lecture06mlm\_ML

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knitr::opts\_chunk$set(echo=TRUE)  
  
# Manually set working directory for all chunks below  
# knitr::opts\_knit$set(root.dir="C:/Dropbox/Papers and Data/SMiP/MLMM2025-main/\_lectures/lecture06\_Introduction\_to\_Multilevel\_Models")

# Multilevel Modeling: Predicting Observed Sum Score for Students Nested in Schools

This example will illustrate the concepts and estimation of multilevel models for observed outcomes. We will predict the sum score of the 10 example items using binary free/reduced lunch status for level-1 students nested in level-2 schools (as clusters). We will use standard ML estimation within the nlme, lme4, and lavaan packages. (We will then conduct the same analyses using Bayesian estimation within Stan using separate files.)

## Preliminary Steps

First (below), we set global options to my preferred versions, and then install and load the R packages to be used. We also load custom functions that perform convenient computations (written by Jonathan Templin). Notably, you may have to re-install the lme4 package if you have problems with matrix incompatibility. If you do, uncomment the lines below to re-install it.

# Set width of output and number of significant digits printed,  
# number of digits before using scientific notation, shut off significance stars  
options(width=120, digits=8, scipen=9, show.signif.stars=FALSE)  
  
##### Check to see if packages are downloaded, install if not, then load #####  
  
# To get compact data description  
if (!require("psych")) install.packages("psych"); library(psych)

Loading required package: psych

# To estimate MLMs using gls or lme  
if (!require("nlme")) install.packages("nlme"); library(nlme)

Loading required package: nlme

# To estimate MLMs using lmer  
# Re-install as needed to fix problem of matrix incompatibility  
#oo <- options(repos = "https://cran.r-project.org/")  
#utils::install.packages("Matrix")  
#utils::install.packages("lme4")  
#options(oo)  
library(Matrix); library(lme4)

Attaching package: 'lme4'

The following object is masked from 'package:nlme':  
  
 lmList

# To get Satterthwaite DDF in lmer  
if (!require("lmerTest")) install.packages("lmerTest"); library(lmerTest)

Loading required package: lmerTest

Attaching package: 'lmerTest'

The following object is masked from 'package:lme4':  
  
 lmer

The following object is masked from 'package:stats':  
  
 step

# To get ICC in lmer  
if (!require("performance")) install.packages("performance"); library(performance)

Loading required package: performance

# To estimate multivariate MLM using multilevel SEM  
if (!require("lavaan")) install.packages("lavaan"); library(lavaan)

Loading required package: lavaan

This is lavaan 0.6-19  
lavaan is FREE software! Please report any bugs.

Attaching package: 'lavaan'

The following object is masked from 'package:psych':  
  
 cor2cov

# Clear workspace (re-run as needed for troubleshooting purposes)  
#rm(list = ls())  
  
# Load R functions for this example from folder within working directory  
functions = paste0("functions/",dir("functions/"))  
temp = lapply(X = functions, FUN = source)

## Data Import, Manipulation, and Description

Next (below), we import the R data file for this example. We also use one of our custom functions to create two new school-level means: one for the sum score student outcome, and one for the binary free/reduced lunch student predictor.

# Load example data  
load("modelingData.RData")  
  
# Create school means for sum score student outcome and free/reduced lunch student predictor   
modelingData = addUnitMeans(data=modelingData, unitVariable="schoolID",   
 meanVariables=c("sumScore","frlunch"),   
 newNames=c("SMsumScore","SMfrlunch"))  
  
# Descriptive statistics for student variables and new school means  
print(describe(modelingData[c("sumScore","frlunch","SMsumScore","SMfrlunch")]), digits=3)

vars n mean sd median trimmed mad min max range skew kurtosis se  
sumScore 1 3100 5.724 2.575 6.00 5.761 2.965 0.00 10.00 10.0 -0.122 -0.832 0.046  
frlunch 2 3100 0.298 0.457 0.00 0.247 0.000 0.00 1.00 1.0 0.884 -1.219 0.008  
SMsumScore 3 3100 5.724 1.078 5.93 5.803 1.112 3.08 7.58 4.5 -0.521 -0.377 0.019  
SMfrlunch 4 3100 0.298 0.224 0.28 0.280 0.208 0.00 0.80 0.8 0.640 -0.561 0.004

As shown by the “SM” variables above, the range across schools in the sum scores is 3.08 to 7.58 on a 0 to 10 scale, and the range across schools in the proportion of students receiving free/reduced lunch is 0 to .80 – there is quite a bit of school-level variability in both variables! In preparation for using school mean free/reduced lunch as a predictor (stay tuned!), we center it near the sample mean. We also create a purely within-school version of student free/reduced lunch using cluster-mean-centering.

# Constant-center school lunch near sample mean to use as observed level-2 predictor  
modelingData$SMfrlunch30 = modelingData$SMfrlunch - .30  
  
# Cluster-mean-center student lunch at school mean to use as observed level-1 predictor  
modelingData$WSfrlunch = modelingData$frlunch - modelingData$SMfrlunch

## Models for Partitioning Student-Level from School-Level Variance

Next, we will estimate and compare two “empty means” (i.e., no-predictor) models for each variable. The first is a single-level model with only a residual variance, and the second is a two-level model that adds a random intercept variance.

### Partitioning Variance in the Sum Score Outcome using General Models

In the empty means, single-level model below, is a placeholder for the cluster-level intercept for each school, which gets defined (so far) by only a fixed intercept, .

$$ \begin{array}
ssumScore\_{pc} = \beta\_{0c} + e\_{pc} \\
\beta\_{0c} = \gamma\_{00} \\
\end{array}
$$

# Single-level empty model predicting observed sum score ignoring school  
# Using gls instead of lm to get model log-likelihood for model comparison  
modelEmptyGLM = gls(data=modelingData, method="ML", model=sumScore~1)  
summary(modelEmptyGLM)

Generalized least squares fit by maximum likelihood  
 Model: sumScore ~ 1   
 Data: modelingData   
 AIC BIC logLik  
 14664.048 14676.127 -7330.0241  
  
Coefficients:  
 Value Std.Error t-value p-value  
(Intercept) 5.7235484 0.046243609 123.7695 0  
  
Standardized residuals:  
 Min Q1 Med Q3 Max   
-2.22332446 -0.66951601 0.10738821 0.88429244 1.66119666   
  
Residual standard error: 2.5743199   
Degrees of freedom: 3100 total; 3099 residual

As shown above, the single-level empty model perfectly reproduces the original mean of the sum score outcome as the fixed intercept. The ML estimate of the outcome SD (with total variance = 6.629) is very close to that of the original data as expected.

In the empty means, two-level model below, we add a random intercept for each school, , whose variance across schools is then estimated to form its random intercept variance.

$$ \begin{array}
ssumScore\_{pc} = \beta\_{0c} + e\_{pc} \\
\beta\_{0c} = \gamma\_{00} + U\_{0c} \\
\end{array}
$$

# Two-level empty model predicting observed sum score with students nested in schools  
modelEmptyRI = lmer(data=modelingData, REML=FALSE, sumScore~1+(1|schoolID))  
summary(modelEmptyRI)

Linear mixed model fit by maximum likelihood . t-tests use Satterthwaite's method ['lmerModLmerTest']  
Formula: sumScore ~ 1 + (1 | schoolID)  
 Data: modelingData  
  
 AIC BIC logLik -2\*log(L) df.resid   
 14276.1 14294.2 -7135.1 14270.1 3097   
  
Scaled residuals:   
 Min 1Q Median 3Q Max   
-2.897172 -0.764468 0.026546 0.784860 2.493814   
  
Random effects:  
 Groups Name Variance Std.Dev.  
 schoolID (Intercept) 1.0509 1.0251   
 Residual 5.5762 2.3614   
Number of obs: 3100, groups: schoolID, 62  
  
Fixed effects:  
 Estimate Std. Error df t value Pr(>|t|)  
(Intercept) 5.72355 0.13693 62.00000 41.8 < 2.2e-16

As shown above, the two-level empty model returns a fixed intercept that is nearly identical to the mean of the original sum score outcome, but it now represents the sample mean of the school means (and is thus a weighted mean). The sum of the two estimated variances is close to the model-estimated variance from the single-level model.

# Show intraclass correlation and its likelihood ratio test  
icc(modelEmptyRI); ranova(modelEmptyRI)

# Intraclass Correlation Coefficient  
  
 Adjusted ICC: 0.159  
 Unadjusted ICC: 0.159

ANOVA-like table for random-effects: Single term deletions  
  
Model:  
sumScore ~ (1 | schoolID)  
 npar logLik AIC LRT Df Pr(>Chisq)  
<none> 3 -7135.06 14276.1   
(1 | schoolID) 2 -7330.02 14664.0 389.925 1 < 2.22e-16

As shown above, the two-level empty model partitions the sum score’s observed variance into between-school mean differences (16.1% as given by the intraclass correlation, ICC = .161) and within-school student deviations from their school mean (the remaining 83.9%). The sum score ICC was computed (using icc from the performance package) as follows:

The ranova command then conducts a likelihood ratio test comparing the log-likelihood from the empty models with vs without a random intercept variance. As expected, the ICC = .161 is significantly > 0.

### Partitioning Variance in the Binary Free/Reduced Lunch Predictor using Generalized Models

Now let’s do the same thing for the binary free/reduced student lunch variable, but we will predict it using a logit link function and a Bernoulli conditional distribution instead. First, a single-level model (equivalent to a logistic regression):

$$ \begin{array}
LLogit(frlunch\_{pc}=1) = \beta\_{0c} \\
\beta\_{0c} = \gamma\_{00} \\
\end{array}
$$

# Single-level empty model predicting observed free/reduced lunch ignoring school  
modelEmptyGLMfr = glm(data=modelingData, family=binomial(link="logit"), formula=frlunch~1)  
summary(modelEmptyGLMfr) # Null deviance= -2LL already

Call:  
glm(formula = frlunch ~ 1, family = binomial(link = "logit"),   
 data = modelingData)  
  
Coefficients:  
 Estimate Std. Error z value Pr(>|z|)  
(Intercept) -0.858074 0.039278 -21.846 < 2.2e-16  
  
(Dispersion parameter for binomial family taken to be 1)  
  
 Null deviance: 3775.42 on 3099 degrees of freedom  
Residual deviance: 3775.42 on 3099 degrees of freedom  
AIC: 3777.42  
  
Number of Fisher Scoring iterations: 4

# Convert logit intercept into probability  
modelEmptyGLMfrProb=1/(1+exp(-1\*coefficients(modelEmptyGLMfr))); modelEmptyGLMfrProb

(Intercept)   
 0.29774194

As shown above, the single-level model perfectly reproduces the mean of the original binary outcome after converting the logit intercept into probability (via the inverse logit link function):

Next, in a two-level model, we add a random intercept for each school, , whose variance across schools is then estimated to form its random intercept variance.

$$ \begin{array}
LLogit(frlunch\_{pc}=1) = \beta\_{0c} \\
\beta\_{0c} = \gamma\_{00} + U\_{0c} \\
\end{array}
$$

# Two-level empty model predicting observed free/reduced lunch with students nested in schools  
modelEmptyRIfr = glmer(data=modelingData, family=binomial(link="logit"), frlunch~1+(1|schoolID))  
summary(modelEmptyRIfr) # deviance = -2LL already

Generalized linear mixed model fit by maximum likelihood (Laplace Approximation) ['glmerMod']  
 Family: binomial ( logit )  
Formula: frlunch ~ 1 + (1 | schoolID)  
 Data: modelingData  
  
 AIC BIC logLik -2\*log(L) df.resid   
 3197.3 3209.4 -1596.6 3193.3 3098   
  
Scaled residuals:   
 Min 1Q Median 3Q Max   
-1.85329 -0.61990 -0.35086 0.68664 4.84803   
  
Random effects:  
 Groups Name Variance Std.Dev.  
 schoolID (Intercept) 1.8966 1.3772   
Number of obs: 3100, groups: schoolID, 62  
  
Fixed effects:  
 Estimate Std. Error z value Pr(>|z|)  
(Intercept) -1.18370 0.18387 -6.4379 0.0000000001212

# Convert logit intercept into probability (sub-object beta holds fixed intercept)  
modelEmptyRIfrProb=1/(1+exp(-1\*(modelEmptyRIfr@beta))); modelEmptyRIfrProb

[1] 0.23438689

# Compute ICC using pi^2/3 = 3.29 as residual variance (sub-object theta holds random intercept variance)  
modelEmptyRIfr@theta^2/(modelEmptyRIfr@theta^2+(pi^2/3))

[1] 0.36567891

# Likelihood Ratio Test for Addition of Random Intercept Variance  
DevTest=-2\*(logLik(modelEmptyGLMfr)-logLik(modelEmptyRIfr))  
Pvalue=pchisq((DevTest), df=1, lower.tail=FALSE)  
# Test Statistic and P-values for DF=1   
DevTest; Pvalue

'log Lik.' 582.13455 (df=1)

'log Lik.' 1.2875812e-128 (df=1)

As shown above, we note the logit fixed intercept has changed: from -0.858074 to -1.18370 (corresponding to prob = .300 vs. prob = .234). This is because the fixed intercept takes on a different “unit-specific” interpretation – it is now specifically the logit of receiving free/reduced lunch for a student in a school with random intercept = 0.

The two-level model partitions its variance into between-school mean differences (36.6.1% as given by the intraclass correlation, ICC = .366) and within-school student deviations from their school mean (the remaining 63.4%). The ICC for binary free/reduced lunch was computed as follows:

Because 36.6% of the variance in student free/reduced lunch reflects school mean differences, this means it can potentially predict both student-level variance and school-level variance in the sum score outcome.

## Models Predicting the Observed Sum Score from Free/Reduced Lunch for Students Nested in Schools

### Smushed Level-1 Slope for Free/Reduced Lunch

Next, we add a level-1 predictor for student free/reduced lunch:

$$ \begin{array}
ssumScore\_{pc} = \beta\_{0c} + \beta\_{1c}(frlunch\_{pc}) + e\_{pc} \\
\beta\_{0c} = \gamma\_{00} + U\_{0c} \\
\beta\_{1c} = \gamma\_{10} \\
\end{array}
$$

# Add smushed level-1 slope for frlunch  
modelSmushed = lmer(data=modelingData, REML=FALSE, sumScore~1+frlunch+(1|schoolID))  
summary(modelSmushed, ddf="Satterthwaite")

Linear mixed model fit by maximum likelihood . t-tests use Satterthwaite's method ['lmerModLmerTest']  
Formula: sumScore ~ 1 + frlunch + (1 | schoolID)  
 Data: modelingData  
  
 AIC BIC logLik -2\*log(L) df.resid   
 14154.2 14178.4 -7073.1 14146.2 3096   
  
Scaled residuals:   
 Min 1Q Median 3Q Max   
-2.955887 -0.711374 0.007208 0.759175 2.625617   
  
Random effects:  
 Groups Name Variance Std.Dev.  
 schoolID (Intercept) 0.63057 0.79408   
 Residual 5.40310 2.32446   
Number of obs: 3100, groups: schoolID, 62  
  
Fixed effects:  
 Estimate Std. Error df t value Pr(>|t|)  
(Intercept) 6.07500 0.11333 67.72498 53.606 < 2.2e-16  
frlunch -1.18040 0.10239 3005.32542 -11.528 < 2.2e-16  
  
Correlation of Fixed Effects:  
 (Intr)  
frlunch -0.269

# Proportion explained of each variance component relative to empty model  
pseudoRSquaredinator(smallerModel=modelEmptyRI, largerModel=modelSmushed)

Pseudo R2 Estimates  
R2 Random.(Intercept): 0.39999  
R2 L1.sigma2: 0.03104

R2 Random.(Intercept) R2 L1.sigma2   
 0.399986894 0.031042888

As shown above, the slope for the new level-1 predictor is significantly negative, indicating a deficit for students who receive free/reduced lunch relative to those who don’t. However, the proportion of variance explained at each level gives us a clue that the model is mis-specified: The only way that a level-1 student predictor can explain level-2 school variance is through the implied level-2 predictor inside the level-1 predictor. The fact that there is only one slope for both implied parts of the predictor indicates the slope is “smushed” – it is a conflated effect that assumes the level-1 within-school and level-2 between-school slopes are of equal magnitude.

### Add Centered School Mean to Unsmush the Level-1 Slope for Free/Reduced Lunch

To fix the misspecification, we add a separate level-2 predictor for the centered school mean of free/reduced lunch:

$$ \begin{array}
ssumScore\_{pc} = \beta\_{0c} + \beta\_{1c}(frlunch\_{pc}) + e\_{pc} \\
\beta\_{0c} = \gamma\_{00} + \gamma\_{01}(SMfrlunch\_c - .30) + U\_{0c} \\
\beta\_{1c} = \gamma\_{10} \\
\end{array}
$$

# Add centered school mean to unsmush the level-1 slope for frlunch  
modelLunch = lmer(data=modelingData, REML=FALSE, sumScore~1+frlunch+SMfrlunch30+(1|schoolID))  
summary(modelLunch, ddf="Satterthwaite")

Linear mixed model fit by maximum likelihood . t-tests use Satterthwaite's method ['lmerModLmerTest']  
Formula: sumScore ~ 1 + frlunch + SMfrlunch30 + (1 | schoolID)  
 Data: modelingData  
  
 AIC BIC logLik -2\*log(L) df.resid   
 14096.7 14126.9 -7043.3 14086.7 3095   
  
Scaled residuals:   
 Min 1Q Median 3Q Max   
-2.987114 -0.728208 0.025203 0.760953 2.687391   
  
Random effects:  
 Groups Name Variance Std.Dev.  
 schoolID (Intercept) 0.18282 0.42758   
 Residual 5.40001 2.32379   
Number of obs: 3100, groups: schoolID, 62  
  
Fixed effects:  
 Estimate Std. Error df t value Pr(>|t|)  
(Intercept) 6.026853 0.075351 90.737526 79.9839 < 2.2e-16  
frlunch -1.042386 0.104699 3038.000001 -9.9561 < 2.2e-16  
SMfrlunch30 -3.125252 0.323165 77.372581 -9.6708 5.875e-15  
  
Correlation of Fixed Effects:  
 (Intr) frlnch  
frlunch -0.417   
SMfrlunch30 0.144 -0.324

# Compute full between level-2 effect  
contest1D(modelLunch, L=c(0,1,1))

Estimate Std. Error df t value Pr(>|t|)  
1 -4.1676375 0.30573522 61.999998 -13.631526 2.5720335e-20

# Proportion explained of each variance component relative to smushed model  
pseudoRSquaredinator(smallerModel=modelSmushed, largerModel=modelLunch)

Pseudo R2 Estimates  
R2 Random.(Intercept): 0.71006  
R2 L1.sigma2: 0.00057

R2 Random.(Intercept) R2 L1.sigma2   
 0.71006397170 0.00057162321

# Proportion explained of each variance component relative to empty model  
pseudoRSquaredinator(smallerModel=modelEmptyRI, largerModel=modelLunch)

Pseudo R2 Estimates  
R2 Random.(Intercept): 0.82603  
R2 L1.sigma2: 0.0316

R2 Random.(Intercept) R2 L1.sigma2   
 0.826034583 0.031596766

# Total R2 relative to empty model  
totalRSquaredinator(model=modelLunch, dvName="sumScore", data=modelingData)

Total R2 Estimate  
Total R2: 0.15758

total R2   
0.15757779

As shown above, the slope for the new level-2 predictor is significantly negative. It represents the contextual effect of free/reduced lunch: the incremental contribution of the school mean predictor after controlling for the student-level effect.

More specifically, the level-1 slope for frlunch (which is now slightly less negative after getting un-smushed by the contextual level-2 effect) now indicates the lunch-related difference between students who attend the same school. In contrast to the previous model, it is now purely a within-school effect that explained 3.16% of the level-1 residual variance representing within-school differences.

The level-2 contextual slope for SMfrlunch30 indicates the incremental effect of the proportion of students who receive free/reduced lunch on school mean sum scores. Given that the level-2 predictor is a proportion ranging from 0 to 1, the slope for its “one unit” change refers to the entire span of the variable (i.e., from 0 to 100% of students). So to make the slope more meaningful, we can divide it by 10: For every 10% more students receiving free/reduced lunch, school mean sum scores are lower by .313.

To get the full model-implied level-2 between effect, we can use contest1D to ask for the sum of the level-1 within-school and level-2 contextual slopes = + = -1.042 + -3.125 = -4.167. It explained 82.0% of the level-2 random intercept variance representing school mean differences.

In total, the model fixed effects explained 15.75% of the total variance in the observed sum scores.

### Switching to Cluster-Mean-Centered version of Level-1 Slope for Free/Reduced Lunch

To obtain the level-2 between effect directly as a model parameter, we can instead use the cluster-mean-centered version of the level-1 student free/reduced lunch predictor (keeping the constant-centered school mean lunch level-2 predictor):

$$ \begin{array}
ssumScore\_{pc} = \beta\_{0c} + \beta\_{1c}(frlunch\_{pc} - SMfrlunch\_{c}) + e\_{pc} \\
\beta\_{0c} = \gamma\_{00} + \gamma\_{01}(SMfrlunch\_c - .30) + U\_{0c} \\
\beta\_{1c} = \gamma\_{10} \\
\end{array}
$$

# Cluster-mean-centered version of modelLunch  
modelLunchCMC = lmer(data=modelingData, REML=FALSE, sumScore~1+WSfrlunch+SMfrlunch30+(1|schoolID))  
summary(modelLunchCMC, ddf="Satterthwaite")

Linear mixed model fit by maximum likelihood . t-tests use Satterthwaite's method ['lmerModLmerTest']  
Formula: sumScore ~ 1 + WSfrlunch + SMfrlunch30 + (1 | schoolID)  
 Data: modelingData  
  
 AIC BIC logLik -2\*log(L) df.resid   
 14096.7 14126.9 -7043.3 14086.7 3095   
  
Scaled residuals:   
 Min 1Q Median 3Q Max   
-2.987114 -0.728208 0.025203 0.760953 2.687391   
  
Random effects:  
 Groups Name Variance Std.Dev.  
 schoolID (Intercept) 0.18282 0.42758   
 Residual 5.40001 2.32379   
Number of obs: 3100, groups: schoolID, 62  
  
Fixed effects:  
 Estimate Std. Error df t value Pr(>|t|)  
(Intercept) 5.714138 0.068492 62.000000 83.4275 < 2.2e-16  
WSfrlunch -1.042386 0.104699 3038.000000 -9.9561 < 2.2e-16  
SMfrlunch30 -4.167637 0.305735 62.000000 -13.6315 < 2.2e-16  
  
Correlation of Fixed Effects:  
 (Intr) WSfrln  
WSfrlunch 0.000   
SMfrlunch30 0.010 0.000

# Compute contextual level-2 effect  
contest1D(modelLunchCMC, L=c(0,-1,1))

Estimate Std. Error df t value Pr(>|t|)  
1 -3.1252519 0.32316535 77.372583 -9.6707518 5.8754771e-15

# Total R2 relative to empty model  
totalRSquaredinator(model=modelLunchCMC, dvName="sumScore", data=modelingData)

Total R2 Estimate  
Total R2: 0.15758

total R2   
0.15757779

As shown above, the slope for the level-2 predictor is now the significantly negative level-2 between effect (found as a linear combination previously). It represents the between-school effect of free/reduced lunch: the total contribution of the school mean predictor without controlling for the student-level effect. The contextual level-2 effect can then be found as a linear combination of (between minus within). The within-school slope given by is exactly the same, despite the fact that it is multiplying the cluster-mean-centered version of the level-2 predictor.

### Multivariate MLM using Latent Centering, Pretending frlunch is Continuous

For our next model, we predict both student free/reduced lunch and student sum scores in a multivariate model. However, because lavaan cannot do multilevel models for categorical outcomes, we must pretend that frlunch is continuous.

$$

$$

# Multivariate MLM using Latent Centering, Still Pretending frlunch is Continuous  
MultivSyntax = "  
level: 1  
 # Level-1 residual variance for SumScore only  
 sumScore ~~ sumScore  
 # frlunch predicts sumScore: level-1 within-school slope  
 sumScore ~ (within)\*frlunch  
level: 2  
 # Fixed intercepts  
 sumScore ~ 1; frlunch ~ 1  
 # Level-2 random intercept variances  
 sumScore ~~ sumScore  
 frlunch ~~ frlunch  
 # frlunch predicts sumScore: level-2 between slope  
 sumScore ~ (between)\*frlunch  
 # Compute contextual level-2 effect  
 context := between - within  
"  
modelMultiv = lavaan(model=MultivSyntax, data=modelingData, cluster="schoolID",   
 mimic="mplus", std.lv=FALSE, estimator="ML")

Warning: lavaan->lav\_data\_full():   
 Level-1 variable "frlunch" has no variance within some clusters . The cluster ids with zero within variance are:   
 5, 8, 20, 33.

summary(object=modelMultiv)

lavaan 0.6-19 ended normally after 48 iterations  
  
 Estimator ML  
 Optimization method NLMINB  
 Number of model parameters 7  
  
 Number of observations 3100  
 Number of clusters [schoolID] 62  
 Number of missing patterns -- level 1 1  
  
Model Test User Model:  
   
 Test statistic 0.000  
 Degrees of freedom 0  
  
Parameter Estimates:  
  
 Standard errors Standard  
 Information Observed  
 Observed information based on Hessian  
  
  
Level 1 [within]:  
  
Regressions:  
 Estimate Std.Err z-value P(>|z|)  
 sumScore ~   
 frlunch (wthn) -1.042 0.105 -9.956 0.000  
  
Variances:  
 Estimate Std.Err z-value P(>|z|)  
 .sumScore 5.400 0.139 38.974 0.000  
  
  
Level 2 [schoolID]:  
  
Regressions:  
 Estimate Std.Err z-value P(>|z|)  
 sumScore ~   
 frlunch (btwn) -4.384 0.330 -13.301 0.000  
  
Intercepts:  
 Estimate Std.Err z-value P(>|z|)  
 .sumScore 7.029 0.120 58.661 0.000  
 frlunch 0.298 0.028 10.466 0.000  
  
Variances:  
 Estimate Std.Err z-value P(>|z|)  
 .sumScore 0.149 0.053 2.823 0.005  
 frlunch 0.047 0.009 5.208 0.000  
  
Defined Parameters:  
 Estimate Std.Err z-value P(>|z|)  
 context -3.341 0.348 -9.602 0.000

The results for the multivariate MLM are the same as for the univariate MLM with respect to level 1, but they differ at level 2 due to the use of latent centering. Rather than compute the observed school mean to use as the level-2 predictor (after centering it at .30), the multivariate model uses latent centering: It estimates a random intercept variance for free/reduced lunch to use as a level-2 predictor instead. Limitations of lavaan's multilevel modeling require us to pretend frlunch is continuous rather than model it properly as binary, however.

Consequently, the sumScore fixed intercept is now specifically for a student who does not receive free/reduced lunch in a school where no students receive free/reduced lunch (rather than in a school where 30% receive free/reduced lunch, as before). The level-2 between slope is a little stronger (-4.384 vs. -4.167) in the latent-centered multivariate model, as has often been found for these models in comparing the recovery of level-2 effects using observed vs. latent level-2 mean variables.

## Models with a Random Slope across Schools for Cluster-Mean-Centered Student Free/Reduced Lunch

Using the cluster-mean-centered version of the level-1 student free/reduced lunch predictor, we allow its slope to vary randomly across schools in the model below:

$$ \begin{array}
ssumScore\_{pc} = \beta\_{0c} + \beta\_{1c}(frlunch\_{pc} - SMfrlunch\_{c}) + e\_{pc} \\
\beta\_{0c} = \gamma\_{00} + \gamma\_{01}(SMfrlunch\_c - .30) + U\_{0c} \\
\beta\_{1c} = \gamma\_{10} + U\_{1c} \\
\end{array}
$$

# Add random level-1 slope for CMC frlunch  
modelRandSlope = lmer(data=modelingData, REML=FALSE,   
 sumScore~1+WSfrlunch+SMfrlunch30+(1+WSfrlunch|schoolID))  
summary(modelRandSlope, ddf="Satterthwaite")

Linear mixed model fit by maximum likelihood . t-tests use Satterthwaite's method ['lmerModLmerTest']  
Formula: sumScore ~ 1 + WSfrlunch + SMfrlunch30 + (1 + WSfrlunch | schoolID)  
 Data: modelingData  
  
 AIC BIC logLik -2\*log(L) df.resid   
 14099.1 14141.3 -7042.5 14085.1 3093   
  
Scaled residuals:   
 Min 1Q Median 3Q Max   
-2.99374 -0.73015 0.01660 0.76986 2.69446   
  
Random effects:  
 Groups Name Variance Std.Dev. Corr   
 schoolID (Intercept) 0.183080 0.42788   
 WSfrlunch 0.068056 0.26087 -0.570  
 Residual 5.388771 2.32137   
Number of obs: 3100, groups: schoolID, 62  
  
Fixed effects:  
 Estimate Std. Error df t value Pr(>|t|)  
(Intercept) 5.714185 0.068496 61.988448 83.4238 < 2.2e-16  
WSfrlunch -1.038722 0.110755 54.836271 -9.3785 0.0000000000005432  
SMfrlunch30 -4.146611 0.303534 62.363714 -13.6611 < 2.2e-16  
  
Correlation of Fixed Effects:  
 (Intr) WSfrln  
WSfrlunch -0.136   
SMfrlunch30 0.010 -0.048

# Likelihood ratio test for significance of random slope variance  
ranova(modelRandSlope)

ANOVA-like table for random-effects: Single term deletions  
  
Model:  
sumScore ~ WSfrlunch + SMfrlunch30 + (1 + WSfrlunch | schoolID)  
 npar logLik AIC LRT Df Pr(>Chisq)  
<none> 7 -7042.54 14099.1   
WSfrlunch in (1 + WSfrlunch | schoolID) 5 -7043.34 14096.7 1.60513 2 0.44818

As shown above, there is a new variance of the random slopes across schools (and the correlation of the random intercept and random slope across schools). The likelihood ratio test indicates the new slope variance is not significant, however, indicating that schools do not vary in the size of their within-school difference due to free/reduced lunch status.

For demonstration purposes, below we estimate a model adding a cross-level interaction between the two predictors, whose purpose is to predict why some schools have greater lunch-related deficits:

$$ \begin{array}
ssumScore\_{pc} = \beta\_{0c} + \beta\_{1c}(frlunch\_{pc} - SMfrlunch\_{c}) + e\_{pc} \\
\beta\_{0c} = \gamma\_{00} + \gamma\_{01}(SMfrlunch\_c - .30) + U\_{0c} \\
\beta\_{1c} = \gamma\_{10} + \gamma\_{11}(SMfrlunch\_c - .30) + U\_{1c} \\
\end{array}
$$

# Add cross-level interaction predicting random level-1 slope for CMC frlunch (demo purposes only)  
modelCrossLevel = lmer(data=modelingData, REML=FALSE,   
 sumScore~1+WSfrlunch+SMfrlunch30+WSfrlunch:SMfrlunch30+(1+WSfrlunch|schoolID))  
summary(modelCrossLevel, ddf="Satterthwaite")

Linear mixed model fit by maximum likelihood . t-tests use Satterthwaite's method ['lmerModLmerTest']  
Formula: sumScore ~ 1 + WSfrlunch + SMfrlunch30 + WSfrlunch:SMfrlunch30 + (1 + WSfrlunch | schoolID)  
 Data: modelingData  
  
 AIC BIC logLik -2\*log(L) df.resid   
 14100.7 14149.1 -7042.4 14084.7 3092   
  
Scaled residuals:   
 Min 1Q Median 3Q Max   
-2.996097 -0.728107 0.012894 0.769974 2.681867   
  
Random effects:  
 Groups Name Variance Std.Dev. Corr   
 schoolID (Intercept) 0.183051 0.42784   
 WSfrlunch 0.067516 0.25984 -0.559  
 Residual 5.388185 2.32125   
Number of obs: 3100, groups: schoolID, 62  
  
Fixed effects:  
 Estimate Std. Error df t value Pr(>|t|)  
(Intercept) 5.714138 0.068491 62.002403 83.4288 < 2.2e-16  
WSfrlunch -1.063158 0.118599 69.168157 -8.9643 0.0000000000003396  
SMfrlunch30 -4.167637 0.305730 62.002403 -13.6317 < 2.2e-16  
WSfrlunch:SMfrlunch30 0.319047 0.556577 69.939471 0.5732 0.5683  
  
Correlation of Fixed Effects:  
 (Intr) WSfrln SMfr30  
WSfrlunch -0.123   
SMfrlunch30 0.010 -0.001   
WSfrln:SM30 -0.001 -0.359 -0.117

As shown above, the cross-level interaction between the lunch predictors indicates that the within-school lunch slope is nonsignificantly less negative (smaller) in schools with a greater proportion of students who receive free/reduced lunch.