

Basics On Optimisation - COMP24112

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1. General Form of Optimisation

The problem of **optimisation** finds the input variables that can give the smallest (minimum) or largest (maximum) output value of a real-valued function. It has the following general form:

$$\begin{aligned} \min \quad & O(x_1, x_2, \dots, x_n), \\ \text{subject to} \quad & f_1(x_1, x_2, \dots, x_n) \leq 0, \\ & f_2(x_1, x_2, \dots, x_n) \leq 0, \\ & \vdots \\ & f_m(x_1, x_2, \dots, x_n) \leq 0. \end{aligned} \tag{1}$$

The real-valued function $O(x_1, x_2, \dots, x_n)$ that takes n real-valued variables as the input is called the **optimisation objective function**. The different real-valued functions $f_i(x_1, x_2, \dots, x_n) \leq 0$ ($i = 1, 2, \dots, m$) are called the **constraints**. They restrict the sets from which the input variables are allowed to choose their values. If all the input variables of the objective function $O(x_1, x_2, \dots, x_n)$ are allowed to be chosen from the set of all real numbers such that $x_i \in R$ for $i = 1, 2, \dots, n$, an **unconstrained optimisation** problem is to be solved, simply written as

$$\min O(x_1, x_2, \dots, x_n). \tag{2}$$

2. Notations

Storing the n input variables in a vector such as $\mathbf{x} = [x_1, x_2, \dots, x_n]$, the optimisation problem in Eq. (1) can be expressed as

$$\begin{aligned} \min \quad & O(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \leq 0, i = 1, 2, \dots, m. \end{aligned} \tag{3}$$

This notation can also be simplified as

$$\min_{f_i(\mathbf{x}) \leq 0, i=1,2,\dots,m} O(\mathbf{x}).$$

The unconstrained optimisation in Eq. (2) is simplified as $\min O(\mathbf{x})$.

3. Example

We look at the example of finding the maximum of the function $(x+1)^2 \sin(y)$, where the input x is allowed to be chosen from the set of real numbers between 0 and 3, expressed as $x \in [0, 5]$ or $0 \leq x \leq 5$. The input y is allowed to be chosen from the set of real numbers between 0 and 3, expressed as $y \in [0, 3]$ or $0 \leq y \leq 3$.

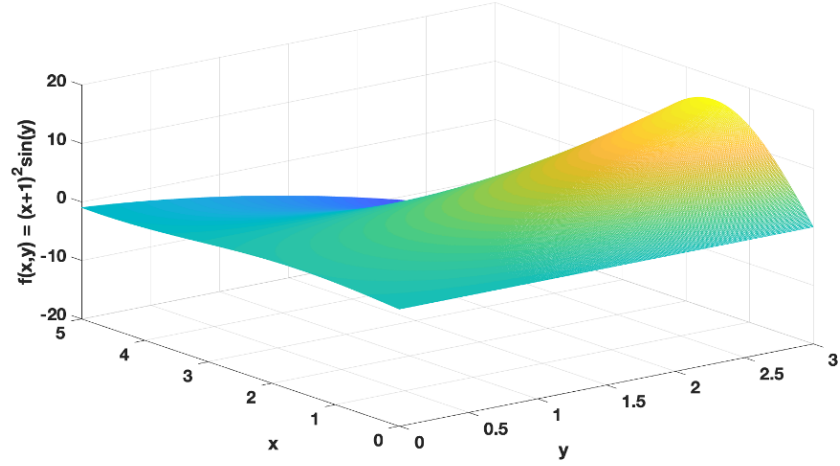


Figure 1: Example function plot for $f(x, y) = (x+1)^2 \sin(y)$ where $0 \leq x \leq 5$ and $0 \leq y \leq 3$.

To formulate it as an optimisation problem, the objective function is set as

$$O(x, y) = (x+1)^2 \sin(y).$$

The input x and y are restricted to the two sets $[0, 5]$ and $[0, 3]$, which can be converted to four constraint functions:

$$\begin{aligned} -x &\leq 0, \\ x - 5 &\leq 0, \\ -y &\leq 0, \\ y - 3 &\leq 0. \end{aligned}$$

Together, it gives the following general-form expression:

$$\begin{aligned} &\max (x+1)^2 \sin(y), \\ &\begin{matrix} -x \leq 0 \\ x - 5 \leq 0 \\ -y \leq 0 \\ y - 3 \leq 0 \end{matrix} \end{aligned}$$

or

$$\begin{aligned} &\min -(x+1)^2 \sin(y). \\ &\begin{matrix} -x \leq 0 \\ x - 5 \leq 0 \\ -y \leq 0 \\ y - 3 \leq 0 \end{matrix} \end{aligned}$$

Usually, you can just simply write it as

$$\max_{\substack{x \in [0,5] \\ y \in [0,3]}} (x+1)^2 \sin(y),$$

and

$$\max_{\substack{0 \leq x \leq 5 \\ 0 \leq y \leq 3}} (x+1)^2 \sin(y),$$