CSC373 Tutorial 1
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Master Theorem

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

1) $f(n) \leq n^{\log_b a - \epsilon}$, then $T(n) = \theta\left(n^{\log_b a}\right)$
2) $f(n) = \theta(n^{\log_b a})$, then $T(n) = \theta\left(n^{\log_b a} \log n\right)$
   2*) $f(n) = \theta\left(n^{\log_b k} \log^k n\right)$, then $T(n) = \theta\left(n^{\log_b a} \log^{k+1} n\right)$
3) $f(n) \geq n^{\log_b a + \epsilon}$, then $T(n) = \theta\left(f(n)\right)$

1)
A:

$$\begin{aligned} T(n) &= aT\left(\frac{n}{b}\right) = a^{\log_b n} \\ &= a\left(aT\left(\frac{n}{b^2}\right)\right) \\ &= a^3 T\left(\frac{n}{b^3}\right) \\ &\vdots \\ &= a^i T\left(\frac{n}{b^i}\right) \end{aligned}$$

$$i = \log_b n$$

Note: $a^{\log_b n} = \left(b^{\log_b a}\right)^{\log_b a}$

$$\begin{aligned} &= b^{\log_b a \log_b n} \\ &= \left(b^{\log_b n}\right)^{\log_b a} \\ &= n^{\log_b a} \end{aligned}$$

B:

$$\begin{aligned} T(n) &= aT\left(\frac{n}{b}\right) + f(n) \\ &= af\left(\frac{n}{b}\right) + f(n) + a^2 T\left(\frac{n}{b^2}\right) \\ &= a^2 f\left(\frac{n}{b^2}\right) + af\left(\frac{n}{b}\right) + f(n) + \cdots \\ &= \sum_{i=0}^{\log_b n} a^i f\left(\frac{n}{b^i}\right) \end{aligned}$$

2)

$$\begin{aligned} &= \sum_{i=0}^{\log_b n} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \\ &= \sum_{i=0}^{\log_b n} \frac{a^i}{\left(b^i\right)^{\log_b a}} n^{\log_b a} \end{aligned}$$

$= \sum_{i=0}^{\log_b n} n^{\log_b a} = n^{\log_b a} \log_b n$

3) $= \sum_{i=0}^{\log_b n} a^i \left(\frac{n}{b^i}\right)^{\log_b a+\epsilon}$

$= \sum_{i=0}^{\log_b n} \frac{a^i}{\left((b^i)^{\log_b a}\right)(b^i)^{\epsilon}} n^{\log_b a+\epsilon}$

$\leq n^{\log_b a+\epsilon} \sum_{i=0}^{\infty} c^i \qquad c < 1$

$= \theta\left(n^{\log_b a+\epsilon}\right) = \theta\left(f(n)\right)$

Practice Problems

for any $\epsilon > 0$

base case $T(1) = 1$

$T(n) =$ a) $2T\left(\frac{n}{2}\right) + n^4 = n^4$

b) $T\left(\frac{7n}{10}\right) + n = n$

c) $2T\left(\frac{n}{4}\right) + \sqrt{n} = \sqrt{n}\log n$

d) $T(n-2) + n^2 = n^3$

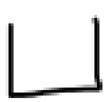
e) $4T\left(\frac{n}{3}\right) + n\log n = n^{\log_3 4}$

f) $16T\left(\frac{n}{4}\right) + n^2\log n = n^2\log^2 n$

g) $4T\left(\frac{n}{2}\right) + n^2\sqrt{n} = n^{2.5}$

h*) $2T\left(\sqrt{n}\right) + \log n = \log n \log\log n$

Knapsack Problem



40lbs

- 1) 10lbs \$60
- 2) 20lbs \$100

1/3 - 3) 30lbs \$120

Caching