Finite Dimensional FRI

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- Traditional Finite Rate of Innovation (FRI) theory considers the problem of sampling continuous-time signals.
- This framework can be naturally extended to the discrete-time case.
- We present a novel approach based on the traditional FRI sampling scheme that takes advantage of the fact that the null space of the problem is finite dimensional.

Introduction

- FRI sampling theory [1, 2] has shown that it is possible to sample and reconstruct classes of non-bandlimited signals.
- FRI methods are based on the fact that the Fourier transform of a sum of Diracs is given by a sum of exponentials. The reconstruction is then based on estimating exponentials from a sequence of samples, which is a classical problem in spectral estimation [3].
- Sampling of discrete-time signals process can be modelled with a matrix multiplication.
  - The input signal is given by a high dimensional vector with few non-zero elements.
  - The acquired signal is a vector of lower dimension which is given by the product of a fat matrix with the input signal.
- The goal is to reconstruct the sparse input vector from the acquired samples.

Traditional FRI

- Let $\boldsymbol{x} \in \mathbb{C}^N$ be a discrete time signal formed by a stream of K Diracs:

$$x[n] = \sum_{k=1}^{K} a_k \, \delta[n - n_k], \qquad n = 0, 1, \ldots, N-1. \tag{1}$$

- The signal $\boldsymbol{x}$ has $2K$ degrees of freedom:
  - Delays $n_k \in \{0, 1, \ldots, N-1\}$, for $k = 1, \ldots, K$,
  - Amplitudes $a_k \in \mathbb{C} \setminus \{0\}$, for $k = 1, \ldots, K$.
- We have access to $M < N$ coefficients of the DFT of $\boldsymbol{x}$.
- We can express the sampling process in matricial form as follows

$$\boldsymbol{y} = \boldsymbol{D}\,\boldsymbol{x}, \tag{2}$$

where $\boldsymbol{y} \in \mathbb{C}^M$ are the available samples and $\boldsymbol{D} \in \mathbb{C}^{M \times N}$ is a partial Fourier matrix: $(\boldsymbol{D})_{m,n} = \exp(-j2\pi mn/N)/\sqrt{N}$, with $m = 0, \ldots, M-1$ and $n = 0, \ldots, N-1$.

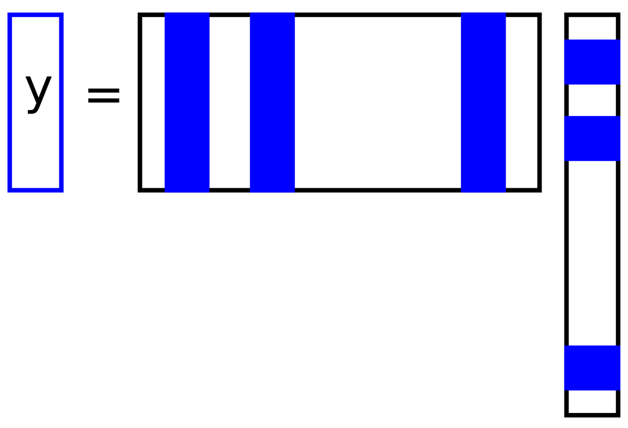


Figure 1: Vector $\boldsymbol{y}$ is the result of multiplying a fat matrix by a high dimensional sparse vector.

- The unitary DFT of $\boldsymbol{x}$ consists of the sum of K exponentials with frequencies $\omega_k = 2\pi n_k/N$:

$$\hat{x}[m] = \frac{1}{\sqrt{N}} \sum_{k=1}^{K} a_k \, e^{-j\omega_k m}, \qquad m = 0, 1, \ldots, N-1.$$

Annihilating filter method

- Sequence $\hat{x}[m]$ is annihilated by a $K+1$ taps filter $h[m]$: $(\hat{x} * h)[m] = 0$.
- Filter with zeros at $z = e^{-j\omega_k}$: $H(z) = \prod_{k=1}^{K}\left(1 - e^{-j\omega_k} z^{-1}\right) = 1 + \sum_{m=1}^{K} h[m]\, z^{-m}$.
- The method is based on finding the $h[m]$ coefficients, and estimating the frequencies ω_k from the roots of $H(z)$. Coefficients $h[m]$ are obtained by building a Toeplitz matrix and establishing the following system:

$$\begin{bmatrix} \hat{x}[K] & \hat{x}[K-1] & \ldots & \hat{x}[0] \\ \hat{x}[K+1] & \hat{x}[K] & \ldots & \hat{x}[1] \\ \vdots & \vdots & \ddots & \vdots \\ \hat{x}[2K] & \hat{x}[2K-1] & \ldots & \hat{x}[K] \end{bmatrix} \cdot \begin{bmatrix} 1 \\ h[1] \\ \vdots \\ h[K] \end{bmatrix} = 0 \quad \Longrightarrow \quad \boldsymbol{S}\,\boldsymbol{h} = \boldsymbol{0}.$$

- We can recover the parameters a_k and n_k from $2K$ samples of $\hat{x}[m]$. The problem is:
  - Nonlinear in ω_k.
  - Linear in a_k.

Finite dimensional FRI: new approach

- Avoid the root finding step and recover the K-sparse vector by solving two linear systems.
- We can reconstruct $\boldsymbol{x}$ from $\boldsymbol{y}$ applying the pseudoinverse of $\boldsymbol{D}$, up to its null space:

$$\boldsymbol{x} = \boldsymbol{D}^H \boldsymbol{y} + \sum_{l=1}^{L} \beta_l \, \boldsymbol{n}_l,$$

where β_l are unknown coefficients, $L = N - M$ is the size of the null space and $\boldsymbol{n}_l$ are L vectors that span the null space of $\boldsymbol{D}$: $\boldsymbol{n}_l \in N(\boldsymbol{D})$ where $N(\boldsymbol{D}) = \{\boldsymbol{n} \in \mathbb{C}^M \,|\, \boldsymbol{D}\,\boldsymbol{n} = \boldsymbol{0}\}$.

- If we premultiply the previous equation by $\boldsymbol{F}_N$ we obtain the Fourier transform of $\boldsymbol{x}$:

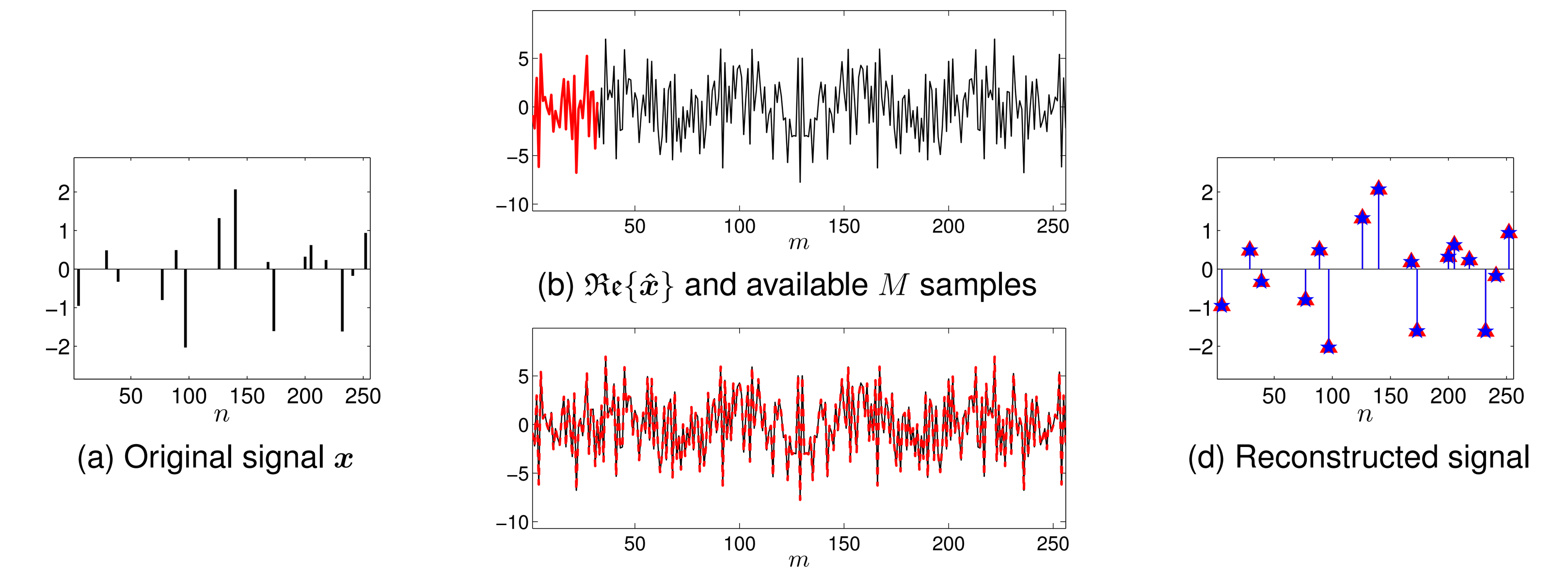
$$\hat{\boldsymbol{x}} = \boldsymbol{F}_N\,\boldsymbol{x} = \boldsymbol{z} + \sum_{l=1}^{L} \beta_l \, \boldsymbol{F}_N\,\boldsymbol{n}_l \quad \overset{\text{Annihilating filter}}{\Longrightarrow} \quad \left(\boldsymbol{Z} + \sum_{l=1}^{L} \beta_l \boldsymbol{E}_{M+l}\right) \boldsymbol{h} = \boldsymbol{0}.$$

- By building the Toeplitz matrices and assuming the annihilating filter $\boldsymbol{h}$ is known, we can establish a determined linear system to find the coefficients β_l:

$$\left[\boldsymbol{E}_{M+1}\,\boldsymbol{h}\ \ldots\ \boldsymbol{E}_N\,\boldsymbol{h}\right] \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_L \end{bmatrix} = -\boldsymbol{Z}\,\boldsymbol{h}.$$

- The annihilating filter can be obtained from $2K$ elements of vector $\boldsymbol{y}$.
- The coefficients β_l, $l = 1, \ldots, L$, correspond to the missing part of $\hat{\boldsymbol{x}}$. We can thus reconstruct $\boldsymbol{x}$ by applying the inverse Fourier transform.

Proposition 1. Let $\boldsymbol{x} \in \mathbb{C}^N$ and $\boldsymbol{y} \in \mathbb{C}^M$ be given as in (1) and (2) respectively. If $M \geq 2K$, the solution to (2) is unique and can be found by solving two linear inversions.



(a) Original signal $\boldsymbol{x}$

(b) $\mathfrak{Re}\{\hat{\boldsymbol{x}}\}$ and available M samples

(c) $\mathfrak{Re}\{\hat{\boldsymbol{x}}\}$ and extrapolated coefficients

(d) Reconstructed signal

Figure 2: $N = 256$, $M = 32$ and $K = 16$. (a) Signal with $K = 16$ Diracs. (b) Real part of the Fourier coefficients in black and available samples in red. (c) Extrapolated coefficients in red. (d) Perfect reconstruction of the signal, in red the original signal and in blue the reconstruction.

Noisy case algorithm and Simulations

- In the presence of noise, the reconstruction algorithm is based on solving two Total Least Squares problems.

Input: Samples $\boldsymbol{y}$, number of Diracs K

Output: Dirac locations and amplitudes: $\{(n_k, a_k)\}_{k=1}^{K}$

1. Denoise samples $\boldsymbol{y}$ with Cadzow algorithm
2. Compute the annihilating filter: $\quad \boldsymbol{h} = \arg\min_{\boldsymbol{h}} \|\boldsymbol{Y}^{toe}\,\boldsymbol{h}\|_2^2$
3. Estimate $\hat{\boldsymbol{x}} = [\boldsymbol{y} \quad \boldsymbol{\beta}]^T$,
   where $\boldsymbol{\beta} = \arg\min_{\boldsymbol{\beta}} \left\|\left[\boldsymbol{Z}\,\boldsymbol{h}\ \boldsymbol{E}_{M+1}\,\boldsymbol{h}\ \ldots\ \boldsymbol{E}_N\,\boldsymbol{h}\right] \begin{bmatrix} 1 \\ \boldsymbol{\beta} \end{bmatrix}\right\|_2^2$
4. Esimate locations n_k from K largest local maxima of IDFT$\{\hat{\boldsymbol{x}}\}$
5. Estimate amplitudes solving a least squares problem from (2) selecting K columns

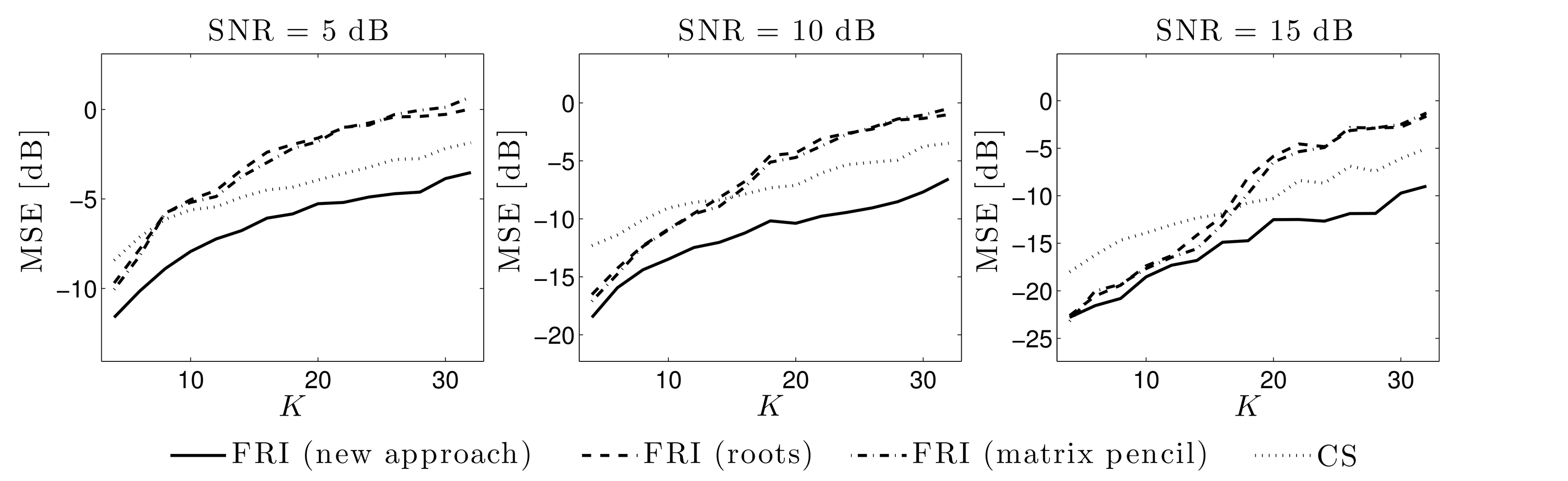


Figure 3: Simulation results showing new finite dimensional FRI approach outperforming traditional FRI methods (root finding of the annihilating filter or matrix pencil) and compressed sensing reconstruction. $N = 256$ and $M = 64$. Simulations performed at different levels of noise (SNR of 5, 10 and 15 dB) and different levels of sparsity K (horizontal axis). The vertical axis shows the normalized average value of the MSE of the reconstructed sparse vector compared to the true $\boldsymbol{x}$.

Conclusions

- Novel method to reconsctruct a finite dimensional sparse vector from partial knowledge of its discrete Fourier transform.
- In the noiseless scenario, perfect reconstruction is achieved with the critical number of samples.
- In the noisy case, this method is more stable and outperforms traditional FRI approaches and CS because it takes advantage of the fact that the null space of the underdetermined system is finite dimensional.

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