_{00} + \gamma_{01}(L2x1_c) + \gamma_{02}(L2x2_c) + U_{0c}$

$\tau_{U_0}^2$ → L2 random intercept variance **leftover** now

  - First subscript = which beta in level-1 model
    Second subscript = order of predictor in level-2 model

Effect Size for Level-2 Cluster Predictors

- Direct: convert t-statistic for fixed effect into d or partial r
  - $d = \frac{2t}{\sqrt{DF_{den}}}, \quad r = \frac{t}{\sqrt{t^2 + DF_{den}}}$

Note: These formulas can be used with any model (multilevel or not)

- Indirect: explained variance of two complementary kinds
  - **Pseudo-R^2**: amount of variance explained *per variance component*
    - Pseudo-$R^2 = \frac{variance_{fewer} - variance_{more}}{variance_{fewer}}$

"fewer" = model with fewer parameters
"more" = model with more parameters

    - It can go negative if adding useless predictors or if the level-1 model is mis-specified (stay tuned!); these problems can be remedied by calculating it with model-implied total variance instead (see Rights & Sterba, 2019; 2020)
    - Only pseudo-R^2 for the L2 random intercept var is relevant for L2 predictors
  - **Total-R^2**: amount of total variance explained (across piles)
    - Generate model-predicted $\hat{y}_{pc}$ values from fixed effects ONLY and correlate them with observed outcomes; square that correlation to get total-R^2

Part 1: Summary

- MLMs begin with an empty model to determine how much outcome variance is attributable to each dimension of sampling:
  - Level-2 between-cluster mean differences → random intercept ($\boldsymbol{\tau^2_{U_0}}$)
  - Level-1 within-cluster person differences → residual ($\boldsymbol{\sigma^2_e}$)
  - Dependency effect size via Intraclass Correlation: $\mathbf{ICC} = \boldsymbol{\tau^2_{U_0}}$ **/** $(\boldsymbol{\tau^2_{U_0}} + \boldsymbol{\sigma^2_e})$
    - ICC = proportion of total variance due to cluster mean differences
    - ICC = average correlation of persons from same cluster
    - Higher ICC and level-1 sample size → larger design effect → smaller effective N

- Modeling cluster differences using random effects (by including $\boldsymbol{\tau^2_{U_0}}$ at a minimum, possibly random slope variances, stay tuned!) allows us to test the effects of level-2 between-cluster predictors
  - Significance tests via Wald tests (usually with denominator DF) as usual
  - Adding fixed slopes for level-2 predictors (cluster characteristics) can explain level-2 random intercept variance (cluster mean differences)
    - Reduction in **level-2 intercept variance** can be quantified by **pseudo-$\boldsymbol{R^2_{U_0}}$**
    - Reduction in **total variance** can be quantified by **total-$\boldsymbol{R^2}$** (≈ pseudo-$R^2_{U_0}$* ICC)

Level-1 Predictors: What **Not** to Do!

- Level-2 predictors ($\boldsymbol{L2x_c}$ below) are **cluster** characteristics
- Level-1 predictors ($\boldsymbol{L1x_{pc}}$ below) are **person** characteristics
  - *What if we added a L1 predictor directly (as we did before at L2)?*

Level-1: $\boldsymbol{y_{pc} = \beta_{0c} + \beta_{1c}(L1x_{pc}) + e_{pc}}$

Level-2: $\boldsymbol{\beta_{0c} = \gamma_{00} + \gamma_{01}(L2x_c) + U_{0c}}$

$\boldsymbol{\beta_{1c} = \gamma_{10}}$

$\boldsymbol{\gamma_{00}}$ = fixed intercept (at pred=0)
$\boldsymbol{\gamma_{01}}$ = fixed slope of $L2x_c$
$\boldsymbol{\gamma_{10}}$ = fixed slope of $L1x_{pc}$
$\boldsymbol{U_{0c}}$ = level-2 random intercept
$\boldsymbol{e_{pc}}$ = level-1 residual

  - First subscript = which beta in level-1 model
    Second subscript = order of predictor in level-2 model

- All good, right? Many researchers mistakenly think so, but this model is **VERY LIKELY to be mis-specified**…
  - … For the **exact same reasons** we need MLM in the first place!

Level-1 (Person-Level) Predictors

- Modeling level-1 predictors is complicated (and often done incorrectly) because **each level-1 predictor is usually really 2 predictor variables** (each with their own slope), **not 1**

- Textbook example: Student Socioeconomic Status (SES)
  - Some **kids** have higher SES than others in their school:
    - **L1 WC variation in SES** *(represented directly as deviation from school mean)*
  - Some **schools** have more high-SES students than other schools:
    - **L2 BC variation in SES** *(represented as school mean or via external info)*

- Can quantify each source of variance with an empty model ICC
  - ICC = (L2 between variance) / (L2 between variance + L1 within variance)
  - **ICC < 1?** L1 predictor has **WC** variation (so it *could* have a **L1 WC** slope)
  - **ICC > 0?** L1 predictor has **BC** variation (so it *could* have a **L2 BC** slope)

Between- vs. Within-Cluster Effects

- Between- and within-cluster slopes in SAME direction
  - SES → Achievement in students
    - **WC: Kids with more money than other kids in their school may have greater achievement than other kids in their school (regardless of school mean SES)**
    - **BC: Schools with more money than other schools may have greater mean achievement than schools with less money**

- Between- and within-cluster slopes in OPPOSITE directions
  - Body mass → life expectancy in animals (Curran and Bauer, 2011)
    - **WC: Within a species, relatively bigger animals have shorter life expectancy (e.g., over-weight ducks die sooner than healthy-weight ducks)**
    - **BC: Larger species tend to have longer life expectancies than smaller species (e.g., whales live longer than cows, cows live longer than ducks)**

- L1 within-cluster and L2 between-cluster slopes usually differ
  - Why? Because variables have different **meanings** at each level!
  - Why? Because variables have different **scales** at each level!

What **Not** to Do: Smushed Effects!

Level-1: $y_{pc} = \beta_{0c} + \beta_{1c}(L1x_{pc}) + e_{pc}$

Level-2: $\beta_{0c} = \gamma_{00} + U_{0c}$

$\beta_{1c} = \gamma_{10}$

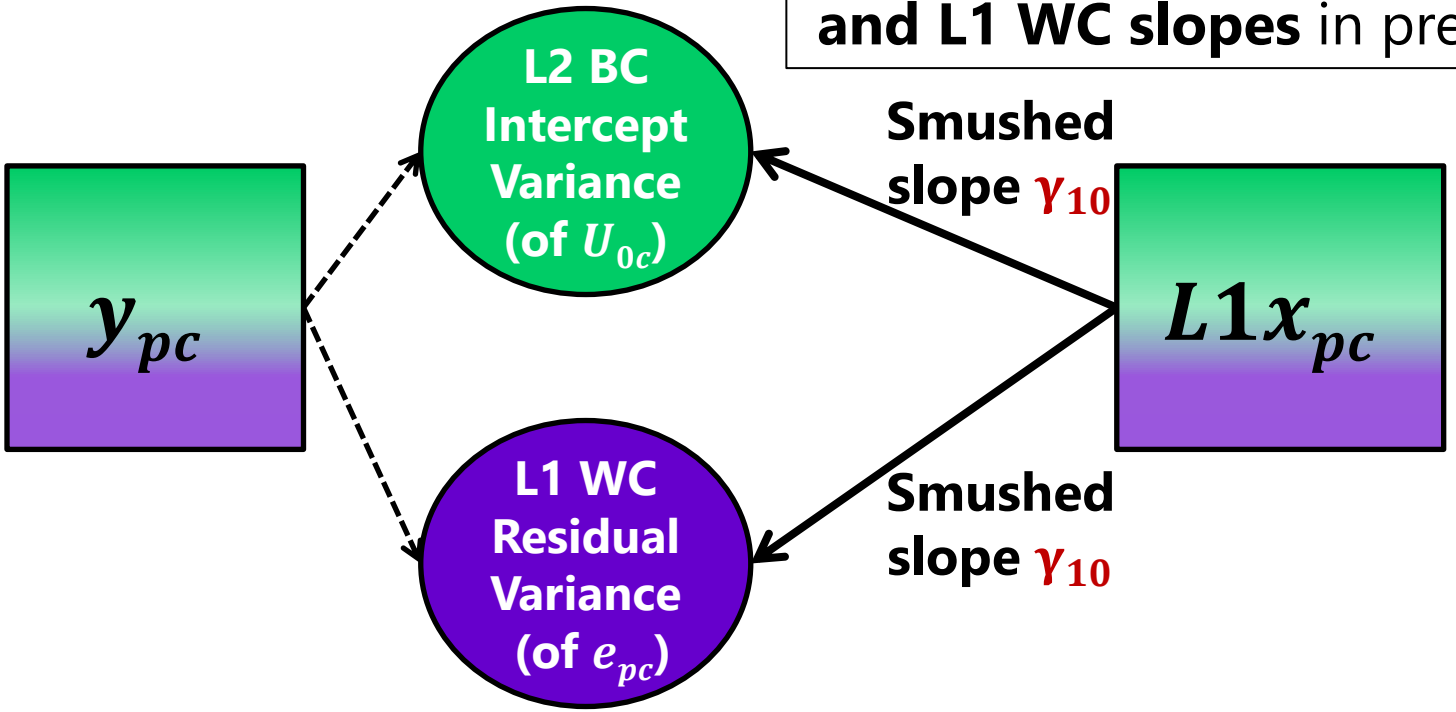
γ_{10} = **smushed effect** (see also *conflated*, *convergence*, or *composite* effect) that assumes equal within- and between-cluster slopes

- If level-1 predictor has both level-2 between and level-1 within variation, then its **one fixed slope has to do the work of two predictors**!
- A **smushed effect** is a **weighted combination of the L1 within and L2 between slopes,** usually **closer to the L1 within slope** (due to larger $L1n$), and thus the L2 between model will be more affected by smushing
- Btw, **smushing** is seen in econometrics (aka, "**endogeneity**" problem) in the context of when to model cluster dependency using fixed effects (i.e., turn cluster ID into a categorical predictor) instead of a random intercept
  - A **smushed effect creates correlation** between the L1 predictor and the L2 random intercept (because the **predictor's L2 effect is modeled wrong**)
  - Smushing is solved when using **fixed effects for cluster ID,** such that the L2 effect of the L1 predictor is then controlled for in "common" variance
  - But we can still avoid smushed effects when using a cluster random intercept…. Next are the 3 main ways to do so!

Univariate MLM: Adding a Level-1 Predictor Without Addressing Level-2 Part = Smushing

BC and WC variance in the **observed level-1 y_{pc} outcome** is partitioned by the **model** into estimated **variance components**

Observed level-1 $L1x_{pc}$ predictor still has both BC and WC variance. AND given that $L1x_{pc}$ has only **one fixed slope**, it captures a smushed effect that presumes **equal L2 BC and L1 WC slopes** in predicting y_{pc}!



3 Kinds of Fixed Slopes for L1 Predictors

- **Is there a Level-1 Within-Cluster (WC) slope?**
  - If you have a higher $\boldsymbol{L1x_{pc}}$ predictor value *than others in your cluster*, do you also have a higher (or lower) $\boldsymbol{y_{pc}}$ outcome value *than others in your cluster*?
  - If so, the **level-1 within-cluster *part* of the L1 predictor** will reduce the level-1 residual variance ($\boldsymbol{\sigma_e^2}$) of the $\boldsymbol{y_{pc}}$ outcome

- **Is there a Level-2 Between-Cluster (BC) slope?**
  - Do clusters with higher average $\boldsymbol{L1x_{pc}}$ predictor values *than other clusters* also have higher (or lower) average $\boldsymbol{y_{pc}}$ outcomes *than other clusters*?
  - If so, the **level-2 between-cluster *part* of the L1 predictor** will reduce level-2 random intercept variance ($\boldsymbol{\tau_{U_0}^2}$) of the $\boldsymbol{y_{pc}}$ outcome

- **Is there a Level-2 Contextual slope: Do the L2 BC and L1 WC slopes differ?**
  - After controlling for the actual value of L1 predictor, is there still **an incremental contribution** from the **level-2 between-cluster *part* of the L1 predictor** (i.e., does a cluster's general tendency matter beyond a person's $\boldsymbol{L1x_{pc}}$ value)?
  - Equivalently, the **Level-2 Contextual slope = L2 BC slope – L1 WC slope**, so the Level-2 Contextual slope directly tests **if a smushed slope is ok (pry not!)**

3 Options to Prevent Smushed Slopes

- Within Univariate MLM framework (predict only one outcome):
  1. **Cluster-mean-centering**: manually carve up L1 predictor into its level-specific parts using observed variables (1 predictor per level)
     - More generally, this is "**variable-centering**" because you are **subtracting a variable** (e.g., the cluster mean here; could use other cluster variables)
     - Will always yield **level-1 within slopes** and **level-2 between slopes**!
  2. **Grand-mean-centering**: do NOT carve up L1 predictor into its level-specific parts, but add level-2 mean to distinguish level-specific slopes
     - More generally, this is "**constant-centering**" because you are **subtracting a constant** while still keeping all levels of variance in the L1 predictor
     - **Choice of constant is irrelevant** (changes where 0 is, not what variance it has)
     - Will always yield **level-1 within slopes** and **level-2 contextual slopes**!
- Within Multivariate MLM framework (i.e., via Multilevel-SEM):
  3. **Latent-centering**: Treat the L1 predictor as another outcome → let the model carve it up into **level-specific latent variables**
     - Best in theory, but the type of level-2 slope (between or contextual) depends on model type, syntax type, and the estimator in Mplus! (Hoffman, 2019)

Option 1. Cluster-Mean-Centering (C-MC)

- We partition the L1 predictor $\boldsymbol{L1x_{pc}}$ into two variables that directly represent its **L2 between**-cluster (BC) and **L1 within**-cluster (WC) sources of variation, and **include these variables as the predictors:**

- **Level-2 Between predictor = cluster mean of $L1x_{pc}$**
  - $\boldsymbol{CMx_c = \overline{L1x_c} - C_2}$
  - CMx_c is centered at constant C_2, chosen for meaningful 0 (e.g., sample mean)
  - CMx_c is positive? Above sample mean → "more than other clusters"
  - CMx_c is negative? Below sample mean → "less than other clusters"

- **Level-1 Within predictor = deviation from cluster mean of $L1x_{pc}$**
  - $\boldsymbol{WCx_{pc} = L1x_{pc} - \overline{L1x_c}}$ *(uncentered cluster mean $\overline{L1x_c}$ is used)*
  - WCx_{pc} is NOT centered at a constant – **we subtract a VARIABLE instead**
  - WCx_{pc} is positive? Above your cluster mean → "more than my cluster"
  - WCx_{pc} is negative? Below your cluster mean → "less than my cluster"

Cluster-MC L1 Predictor + Cluster Mean

→ WC and BC effects directly through **separate** parameters

$L1x_{pc}$ is cluster-mean-centered into WCx_{pc}, with CMx_c at L2:

Level-1: $y_{pc} = \beta_{0c} + \beta_{1c}(WCx_{pc}) + e_{pc}$

$WCx_{pc} = L1x_{pc} - \overline{L1x_c}$ → only has L1 within variation

Level-2: $\beta_{0c} = \gamma_{00} + \gamma_{01}(CMx_c) + U_{0c}$

$\beta_{1c} = \gamma_{10}$

$CMx_c = \overline{L1x_c} - C_2$ → only has L2 between variation

γ_{10} = within effect of having more $L1x_{pc}$ *than others in your cluster*

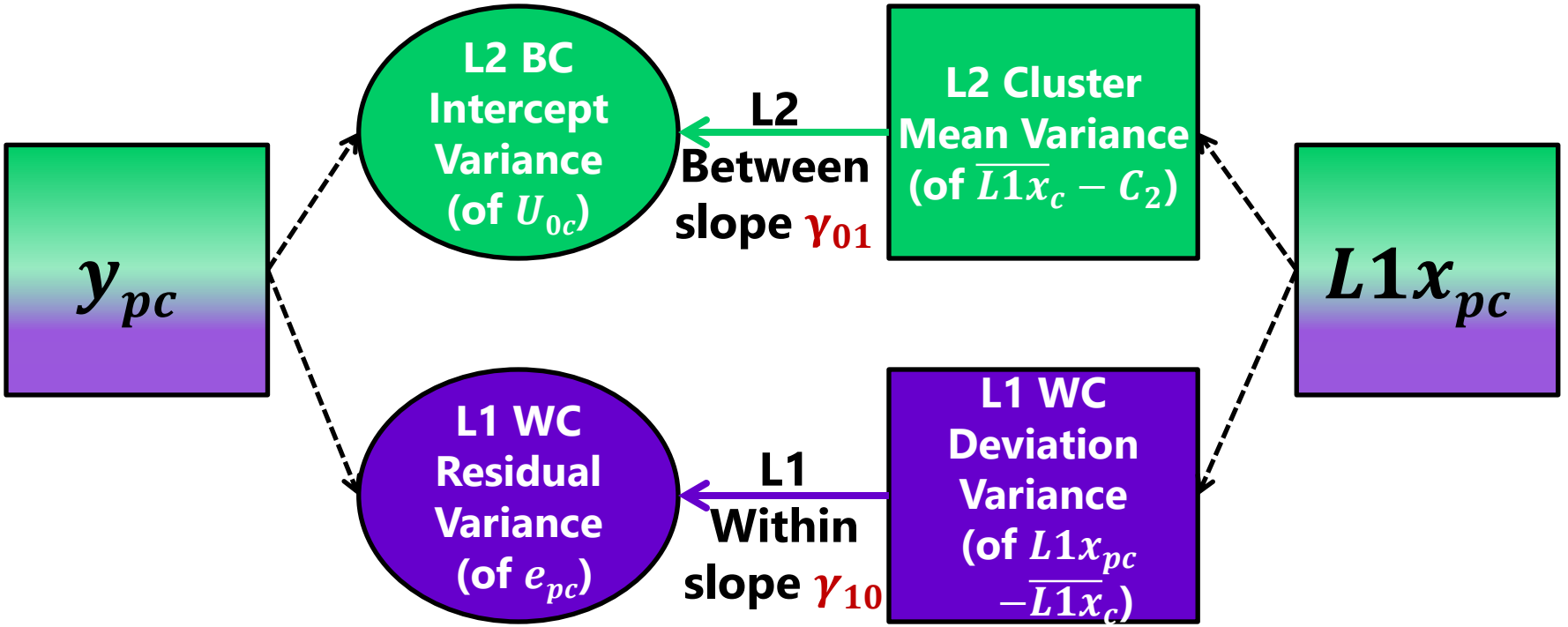
γ_{01} = between effect of having more $\overline{L1x_c}$ *than other clusters*

Because WCx_{pc} and CMx_c are uncorrelated, each gets the total effect for its level (L1 = within, L2 = between)

Univariate MLM: Cluster-Mean-Centering

Model-based partitioning of level-1 y_{pc} outcome into level-specific **latent variables**

Manual partitioning of level-1 $L1x_{pc}$ predictor into level-specific **observed variables**



Why not let the model make variance components for $L1x_{pc}$, too? That is option 3, multivariate MLM (or "multilevel SEM"): stay tuned…

3 Kinds of Fixed Slopes for L1 Predictors

- **2 kinds of slopes Cluster-Mean-Centering tells us *directly*:**

- **Is there a Level-1 Within-Cluster (WC) slope?**
  - If you have higher predictor values than the rest of your cluster, do you also have higher outcomes values than the rest of your cluster, such that the within-cluster deviation of the L1 predictor accounts for L1 residual outcome variance ($\boldsymbol{\sigma_e^2}$)?
  - **Given directly by fixed slope of $\boldsymbol{WCx_{pc}}$ regardless of whether $\boldsymbol{CMx_c}$ is there**
  - Note: L1 slope multiplies the **relative** value of $\boldsymbol{L1x_{pc}}$, NOT the **original** $\boldsymbol{L1x_{pc}}$

- **Is there a Level-2 Between-Cluster (BC) slope?**
  - Do clusters with higher predictor values than other clusters (*on average*) also have higher outcomes than other clusters (*on average*), such that the cluster mean of the L1 predictor accounts for level-2 random intercept variance ($\boldsymbol{\tau_{U_0}^2}$)?
  - **Given directly by fixed slope of $\boldsymbol{CMx_c}$ regardless of whether $\boldsymbol{WCx_{pc}}$ is there**
  - Note: BC slope is NOT controlling for the original $\boldsymbol{L1x_{pc}}$ for each person

3rd Kind of Slope for L1 Predictors

- **What Cluster-Mean-Centering DOES NOT tell us *directly*:**

- **Is there a Level-2 Contextual effect: Do the BC and WC slopes differ?**
  - After controlling for the original value of the L1 predictor per person, is there still **an incremental contribution from having a higher cluster mean** of the L1 predictor (i.e., does a cluster's general tendency for the predictor explain more $\tau^2_{U_0}$ above and beyond just the person-specific value of the L1 predictor)?
  - If there is no contextual effect, then the L1 predictor's **L2 BC** and **L1 WC** slopes show **convergence**, which means their effects are of equivalent magnitude

- **To answer this question about the Level-2 Contextual effect for the incremental contribution of the cluster mean, we have two options:**
  - Still use Cluster-MC, and ask for the **contextual slope = between – within** (via SAS ESTIMATE, R contest1D, SPSS TEST, STATA LINCOM, Mplus NEW…)
  - Use **"constant-centering"** for the L1 predictor: $L1x_{pc} = L1x_{pc} - C_1$
    → **centered at CONSTANT C_1, NOT A LEVEL-2 VARIABLE**
    - Which constant only matters for the reference point; it could be the grand mean or any (even 0)

Why the Difference in the Level-2 Effect? Remember Regular Old Regression…

- In this model: $y_p = \beta_0 + \beta_1(x1_p) + \beta_2(x2_p) + e_p$
- If $x1_p$ and $x2_p$ **ARE NOT** correlated:
  - β_1 carries **ALL the relationship** between $x1_p$ and y_p
  - β_2 carries **ALL the relationship** between $x2_p$ and y_p
- If $x1_p$ and $x2_p$ **ARE** correlated:
  - β_1 is **different than** the bivariate relationship between $x1_i$ and y_i
    - "Unique" effect of $x1_p$ *controlling for* $x2_p$ (i.e., *holding* $x2_p$ *constant)*
  - β_2 is **different than** the bivariate relationship between $x2_i$ and y_i
    - "Unique" effect of $x2_p$ *controlling for* $x1_p$ (i.e., *holding* $x1_p$ *constant)*
- **Hang onto that idea…**

Cluster-Mean-Centering vs. Constant-Centering for Level-1 Predictors

Level 2		Original	Cluster-MC Level 1	Grand-MC Level 1
$\overline{L1x_c}$	$CMx_c = \overline{L1x_c} - 5$	$L1x_{pc}$	$WCx_{pc} = L1x_{pc} - \overline{L1x_c}$	$L1x_{pc} = L1x_{pc} - 5$
3	−2	2	−1	−3
3	−2	4	1	−1
7	2	6	−1	1
7	2	8	1	3

Same L2 CMx_c goes into the model given either way of centering the L1 predictor $L1x_{pc}$

In **variable-centering** (C-MC), the level-2 BC mean variation is gone from WCx_{pc}, so it is NOT CORRELATED with CMx_c

In **constant-centering**, the level-2 BC mean variation is still inside $L1x_{pc}$, so it IS STILL CORRELATED with CMx_c

So the effects of CMx_c and $L1x_{pc}$ when included together under constant-centering will be different than if either predictor were included by itself…

Level-1 Predictor + Cluster Mean

→ Model tests difference of WC vs. BC effects

$L1x_{pc}$ is constant-centered, but <u>WITH</u> CMx_c at Level 2:

Level-1: $y_{pc} = \beta_{0c} + \beta_{1c}(L1x_{pc}) + e_{pc}$

Level-2: $\beta_{0c} = \gamma_{00} + \gamma_{01}(CMx_c) + U_{0c}$

$\beta_{1c} = \gamma_{10}$

$L1x_{pc} = L1x_{pc} - C_1$ → still has both L2 between and L1 within variation

$CMx_c = \overline{L1x_c} - C_2$ → only has L2 between variation

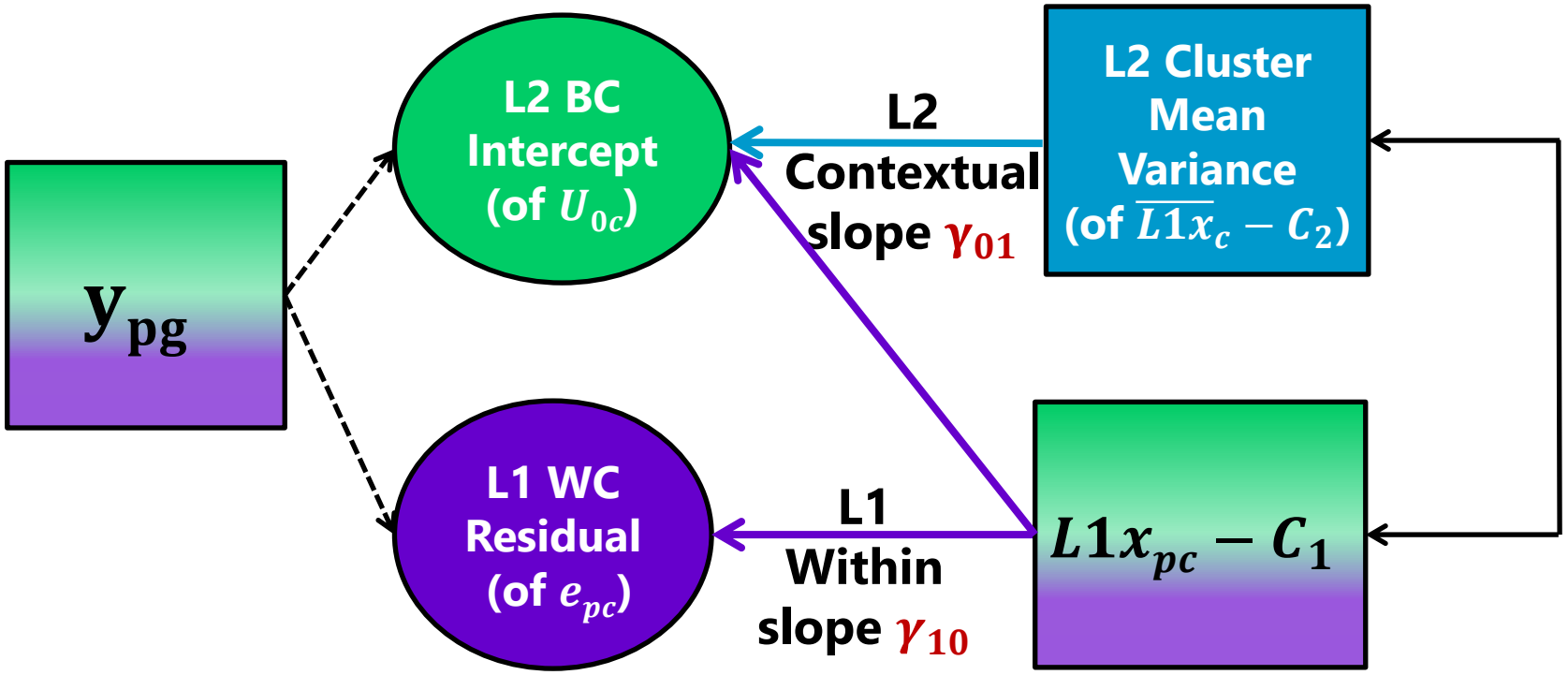
γ_{01} becomes the **within effect** → *unique* L1 effect after controlling for L2 CMx_c

γ_{01} *becomes* the **L2 Contextual slope** that indicates <u>how the L2 BC effect differs from the L1 WC effect</u>
→ *unique* level-2 slope after controlling for $L1x_{pc}$
→ does cluster mean matter beyond person value?

Constant-Centering + Cluster Mean

Model-based partitioning of y_{pc} outcome into level-specific **latent variables**

$L1x_{pc}$ **is still NOT partitioned**, but cluster mean $\overline{L1x_c} - C_2$ is added to allow an **incremental L2 effect**



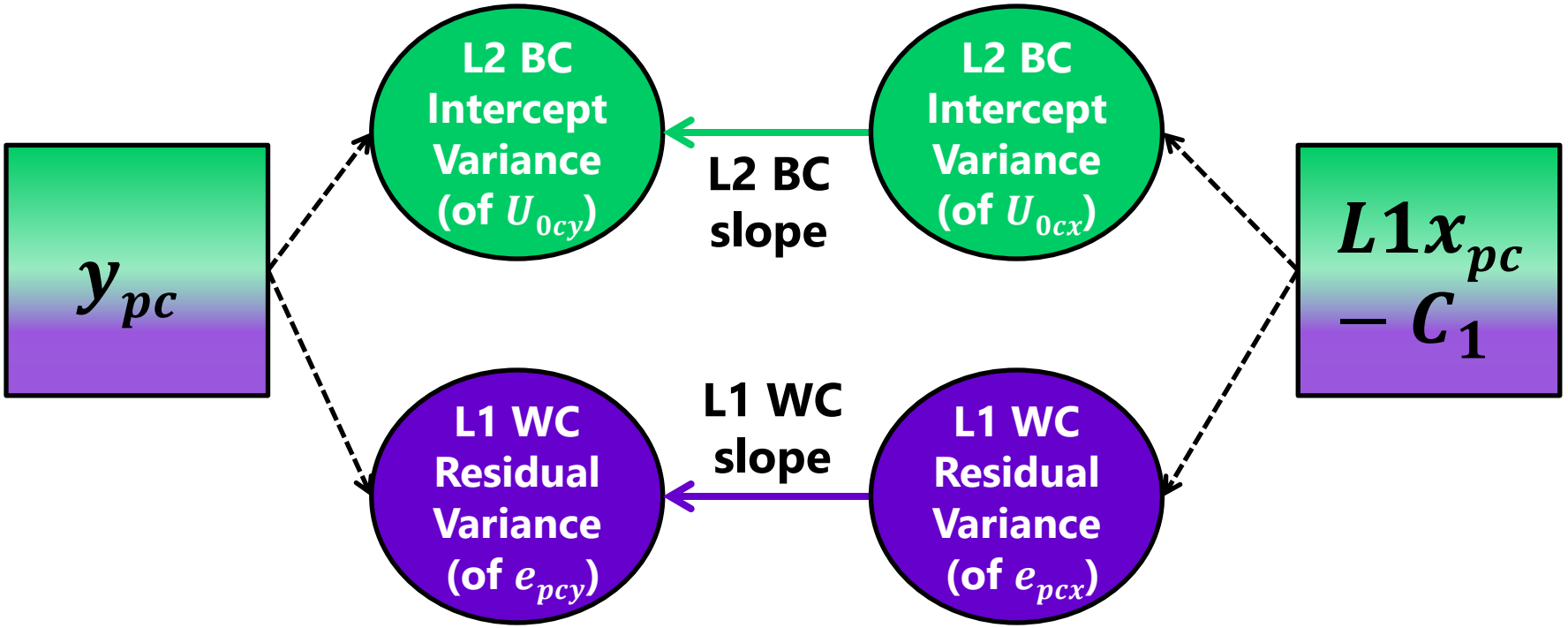
L2 BC slope = **L1 WC slope** + **Level-2 Contextual slope**

Because original $L1x_{pc}$ still has L2 BC variance, it still carries ***some*** of the L2 BC effect…

Option 3: Latent-Centering in Multivariate MLM

Model-based partitioning of level-1 y_{pc} outcome into level-specific **latent variables**

Model-based partitioning of level-1 $L1x_{pc}$ predictor (= outcome now) into level-specific **latent variables**



Univariate MLM software can be tricked into multivariate MLM if the relationships between X and Y at each level are phrased as covariances, but not if you want directed regressions (or moderators thereof)

I *Usually* Prefer Variable-Centering (using observed or latent variables)…

- **…because constant-centering is much easier to screw up!** ☺
- Table 1 below from: Hoffman, L., & Walters, R. W. (2022). Catching up on multilevel modeling. *Annual Review of Psychology, 73*, 629-658.

Table 1 Predictor effect type by model specification

Centering strategy for level-1 predictor (constant-centered level-2 predictor)	Fixed effect type by predictors included		
	Level-1 only	Level-2 only	Both levels
Variable-centered level-1			
Level-1 predictor: $L1x_{wb} = x_{wb} - \bar{x}_b$	Within	(= 0)	Within
Level-2 predictor: $L2x_b = \bar{x}_b - C_2$	(= 0)	Between	Between
Constant-centered level-1			
Level-1 predictor: $L1x_{wb} = x_{wb} - C_1$	Smushed	(= 0)	Within
Level-2 predictor: $L2x_{wb} = \bar{x}_b - C_2$	(= Within)	Between	Contextual

Abbreviations: w, within; b, between; C_1, level-1 centering constant; C_2, level-2 centering constant.
Parentheses indicate assumptions about the fixed slopes of omitted predictors.

Constant-Centering for L1 predictors (+ Cluster Mean) may be preferable when:

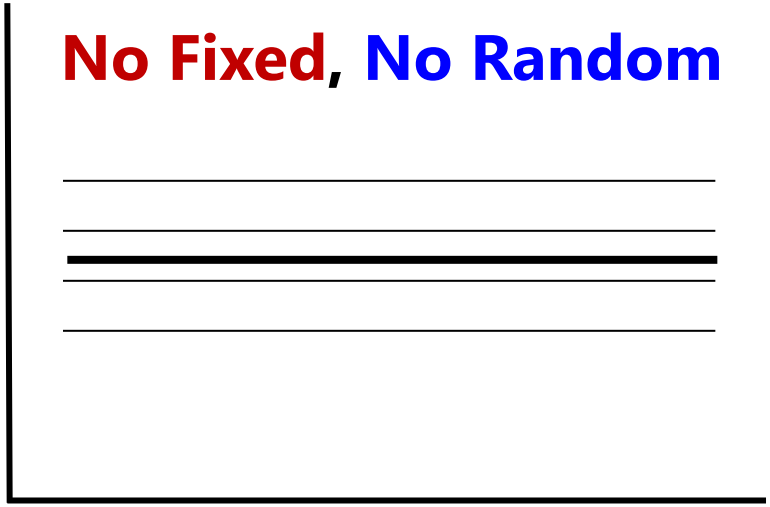
- **You really do want level-2 contextual effects**
  - Directly model the incremental contribution of the cluster mean after controlling for a person's actual (not relative) predictor
- **For *categorical* level-1 predictors**
  - e.g., 0/1 predictors when cluster-MC → impossible values
- **When the cluster mean is not a reliable cluster-level predictor**
  - When the sample of persons within clusters is not complete enough to form a useful cluster mean, using externally-provided info may do a better job of representing the cluster (in which case cluster-MC doesn't really make sense without the cluster mean to go in with it)

- But cluster-MC or latent-centering is needed instead to prevent a L1 predictor's **random slope** from being smushed…
  - **Fixed slope** → every cluster gets the **same**
  - **Random slope** → every cluster gets their **own!**

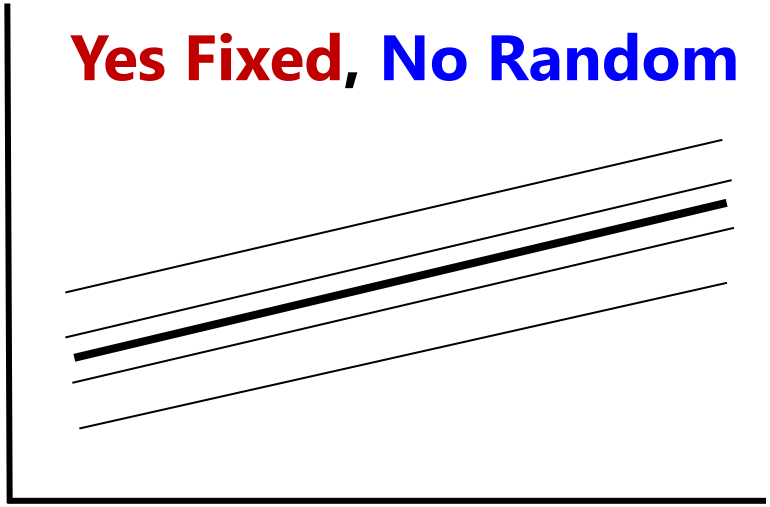
Fixed and Random Effects of L1 Predictor

(Note: The cluster intercept is random in every figure)

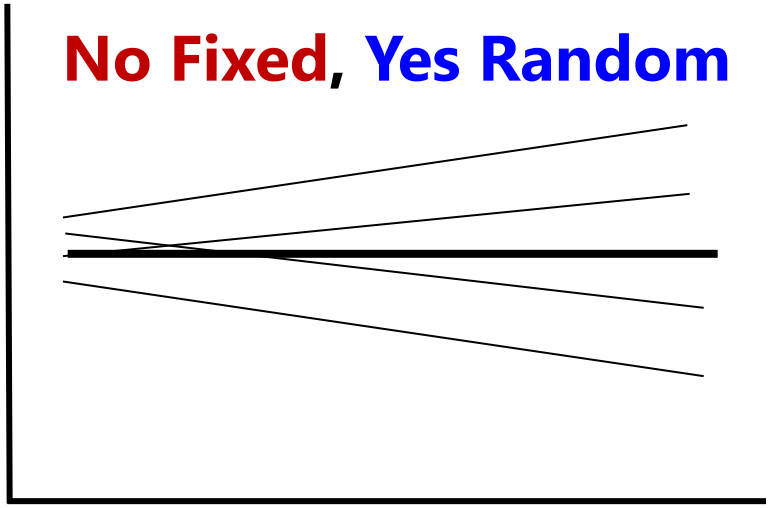
No Fixed, No Random



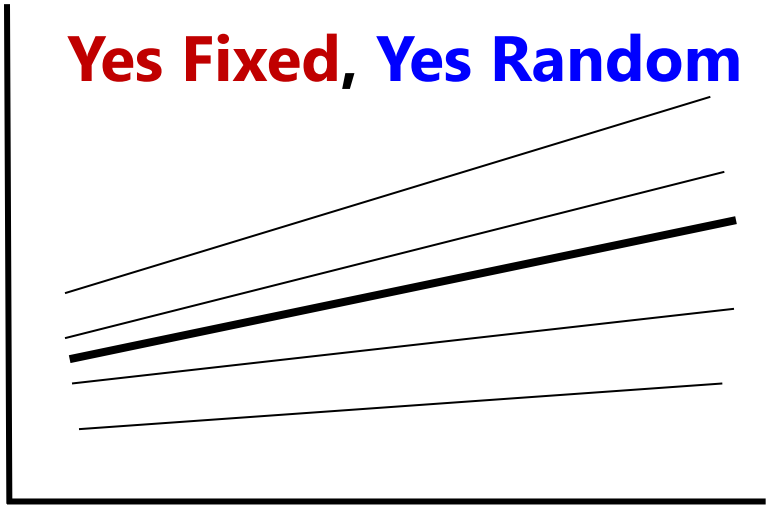
Yes Fixed, No Random



No Fixed, Yes Random



Yes Fixed, Yes Random



Cluster-MC Predictor* with Random Slope

$L1x_{pc}$ is cluster-mean-centered into WCx_{pc}, with CMx_c at L2:

Level-1: $y_{pc} = \beta_{0c} + \beta_{1c}(WCx_{pc}) + e_{pc}$

$WCx_{pc} = L1x_{pc} - \overline{L1x}_c$ → only has L1 within variation

Level-2: $\beta_{0c} = \gamma_{00} + \gamma_{01}(CMx_c) + U_{0c}$

$\beta_{1c} = \gamma_{10} + U_{1c}$

$CMx_c = \overline{L1x}_c - C_2$ → only has L2 between variation

U_{1c} is a random slope for the WC effect of WCx_{pc}

γ_{10} = within effect of having more $L1x_{pc}$ *than others in your cluster*

γ_{01} = between effect of having more $\overline{L1x}_c$ *than other clusters*

Because WCx_{pc} and CMx_c are uncorrelated, each gets the <u>total</u> effect for its level (L1 = within, L2 = between)

* If a constant-centered L1 predictor were used instead, the U_{1c} random slope would also multiply its L2 between part, creating bias in the estimated random slope variance. **To avoid such a smushed random slope, we need to use either cluster-MC (in univariate MLM) or latent-centering (in multivariate MLM).**

Implications of Random Slopes

- **L2 random slopes** capture a second, distinct source of cluster **dependency**—differences in **slope of a L1 person predictor**
  - Beyond **constant** covariance for persons from same L2 cluster (as created by the **L2 random intercept**), the **L2 random slope** adds **non-constant** covariance across values of its L1 predictor (e.g., WCx_{pc})
  - Also adds **quadratic heterogeneity** of variance across L1 predictor: $Var(y_{pc}) = \tau^2_{U_0} + (WCx^2_{pc} * \tau^2_{U_1}) + (2WCx_{pc} * \tau_{U_{01}}) + \sigma^2_e$

- **Random slopes do NOT*** explain variance (like **fixed slopes** do) because cluster slope differences are still "**error**" conceptually
  - We know **THAT** clusters need different slopes of **L1** WCx_{pc} but not **WHY**
- Therefore, random slopes imply another role for level-2 cluster predictors—to explain cluster differences in slope of **L1** WCx_{pc}
  - To do so, we need "**cross-level interactions**" of L2 by L1 predictors!

***** *Hill that I will die on, but others disagree (i.e., marginal vs. conditional R^2)*

Explained Variance by Fixed Slopes

- **Fixed slopes of level-2 cluster predictors *by themselves*:**
  - L2 BC main effects or interactions reduce L2 random intercept variance
- **Fixed slopes of *cross-level interactions* (level-1 * level-2):**
  - If the **L1 person predictor also has a random slope**, its cross-level interaction will reduce its corresponding **L2 random slope variance**
    - So make sure you test the random slope before any cross-level interactions!
  - If the **L1 person predictor does NOT have a random slope**, its cross-level interaction will reduce the **L1 residual variance** instead
    - This condition creates a "systematically varying" L1 slope instead, in which the slope varies only by interacting predictors (but not randomly otherwise)
- **Fixed slopes of level-1 person predictors *without L2 variance*:**
  - L1 WC main effects or interactions reduce L1 residual variance
- **Fixed slopes of level-1 person predictors *with L2 variance*:**
  - L1 WC main effects or interactions reduce both L1 residual variance and L2 random intercept variance; need to add corresponding L2 main effects, L2 interactions, or cross-level interactions in order to prevent smushing!
    - See Hoffman & Walters (2022) and Hoffman (2019) for elaboration

Part 2: Summary

- **Level-1 predictors** are person characteristics, but they **almost always contain cluster mean differences** (level-2 variance) as well
  - **Variance** at each level → **different slope** at each level!

- **3 options** for specifying fixed slopes of a L1 predictor in order to distinguish its level-specific effects (i.e., **avoid smushed effects**):
  1. **Cluster-Mean-Centering**: Manually carve up into L2 BC (cluster mean → **L2 Between slope**) and L1 WC deviation (→ **L1 Within slope**)
  2. **Constant-Centering**: Add cluster mean to become **L2 Contextual slope**, then L1 predictor's unique effect is **L1 Within slope**
  3. **Latent-Centering**: Let multivariate MLM estimate L2 and L1 variance components, same as for the outcome → analogous to Cluster-MC

- A **level-2 random slope** variance allows cluster differences in the effect of a L1 person predictor (using only options 1 or 3)
  - Implies **heterogeneity** of variance and covariance across L1 predictor
  - Implies **another way clusters differ** (to be explained by **cross-level interactions** between that L1 predictor and L2 predictors)

Begin Bonus Material

- Significance testing for each side of the model
- Likelihood ratio tests and information criteria
- Maximum likelihood (ML) vs.
  residual maximum likelihood (REML)
- Model comparisons in ML vs. REML
- Why explaining level-1 residual variance will increase level-2 random intercept variance (and the design effect)

- Comparison of between vs contextual effects
- More depictions of level-2 between, level-2 contextual, and level-1 within slopes
  - Example variables: How often students are read to by their parents predict student math outcomes

Relative Model Fit by Model Side

- Nested models (i.e., in which one is a subset of the other) can now differ from each other in two distinct ways

- **Model for the Means** → which predictors and which fixed slopes for them are included in the model
  - **Does not** require assessment of relative model fit using −2LL because we can still use univariate or multivariate Wald tests for this (although we have more choices for denominator degrees of freedom)

- **Model for the Variance** → what the pattern of variance and covariance of residuals from the same sampling unit should be
  - **DOES** require assessment of relative model fit using −2LL
  - Cannot use the Wald test *p*-values (even if they show up on the output) for testing the significance of variances because those *p*-values use a two-sided sampling distribution for what the variance could be (but variances cannot be negative, so those *p*-values are not valid)

Significance of Fixed Effects in MLM

	Denominator DF is infinite (Proper Wald test)	**Denominator DF is estimated instead** ("Modified" Wald test)
Numerator DF = 1 *(test one fixed effect)* is **Univariate Wald Test**	use $\boldsymbol{z}$ distribution (Mplus, STATA default)	use $\boldsymbol{t}$ distribution (SAS, SPSS, STATA with dfmethod option)
Numerator DF > 1 *(test 2+ fixed effects)* is **Multivariate Wald Test**	use $\boldsymbol{\chi^2}$ distribution (Mplus, STATA default)	use $\boldsymbol{F}$ distribution (SAS, SPSS, STATA with dfmethod option)
Options for estimating Denominator DF (DDF)	not applicable	SAS, STATA: Kenward-Roger SAS, STATA, SPSS: **Satterthwaite**

In R, the default and optional DDF behavior vary across packages:

- Kenward-Roger and Satterthwaite are available through the lmerTest package (for use with the lmer function from the lme4 package)
- Satterthwaite DDF may not always work in nlme package (gls or lme functions)

Denominator DF (DDF) Methods

- **Between-Within** (DDFM=BW in SAS, REPEATED in STATA):
  - Total DDF comes from total number of observations, separated into level-2 for L2*n* clusters and level-1 for L1*n* persons (like in RM ANOVA)
    - **Level-2 DDF** = L2*n* – #level-2 fixed effects
    - **Level-1 DDF** = Total DDF – Level-2 DDF – #level-1 fixed effects
    - Level-1 effects with random slopes still get level-1 DDF

- **Satterthwaite** (DDFM=Satterthwaite in SAS and STATA, available in LME and LMER in R, default in SPSS):
  - More complicated, but analogous to two-group *t*-test given unequal residual variances and unequal group sizes
  - Incorporates contribution of variance components at each level
    - Level-2 DDF will resemble Level-2 DDF from BW method
    - Level-1 DDF will resemble Level-2 DDF from BW method if the level-1 effect also has a random slope, but it will resemble level-1 DDF otherwise

Denominator DF (DDF) Methods

- **Kenward-Roger** (DDFM=KR in SAS, KROGER in STATA, available in LME and LMER in R, available in SPSS 26+):
  - Adjusts the asymptotic covariance matrix of the fixed effects to reflect the uncertainty introduced by using large-sample techniques of maximum likelihood estimation in small L2*n* samples
  - This creates different (larger) SEs for the fixed effects
  - Then uses Satterthwaite DDF, new SEs, and *t* to get *p*-values

- Differences in inference not likely to matter often in practice unless sample sizes are very small
  - e.g., critical *t*-value at DDF=20 is 2.086, at infinite DDF is 1.960 instead

- When in doubt, use KR (is overkill at worst, becomes Satterthwaite)
  - I use Satterthwaite in my teaching for comparability across programs

Comparing Models for the Variance

- Unlike **fixed effects** (which can always use Wald-type tests), testing **random effects** requires assessment of **relative model fit**: how well does the model fit relative to other possible models?

- Model fit is indexed by overall model **log-likelihood** (**LL**):
  - Multivariate height for each cluster's outcomes given model parameters
  - Sum heights across all (independent) clusters = **model LL**
  - Two flavors in MLM: Maximum Likelihood (ML) or Restricted ML (REML)

- What you get for this on your output varies by software…

- Given as −2*log likelihood (−2LL) in SAS or SPSS MIXED, some R: **−2LL** gives BADNESS of fit, so **smaller** value = better model

- Given as just log-likelihood (LL) in STATA MIXED and Mplus, some R: **LL** gives GOODNESS of fit, so **bigger** value = better model

Comparing Models for the Variance

- Nested models are compared using a **"likelihood ratio test": −2ΔLL test** (aka, "χ^2 test" in SEM; "deviance difference test" in MLM)

"fewer" = from model with fewer parameters
"more" = from model with more parameters

Results of 1. & 2. must be positive values!

1. Calculate **−2ΔLL:** if given −2LL, use $-2\Delta LL = (-2LL_{fewer}) - (-2LL_{more})$
   if given LL, use $-2\Delta LL = -2 * (LL_{fewer} - LL_{more})$
2. Calculate **ΔDF =** $(\#\ Parms_{more}) - (\#\ Parms_{fewer})$
3. **Compare −2ΔLL to χ^2 distribution with numerator DF = ΔDF**
4. Get *p*-value (from CHIDIST in excel, LRTEST in STATA, R/ANOVA in R)

- When testing random effect variances (that can't be negative), a "**mixture**" χ^2 distribution should be used (otherwise is conservative)
  - e.g., Add random intercept? DF is mixture of 1 (when positive) and 0 (when it would have been negative), so you can just cut the *p*-value in half
  - e.g., Add random slope variance? DF is mixture of 2 (when positive) and 1 (when it would have been negative; just covariance), so critical value is lower

Comparing Models for the Variance

- What your *p*-value for the −2ΔLL test means:
  - If you **ADD** parameters, then your model can get **better** (if −2ΔLL test is significant) or **not better** (not significant)
  - If you **REMOVE** parameters, then your model can get **worse** (if −2ΔLL test is significant) or **not worse** (not significant)

- Nested or non-nested models can also be compared by **Information Criteria** that also reflect model parsimony
  - No significance tests or critical values, just "smaller is better"
  - **AIC** = Akaike IC = −2LL + 2 *(#parameters)
  - **BIC** = Bayesian IC = −2LL + log(N)*(#parameters)
  - What "parameters" means depends on flavor (not in R or STATA!):
    - ML = ALL parameters; REML = variance model parameters only

Flavors of Maximum Likelihood

- For MLMs, maximum likelihood estimation comes in 2 flavors:
- **"Restricted (or residual) maximum likelihood"**
  - Only available for general linear models or general linear mixed models (key: based on normally distributed residuals at all levels of analysis)
  - **REML = OLS** given complete outcomes, but it doesn't require them
  - Estimates variances the same way as in OLS (accurate) → $\frac{\sum(y_{pc} - \hat{y}_{pc})^2}{N - k}$
- **"Maximum likelihood" (ML; also called FIML*)**
  - Is more general, is available for all of the above, as well as for non-normal outcomes and models with latent variables (CFA/SEM/IRT/DCM)
  - Is NOT equivalent to OLS: It under-estimates variances by not accounting for number of estimated fixed effects → $\frac{\sum(y_{pc} - \hat{y}_{pc})^2}{N}$
- **FI = Full information → it uses all original data (they both do)*

LRTs using ML vs. REML in a nutshell

All comparisons must use exact same sample to be valid!!!	**ML**	**REML**
To select, type…	METHOD=ML (-2 log likelihood)	METHOD=REML *default* (-2 res log likelihood)
In estimating variances, it treats fixed effects as…	**Known** (DF for having to also estimate fixed effects is **not** factored in)	**Unknown** (DF for having to also estimate fixed effects **is** factored in)
So, in small samples, L2 variances will be…	**Too small** (but less of a difference after Level-2 sample size = 100 or so)	**Unbiased** (correct)
But because it indexes the fit of the…	**Entire model** (means + variances)	**Variance model only**
You can compare models differing in…	**Fixed and/or random effects** (either/both)	**Random effects only** (same fixed effects)

Summary of Rules for Comparing Models

All observations must be the same across models!

Compare Models Differing In:

Type of Comparison:	**Means Model (Fixed Effects) Only**	**Variance Model (Random Effects) Only**	**Both Means and Variance Model (Fixed and Random)**
Nested? YES, can do significance tests via…	Fixed effect *p*-values from ML or REML -- *OR* -- ML −2ΔLL only (NO REML −2ΔLL)	NO *p*-values REML −2ΔLL (ML −2ΔLL is ok if big N)	ML −2ΔLL only (NO REML −2ΔLL)
Non-Nested? NO signif. tests, instead see…	ML AIC, BIC (NO REML AIC, BIC)	REML AIC, BIC (ML ok if big N)	ML AIC, BIC only (NO REML AIC, BIC)

Nested = one model is a direct subset of the other

Non-Nested = one model is not a direct subset of the other

Increases in Random Intercept Variance

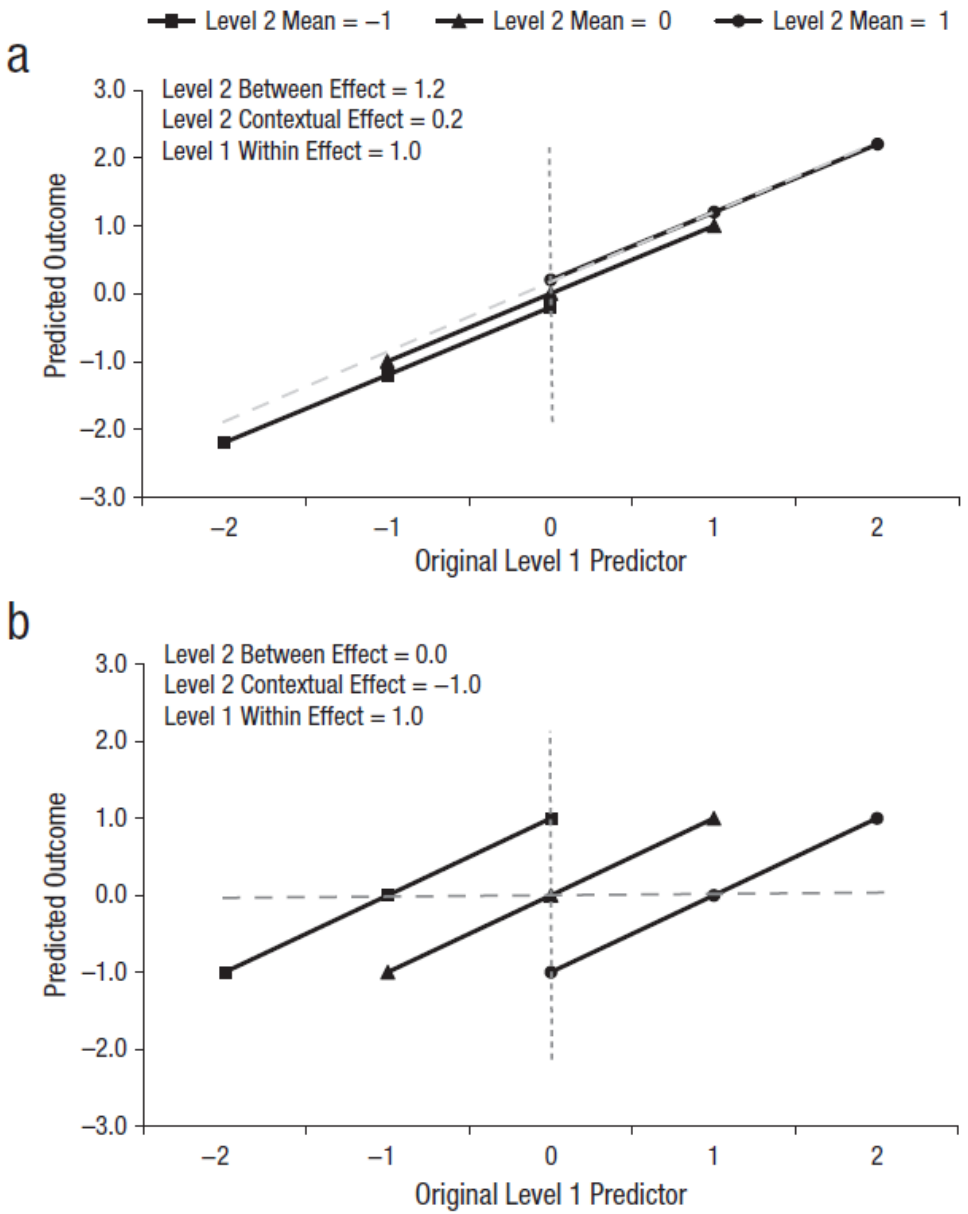
- Level-2 random intercept variance $\tau^2_{U_0}$ will often increase as a consequence of reducing level-1 residual variance σ^2_e
- **Observed level-2 $\tau^2_{U_0}$** is NOT just between-cluster variance
  - Also has a small part of within-cluster variance (**level-1 σ^2_e**), or:

    Observed $\tau^2_{U_0}$ = True $\tau^2_{U_0}$ + $(\sigma^2_e/L1n)$
    - With increasing $L1n$ perons, bias in true $\tau^2_{U_0}$ due to level-1 σ^2_e is minimized
  - Model estimates of "**True**" $\tau^2_{U_0}$ use **$(\sigma^2_e/L1n)$** as correction factor:

    True $\tau^2_{U_0}$ = Observed $\tau^2_{U_0}$ – $(\sigma^2_e/L1n)$

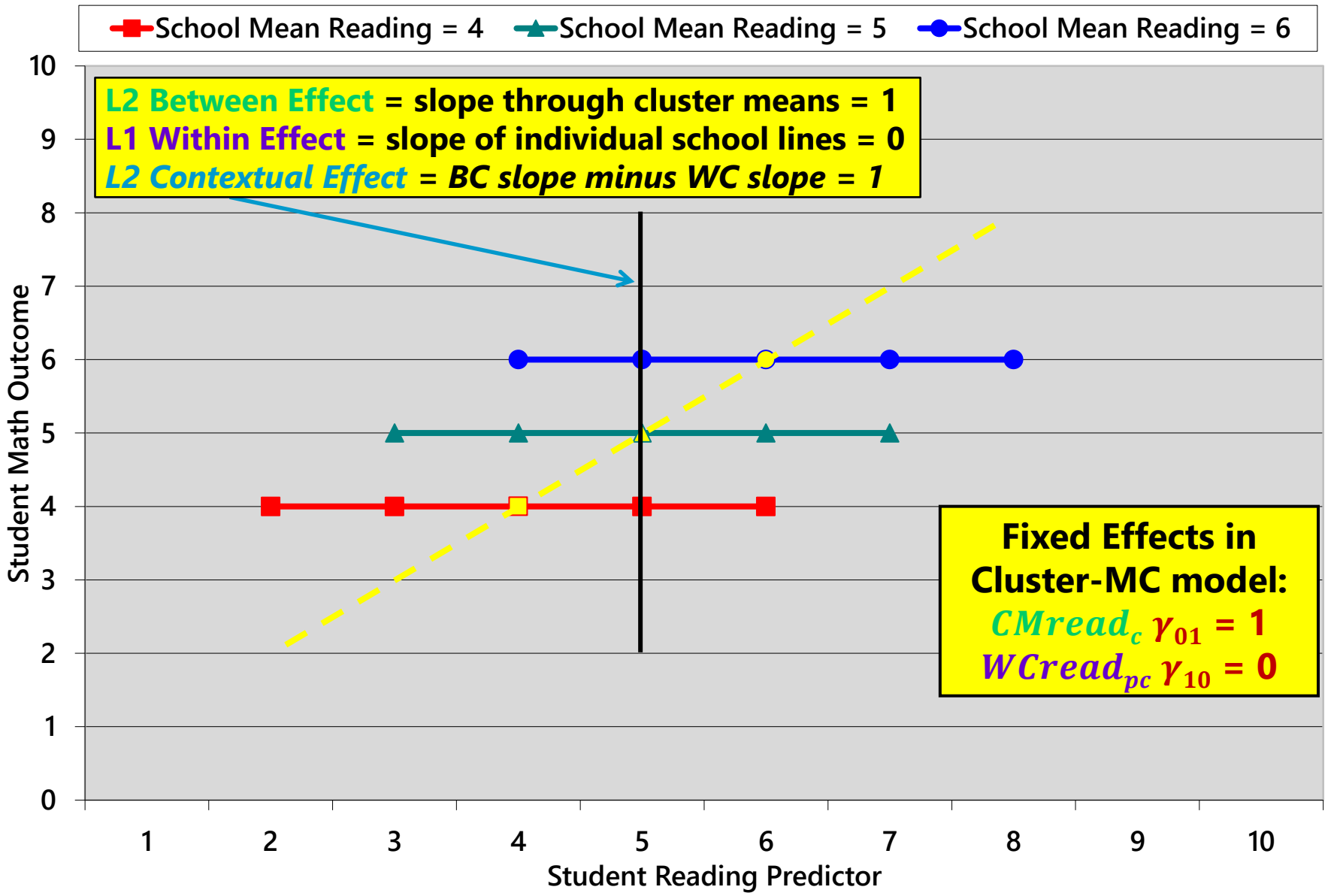
- e.g., **Observed level-2 $\tau^2_{U_0}$ = 4.65**, **level-1 σ^2_e = 7.06**, **$L1n$ = 4**
  - **True $\tau^2_{U_0}$** = **4.65** –(**7.06/4**) = **2.88** in empty means model
  - Add L1 within-cluster predictor → reduce σ^2_e from **7.06** to **2.17**
  - But now **True $\tau^2_{U_0}$** = **4.65** –(**2.17/4**) = **4.10** → more dependency!

Bonus: Between vs. Contextual Effects

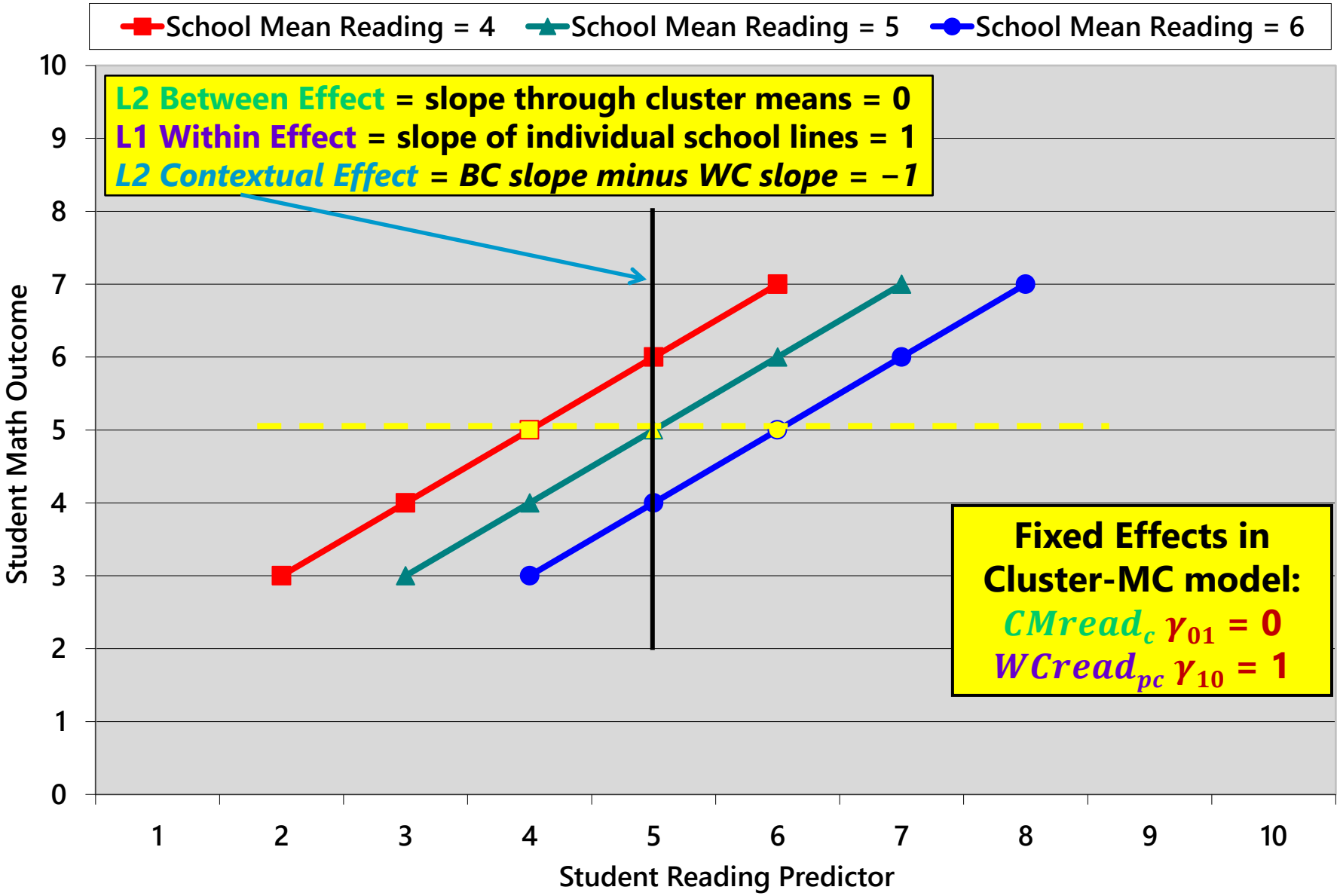


- Image from Hoffman (2019), example using student SES
- *Top:* Contextual effect is minimal—there is no added benefit to going to a high-SES school when comparing across schools *at same level of student SES*
- *Bottom:* Contextual effect is negative—at the same student SES level, relatively high students from low-SES schools do better than relatively low students from high-SES schools

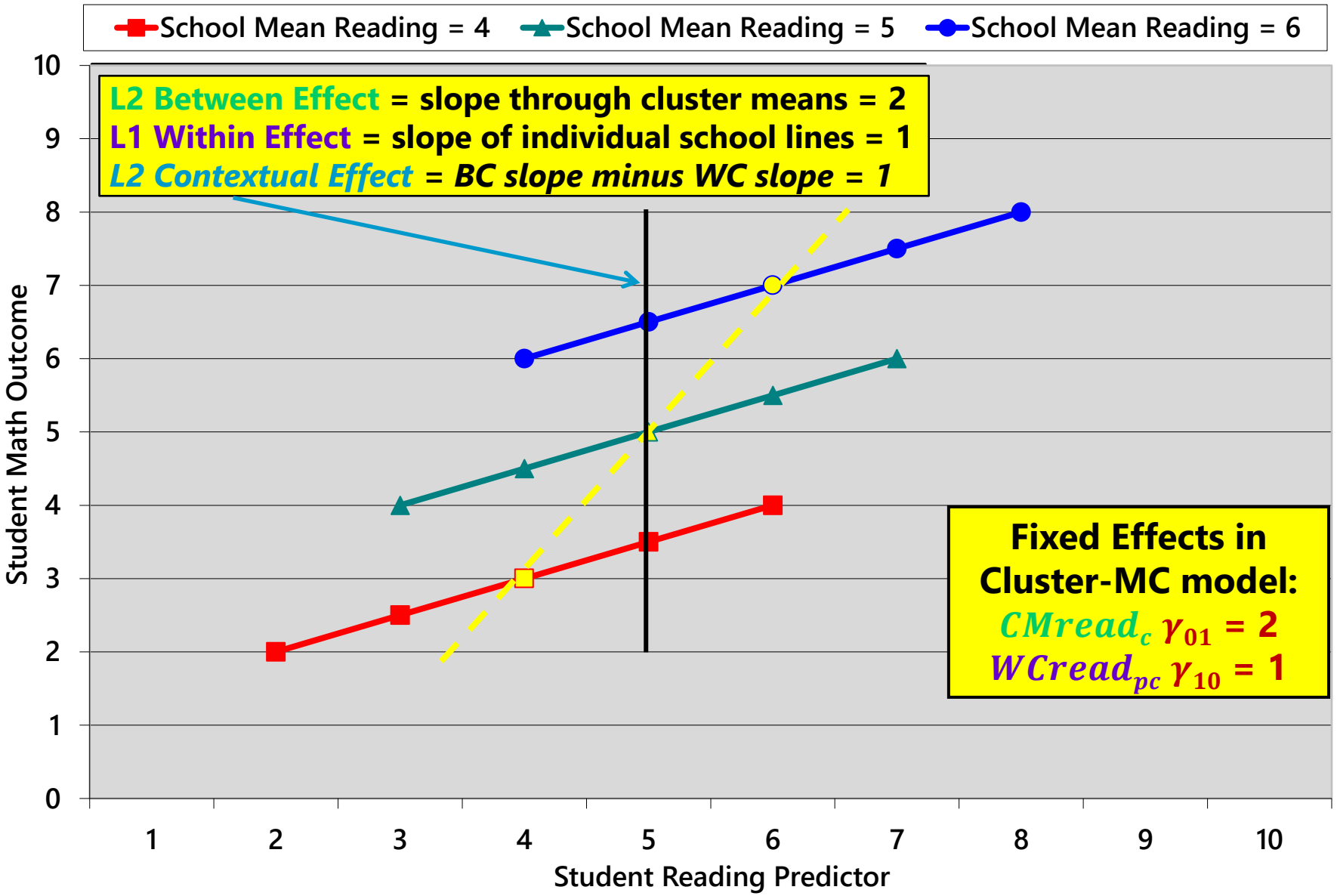
ALL Between Effect, **NO** Within Effect



NO Between Effect, **ALL** Within Effect



Between Effect > Within Effect



Between, Within, and Contextual Effects